Estimation of Coupled Exponential Distribution

Plot of equations of GeoMean, mean of pairs, second moment of triplets.

From notes by Amaneh Al-Najafi

Correction: the equation for G (geometric mean), the sign of κ needs to be reversed. This changes the equation to reference from the paper by Vogel to:
$G = \frac{\sigma\,\mu}{\kappa}$ Exp[PolyGamma[1] – PolyGamma[$1 + \frac{1}{\kappa}$] + κ]

$$\begin{bmatrix} G = \dfrac{\sigma\mu}{\kappa}\exp(\psi(1) - \psi(1 + \dfrac{1}{\kappa})) \\ \mu_1 = \mu + \dfrac{\sigma}{2} \\ \mu_1^{(2)} = \mu^2 + \dfrac{2\mu\sigma}{3+\kappa} + \dfrac{2\sigma^2}{3(3+\kappa)} \end{bmatrix}.$$

$$\mu = \mu_1 - \frac{\sigma}{2}$$

$$\kappa = \left[2\sigma(\mu_1 - \frac{\sigma}{2}) + \frac{2\sigma^2}{3} - 3\Big(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2\Big)\right]\left(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2\right)^{-1}$$

$$\sigma = \mu_1 - \sqrt{\mu_1^2 - 2G\frac{\left[2\sigma(\mu_1 - \frac{\sigma}{2}) + \frac{2\sigma^2}{3} - 3\Big(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2\Big)\right]}{(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2)^{-1}}\left[\exp\left(\psi\Big(1 + \frac{\mu_2 - (\mu_1 - \frac{\sigma}{2})^2}{2\sigma(\mu_1 - \frac{\sigma}{2}) + \frac{2\sigma^2}{3} - 3\Big(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2\Big)}\Big) + \gamma\right)\right]}$$

Where $\gamma = 0.5772$

My estimates of the GM, the 1st moment of the pairs, and the 2nd moment of the triplets for several examples of the GPD:

\kappa	GM	1st moment	2nd moment
0.5	2.716987	2.2497471	0.424
1	3.17177	2.249486	4.105583
2	4.9799	2.25	5589.1

The results below are based on Amenah’s derivation; however, the Mathematica derivation showed a difference, so this section will eventually be modified.

```
In[ ]:= $Assumptions == {μ, σ, κ} ∈ Reals && 0 < σ < ∞ && 0 < κ < ∞ && 0 ≤ p ≤ 1
Out[ ]= True == {μ, σ, κ} ∈ ℝ && 0 < σ < ∞ && 0 < κ < ∞ && 0 ≤ p ≤ 1
```

In[•]:=
```
Clear[CoupledExponentialEstimators, GeoMean, MeanPairs, SecondMomentTriplets];
CoupledExponentialEstimators[GeoMean_, MeanPairs_, SecondMomentTriplets_] :=
 {
  GeoMean == σ μ/κ Exp[PolyGamma[1] - PolyGamma[1 + 1/κ] + κ],
  MeanPairs == μ + σ/2,
  SecondMomentTriplets == μ^2 + 2 μ σ/(3 + κ) + 2 σ^2/(3 (3 + κ))
 }
```

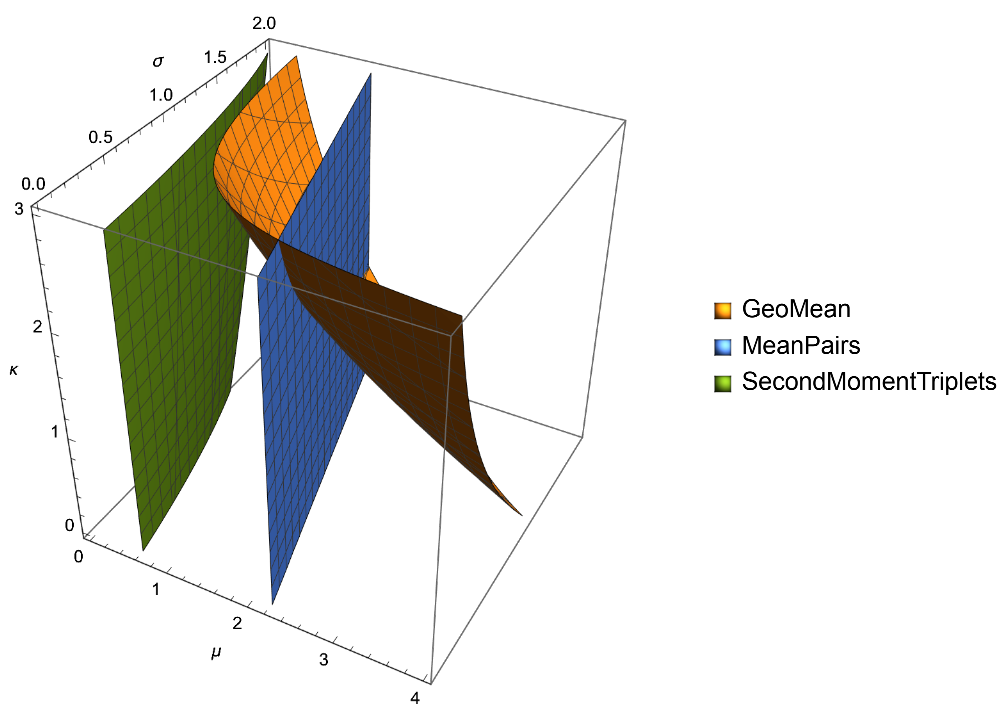
$$\text{GeoMean} == \frac{\sigma\,\mu}{\kappa}\,\text{Exp}\left[\text{PolyGamma}[1] - \text{PolyGamma}\left[1 + \frac{1}{\kappa}\right] + \kappa\right],$$

$$\text{MeanPairs} == \mu + \frac{\sigma}{2},$$

$$\text{SecondMomentTriplets} == \mu^2 + \frac{2\,\mu\,\sigma}{3 + \kappa} + \frac{2\,\sigma^2}{3\,(3 + \kappa)}$$

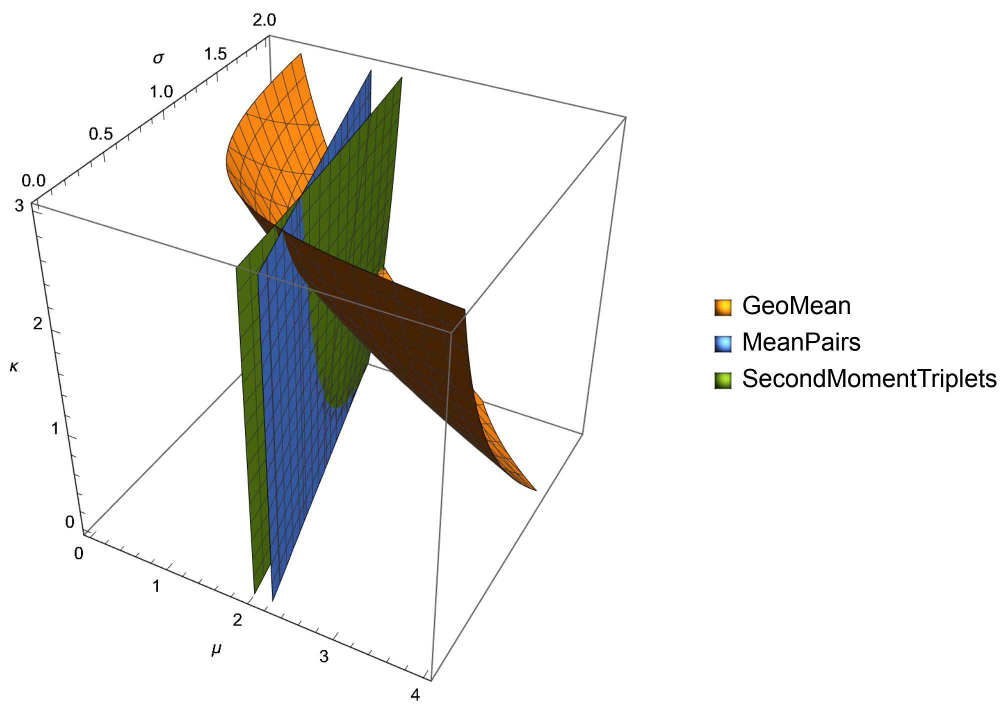
In[•]:=
```
ContourPlot3D[Evaluate@CoupledExponentialEstimators[2.72, 2.25, 0.424],
 {μ, 0, 4}, {σ, 0, 2}, {κ, 0, 3},
 AxesLabel → {μ, σ, κ},
 PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]
```

