

Estimation of Coupled Exponential Distribution

Plot of equations of GeoMean, mean of pairs, second moment of triplets.

From notes by Amaneh Al-Najafi

Correction: the equation for G (geometric mean), the sign of κ needs to be reversed. This changes the equation to reference from the paper by Vogel to:

$$G = \frac{\sigma\mu}{\kappa} \text{Exp}\left[\text{PolyGamma}[1] - \text{PolyGamma}\left[1 + \frac{1}{\kappa}\right] + \kappa\right]$$

$$\begin{bmatrix} G = \frac{\sigma\mu}{\kappa} \exp\left(\psi(1) - \psi\left(1 + \frac{1}{\kappa}\right)\right) \\ \mu_1 = \mu + \frac{\sigma}{2} \\ \mu_1^{(2)} = \mu^2 + \frac{2\mu\sigma}{3 + \kappa} + \frac{2\sigma^2}{3(3 + \kappa)} \end{bmatrix}.$$

$$\mu = \mu_1 - \frac{\sigma}{2}$$

$$\kappa = \left[2\sigma\left(\mu_1 - \frac{\sigma}{2}\right) + \frac{2\sigma^2}{3} - 3\left(\mu_2 - \left(\mu_1 - \frac{\sigma}{2}\right)^2\right) \right] \left(\mu_2 - \left(\mu_1 - \frac{\sigma}{2}\right)^2 \right)^{-1}$$

$$\sigma = \mu_1 - \sqrt{\mu_1^2 - 2G \frac{\left[2\sigma\left(\mu_1 - \frac{\sigma}{2}\right) + \frac{2\sigma^2}{3} - 3\left(\mu_2 - \left(\mu_1 - \frac{\sigma}{2}\right)^2\right) \right]}{\left(\mu_2 - \left(\mu_1 - \frac{\sigma}{2}\right)^2\right)^{-1}} \left[\exp\left(\psi\left(1 + \frac{\mu_2 - \left(\mu_1 - \frac{\sigma}{2}\right)^2}{2\sigma\left(\mu_1 - \frac{\sigma}{2}\right) + \frac{2\sigma^2}{3} - 3\left(\mu_2 - \left(\mu_1 - \frac{\sigma}{2}\right)^2\right)}\right) + \gamma \right) \right]}$$

Where $\gamma = 0.5772$

My estimates of the GM, the 1st moment of the pairs, and the 2nd moment of the triplets for several examples of the GPD:

\kappa	GM	1st moment	2nd moment
0.5	2.716987	2.2497471	0.424
1	3.17177	2.249486	4.105583
2	4.9799	2.25	5589.1

The results below are based on Amenah's derivation; however, the Mathematica derivation showed a difference, so this section will eventually be modified.

$In[] := \$Assumptions = \{\mu, \sigma, \kappa\} \in \text{Reals} \ \&\& \ 0 < \sigma < \infty \ \&\& \ 0 < \kappa < \infty \ \&\& \ 0 \leq p \leq 1$

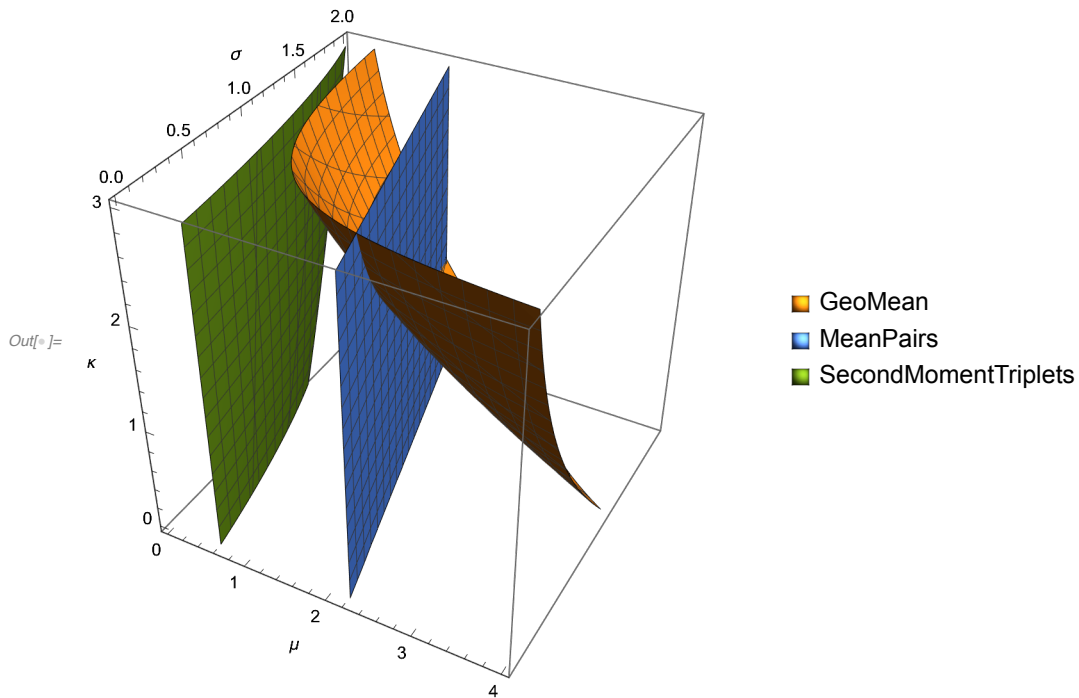
$Out[] = \text{True} = \{\mu, \sigma, \kappa\} \in \mathbb{R} \ \&\& \ 0 < \sigma < \infty \ \&\& \ 0 < \kappa < \infty \ \&\& \ 0 \leq p \leq 1$

```

In[ ]:= Clear[CoupledExponentialEstimators, GeoMean, MeanPairs, SecondMomentTriplets];
CoupledExponentialEstimators[GeoMean_, MeanPairs_, SecondMomentTriplets_] :=
{
  GeoMean ==  $\frac{\sigma \mu}{\kappa} \text{Exp}\left[\text{PolyGamma}[1] - \text{PolyGamma}\left[1 + \frac{1}{\kappa}\right] + \kappa\right],$ 
  MeanPairs ==  $\mu + \frac{\sigma}{2},$ 
  SecondMomentTriplets ==  $\mu^2 + \frac{2 \mu \sigma}{3 + \kappa} + \frac{2 \sigma^2}{3 (3 + \kappa)}$ 
}

In[ ]:= ContourPlot3D[Evaluate@CoupledExponentialEstimators[2.72, 2.25, 0.424],
  {μ, 0, 4}, {σ, 0, 2}, {κ, 0, 3},
  AxesLabel → {μ, σ, κ},
  PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]

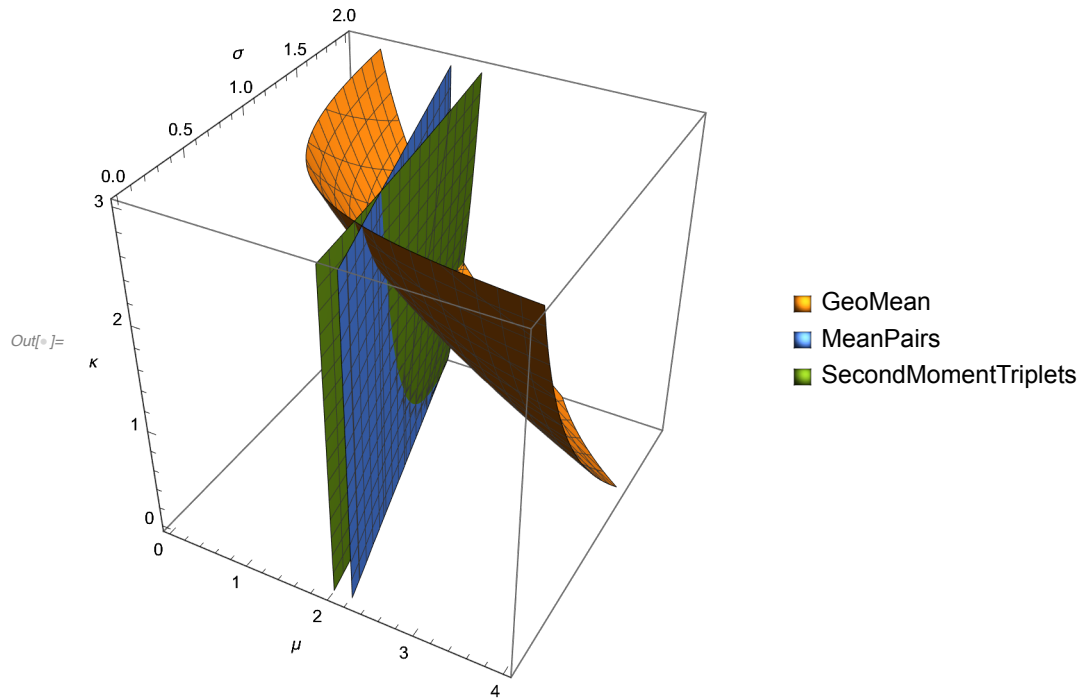
```



```

In[ ]:= ContourPlot3D[Evaluate@CoupledExponentialEstimators[3.17, 2.25, 4.15],
  { $\mu$ , 0, 4}, { $\sigma$ , 0, 2}, { $\kappa$ , 0, 3},
  AxesLabel  $\rightarrow$  { $\mu$ ,  $\sigma$ ,  $\kappa$ },
  PlotLegends  $\rightarrow$  {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]

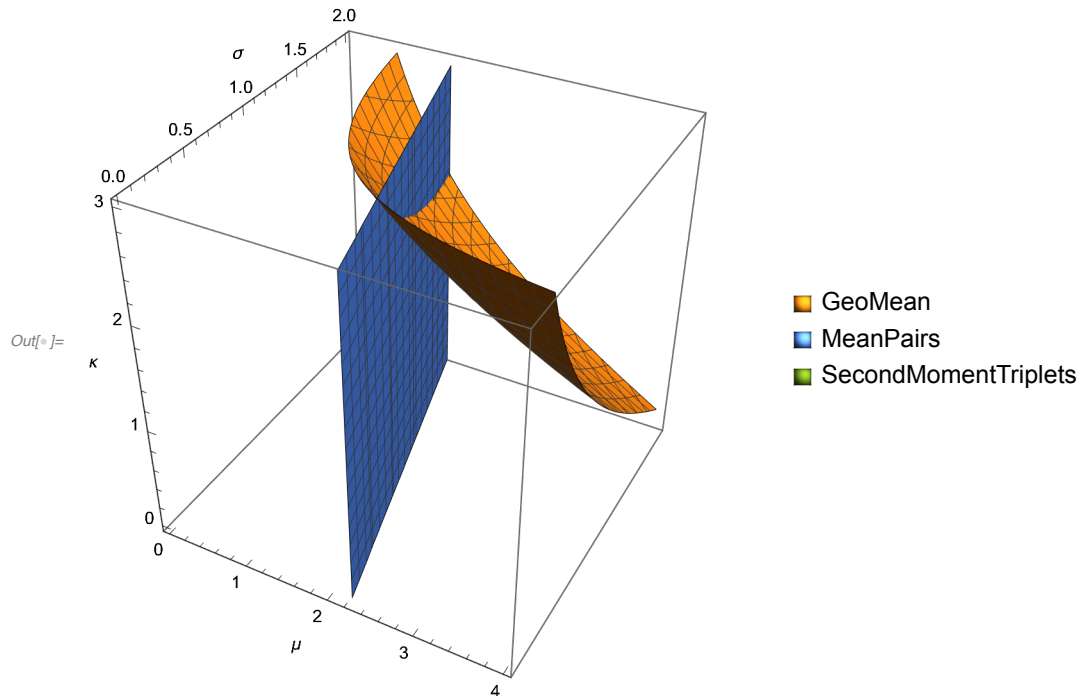
```



```

In[ ]:= ContourPlot3D[Evaluate@CoupledExponentialEstimators[4.98, 2.25, 5589],
  {μ, 0, 4}, {σ, 0, 2}, {κ, 0, 3},
  AxesLabel → {μ, σ, κ},
  PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]

