Executive Summary

This week's portfolio optimization initiative focused on advancing the analytical framework for **option pricing and market modeling** beyond the traditional **Black–Scholes–Merton (BSM)** paradigm.
The objective was to enhance pricing accuracy and market realism by integrating **stochastic volatility**, **jump-diffusion**, and **numerical PDE** techniques into the modeling architecture.

Key highlights from Days 50 to 54 include:

- **Model Evolution:** Transition from the constant-volatility BSM model to **Heston Stochastic Volatility** and **Merton Jump Diffusion** frameworks to address volatility smiles, fat tails, and jump risks.

- **Numerical Validation:** Implementation and convergence testing using **Monte Carlo (MC)** and **Finite Difference (FD)** methods to ensure model robustness and computational accuracy.

- **Instability Diagnostics:** Identification of numerical divergence in the FD scheme and development of refinements in grid design, boundary conditions, and time-stepping stability.

- **Risk Premium Insights:** Quantification of **jump risk premiums** and their impact on call and put prices, highlighting the real-world deviations from theoretical baselines.

- **Integrated Framework:** Development of a unified pricing comparison across **BSM, FD, MC GBM, Heston, and Jump Diffusion** methods, providing a benchmark for model behavior under various volatility and market regimes.

The week's work establishes a strong foundation for **multi-model calibration**, **scenario testing**, and **risk-adjusted performance assessment**, paving the way for more resilient and adaptive portfolio strategies in subsequent phases.

Weekly Portfolio Management and Optimization Report (Days 50–54)

Overview

This report summarizes the portfolio management and optimization activities conducted over **Days 50 to 54**.
The week's focus was to move beyond the classical **Black–Scholes–Merton (BSM)** framework by implementing **advanced stochastic and numerical models** for more realistic option pricing and market analysis.

Day 50: Limitations of BSM and the Necessity of Stochastic Volatility (SV) Models

Day 50 established the **Black–Scholes–Merton (BSM)** model as the foundational analytical model for option pricing.
Its primary assumption of constant volatility (σ) was critically examined, revealing several practical shortcomings:

- **Volatility Smile/Skew:** Market-implied volatility varies with strike price and maturity, forming the volatility smile or skew.
- **Return Distribution:** BSM assumes lognormal returns, while real markets exhibit leptokurtic (fat-tailed) distributions.
- **Dynamic Effects:** The model ignores volatility clustering and the leverage effect, where volatility rises as prices fall.
- **Jumps:** BSM assumes continuous paths and cannot capture sudden jumps or shocks in market prices.

These limitations motivated the transition to **Stochastic Volatility (SV)** models, where volatility itself is modeled as a **dynamic, random, and mean-reverting process** correlated with the asset.
The **Heston model (1993)** was emphasized as a canonical framework capable of capturing volatility clustering and realistic smile behavior while retaining analytical tractability.

Day 51: Heston Stochastic Volatility Model and Numerical Convergence

Day 51 focused on the **implementation and validation** of the Heston model through **Monte Carlo (MC)** simulations.

The model defines asset prices and their variances using **Stochastic Differential Equations (SDEs)**. Key parameters include:

- Mean reversion speed (κ)
- Long-term variance (θ)
- Volatility of volatility (ξ)
- Correlation (ρ), typically negative to capture the leverage effect

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Simulation results for the **M&M ticker** confirmed **numerical convergence**: as grid resolution increased (M=N=100, 200, 400, 800), **Heston MC prices approached analytical BSM prices**.
This validated both the **accuracy and stability** of the numerical implementation and showed that the Heston model allows greater flexibility in capturing market-implied volatility structures.

Day 52: Finite Difference (PDE Pricing) and Instability Analysis

Day 52 implemented the **Finite Difference (FD)** method using the **Crank–Nicolson scheme** with **Rannacher smoothing** to numerically solve the Black–Scholes PDE.
This approach is essential for pricing options with complex features such as **American-style exercise** or **variable volatility regimes**, where no closed-form solution exists.

Testing against BSM benchmarks for the **M&M ticker** revealed **numerical instability**.
Errors **increased with grid refinement** — the mean absolute error for calls rose from **8.435 at M100 to 97.459 at M800**, producing **negative convergence orders** (e.g., -1.1559 for calls).
This highlighted issues with **boundary conditions**, **domain truncation (Smax)**, and **time-stepping ratios**, necessitating further refinement of the FD numerical engine.

Day 53: Jump Diffusion Models and Jump Risk Premium

Day 53 introduced the **Merton Jump Diffusion (MJD)** model, which augments standard diffusion dynamics with a **Poisson jump process** to model discontinuous price movements and fat-tailed return distributions.

Model calibration was performed by minimizing the **Mean Squared Error (MSE)** between model prices and observed market data.

Results for the **M&M ticker** showed that **MJD prices exceeded BSM values**, with the largest difference of **+2.16 for the 3700 call**, reflecting the **jump risk premium**.

Sensitivity analyses demonstrated how option prices respond to changes in:

- Jump frequency (λ)
- Mean jump size (μ□)
- Jump volatility (σ□)

These analyses confirmed the MJD model's ability to capture **event-driven risks** and **tail behaviors** observed in real markets.

Day 54: Unified Stochastic Models Framework and Comparative Analysis

Day 54 consolidated the week's work into a **comparative analytical framework** encompassing five pricing approaches:
BSM, Finite Difference (FD), Monte Carlo GBM, Monte Carlo Heston, and Monte Carlo Jump Diffusion.

Comparative Findings:

- **Finite Difference (FD):** Stable FD implementations align closely with BSM (typically <0.3% deviation).
- **Monte Carlo GBM:** Slightly lower prices than BSM due to path sampling variance.
- **Monte Carlo Heston:** Produces lower values reflecting stochastic volatility and mean reversion.
- **Monte Carlo Jump Diffusion:** Generates higher call prices (+14.44 for the 3600 call) and marginally lower put prices, capturing tail risk premiums.

Conclusion

Each model serves a distinct analytical purpose:

- **BSM and FD** are ideal for **benchmarking and accuracy validation**.
- **Heston and Jump Diffusion** models are well-suited for **stress testing and scenario analysis**.

Collectively, these models provide a **multi-layered understanding of market efficiency**, volatility structures, and the complex dynamics underlying option pricing.