Out[•]=



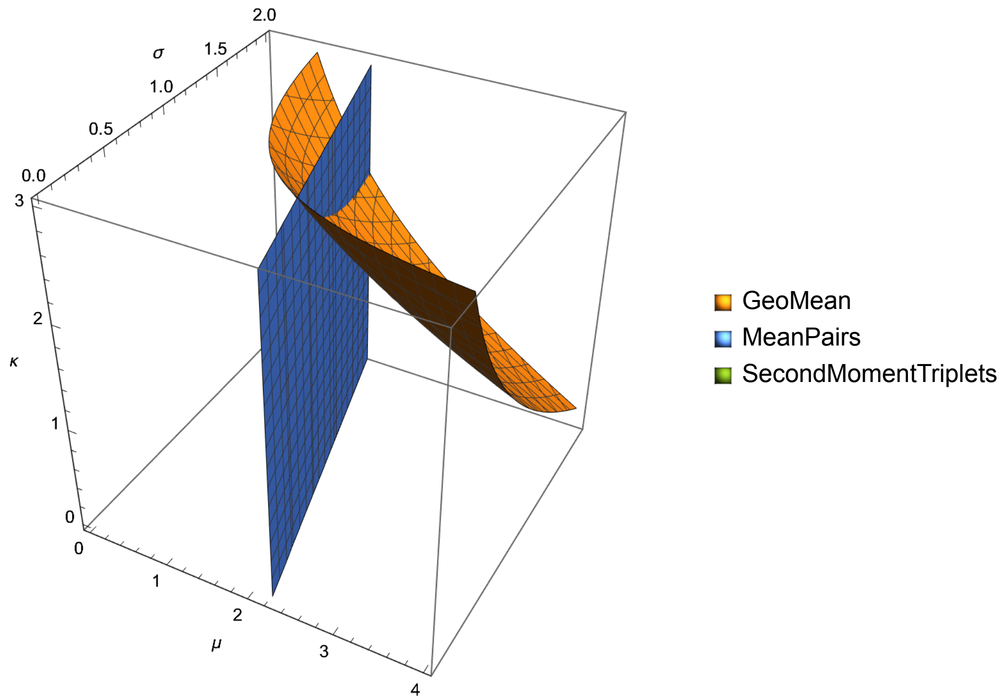
In[ ]:=
```
ContourPlot3D[Evaluate@CoupledExponentialEstimators[3.17, 2.25, 4.15],
  {μ, 0, 4}, {σ, 0, 2}, {κ, 0, 3},
  AxesLabel → {μ, σ, κ},
  PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]
```

Out[ ]=



In[ ]:=
```
ContourPlot3D[Evaluate@CoupledExponentialEstimators[4.98, 2.25, 5589],
 {μ, 0, 4}, {σ, 0, 2}, {κ, 0, 3},
 AxesLabel → {μ, σ, κ},
 PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]
```

Out[ ]=



Derivation of the GeoMean of the Coupled Exponential Distribution

Assuming $\mu = 0$

The Quantile function of the Coupled Exponential Distribution

$$\text{If}\left[\kappa \neq 0,\ \frac{-\sigma}{\kappa}\,(1 - (1 - p)^{-\kappa}),\ -\sigma\,\text{Log}[1 - p]\right]$$

simplifies to σ CoupledLogarithm$\left[(1-p)^{-1}, \kappa\right]$

```
ClearAll[p, κ];
```

```
ClearAll[CoupledExponentialQuantileFunction];
CoupledExponentialQuantileFunction[p_, μ_ : 0, σ_, κ_] :=
 CoupledExponentialQuantileFunction[p, μ, σ, κ] = σ CoupledLogarithm[(1 - p)^-1, κ, 0]
```

Test Quantile Function

In[ ]:=
```
CoupledExponentialQuantileFunction[0.999, 0, 1, -0.5]
```

Out[ ]=
```
1.93675
```

In[ ]:=
```
If[κ ≠ 0, -σ/κ (1 - (1 - p)^-κ),
  -σ Log[1 - p]
 ] /. {p → 0.999, σ → 1, κ → -0.5}
```

Out[ ]= 1.93675

Integration of Quantile Function to form Geometric Mean

In[ ]:=
```
FullSimplify@
 Exp[∫_0^1 FullSimplify@Log@CoupledExponentialQuantileFunction[p, 0, σ, κ] ⅆp]
```

Out[ ]= $e^{\int_0^1 \left\{\begin{array}{ll} \mathrm{Log}\left[\frac{(-1+(1-p)^{-\kappa})\sigma}{\kappa}\right] & \kappa\neq 0 \\ \mathrm{Log}[-\sigma\,\mathrm{Log}[1-p]] & \mathrm{True} \end{array}\right. \text{ if } p<1 \; dp}$

In[ ]:=
```
Assuming[0 < κ < ∞, FullSimplify@Exp[∫_0^1 FullSimplify@Log[((-1 + (1 - p)^-κ) σ)/κ] ⅆp]]
```

Out[ ]= $\frac{e^{-\mathrm{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]}\sigma}{\kappa}$

In[ ]:=
```
Assuming[-1 < κ < 0, FullSimplify@Exp[∫_0^1 FullSimplify@Log[((-1 + (1 - p)^-κ) σ)/κ] ⅆp]]
```

Out[ ]= $-\frac{e^{\kappa-\mathrm{HarmonicNumber}\left[-\frac{1+\kappa}{\kappa}\right]}\sigma}{\kappa}$

In[ ]:=
```
FullSimplify@Exp[∫_0^1 Log[- σ Log[1 - p]] ⅆp]
```

Out[ ]= $e^{-\mathrm{EulerGamma}}\sigma$

The Harmonic number and the Digamma functions have the following relationship.
$H_z = \psi(z+1) - \gamma$ where γ is the Euler gamma constant 0.5172216...

See this Wolfram Research article on the history.
https://functions.wolfram.com/GammaBetaErf/HarmonicNumber2/introductions/DifferentiatedGammas/ShowAll.html

Summarizing Result

GeometricMean of Coupled Exponential =

$$\begin{cases} \frac{e^{-\mathrm{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]}\sigma}{\kappa} & \kappa > 0 \\ -\frac{e^{\kappa-\mathrm{HarmonicNumber}\left[-\frac{1+\kappa}{\kappa}\right]}\sigma}{\kappa} & -1 < \kappa < 0 \\ e^{-\mathrm{EulerGamma}}\sigma & \kappa = 0 \end{cases}$$

Assuming $\mu \neq 0$

In[ ]:=
```
ClearAll[CoupledExponentialQuantileFunction];
CoupledExponentialQuantileFunction[p_, μ_, σ_, κ_] :=
 CoupledExponentialQuantileFunction[p, μ, σ, κ] =
  μ + σ CoupledLogarithm[(1 - p)^-1, κ, 0]
```

In[ ]:=
```
Assuming[0 < κ < ∞ && μ ∈ Reals,
 FullSimplify@Exp[∫_0^1 FullSimplify@Log[μ + ((-1 + (1 - p)^-κ) σ)/κ] dp]]
```

Out[ ]= $e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu \;\text{ if }\; \mu \geq 0$

In[ ]:=
```
Assuming[-1 < κ < 0 && μ ∈ Reals,
 FullSimplify@Exp[∫_0^1 FullSimplify@Log[μ + ((-1 + (1 - p)^-κ) σ)/κ] dp]]
```

Out[ ]= $e^{-\pi\left(-1+\frac{\kappa\mu}{\sigma}\right)^{-1/\kappa}\text{Csc}\left[\frac{\pi}{\kappa}\right]+\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu \;\text{ if }\; \mu \geq 0$