```



Derivation of the GeoMean of the Coupled Exponential Distribution

Assuming $\mu = 0$

The Quantile function of the Coupled Exponential Distribution

If $\left[\kappa \neq 0, \frac{-\sigma}{\kappa} (1 - (1 - p)^{-\kappa}) \right]$,

$-\sigma \text{Log}[1 - p]$

]

simplifies to $\sigma \text{CoupledLogarithm}[(1 - p)^{-1}, \kappa]$

```
ClearAll[p, κ];
```

```
ClearAll[CoupledExponentialQuantileFunction];
```

```
CoupledExponentialQuantileFunction[p_, μ_ : 0, σ_, κ_] :=
```

```
  CoupledExponentialQuantileFunction[p, μ, σ, κ] = σ CoupledLogarithm[(1 - p)-1, κ, 0]
```

Test Quantile Function

```
In[ ]:= CoupledExponentialQuantileFunction[0.999, 0, 1, -0.5]
```

```
Out[ ]:= 1.93675
```

```
In[ ]:= If[κ ≠ 0,  $\frac{-\sigma}{\kappa} (1 - (1 - p)^{-\kappa})$ ,  

    -σ Log[1 - p]  

    ] /. {p → 0.999, σ → 1, κ → -0.5}
```

Out[]:= 1.93675

Integration of Quantile Function to form Geometric Mean

```
In[ ]:= FullSimplify@  

    Exp[ $\int_0^1 \text{FullSimplify@Log@CoupledExponentialQuantileFunction}[p, 0, \sigma, \kappa] \, dp$ ]
```

Out[]:= $e^{\int_0^1 \begin{cases} \text{Log}\left[\frac{(-1 + (1 - p)^{-\kappa}) \sigma}{\kappa}\right] & \kappa \neq 0 \\ \text{Log}[-\sigma \text{Log}[1 - p]] & \text{True} \end{cases} \, dp}$

```
In[ ]:= Assuming[0 < κ < ∞, FullSimplify@Exp[ $\int_0^1 \text{FullSimplify@Log}\left[\frac{(-1 + (1 - p)^{-\kappa}) \sigma}{\kappa}\right] \, dp$ ]]]
```

Out[]:= $\frac{e^{-\text{HarmonicNumber}\left[-1 + \frac{1}{\kappa}\right] \sigma}}{\kappa}$

```
In[ ]:= Assuming[-1 < κ < 0, FullSimplify@Exp[ $\int_0^1 \text{FullSimplify@Log}\left[\frac{(-1 + (1 - p)^{-\kappa}) \sigma}{\kappa}\right] \, dp$ ]]]
```

Out[]:= $-\frac{e^{\kappa - \text{HarmonicNumber}\left[-\frac{1 + \kappa}{\kappa}\right] \sigma}}{\kappa}$

```
In[ ]:= FullSimplify@Exp[ $\int_0^1 \text{Log}[-\sigma \text{Log}[1 - p]] \, dp$ ]
```

Out[]:= $e^{-\text{EulerGamma} \sigma}$

The Harmonic number and the Digamma functions have the following relationship.

$H_z = \psi(z + 1) - \gamma$ where γ is the Euler gamma constant 0.5172216...

See this Wolfram Research article on the history.

<https://functions.wolfram.com/GammaBetaErf/HarmonicNumber2/introductions/DifferentiatedGamma/ShowAll.html>

Summarizing Result

GeometricMean of Coupled Exponential =

$$\begin{cases} \frac{e^{-\text{HarmonicNumber}\left[-1 + \frac{1}{\kappa}\right] \sigma}}{\kappa} & \kappa > 0 \\ -\frac{e^{\kappa - \text{HarmonicNumber}\left[-\frac{1 + \kappa}{\kappa}\right] \sigma}}{\kappa} & -1 < \kappa < 0 \\ e^{-\text{EulerGamma} \sigma} & \kappa = 0 \end{cases}$$

Assuming $\mu \neq 0$

```
In[6]:= ClearAll[CoupledExponentialQuantileFunction];
CoupledExponentialQuantileFunction[p_,  $\mu$ _,  $\sigma$ _,  $\kappa$ _] :=
CoupledExponentialQuantileFunction[p,  $\mu$ ,  $\sigma$ ,  $\kappa$ ] =
 $\mu + \sigma \text{CoupledLogarithm}[(1 - p)^{-1}, \kappa, 0]$ 
```

```
In[7]:= Assuming[ $0 < \kappa < \infty \ \&\& \ \mu \in \text{Reals}$ ,
FullSimplify@Exp[ $\int_0^1 \text{FullSimplify@Log}\left[\mu + \frac{(-1 + (1 - p)^{-\kappa}) \sigma}{\kappa}\right] \text{d}p$ ]]
```

```
Out[6]:=  $e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]} \mu$  if  $\mu \geq 0$ 
```

```
In[8]:= Assuming[ $-1 < \kappa < 0 \ \&\& \ \mu \in \text{Reals}$ ,
FullSimplify@Exp[ $\int_0^1 \text{FullSimplify@Log}\left[\mu + \frac{(-1 + (1 - p)^{-\kappa}) \sigma}{\kappa}\right] \text{d}p$ ]]
```

```
Out[7]:=  $e^{-\pi \left(-1 + \frac{\kappa \mu}{\sigma}\right)^{-1/\kappa} \text{Csc}\left[\frac{\pi}{\kappa}\right] + \kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]} \mu$  if  $\mu \geq 0$ 
```

```
In[179]:= Assuming[ $0 < \mu < \infty \ \&\& \ \mu < \sigma < \infty$ , FullSimplify@Exp[ $\int_0^1 \text{Log}[\mu - \sigma \text{Log}[1 - p]] \text{d}p$ ]]
```

```
Out[179]:=  $e^{-e^{\mu/\sigma} \text{ExpIntegralEi}\left[-\frac{\mu}{\sigma}\right]} \mu$ 
```

