Version 1.0

Mathematical Basis

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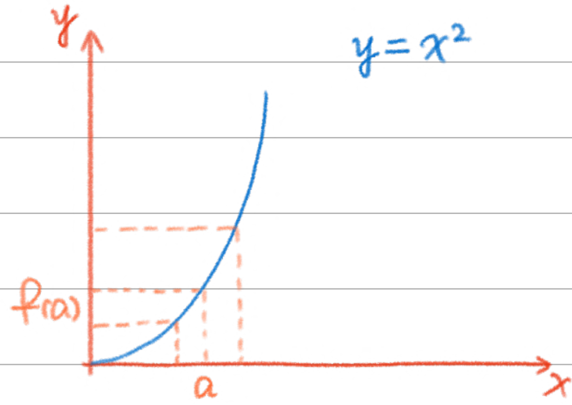
Lecture 0a Introduction to Calculus I

I Derivative

1 limit example $f(x) = x^2$



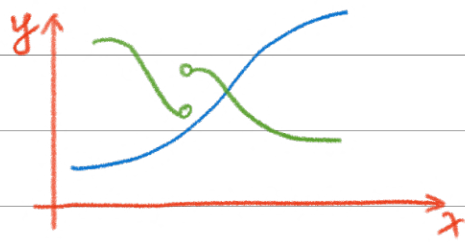
when $1 < x < 5 \Rightarrow 1 < y < 25$

$2 < x < 4 \Rightarrow 2 < y < 16$

$\vdots$

$x \to 3 \Rightarrow y = 9$

we mark $\lim_{x\to a} f(x) = L$



Blue : Continuous

Green : Piecewise continuous

If the function $f(x)$ is defined and continuous at its domain

we write $\lim_{x\to a} f(x) = f(a)$

If the function $f(x)$ is not defined or continuous at $x=a$

Such as $f(\theta) = \frac{\sin\theta}{\theta}$ is not defined at $\theta = 0$

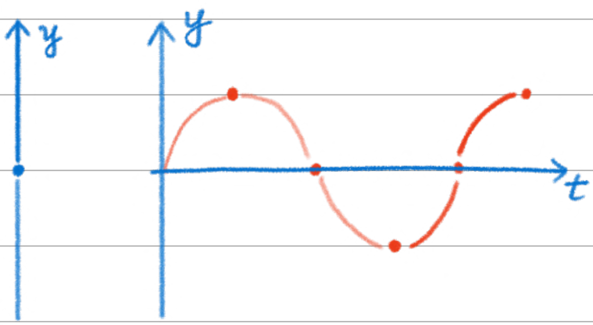
$g(x) = \frac{x^2-1}{x-1}$ is not defined at $x=1$

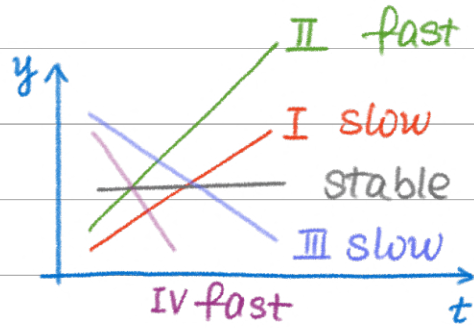
Then, we may fined other way to find its limit.

Example $\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1)$

$= 2$

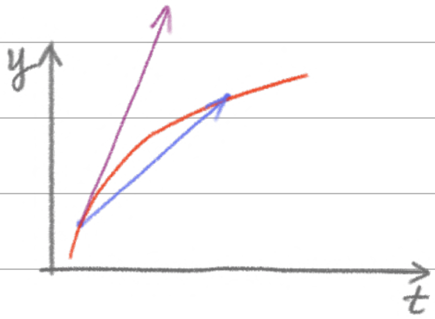
Speed





Average speed $\langle v \rangle := \dfrac{y_2 - y_1}{t_2 - t_1}$ (Slop)

Instant speed $v := \lim\limits_{\Delta t \to 0} \dfrac{\Delta y}{\Delta t}$



For example $y = kt^2$

$$\Delta y = y(t+\Delta t) - y(t) = k(t+\Delta t)^2 - kt^2$$
$$= k(2t\Delta t + \Delta t^2)$$
$$v = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{k(2t\Delta t + \Delta t^2)}{\Delta t}$$
$$= 2kt$$

If $y = \quad kt^2, \quad kt^n$

$v = \dfrac{dy}{dt} = 2kt, \quad nkt^{n-1}$

Derivative of Trigonometric function

$\dfrac{d}{d\theta}\sin\theta = \cos\theta \qquad \dfrac{d}{d\theta}\cos\theta = -\sin\theta$

Derivative: $\dfrac{d}{dt}y(t) = \lim\limits_{\Delta t \to 0} \dfrac{\Delta y}{\Delta t}$

Given $u(t)$, $v(t)$, $w(u(t))$, we have

$$\frac{d}{dt}(u+v) = \frac{d}{dt}u + \frac{d}{dt}v$$
$$\frac{d}{dt}(u \cdot v) = u\frac{d}{dt}v + v\frac{d}{dt}u$$
$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{d}{dt}\left(u\frac{1}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dt} - u\frac{d}{dt}v\right)$$
$$\frac{d}{dt}w = \frac{dw}{du}\frac{du}{dt}$$ Chain Rule

y	$\frac{d}{dt}y$
kt^n	nkt^{n-1}
k	0
kt	k
kt^2	$2kt$
$kt^{\frac{1}{2}}$	$\frac{1}{2}kt^{-\frac{1}{2}}$
a^x	$a^x \ln(a)$
e^x	e^x
$\ln(t)$	$\frac{1}{x}$
$\sin\theta$	$\cos\theta$
$\cos\theta$	$-\sin\theta$
$\tan\theta$	$\sec^2\theta$
$\cot\theta$	$-\csc^2\theta$
$\ln\lvert\sec\theta\rvert$	$\tan\theta$
$\ln\lvert\sin\theta\rvert$	$\cot\theta$

example

1° $\frac{d}{dt}(t+t^2)^2 = 2(t+t^2)\frac{d}{dt}(t+t^2) = 2(t+t^2)(1+2t)$

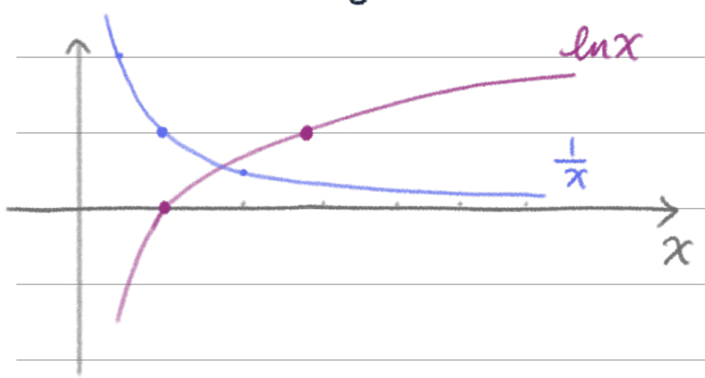
2° $\frac{d}{dt}\sin(t+t^2) = \cos(t+t^2)\frac{d}{dt}(t+t^2) = \cos(t+t^2)(1+2t)$

3° $\frac{d}{dt}\cos[\sin(t+t^2)] = -\sin[\sin(t+t^2)]\cos(t+t^2)(1+2t)$

Exponential $e \approx 2.71828$

$\log x$: $10^? = x$ $\quad \log_{10}10 = 1$, $\log_{10}100 = 2$, $\log_{10}1000 = 3$

$\log_e x = \frac{\log_{10} x}{\log_{10} e} := \ln x$ $\quad e^? = x$



$\frac{d}{dx}\ln x = \frac{1}{x}$

Now $y = a^x$, we have $\ln(y) = \ln(a^x) = x\ln(a)$

Then: $\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x\ln(a))$

$\frac{1}{y}\frac{d}{dx}y = \ln(a)$

$\frac{d}{dx}y = y\ln(a)$ $\qquad \frac{d}{dx}y = a^x \cdot \ln(a)$

Extrema $f(x_0)$: $\frac{d}{dx}f(x_0) = 0$

Maximum $\frac{d^2}{dx^2}f(x_0) < 0$

Minimum $\frac{d^2}{dx^2}f(x_0) > 0$

Taylor's expension

at $x=x_0$:

$$f(x) = f(x_0) + \frac{x-x_0}{1} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \cdots$$

at $x=0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

for $\sin(\theta)$

$$\sin(\theta) = \sin(\theta_0) + (\theta-\theta_0)\cos(\theta_0) - \frac{(\theta-\theta_0)^2}{2!}\sin(\theta_0) - \frac{(\theta-\theta_0)^3}{3!}\cos(\theta_0) + \cdots$$

$$\cos(\theta) = \cos(\theta_0) - (\theta-\theta_0)\sin(\theta_0) - \frac{(\theta-\theta_0)^2}{2!}\cos(\theta_0) + \frac{(\theta-\theta_0)^3}{3!}\sin(\theta_0) + \cdots$$

at $\theta_0 = 0$, $\sin(0)=0$, $\cos(0)=1$

$$\sin(\theta) = 0 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + (-1)^n \frac{\theta^{2n}}{(2n)!} + \cdots$$

exercise

1° $\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x} = \frac{d}{d\sin x}\frac{1}{\sin x}\frac{d\sin x}{dx} = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x$

2° $\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{d}{d\cos x}\frac{1}{\cos x}\frac{d\cos x}{dx} = \sec x \tan x$

* cosine rule

For a triangle with sides a, b, c and the opposite angle γ, we have

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

* Sine rule

For a triangle with sides a, b, c opposite angles α, β, γ, we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

Lecture 06 Introduction to Calculus II

The reverse operation of derivative is called an integral

Example :

1° $\frac{dF(x)}{dx} = x^2$, $F(x) = \frac{1}{3}x^3 + C$, where C is a const

2° $\frac{dG(x)}{dx} = x + x^2$, $G(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$

$\frac{dF(x)}{dx} = f(x) \Rightarrow F(x) = \int f(x)dx$

$\frac{dG(x)}{dx} = g(x) \Rightarrow G(x) = \int g(x)dx$

$\frac{d}{dx}f(x) = f'(x)$	$df(x) = f'(x)dx$	$\int f'(x)dx = f(x) + c$
$\frac{d}{dx}\sin x = \cos x$	$d\sin x = \cos x\,dx$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}\cos x = -\sin x$	$d\cos x = -\sin x\,dx$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}\tan x = \sec^2 x$	$d\tan x = \sec^2 x\,dx$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}\cot x = -\csc^2 x$	$d\cot x = -\csc^2 x\,dx$	$-\int \csc^2 x\,dx = \cot x + c$
$\frac{d}{dx}\sec x = \sec x\tan x$	$d\sec x = \sec x\tan x\,dx$	$\int \sec x\tan x\,dx = \sec x\tan x + c$
$\frac{d}{dx}\csc x = -\csc x\cot x$	$d\csc x = -\csc x\cot x\,dx$	$\int \csc x\cot x\,dx = -\csc x + c$
$\frac{d}{dx}c = 0$	$dc = 0\cdot dx$	$\int 0\,dx = c$
$\frac{d}{dx}x^n = x^{n-1}$	$dx^n = x^{n-1}dx$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ $(n \neq -1)$
		$\int x^{n-1}dx = \frac{1}{n}x^n + c$ $(n \neq 0)$
$\frac{d}{dx}e^x = e^x$	$de^x = e^x dx$	$\int e^x dx = e^x + c$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$d\ln x = \frac{1}{x}dx$	$\int \frac{1}{x}dx = \ln\|x\| + c$

$\int \tan x\,dx = \ln|\sec x| + c$

$\int \cot x\,dx = \ln|\sin x| + c$

Let $u = u(x)$, $v = v(x)$

$\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v$ $\quad \int (u+v)dx = \int u dx + \int v dx$

$\frac{d}{dx}[\alpha \cdot u(x)] = \alpha \frac{d}{dx} u$ $\quad \int \alpha u(x) dx = a \int u dx$

example:

1° $\int a dx = \int a x^0 dx = a \int x^0 dx = ax + c$

2° $\int ax dx = \frac{a}{2} x^2 + c$

3° $\int (ax^2 + bx + d) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + dx + c \Rightarrow \int ax^n dx = a \frac{1}{n+1} x^{n+1} + c$

4° $\int \sin^3 x dx = \int \sin^2 x \cdot \sin x dx$, let $u(x) = \cos x$, $du(x) = -\sin x dx$.