In[ ]:=
```
Assuming[0 < μ < ∞ && μ < σ < ∞, FullSimplify@Exp[∫_0^1 Log[μ - σ Log[1 - p]] dp]]
```

Out[ ]= $e^{-e^{\mu/\sigma}\,\text{ExpIntegralEi}\left[-\frac{\mu}{\sigma}\right]}\,\mu$

There is no simplification to the solution by assuming `0<μ<∞ && μ<σ<∞`

In[ ]:=
```
Assuming[0 < κ < ∞ && 0 < μ < ∞ && μ < σ < ∞,
 FullSimplify@Exp[∫_0^1 FullSimplify@Log[μ + ((-1 + (1 - p)^-κ) σ)/κ] dp]]
```

Out[ ]= $e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu$

In[ ]:=
```
Assuming[-1 < κ < 0 && 0 < μ < ∞ && μ < σ < ∞,
 FullSimplify@Exp[∫_0^1 FullSimplify@Log[μ + ((-1 + (1 - p)^-κ) σ)/κ] dp]]
```

Out[ ]= $e^{\int_0^1 \text{Log}\left[\mu+\frac{\left(-1+(1-p)^{-\kappa}\right)\sigma}{\kappa}\right]dp}$

Check relationship with equation solved by Amenah; there does seem to be a difference

In[ ]:=
```
FullSimplify[(μ σ)/κ Exp[PolyGamma[1] - PolyGamma[1 + 1/κ]], 0 < κ < ∞ && 0 ≤ μ < ∞ && 0 < σ < ∞]
```

Out[ ]= $\frac{e^{-\text{HarmonicNumber}\left[\frac{1}{\kappa}\right]}\,\mu\,\sigma}{\kappa}$

In[ ]:= $\texttt{FullSimplify}\left[e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]},\ 0<\kappa<\infty\right]$

Out[ ]= $e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}$

In[ ]:= $\texttt{FullSimplify}\left[\texttt{Limit}\left[e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu,\ \mu\to 0\right],\ 0<\kappa<\infty\right]$

Out[ ]= $\frac{e^{-\texttt{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]}\sigma}{\kappa}$

In[ ]:= $\texttt{FullSimplify}\left[\texttt{Limit}\left[e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu,\ \kappa\to 0\right],\ \mu\geq 0\right]$

Out[ ]= $\lim_{\kappa\to 0} e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu$

Summarizing Result

GeometricMean of Coupled Exponential

Assuming $\mu\geq 0$ && $\sigma>\mu$

$$\begin{cases} e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu\ \text{ if }\ \mu\geq 0 & \kappa>0 \\ e^{-\pi\left(-1+\frac{\kappa\mu}{\sigma}\right)^{-1/\kappa}\texttt{Csc}\left[\frac{\pi}{\kappa}\right]+\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu\ \text{ if }\ \mu\geq 0 & -1<\kappa<0 \\ e^{-e^{\mu/\sigma}\texttt{ExpIntegralEi}\left[-\frac{\mu}{\sigma}\right]}\mu & \kappa=0 \end{cases}$$

Reduction of Equations

In[ ]:= $\texttt{Solve}\Big[\Big\{$

$\texttt{GeoMean} == e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu,$

$\texttt{MeanPairs} == \mu+\frac{\sigma}{2},$

$\texttt{SecondMomentTriplets} == \mu^2+\frac{2\mu\sigma}{3+\kappa}+\frac{2\sigma^2}{3(3+\kappa)}$

$\Big\},\ \{\mu,\sigma,\kappa\},\ \texttt{Reals}\Big]$

Solve : This system cannot be solved with the methods available to Solve.

Out[ ]= $\texttt{Solve}\left[\left\{\texttt{GeoMean} == e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu,\ \texttt{MeanPairs} == \mu+\frac{\sigma}{2},\ \texttt{SecondMomentTriplets} == \mu^2+\frac{2\mu\sigma}{3+\kappa}+\frac{2\sigma^2}{3(3+\kappa)}\right\},\ \{\mu,\sigma,\kappa\},\ \mathbb{R}\right]$

Fix mistake in next equation

$$\mu == \text{MeanPairs} - \frac{\sigma}{2}$$

In[ ]:=

$$\text{Solve}\Big[\Big\{\text{GeoMean} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\text{MeanPairs}-\frac{\sigma}{2}\right)}{\sigma}\right]}\left(\text{MeanPairs}-\frac{\sigma}{2}\right),\ \text{SecondMomentTriplets} == \left(\text{MeanPairs}-\frac{\sigma}{2}\right)^2+\frac{2\left(\text{MeanPairs}-\frac{\sigma}{2}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\Big\},\ \{\sigma,\kappa\},\ \text{Reals}\Big]$$

Solve : This system cannot be solved with the methods available to Solve.

Out[ ]=

$$\text{Solve}\Big[\Big\{\text{GeoMean} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\text{MeanPairs}-\frac{\sigma}{2}\right)}{\sigma}\right]}\left(\text{MeanPairs}-\frac{\sigma}{2}\right),\ \text{SecondMomentTriplets} == \left(\text{MeanPairs}-\frac{\sigma}{2}\right)^2+\frac{2\left(\text{MeanPairs}-\frac{\sigma}{2}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\Big\},\ \{\sigma,\kappa\},\ \mathbb{R}\Big]$$

In[ ]:=

$$\text{SolveValues}\Big[\Big\{\text{GeoMean} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\text{MeanPairs}-\frac{\sigma}{2}\right)}{\sigma}\right]}\left(\text{MeanPairs}-\frac{\sigma}{2}\right),\ \text{SecondMomentTriplets} == \left(\text{MeanPairs}-\frac{\sigma}{2}\right)^2+\frac{2\left(\text{MeanPairs}-\frac{\sigma}{2}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\Big\},\ \{\sigma,\kappa\},\ \text{Reals}\Big]$$

SolveValues : This system cannot be solved with the methods available to SolveValues.

Out[ ]=

$$\text{SolveValues}\Big[\Big\{\text{GeoMean} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\text{MeanPairs}-\frac{\sigma}{2}\right)}{\sigma}\right]}\left(\text{MeanPairs}-\frac{\sigma}{2}\right),\ \text{SecondMomentTriplets} == \left(\text{MeanPairs}-\frac{\sigma}{2}\right)^2+\frac{2\left(\text{MeanPairs}-\frac{\sigma}{2}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\Big\},\ \{\sigma,\kappa\},\ \mathbb{R}\Big]$$

In[ ]:=

$$\text{SolveValues}\Big[\text{GeoMean} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\text{MeanPairs}-\frac{\sigma}{2}\right)}{\sigma}\right]}\left(\text{MeanPairs}-\frac{\sigma}{2}\right),\ \sigma,\ \text{Reals}\Big]$$

SolveValues : This system cannot be solved with the methods available to SolveValues.

Out[ ]=

$$\text{SolveValues}\Big[\text{GeoMean} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\text{MeanPairs}-\frac{\sigma}{2}\right)}{\sigma}\right]}\left(\text{MeanPairs}-\frac{\sigma}{2}\right),\ \sigma,\ \mathbb{R}\Big]$$