There is no simplification to the solution by assuming $0 < \mu < \infty \ \&\& \ \mu < \sigma < \infty$

```
In[180]:= Assuming[ $0 < \kappa < \infty \ \&\& \ 0 < \mu < \infty \ \&\& \ \mu < \sigma < \infty$ ,
FullSimplify@Exp[ $\int_0^1 \text{FullSimplify@Log}\left[\mu + \frac{(-1 + (1 - p)^{-\kappa}) \sigma}{\kappa}\right] \text{d}p$ ]]
```

```
Out[180]:=  $e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]} \mu$ 
```

```
In[181]:= Assuming[ $-1 < \kappa < 0 \ \&\& \ 0 < \mu < \infty \ \&\& \ \mu < \sigma < \infty$ ,
FullSimplify@Exp[ $\int_0^1 \text{FullSimplify@Log}\left[\mu + \frac{(-1 + (1 - p)^{-\kappa}) \sigma}{\kappa}\right] \text{d}p$ ]]
```

```
Out[181]:=  $e^{\int_0^1 \text{Log}\left[\mu + \frac{(-1 + (1 - p)^{-\kappa}) \sigma}{\kappa}\right] \text{d}p}$ 
```

Check relationship with equation solved by Amenah; there does seem to be a difference

```
In[6]:= FullSimplify[ $\frac{\mu \sigma}{\kappa} \text{Exp}\left[\text{PolyGamma}[1] - \text{PolyGamma}\left[1 + \frac{1}{\kappa}\right]\right]$ ,  $0 < \kappa < \infty \ \&\& \ 0 \leq \mu < \infty \ \&\& \ 0 < \sigma < \infty$ ]
```

```
Out[6]:=  $\frac{e^{-\text{HarmonicNumber}\left[\frac{1}{\kappa}\right]} \mu \sigma}{\kappa}$ 
```

In[6]:= FullSimplify[$e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]}$, $0 < \kappa < \infty$]

Out[6]:= $e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]}$

In[7]:= FullSimplify[Limit[$e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]}$ μ , $\mu \rightarrow 0$], $0 < \kappa < \infty$]

Out[7]:= $\frac{e^{-\text{HarmonicNumber}\left[-1 + \frac{1}{\kappa}\right]} \sigma}{\kappa}$

In[8]:= FullSimplify[Limit[$e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]}$ μ , $\kappa \rightarrow 0$], $\mu \geq 0$]

Out[8]:= $\lim_{\kappa \rightarrow 0} e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]}$ μ

Summarizing Result

GeometricMean of Coupled Exponential

Assuming $\mu \geq 0$ && $\sigma > \mu$

$$\begin{cases} e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]}$$
 μ if $\mu \geq 0$ & $\kappa > 0$ \\ $e^{-\pi \left(-1 + \frac{\kappa \mu}{\sigma}\right)^{-1/\kappa} \text{Csc}\left[\frac{\pi}{\kappa}\right] + \kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]}$ μ if $\mu \geq 0$ & $-1 < \kappa < 0$ \\ $e^{-e^{\mu/\sigma} \text{ExpIntegralEi}\left[-\frac{\mu}{\sigma}\right]}$ μ & $\kappa = 0$ \end{cases}

Solution for $\kappa = 0$

Examine application of L'Hopitals Rule

Assume that $\mu > 0$

The issue must be that $\lim_{\kappa \rightarrow 0} \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right] \rightarrow \infty$

In[166]:= Assuming[$0 < \mu < \infty$ && $\mu < \sigma < \infty$, Limit[$\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]$, $\kappa \rightarrow 0$]]

Out[166]:= $\lim_{\kappa \rightarrow 0} \kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]$

In[167]:= D[1 / κ , κ]

Out[167]:= $-\frac{1}{\kappa^2}$

In[170]:= DerHypGeoTerm[$\kappa_$, $\mu_$, $\sigma_$] := D[$\text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]$, κ]

In[171]:= Assuming[$0 < \mu < \infty \&\& \mu < \sigma < \infty$, Limit[$- \kappa^2 \text{DerHypGeoTerm}[\kappa, \mu, \sigma], \kappa \rightarrow 0$]]

$$\text{Out[171]} = \lim_{\kappa \rightarrow 0} -\kappa^2 \left(-\frac{\mu \left(\frac{\sigma}{\kappa \mu} - \text{Hypergeometric2F1} \left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma} \right] \right)}{\kappa \left(1 - \frac{\kappa \mu}{\sigma} \right) \sigma} - \frac{\text{Hypergeometric2F1}^{(0,0,1,0)} \left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma} \right]}{\kappa^2} - \frac{\text{Hypergeometric2F1}^{(0,1,0,0)} \left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma} \right]}{\kappa^2} \right)$$

Check case when $\mu = 0$

In[175]:= Assuming[$0 < \mu < \infty \&\& \mu < \sigma < \infty$, Limit[$\frac{e^{-\text{HarmonicNumber} \left[-1 + \frac{1}{\kappa} \right] \sigma}}{\kappa}, \kappa \rightarrow 0$]]

$$\text{Out[175]} = \lim_{\kappa \rightarrow 0} \frac{e^{-\text{HarmonicNumber} \left[-1 + \frac{1}{\kappa} \right] \sigma}}{\kappa}$$

In[178]:= $D\left[e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]}, \{\kappa, \#\}\right] \& /@ \{1, 2, 3\}$

$$\begin{aligned} \text{Out[178]} = & \left\{ \frac{e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(\frac{\pi^2}{6} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 2\right] \right)}{\kappa^2}, \right. \\ & - \frac{2 e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(\frac{\pi^2}{6} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 2\right] \right)}{\kappa^3} + \\ & \frac{e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(\frac{\pi^2}{6} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 2\right] \right)^2}{\kappa^4} + \\ & \frac{2 e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 3\right] + \text{Zeta}[3] \right)}{\kappa^4}, \\ & \frac{6 e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(\frac{\pi^2}{6} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 2\right] \right)}{\kappa^4} - \\ & \frac{6 e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(\frac{\pi^2}{6} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 2\right] \right)^2}{\kappa^5} + \\ & \frac{e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(\frac{\pi^2}{6} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 2\right] \right)^3}{\kappa^6} + \\ & \frac{6 e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(\frac{\pi^4}{90} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 4\right] \right)}{\kappa^6} - \\ & \frac{12 e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 3\right] + \text{Zeta}[3] \right)}{\kappa^5} + \frac{1}{\kappa^6} e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \\ & \left. \left(\frac{\pi^2}{6} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 2\right] \right) \left(-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 3\right] + \text{Zeta}[3] \right) \right\} \end{aligned}$$

In[177]:= $\frac{e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]} \left(\frac{\pi^2}{6} - \text{HarmonicNumber}\left[-1+\frac{1}{\kappa}, 2\right] \right)}{\kappa^2} \sigma / . \kappa \rightarrow 0$

Power : Infinite expression $\frac{1}{0}$ encountered.

Power : Infinite expression $\frac{1}{0^2}$ encountered.

Power : Infinite expression $\frac{1}{0}$ encountered.

General : Further output of Power::infy will be suppressed during this calculation.

