V. Conclusion

In this correspondence, we have shown that LLL-aided zero-forcing, which is a polynomial-time algorithm, achieves the maximum receive diversity in MIMO systems. By using LLL reduction before zero-forcing, the complexity of the MIMO decoding is equal to the complexity of the zero-forcing method with just an additional polynomial-time preprocessing for the whole fading block. Also, it is shown that by using the naive lattice decoding, instead of ML decoding, we do not loose the receive diversity order.

Acknowledgment

The authors would like to thank M. O. Damen for helpful discussions.

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Generalized Box–Müller Method for Generating q-Gaussian Random Deviates

William J. Thistleton, John A. Marsh, *Member, IEEE*, Kenric Nelson, *Member, IEEE*, and Constantino Tsallis

***Abstract*—The q-Gaussian distribution is known to be an attractor of certain correlated systems and is the distribution which, under appropriate constraints, maximizes a generalization of the familiar Shannon entropy. This generalized entropy, or q-entropy, provides the basis of nonextensive statistical mechanics, a theory which is postulated as a natural extension of the standard (Boltzmann–Gibbs) statistical mechanics, and which may explain the ubiquitous appearance of heavy-tailed distributions in both natural and man-made systems. The q-Gaussian distribution is also used as a numerical tool, for example as a visiting distribution in Generalized Simulated Annealing. A simple, easy to implement numerical method for generating random deviates from a q-Gaussian distribution based upon a generalization of the well known Box-Müller method is developed and presented. This method is suitable for a larger range of q values, $-\infty < q < 3$, than has previously appeared in the literature, and can generate deviates from q-Gaussian distributions of arbitrary width and center. MATLAB code showing a straightforward implementation is also included.**

***Index Terms*—Entropy maximizing distribution, q-Gaussian distribution, random number generation.**

I. Introduction

The Gaussian (normal) distribution is ubiquitous in probability and statistics due to its role as an attractor of independent systems with finite variance. It is also the distribution which maximizes the Boltzmann–Gibbs entropy $S_{BG} \equiv E[-\log(f(X))] = -\int_{-\infty}^{\infty} \log[f(x)]f(x)dx$ under appropriate constraints [1]. There are many methods available to transform computed pseudorandom uniform deviates into normal deviates, for example the "ziggurat method" of Marsaglia and Tsang [2]. One of the most popular methods is that of Box and Müller [3]. Though not optimal in its properties, it is easy to understand and to implement numerically.

Since q-Gaussian distributions are ubiquitous within the framework of nonextensive statistical mechanics [4] and are applicable to a variety of complex signals and systems, there is a need for a q-Gaussian generator. The q-Gaussian distribution was originally defined as the maximizing distribution for the entropic form

$$S_q = \frac{1 - \int_{-\infty}^{\infty} [p(x)]^q \, dx}{q-1} (q \in \Re) \tag{1}$$

under appropriate constraints. [5] Generalizations of Gauss' Law of Errors [6] and the Central Limit Theorem [7] further demonstrate the underlying prevalence for complex systems to exhibit q-Gaussian

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characteristics. Mechanisms which have been shown to produce q-Gaussian distributions are reviewed in [8] and include multiplicative noise, weakly chaotic dynamics, correlated anomalous diffusion, preferential growth of networks, and asymptotically scale-invariant correlations. The q-Gaussian distributions have been applied in a broad range of fields [9] including thermodynamics, biology, economics, and quantum mechanics. Within electrical and computing systems, applications have included wavelet signal processing [10], generalized simulated annealing [11], quantum information theory [12], and image fusion [13]. Other areas worth further investigation are communications systems and coding techniques which are influenced by multiplicative noise or exhibit power-law distributions.

A recent paper by Deng *et al.* [14] specifies an algorithm to generate q-Gaussian random deviates as the ratio of a standard normal distribution and square root of an independent χ^2 scaled by its degrees of freedom. However, since the resulting Gosset's Student-t distribution is a special case of the q-Gaussian for q in the range 1 to 3, our method allows the generation of a fuller set of distributions [15]. Bailey's Method [16] for generating the Student-t agrees with our method for positive values of the degrees of freedom parameter but does not produce values for compact support q-Gaussian distributions.

In this correspondence, we generalize the Box–Müller algorithm to produce q-Gaussian deviates. Our technique works over the full range of interesting q values, from $-\infty$ to 3. In what follows, we show analytically that the algorithm produces random numbers drawn from the correct q-Gaussian distribution and we demonstrate the algorithm numerically for several interesting values of q. We note here the fact that, while the original Box–Müller algorithm produces pairs of independent Gaussian deviates, our technique produces q-Gaussian deviate pairs that are uncorrelated but not independent. Indeed, for $q < 1$ the joint probability density function of the q-Gaussian has support on a circle while the marginal distributions have support on intervals. Thus the joint distribution may not be recovered simply as the product of the marginal distributions.