$\int \sin^3 x dx = -\int \sin^2 x \, d\cos x = -\int (1-\cos^2 x) d\cos x = -\int (1-u^2) du = -u + \frac{1}{3}u^3 + c$

$= -\cos x + \frac{1}{3}\cos^3 x + c$

5° $\int \sin x \cos x dx = \int \sin x \, d\sin x = \frac{1}{2}\sin^2 x + c$

6° $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x} d\cos x = -\ln(\cos x) + c$

7° $\int x^3 \sin(x^4+2) dx = \frac{1}{4}\int \sin(x^4+2) dx^4 = \frac{1}{4}\int \sin(x^4+2) d(x^4+2)$

$= -\frac{1}{4}\cos(x^4+2) + c$

8° $\int \frac{\ln x}{x} dx = \int \ln x \, d(\ln x) = \frac{1}{2}(\ln x)^2 + c$

9° $\int \frac{x}{\sqrt{x-1}} dx$ we define $u(x) := x-1$, $dx = du$

$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u+2}{\sqrt{u}} du = \int u^{\frac{1}{2}} du + 2\int u^{-\frac{1}{2}} du$

$= \frac{2}{3} u^{\frac{3}{2}} + 4u^{\frac{1}{2}} + c$

10° $\int \csc x\,dx = \int \frac{\csc x(\csc x+\cot x)}{\csc x+\cot x}dx = \int \frac{\csc^2 x+\csc x\cot x}{\csc x+\cot x}dx$

let $u=\csc x+\cot x$, then $du=-(\csc x\cot x+\csc^2 x)dx$

$\int \csc x\,dx = -\int \frac{1}{u}du = -\ln|u|+C = -\ln|\csc x+\cot x|+C$

11° $\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx$

let $u=\sec x+\tan x$, then $du=(\sec x\tan x+\sec^2 x)dx$

$\int \sec x\,dx = \int \frac{1}{u}du = \ln|u|+C = \ln|\sec x+\tan x|+C$

Hyperbolic functions

1° hyperbolic sine: the odd part of the exponential function

$\sinh x = \frac{1}{2}(e^x-e^{-x}) = \frac{e^{2x}-1}{2e^x}$ $\qquad \sinh x = -i\sin(ix)$

2° hyperbolic cosine: the even part of the exponential function

$\cosh x = \frac{1}{2}(e^x+e^{-x}) = \frac{e^{2x}+1}{2e^x}$ $\qquad \cosh x = \cos(ix)$

3° hyperbolic tangent: $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$, $\tanh x = -i\tan(ix)$

4° hyperbolic cotangent for $x\neq 0$: $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$, $\coth x = i\cot(ix)$

5° hyperbolic secant $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}} = \frac{2e^x}{2e^x+1}$, $\operatorname{sech} x = \operatorname{sech}(ix)$

6° hyperbolic cossecant $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x-e^{-x}} = \frac{2e^x}{2e^x-1}$ $\operatorname{csch} x = i\csc(ix)$

$\frac{d}{dx}\sinh x = \cosh x$ $\qquad d\sinh x = \cosh x\,dx$ $\qquad \sinh x = \int \cosh x\,dx + C$

$\frac{d}{dx}\cosh x = \sinh x$ $\qquad d\cosh x = \sinh x\,dx$ $\qquad \cosh x = \int \sinh x\,dx + C$

Distribution integral method

we have $u=u(x)$, $v=v(x)$, then $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

we get $\int\frac{duv}{dx}dx=\int u\frac{dv}{dx}dx+\int v\frac{du}{dx}dx$

$u\cdot v=\int udv+\int vdu$ or $\int udv=u\cdot v-\int vdu$

example

1° $\int x\sin x\,dx=-\int x\,d\cos-=-x\cos x+\int\cos x\,dx=\sin x-x\cdot\cos x+C$

$u=x \quad + \quad v''=\sin x$

$u'=1 \quad - \quad v'=-\cos x$

$u''=0 \quad \xrightarrow{\int} \quad v=-\sin x$

$\int uv''dx=\int udv'=uv'-\int v'du=uv'-\int u'v'dx=uv'-\int u'dv$

$=uv'-u'v+\int vdu'$

$=uv'-u'v+\int vu''dx$

$\int x\sin x\,dx=-x\cos x+\sin x+C$

2° $\int x^2e^xdx=\int x^2de^x=x^2e^x-\int e^xdx^2$

$=x^2e^x-2\int e^xx\,dx=x^2e^x-2\int x\,de^x=x^2e^x-2xe^x+2e^x$

$u=x^2 \quad + \quad v^{(3)}=e^x$

$u'=2x \quad - \quad v''=e^x$

$u''=2 \quad + \quad v'=e^x$

$u^{(3)}=0 \quad \xrightarrow{\int} \quad v=e^x$

$\int x^2e^xdx=x^2e^x-2xe^x+2e^x+C=e^x(x^2-2x+2)+C$

$\frac{d}{dx}(f(x)g(x))=g(x)\frac{d}{dx}f(x)+f(x)\frac{d}{dx}g(x)$

chain rule

$\frac{d}{dx}f(g(h(x)))=\frac{d}{g(h(x))}f(g(h(x)))\cdot\frac{d}{dh(x)}g(h(x))\cdot\frac{d}{dx}h(x)$

3° $\int e^{ax}\cos 3x\,dx$

$= \frac{1}{3}e^{ax}\sin 3x + \frac{ae^{ax}}{9}\cos 3x - \frac{a^2}{9}\int e^{ax}\cos 3x\,dx$

$= \frac{3}{9+a^2}e^{ax}\sin 3x + \frac{ae^{ax}}{9+a^2}\cos 3x$

$u = e^{ax}$ $\quad v^{(n)} = \cos 3x$

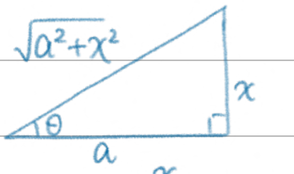
$u' = ae^{ax}$ $\quad v^{(n-1)} = \frac{1}{3}\sin 3x$

$u'' = a^2 e^{ax}$ $\quad v^{(n-2)} = -\frac{1}{3^2}\cos 3x$

Trigonometric substitution

example

1° $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$



$\tan\theta = \frac{x}{a}$, $\tan^{-1}\frac{x}{a} = \theta$

$= \int \frac{1}{a^2(1+x^2/a^2)}dx = \frac{1}{a^2}\int \frac{1}{1+x^2/a^2}dx$

$= \frac{1}{a^2}\int \frac{1}{1+\tan^2\theta}dx$

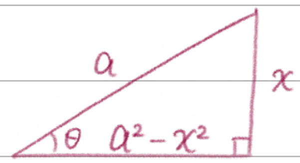
where $\tan\theta = \frac{x}{a}$, $d\tan\theta = \frac{1}{a}dx$, $\sec^2\theta\,d\theta = \frac{1}{a}dx$

$dx = a\sec^2\theta\,d\theta$,

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\int \frac{1}{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta = \frac{1}{a}\int \cos^2\theta\cdot\sec^2\theta\,d\theta = \frac{1}{a}\int d\theta$

$= \frac{1}{a}\theta + C = \frac{1}{a}\arctan\frac{x}{a} + C$

2° $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$



$\sin\theta = \frac{x}{a}$, $\arcsin\frac{x}{a} = \theta$

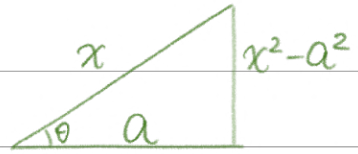
$= \int \frac{1}{\sqrt{a^2(1-x^2/a^2)}}dx = \frac{1}{a}\int \frac{1}{\sqrt{1-x^2/a^2}}dx$

let $\sin\theta = \frac{x}{a}$, $d\sin\theta = \frac{1}{a}dx$, $a\cos\theta\,d\theta = dx$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int d\theta$

$= \theta + C = \arcsin\frac{x}{a} + C$

3° $\int \frac{1}{\sqrt{x^2-a^2}}dx$



$\sec\theta = \frac{x}{a}$, $\operatorname{arcsec}\frac{x}{a} = \theta$

$= \int \frac{1}{a\sqrt{x^2/a^2-1}}dx = \frac{1}{a}\int \frac{1}{\sqrt{x^2/a^2-1}}dx$

let $\sec\theta = \frac{x}{a}$, $d\sec\theta = \frac{1}{a}dx$, $a\sec\theta\tan\theta\,d\theta = dx$

$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta = \int \sec\theta\,d\theta$

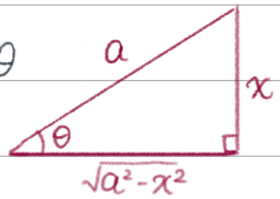
$= \ln|\sec\theta + \tan\theta| + C$

$= \ln|x + \sqrt{x^2-a^2}| + C$

Note

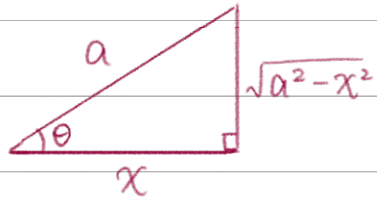
$\frac{1}{\sqrt{a^2-x^2}} \rightarrow \frac{1}{a\sqrt{1-x^2/a^2}}$ we consider $\frac{x}{a} = \sin\theta$

$dx = a d\sin\theta \quad dx = a\cos\theta d\theta$



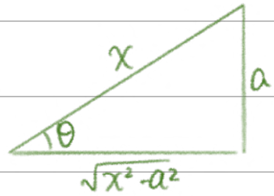
here, we can also consider $\frac{x}{a} = \cos\theta$