Out[177]= ComplexInfinity

Reduction of Equations

```
In[69]:= Solve[
{
  GeoMean == eκ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1-κμ/σ] μ,
  MeanPairs == μ + σ/2,
  SecondMomentTriplets == μ2 + (2 μ σ)/(3 + κ) + (2 σ2)/(3 (3 + κ))
}, {μ, σ, κ}, Reals]
```

... Solve : This system cannot be solved with the methods available to Solve.

```
Out[69]:= Solve[{GeoMean == eκ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1-κμ/σ] μ, MeanPairs == μ + σ/2,
  SecondMomentTriplets == μ2 + (2 μ σ)/(3 + κ) + (2 σ2)/(3 (3 + κ))}, {μ, σ, κ}, ℝ]
```

Fix mistake in next equation

$$\mu == \text{MeanPairs} - \frac{\sigma}{2}$$

```
In[153]:= Solve[
{
  GeoMean == eκ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1-κ (MeanPairs - σ/2)/σ] (MeanPairs - σ/2),
  SecondMomentTriplets == (MeanPairs - σ/2)2 + (2 (MeanPairs - σ/2) σ)/(3 + κ) + (2 σ2)/(3 (3 + κ))
}, {σ, κ}, Reals]
```

... Solve : This system cannot be solved with the methods available to Solve.

```
Out[153]:= Solve[{GeoMean == eκ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1-κ (MeanPairs - σ/2)/σ] (MeanPairs - σ/2),
  SecondMomentTriplets == (MeanPairs - σ/2)2 + (2 (MeanPairs - σ/2) σ)/(3 + κ) + (2 σ2)/(3 (3 + κ))}, {σ, κ}, ℝ]
```

```
In[154]:= SolveValues[
{
  GeoMean == eκ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1 - (κ (MeanPairs - σ/2))/σ] (MeanPairs - σ/2),
  SecondMomentTriplets == (MeanPairs - σ/2)2 + (2 (MeanPairs - σ/2) σ) / (3 + κ) + (2 σ2) / (3 (3 + κ))
}, {σ, κ}, Reals]
```

... SolveValues : This system cannot be solved with the methods available to SolveValues.

```
Out[154]= SolveValues[ { GeoMean == eκ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1 - (κ (MeanPairs - σ/2))/σ] (MeanPairs - σ/2),
  SecondMomentTriplets == (MeanPairs - σ/2)2 + (2 (MeanPairs - σ/2) σ) / (3 + κ) + (2 σ2) / (3 (3 + κ)) }, {σ, κ}, ℝ]
```

```
In[155]:= SolveValues[
  GeoMean == eκ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1 - (κ (MeanPairs - σ/2))/σ] (MeanPairs - σ/2),
  σ, Reals]
```

... SolveValues : This system cannot be solved with the methods available to SolveValues.

```
Out[155]= SolveValues[ GeoMean == eκ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1 - (κ (MeanPairs - σ/2))/σ] (MeanPairs - σ/2), σ, ℝ]
```

```
In[156]:= Solve[
  SecondMomentTriplets == (MeanPairs - σ/2)2 + (2 (MeanPairs - σ/2) σ) / (3 + κ) + (2 σ2) / (3 (3 + κ)),
  σ, Reals]
```

$$\left\{ \left\{ \sigma \rightarrow \frac{6 (\text{MeanPairs} + \text{MeanPairs} \kappa)}{5 + 3 \kappa} - \right. \right. \\ \left. 2 \sqrt{3} \sqrt{\left(\frac{1}{(5 + 3 \kappa)^2} (-12 \text{MeanPairs}^2 + 15 \text{SecondMomentTriplets} - \right. \right. \\ \left. \left. 8 \text{MeanPairs}^2 \kappa + 14 \text{SecondMomentTriplets} \kappa + 3 \text{SecondMomentTriplets} \kappa^2) \right)} \right. \\ \left. \text{if} \left(\text{SecondMomentTriplets} > \frac{12 \text{MeanPairs}^2 + 8 \text{MeanPairs}^2 \kappa}{15 + 14 \kappa + 3 \kappa^2} \ \&\& \ \kappa > -\frac{5}{3} \right) || \right. \\ \left(-3 < \kappa < -\frac{5}{3} \ \&\& \ \text{SecondMomentTriplets} < \frac{12 \text{MeanPairs}^2 + 8 \text{MeanPairs}^2 \kappa}{15 + 14 \kappa + 3 \kappa^2} \right) || \\ \left(\kappa < -3 \ \&\& \ \text{SecondMomentTriplets} > \frac{12 \text{MeanPairs}^2 + 8 \text{MeanPairs}^2 \kappa}{15 + 14 \kappa + 3 \kappa^2} \right) \left. \right\} \left. \right\},$$

$$\left\{ \sigma \rightarrow \frac{6 (\text{MeanPairs} + \text{MeanPairs} \kappa)}{5 + 3 \kappa} + \right. \\ \left. 2 \sqrt{3} \sqrt{\left(\frac{1}{(5 + 3 \kappa)^2} (-12 \text{MeanPairs}^2 + 15 \text{SecondMomentTriplets} - \right. \right. \\ \left. \left. 8 \text{MeanPairs}^2 \kappa + 14 \text{SecondMomentTriplets} \kappa + 3 \text{SecondMomentTriplets} \kappa^2) \right)} \right. \\ \left. \text{if} \left(\text{SecondMomentTriplets} > \frac{12 \text{MeanPairs}^2 + 8 \text{MeanPairs}^2 \kappa}{15 + 14 \kappa + 3 \kappa^2} \ \&\& \ \kappa > -\frac{5}{3} \right) || \right. \\ \left(-3 < \kappa < -\frac{5}{3} \ \&\& \ \text{SecondMomentTriplets} < \frac{12 \text{MeanPairs}^2 + 8 \text{MeanPairs}^2 \kappa}{15 + 14 \kappa + 3 \kappa^2} \right) || \\ \left(\kappa < -3 \ \&\& \ \text{SecondMomentTriplets} > \frac{12 \text{MeanPairs}^2 + 8 \text{MeanPairs}^2 \kappa}{15 + 14 \kappa + 3 \kappa^2} \right) \left. \right\} \left. \right\}$$

Simplify expression in terms of κ

In[157]:= FullSimplify[GeoMean ==

$$\begin{aligned} & \kappa \text{MeanPairs} - \frac{\left(\frac{6 (\text{MeanPairs} + \text{MeanPairs} \kappa)}{5 + 3 \kappa} - 2 \sqrt{3} \sqrt{\left(\frac{1}{(5 + 3 \kappa)^2} (-12 \text{MeanPairs}^2 + 15 \text{SecondMomentTriplets} - 8 \text{MeanPairs}^2 \kappa + 14 \text{SecondMomentTriplets} \kappa + 3 \text{SecondMomentTriplets} \kappa^2) \right)} \right)}{2} \\ & \times \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\left(\frac{6 (\text{MeanPairs} + \text{MeanPairs} \kappa)}{5 + 3 \kappa} - 2 \sqrt{3} \sqrt{\left(\frac{1}{(5 + 3 \kappa)^2} (-12 \text{MeanPairs}^2 + 15 \text{SecondMomentTriplets} - 8 \text{MeanPairs}^2 \kappa + 14 \text{SecondMomentTriplets} \kappa + 3 \text{SecondMomentTriplets} \kappa^2) \right)} \right)}{2} \right] \\ & e^{\left(\text{MeanPairs} - \frac{1}{2} \left(\frac{6 (\text{MeanPairs} + \text{MeanPairs} \kappa)}{5 + 3 \kappa} - 2 \sqrt{3} \sqrt{\left(\frac{1}{(5 + 3 \kappa)^2} (-12 \text{MeanPairs}^2 + 15 \text{SecondMomentTriplets} - 8 \text{MeanPairs}^2 \kappa + 14 \text{SecondMomentTriplets} \kappa + 3 \text{SecondMomentTriplets} \kappa^2) \right)} \right) \right)} \end{aligned}$$

$$\begin{aligned} & \kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, \frac{-2 \text{MeanPairs} (3 + 2 \kappa) + (2 + \kappa) (5 + 3 \kappa) \sqrt{\text{SecondMomentTriplets} + \frac{4 \text{MeanPairs}^2}{(5 + 3 \kappa)^2} + \frac{4 (-2 \text{MeanPairs}^2 + \text{SecondMomentTriplets})}{5 + 3 \kappa}}}{-6 \text{MeanPairs} (1 + \kappa) + 2 (5 + 3 \kappa) \sqrt{\text{SecondMomentTriplets} + \frac{4 \text{MeanPairs}^2}{(5 + 3 \kappa)^2} + \frac{4 (-2 \text{MeanPairs}^2 + \text{SecondMomentTriplets})}{5 + 3 \kappa}}} \right] \\ & \frac{1}{5 + 3 \kappa} e^{\left(2 \text{MeanPairs} + (5 + 3 \kappa) \sqrt{\text{SecondMomentTriplets} + \frac{4 \text{MeanPairs}^2}{(5 + 3 \kappa)^2} + \frac{4 (-2 \text{MeanPairs}^2 + \text{SecondMomentTriplets})}{5 + 3 \kappa}} \right)} : \\ & \text{GeoMean} \end{aligned}$$

