Problem: estimating parameters of the Independent Approximating distribution μ, σ and κ.

we will use two methods to estimate the parameters (MOM) and (PWM):

- Method of moments (MOM)

  We have the system:

$$\begin{aligned}\mu_1 =& \mu + \frac{\sigma}{2}\\ \mu_2 =& \mu^2 + \frac{2\mu\sigma}{3+\kappa} + \frac{2\sigma^2}{3(3+\kappa)}\\ \mu_3 =& \mu^3 + \frac{3\mu^2\sigma}{(4+2\kappa)} + \frac{12\mu\sigma^2 + 3\sigma^3}{2(4+\kappa)(4+2\kappa)}\end{aligned}$$

  From first equation we find

$$\mu = \mu_1^{(2)} - \frac{\sigma}{2}, \tag{0.1}$$

  and from second equation

$$(3+\kappa)(\mu_2^{(3)} - \mu^2) = 2\mu\sigma + \frac{2\sigma^2}{3} \tag{0.2}$$

  then

$$\kappa = (2\mu\sigma + \frac{2\sigma^2}{3} - 3(\mu_2^{(3)} - \mu^2))(\mu_2^{(3)} - \mu^2)^{-1}. \tag{0.3}$$

  Using (0.1) and (0.3) we find

$$\kappa = (2(\mu_1^{(2)} - \frac{\sigma}{2})\sigma + \frac{2\sigma^2}{3} - 3(\mu_2^{(3)} - (\mu_1^{(2)} - \frac{\sigma}{2})^2))(\mu_2^{(3)} - (\mu_1^{(2)} - \frac{\sigma}{2})^2)^{-1}. \tag{0.4}$$

  From last equation of the system we find

$$(\mu_3^{(4)} - \mu^3)(4+2\kappa)(4+\kappa) = 3\mu^2\sigma(4+k) + 6\mu\sigma^2(4+2\kappa).$$

  By (0.1) and (0.4) we find

  $(\mu_3^{(4)} - (\mu_1^{(2)} - \frac{\sigma}{2})^3)(4 + 2(2(\mu_1^{(2)} - \frac{\sigma}{2})\sigma + (2\sigma^2)/3 - 3(\mu_2^{(3)} - (\mu_1^{(2)} - \sigma/2)^2))(\mu_2^{(3)} - (\mu_1^{(2)} - \sigma/2)^2)^{-1})(4 + (2(\mu_1^{(2)} - \sigma/2)\sigma + (2\sigma^2)/3 - 3(\mu_2^{(3)} - (\mu_1^{(2)} - \sigma/2)^2))(\mu_2^{(3)} - (\mu_1^{(2)} - \sigma/2)^2)^{-1}) = 3(\mu_1^{(2)} - \sigma/2)^2\sigma(4 + (2(\mu_1^{(2)} - \sigma/2)\sigma + (2\sigma^2)/3 - 3(\mu_2^{(3)} -$

$(\mu_1^{(2)}-\sigma/2)^2))(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)^{-1})+6(\mu_1^{(2)}-\sigma/2)\sigma^2(4+2(2(\mu_1^{(2)}-\sigma/2)\sigma+(2\sigma^2)/3-3(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2))(\mu_2^{(3)}-(mu_1^{(2)}-\sigma/2)^2)^{-1})$

Multiply by $(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)^2$ we find

$\hat{\sigma}=(\mu_3^{(4)}-(\mu_1^{(2)}-\sigma/2)^3)(4(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)+2(2(\mu_1^{(2)}-\sigma/2)\sigma+(2\sigma^2)/3-3(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)))(4(\mu_2^{(3)}-(\mu_1^{(2)}-\mu/2)^2)+(2(\mu_1^{(2)}-\sigma/2)\sigma+(2\sigma^2)/3-3(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)))-3(mu_1^{(2)}-\sigma/2)^2\sigma(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)(4(mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)-(2(\mu_1^{(2)}-\sigma/2)\sigma+(2\sigma^2)/3-3(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)))-6(\mu_1^{(2)}-\sigma/2)\sigma^2(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)(4(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2)-2(2(\mu_1^{(2)}-\sigma/2)\sigma+(2\sigma^2)/3-3(\mu_2^{(3)}-(\mu_1^{(2)}-\sigma/2)^2))).$

Last equation is polynomial equation with degree 7, that mean we will get 7 root for $\hat{\sigma}$ some of them real values and some imaginary.

- Probability-Weighted Moments Method (PWM)

  This method uses the cumulative distribution function, I used this equation

$$f^{(n+1)}(x)=\frac{\frac{1}{\sigma^{n+1}}(1+\kappa\frac{x-\mu}{\sigma})^{-(1/\kappa+1)(n+1)}}{\int_\mu^\infty\frac{1}{\sigma^{n+1}}(1+\kappa\frac{x-\mu}{\sigma})^{-(1/\kappa+1)(n+1)}dx}$$

to find the (cdf), where

$$F^{(n+1)}(x)=1-(1+\kappa\frac{x-\mu}{\sigma})^{-(\frac{1}{\kappa}+1)(n+1)+1},\quad x>\mu.$$

the quantile function under assumption $\mu=0$ is

$$\frac{\sigma}{\kappa}((1-u)^{\overline{\frac{-\kappa}{n\kappa+n+1}}}-1).$$

apply the method (PWM)

$$\beta_s = E(x\{1-F(x)\}^s) \tag{0.5}$$
$$= \frac{\sigma}{(s+1)(s(n+n\kappa+1)+n\kappa+n+1-\kappa)} \tag{0.6}$$

From (0.6) we have

$$\beta_0 = \frac{\sigma}{n\kappa+n+1-\kappa} \tag{0.7}$$
$$\beta_1 = \frac{\sigma}{2(2n+2n\kappa+2-\kappa)} \tag{0.8}$$

The PWM estimators $\hat{\sigma}, \hat{\kappa}$ of the parameters are the solutions of (0.7) and (0.8) for σ, κ as

$$\hat{\kappa} = -\frac{(n+1)(4\beta_1-\beta_0)}{\beta_1(4n-2)+\beta_0(-n)+\beta_0} \tag{0.9}$$
$$\hat{\sigma} = \frac{2\beta_1\beta_0(n+1)}{\beta_1(4n-2)+\beta_0(-n)+\beta_0} \tag{0.10}$$

Using the same method we can estimate the location parameter where we will have three equations instead of two, but there is a lot of computation also, I think it is possible.