II. Review of the Box–Müller Technique

We first review the well known method of Box–Müller for generating Gaussian deviates from a standard normal distribution. We then present a simple variation of this method useful for generating q-Gaussian deviates. Our method is easily understood and can be coded very simply using the pseudorandom number generator for uniform deviates available in almost all computational environments.

Box and Müller built their method from the following observations. Given $Z_1, Z_2 \overset{iid}{\approx} N(0,1)$ uncorrelated (hence in this case independent) standard normal distributions with mean 0 and variance 1, contours of their joint probability density function are circles with $\Theta \equiv \arctan(Z_1/Z_2)$ uniformly distributed. Also, the square of a standard normal distribution has a χ^2 distribution with 1 degree of freedom. Denote a χ^2 random variable with ν degrees of freedom as χ^2_ν and write $Z^2 \sim \chi^2_1$. Because the sum of two independent χ^2 distributions each with 1 degree of freedom is again χ^2 but with 2 degrees of freedom, the random variable defined as $R^2 = Z_1^2 + Z_2^2$ has a χ^2_2 distribution. Related to this point, if a random variable R^2 has a χ^2_2 distribution then $U \equiv \exp(-\frac{1}{2}R^2)$ is uniformly distributed. We are led to the Box–Müller transformations

$$\begin{aligned} U_1 &= \exp\left(-\frac{1}{2}\left(Z_1^2+Z_2^2\right)\right) \\ U_2 &= \frac{1}{2\pi}\operatorname{atan}\left(\frac{Z_2}{Z_1}\right) \end{aligned} \tag{2a,b}$$

and inverse transformations

$$Z_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \tag{3}$$

and

$$Z_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2). \tag{4}$$

We now apply these transformations assuming U_1 and U_2 are independent, uniformly distributed on the interval $(0,1)$, and show that Z_1 and Z_2 are normally distributed. Indeed, given a transformation $(Z_1, Z_2) = T(U_1, U_2)$ between pairs of random variables, we construct the joint density of (Z_1, Z_2), denoted $f_{Z_1,Z_2(z_1,z_2)}$, as

$$f_{Z_1,Z_2}(z_1,z_2) = f_{U_1,U_2}(z_1,z_2)\left|J(z_1,z_2)\right| \tag{5}$$

where the Jacobian is given by

$$J(z_1,z_2) \equiv \frac{\partial u_1}{\partial z_1}\frac{\partial u_2}{\partial z_2} - \frac{\partial u_1}{\partial z_2}\frac{\partial u_2}{\partial z_1}. \tag{6}$$

Since U_1 and U_2 are independent, uniformly distributed random variables on $(0,1)$ we immediately have $f_{U_1,U_2}(z_1,z_2) = 1$. Evaluation of the Jacobian yields

$$\begin{aligned} f_{Z_1,Z_2}(z_1,z_2) &= \frac{1}{2\pi}\exp\left(-\left(z_1^2+z_2^2\right)/2\right) \\ &= \left[\frac{1}{\sqrt{2\pi}}\exp\left(-z_1^2/2\right)\right]\left[\frac{1}{\sqrt{2\pi}}\exp\left(-z_2^2/2\right)\right]. \end{aligned} \tag{7}$$

The Box-Müller method is thus very simple: start with two independent random numbers drawn from the range $(0,1)$, apply the transformations of (3)–(4), and one arrives at two random numbers drawn from a standard Gaussian distribution (mean 0 and variance 1). The two random deviates thus generated are independent due to the separation of the terms in (7).

III. The Q-Gaussian Distribution

Before presenting our generalization of the Box-Müller algorithm, we briefly review the definition and properties of the q-Gaussian function. First, we introduce the q-logarithm and its inverse, the q-exponential [17], as

$$\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q} \qquad x>0 \tag{8}$$

and

$$e_q^x \equiv \begin{cases} [1+(1-q)x]^{\frac{1}{(1-q)}}, & 1+(1-q)x \geq 0 \\ 0, & else \end{cases}. \tag{9}$$