Contour Plots

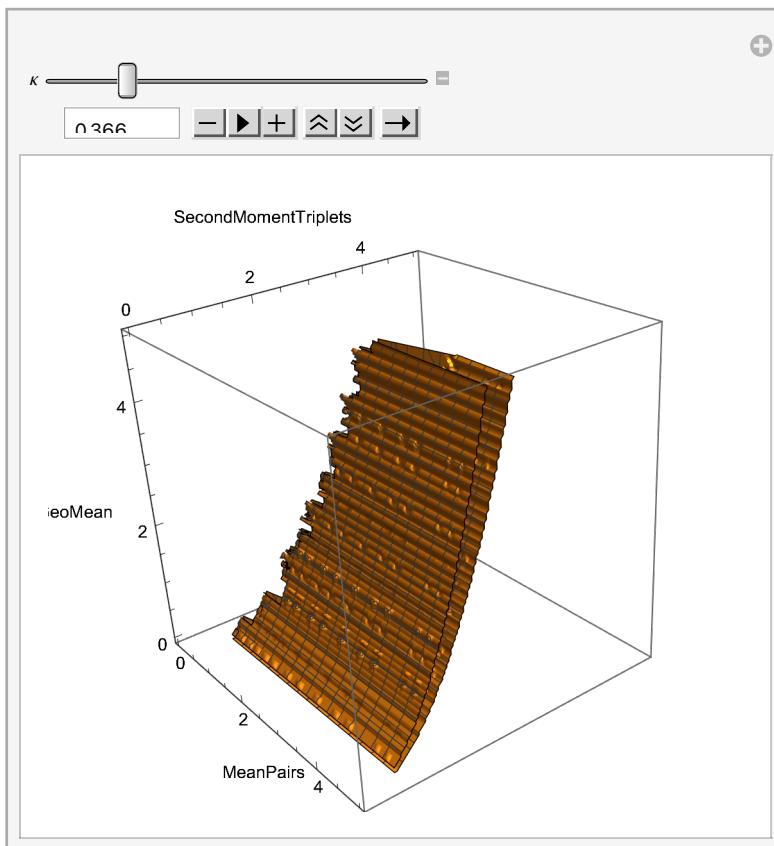
```
In[158]:= Manipulate[
  ContourPlot3D[

$$\frac{1}{5 + 3 \kappa} e^{\frac{\kappa \operatorname{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, \frac{-2 \operatorname{MeanPairs} (3 + 2 \kappa) + (2 + \kappa) (5 + 3 \kappa) \sqrt{\operatorname{SecondMomentTriplets} + \frac{4 \operatorname{MeanPairs}^2}{(5 + 3 \kappa)^2} + \frac{4 (-2 \operatorname{MeanPairs}^2 + \operatorname{SecondMomentTriplets})}{5 + 3 \kappa}}{-6 \operatorname{MeanPairs} (1 + \kappa) + 2 (5 + 3 \kappa) \sqrt{\operatorname{SecondMomentTriplets} + \frac{4 \operatorname{MeanPairs}^2}{(5 + 3 \kappa)^2} + \frac{4 (-2 \operatorname{MeanPairs}^2 + \operatorname{SecondMomentTriplets})}{5 + 3 \kappa}}}\right]}{2 \operatorname{MeanPairs} + (5 + 3 \kappa) \sqrt{\left(\operatorname{SecondMomentTriplets} + \frac{4 \operatorname{MeanPairs}^2}{(5 + 3 \kappa)^2} + \frac{4 (-2 \operatorname{MeanPairs}^2 + \operatorname{SecondMomentTriplets})}{5 + 3 \kappa}\right)}}} = \operatorname{GeoMean},$$

    {GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets, 0, 5},
    AxesLabel -> {"MeanPairs", "SecondMomentTriplets", "GeoMean"}],
    {{κ, 1}, 0, 2}
  ]

```

Out[158]=

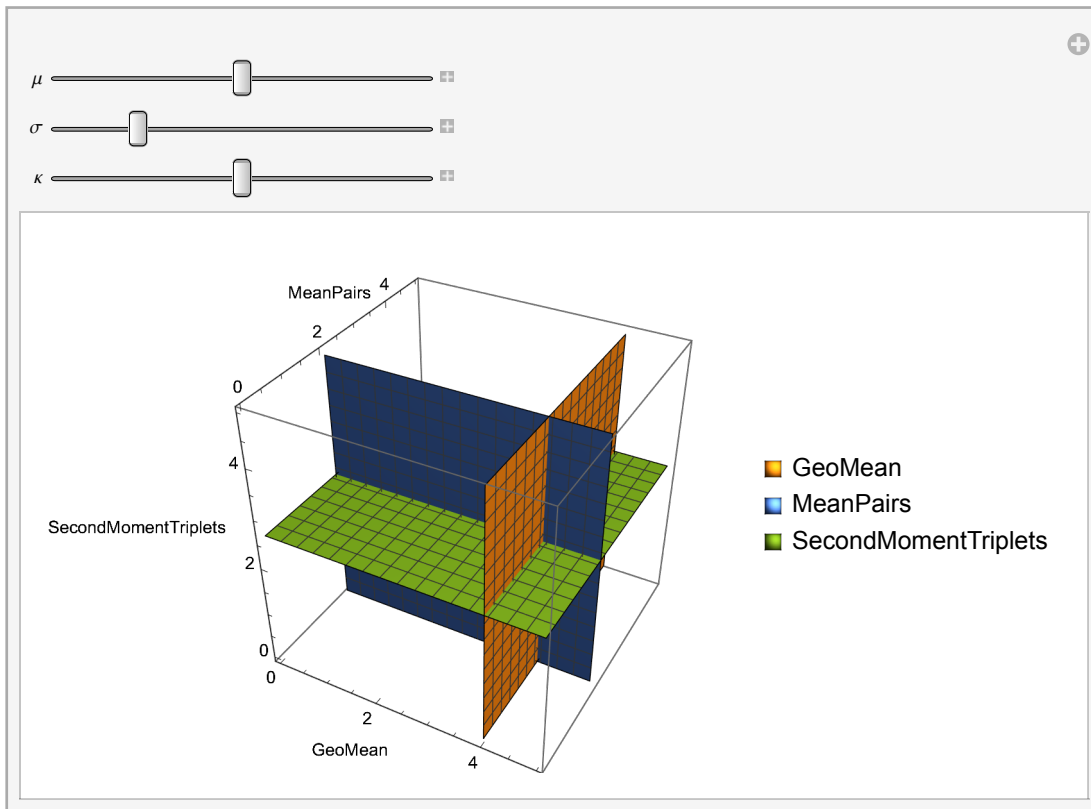


```

In[97]:= Manipulate[
  ContourPlot3D[
    Evaluate[{
      GeoMean == e $\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]$   $\mu$ ,
      MeanPairs ==  $\mu + \frac{\sigma}{2}$ ,
      SecondMomentTriplets ==  $\mu^2 + \frac{2 \mu \sigma}{3 + \kappa} + \frac{2 \sigma^2}{3 (3 + \kappa)}$ 
    }],
    {GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets, 0, 5},
    AxesLabel → {"GeoMean", "MeanPairs", "SecondMomentTriplets"},
    PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}],
  {{μ, 0.1}, 0, 2}, {{σ, 1}, 0, 10}, {{κ, 0.5}, 0, 2}
]

```

Out[97]=



Contour Plots for distribution parameters given moment estimations

Attempts to use the Manipulate control result in aborted computation

```

In[132]:= Manipulate[
  ContourPlot3D[
    Evaluate[{
      
$$e^{\kappa S} \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa S}, 1 + \frac{1}{\kappa S}, 1 - \frac{\kappa S \mu S}{\sigma S}\right] \mu S = e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right] \mu},$$

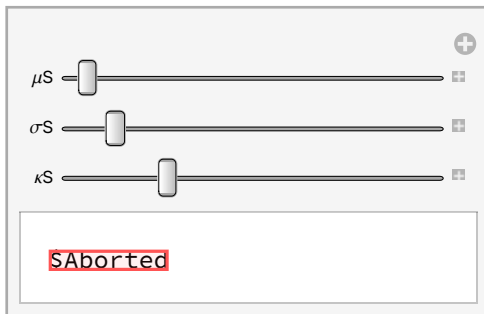
      
$$\mu S + \frac{\sigma S}{2} = \mu + \frac{\sigma}{2},$$

      
$$\mu S^2 + \frac{2 \mu S \sigma S}{3 + \kappa S} + \frac{2 \sigma S^2}{3 (3 + \kappa S)} = \mu^2 + \frac{2 \mu \sigma}{3 + \kappa} + \frac{2 \sigma^2}{3 (3 + \kappa)}$$

    }],
    {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
    AxesLabel → {"μ", "σ", "κ"},
    PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}],
  {{μS, 0.1}, 0, 5}, {{σS, 1}, 0, 10}, {{κS, 0.5}, 0, 2}
]

```

Out[132]=



... General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

... General : $-\frac{4.94477 \times 10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

... General : $-\frac{4.85853 \times 10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

... General : Further output of General::munfl will be suppressed during this calculation.

