Estimation of Coupled Exponential Distribution

Plot of equations of GeoMean, mean of pairs, second moment of triplets.

From notes by Amaneh Al-Najafi

Correction: the equation for G (geometric mean), the sign of κ needs to be reversed. This changes the equation to reference from the paper by Vogel to:
$G = \frac{\sigma\,\mu}{\kappa}$ Exp[PolyGamma[1] – PolyGamma[$1 + \frac{1}{\kappa}$] + κ]

$$\begin{bmatrix} G = \dfrac{\sigma\mu}{\kappa}\exp(\psi(1) - \psi(1 + \dfrac{1}{\kappa})) \\ \\ \mu_1 = \mu + \dfrac{\sigma}{2} \\ \\ \mu_1^{(2)} = \mu^2 + \dfrac{2\mu\sigma}{3+\kappa} + \dfrac{2\sigma^2}{3(3+\kappa)} \end{bmatrix}.$$

$$\mu = \mu_1 - \frac{\sigma}{2}$$

$$\kappa = \left[2\sigma(\mu_1 - \frac{\sigma}{2}) + \frac{2\sigma^2}{3} - 3\Big(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2\Big)\right]\Big(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2\Big)^{-1}$$

$$\sigma = \mu_1 - \sqrt{\mu_1^2 - 2G\frac{\left[2\sigma(\mu_1 - \frac{\sigma}{2}) + \frac{2\sigma^2}{3} - 3\Big(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2\Big)\right]}{(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2)^{-1}}\left[\exp\left(\psi\Big(1 + \frac{\mu_2 - (\mu_1 - \frac{\sigma}{2})^2}{2\sigma(\mu_1 - \frac{\sigma}{2}) + \frac{2\sigma^2}{3} - 3\Big(\mu_2 - (\mu_1 - \frac{\sigma}{2})^2\Big)}\Big) + \gamma\right)\right]}$$

Where $\gamma = 0.5772$

My estimates of the GM, the 1st moment of the pairs, and the 2nd moment of the triplets for several examples of the GPD:

\kappa	GM	1st moment	2nd moment
0.5	2.716987	2.2497471	0.424
1	3.17177	2.249486	4.105583
2	4.9799	2.25	5589.1

```
In[]:= $Assumptions = {μ, σ, κ} ∈ Reals && 0 < σ < ∞ && 0 < κ < ∞ && 0 ≤ p ≤ 1
Out[]= True == {μ, σ, κ} ∈ ℝ && 0 < σ < ∞ && 0 < κ < ∞ && 0 ≤ p ≤ 1
```

In[ ]:=
```
Clear[CoupledExponentialEstimators, GeoMean, MeanPairs, SecondMomentTriplets];
CoupledExponentialEstimators[GeoMean_, MeanPairs_, SecondMomentTriplets_] :=
 {
  GeoMean == σ μ/κ Exp[PolyGamma[1] - PolyGamma[1 + 1/κ] + κ],
  MeanPairs == μ + σ/2,
  SecondMomentTriplets == μ^2 + (2 μ σ)/(3 + κ) + (2 σ^2)/(3 (3 + κ))
 }
```

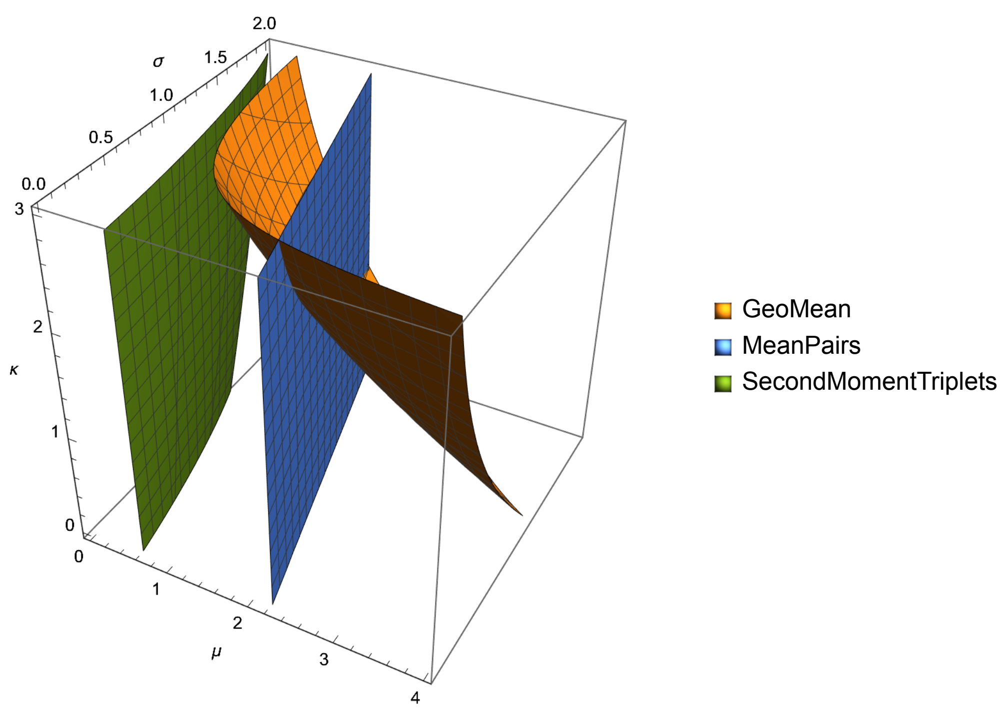
$$\text{GeoMean} == \frac{\sigma\,\mu}{\kappa}\,\text{Exp}\left[\text{PolyGamma}[1] - \text{PolyGamma}\left[1+\frac{1}{\kappa}\right] + \kappa\right],$$

$$\text{MeanPairs} == \mu + \frac{\sigma}{2},$$

$$\text{SecondMomentTriplets} == \mu^2 + \frac{2\,\mu\,\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)}$$

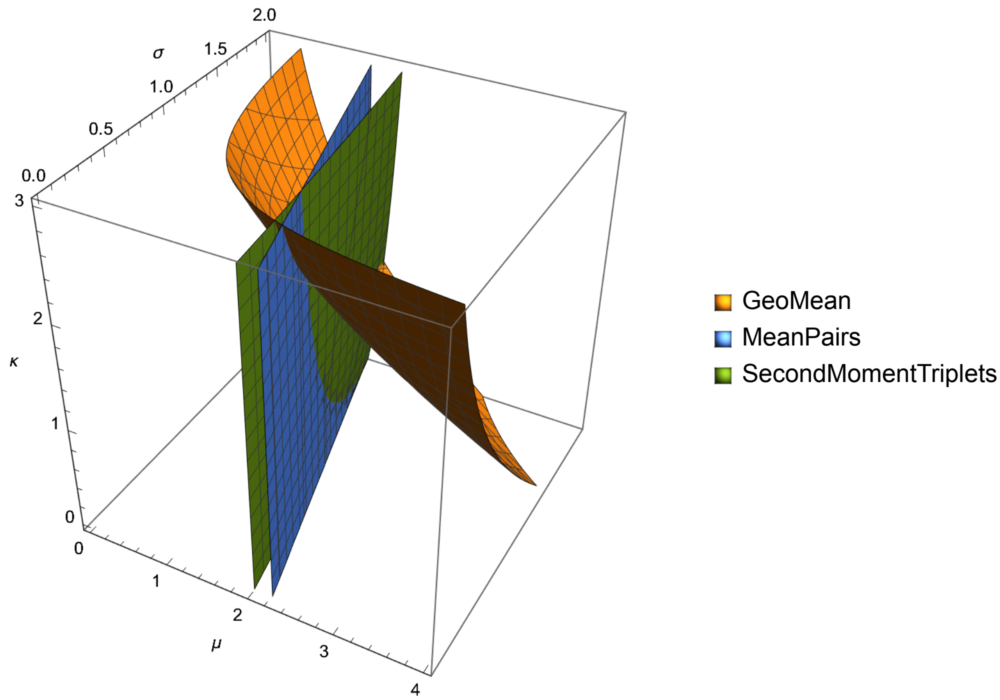
In[ ]:=
```
ContourPlot3D[Evaluate@CoupledExponentialEstimators[2.72, 2.25, 0.424],
 {μ, 0, 4}, {σ, 0, 2}, {κ, 0, 3},
 AxesLabel → {μ, σ, κ},
 PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]
```