In[ ]:=
```
Solve[
 SecondMomentTriplets == (MeanPairs - σ/2)^2 + (2 (MeanPairs - σ/2) σ)/(3 + κ) + (2 σ^2)/(3 (3 + κ)),
 σ, Reals]
```

$$\left\{\left\{\sigma \to \frac{6\,(\text{MeanPairs} + \text{MeanPairs}\,\kappa)}{5 + 3\,\kappa} - 2\,\sqrt{3}\,\sqrt{\left(\frac{1}{(5 + 3\,\kappa)^2}\,\left(-12\,\text{MeanPairs}^2 + 15\,\text{SecondMomentTriplets} - 8\,\text{MeanPairs}^2\,\kappa + 14\,\text{SecondMomentTriplets}\,\kappa + 3\,\text{SecondMomentTriplets}\,\kappa^2\right)\right)}\right.\right.$$

$$\text{if }\left(\text{SecondMomentTriplets} > \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\,\kappa}{15 + 14\,\kappa + 3\,\kappa^2}\ \&\&\ \kappa > -\frac{5}{3}\right)\ ||\ \left(-3 < \kappa < -\frac{5}{3}\ \&\&\ \text{SecondMomentTriplets} < \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\,\kappa}{15 + 14\,\kappa + 3\,\kappa^2}\right)\ ||\ \left(\kappa < -3\ \&\&\ \text{SecondMomentTriplets} > \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\,\kappa}{15 + 14\,\kappa + 3\,\kappa^2}\right)\Bigg\},$$

$$\left\{\sigma \to \frac{6\,(\text{MeanPairs} + \text{MeanPairs}\,\kappa)}{5 + 3\,\kappa} + 2\,\sqrt{3}\,\sqrt{\left(\frac{1}{(5 + 3\,\kappa)^2}\,\left(-12\,\text{MeanPairs}^2 + 15\,\text{SecondMomentTriplets} - 8\,\text{MeanPairs}^2\,\kappa + 14\,\text{SecondMomentTriplets}\,\kappa + 3\,\text{SecondMomentTriplets}\,\kappa^2\right)\right)}\right.$$

$$\text{if }\left(\text{SecondMomentTriplets} > \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\,\kappa}{15 + 14\,\kappa + 3\,\kappa^2}\ \&\&\ \kappa > -\frac{5}{3}\right)\ ||\ \left(-3 < \kappa < -\frac{5}{3}\ \&\&\ \text{SecondMomentTriplets} < \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\,\kappa}{15 + 14\,\kappa + 3\,\kappa^2}\right)\ ||\ \left(\kappa < -3\ \&\&\ \text{SecondMomentTriplets} > \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\,\kappa}{15 + 14\,\kappa + 3\,\kappa^2}\right)\Bigg\}\Bigg\}$$

Simplify expression in terms of κ

In[ ]:=

$$\text{FullSimplify}\Big[\text{GeoMean} == \mathbb{e}^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\text{MeanPairs}-\frac{\left(\frac{6\,(\text{MeanPairs}+\text{MeanPairs}\,\kappa)}{5+3\kappa}-2\sqrt{3}\sqrt{\left(\frac{1}{(5+3\kappa)^2}\left(-12\,\text{MeanPairs}^2+15\,\text{SecondMomentTriplets}-8\,\text{MeanPairs}^2\,\kappa+14\,\text{SecondMomentTriplets}\,\kappa+3\,\text{Sec}\cdots\right.\right.}\right.}{2}\right.}{\left(\frac{6\,(\text{MeanPairs}+\text{MeanPairs}\,\kappa)}{5+3\kappa}-2\sqrt{3}\sqrt{\left(\frac{1}{(5+3\kappa)^2}\left(-12\,\text{MeanPairs}^2+15\,\text{SecondMomentTriplets}-8\,\text{MeanPairs}^2\,\kappa+14\,\text{SecondMomentTriplets}\,\kappa+3\,\text{S}\cdots\right.\right.}\right.}\right.}$$

$$\left(\text{MeanPairs}-\frac{1}{2}\left(\frac{6\,(\text{MeanPairs}+\text{MeanPairs}\,\kappa)}{5+3\kappa}-2\sqrt{3}\sqrt{\left(\frac{1}{(5+3\kappa)^2}\left(-12\,\text{MeanPairs}^2+15\,\text{SecondMomentTriplets}-8\,\text{MeanPairs}^2\,\kappa+14\,\text{SecondMomentTriplets}\,\kappa+3\,\text{SecondMomentTriplets}\,\kappa^2\right)\right)}\right)\right)\Big]$$

Out[ ]=

$$\frac{1}{5+3\kappa}\,\mathbb{e}^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},\frac{-2\,\text{MeanPairs}\,(3+2\kappa)+(2+\kappa)\,(5+3\kappa)\sqrt{\text{SecondMomentTriplets}+\frac{4\,\text{MeanPairs}^2}{(5+3\kappa)^2}+\frac{4\left(-2\,\text{MeanPairs}^2+\text{SecondMomentTriplets}\right)}{5+3\kappa}}}{-6\,\text{MeanPairs}\,(1+\kappa)+2\,(5+3\kappa)\sqrt{\text{SecondMomentTriplets}+\frac{4\,\text{MeanPairs}^2}{(5+3\kappa)^2}+\frac{4\left(-2\,\text{MeanPairs}^2+\text{SecondMomentTriplets}\right)}{5+3\kappa}}}\right]}$$

$$\left(2\,\text{MeanPairs}+(5+3\kappa)\sqrt{\text{SecondMomentTriplets}+\frac{4\,\text{MeanPairs}^2}{(5+3\kappa)^2}+\frac{4\left(-2\,\text{MeanPairs}^2+\text{SecondMomentTriplets}\right)}{5+3\kappa}}\right)$$

GeoMean

Contour Plots

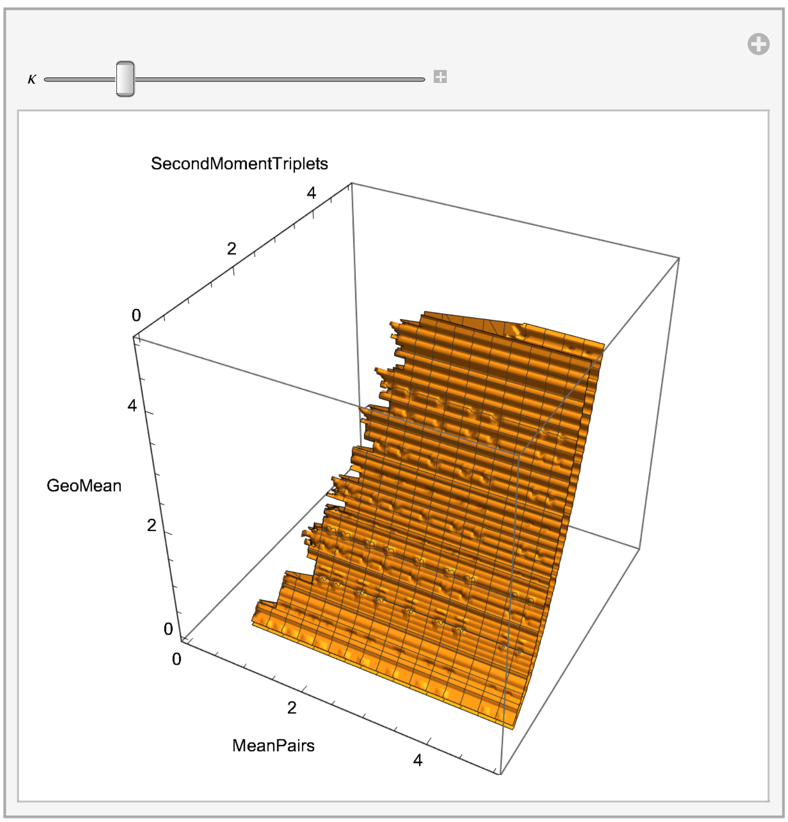
In[ ]:=
```
Manipulate[
 ContourPlot3D[
```

$$\frac{1}{5+3\,\kappa}\,e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},\frac{-2\,\text{MeanPairs}\,(3+2\,\kappa)+(2+\kappa)\,(5+3\,\kappa)\,\sqrt{\text{SecondMomentTriplets}+\frac{4\,\text{MeanPairs}^2}{(5+3\,\kappa)^2}+\frac{4\,\left(-2\,\text{MeanPairs}^2+\text{SecondMomentTriplets}\right)}{5+3\,\kappa}}}{-6\,\text{MeanPairs}\,(1+\kappa)+2\,(5+3\,\kappa)\,\sqrt{\text{SecondMomentTriplets}+\frac{4\,\text{MeanPairs}^2}{(5+3\,\kappa)^2}+\frac{4\,\left(-2\,\text{MeanPairs}^2+\text{SecondMomentTriplets}\right)}{5+3\,\kappa}}}\right]}$$

$$\left(2\,\text{MeanPairs}+(5+3\,\kappa)\,\sqrt{\left(\text{SecondMomentTriplets}+\frac{4\,\text{MeanPairs}^2}{(5+3\,\kappa)^2}+\frac{4\,\left(-2\,\text{MeanPairs}^2+\text{SecondMomentTriplets}\right)}{5+3\,\kappa}\right)}\right) == \text{GeoMean},$$

