Condensed Matter Physics 2023
Quiz 2 (Week 9)

1. The Fermi temperature T_F of a metal is 81815 K. Calculate the Fermi energy. Do you recognise which metal this is?

The Fermi energy is:

$$E_F = \frac{1}{2}mv_F^2 = k_BT_F = 1.38 \times 10^{-23} \cdot 81815 = 1.13 \times 10^{-18} \text{ J or } 7.05 \text{ eV}$$

As seen in the lectures, this is the Fermi energy of copper.

2. For the same metal of Question 1, calculate the density of free electrons. Then, find the drift velocity of a wire when it carries a current (density) of $j = 1.5 \times 10^7$ A/m^2.

Given that:

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

we can invert the formula and obtain:

$$n = \frac{1}{3\pi^2}\left(\frac{2mE_F}{\hbar^2}\right)^{3/2} = 8.5 \times 10^{28} \text{ m}^{-3}$$

The magnitude of the current density is $j = nev_d$, so we can find the drift velocity:

$$v_d = \frac{j}{ne} = \frac{1.5 \times 10^7}{8.5 \times 10^{28} \cdot 1.602 \times 10^{-19}} = 1.1 \times 10^{-3} \text{ m/s}$$

Note: one can also wok out n from knowing the crystal structure of copper, fcc with a lattice constant $a = 3.6149$ Å, and the fact that copper is monovalent. Then, $n = 4/a^3$ which yields the same result as above.

3. (a) Using the free electron model, calculate the Fermi velocity for gold given that the Fermi energy is 5.5 eV.
   (b) Using now kinetic theory, calculate the root-mean-square velocity at room temperature (from the average kinetic energy).

   How do the two values compare?

The Fermi velocity is:

$$v_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2 \cdot 5.5 \cdot 1.602 \times 10^{-19}}{9.11 \times 10^{-31}}} = 1.39 \times 10^6 \text{ m/s}$$

The root mean square velocity is obtained from the relation

$$\frac{1}{2}mv_{\text{rms}}^2 = \frac{3}{2}k_BT$$

so that

$$v_{\text{rms}} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3 \cdot 1.38 \times 10^{-23} \cdot 300}{9.11 \times 10^{-31}}} = 1.17 \times 10^5 \text{ m/s}$$

As we know, an approximation based on kinetic theory underestimates the free electron velocity by an order of magnitude.

4. The experimental specific heat of potassium metal at low temperature has the form $C = \gamma T + \alpha T^3$. Its Fermi temperature is 37368 K.
   (a) Explain the origin of each of the two terms in this expression.
   (b) Mainstream only. Estimate γ for a mole of potassium from Sommerfeld theory.
   (c) Advanced only. Estimate what fraction of electrons can contribute to the heat capacity of potassium at room temperature.

(a) The cubic term αT^3 is due to the lattice, while the linear term γT is the specific heat of free electrons. The electron contribution dominates at sufficiently low temperatures.

(b) We estimate γ as:

$$\gamma = \frac{\pi^2}{3}\frac{3N_A k_B}{2T_F} = \frac{\pi^2 \cdot 6.022 \times 10^{23} \cdot 1.38 \times 10^{-23}}{2 \cdot 37368} = 1.1 \times 10^{-3}\ \mathrm{J/(mol\ K^2)}$$

where N_A is Avogadro's number.
Note: the Fermi temperature I gave actually corresponds to sodium rather than potassium. This is why this estimate agrees with sodium on slide 27 of Lecture 2, and potassium is slightly different.

(c) The fraction of electrons that can contribute to the heat capacity is approximately

$$\text{Fraction} = \frac{3}{2}\frac{k_B T}{E_F} = \frac{3}{2}\frac{T}{T_F} = \frac{3 \cdot 300}{2 \cdot 37368} = 0.012 = 1.2\%$$

Note: the Fermi temperature I gave actually corresponds to sodium rather than potassium.