Out[ ]=



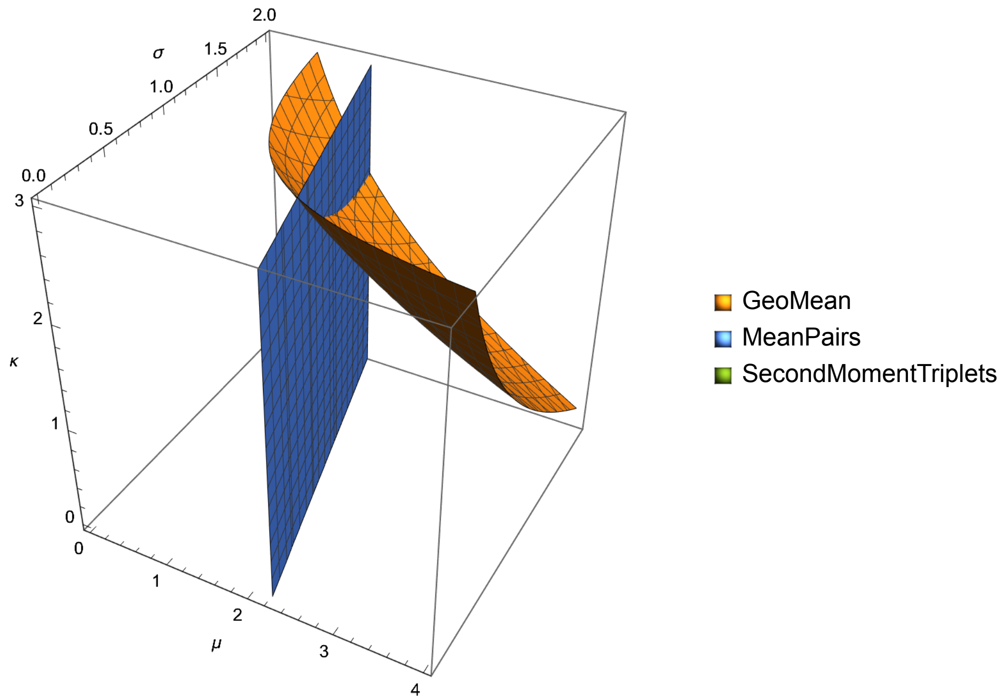
In[∘]:=
```
ContourPlot3D[Evaluate@CoupledExponentialEstimators[3.17, 2.25, 4.15],
  {μ, 0, 4}, {σ, 0, 2}, {κ, 0, 3},
  AxesLabel → {μ, σ, κ},
  PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]
```

Out[∘]=



In[ ]:=
```
ContourPlot3D[Evaluate@CoupledExponentialEstimators[4.98, 2.25, 5589],
 {μ, 0, 4}, {σ, 0, 2}, {κ, 0, 3},
 AxesLabel → {μ, σ, κ},
 PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]
```

Out[ ]=



Derivation of the GeoMean of the Coupled Exponential Distribution

Assuming $\mu = 0$

The Quantile function of the Coupled Exponential Distribution

$$\text{If}\left[\kappa \neq 0,\ \frac{-\sigma}{\kappa}\left(1 - (1-p)^{-\kappa}\right),\ -\sigma\,\text{Log}[1-p]\right]$$

simplifies to $\sigma\,\text{CoupledLogarithm}\left[(1-p)^{-1}, \kappa\right]$

```
ClearAll[p, κ];
```

```
ClearAll[CoupledExponentialQuantileFunction];
CoupledExponentialQuantileFunction[p_, μ_ : 0, σ_, κ_] :=
 CoupledExponentialQuantileFunction[p, μ, σ, κ] = σ CoupledLogarithm[(1 - p)^-1, κ, 0]
```

Test Quantile Function

In[ ]:=
```
CoupledExponentialQuantileFunction[0.999, 0, 1, -0.5]
```

Out[ ]= 1.93675

In[∘]:=
```
If[κ ≠ 0, -σ/κ (1 - (1 - p)^-κ),
  -σ Log[1 - p]
 ] /. {p → 0.999, σ → 1, κ → -0.5}
```

Out[∘]= 1.93675

Integration of Quantile Function to form Geometric Mean

In[∘]:=
```
FullSimplify@
 Exp[∫_0^1 FullSimplify@Log@CoupledExponentialQuantileFunction[p, 0, σ, κ] ⅆp]
```

Out[∘]= $e^{\int_0^1 \begin{cases} \mathrm{Log}\left[\frac{(-1+(1-p)^{-\kappa})\,\sigma}{\kappa}\right] & \kappa\neq 0 \\ \mathrm{Log}[-\sigma\,\mathrm{Log}[1-p]] & \mathrm{True} \end{cases}\ \text{if } p<1\ dp}$

In[∘]:=
```
Assuming[0 < κ < ∞, FullSimplify@Exp[∫_0^1 FullSimplify@Log[((-1 + (1 - p)^-κ) σ)/κ] ⅆp]]
```

Out[∘]= $\frac{e^{-\mathrm{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]}\,\sigma}{\kappa}$

In[∘]:=
```
Assuming[-1 < κ < 0, FullSimplify@Exp[∫_0^1 FullSimplify@Log[((-1 + (1 - p)^-κ) σ)/κ] ⅆp]]
```

Out[∘]= $-\frac{e^{\kappa-\mathrm{HarmonicNumber}\left[-\frac{1+\kappa}{\kappa}\right]}\,\sigma}{\kappa}$

In[∘]:=
```
FullSimplify@Exp[∫_0^1 Log[- σ Log[1 - p]] ⅆp]
```

Out[∘]= $e^{-\mathrm{EulerGamma}}\,\sigma$

The Harmonic number and the Digamma functions have the following relationship.
$H_z = \psi(z+1) - \gamma$ where γ is the Euler gamma constant 0.5172216...