$dx = a d\cos\theta \quad dx = -a\sin\theta d\theta$



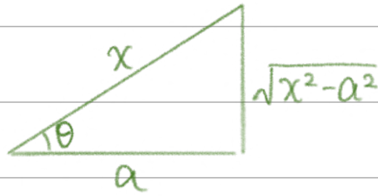
$\frac{1}{\sqrt{x^2-a^2}} \rightarrow \frac{1}{a\sqrt{x^2/a^2-1}}$ we consider $\frac{x}{a} = \sec\theta$

$dx = a d\sec\theta \quad dx = a\sec\theta\tan\theta d\theta$



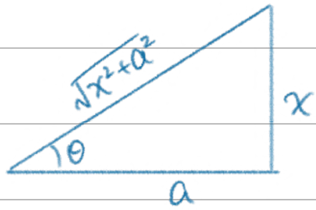
here, we can also consider $\frac{x}{a} = \csc\theta$

$dx = a d\csc\theta \quad dx = -\csc\theta\cot\theta d\theta$



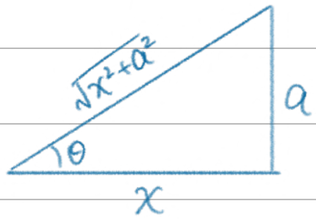
$\frac{1}{\sqrt{a^2+x^2}} \rightarrow \frac{1}{a\sqrt{1+x^2/a^2}}$ we consider $\frac{x}{a} = \tan\theta$

$dx = a d\tan\theta \qquad dx = a\sec^2\theta d\theta$



we can also consider $\frac{x}{a} = \cot\theta$

$dx = a d\cot\theta \qquad dx = -a\csc^2\theta d\theta$



Trigonometric function

1° Basic identities

$\sin^2\alpha + \cos^2\alpha = 1, \qquad 1 + \tan^2\alpha = \sec^2\alpha, \qquad 1 + \cot^2\alpha = \csc^2\alpha$

2° Double angle formulas

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$\tan(2\alpha) = 2\tan\alpha / (1 - \tan^2\alpha) \quad (\tan^2\alpha \neq 1)$

3° Half angle formula (derived from double angle formula)

$\sin^2\alpha = \frac{1}{2}(1 - \cos(2\alpha))$

$\cos^2\alpha = \frac{1}{2}(1 + \cos(2\alpha))$

$\tan^2\alpha = (1 - \cos(2\alpha))/(1 + \cos(2\alpha))$

4 Sum and difference formulas

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = (\tan\alpha \pm \tan\beta)/(1 \mp \tan\alpha\tan\beta)$

5° Producto to sum formulas

$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$

$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$

$\sin\alpha\sin\beta = -\frac{1}{2}(\cos(\alpha+\beta) - \cos(\alpha-\beta))$

6° Sum to product formula

$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$

$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

Partial fraction decomposition

example

1° $\int \frac{1}{x^2+x} dx = \int \frac{1}{x(x+1)} dx$

let $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{(A+B)x+A}{x(x+1)}$, we get $A=1$, $B=-1$

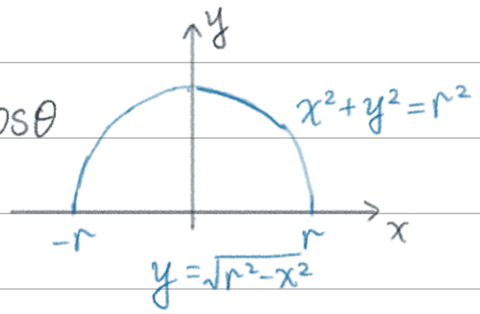
$$\int \frac{1}{x^2+x} dx = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \ln|x| - \ln|x+1| + C$$

2° $\int \frac{1}{x^2-1} dx = \int \frac{1}{(x+1)(x-1)} dx$

let $\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$, we get $A = -\frac{1}{2}$, $B = \frac{1}{2}$

$$\int \frac{1}{x^2-1} dx = -\frac{1}{2}\int \frac{1}{x+1} dx + \frac{1}{2}\int \frac{1}{x-1} dx = -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

3° $\int_{-r}^{r} \sqrt{r^2-x^2}\, dx$



$= \int_{-r}^{r} r\sqrt{1-x^2/r^2}\, dx$, let $\frac{x}{r} = \cos\theta$, $dx = r\,d\cos\theta$

we have $dx = -r\sin\theta\, d\theta$

when $x=-r$, $\cos\theta = -1$, $\theta = -\pi$

when $x = r$, $\cos\theta = 1$, $\theta = 0$ we have $\cos 2\theta = 1 - 2\sin\theta$

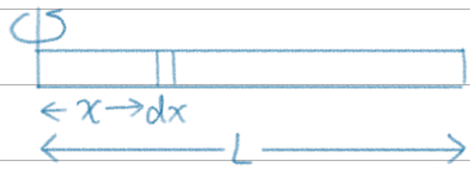
$$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx = -r^2 \int_{-\pi}^{0} \sin^2\theta\, d\theta$$

$$= -\frac{1}{2}r^2 \int_{-\pi}^{0} (1-\cos 2\theta)\, d\theta = -\frac{1}{2}r^2\left(\int_{-\pi}^{0} d\theta - \int_{-\pi}^{0} \cos 2\theta\, d\theta\right)$$

$$= -\frac{1}{2}r^2(-\pi - 0) = \frac{1}{2}\pi r^2$$

$$\int_{x_1}^{x_2} f(x)\,dx = F(x_2) - F(x_1) = \lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x \ : \text{AREA}$$

4° Moment of inertia $I = mr^2$



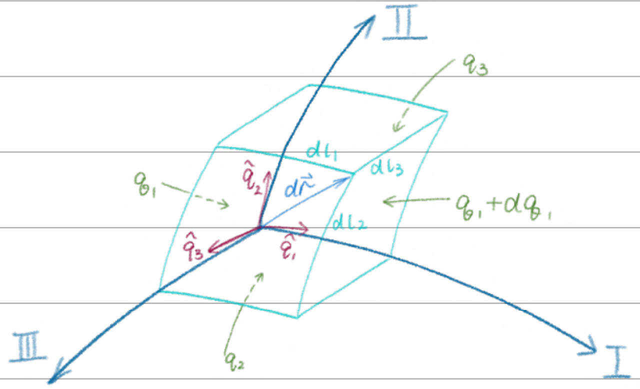
$dI = x^2\, dm$

$\quad = x^2 \rho\, dx \qquad \rho = M/L \ : dm = \rho\, dx$

$I = \rho \int_0^L x^2\, dx = \frac{1}{3}ML^2$

Lecture 01: Coordinate system I

we image a coordinate system as



this is a general coordinate system

A coordinate system is formed by

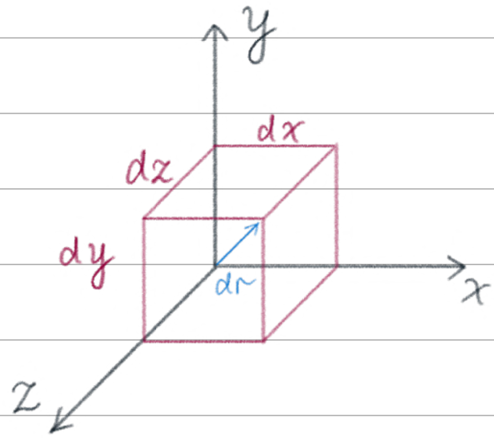
$\hat{q}_1\ \hat{q}_2\ \hat{q}_3,\ \hat{q}_1 \perp q_1,\ \hat{q}_2 \perp q_2, \hat{q}_3 \perp q_3$

where $\begin{cases} dl_1 = h_1 dq_1 \\ dl_2 = h_2 dq_2 \\ dl_3 = h_3 dq_3 \end{cases}$

$$d\vec{r} = h_1 dq_1 \hat{q}_1 + h_2 dq_2 \hat{q}_2 + h_3 dq_3 \hat{q}_3$$

$$dV = h_1 h_2 h_3\, dq_1 dq_2 dq_3$$

For Cartesian coordinate system



$dV = dx\,dy\,dz$

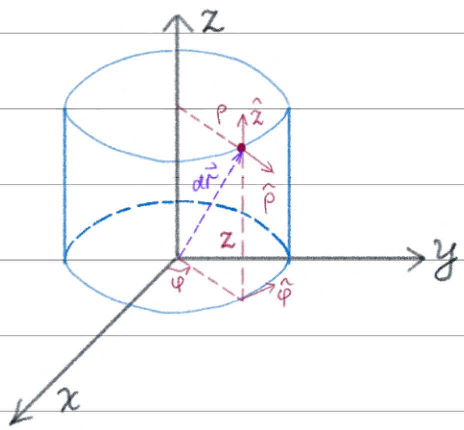
$d\vec{r} = dx\cdot\hat{x} + dy\cdot\hat{y} + dz\cdot\hat{z}$

$\hat{q}_1 \to \hat{x} \quad q_1 \to x \quad h_1 = 1$

$\hat{q}_2 \to \hat{y} \quad q_2 \to y \quad h_2 = 1$

$\hat{q}_3 \to \hat{z} \quad q_3 \to z \quad h_3 = 1$

For cylindrical coordinate system



$(x, y, z) \to (\rho, \varphi, z)$

$dV = dz\cdot d\rho\cdot \rho d\varphi \quad dV = \rho\, d\rho\, d\varphi\, dz$

$d\vec{r} = d\rho\,\hat{\rho} + \rho\, d\varphi\,\hat{\varphi} + dz\,\hat{z}$

$\hat{q}_1 \to \hat{\rho} \quad q_1 \to \rho \quad h_1 = 1$

$\hat{q}_2 \to \hat{\varphi} \quad q_2 \to \varphi \quad h_2 = \rho$

$\hat{q}_3 \to \hat{z} \quad q_3 \to z \quad h_3 = 1$

For Spherical coordinate system

$(x, y, z) \rightarrow (r, \theta, \varphi)$

$dV = dr \cdot r d\theta \cdot r\sin\theta d\varphi$

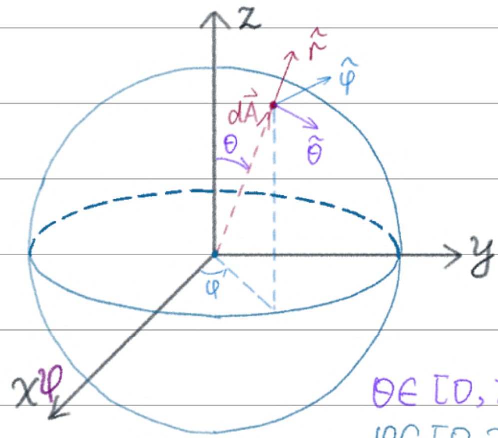
$dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$

$d\vec{A} = dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta d\varphi\,\hat{\varphi}$

$\hat{q}_1 \rightarrow \hat{r} \quad q_1 \rightarrow r \quad h_1 = 1$

$\hat{q}_2 \rightarrow \hat{\theta} \quad q_2 \rightarrow \theta \quad h_2 = r$

$\hat{q}_3 \rightarrow \hat{\varphi} \quad q_3 \rightarrow \varphi \quad h_3 = r\sin\theta$