These functions reduce to the usual logarithm and exponential functions when $q=1$. The q-Gaussian density is defined for $-\infty < q < 3$ as

$$\begin{aligned} p\left(x;\overline{\mu}_q,\overline{\sigma}_q\right) &= A_q\sqrt{B_q}\left[1+(q-1)B_q\left(x-\overline{\mu}_q\right)^2\right]^{1/(1-q)} \\ &= A_q\sqrt{B_q}e_q^{-B_q\left(x-\overline{\mu}_q\right)^2}. \end{aligned} \tag{10}$$

where the parameters $\overline{\mu}_q$, $\overline{\sigma}_q^2$, A_q, and B_q are defined as follows. First, the q-mean $\overline{\mu}_q$ is defined analogously to the usual mean, except using the so-called q-expectation value (based on the escort distribution), as follows:

$$\overline{\mu}_q \equiv \langle x\rangle_q \equiv \frac{\int x\,[p(x)]^q\,dx}{\int [p(x)]^q\,dx}. \tag{11}$$

Similarly, the q-variance, $\overline{\sigma}_q^2$ is defined analogously to the usual second order central moment, as

$$\overline{\sigma}_q^2 \equiv \left\langle\left(x-\overline{\mu}_q\right)^2\right\rangle_q \equiv \frac{\int\left(x-\overline{\mu}_q\right)^2[p(x)]^q\,dx}{\int [p(x)]^q\,dx}. \tag{12}$$

When $q = 1$, these expressions reduce to the usual mean and variance. The normalization factor is given by

$$A_q = \begin{cases} \frac{\Gamma\left[\frac{5-3q}{2(1-q)}\right]}{\Gamma\left[\frac{2-q}{1-q}\right]}\sqrt{\frac{1-q}{\pi}}, & q < 1 \\ \frac{1}{\sqrt{\pi}}, & q = 1 \\ \frac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma\left[\frac{3-q}{2(q-1)}\right]}\sqrt{\frac{q-1}{\pi}}, & 1 < q < 3 \end{cases}. \tag{13}$$

Finally, the width of the distribution is characterized by

$$B_q = \left[(3-q)\,\overline{\sigma}_q^2\right]^{-1} \qquad q \in (-\infty, 3). \tag{14}$$

Denote a general q-Gaussian random variable X with q-mean $\overline{\mu}_q$ and q-variance $\overline{\sigma}_q$ as $X \sim N_q\left(\overline{\mu}_q, \overline{\sigma}_q^2\right)$, and call the special case of $\overline{\mu}_q \equiv 0$ and $\overline{\sigma}_q^2 \equiv 1$ a *standard q-Gaussian,* $Z \sim N_q(0,1)$. The density of the standard q-Gaussian distribution may then be written as

$$\begin{aligned} &p\left(x; \overline{\mu}_q = 0, \overline{\sigma}_q = 1\right) \\ &\quad = A_q\sqrt{B_q}\,e_q^{-B_q x^2} = \frac{A_q}{\sqrt{3-q}}\left[1 + \frac{q-1}{3-q}x^2\right]^{\frac{1}{1-q}}. \end{aligned} \tag{15}$$

We show below (23) that the marginal distributions associated with our transformed random variables recover this form with new parameter q'.

The q-Gaussian distribution reproduces the usual Gaussian distribution when $q = 1$. Its density has compact support for $q < 1$, and decays asymptotically as a power law for $1 < q < 3$. For $3 \le q$, the form given in (10) is not normalizable. The usual variance (second order moment) is finite for $q < 5/3$, and, for the standard q-Gaussian, is given by $\sigma^2 = (3-q)/(5-3q)$. The usual variance of the q-Gaussian diverges for $5/3 \le q < 3$, however the q-variance remains finite for the full range $-\infty < q < 3$, equal to unity for the standard q-Gaussian.

IV. Generalized Box Müller for Q-Gaussian Deviates

Our generalization of the Box–Müller technique is based on preserving its circular symmetry, while changing the behavior in the radial direction. Our algorithm starts with the same two uniform deviates as the original Box–Müller technique, and applies a similar transformation to yield q-Gaussian deviates. Thus it preserves the simplicity of implementation that makes the original Box–Müller technique so useful.

Indeed, given independent uniform distributions U_1 and U_2 defined on (0,1), we define two new random variables Z_1 and Z_2 as

$$Z_1 \equiv \sqrt{-2\ln_q(U_1)}\cos(2\pi U_2) \tag{16}$$

and

$$Z_2 \equiv \sqrt{-2\ln_q(U_1)}\sin(2\pi U_2). \tag{17}$$

We now show that each of Z_1 and Z_2 is a standard q-Gaussian characterized by a new parameter $q' = \frac{3q-1}{q+1}$. Thus as q (used in the q-log of the transformation) ranges over $(-1, \infty)$, the q-Gaussian is characterized by q' in the range $(-\infty, 3)$, which is the interesting range for the q-Gaussian. For $q' \ge 3$ the distribution can not be normalized.