See this Wolfram Research article on the history.
https://functions.wolfram.com/GammaBetaErf/HarmonicNumber2/introductions/DifferentiatedGammas/ShowAll.html

Summarizing Result

GeometricMean of Coupled Exponential =

$$\begin{cases} \frac{e^{-\mathrm{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]}\,\sigma}{\kappa} & \kappa > 0 \\ -\frac{e^{\kappa-\mathrm{HarmonicNumber}\left[-\frac{1+\kappa}{\kappa}\right]}\,\sigma}{\kappa} & -1<\kappa<0 \\ e^{-\mathrm{EulerGamma}}\,\sigma & \kappa = 0 \end{cases}$$

Assuming $\mu \neq 0$

In[ ]:=
```
ClearAll[CoupledExponentialQuantileFunction];
CoupledExponentialQuantileFunction[p_, μ_, σ_, κ_] :=
 CoupledExponentialQuantileFunction[p, μ, σ, κ] =
  μ + σ CoupledLogarithm[(1 - p)^-1, κ, 0]
```

In[ ]:=
```
Assuming[0 < κ < ∞ && μ ∈ Reals,
 FullSimplify@Exp[∫_0^1 FullSimplify@Log[μ + ((-1 + (1 - p)^-κ) σ)/κ] ⅆp]]
```

Out[ ]= $e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu \quad \text{if} \quad \mu \geq 0$

In[ ]:=
```
Assuming[-1 < κ < 0 && μ ∈ Reals,
 FullSimplify@Exp[∫_0^1 FullSimplify@Log[μ + ((-1 + (1 - p)^-κ) σ)/κ] ⅆp]]
```

Out[ ]= $e^{-\pi\left(-1+\frac{\kappa\mu}{\sigma}\right)^{-1/\kappa}\text{Csc}\left[\frac{\pi}{\kappa}\right]+\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu \quad \text{if} \quad \mu \geq 0$

In[ ]:=
```
Assuming[μ ∈ Reals, FullSimplify@Exp[∫_0^1 Log[μ - σ Log[1 - p]] ⅆp]]
```

Out[ ]= $Aborted

Check relationship with equation solved by Amenah

In[ ]:=
```
FullSimplify[(μ σ)/κ Exp[PolyGamma[1] - PolyGamma[1 + 1/κ]], 0 < κ < ∞ && 0 ≤ μ < ∞ && 0 < σ < ∞]
```

Out[ ]= $\dfrac{e^{-\text{HarmonicNumber}\left[\frac{1}{\kappa}\right]}\,\mu\,\sigma}{\kappa}$

In[ ]:=
```
FullSimplify[e^(κ Hypergeometric2F1[1, 1/κ, 1 + 1/κ, 1 - (κ μ)/σ]), 0 < κ < ∞]
```

Out[ ]= $e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}$

In[ ]:=
```
FullSimplify[Limit[e^(κ Hypergeometric2F1[1, 1/κ, 1 + 1/κ, 1 - (κ μ)/σ]) μ, μ → 0], 0 < κ < ∞]
```

Out[ ]= $\dfrac{e^{-\text{HarmonicNumber}\left[-1+\frac{1}{\kappa}\right]}\,\sigma}{\kappa}$

In[ ]:=
```
FullSimplify[Limit[e^(κ Hypergeometric2F1[1, 1/κ, 1 + 1/κ, 1 - (κ μ)/σ]) μ, κ → 0], μ ≥ 0]
```

Out[ ]= $\lim\limits_{\kappa\to 0} e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu$

Summarizing Result

GeometricMean of Coupled Exponential =

$$\begin{cases} e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu \text{ if } \mu \geq 0 & \kappa > 0 \\ e^{-\pi\left(-1+\frac{\kappa\mu}{\sigma}\right)^{-1/\kappa}\text{Csc}\left[\frac{\pi}{\kappa}\right]+\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu \text{ if } \mu \geq 0 & -1 < \kappa < 0 \\ ? & \kappa = 0 \end{cases}$$

Reduction of Equations

In[69]:=

$$\text{Solve}\Big[\Big\{\text{GeoMean} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu,\ \text{MeanPairs} == \mu + \frac{\sigma}{2},\ \text{SecondMomentTriplets} == \mu^2 + \frac{2\,\mu\,\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)}\Big\}, \{\mu, \sigma, \kappa\}, \text{Reals}\Big]$$

··· Solve : This system cannot be solved with the methods available to Solve.

Out[69]=

$$\text{Solve}\Big[\Big\{\text{GeoMean} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\,\mu,\ \text{MeanPairs} == \mu + \frac{\sigma}{2},\ \text{SecondMomentTriplets} == \mu^2 + \frac{2\,\mu\,\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)}\Big\}, \{\mu, \sigma, \kappa\}, \mathbb{R}\Big]$$

$$\mu == \frac{\sigma}{2} - \text{MeanPairs},$$

In[70]:= Solve[

$$\{\mathrm{GeoMean} == e^{\kappa\,\mathrm{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right)}{\sigma}\right]}\left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right),$$

$$\mathrm{SecondMomentTriplets} == \left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right)^2+\frac{2\left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}$$

$$\},\ \{\sigma,\kappa\},\ \mathrm{Reals}]$$

··· Solve : This system cannot be solved with the methods available to Solve.

Out[70]= Solve[

$$\left\{\mathrm{GeoMean} == e^{\kappa\,\mathrm{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right)}{\sigma}\right]}\left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right),\ \mathrm{SecondMomentTriplets} == \left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right)^2+\frac{2\left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\right\},\ \{\sigma,\kappa\},\ \mathbb{R}]$$

In[71]:= SolveValues[

$$\{\mathrm{GeoMean} == e^{\kappa\,\mathrm{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right)}{\sigma}\right]}\left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right),$$

$$\mathrm{SecondMomentTriplets} == \left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right)^2+\frac{2\left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}$$

$$\},\ \{\sigma,\kappa\},\ \mathrm{Reals}]$$

··· SolveValues : This system cannot be solved with the methods available to SolveValues.

Out[71]= SolveValues[

$$\left\{\mathrm{GeoMean} == e^{\kappa\,\mathrm{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right)}{\sigma}\right]}\left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right),\ \mathrm{SecondMomentTriplets} == \left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right)^2+\frac{2\left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\right\},\ \{\sigma,\kappa\},\ \mathbb{R}]$$

In[75]:= SolveValues[

$$\mathrm{GeoMean} == e^{\kappa\,\mathrm{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right)}{\sigma}\right]}\left(\frac{\sigma}{2}-\mathrm{MeanPairs}\right),$$

$$\sigma,\ \mathrm{Reals}]$$

··· SolveValues : This system cannot be solved with the methods available to SolveValues.