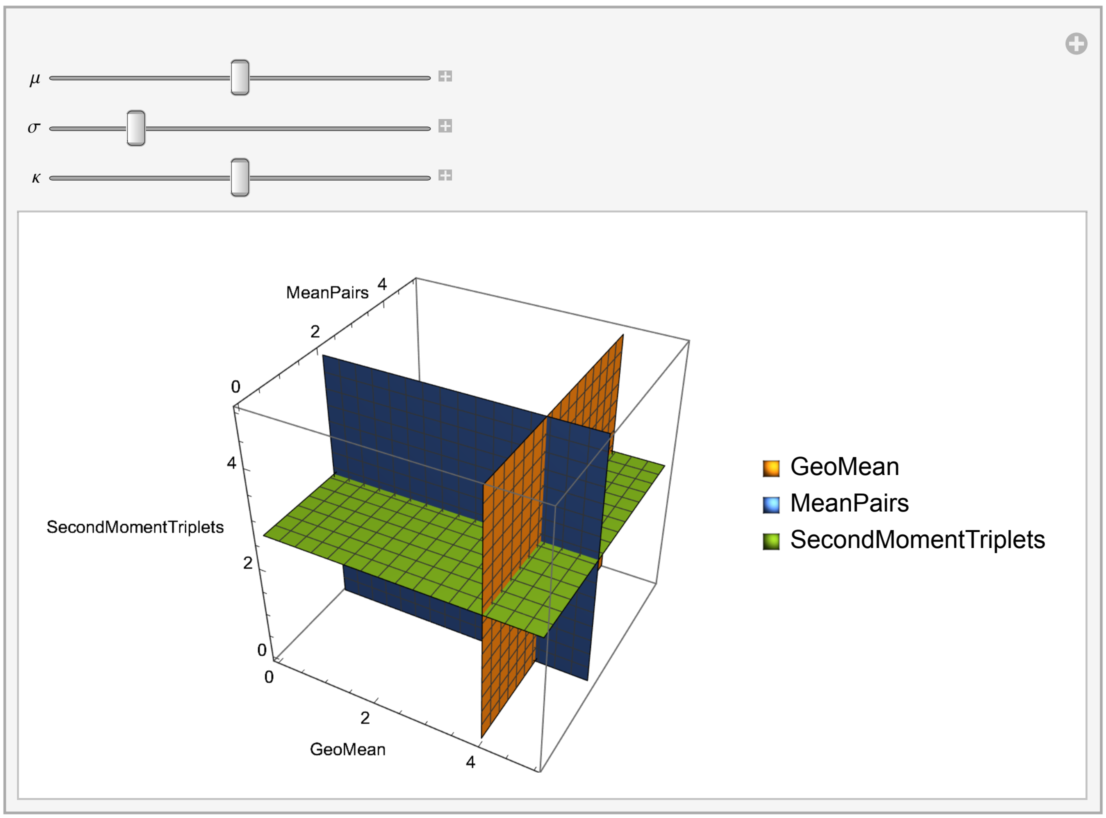
```
  {GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets, 0, 5},
  AxesLabel → {"MeanPairs", "SecondMomentTriplets", "GeoMean"}],
 {{κ, 1}, 0, 2}
]
```

Out[ ]=



In[∘]:=
```
Manipulate[
 ContourPlot3D[
  Evaluate[{
    GeoMean == ⅇ^(κ Hypergeometric2F1[1, 1/κ, 1 + 1/κ, 1 - κ μ/σ]) μ,
    MeanPairs == μ + σ/2,
    SecondMomentTriplets == μ^2 + (2 μ σ)/(3 + κ) + (2 σ^2)/(3 (3 + κ))
   }],
  {GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets, 0, 5},
  AxesLabel → {"GeoMean", "MeanPairs", "SecondMomentTriplets"},
  PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}],
 {{μ, 0.1}, 0, 2}, {{σ, 1}, 0, 10}, {{κ, 0.5}, 0, 2}
]
```

Out[∘]=



Contour Plots for distribution parameters given moment estimations

Attempts to use the Manipulate control result in aborted computation

In[ ]:=

```
Manipulate[
 ContourPlot3D[
  Evaluate[{
```

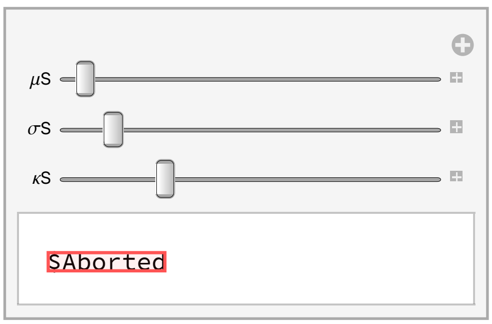
$$e^{\kappa S\, \text{Hypergeometric2F1}\left[1,\frac{1}{\kappa S},1+\frac{1}{\kappa S},1-\frac{\kappa S\,\mu S}{\sigma S}\right]}\,\mu S == e^{\kappa\, \text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\,\mu}{\sigma}\right]}\,\mu,$$

$$\mu S+\frac{\sigma S}{2} == \mu+\frac{\sigma}{2},$$

$$\mu S^2+\frac{2\,\mu S\,\sigma S}{3+\kappa S}+\frac{2\,\sigma S^2}{3\,(3+\kappa S)} == \mu^2+\frac{2\,\mu\,\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}$$

```
   }],
  {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
  AxesLabel → {"μ", "σ", "κ"},
  PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}],
 {{μS, 0.1}, 0, 5}, {{σS, 1}, 0, 10}, {{κS, 0.5}, 0, 2}
]
```

Out[ ]=



General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

General : $\frac{-4.94477\times10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

General : $-\frac{4.85853\times10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

General : Further output of General::munfl will be suppressed during this calculation.

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General : $-\frac{4.85853\times 10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

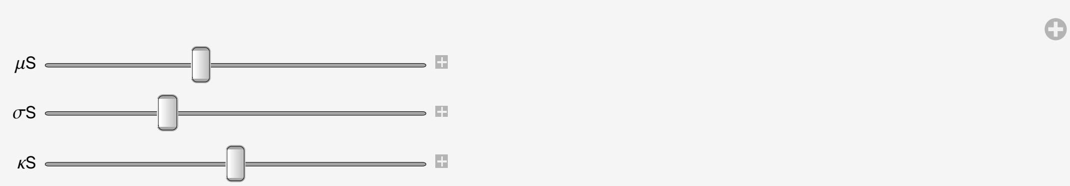
General : Further output of General::munfl will be suppressed during this calculation.

First compute a set of moments

In[ ]:=

```
Manipulate[
 Evaluate[{
   e^(κS Hypergeometric2F1[1, 1/κS, 1+1/κS, 1-(κS μS)/σS]) μS == e^(κ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1-(κ μ)/σ]) μ,
   μS + σS/2 == μ + σ/2,
   μS^2 + (2 μS σS)/(3 + κS) + (2 σS^2)/(3 (3 + κS)) == μ^2 + (2 μ σ)/(3 + κ) + (2 σ^2)/(3 (3 + κ))
  }],
 {{μS, 0.1}, 0, 5}, {{σS, 1}, 0, 10}, {{κS, 0.5}, 0, 2}
]
```

Out[ ]=



$$\left\{\frac{27}{4} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu,\ \frac{7}{2} == \mu+\frac{\sigma}{2},\ \frac{17}{2} == \mu^2+\frac{2\mu\sigma}{3+\kappa}+\frac{2\sigma^2}{3(3+\kappa)}\right\}$$