Proceed as follows. Obtain the inverse transformations as

$$U_1 = \exp_q\left(-\frac{1}{2}\left(Z_1^2 + Z_2^2\right)\right) \tag{18}$$

and

$$U_2 = \frac{1}{2\pi}\operatorname{atan}\left(\frac{Z_2}{Z_1}\right) \tag{19}$$

so that the Jacobian is

$$J(z_1, z_2) = -\frac{1}{2\pi}\left[\exp_q\left(-\frac{1}{2}\left(z_1^2 + z_2^2\right)\right)\right]^q. \tag{20}$$

And, since $f_{U_1,U_2}(u_1, u_2) = 1$, we obtain the joint density of Z_1 and Z_2 as

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi}\left(e_q^{-\frac{1}{2}\left(z_1^2+z_2^2\right)}\right)^q = \frac{1}{2\pi}e_{(2-1/q)}^{-\frac{q}{2}\left(z_1^2+z_2^2\right)}. \tag{21}$$

Note that for the second part of the above equality we have manipulated the definition of the q-exponential as $\left[e_q^x\right]^q = [1 + (1-q)x]^{\frac{q}{1-q}} = [1 + (1-p)qx]^{\frac{1}{1-p}}$ for $p = 2 - \frac{1}{q}$. Note also that in the limit as $q \to 1$ we recover the product of independent standard normal distributions as required.

The marginal distributions resulting from this joint density are standard q-Gaussian. As illustrated below for the case $q > 1$, one obtains the marginal distributions by integrating (with the aid of the Maple symbolic integration facility) the joint density

$$\begin{aligned} I &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{Z_1,Z_2}(z_1, z_2)\,dz_1 \\ &= \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{q+1}{2(q-1)}\right)}{\Gamma\left(\frac{1}{q-1}\right)}\sqrt{\frac{1}{2}(q-1)}\left(1 - \frac{1}{2}(1-q)z_2^2\right)^{\frac{1}{2}\left(\frac{1+q}{1-q}\right)}. \end{aligned} \tag{22}$$

Note that, by the definition of q', we have $\frac{1}{2}(q-1) = \frac{q'-1}{3-q'}$ and $\frac{1}{2}\left(\frac{1+q}{1-q}\right) = \frac{1}{1-q'}$, and so we obtain

$$\begin{aligned} I &= \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{q'-1}\right)}{\Gamma\left(\frac{3-q'}{2(q'-1)}\right)}\sqrt{\frac{q'-1}{3-q'}}\left(1 + \frac{q'-1}{3-q'}x^2\right)^{\frac{1}{1-q'}} \\ &= A_{q'}\sqrt{B_{q'}}\,e_{q'}^{-B_{q'}x^2}. \end{aligned} \tag{23}$$

As seen from (15) this is a standard q-Gaussian with parameter q'.

An important result is that the standard q-Gaussian recovers the Student's-t distribution with η degrees of freedom when $q \equiv (3+\eta)/(1+\eta)$. Therefore, our method may also be used to generate random deviates from a Student's-t distribution by taking $q' \equiv \frac{3+\eta}{1+\eta}$, or $q = \frac{q'+1}{3-q'} = \frac{2+\eta}{\eta}$. Also, it is interesting to note that, in the limit as $q' \to -\infty$ the q-Gaussian recovers a uniform distribution on the interval $(-1, 1)$.

To illustrate our method, we consider the following empirical probability density functions (relative frequency histograms normalized to have area equal to 1) of generated deviates, with q-Gaussian density functions superimposed. Each histogram is the result of 5×10^6 deviates. We present first two typical histograms for $q' < 1$. Fig. 1 shows the case $q' = -5.0$ and Fig. 2 shows the case $q' = 1/7$. Both the empirical densities show excellent agreement with the theoretical q-Gaussian densities, which have compact support since $q' < 1$. For $q' > 1$, the density has heavy tails and so a standard empirical probability density function is of little value. We show three different plots of the empirical density for the case $q' = 2.75$. In Fig. 3 we show a semi-log plot of the empirical density function, which helps to illustrate the behavior in the near tails of the distribution. In Fig. 4 we plot the q-log of the (appropriately scaled) density versus x^2, so the resulting density function appears as a straight line with slope equal to $-B_q$. Fig. 5 provides another validation of the method. We transform as $Y = \log(|X|)$ to produce a random variable without heavy tails, which may be treated with a traditional histogram. This random variable has density

$$f_Y(y) = 2e^y A_q\sqrt{B_q}\left[1 - \frac{1}{2}(1-q)e^{2y}\right]^{\frac{1}{2}\cdot\left(\frac{1+q}{1-q}\right)}. \tag{24}$$