Out[75]= SolveValues[

$$\mathrm{GeoMean} == e^{\kappa\,\mathrm{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right)}{\sigma}\right]}\left(-\mathrm{MeanPairs}+\frac{\sigma}{2}\right),\ \sigma,\ \mathbb{R}]$$

In[76]:=
$$\text{Solve}\left[\text{SecondMomentTriplets} == \left(\frac{\sigma}{2} - \text{MeanPairs}\right)^2 + \frac{2\left(\frac{\sigma}{2} - \text{MeanPairs}\right)\sigma}{3+\kappa} + \frac{2\sigma^2}{3(3+\kappa)},\ \sigma,\ \text{Reals}\right]$$

$$\left\{\left\{\sigma \to \frac{6(5\,\text{MeanPairs} + \text{MeanPairs}\,\kappa)}{29+3\kappa} - 2\sqrt{3}\sqrt{\left(\frac{1}{(29+3\kappa)^2}\left(-12\,\text{MeanPairs}^2 + 87\,\text{SecondMomentTriplets} - 8\,\text{MeanPairs}^2\kappa + 38\,\text{SecondMomentTriplets}\,\kappa + 3\,\text{SecondMomentTriplets}\,\kappa^2\right)\right)}\right.\right.$$
$$\text{if}\ \left(\text{SecondMomentTriplets} > \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\kappa}{87+38\kappa+3\kappa^2}\ \&\&\ \kappa > -3\right) \,||\, \left(-\frac{29}{3} < \kappa < -3\ \&\&\ \text{SecondMomentTriplets} < \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\kappa}{87+38\kappa+3\kappa^2}\right) \,||\, \left(\kappa < -\frac{29}{3}\ \&\&\ \text{SecondMomentTriplets} > \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\kappa}{87+38\kappa+3\kappa^2}\right)\Big\},$$

$$\left\{\sigma \to \frac{6(5\,\text{MeanPairs} + \text{MeanPairs}\,\kappa)}{29+3\kappa} + 2\sqrt{3}\sqrt{\left(\frac{1}{(29+3\kappa)^2}\left(-12\,\text{MeanPairs}^2 + 87\,\text{SecondMomentTriplets} - 8\,\text{MeanPairs}^2\kappa + 38\,\text{SecondMomentTriplets}\,\kappa + 3\,\text{SecondMomentTriplets}\,\kappa^2\right)\right)}\right.$$
$$\text{if}\ \left(\text{SecondMomentTriplets} > \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\kappa}{87+38\kappa+3\kappa^2}\ \&\&\ \kappa > -3\right) \,||\, \left(-\frac{29}{3} < \kappa < -3\ \&\&\ \text{SecondMomentTriplets} < \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\kappa}{87+38\kappa+3\kappa^2}\right) \,||\, \left(\kappa < -\frac{29}{3}\ \&\&\ \text{SecondMomentTriplets} > \frac{12\,\text{MeanPairs}^2 + 8\,\text{MeanPairs}^2\kappa}{87+38\kappa+3\kappa^2}\right)\Big\}\Big\}$$

Similify expression in terms of κ

In[77]:=

$$\text{FullSimplify}\Big[\text{GeoMean} == \mathbb{e}^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\frac{\left(\frac{6\,(5\,\text{MeanPairs}+\text{MeanPairs}\,\kappa)}{29+3\,\kappa}-2\,\sqrt{3}\,\sqrt{\left(\frac{1}{(29+3\,\kappa)^2}\,\left(-12\,\text{MeanPairs}^2+87\,\text{SecondMomentTriplets}-8\,\text{MeanPairs}^2\,\kappa+38\,\text{SecondMomentTriplets}\,\kappa+3\,\text{SecondMomentTr}\right.\right.}}{2}}{\left(\frac{6\,(5\,\text{MeanPairs}+\text{MeanPairs}\,\kappa)}{29+3\,\kappa}-2\,\sqrt{3}\,\sqrt{\left(\frac{1}{(29+3\,\kappa)^2}\,\left(-12\,\text{MeanPairs}^2+87\,\text{SecondMomentTriplets}-8\,\text{MeanPairs}^2\,\kappa+38\,\text{SecondMomentTriplets}\,\kappa+\right.\right.}}\right.}$$

$$\left(\frac{1}{2}\left(\frac{6\,(5\,\text{MeanPairs}+\text{MeanPairs}\,\kappa)}{29+3\,\kappa}-2\,\sqrt{3}\,\sqrt{\left(\frac{1}{(29+3\,\kappa)^2}\,\left(-12\,\text{MeanPairs}^2+87\,\text{SecondMomentTriplets}-8\,\text{MeanPairs}^2\,\kappa+38\,\text{SecondMomentTriplets}\,\kappa+3\,\text{SecondMomentTriplets}\,\kappa^2\right)\right)}\right)-\text{MeanPairs}\right)\Big]$$

Out[77]=

$$\text{GeoMean}+\frac{1}{29+3\,\kappa}\,\mathbb{e}^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},-\frac{10\,\text{MeanPairs}\,(3+2\,\kappa)+\sqrt{3}\,(-2+\kappa)\,(29+3\,\kappa)\,\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\,\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\,\kappa)}{(29+3\,\kappa)^2}}}{-6\,\text{MeanPairs}\,(5+\kappa)+2\,\sqrt{3}\,(29+3\,\kappa)\,\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\,\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\,\kappa)}{(29+3\,\kappa)^2}}}\right]}$$

$$\left(14\,\text{MeanPairs}+\sqrt{3}\,(29+3\,\kappa)\,\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\,\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\,\kappa)}{(29+3\,\kappa)^2}}\right) == 0$$

Contour Plots

In[79]:=
```
Manipulate[
 ContourPlot3D[
  GeoMean +
```

$$\frac{1}{29+3\,\kappa}\,e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},-\frac{10\,\text{MeanPairs}\,(3+2\,\kappa)+\sqrt{3}\,(-2+\kappa)\,(29+3\,\kappa)\,\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\,\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\,\kappa)}{(29+3\,\kappa)^2}}}{-6\,\text{MeanPairs}\,(5+\kappa)+2\,\sqrt{3}\,(29+3\,\kappa)\,\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\,\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\,\kappa)}{(29+3\,\kappa)^2}}}\right]}\left(14\,\text{MeanPairs}+\sqrt{3}\,(29+3\,\kappa)\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\,\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\,\kappa)}{(29+3\,\kappa)^2}}\right) == 0,$$

```
  {GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets, 0, 5},
  AxesLabel → {"MeanPairs", "SecondMomentTriplets", "GeoMean"}],
 {κ, 0, 2}
]
```

Out[79]=

κ

```
$Aborted
```

Power : Infinite expression $\frac{1}{0}$ encountered.

Power : Infinite expression $\frac{1}{0}$ encountered.

Power : Infinite expression $\frac{1}{0}$ encountered.

General : Further output of Power::infy will be suppressed during this calculation.

Infinity : Indeterminate expression 0 ComplexInfinity encountered.

Infinity : Indeterminate expression 0 ComplexInfinity encountered.

Infinity : Indeterminate expression 0 ComplexInfinity encountered.

General : Further output of Infinity::indet will be suppressed during this calculation.

