
Coupled Exponentials & Logarithms

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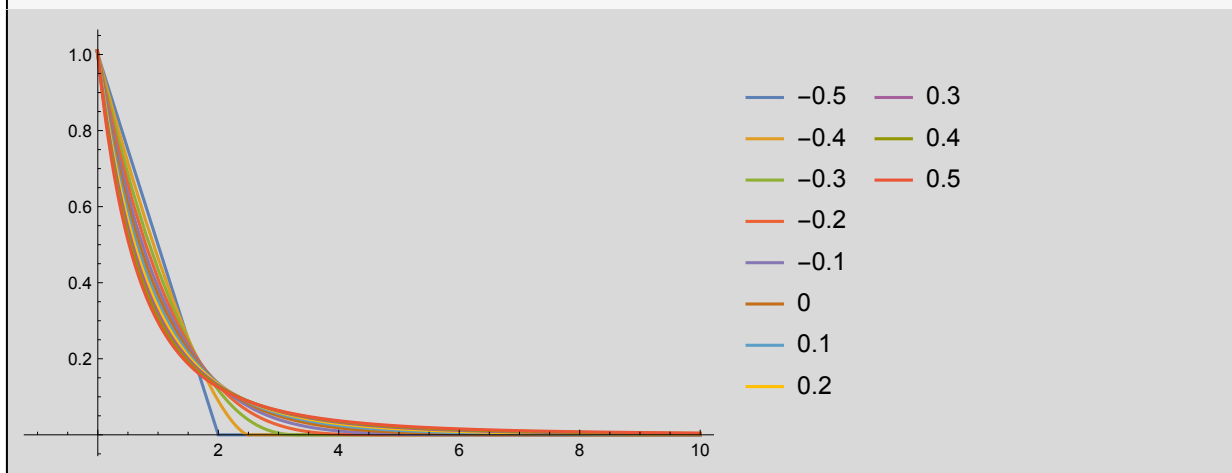
Graphic of Coupled Exponential

Graph shows curves from linear ($\kappa = -0.5$) to exponential ($\kappa = 0$)

In[]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};  
Plot[CoupledExponential[x, #]^-1 & /@ CouplingValues // Evaluate,  
{x, -1, 10}, PlotLegends -> CouplingValues, PlotRange -> Automatic]
```

Out[]:=



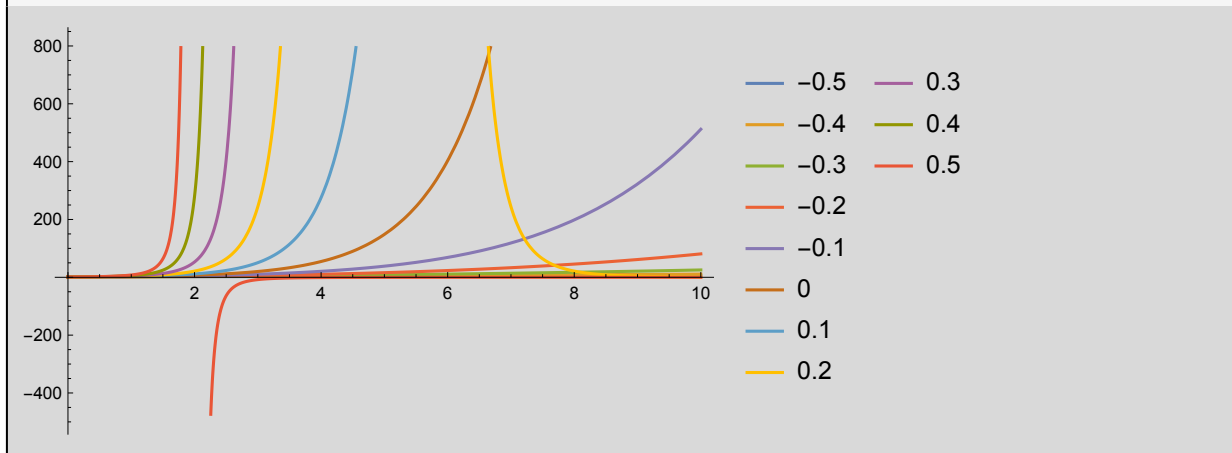
The curves are produced by the Coupled Exponential Function

$$(1 + \kappa x)^{-\frac{1+\kappa}{\kappa}}$$

In[]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Plot[CoupledExponential[-x, #]^-1 & /@ CouplingValues // Evaluate,
{x, 0, 10}, PlotLegends -> CouplingValues]
```