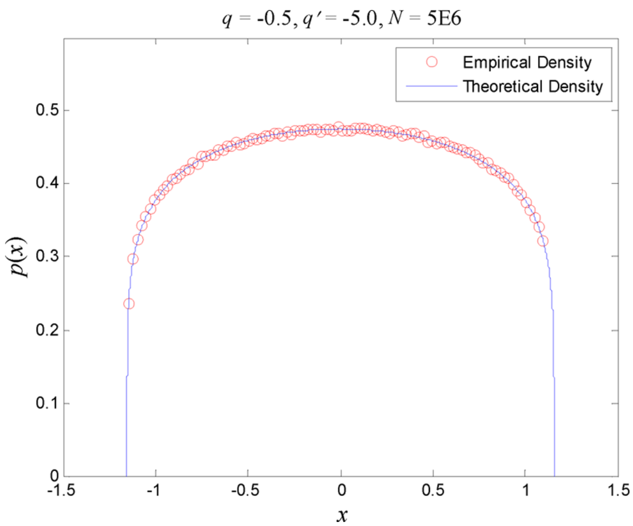

Fig. 1. Theoretical density $p(x)$ and histogram of simulated data for a standard q-Gaussian distribution with $q' < 0$. The distribution has compact support for this q'.

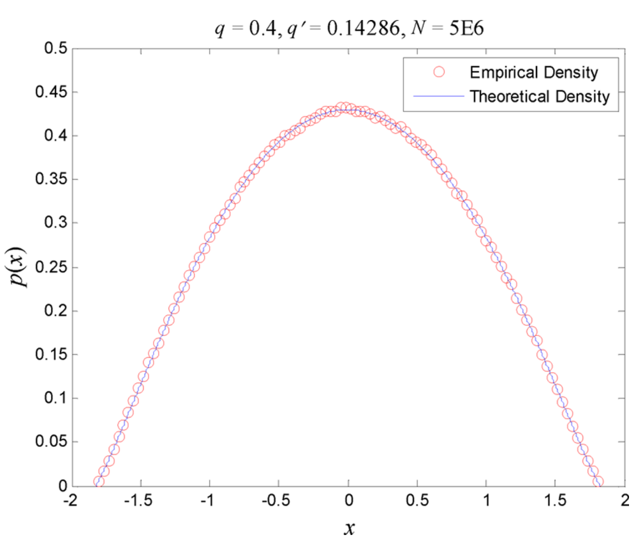

Fig. 2. Theoretical density $p(x)$ and histogram of simulated data for a standard q-Gaussian distribution with $0 < q' < 1$. The distribution has compact support for this q'.

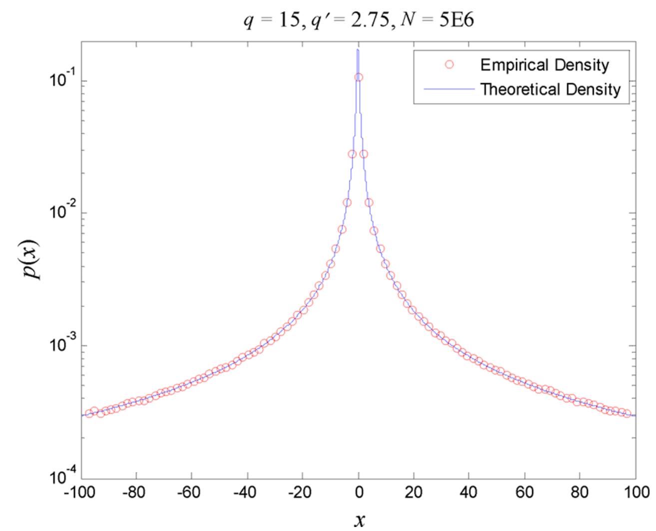

Fig. 3. Theoretical density $p(x)$ and histogram of simulated data for a standard q-Gaussian distribution with $1 < q' < 3$, truncated domain.