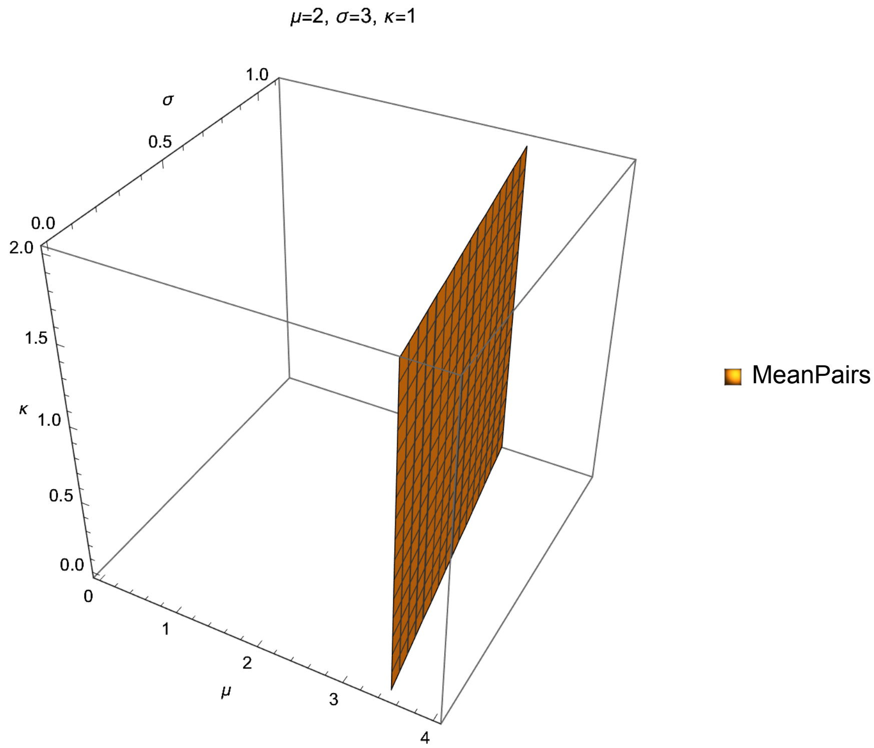
Try Contour Maps from simplest to hardest curve individually

In[ ]:=

```
CEEstimationPlot[Equation_, EquationLabel_] := ContourPlot3D[
  Evaluate[Equation],
  {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
  AxesLabel → {"μ", "σ", "κ"},
  PlotLegends → {EquationLabel},
  PlotLabel → "μ=2, σ=3, κ=1"]
```

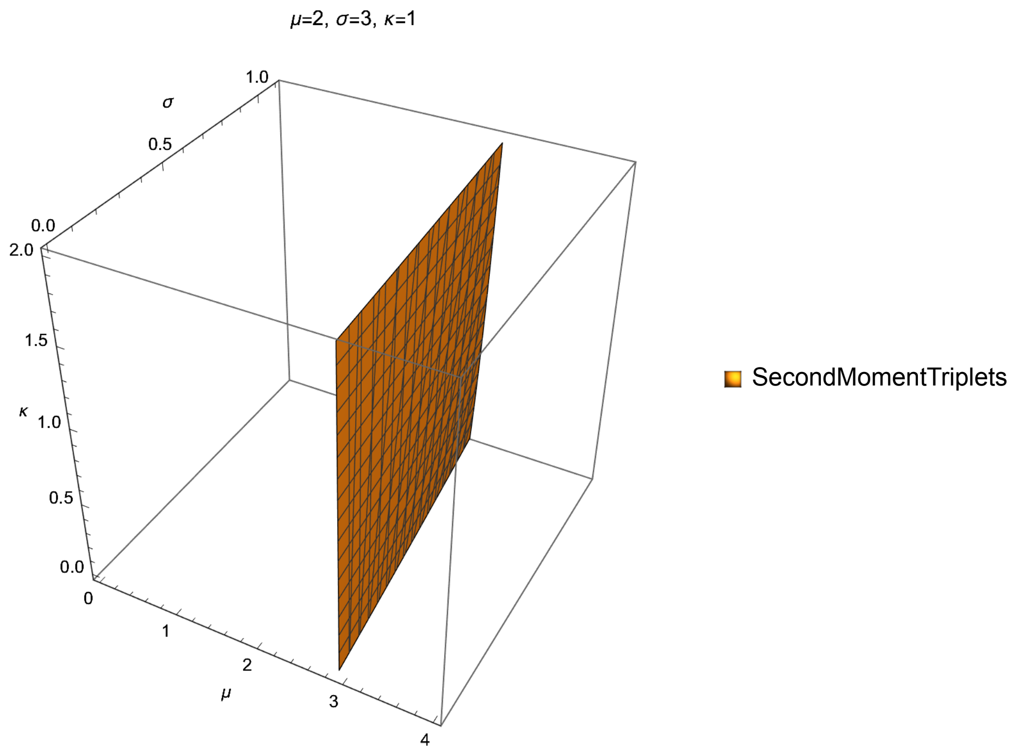
In[ ]:= CEEstimationPlot$\left[\frac{7}{2} == \mu + \frac{\sigma}{2}, \text{"MeanPairs"}\right]$

Out[ ]=



In[ ]:= CEEstimationPlot$\left[\frac{17}{2} == \mu^2 + \frac{2\,\mu\,\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)}, \text{"SecondMomentTriplets"}\right]$

Out[ ]=



In[ ]:= CEEstimationPlot$\left[\frac{27}{4} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\,\mu}{\sigma}\right]}\,\mu, \text{"GeoMean"}\right]$

General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

General : $\frac{-4.94477\times10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

General : $-\frac{4.85853\times10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

General : Further output of General::munfl will be suppressed during this calculation.

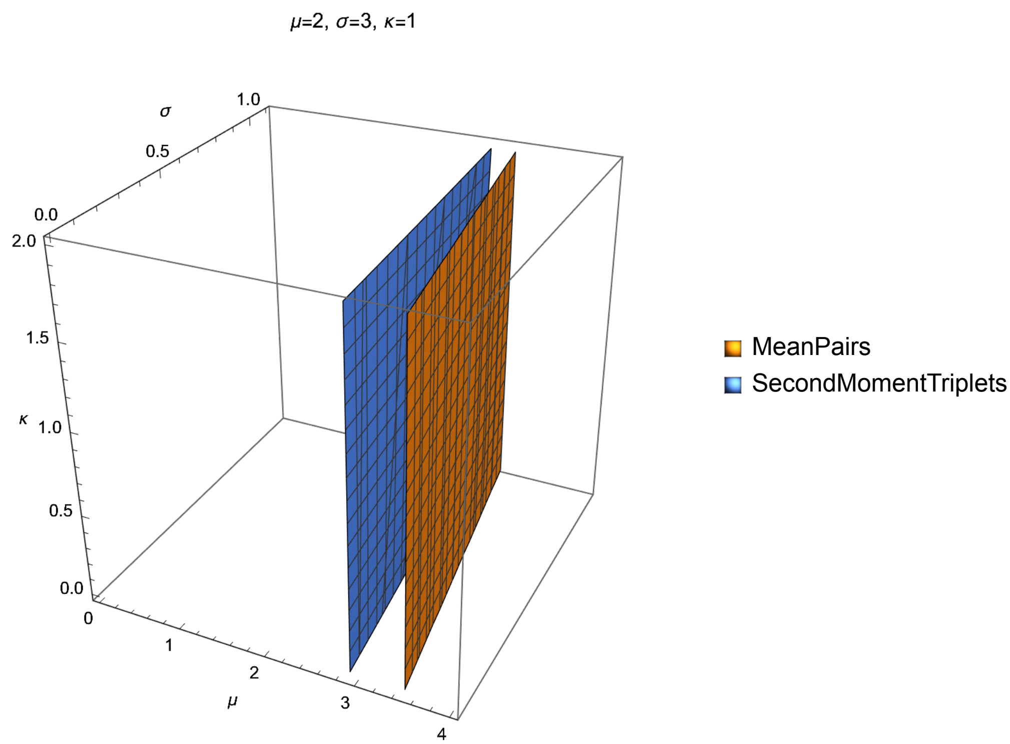
Out[ ]= $Aborted

Not sure why the 3D contour plot for the MeanPair and SecondMomentTriplets does not show an intersection but presumably it has something to do with the internal contour parameter settings