In[82]:=
$$\text{ContourPlot3D}\Big[\text{GeoMean} + \frac{1}{29+3\kappa}\, e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},-\frac{10\,\text{MeanPairs}\,(3+2\kappa)+\sqrt{3}\,(-2+\kappa)\,(29+3\kappa)\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\kappa)}{(29+3\kappa)^2}}}{-6\,\text{MeanPairs}\,(5+\kappa)+2\sqrt{3}\,(29+3\kappa)\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\kappa)}{(29+3\kappa)^2}}}\right]}\left(14\,\text{MeanPairs}+\sqrt{3}\,(29+3\kappa)\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\kappa)}{(29+3\kappa)^2}}\right) ==$$
$$0\ \text{/. }\kappa\to 0.1,$$

```
{GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets, 0, 5},
AxesLabel → {"MeanPairs", "SecondMomentTriplets", "GeoMean"}]
```

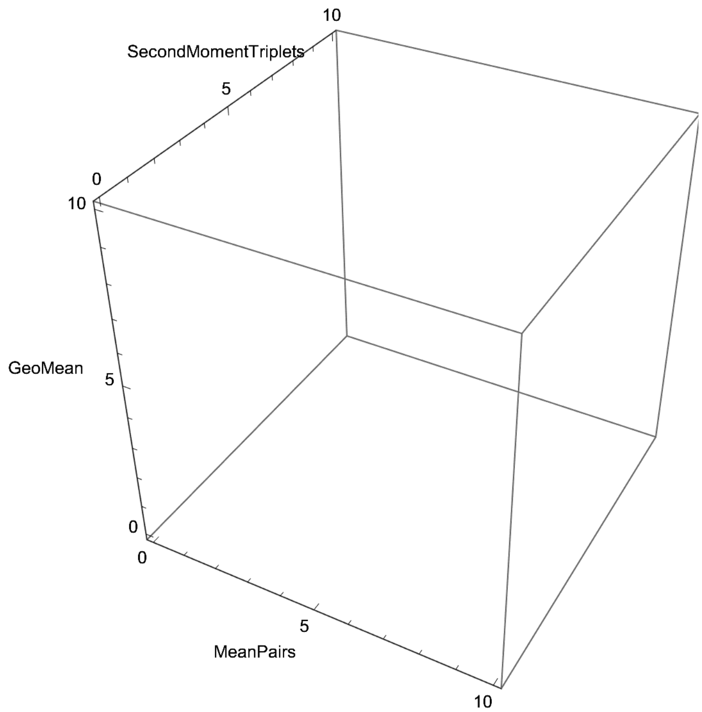
In[86]:=
$$\text{ContourPlot3D}\Big[\text{Evaluate@}\left(\text{GeoMean} + \frac{1}{29+3\kappa}\, e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},-\frac{10\,\text{MeanPairs}\,(3+2\kappa)+\sqrt{3}\,(-2+\kappa)\,(29+3\kappa)\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\kappa)}{(29+3\kappa)^2}}}{-6\,\text{MeanPairs}\,(5+\kappa)+2\sqrt{3}\,(29+3\kappa)\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\kappa)}{(29+3\kappa)^2}}}\right]}\left(14\,\text{MeanPairs}+\sqrt{3}\,(29+3\kappa)\sqrt{\frac{-4\,\text{MeanPairs}^2\,(3+2\kappa)+\text{SecondMomentTriplets}\,(3+\kappa)\,(29+3\kappa)}{(29+3\kappa)^2}}\right) == 0\right)\ \text{/. }\kappa\to 0.1,$$

```
{GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets, 0, 5},
AxesLabel → {"MeanPairs", "SecondMomentTriplets", "GeoMean"}]
```

Out[86]=
```
ContourPlot3D[Evaluate[GeoMean +
  1/(29 + 3 κ) ⅇ^(κ Hypergeometric2F1[1, 1/κ, 1 + 1/κ, -(10 MeanPairs (3 + 2 κ) + Sqrt[3] (-2 + κ) (29 + 3 κ) Sqrt[(-4 MeanPairs^2 (3 + 2 κ) + SecondMomentTriplets (3 + κ) (29 + 3 κ))/(29 + 3 κ)^2])/(-6 MeanPairs (5 + κ) + 2 Sqrt[3] (29 + 3 κ) Sqrt[(-4 MeanPairs^2 (3 + 2 κ) + SecondMomentTriplets (3 + κ) (29 + 3 κ))/(29 + 3 κ)^2])])
  (14 MeanPairs + Sqrt[3] (29 + 3 κ)
   Sqrt[(-4 MeanPairs^2 (3 + 2 κ) + SecondMomentTriplets (3 + κ) (29 + 3 κ))/(29 + 3 κ)^2]) == 0] /.
  κ → 0.1, {GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets,
  0,
  5},
 AxesLabel → {MeanPairs, SecondMomentTriplets, GeoMean}]
```

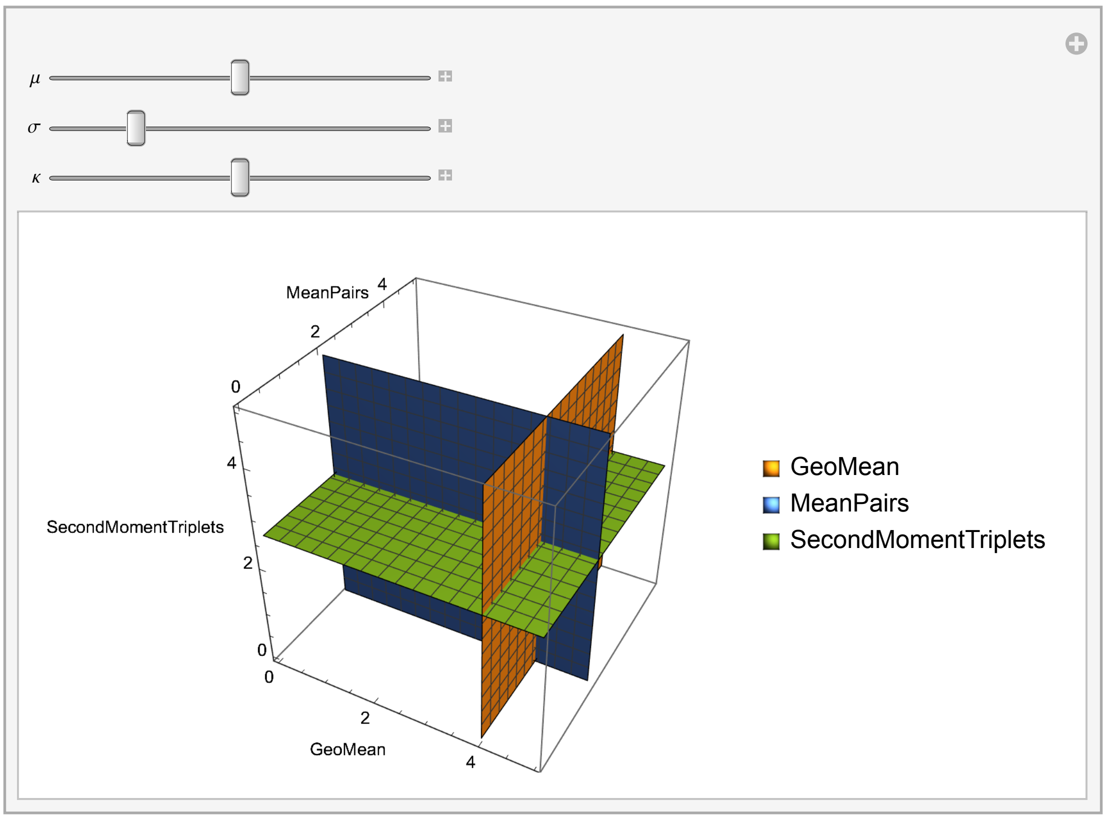
In[88]:=
```
ContourPlot3D[GeoMean + 0.034129692832764506`
  ⅇ^(0.1` Hypergeometric2F1[1, 10.`, 11.`, -(32.` MeanPairs - 3.2908965343808667` Sqrt[-12.8` MeanPairs^2 + 90.83` SecondMomentTriplets])/(-30.599999999999998` MeanPairs + 3.4641016151377544` Sqrt[-12.8` MeanPairs^2 + 90.83` SecondMomentTriplets])])
  (14 MeanPairs + 1.7320508075688772`
    Sqrt[-12.8` MeanPairs^2 + 90.83` SecondMomentTriplets]) == 0,
 {GeoMean, 0, 10}, {MeanPairs, 0, 10}, {SecondMomentTriplets, 0, 10},
 AxesLabel → {"MeanPairs", "SecondMomentTriplets", "GeoMean"}]
```