$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi)$

$dl_1 = dx$	$dl_1 = d\rho$	$dl_1 = dr$
$dl_2 = dy$	$dl_2 = \rho d\varphi$	$dl_2 = r d\theta$
$dl_3 = dz$	$dl_3 = dz$	$dl_3 = r\sin\theta d$

$d\vec{A} = dl_1\hat{q}_1 + dl_2\hat{q}_2 + dl_3\hat{q}_3$

$= h_1 dq_1 \hat{q}_1 + h_2 dq_2 \hat{q}_2 + h_3 dq_3 \hat{q}_3$

$d\vec{A} = \hat{x}dx + \hat{y}dy + \hat{z}dz$

$d\vec{A} = d\rho\,\hat{\rho} + \rho d\varphi\,\hat{\varphi} + dz\,\hat{z}$

$d\vec{A} = dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta d\varphi\,\hat{\varphi}$

$dV = dl_1 \cdot dl_2 \cdot dl_3$

$= h_1 h_2 h_3 \, dq_1 \, dq_2 \, dq_3$

$dV = dx\,dy\,dz$

$dV = \rho\,d\rho\,d\varphi\,dz$

$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi$

we have a salar function f, its Gradient

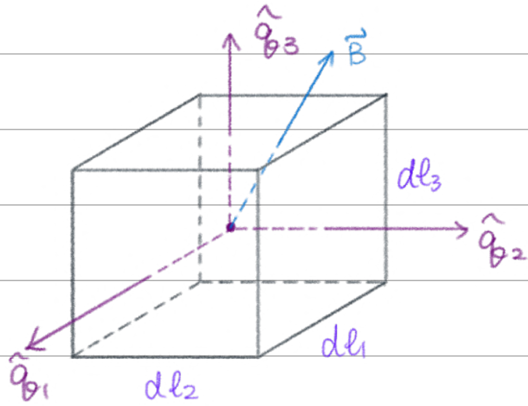
$$\vec{\nabla} f = \left(\frac{\partial}{\partial l_1}\hat{q}_1 + \frac{\partial}{\partial l_2}\hat{q}_2 + \frac{\partial}{\partial l_3}\hat{q}_3\right) f$$

$$= \left(\frac{1}{h_1}\frac{\partial}{\partial q_1}\hat{q}_1 + \frac{1}{h_2}\frac{\partial}{\partial q_2}\hat{q}_2 + \frac{1}{h_3}\frac{\partial}{\partial q_3}\hat{q}_3\right) f$$

$$= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) f$$

$$= \left(\frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\hat{\varphi} + \frac{\partial}{\partial z}\hat{z}\right) f$$

$$= \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\hat{\varphi}\right) f$$



The outgoing flux per unit volume

the flux of $\vec{B}$ via $dl_1 \cdot dl_2$

For the left surface $d\Phi_L = B_L dl_1 dl_3$

$d\Phi_L = B_L h_{1L} dq_1 h_{3L} dq_3$, for centre $B_c = B$

$$= -\left(B_2 - \frac{\partial B_2}{\partial q_2}\cdot\frac{1}{2}dq_2\right)\left(h_1 - \frac{\partial h_1}{\partial q_2}\cdot\frac{1}{2}dq_2\right)dq_1 \cdot \left(h_3 - \frac{\partial h_3}{\partial q_2}\cdot\frac{1}{2}dq_2\right)\cdot dq_3$$

$$d\Phi_L = -B_2 h_1 h_3 dq_1 dq_3 + \frac{\partial}{\partial q_2}(B_2 h_1 h_3)\frac{dq_2}{2} dq_1 dq_3$$

for the right surface $d\Phi_R = B_R dl_1 dl_3$

$$d\Phi_R = B_2 h_1 h_3 dq_1 dq_3 + \frac{\partial}{\partial q_2}(B_2 h_1 h_3)\frac{dq_2}{2} dq_1 dq_3$$

we can get the total flux

$$d\Phi = \left[\frac{\partial}{\partial q_1}(B_1 h_2 h_3) + \frac{\partial}{\partial q_2}(B_2 h_1 h_3) + \frac{\partial}{\partial q_3}(B_3 h_1 h_2)\right] dq_1 dq_2 dq_3$$

we can get the Divergent of $\vec{B}$

$$\frac{d}{dV}\Phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(B_1 h_2 h_3) + \frac{\partial}{\partial q_2}(B_2 h_1 h_3) + \frac{\partial}{\partial q_3}(B_3 h_1 h_2)\right]$$

$$\vec{\nabla}\cdot\vec{B} = \frac{d}{dV}\Phi$$

Divergent of vector function $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(A_1 h_2 h_3) + \frac{\partial}{\partial q_2}(A_2 h_1 h_3) + \frac{\partial}{\partial q_3}(A_3 h_1 h_2)\right] = \frac{1}{h_1 h_2 h_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}(A_i h_j h_k)$$

$$= \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \varphi}A_\varphi + \frac{\partial}{\partial z}A_z$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}A_\varphi$$

Lecture 03: First-order ordinary differential equation I

ODE and its applications for physics

(1) derive ODE from physical modeling

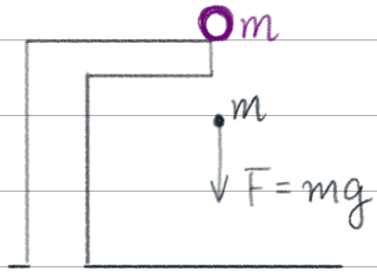
(2) solution of ODE

(3) Interpretation

Def

1^{st} order ODE involves only the 1^{st} derivative of the function

"Ordiniary" means there is no partial derivative equation



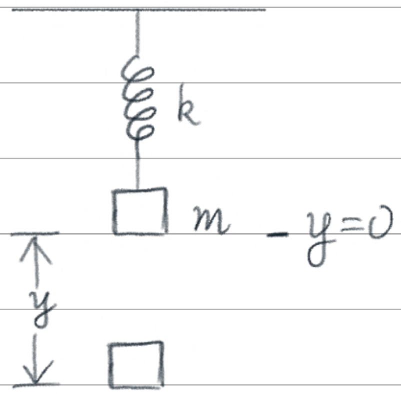
examples:

1° $\Sigma \vec{F} = m\vec{a}$

we have $-F\hat{y} = ma_y$. $a_y = g$, then

$-mg = m\frac{d^2}{dt^2}y$,

then, we build $\frac{d^2}{dt^2}y + g = 0$



2° $\Sigma \vec{F} = F_s\hat{y} - G\hat{y} = ma_y\hat{y}$

we get $kx - mg = m\ddot{y}$

Separable ordiniary differential equations

$$g(y)\frac{dy}{dx} = f(x)$$

we can perform a total differentiation

$$g(y)dy = f(x)dx$$

we can get $\int g(y)dy = \int f(x)dx + C$

Lecture 02: Coordinate system II

Laplace operator

$$\nabla^2 f = \vec{\nabla}\cdot\vec{\nabla} f$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

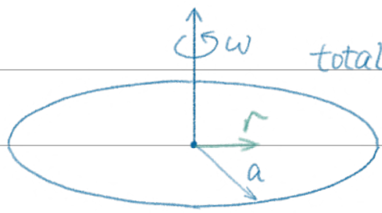
$$= \frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f + \frac{\partial^2}{\partial z^2}f$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}f + \frac{\partial^2}{\partial z^2}f$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}f\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}f\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}f$$

example

Moment of inertia



total mass: M

for a point mass $I = mr^2$

$$I = \lim_{\Delta m\to 0}\sum_i r_i^2\,\Delta m \quad , \quad dm = \sigma dA = \frac{M}{\pi a^2}\,dr\cdot r d\theta$$

we get $dm = \frac{M}{\pi a^2} r\,dr\,d\theta$

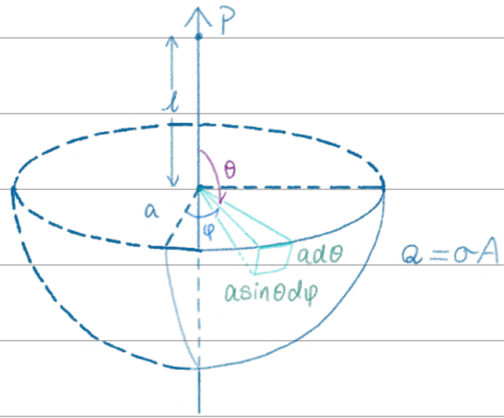
$$I = \int_0^{2\pi}\int_0^a r^3\sigma\,dr\,d\theta$$

$$= \frac{M}{\pi a^2}\int_0^{2\pi}\int_0^a r^3\sigma\,dr\,d\theta$$

$$= \frac{1}{2}Ma^2$$

examples

1° the electric potential at P



for point charge $V = \frac{1}{4\pi\varepsilon_0}\frac{q_0}{r}$

we have $dV = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}dq_0$, $V = \int dV$

$$dq_0 = \sigma dA = \sigma \cdot a d\theta \cdot a\sin\theta d\varphi$$

$$dq_0 = \sigma a^2 \sin\theta d\theta d\varphi$$

$$r = (a^2 + l^2 - 2al\cos\theta)^{1/2}$$

$$V = \int_0^{\pi/2}\int_{\pi/2}^{\pi} \frac{1}{4\pi\varepsilon_0}\frac{\sigma a^2 \sin\theta d\theta d\varphi}{(a^2+l^2-2al\cos\theta)^{1/2}}$$

we calculate it, get

$$V = \frac{\sigma a^2}{4\pi\varepsilon_0} \cdot \int_0^{\pi/2} d\varphi \cdot \int_{\pi/2}^{\pi} \frac{\sin\theta d\theta}{(a^2+l^2-2al\cos\theta)^{1/2}}$$

$$= \frac{\sigma a^2}{8\varepsilon_0} \cdot \left(-\int_{\pi/2}^{\pi} (a^2+l^2-2al\cos\theta)^{-1/2} d\cos\theta\right)$$

$$= \frac{\sigma a^2}{8\varepsilon_0} \cdot \frac{1}{2al}\int_{\pi/2}^{\pi} (a^2+l^2-2al\cos\theta)^{-1/2} d(a^2+l^2-2al\cos\theta)$$

$$= \frac{\sigma a^2}{8\varepsilon_0 l}\int_{\pi/2}^{\pi} (a^2+l^2-2al\cos\theta)^{-1/2} d(a^2+l^2-2al\cos\theta)$$

$$= \frac{\sigma a^2}{8\varepsilon_0 l} \cdot \frac{1}{2al} \cdot 2\left[a^2+l^2-2al\cos\theta\right]^{1/2}\Big|_{\pi/2}^{\pi}$$

$$= \frac{\sigma a}{8\varepsilon_0 l^2}\left[(a^2+l^2+2al)^{1/2} - (a^2+l^2)^{1/2}\right)$$

Lecture 03: First-order ordinary differential equation I

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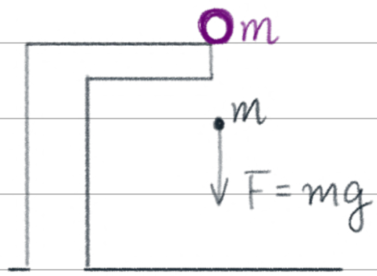