Out[]:=



The curves are produced by the Coupled Exponential Function

$$(1 - \kappa x)^{\frac{1+\kappa}{-\kappa}}$$

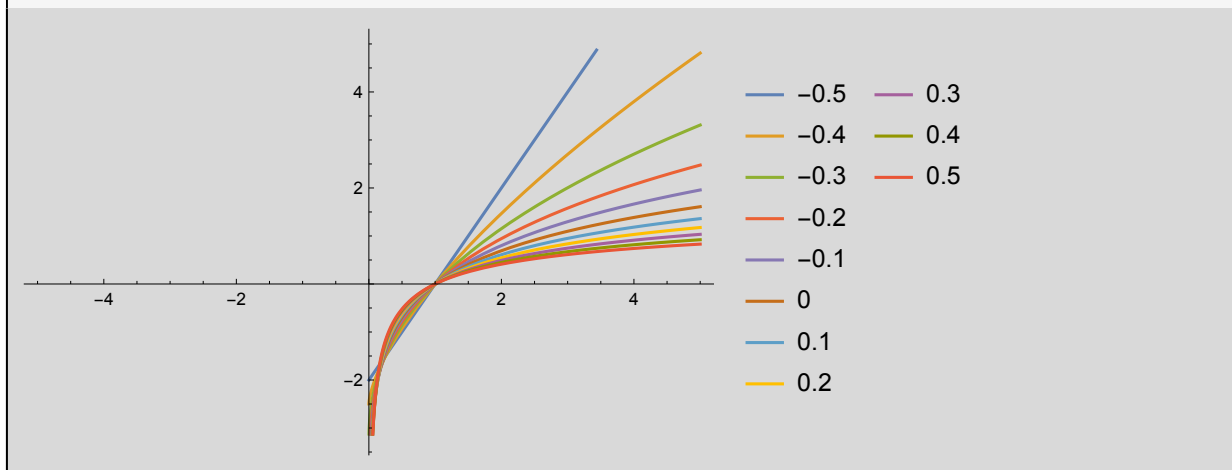
Graphic of Coupled Logarithm

Graph shows curves from linear to logarithmic

In[]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Quiet[Plot[
-CoupledLogarithm[x^-1, #] & /@ CouplingValues // Evaluate, {x, -5, 5},
PlotLegends -> CouplingValues],
{Power::infy}]
```

Out[]:=



The curves are produced by the Coupled Logarithmic Function

$$\frac{1}{-\kappa} \left(x^{\frac{-\kappa}{1+\kappa}} - 1 \right)$$

Coupled Normal Distribution

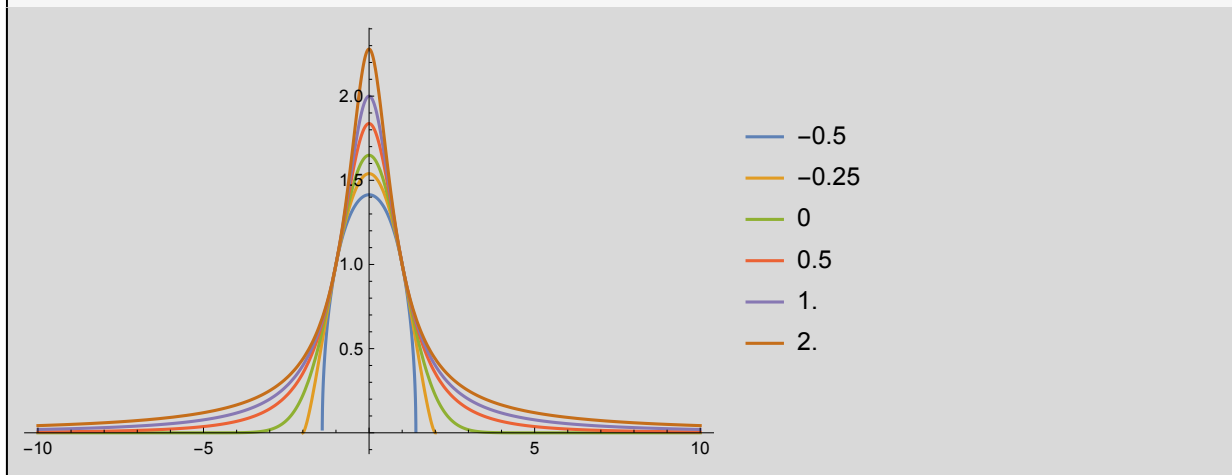
In[]:=