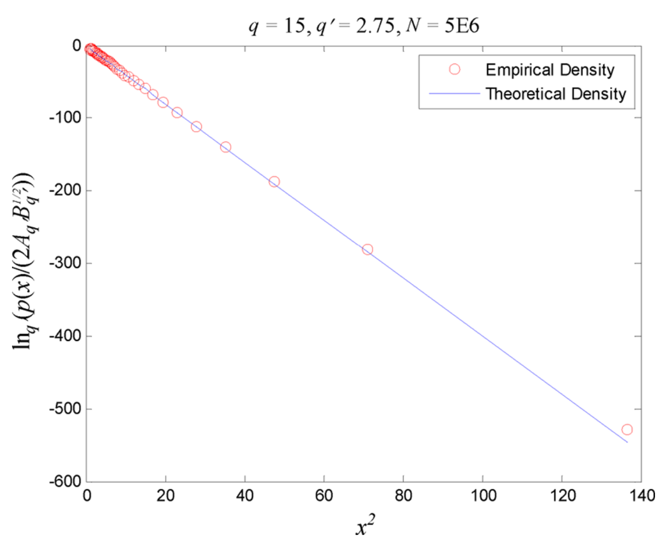

Fig. 4. Theoretical density $\ln_{q'}\left(p(x)/A_{q'}\sqrt{B_{q'}}\right)$ and histogram of simulated data for a standard q-Gaussian distribution, as a function of x^2. Plotted in this form, the q-Gaussian appears as a straight line. The value of q' is the same as in Fig. 3.

We demonstrate the utility of the q-variance $\overline{\sigma}_q^2$ as a characterization of heavy tail q-Gaussian distributions in Fig. 6. For standard q-Gaussian data generated for various values of parameter q, we see that as $q' \rightarrow 5/3$ the second order central moment (i.e., the usual variance) diverges, whereas the q-variance remains steady at unity. For $q' \approx 3$ numerical determination of $\overline{\sigma}_q^2$ is difficult.

The algorithm described in (16)–(17) yields random deviates drawn from a standard q-Gaussian distribution. To produce deviates drawn from a general q-Gaussian distribution with q-mean $\overline{\mu}_q$ and q-variance $\overline{\sigma}_q^2$, one simply performs the usual transformation $X = \overline{\mu}_q + \overline{\sigma}_q Z$. Here $Z \sim N_q(0,1)$ results in $X \sim N_q(\overline{\mu}_q, \overline{\sigma}_q^2)$. Fig. 7 illustrates this transformation, and also further illustrates the utility of the q-variance as a method of characterization. Shown are the variance and q-variance of sample data obtained by transforming standard q-Gaussian data, $Z \sim N_q(0,1)$ with $q'1.4$, via $X = 5+3Z \sim N_q(5,9)$. Data were generated for sample sizes N_S ranging from 50 to 2000. As the sample size grows, the q-variance converges nicely to the expected value $\overline{\sigma}_q^2 = 9$. The sample variances, however, converge relatively slowly to the predicted value $\sigma^2 = [(3-q')/(5-3q')]\,\overline{\sigma}_q^2 = 18$ (where the quantity $(3-q')/(5-3q') = 2$ is the variance of the untransformed data). The utility of the q-variance is illustrated even more dramatically in Fig. 8 where $q' = 2.75$. Note the variance diverges as N_S grows, whereas the q-variance remains at the predicted value of $\overline{\sigma}_q^2 = 9$.

V. Summary and Conclusion

We have presented a method for generating random deviates drawn from a q-Gaussian distribution with arbitrary spread and center, and over the full range of meaningful q values $-\infty < q < 3$. The method presented is simple to code and relatively efficient. The relationship between generating q value and target q' value was developed and verified.

It should be noted that the traditional Box–Müller method produces uncorrelated pairs of bivariate normal deviates. For the case of bivariate normality this implies that the deviates so produced are generated in

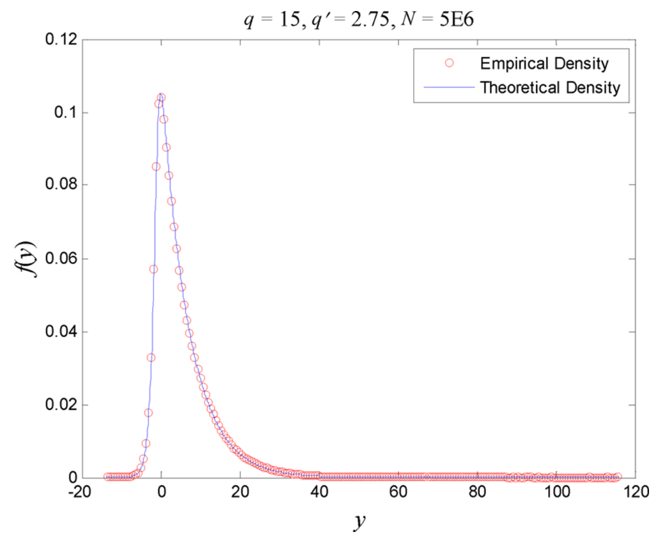

Fig. 5. Theoretical density $f(y)$ of the random variable $Y = \log(|X|)$, and histogram of simulated data, for a standard q-Gaussian random variable X. The value of q' is the same as in Fig. 3.