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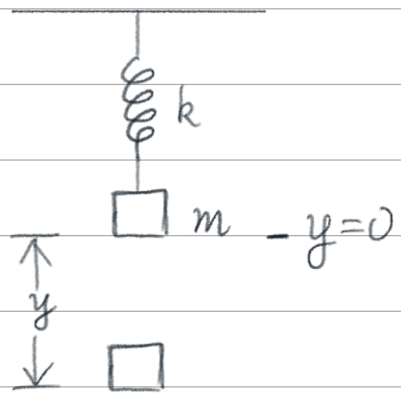

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we can perform a total differentiation

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we can get $\int g(y)dy = \int f(x)dx + C$

3° Radioactive material substance decomposes in a rate k

$\lambda = t_0$: 100% → 50% → 25%

$k \propto$ the amount present

(1) decomposes rate

if $y(t) :=$ the amount of radioactive substance at time t

then $\frac{d}{dt} y(t)$

(2) $\frac{dy}{dt} = k y(t)$

we get

$$\int \frac{dy}{y(t)} = \int k\,dt + C,$$

$$\ln y(t) = kt + C, \quad y(t) = e^{(kt+C)} = De^{kt}$$

4° $9yy' + 4x = 0$

$$\Rightarrow 9yy' = -4x, \quad 9y\frac{dy}{dx} = -4x$$

$$\Rightarrow 9y\,dy = -4x\,dx,$$

$$9y^2 = -4x^2 + 2C, \quad \frac{y^2}{2^2} + \frac{x^2}{3^2} = D^2$$

5° $y' = 1 + y^2$

$$\Rightarrow \frac{dy}{dx} = 1 + y^2, \int \frac{1}{1+y^2}\,dy = \int dx + C$$

$$\arctan y = x + C, \quad y = \tan(x+C)$$

Reduction to separable form

6° $y' = g(y/x)$: $\frac{dy}{dx} = g(\frac{y}{x})$

we can set $u = \frac{y}{x}$, then $y = ux$

$$dy = d(ux), \quad \frac{dy}{dt} = u + x\frac{du}{dx}, \quad y' = u + xu'$$

we have $y' = g(y/x) = g(u)$, then $g(u) = u'x + u$

we get $\frac{du}{g(u) - u} = \frac{dx}{x}$

7° $2xyy' = y^2 - x^2$,

we have $y' = \frac{y}{2x} - \frac{x}{2y}$, let $u = \frac{y}{x}$, then $y = ux$, we get

$y' = u + xu'$

$y' = \frac{u}{2} - \frac{1}{2u}$, $y' = \frac{u^2-1}{2u}$

we have: $u + xu' = (u^2-1)/2u$, then

$xu' = (-u^2-1)/2u$, $x\frac{du}{dx} = -\frac{u^2+1}{2u}$, $\frac{2u}{u^2+1}du = -\frac{1}{x}dx$,

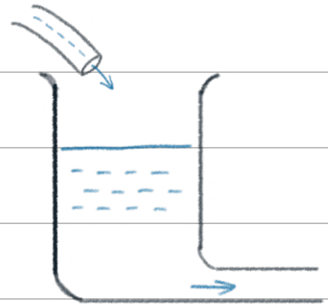
$\int \frac{2u}{u^2+1}du = -\int \frac{1}{x}dx + c$ $\int \frac{1}{u^2+1}du^2 = -\ln|x| + c$

we get $\ln(u^2+1) = -\ln|x| + c$

$u^2 + 1 = e\char`^(-\ln|x| + c) = \frac{1}{|x|}D$ $(e^c = D)$

$u^2 = \frac{1}{|x|}D - 1$, $\frac{y^2}{x^2} = \frac{1}{|x|}D - 1$

$y = \sqrt{D|x| - x^2}$



8° initial: $200\,\ell\ H_2O + 40g\ NaCl$

input: $5\ell/min$ with $10g\ NaCl$

output: $5\ell/min$

we let $y(t)$ is the amount of NaCl in the tank at time t we have

$y(0) = 40$

we let $\frac{dy}{dt}$ is the change of salt per unit time

$\frac{dy}{dt} = 10 - 5 \cdot \frac{y}{200}$

then $\frac{dy}{dt} = 10 - 0.025y$, we can separate this differential equantion

$-40\ln(10 - 0.025y) = t + c$

$y = 40(10 - De^{-0.025t})$

we have $y(0) = 40(10-D) = 40$, $D = 9$,

$y = 40(10 - 9e^{-0.025t})$

$y = 400 - 360e^{-0.025t}$

$\lim_{t \to 0} y = 400$,

Exact differential equations and Integrating factor

If we have $u(x, y)$ which has continuous partial derivatives, we can write

$$du(x, y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

If $u(x, y) = const$, we has $du(x, y) = 0$

we let $du = M(x, y)dx + N(x, y)dy$, $M = \frac{\partial u}{\partial x}$, $N = \frac{\partial u}{\partial y}$

$$\begin{cases} \frac{\partial M}{\partial y} = \frac{\partial^2}{\partial x \partial y}u(x, y) \\ \frac{\partial N}{\partial x} = \frac{\partial^2}{\partial x \partial y}u(x, y), \end{cases} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

we have $u(x, y) = \int M(x, y)dx + k(y) = \int N(x, y)dy + l(x)$

If we have a differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

meets $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$

we call it an exact differential equation $u(x, y) = \begin{cases} const \\ Mdx + k(y) \\ Ndy + l(x) \end{cases}$

9° $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$, $\frac{dy}{dx} = -\frac{x^3 + 3xy^2}{3x^2y + y^3}$

let $M(x, y) = x^3 + 3xy^2$, $N(x, y) = 3x^2y + y^3$

$$\frac{\partial M}{\partial y} = 6xy, \quad \frac{\partial N}{\partial x} = 6xy,$$

the original equantion is an exact differential equation, we can write

$$u(x, y) = const$$

$$u(x, y) = \int M(x, y)dx + k(y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + k(y)$$

$$u(x, y) = \int N(x, y)dy + l(x) = \frac{1}{4}y^3 + \frac{3}{2}x^2y^2 + l(x)$$

we get $u(x, y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + \frac{3}{2}x^2y^2 = const$

Lecture 04: First-order ordinary differential equation II

a first-order differential equation can be written as

$$y'(x)+p(x)y(x)=r(x)$$

if $p(x)$ and $r(x)$ are functions of x
we call it a linear ordinary differential equation
if $r(x)=0$: $y'(x)+p(x)=0$, we call it homogeneous
if $r(x)\neq 0$: $y'(x)+p(x)=r(x)$, we call it non-homogeneous

If we have a separable ODE as follows

$$g(y)\,dy=f(x)\,dx$$

we get $\int g(y)\,dy=\int f(x)\,dx+c$

If we have an exact ODE as follows

$$M(x,y)\,dx+N(x,y)\,dy=0,$$

meets $\frac{\partial}{\partial y}M(x,y)=\frac{\partial}{\partial x}N(x,y)$

$$u(x,y)=\text{const}$$

$$u(x,y)=\int M(x,y)\,dx+k(y)$$

$$u(x,y)=\int N(x,y)\,dy+l(x)$$

we consider $u(x,y)=\text{const}$, $du=0$: $\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$

$$\frac{\partial^2}{\partial x\partial y}u(x,y)=\frac{\partial^2}{\partial y\partial x}u(x,y)$$

we can change the orginial equation to

$$M(x,y)\,dx+N(x,y)\,dy=0$$

meets $\frac{\partial}{\partial y}M(x,y)=\frac{\partial}{\partial x}N(x,y)$, $M=\frac{\partial u}{\partial x}$, $N=\frac{\partial u}{\partial y}$

$du(x,y)=M\,dx$ $\quad u(x,y)=\int M\,dx+k(y)$

$du(x,y)=N\,dy$ $\quad u(x,y)=\int N\,dy+l(x)$

example:

an initial value problem

$(\sin x \cosh y)dx - (\cos x \sinh y)dy = 0,$ ①

$y(x=0) = 3$ ②

let $M(x,y) = \sin x \cosh y$, $\frac{\partial M}{\partial y} = \sin x \sinh y$

$N(x,y) = -\cos x \sinh y$, $\frac{\partial N}{\partial x} = \sin x \sinh y$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\begin{cases} M = \frac{\partial u}{\partial x} \\ N = \frac{\partial u}{\partial y} \end{cases}$

we get the original equation is an exact differential equation, we get

$u(x,y) = \int \sin x \cosh y \, dx + k(y) = -\cos x \cosh y + k(y)$

$u(x,y) = -\int \cos x \sinh y \, dy + l(x) = -\cos x \cosh y + l(x)$

$\frac{\partial u}{\partial y} = -\cos x \sinh y + \frac{\partial}{\partial y} k(y)$,

$\frac{\partial u}{\partial x} = \sin x \cosh y + \frac{\partial}{\partial x} l(x)$,

we have $\frac{\partial}{\partial y} k(y) = 0$, $k(y) = c$

$\frac{\partial}{\partial x} l(x) = 0$, $l(x) = d$

$u = -\cos x \cosh y + c = \cos x \cosh y + d$

we get $\cos x \cosh y = D$, $\cosh y = \frac{1}{2}(e^{y} + e^{-y})$

we have $y(x=0) = 3$, then $\cosh(3) = D$

$\cos x \cosh y - \cosh(3) = 0$

$e^{i\theta} = \cos\theta + i\sin\theta$ $\quad e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, $\sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta})$

$\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$, $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$

$\frac{d}{d\theta}\cosh\theta = \frac{d}{d\theta}\frac{1}{2}(e^{\theta} + e^{-\theta}) = \frac{1}{2}(e^{\theta} - e^{-\theta})$

$= \sinh\theta$

$\frac{d}{d\theta}\sinh\theta = \frac{d}{d\theta}\frac{1}{2}(e^{\theta} - e^{-\theta}) = \frac{1}{2}(e^{\theta} + e^{-\theta})$

$= \cosh\theta$

For $P(x,y)dx+Q(x,y)dy=0$

if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

we employ $F \cdot P(x,y)dx + F \cdot Q(x,y)dy = 0$ for $\frac{\partial FP}{\partial y} = \frac{\partial FQ}{\partial x}$

if $F = F(x)$

for example $-ydx+xdy=0$, $P=-y$, $Q=x$, $\frac{\partial P}{\partial y}=-1$, $\frac{\partial Q}{\partial x}=1$

we employ $F=\frac{1}{x^2}$, we get $-\frac{y}{x^2}dx+\frac{1}{x}dy=0$, $FP=-\frac{y}{x^2}$, $FQ=\frac{1}{x}$,

$$\frac{\partial FP}{\partial y} = -\frac{1}{x^2}, \quad \frac{\partial FQ}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial FP}{\partial y} = \frac{\partial FQ}{\partial x}$$

let $FP=M$, $FQ=N$

we get $-\frac{y}{x^2}dx+\frac{1}{x}dy=0$, $M=-\frac{y}{x^2}$, $N=\frac{1}{x}$

we can use $\begin{cases} \partial M(x,y) = \frac{\partial}{\partial x} u(x,y), & \frac{\partial M(x,y)}{\partial y} = \frac{\partial^2}{\partial x \partial y} u(x,y) \\ \partial N(x,y) = \frac{\partial}{\partial y} u(x,y), & \frac{\partial N(x,y)}{\partial x} = \frac{\partial^2}{\partial y \partial x} u(x,y) \end{cases}$, then

$$\begin{cases} u(x,y) = \int M(x,y)dx + k(y) \\ u(x,y) = \int N(x,y)dy + \ell(x) \end{cases}$$