```

CouplingValues = {-0.5, -0.25, 0, 0.5, 1.0, 2.0};
Quiet[Plot[
  PDF[CoupledNormalDistribution[0, 1, #], {x}] /
    PDF[CoupledNormalDistribution[0, 1, #], {1}] & /@
  CouplingValues // Evaluate, {x, -10, 10},
  PlotLegends → CouplingValues,
  PlotRange → Full],
{Power::infy}]

```

Out[]:=

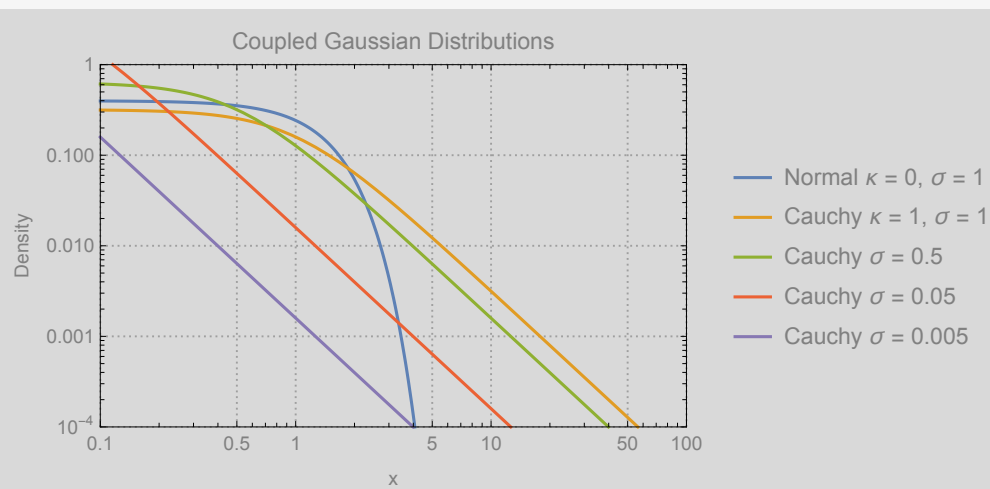


Coupled Gaussian is Scale-Free as $\sigma \rightarrow 0$

In[]:=

```
Parameters = {{1, 1, 0.5, 0.05, 0.005}, {0, 1, 1, 1, 1}};
Quiet[LogLogPlot[MapThread[
  PDF[CoupledNormalDistribution[0, #1, #2], {x}] &, Parameters] // Evaluate,
{x, 0.1, 100},
PlotLegends → {"Normal  $\kappa = 0, \sigma = 1$ ",
  "Cauchy  $\kappa = 1, \sigma = 1$ ", "Cauchy  $\sigma = 0.5$ ",
  "Cauchy  $\sigma = 0.05$ ", "Cauchy  $\sigma = 0.005$ "},
LabelStyle → Directive[Gray, Smaller],
PlotRange → {{0.1, 100}, {10-4, 1}},
PlotTheme → {"Detailed"},
FrameLabel → {"x", "Density"},
PlotLabel → "Coupled Gaussian Distributions"],
{Power::infy}]
```

Out[]:=



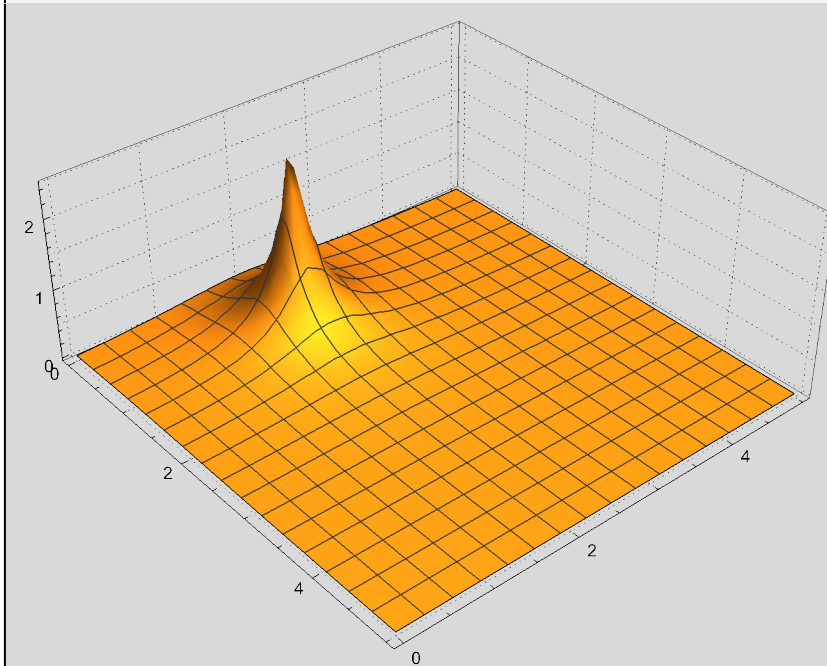
Multivariate Coupled Distribution

Multivariate Coupled Exponential

In[]:=