Out[88]=



In[97]:=
```
Manipulate[
 ContourPlot3D[
  Evaluate[{
    GeoMean == e^(κ Hypergeometric2F1[1, 1/κ, 1 + 1/κ, 1 - κ μ/σ]) μ,
    MeanPairs == μ + σ/2,
    SecondMomentTriplets == μ^2 + (2 μ σ)/(3 + κ) + (2 σ^2)/(3 (3 + κ))
   }],
  {GeoMean, 0, 5}, {MeanPairs, 0, 5}, {SecondMomentTriplets, 0, 5},
  AxesLabel → {"GeoMean", "MeanPairs", "SecondMomentTriplets"},
  PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}],
 {{μ, 0.1}, 0, 2}, {{σ, 1}, 0, 10}, {{κ, 0.5}, 0, 2}
]
```

Out[97]=



Contour Plots for distribution parameters given moment estimations

Attempts to use the Manipulate control result in aborted computation

In[132]:=

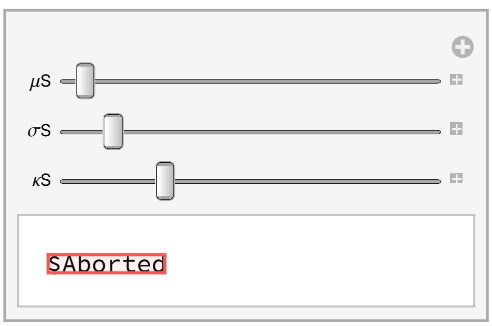
$$\text{Manipulate}\Big[\text{ContourPlot3D}\Big[\text{Evaluate}\Big[\Big\{e^{\kappa S\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa S},1+\frac{1}{\kappa S},1-\frac{\kappa S\,\mu S}{\sigma S}\right]}\,\mu S == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\,\mu}{\sigma}\right]}\,\mu,$$

$$\mu S+\frac{\sigma S}{2} == \mu+\frac{\sigma}{2},$$

$$\mu S^2+\frac{2\,\mu S\,\sigma S}{3+\kappa S}+\frac{2\,\sigma S^2}{3\,(3+\kappa S)} == \mu^2+\frac{2\,\mu\,\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\Big\}\Big],$$

```
  {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
  AxesLabel → {"μ", "σ", "κ"},
  PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}],
 {{μS, 0.1}, 0, 5}, {{σS, 1}, 0, 10}, {{κS, 0.5}, 0, 2}
]
```

Out[132]=



General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

General : $\frac{-4.94477\times10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

General : $-\frac{4.85853\times10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

General : Further output of General::munfl will be suppressed during this calculation.

General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

General : $\frac{-4.94477\times10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

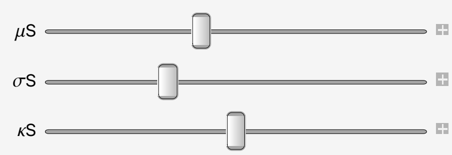
General : $-\frac{4.85853\times10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

General : Further output of General::munfl will be suppressed during this calculation.

First compute a set of moments

In[ ]:=
```
Manipulate[
 Evaluate[{
   ⅇ^(κS Hypergeometric2F1[1, 1/κS, 1 + 1/κS, 1 - (κS μS)/σS]) μS == ⅇ^(κ Hypergeometric2F1[1, 1/κ, 1 + 1/κ, 1 - (κ μ)/σ]) μ,
   μS + σS/2 == μ + σ/2,
   μS^2 + (2 μS σS)/(3 + κS) + (2 σS^2)/(3 (3 + κS)) == μ^2 + (2 μ σ)/(3 + κ) + (2 σ^2)/(3 (3 + κ))
   }],
 {{μS, 0.1}, 0, 5}, {{σS, 1}, 0, 10}, {{κS, 0.5}, 0, 2}
 ]
```

Out[ ]=



$$\left\{\frac{27}{4} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\,\mu}{\sigma}\right]}\mu,\ \frac{7}{2} == \mu+\frac{\sigma}{2},\ \frac{17}{2} == \mu^2+\frac{2\,\mu\,\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\right\}$$

2D Plots are possible

In[131]:= ContourPlot[

$$\left\{\frac{27}{4} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\left(\frac{7}{2}-\frac{\sigma}{2}\right)}{\sigma}\right]}\left(\frac{7}{2}-\frac{\sigma}{2}\right),\ \frac{17}{2} == \left(\frac{7}{2}-\frac{\sigma}{2}\right)^2+\frac{2\left(\frac{7}{2}-\frac{\sigma}{2}\right)\sigma}{3+\kappa}+\frac{2\,\sigma^2}{3\,(3+\kappa)}\right\},$$

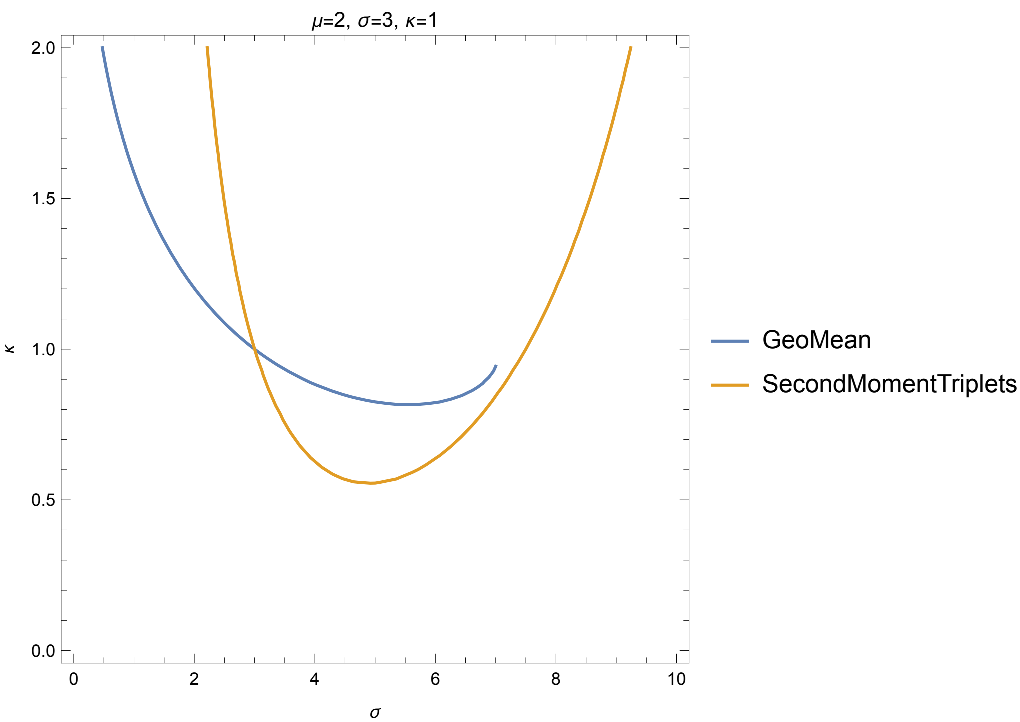
{σ, 0, 10}, {κ, 0, 2},

FrameLabel → {"σ", "κ"},

PlotLegends → {"GeoMean", "SecondMomentTriplets"},

PlotLabel → "μ=2, σ=3, κ=1"]