... General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

... General : $-\frac{4.94477 \times 10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

... General : $-\frac{4.85853 \times 10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

... General : Further output of General::munfl will be suppressed during this calculation.

... General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

... General : $-\frac{4.94477 \times 10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

General : $-\frac{4.85853 \times 10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

General : Further output of General::munfl will be suppressed during this calculation.

First compute a set of moments

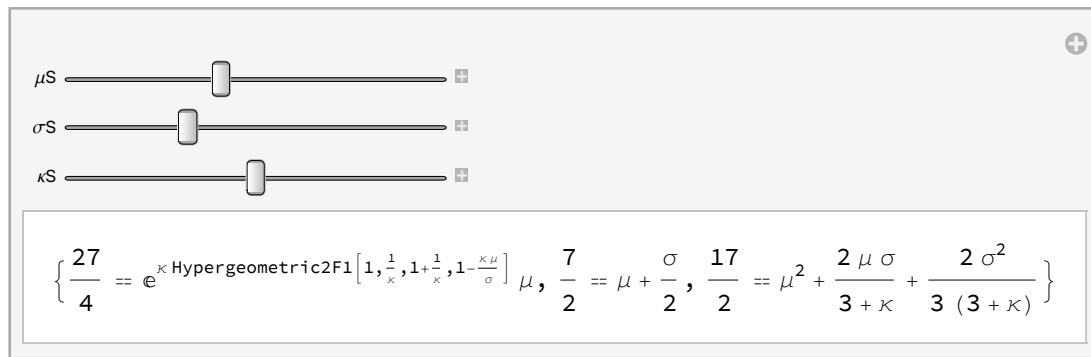
```
In[ ]:= Manipulate[
  Evaluate[ {
    
$$e^{\kappa S \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa S}, 1 + \frac{1}{\kappa S}, 1 - \frac{\kappa S \mu S}{\sigma}\right]} \mu S = e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]} \mu,$$

    
$$\mu S + \frac{\sigma S}{2} = \mu + \frac{\sigma}{2},$$

    
$$\mu S^2 + \frac{2 \mu S \sigma S}{3 + \kappa S} + \frac{2 \sigma S^2}{3 (3 + \kappa S)} = \mu^2 + \frac{2 \mu \sigma}{3 + \kappa} + \frac{2 \sigma^2}{3 (3 + \kappa)}$$

  ]],
  {{μS, 0.1}, 0, 5}, {{σS, 1}, 0, 10}, {{κS, 0.5}, 0, 2}
```

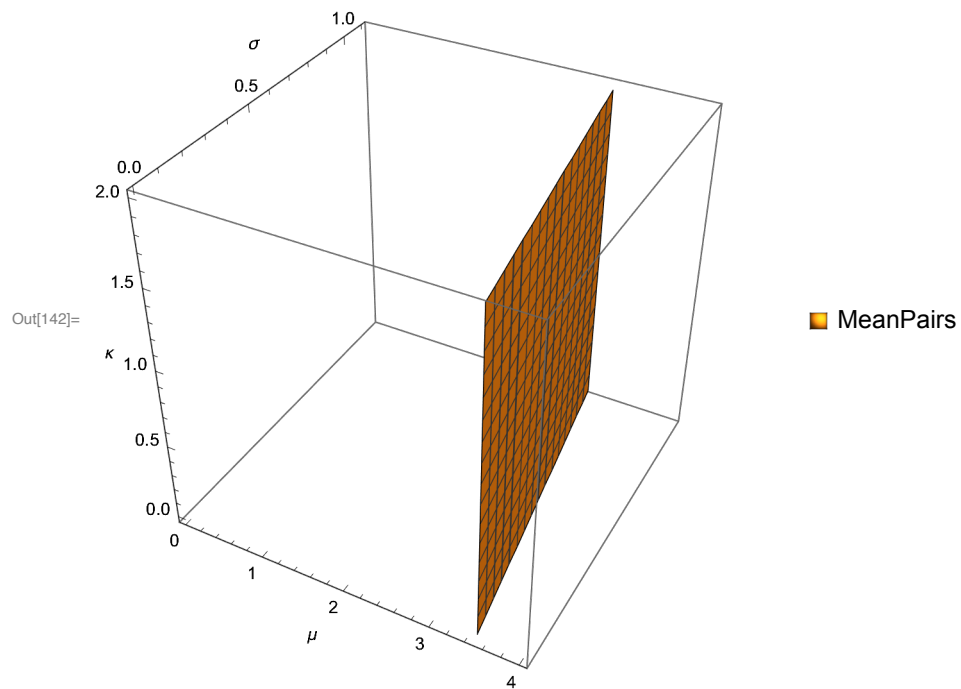
Out[]:=



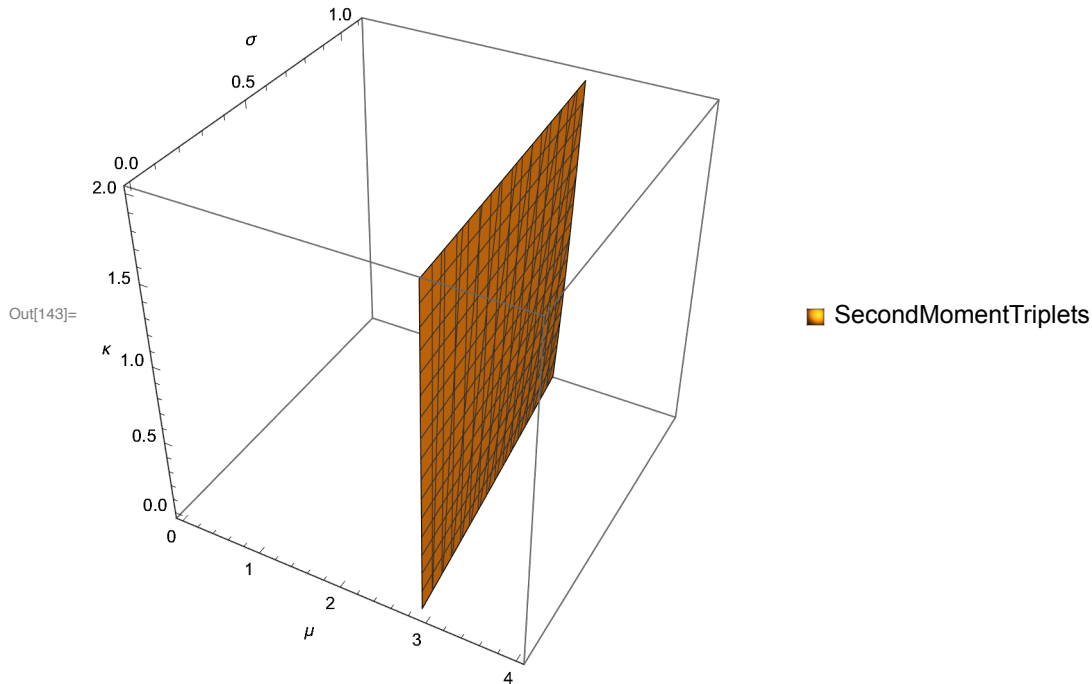
Try Contour Maps from simplest to hardest curve individually

```
In[141]:= CEEstimationPlot[Equation_, EquationLabel_] := ContourPlot3D[
  Evaluate[Equation],
  {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
  AxesLabel → {"μ", "σ", "κ"},
  PlotLegends → {EquationLabel},
  PlotLabel → "μ=2, σ=3, κ=1"]
```

```
In[142]:= CEEstimationPlot[ $\frac{7}{2} == \mu + \frac{\sigma}{2}$ , "MeanPairs"]  
 $\mu=2, \sigma=3, \kappa=1$ 
```



```
In[143]:= CEEstimationPlot[ $\frac{17}{2} == \mu^2 + \frac{2 \mu \sigma}{3 + \kappa} + \frac{2 \sigma^2}{3 (3 + \kappa)}$ , "SecondMomentTriplets"]
           $\mu=2, \sigma=3, \kappa=1$ 
```



```
In[144]:= CEEstimationPlot[ $\frac{27}{4} == e^{\text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]}$   $\mu$ , "GeoMean"]
```

General : 0.1375 6999999 is too small to represent as a normalized machine number; precision may be lost.

General : $\frac{-4.94477 \times 10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

General : $-\frac{4.85853 \times 10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

General : Further output of General::munfl will be suppressed during this calculation.