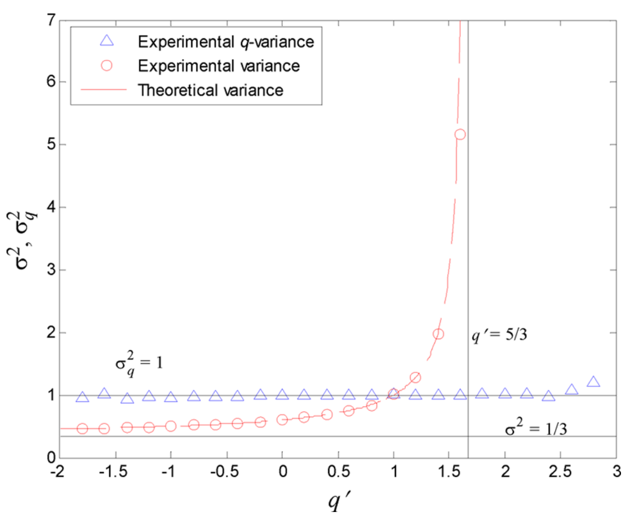

Fig. 6. Comparison of computed variance (circles) and q-variance (triangles). The usual second order central moment is seen to diverge for generated $q' \geq 5/3$. For $q \to -\infty$ the q-Gaussian approaches a uniform distribution on $(-1, 1)$ and so $\sigma^2 \to 1/3$. Very heavy tails for large q renders computation of q-variance difficult.

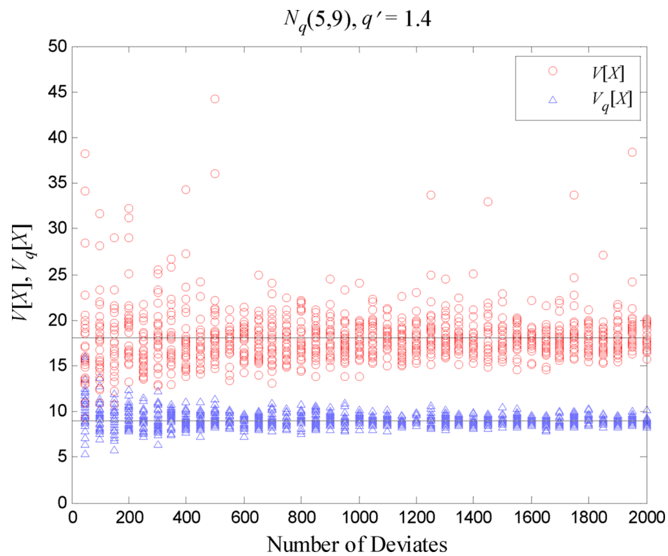

Fig. 7. Computed variance and q-variance for q-Gaussian distributions transformed by $X = 5 + 3Z \sim N_q(5, 9)$. The q-variance is $\overline{\sigma}_q^2 = 9$, and since $q' < 5/3$, the variance is finite, $\sigma^2 = 18$ for this value of q'. Shown are the results from 30 runs at each sample size, where sample sizes range from 50 to 2000. The q-variance shows much faster convergence than the variance.

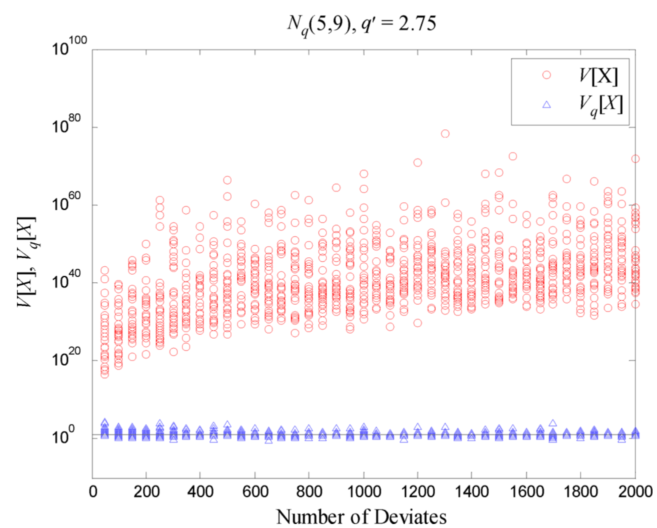

Fig. 8. Computed variance and q-variance for q-Gaussian distributions transformed by $X = 5 + 3Z \sim N_q(5, 9)$. The q-variance remains finite, at the predicted value $\overline{\sigma}_q^2 = 9$. However, since $q' > 5/3$, the variance is infinite. Shown are the results from 30 runs at each sample size, where sample sizes range from 50 to 2000.