With a initial condition, we can solve this differential equation

Moreover

we have $\frac{\partial FP}{\partial y} = \frac{\partial FQ}{\partial x}$, $F\frac{\partial}{\partial y}P = F\frac{\partial}{\partial x}Q + Q\frac{\partial}{\partial x}F$

$$\frac{F}{Q}\left(\frac{\partial}{\partial y}P - \frac{\partial}{\partial x}Q\right) = \frac{\partial F}{\partial x}, \quad \frac{1}{Q}\left(\frac{\partial}{\partial y}P - \frac{\partial}{\partial x}Q\right) = \frac{1}{F} \cdot \frac{\partial F}{\partial x}$$

$$\frac{1}{Q}\left(\frac{\partial}{\partial y}P - \frac{\partial}{\partial x}Q\right)\partial x = \frac{1}{F}\partial F$$

$$\int \frac{1}{Q}\left(\frac{\partial}{\partial y}P - \frac{\partial}{\partial x}Q\right)\partial x + C = \int \frac{1}{F}\partial F$$

we have $\ln F = \int \frac{1}{Q}\left(\frac{\partial}{\partial y}P - \frac{\partial}{\partial x}Q\right)\partial x + C$

$F = \exp\left(\int \frac{1}{Q}\left(\frac{\partial}{\partial y}P - \frac{\partial}{\partial x}Q\right)\partial x\right) \cdot D$, D is a constant

we set $C=0$, then $D=1$:

$$F(x) = \exp\left(\int \frac{1}{Q}\left(\frac{\partial}{\partial y}P - \frac{\partial}{\partial x}Q\right)dx\right)$$

we call this method Integrating factor

This is How to find $F(x)$ make $\frac{\partial FP}{\partial y} = \frac{\partial FQ}{\partial x}$

summary

If we have $P(x,y)dx + Q(x,y)dy = 0$

where $P(x,y)$ and $Q(x,y)$ are known,

if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, then find F, makes $\frac{\partial FP}{\partial y} = \frac{\partial FQ}{\partial x}$, (exact differential equation)

we can set $\begin{cases} F = F(x) = \exp\left(\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\right) & \text{or} \\ F = F(y) = \exp\left(\int \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy\right) \end{cases}$

Linear ODE

Def $y' + p(x)y = r(x)$ is a linear ordinary differential equation

if $r(x)$ are functions of x only. we want solve $y(x)$

If $r(x)=0$, $y' + p(x)y = 0$: homogeneous linear ODE, then

$$\frac{dy}{dx} = -p(x)y, \quad \frac{1}{y}dy = -p(x)dx$$

we get $\ln y(x) = -\int p(x)dx + C$

$$y(x) = D\exp\left(-\int p(x)dx\right)$$, D is a constant to be determined

If $r(x) \neq 0$, $y' + p(x)y = r(x)$: non-homogeneous linear ODE, then

$$\frac{dy}{dx} = (r - py), \quad dy = (r - py)dx$$

we have $(r - py)dx - dy = 0$,

we can address it as $Mdx + Ndy = 0$ (exact differential equation)

$\mathbb{P}dx + \mathbb{Q}dy = 0$. (non-exact differential equation)

and we need to find the integrating factor

we set $\mathbb{P} = r - py$, $\mathbb{Q} = -1$, $\frac{\partial \mathbb{P}}{\partial y} = -p$, $\frac{\partial \mathbb{Q}}{\partial x} = 0$

Assume $F(x)(r - py)dx - F(x)dy = 0$ is an exact differential equation

then $F(x) = \exp\left(\int \frac{1}{\mathbb{Q}}\left(\frac{\partial \mathbb{P}}{\partial y} - \frac{\partial \mathbb{Q}}{\partial x}\right)dx\right)$

$$= \exp\left(\int p(x)dx\right)$$

we get $F(x)(r-Py)dx - F(x)dy = 0$,

$$e^{\int p(x)dx}[(r-Py)dx - dy] = 0$$

here we can solve this equation via the original equation

$$y' + P(x)y = r(x)$$

we get $e^{\int pdx}(y' + Py) = re^{\int pdx}$

$$(e^{\int Pdx} \cdot y)' = re^{\int Pdx}$$

$$\frac{d}{dx}(e^{\int pdx} \cdot y) = re^{\int pdx}$$

$$d(e^{\int pdx} \cdot y) = re^{\int Pdx}dx$$

we get $\int d(e^{\int pdx} \cdot y) = \int re^{\int Pdx} dx + c$

$$e^{\int pdx} y(x) = \int re^{\int pdx} dx + c$$

then $y(x) = e^{-\int pdx}(\int re^{\int pdx} dx + c)$

we can also write $y(x) = e^{-h(x)}(\int re^{h(x)} dx + C)$, where $h(x) = \int p(x)dx$

example

1° $y' + y\tan x = \sin 2x$, $y(0) = 1$

This is a linear non-homogeneous 1st order ODE

we set $P(x) = \tan(x)$, $r(x) = \sin 2x$

then $h(x) = \int p(x)dx = \int \tan(x)dx = \ln|\sec x|$

$$e^{h(x)} = |\sec x|, \quad e^{-h(x)} = \frac{1}{|\sec x|}$$

we employ $y(x) = e^{-h(x)}(\int r(x)e^{h(x)} dx + C)$

$$y(x) = \frac{1}{|\sec x|}(\int \sin(2x)|\sec x| dx + C)$$

$$= \frac{1}{|\sec x|}(\int 2\sin x\cos x|\sec x| dx + c)$$

$$= C\cos x - 2\cos^2 x$$

we have $y(0) = 1$, $y(0) = C - 2$, $C = 3$

we get $y(x) = 3\cos x - 2\cos^2 x$

Lecture 05: First-order ordinary differential equation II

Once we need to solve a First-order ODE

determine if it is separable

\# if yes, solve it directly

\# if no, determine if it is exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

- if yes, solve it by $\begin{cases} u(x,y) = \int M dx + k(y) \\ u(x,y) = \int N dy + l(x) \end{cases}$ and boundary conditions
- if no, build integrating factor $\begin{cases} F(x) = \exp\left(\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\right) \\ F(y) = \exp\left(\int \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy\right) \end{cases}$

  transfer it to an exact ODE, $FPdx + FQdx$, $M = FP$, $N = FQ$

  consider boundary conditions, solve $FPdx + FQdy = 0$
- if we find it is a non-homogeneous ODE: $y' + T(x)y = \lambda(x)$

  we can employ $y(x) = e^{-h(x)}\left(\int \lambda e^{h(x)} d(x) + c\right)$ where $h(x) = \int T(x)dx$

Bernoulli equation

$$y' + p(x)y = g(x)y^a \quad (a \in \mathbb{R})$$

if $a=0$, we have $y' + p(x)y = g(x)$, a linear non-homogeneous ODE

if $a=1$, we have $y' + (p(x) - g(x))y = 0$, a linear homogeneous ODE

if $a \geq 2$, it's a non-linear ODE.

we set $u(x) = [y(x)]^{1-a}$,

$$\begin{aligned} u'(x) &= (1-a)y^{-a}y' \qquad\qquad y' = gy^a - py \\ &= (1-a)y^{-a}(gy^a - py) \\ &= (1-a)(g - py^{1-a}) \end{aligned}$$

we get $u'(x) + (1-a)pu = (1-a)g$,

we can write it as $u' + Tu = \lambda(x)$, $T = (1-a)p$, $\lambda = (1-a)g$

here $u(x) = e^{-h(x)}\left(\int \lambda(x) e^{h(x)} dx + C\right)$ where $h(x) = \int T(x)dx$

example

$1°$ $y' = Ay - By^2$

we can write $y' - Ay = -By^2$

refer Bernoulli equation $y' + p(x)y = g(x)y^a$

here, $p(x) = -A$, $g(x) = -B$, $a = 2$

let $u(x) = [y(x)]^{1-a} = \frac{1}{y(x)}$

employ $u'(x) + (1-a)pu = (1-a)g$,

we have $u'(x) + Au = B$,

$$u(x) = \frac{B}{A} + ce^{-Ax}, \quad \frac{1}{y} = \frac{B}{A} + ce^{-Ax}$$

we get $y(x) = (\frac{B}{A} + ce^{-Ax})^{-1}$

Existence and uniqueness of solution

an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \qquad ①$$

it may a : no solution

b : precisely one solution

c : more than one solution

$2°$ $|y'| + |y| = 0$, $y(0) = 1$: no solution

$3°$ $y' = 2x$, $y(0) = 1$, it has only one solution $y = x^2 + 1$

$4°$ $xy' = y - 1$, $y(0) = 1$, we have $y(x) = cx + 1$,

it has many solutions

Theorem

1° Existance theorem,

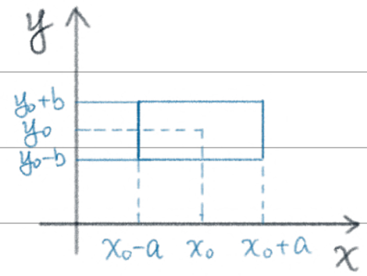
we have $y' = f(x,y)$, $y(x_0) = y_0$

if it is continuous at all points in a rectangular and its boundary

$R: |x - x_0| \leq a,\ |y - y_0| \leq b$

Simultaneously $|f(x,y)| \leq K$

then it has at least one solution



2° Uniqueness theorem

we have $y' = f(x,y)$, $y(x_0) = y_0$

if $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous in rectangular and its boundary

$R: |x - x_0| \leq a,\ |y - y_0| \leq b$

Simultaneously $|f(x,y)| \leq K \ \wedge \ \left|\frac{\partial f(x,y)}{\partial y}\right| \leq M$

then it has an unique solution

Lecture 06: Second order linear differential equation I

2nd order ODE

linear 2nd-order ODE $y'' + p(x)y' + q(x)y = r(x)$,

where $p(x), q(x), r(x)$ are functions of x only

if $r(x) = 0$, it's homogeneous 2nd order ODE

if $r(x) \neq 0$, it's non-homogeneous 2nd order ODE

for example

1° $y'' + 25y' = e^{-x}\cos x$: 2nd order linear, non-homogeneous ODE

2° $y'' + \frac{1}{x}y + y = 0$: 2nd order linear homogeneous ODE

3° $yy'' + y'^2 = 0$: 2nd order non-linear homogeneous ODE

4° $y'' + y' = 0$: 2nd order linear homogeneous ODE

it has two solutions: $y_1 = \cos x$, $y_2 = \sin x$

If we have a solution $y = c_1\cos x + c_2\sin x$, it is also a solution

Superposition principle

For a linear homogeneous ODE, we have this fundamental theorem

any linear combination of two independent solutions on an open interval I is also a solution of the original equation.