Out[144]= \$Aborted

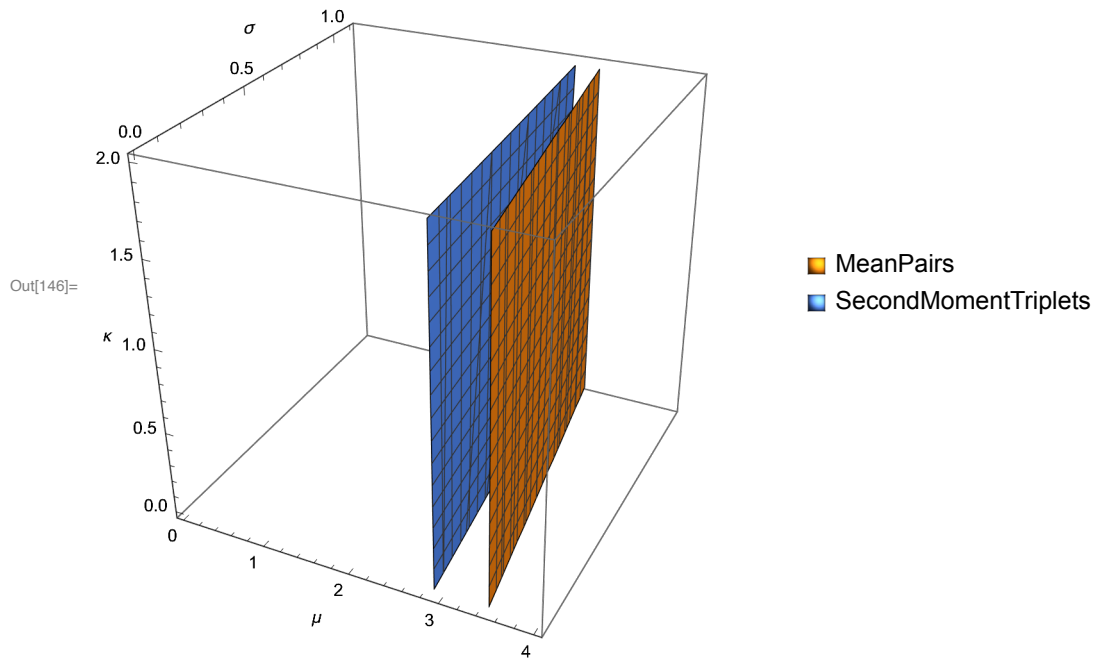
Not sure why the 3D contour plot for the MeanPair and SecondMomentTriplets does not show an intersection but presumably it has something to do with the internal contour parameter settings

```

In[146]:= ContourPlot3D[
  Evaluate[{
     $\frac{7}{2} == \mu + \frac{\sigma}{2},$ 
     $\frac{17}{2} == \mu^2 + \frac{2 \mu \sigma}{3 + \kappa} + \frac{2 \sigma^2}{3 (3 + \kappa)}$ 
  }],
  { $\mu$ , 0, 4}, { $\sigma$ , 0, 1.1}, { $\kappa$ , 0, 2},
  AxesLabel → {" $\mu$ ", " $\sigma$ ", " $\kappa$ "},
  PlotLegends → {"MeanPairs", "SecondMomentTriplets"},
  PlotLabel → " $\mu=2, \sigma=3, \kappa=1$ "
]

```

$\mu=2, \sigma=3, \kappa=1$



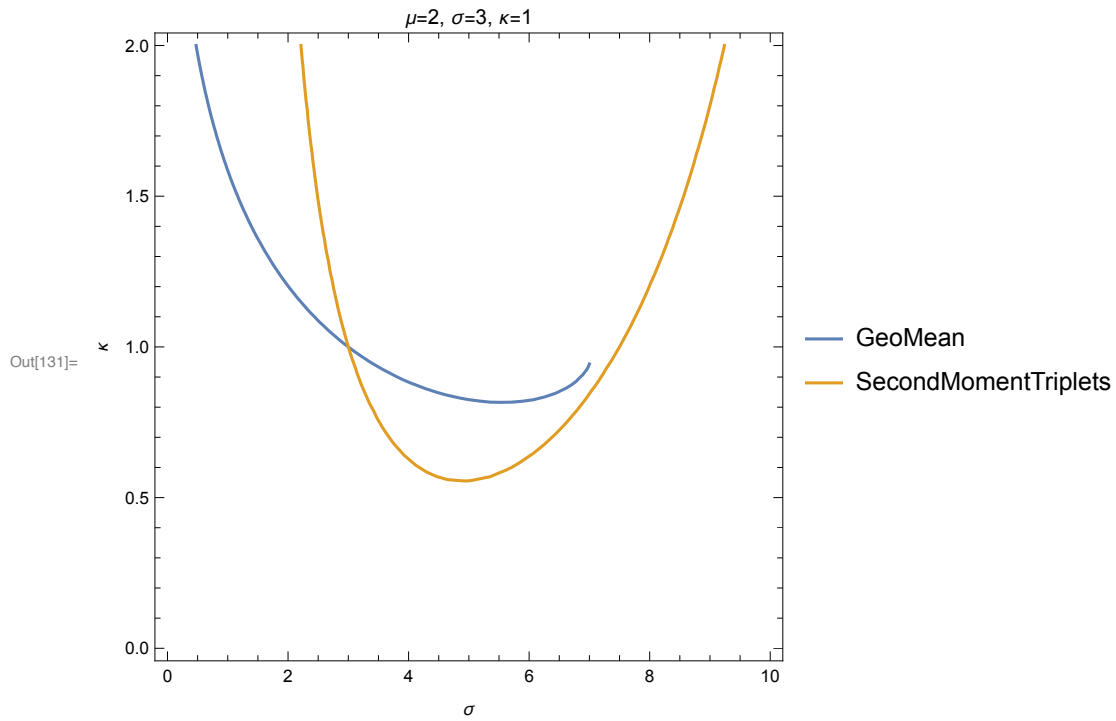
2D Plots do show clear intersections

```

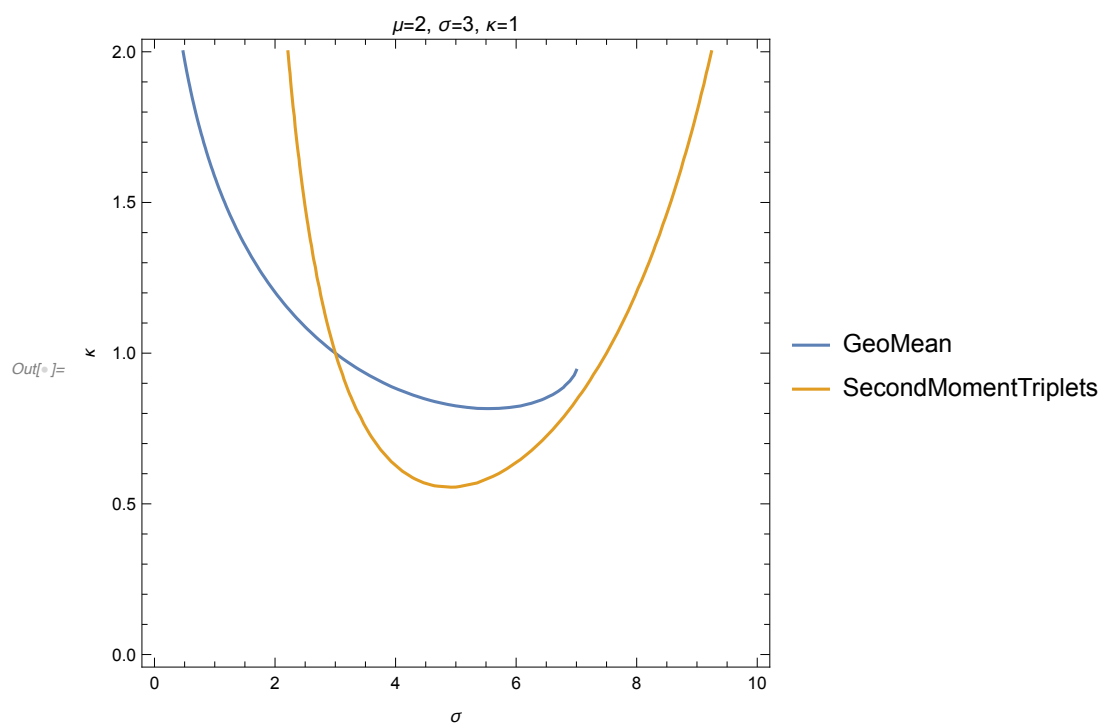
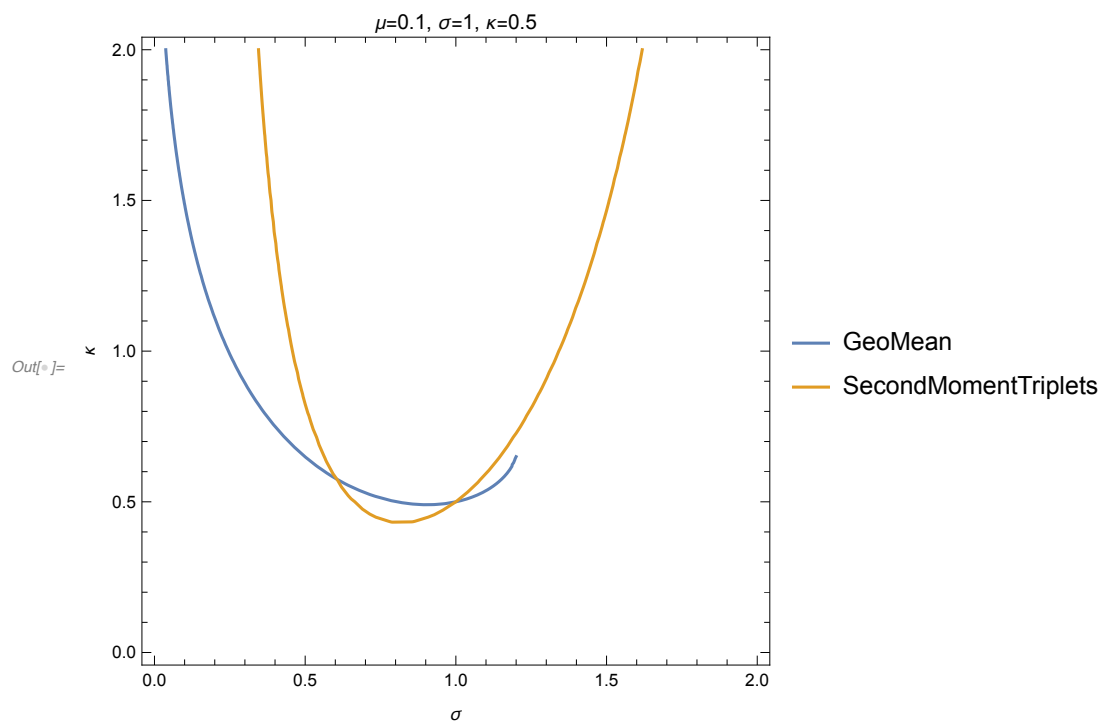
In[131]:= ContourPlot[

$$\left\{ \frac{27}{4} == e^{\kappa \text{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \left(\frac{7}{2} - \frac{\sigma}{2}\right)}{\sigma}\right]} \left(\frac{7}{2} - \frac{\sigma}{2}\right), \frac{17}{2} == \left(\frac{7}{2} - \frac{\sigma}{2}\right)^2 + \frac{2 \left(\frac{7}{2} - \frac{\sigma}{2}\right) \sigma}{3 + \kappa} + \frac{2 \sigma^2}{3 (3 + \kappa)} \right\},$$

    {σ, 0, 10}, {κ, 0, 2},
    FrameLabel → {"σ", "κ"},
    PlotLegends → {"GeoMean", "SecondMomentTriplets"},
    PlotLabel → "μ=2, σ=3, κ=1"]
    
```