In[ ]:=

```
ContourPlot3D[
 Evaluate[{
   7/2 == μ + σ/2,
   17/2 == μ^2 + (2 μ σ)/(3 + κ) + (2 σ^2)/(3 (3 + κ))
  }],
 {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
 AxesLabel → {"μ", "σ", "κ"},
 PlotLegends → {"MeanPairs", "SecondMomentTriplets"},
 PlotLabel → "μ=2, σ=3, κ=1"]
```

Out[ ]=



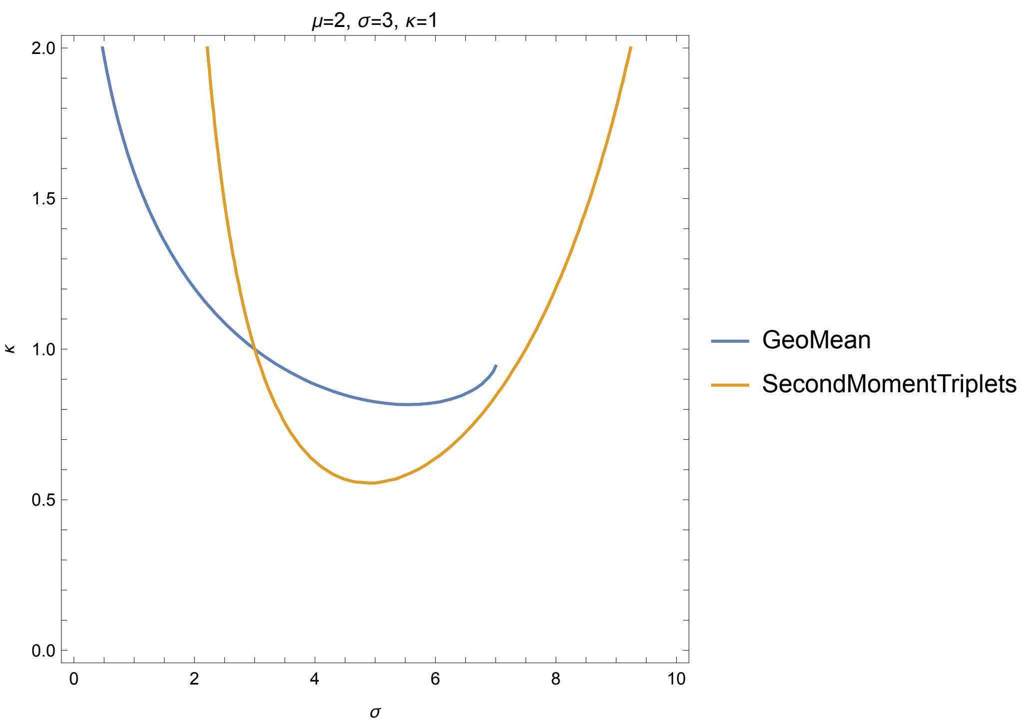
2D Plots do show clear intersections

In[ ]:= ContourPlot[

$$\left\{\frac{27}{4} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\frac{7}{2}-\frac{\sigma}{2}\right)}{\sigma}\right]}\left(\frac{7}{2}-\frac{\sigma}{2}\right),\ \frac{17}{2} == \left(\frac{7}{2}-\frac{\sigma}{2}\right)^2+\frac{2\left(\frac{7}{2}-\frac{\sigma}{2}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\right\},$$

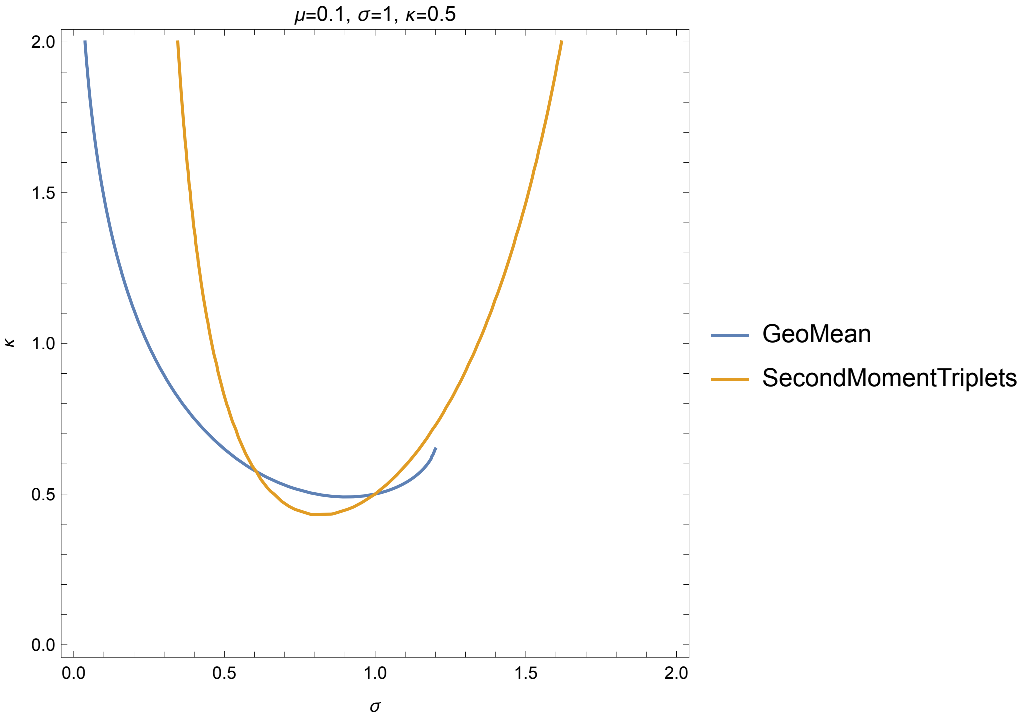
```
{σ, 0, 10}, {κ, 0, 2},
FrameLabel → {"σ", "κ"},
PlotLegends → {"GeoMean", "SecondMomentTriplets"},
PlotLabel → "μ=2, σ=3, κ=1"]
```

Out[ ]=

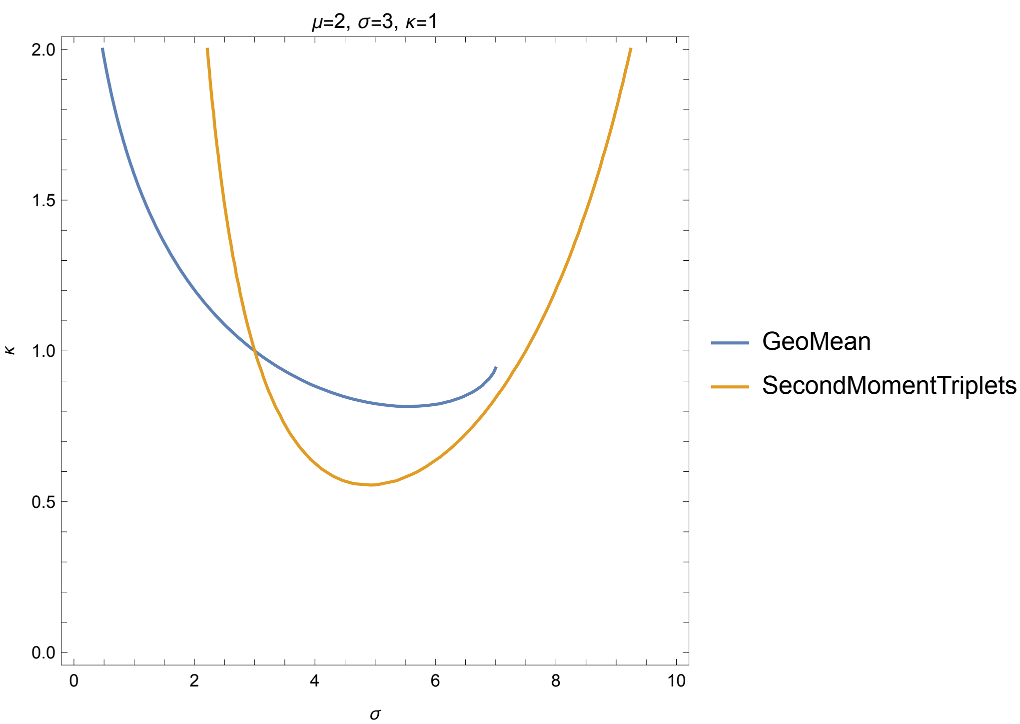


Saved Plots

Out[ ]=



Out[ ]=



Find Minimum of SecondMomentTriples with respect to κ

In[ ]:= Solve$\left[\frac{17}{2} == \left(\frac{7}{2} - \frac{\sigma}{2}\right)^2 + \frac{2\left(\frac{7}{2} - \frac{\sigma}{2}\right)\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)}, \kappa\right]$