Proof if y_1, y_2 are two independent solutions of $y'' + py' + qy = 0$

we have $y = c_1y_1 + c_2y_2$

$y' = c_1y_1' + c_2y_2'$

$y'' = c_1y_1'' + c_2y_2''$

then, $c_1y_1'' + c_2y_2'' + pc_1y_1' + pc_2y_2' + qc_1y_1 + qc_2y_2$

$= c_1y_1'' + pc_1y_1' + qc_1y_1 + c_2y_2'' + pc_2y_2' + qc_2y_2 = 0$

5° $y'' + y = 1$: 2nd order linear non-homogeneous ODE, and we have 2 solutions

$y_1 = 1 + \cos x$, $y_2 = 1 + \sin x$

then $y_3 = C_1 y_1 + C_2 y_2$ (C_1, C_2 are constants) may be not a solution of original equation

because $y_3'' + y_3 = C_1 + C_2$, y_3 is a solution iff $C_1 + C_2 = 1$

Initial value problem of 2nd order linear homogeneous ODE

for $y'' + p(x)y' + q(x)y = 0$

initial conditions (I.C.) $y(x_0) = K_0$

$y'(x_0) = K_1$

General solution : $y = C_1 y_1 + C_2 y_2$, C_1, C_2 are constants to be determinted

Particular solution $y = C_1 y_1 + C_2 y_2$, C_1, C_2 are determinted by IC

6° $y'' + y = 0$, $\begin{cases} y(0) = 3.0 \\ y'(0) = -0.5 \end{cases}$

(1) general solution $y = C_1 \cos x + C_2 \sin x$, $y' = -C_1 \sin x + C_2 \cos x$

(2) by IC-1, $y(0) = C_1$, $C_1 = 3.0$

by IC-2 $y'(0) = C_2$, $C_2 = -0.5$

Basis solution

for $y'' + p(x)y' + q(x) = 0$: 2nd linear homogeneous ODE

it has a general solution : $y = C_1 y_1 + C_1 y_2$.

C_1 and, C_2 are constants to be determined;

y_1 and y_2 are independend, i.e. they are not proportional

we call y_1 and y_2 are a basis of solution on I.

if C_1, C_2 are determined, y becomes a particular solution

If one solution is known, we need to find a basis

7° $(x^2-x)y''-xy'+y=0$,

by observation, we find a solution $y_1=x$

we assume $y_2=uy_1$, u is an unknown function to be determined

$y_2'=u'y_1+uy_1'$,

$y_2''=u''y_1+2u'y_1'+uy_1''$

we have $(x^2-x)(u''y_1+2u'y_1'+uy_1'')-x(u'y_1+uy_1')+uy_1=0$

we employ $y_1=x$, get : $(x^2-x)(u''x+2u')-x(u'x+u)+ux=0$, then

$u''x^3+u'x^2-u''x^2-2u'x=0$

$u''(x^2-x)+u'(x-2)=0$

let $v(x)=u'(x)$, we have $(x^2-x)v'+(x-2)v=0$

we have $\frac{1}{v}dv=-\frac{x-2}{x^2-x}dx$, $\ln|v|=\int\frac{1}{x-1}-\frac{2}{x}dx+c$,

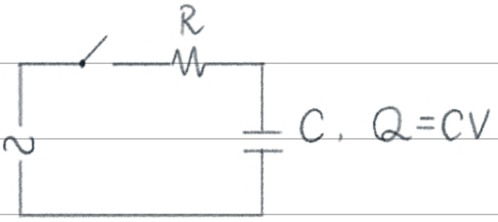
$\ln|v|=\ln|x-1|-2\ln|x|+c=\ln\frac{|x-1|}{x^2}+c$

$=\ln\frac{|x-1|}{x^2}$ by force $c=0$

we get $v=\frac{x-1}{x^2}=\frac{1}{x}-\frac{1}{x^2}$, employ $u=\int v\,dx+c$

$u=\ln|x|-\frac{1}{x}$ by force $c=0$

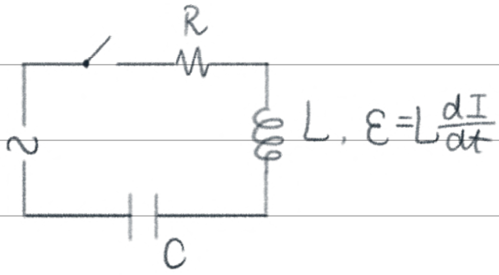
we have $y_1=x$, $y_2=uy_1$, then $y_2=x\ln|x|+1$



8° $V(t)=IR+\frac{1}{C}Q$, $I=\frac{dQ}{dt}$

we have $V=IR+\frac{1}{C}\int I\,dt$, then $V'=I'R+\frac{1}{C}I$,

we get $RI'+\frac{1}{C}I=V'$



9° $V(t)=IR+\varepsilon+\frac{1}{C}Q$, $dQ=Idt$, $\varepsilon=L\frac{dI}{dt}$

we have $V=IR+L\frac{dI}{dt}+\frac{1}{C}\int I\,dt$

we get $LI''+RI'+\frac{1}{C}I=V'$,

$I''+\frac{R}{L}I'+\frac{1}{CL}I=V'$

Lecture 07: Second order linear differential equation II

Review of 2nd ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

linear: if $p(x)$, $q(x)$, and $r(x)$ are functions of x only

homogeneous: if $r(x) = 0$

non-homogeneous: if $r(x) \neq 0$

if it is a 2nd order linear homogeneous ODE: $y'' + p(x)y' + q(x)y = 0$

and $y''(x)$, $y'(x)$, $y(x)$ are continuous in a domain I,

then, it has at least one solution

it has two basis solutions $y_1(x)$, $y_2(x)$, here y_1 and y_2 are independent

then we have a general solution

$$y = C_1 y_1 + C_2 y_2, \quad C_1 \text{ and } C_2 \text{ are constants to be determined}$$

if we offer boundary conditions (initial conditions, IC), we can determine C_1, C_2

then we have at least one particular solution: $y_1 = C_1 y_1 + C_2 y_2$

If one solution y_1 is obtained

by assuming $y_2 = u(x) y_1$

we have a 2nd order linear homogeneous ODE

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

if one solution y_1 is obtained, assume $y_2 = uy_1$

then $y_2'' + py_2' + qy_2' = 0$, $\begin{cases} y_2' = u'y_1 + uy_1' \\ y_2'' = u''y_1 + 2u'y_1' + uy_1'' \end{cases}$

we have $u''y_1 + 2u'y_1' + uy_1'' + pu'y_1 + puy_1' + quy_1 = 0$

$$u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1)u = 0$$

from original equation, $y_1'' + py_1 + q_0y = 0$, we get

$$u''y_1 + u'(2y_1' + py_1) = 0$$

let $v(x) = u'(x)$, we have $v'y_1 + v(2y_1' + py_1) = 0$, then

$$\frac{1}{v}dv = -\frac{2y_1' + py_1}{y_1}dx$$

we get $\ln|v| = -2\int \frac{y_1'}{y_1}dx - \int pdx$

$$\ln|v| = -2\int \frac{1}{y_1}dy_1 - \int pdx = -2\ln|y_1| - \int pdx = \ln|y_1|^{-2} - \int pdx$$

here $v = y_1^{-2} \cdot \exp(-\int pdx)$, $u = \int (y_1^{-2} \cdot \exp(-\int pdx))dx$

$$y_2 = uy_1 = \int vdx\, y_1, \quad y_2 = y_1\int (y_1^{-2} \cdot \exp(-\int pdx))dx$$

for a 2nd order linear homogeneous ODE

$$y_2 = y_1 \int (y_1^{-2} \exp(-\int p(x)dx))dx$$

review $y' + ky = 0$, $y = ce^{-kx}$

2nd order linear homogeneous ODE with constant coefficients

$$y'' + ay' + by = 0$$

Try $y_0(x) = e^{\lambda x}$, here $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$ we have

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0,$$

here, we get the character equation $\lambda^2 + a\lambda + b = 0$, $\begin{cases} \lambda_1 = \frac{1}{2}(-a + \sqrt{a^2-4b}) \\ \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2-4b}) \end{cases}$

case I: when $a^2 - 4b > 0$, we have two different real roots of λ : λ_1, λ_2

we have two basis solution $y_1 = e^{\lambda_1 x}$ $y_2 = e^{\lambda_2 x}$

$$y_1 = e^{\frac{1}{2}(-a+\sqrt{a^2-4b})x}, \quad y_2 = e^{\frac{1}{2}(-a-\sqrt{a^2-4b})x}$$

the general solution is $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$$y = C_1 \exp(\tfrac{1}{2}(-a+\sqrt{a^2-4b})x) + C_2 \exp(\tfrac{1}{2}(-a-\sqrt{a^2-4b})x)$$

C_1 and C_2 are constants to be determined, here $\Delta = a^2 - 4b > 0$

case II: when $a^2 - 4b = 0$, we have a real double root of λ, $\lambda = -\frac{a}{2}$

we can get one basis solution $y_1 = e^{-\frac{a}{2}x}$.

employ $y_2 = y_1 \int (y_1^{-2} \cdot \exp(-\int p\,dx))dx$, $p = a$

$$y_2 = e^{-\frac{a}{2}x} \int (e^{ax} \cdot \exp(-\int a\,dx))dx$$

$$= e^{-\frac{a}{2}x} \int e^{ax} \cdot \exp(-ax + n)\,dx, \quad n \text{ is a constant to be determined}$$

$$= e^{-\frac{a}{2}x} \int D\,dx, \quad D = e^n$$

we get, $y_2 = e^{-\frac{a}{2}x}(Dx + F)$, D, F are constants to be determined

if we force $D = 1$, $F = 0$, then, $y_2 = xe^{-\frac{a}{2}x}$

then, we get two basis solution $y_1 = e^{-\frac{a}{2}x}$, $y_2 = xe^{-\frac{a}{2}x}$,

here $y_2 = xy_1$

and the general solution $y = C_1 e^{-\frac{a}{2}x} + C_2 x e^{-\frac{a}{2}x}$

C_1 and C_2 are constants to be determined, here $\Delta = a^2 - 4b = 0$

case III: when $a^2-4b<0$, we have two different complex roots of λ: λ_1, λ_2

$$\begin{cases}\lambda_1=\frac{1}{2}(-a+\sqrt{a^2-4b})=\frac{1}{2}(-a+i\sqrt{4b-a^2}) \\ \lambda_2=\frac{1}{2}(-a-\sqrt{a^2-4b})=\frac{1}{2}(-a-i\sqrt{4b-a^2})\end{cases}$$

employ $\omega=\frac{1}{2}\sqrt{4b-a^2}$, then $\omega=\frac{1}{2}\sqrt{-\Delta}$, we have $\begin{cases}\lambda_1=-\frac{1}{2}a+i\omega \\ \lambda_2=-\frac{1}{2}a-i\omega\end{cases}$

we get the basis solutions $\begin{cases}y_1=e^{(-\frac{1}{2}a+i\omega)x} \\ y_2=e^{(-\frac{1}{2}a-i\omega)x}\end{cases}$

and the general solution $y=e^{-\frac{1}{2}ax}(c_1e^{i\omega x}+c_2e^{-i\omega x})$

$$y=e^{-\frac{a}{2}x}(c_1(\cos\omega x+i\sin\omega x)+c_2(\cos\omega x-i\sin\omega x))$$

$$y=e^{-\frac{a}{2}x}((c_1+c_2)\cos\omega x+i(c_1-c_2)\sin\omega x)$$

we use $A=c_1+c_2$, $B=i(c_1-c_2)$, A and B are constants to be determined

the basis $\begin{cases}y_1=e^{(-\frac{1}{2}a+i\omega)x}, \quad y_2=e^{(-\frac{1}{2}a-i\omega)x} \\ y_1=e^{-\frac{ax}{2}}\cos\omega x, \quad y_2=e^{-\frac{ax}{2}}\sin\omega x\end{cases}$

$$y=\begin{cases}e^{-\frac{a}{2}x}(c_1e^{i\omega x}+c_2e^{-i\omega x}) \\ e^{-\frac{a}{2}x}(A\cos\omega x+B\sin\omega x)\end{cases}$$