Out[131]=



In[127]:=

```
ContourPlot3D[
 {0.9647891183656532` == ℯ^(κ Hypergeometric2F1[1, 1/κ, 1+1/κ, 1-κμ/σ]) μ,
  0.6` == μ + σ/2, 0.25761904761904764` == μ^2 + 2 μ σ/(3 + κ) + 2 σ^2/(3 (3 + κ))},
 {μ, 0, 2}, {σ, 0, 1.5}, {κ, 0, 2},
 AxesLabel → {"μ", "σ", "κ"},
 PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"}]
```

General : $0.375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

General : $\frac{-2.35703\times10^{-302}}{-363997244}$ is too small to represent as a normalized machine number; precision may be lost.

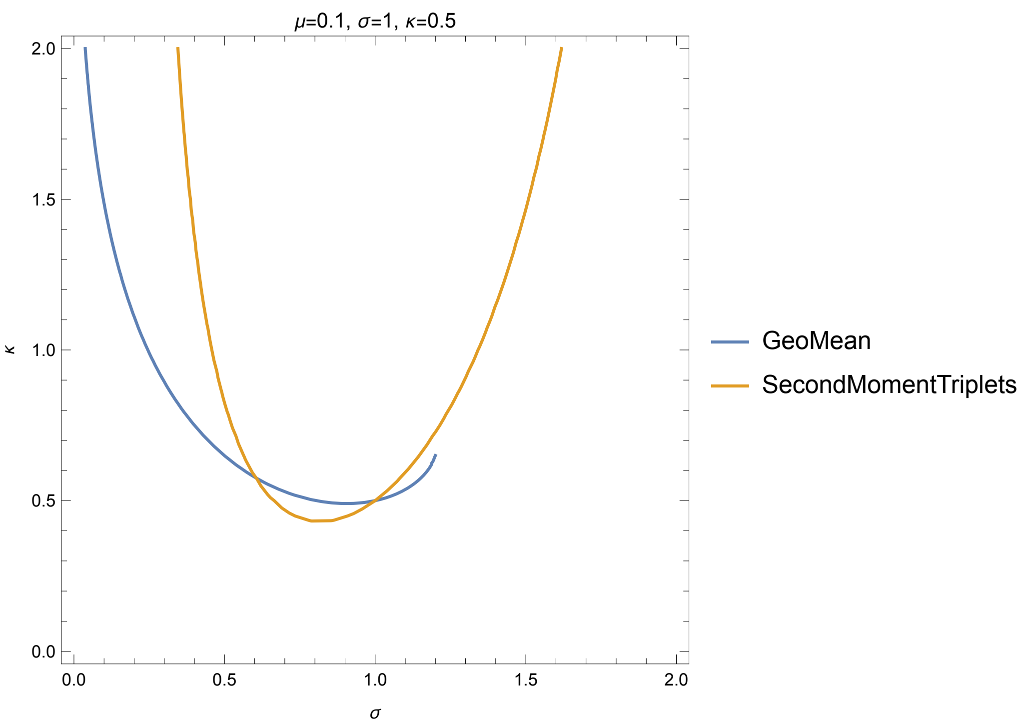
General : $-\frac{6.82102\times10^{-308}}{370997138}$ is too small to represent as a normalized machine number; precision may be lost.

General : Further output of General::munfl will be suppressed during this calculation.

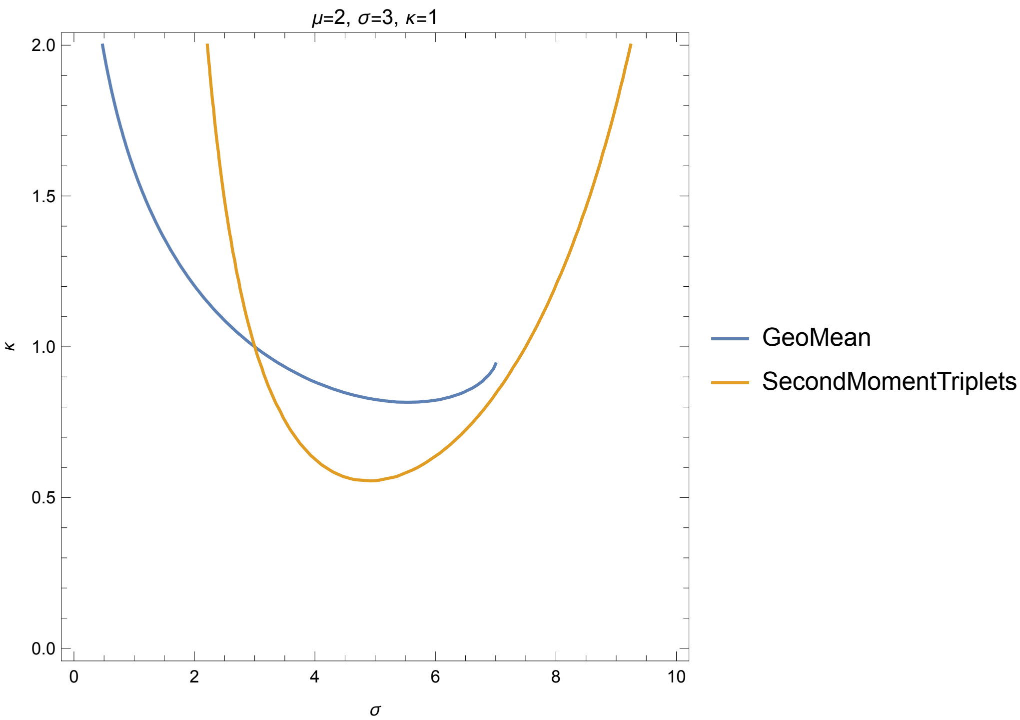
Out[127]= $Aborted

Saved Plots

Out[•]=



Out[ ]=



In[133]:=

ContourPlot3D[
 Evaluate[{

$$\frac{27}{4} == e^{\kappa\,\text{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\mu}{\sigma}\right]}\mu,$$

$$\frac{7}{2} == \mu + \frac{\sigma}{2},$$

$$\frac{17}{2} == \mu^2 + \frac{2\,\mu\,\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)}$$

 }],
 {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
 AxesLabel → {"μ", "σ", "κ"},
 PlotLegends → {"GeoMean", "MeanPairs", "SecondMomentTriplets"},
 PlotLabel → "μ=2, σ=3, κ=1"]

General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

General : $\frac{-4.94477\times10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

General : $-\frac{4.85853\times10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

General : Further output of General::munfl will be suppressed during this calculation.