independent pairs. For the q-Gaussian, however, even when the second-order moment exists (i.e., $q < 5/3$) the deviate pairs are uncorrelated but not independent. It is an interesting question, unresolved at this point, as to whether these pairs are q-correlated [7]. Random variables which are q-correlated have a special correlation structure in that the q-Fourier Transform of the sum of the random variables must be the q-product of the individual q-Fourier Transforms.

Applications of this q-Gaussian random number generator include many numerical techniques for which a heavy-tailed ($q > 1$) or compact-support ($q < 1$) distribution is required. Examples include generating the visiting step sizes in generalized simulated annealing and generating noise sources with long-range correlations.

APPENDIX

We present, for convenience of implementation, a MATLAB code for the generalized Box-Müller method presented herein. The code shown is intended to demonstrate the algorithm, and is not optimized for speed.

The algorithm is straightforward to implement, and is shown below as two functions. The first function qGaussianDist generates the q-Gaussian random deviates, and calls the second function, log_q, which calculates the q-log. The method relies on a high quality uniform random number generator that produces deviates in the range $(0, 1)$. The MATLAB command producing these deviates is called rand, and in the default implementation in version 7.1, uses a method due to Marsagalia [18]. The other MATLAB-specific component to the code given below is the built-in value eps, whose value is approximately 1E-16 and which represents the limit of double precision.

```
function qGaussian = qGaussianDist(nSamples, qDist)
%
% Returns random deviates drawn from a q-Gaussian
% distribution.
% The number of samples returned is nSamples.
% The q that characterizes the q-Gaussian is given
% by qDist
%
  % Check that q < 3
  if qDist < 3
    % Calculate the q to be used on the q-log
    qGen = (1 + qDist)/(3 - qDist);

    % Initialize the output vector
    qGaussian = zeros(1, nSamples);

    % Loop through, populate the output vector
    for k = 1 : nSamples
        % Get two uniform random deviates
        % from built-in rand function
        u1 = rand;
        u2 = rand;

        % Apply the q-Box-Muller algorithm,
        % taking only one of two possible values
        R = sqrt(-2 * log_q(u1, qGen))
        qGaussian(k) = R * sin(2 * pi * u2);
    end
  % Return 0 and give a warning if q >= 3
  else
        warning('q value must be less than 3')
        qGaussian = 0;
  end
end
```

```
function a = log_q(x, q)
%
% Returns the q-log of x, using q
%

  % Check to see if q = 1 (to double precision)
  if abs(q - 1) < 10 * eps
    % If q is 1, use the usual natural logarithm
    a = log(x);
  else
    % If q differs from 1, use def of the q-log
    a = (x.^(1 - q) - 1)./(1 - q);
  end
end
```

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Kraft Inequality and Zero-Error Source Coding With Decoder Side Information

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***Abstract*—For certain source coding problems, it is well known that the Kraft inequality provides a simple sufficient condition for the existence of codes with given codeword lengths. Motivated by this fact, a sufficient condition based on the Kraft inequality can also be sought for the problem of zero-error instantaneous coding with decoder side information. More specifically, it can be envisioned that a sufficient condition for the existence of such codes with codeword lengths $\{l_x\}$ is that for some $0 < \alpha \leq 1$**

$$\sum_{x \in \mathcal{F}} 2^{-l_x} \leq \alpha$$

for each clique $\mathcal{F}$ in the characteristic graph G of the source-side information pair. In this correspondence, it is shown that 1) if the above is indeed a sufficient condition for a class $\mathcal{G}$ of graphs, then it is possible to come as close as $1 - \log_2 \alpha$ bits to the asymptotic limits for each graph in $\mathcal{G}$, 2) there exist graph classes of interest for which such α can indeed be found, and finally 3) no such α can be found for the class of all graphs.

***Index Terms*—Clique entropy, confusability graph, Kraft inequality, rectangle packing, redundancy, side information, zero-error.**

I. INTRODUCTION

In the classical problem of instantaneous (i.e., prefix-free) coding of a sequence drawn from the alphabet $\mathcal{X}$, the Kraft inequality

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