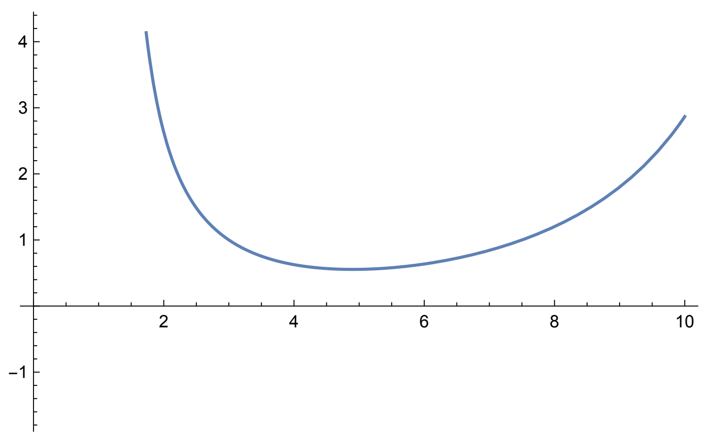
Out[ ]= $\left\{\left\{\kappa \to \frac{-135 + 42\,\sigma - 5\,\sigma^2}{3\,\left(15 - 14\,\sigma + \sigma^2\right)}\right\}\right\}$

In[ ]:= FindMinimum$\left[\frac{-135 + 42\,\sigma - 5\,\sigma^2}{3\,\left(15 - 14\,\sigma + \sigma^2\right)}, \{\sigma, 1\}\right]$

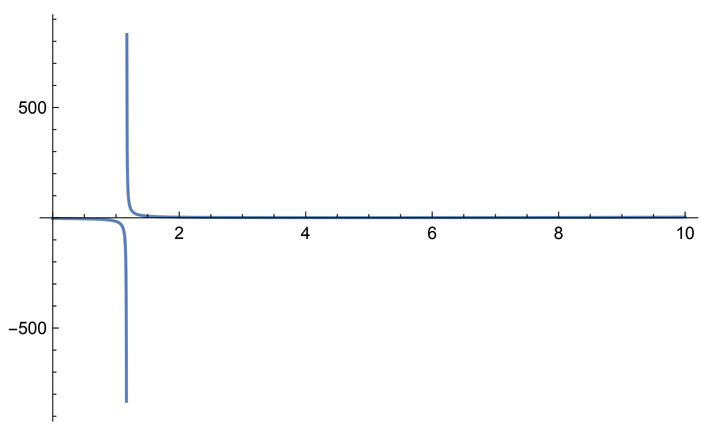
FindMinimum : Line search unable to find a sufficient decrease in the function value with MachinePrecision digit precision.

Out[ ]= $\{-4.3368 \times 10^{14}, \{\sigma \to 1.16905\}\}$

In[ ]:= Plot$\left[\frac{-135 + 42\,\sigma - 5\,\sigma^2}{3\,\left(15 - 14\,\sigma + \sigma^2\right)}, \{\sigma, 0, 10\}, \text{PlotRange} \to \text{Automatic}\right]$

Out[ ]=


In[ ]:= Plot$\left[\frac{-135 + 42\,\sigma - 5\,\sigma^2}{3\,\left(15 - 14\,\sigma + \sigma^2\right)}, \{\sigma, 0, 10\}, \text{PlotRange} \to \text{Full}\right]$

Out[ ]=


In[ ]:= FindMinimum$\left[\dfrac{-135+42\,\sigma-5\,\sigma^2}{3\left(15-14\,\sigma+\sigma^2\right)}, \{\sigma, 2\}\right]$

Out[ ]= {0.554805, {σ → 4.89929}}

So there are some challenges with the search for σ and κ if the search does not seed a value close to the solution. In particular the zero of $\left(15 - 14\,\sigma + \sigma^2\right)$ causes κ to go to infinity. It's important to be the side that produces a positive kappa

Algorithm Prototype for Estimating Coupled Exponentials

Compute Moments

This section will be replaced with drawing of samples and estimation of the moments

In[119]:=
```
Clear[GeoMeanCE, MeanPairsCE, SecondMomentTripletsCE];
SetAttributes[{"GeoMeanCE", "MeanPairsCE", "SecondMomentTripletsCE"}, Listable];
GeoMeanCE[μ_, σ_, κ_] := GeoMeanCE[μ, σ, κ] =
```

$$\begin{cases} \begin{cases} e^{\kappa\,\mathrm{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu & \kappa>0 \\ e^{-\pi\left(-1+\frac{\kappa\mu}{\sigma}\right)^{-1/\kappa}\mathrm{Csc}\left[\frac{\pi}{\kappa}\right]+\kappa\,\mathrm{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu & -1\le\kappa<0 \\ e^{-e^{\mu/\sigma}\,\mathrm{ExpIntegralEi}\left[-\frac{\mu}{\sigma}\right]}\,\mu & \kappa==0 \\ \text{"Not a Distribution"} & \mathrm{True} \end{cases} & \mu>0 \\ \begin{cases} \dfrac{e^{-\mathrm{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]}\,\sigma}{\kappa} & \kappa>0 \\ -\dfrac{e^{\kappa-\mathrm{HarmonicNumber}\left[-\frac{1+\kappa}{\kappa}\right]}\,\sigma}{\kappa} & -1\le\kappa<0 \\ e^{-\mathrm{EulerGamma}}\,\sigma & \kappa==0 \\ \text{"Not a Distribution"} & \mathrm{True} \end{cases} & \mu==0 \\ \text{"Not a Distribution"} & \mathrm{True} \end{cases};$$

```
MeanPairsCE[μ_, σ_] := MeanPairsCE[μ, σ] =
```

$$\mu + \frac{\sigma}{2};$$

```
SecondMomentTripletsCE[μ_, σ_, κ_] := SecondMomentTripletsCE[μ, σ, κ] =
```

$$\mu^2 + \frac{2\,\mu\,\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)};$$

In[125]:=
```
CEMoments =
  {GeoMeanCE[#Location, #Scale, #Shape], MeanPairsCE[#Location, #Scale],
     SecondMomentTripletsCE[#Location, #Scale, #Shape]} &[<|
    "Location" → {0, 1, 2},
    "Scale" → {1, 2, 3},
    "Shape" → {0.5, 1, 2}
    |>
  ]
```

Out[125]= $\left\{\left\{0.735759, 4, 2\, e^{\frac{\pi}{\sqrt{3}}}\right\}, \left\{\frac{1}{2}, 2, \frac{7}{2}\right\}, \left\{0.190476, \frac{8}{3}, \frac{38}{5}\right\}\right\}$

Determine Initial Guess of κ using SecondMomentTriplets equation

The purpose of this step is to find a seed for the Geometric Mean equation. The minimum of the Second Moment Triplets equation does not work as a requirement since there are values of σ where the equation goes through zero and/or infinity. Finding the root of the derivative can work but values of the σ that are negative and/or going to infinity have to be rejected.

In[ ]:= $\text{Solve}\left[\text{SecondMomentTripletsCEValue} == \mu^2 + \frac{2\,\mu\,\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)} \text{ /. } \mu \to \text{MeanPairsCEValue} - \frac{\sigma}{2}, \kappa\right]$

Out[ ]= $\{\{\kappa \to (-36\,\text{MeanPairsCEValue}^2 + 36\,\text{SecondMomentTripletsCEValue} + 12\,\text{MeanPairsCEValue}\,\sigma - 5\,\sigma^2) / (3\,(4\,\text{MeanPairsCEValue}^2 - 4\,\text{SecondMomentTripletsCEValue} - 4\,\text{MeanPairsCEValue}\,\sigma + \sigma^2))\}\}$

Difficulty defining the region to search over in the Second Moment Triplets equation