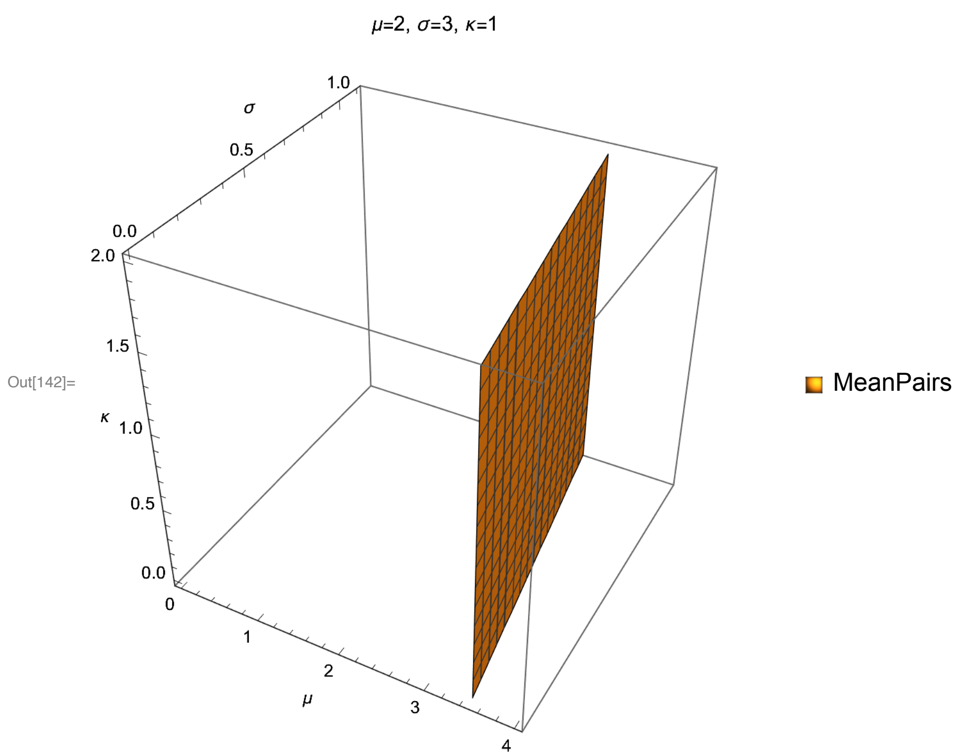
Out[133]= $Aborted

Try Contour Maps from simplist to hardest curve individually

In[141]:=
```
CEEstimationPlot[Equation_, EquationLabel_] := ContourPlot3D[
  Evaluate[Equation],
  {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
  AxesLabel → {"μ", "σ", "κ"},
  PlotLegends → {EquationLabel},
  PlotLabel → "μ=2, σ=3, κ=1"]
```

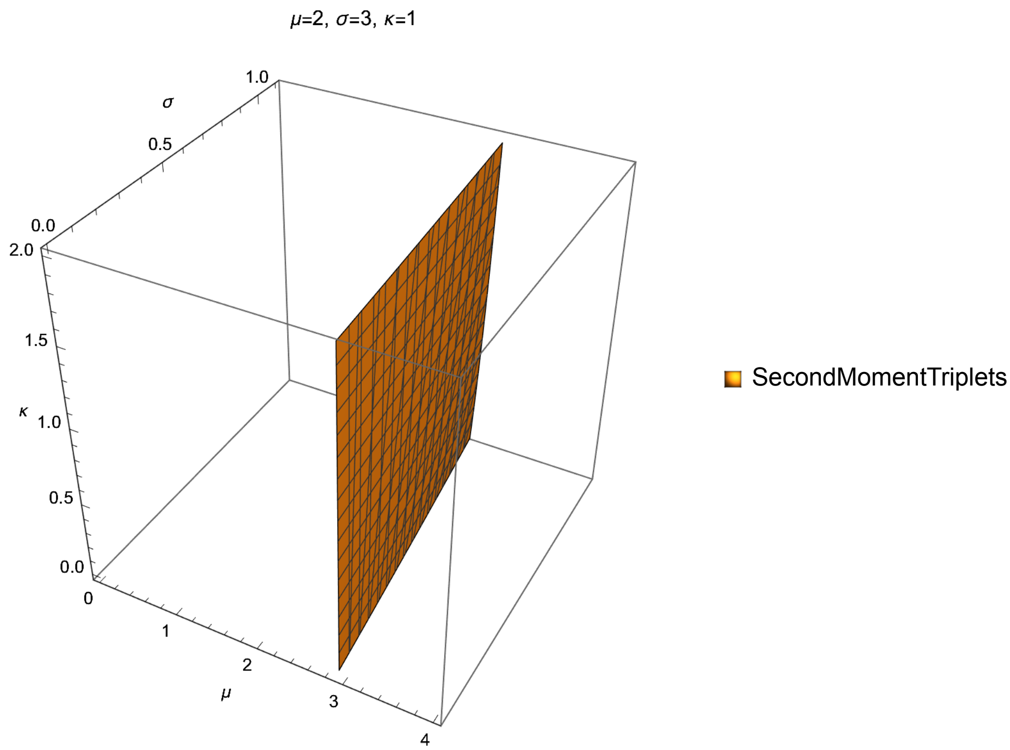
In[142]:= CEEstimationPlot$\left[\frac{7}{2} == \mu + \frac{\sigma}{2}, \text{"MeanPairs"}\right]$

Out[142]=



In[143]:= $\texttt{CEEstimationPlot}\left[\frac{17}{2} == \mu^2 + \frac{2\,\mu\,\sigma}{3+\kappa} + \frac{2\,\sigma^2}{3\,(3+\kappa)}, \texttt{"SecondMomentTriplets"}\right]$

Out[143]=



In[144]:= $\texttt{CEEstimationPlot}\left[\frac{27}{4} == e^{\kappa\,\texttt{Hypergeometric2F1}\left[1,\frac{1}{\kappa},1+\frac{1}{\kappa},1-\frac{\kappa\,\mu}{\sigma}\right]}\,\mu, \texttt{"GeoMean"}\right]$

··· General : $0.1375^{6999999}$ is too small to represent as a normalized machine number; precision may be lost.

··· General : $\frac{-4.94477\times10^{-301}}{-335997648}$ is too small to represent as a normalized machine number; precision may be lost.

··· General : $-\frac{4.85853\times10^{-307}}{342997550}$ is too small to represent as a normalized machine number; precision may be lost.

··· General : Further output of General::munfl will be suppressed during this calculation.

Out[144]= $Aborted

In[146]:=
```
ContourPlot3D[
  Evaluate[{
    7/2 == μ + σ/2,
    17/2 == μ^2 + (2 μ σ)/(3 + κ) + (2 σ^2)/(3 (3 + κ))
  }],
  {μ, 0, 4}, {σ, 0, 1.1}, {κ, 0, 2},
  AxesLabel → {"μ", "σ", "κ"},
  PlotLegends → {"MeanPairs", "SecondMomentTriplets"},
  PlotLabel → "μ=2, σ=3, κ=1"]
```

Out[146]=