```
Plot3D[PDF[MultivariateCoupledDistribution[{1, 2}, {{1, 0}, {0, 1}}, 2, 1],  
  {x, y}],  
  {x, 0, 5}, {y, 0, 5},  
  PlotLegends → None,  
  PlotTheme → "Detailed",  
  PlotRange → Full  
]
```

Out[]:=

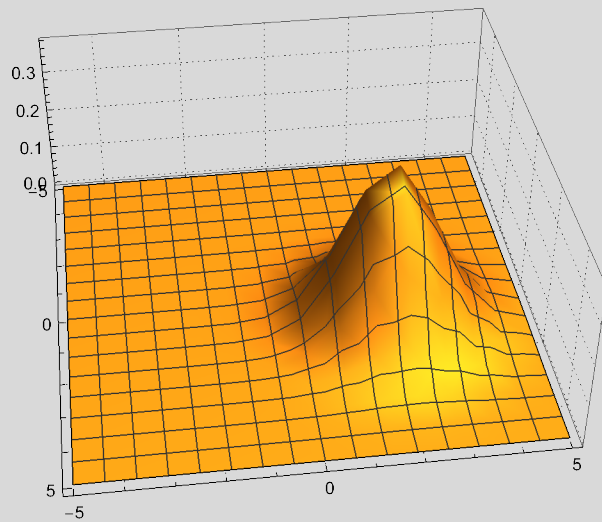


Multivariate Coupled Gaussian

In[]:=

```
Plot3D[
  PDF[MultivariateCoupledDistribution[{1, 2}, {{1, -0.01}, {0.01, 1}}, 0.01, 2],
    {x, y}],
  {x, -5, 5}, {y, -5, 5},
  PlotLegends → None,
  PlotTheme → "Detailed",
  PlotRange → Full
]
```

Out[]:=



Test Normalization of Coupled Multivariate Gaussian

In[]:=

```
Assuming[-1/2 < κ < ∞,
  Integrate[PDF[MultivariateCoupledDistribution[{0, 0}, {{1, 0}, {0, 1}}, κ, 2],
    {x, y}],
  {x, -∞, ∞}, {y, -∞, ∞}
] // FullSimplify
```

Out[]:=

1

In[8]:=

```
Assuming[-1/3 < κ < ∞, Integrate[PDF[MultivariateCoupledDistribution[
  {0, 0, 0}, {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}, κ, 2],
  {x, y, z}],
  {x, -∞, ∞}, {y, -∞, ∞}, {z, -∞, ∞}
]] // FullSimplify
```

Out[8]=

$$\frac{1}{2\pi \text{Beta}\left[-\frac{1+\kappa}{2\kappa}, \frac{3}{2}\right]} \sqrt{-\kappa} \kappa \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(1+x^2\kappa+y^2\kappa+z^2\kappa\right)^{3+\frac{1}{\kappa}} \left(x^2+y^2+z^2\right)^{\kappa\geq-1}}} dx dy dz, \quad \kappa \geq 0$$

True

Assumptions $\rightarrow -\frac{1}{3} < \kappa < \infty \&\& \left(-\frac{1}{3} < \kappa < 0 \mid \mid \kappa \leq -\frac{1}{3}\right)$

In[9]:=

```
Assuming[-1/4 < κ < ∞,
  Integrate[PDF[MultivariateCoupledDistribution[{0, 0, 0, 0},
    {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}, κ, 2],
    {w, x, y, z}],
    {w, -∞, ∞}, {x, -∞, ∞}, {y, -∞, ∞}, {z, -∞, ∞}
  ] // FullSimplify
```

Out[9]=

$$\frac{1}{\pi^2 \text{Beta}\left[-1-\frac{1}{2\kappa}, 2\right]} \kappa^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(1+w^2\kappa+x^2\kappa+y^2\kappa+z^2\kappa\right)^{4+\frac{1}{\kappa}} \left(w^2+x^2+y^2+z^2\right)^{\kappa\geq-1}}} dw dx dy dz, \quad \kappa \geq 0$$

True

Assumptions $\rightarrow -\frac{1}{4} < \kappa < \infty \&\& \left(-\frac{1}{4} < \kappa < 0 \mid \mid \kappa \leq -\frac{1}{4}\right)$