Saved Plots



Find Minimum of SecondMomentTriplets with respect to κ

In[159]:= **Solve** $\left[\frac{17}{2} = \left(\frac{7}{2} - \frac{\sigma}{2}\right)^2 + \frac{2\left(\frac{7}{2} - \frac{\sigma}{2}\right)\sigma}{3 + \kappa} + \frac{2\sigma^2}{3(3 + \kappa)}, \kappa\right]$

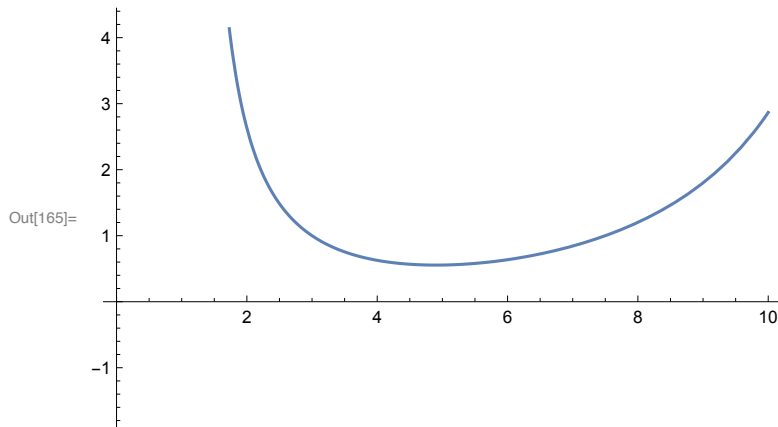
Out[159]:= $\left\{\left\{\kappa \rightarrow \frac{-135 + 42\sigma - 5\sigma^2}{3(15 - 14\sigma + \sigma^2)}\right\}\right\}$

In[161]:= **FindMinimum** $\left[\frac{-135 + 42\sigma - 5\sigma^2}{3(15 - 14\sigma + \sigma^2)}, \{\sigma, 1\}\right]$

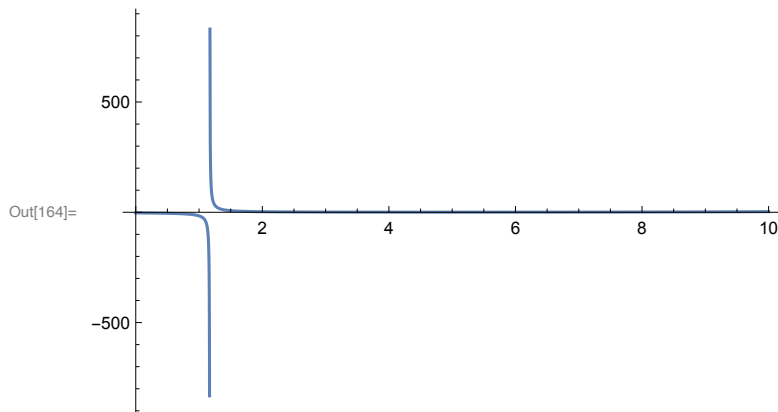
FindMinimum : Line search unable to find a sufficient decrease in the function value with MachinePrecision digit precision.

Out[161]:= $\{-4.3368 \times 10^{14}, \{\sigma \rightarrow 1.16905\}\}$

In[165]:= **Plot** $\left[\frac{-135 + 42\sigma - 5\sigma^2}{3(15 - 14\sigma + \sigma^2)}, \{\sigma, 0, 10\}, \text{PlotRange} \rightarrow \text{Automatic}\right]$



In[164]:= **Plot** $\left[\frac{-135 + 42\sigma - 5\sigma^2}{3(15 - 14\sigma + \sigma^2)}, \{\sigma, 0, 10\}, \text{PlotRange} \rightarrow \text{Full}\right]$



```
In[163]:= FindMinimum[ $\frac{-135 + 42 \sigma - 5 \sigma^2}{3 (15 - 14 \sigma + \sigma^2)}$ , { $\sigma$ , 2}]
```

```
Out[163]:= {0.554805, { $\sigma \rightarrow 4.89929$ }}
```

So there are some challenges with the search for σ and κ if the search does not seed a value close to the solution. In particular the zero of $(15 - 14 \sigma + \sigma^2)$ causes κ to go to infinity. It's important to be the side that produces a positive kappa

Algorithm Prototype for Estimating Coupled Exponentials

Compute Moments

This section will be replaced with drawing of samples and estimation of the moments

```
In[163]:= Clear[GeoMeanCE];
GeoMeanCE[ $\mu$ _,  $\sigma$ _,  $\kappa$ _] := GeoMeanCE[ $\mu$ ,  $\sigma$ ,  $\kappa$ ] =
```

$$\left\{ \begin{array}{ll} e^{\kappa \operatorname{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]} \mu & \kappa > 0 \\ e^{-\pi \left(-1 + \frac{\kappa \mu}{\sigma}\right)^{-1/\kappa} \operatorname{Csc}\left[\frac{\pi}{\kappa}\right] + \kappa \operatorname{Hypergeometric2F1}\left[1, \frac{1}{\kappa}, 1 + \frac{1}{\kappa}, 1 - \frac{\kappa \mu}{\sigma}\right]} \mu & -1 \leq \kappa < 0 \\ e^{-e^{\mu/\sigma} \operatorname{ExpIntegralEi}\left[-\frac{\mu}{\sigma}\right]} \mu & \kappa = 0 \\ \text{"Not a Distribution"} & \text{True} \end{array} \right.$$