Summary: for a 2nd order linear homogeneous ODE with constant coefficients

$$y''+ay'+by=0$$

case	Roots	Basis solutions	General solution
I	distinct roots λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$
II	real double root $\lambda=-\frac{a}{2}$	$e^{-\frac{a}{2}x}, xe^{-\frac{a}{2}x}$	$c_1e^{-\frac{a}{2}x}+c_2xe^{-\frac{a}{2}x}$
III	complex conjugate $\lambda_1=-\frac{a}{2}+i\omega, \lambda_2=-\frac{a}{2}-i\omega$	$e^{(-\frac{1}{2}a+i\omega)x}, e^{(-\frac{1}{2}a-i\omega)x}$; $e^{-\frac{ax}{2}}\cos\omega x, e^{-\frac{ax}{2}}\sin\omega x$; $\omega=\frac{1}{2}\sqrt{-\Delta}$	$e^{\frac{a}{2}x}(c_1e^{i\omega x}+c_2e^{-i\omega x})$; $e^{-\frac{a}{2}x}(A\cos\omega x+B\sin\omega x)$; $A=c_1+c_2$ $B=i(c_1-c_2)$

examples

1° $y'' - y = 0$,

try $y = e^{\lambda x}$: $\lambda^2 e^{\lambda x} - e^{\lambda x} = 0$,

we get the character equation $\lambda^2 - 1 = 0$,

$\lambda_1 = 1$, $\lambda_2 = -1$

the general solution $y = C_1 e^{x} + C_2 e^{-x}$

C_1 and C_2 are constants to be determined

2° $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$

try $y = e^{\lambda x}$, $\lambda^2 + \lambda - 2 = 0$, $\lambda_1 = -2$, $\lambda_2 = 1$

the general solution $y = C_1 e^{-2x} + C_2 e^{x}$, $y' = -2C_1 e^{-2x} + C_2 e^{x}$

C_1 and C_2 are constants to be determined, use IC:

$y(0) = C_1 + C_2 = 4$,

$y'(0) = -2C_1 + C_2 = -5$,

then $C_1 = 3$, $C_2 = 1$,

we get the particular solution $y = 3e^{-2x} + e^{x}$

3° $y'' + 6y' + 9y = 0$

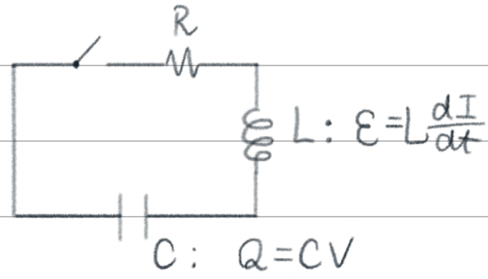
try $y = e^{\lambda x}$: $\lambda^2 + 6\lambda + 9 = 0$, $\lambda = -3$

we have one basis solution $y_1 = e^{-3x}$, and another one $y_2 = xe^{-3x}$

we get a general solution $y = C_1 e^{-3x} + C_2 x e^{-3x}$

C_1 and C_2 are constants to be determined

4° connected circuit at $t=0$



R, L, C are known constants

$V=\frac{Q}{c}=-(IR+\mathcal{E}) \qquad \frac{Q}{c}=\frac{1}{c}\int I dt$

$-\frac{1}{c}\int I dt = IR + L\frac{dI}{dt}$

we process $\frac{d}{dt}(-\frac{1}{c}\int I dt)=\frac{d}{dt}(IR+L\frac{dI}{dt})$, get

$I''+\frac{R}{L}I'+\frac{1}{LC}I=0$

let $a=\frac{R}{L}$, $b=\frac{1}{LC}$, then $a^2-4b=\frac{R^2}{L^2}-\frac{4}{LC}$

If $R=0$: $I''+\frac{1}{LC}I=0$, $a^2-4b=-\frac{4}{LC}$

try $e^{\lambda x}=y$, $\lambda^2+\frac{1}{LC}=0$, we get $\lambda=\pm i\sqrt{\frac{4}{LC}}$, $\omega=\sqrt{\frac{4}{LC}}$

here $I(t)=A\cos(\sqrt{\frac{4}{LC}}t)+iB\sin(\sqrt{\frac{4}{LC}}t)$

if $R=1$, $L=2$, $C=2$

presently $I''+\frac{1}{2}I'+\frac{1}{4}I=0$, $a=\frac{1}{2}$, $b=\frac{1}{4}$

then $a^2-4b=-\frac{3}{4}$, $\omega=\frac{1}{2}\sqrt{4b-a^2}=\frac{\sqrt{3}}{4}$

we employ $y(x)=e^{-\frac{a}{2}x}(A\cos\omega x+B\sin\omega x)$, get

$I(t)=e^{-\frac{t}{4}}(A\cos(\frac{\sqrt{3}}{4}t)+B\cos(\frac{\sqrt{3}}{4}t))$

Lecture 08: Second order linear differential equation III

For a 2nd linear homogeneous ODE:

$$x^2y'' + xay' + by = 0, \qquad a, b \text{ are constant}$$

we call it Eulor-Cauchy equation

Try $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$, we have

$$m(m-1)x^m + amx^m + bx^m = 0$$

we get the following character equation

$$m(m-1) + am + b = 0$$

$$m^2 + (a-1)m + b = 0, \qquad \Delta = (a-1)^2 - 4b$$

case	Roots	Basis solutions	General solution
I: $\Delta>0$	distinct roots m_1, m_2	x^{m_1}, x^{m_2}	$y = C_1x^{m_1} + C_2x^{m_2}$
II: $\Delta=0$	double root $m = \frac{1}{2}(1-a)$	x^m, $\ln\lvert x\rvert \cdot x^m$	$y = (C_1 + C_2\ln\lvert x\rvert)x^m$
III: $\Delta<0$	complex conjugate $m_1 = \frac{1}{2}(1-a) + i\omega$, $m_2 = \frac{1}{2}(1-a) - i\omega$	$\begin{cases} x^{m_1} = x^{\frac{1}{2}(1-a)} \cdot x^{i\omega} \\ x^{m_2} = x^{\frac{1}{2}(1-a)} \cdot x^{-i\omega} \end{cases}$	$y = x^{\frac{1-a}{2}}(C_1x^{i\omega} + C_2x^{-i\omega})$

$$\omega = \frac{1}{2}\sqrt{-\Delta} = \frac{1}{2}\sqrt{4b-(a-1)^2}$$

examples

1° $x^2y'' + \frac{3}{2}xy' - \frac{1}{2}y = 0$

Try $y = x^m$, we have $m(m-1) + \frac{3}{2}m - \frac{1}{2} = 0$, $(2m-1)(m+1) = 0$,

we get $m_1 = -1$, $m_2 = \frac{1}{2}$, $y_1 = x^{-1}$. $y_2 = x^{\frac{1}{2}}$

the general solution $y(x) = C_1\frac{1}{x} + C_2\sqrt{x}$

C_1 and C_2 are constants to be determined

2° $x^2y'' - 5xy' + 9y = 0$

try $y = x^m$, we have $m(m-1) - 5m + 9 = 0$, $(m-3)^2 = 0$: double root case

we get $y_1 = x^3$. $y_2 = \ln|x|x^3$

the general solution $y = C_1x^3 + C_2x^3\ln|x|$

C_1 and C_2 are constants to be determined

veritication

let $y_2 = uy_1 = ux^3$, we have $\begin{cases} y_2' = 3ux^2 + u'x^3 \\ y_2'' = 6ux + 6u'x^2 + u''x^3 \end{cases}$

we have $x^2(6ux + 6u'x^2 + u''x^3) - 5x(3ux^2 + u'x^3) + 9ux^3 = 0$

$6ux^3 + 6u'x^4 + u''x^5 - 15ux^3 - 5u'x^4 + 9ux^3 = 0$

$6u + 6u'x + u''x^2 - 15u - 5u'x + 9u = 0$

$u' + u''x = 0$

let $v = u'$, then $v + v'x = 0$,

$\ln|v| = -\ln|x| + C_3$

$v = \frac{1}{x}$, we force $C_3 = 0$

we get $u = \ln|x| + C$, we force $C_4 = 0$,

we get $y_2 = ux^3 = x^3\ln|x|$

3° $x^2y'' + 0.6xy' + 16.04y = 0$

Try $y = x^m$, we have $m^2 - 0.4m + 16.04 = 0$

$\Delta < 0$, $m_1 = 0.2 + 4i$, $m_2 = 0.2 - 4i$

we get the basis $y_1 = x^{0.2}x^{4i}$, $y_2 = x^{0.2}x^{-4i}$

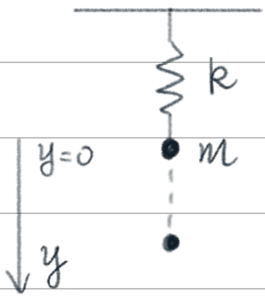
$y_1 = x^{\frac{1}{5}}(e^{\ln x})^{4i} = x^{\frac{1}{5}}e^{i4\ln x} = x^{\frac{1}{5}}(\cos(4\ln x) + i\sin(4\ln x))$

$y_2 = x^{\frac{1}{5}}(e^{\ln x})^{-4i} = x^{\frac{1}{5}}e^{-i4\ln x} = x^{\frac{1}{5}}(\cos(4\ln x) - i\sin(4\ln x))$

we can get the general solution $y = x^{\frac{1}{5}}(A\cos(4\ln x) - B\sin(4\ln x))$

A and B are constants to be determined

4° Spring oscillator



From Newton's law, $\Sigma\vec{F} = ma$

$\vec{F_s} = -ky = my''$, $y = y(t)$

we have $y'' + \frac{k}{m}y = 0$

Try $y = e^{\lambda t}$, $\lambda^2 + \frac{k}{m} = 0$, $\lambda = \pm i\sqrt{\frac{k}{m}}$

then we have the general solution $y = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t$

A and B are constants to be determined