Coupled Exponentials & Logarithms

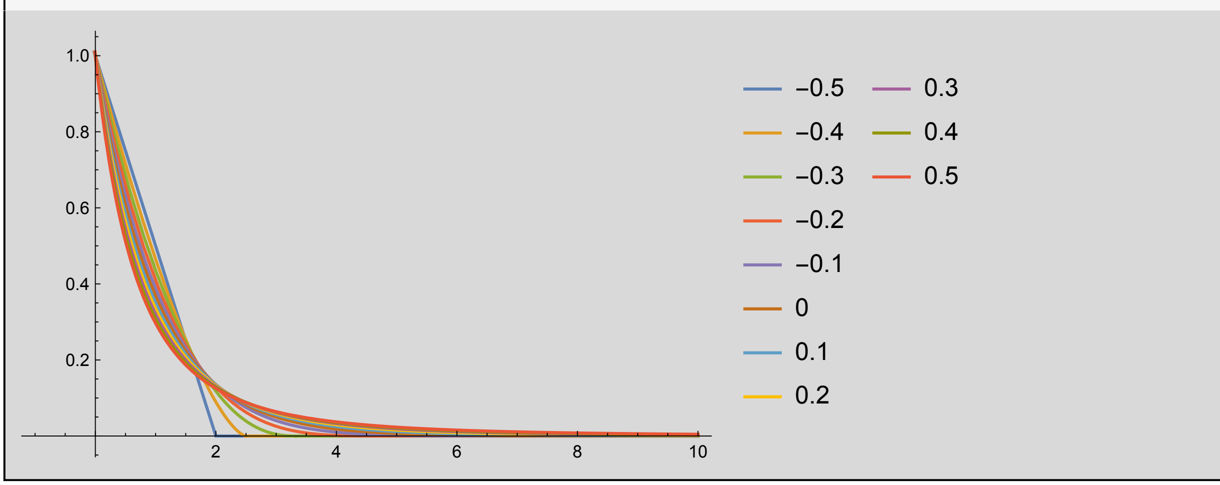
Graphic of Coupled Exponential

Graph shows Coupled Exponential decay using the inverse of the CoupledExponential Function with coupling κ values with compact-support (-0.5 to -0.1), exponential (0), and heavy-tail (0.1 to 0.5).

In[∘]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Plot[CoupledExponential[x, #]^-1 & /@ CouplingValues // Evaluate,
 {x, -1, 10}, PlotLegends → CouplingValues, PlotRange → Automatic]
```

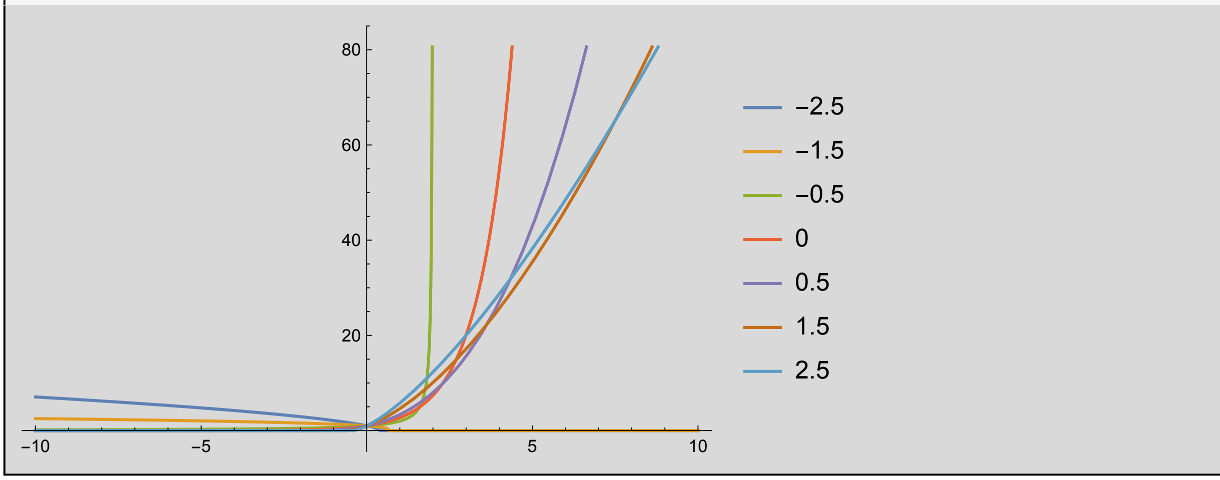
Out[∘]=



Graph shows Coupled Exponential over a broad range of coupling κ and variable x values.

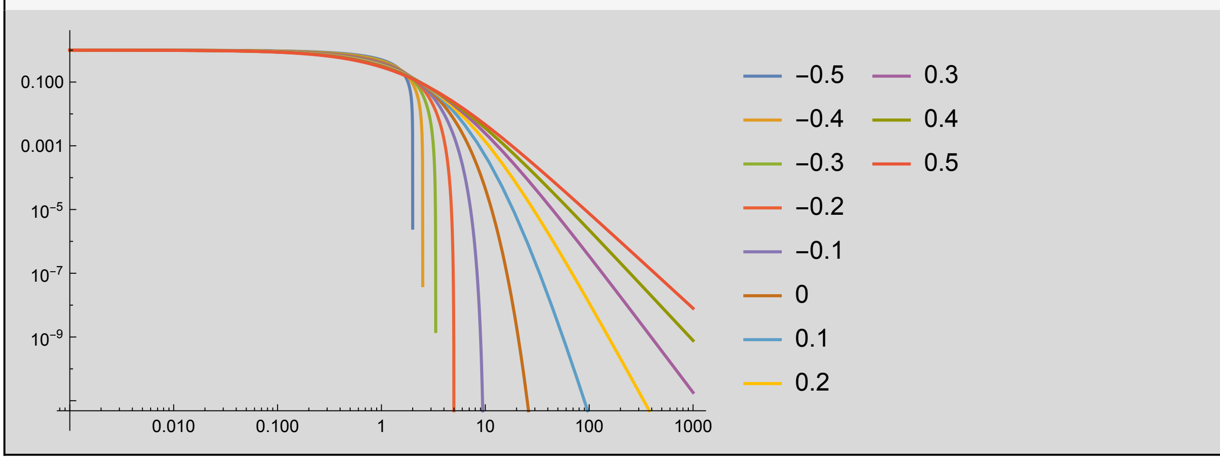
In[∘]:=
```
CouplingValues = {-2.5, -1.5, -0.5, 0, 0.5, 1.5, 2.5};
Plot[CoupledExponential[x, #] & /@ CouplingValues // Evaluate,
 {x, -10, 10}, PlotLegends → CouplingValues, PlotRange → Automatic]
```

Out[∘]=



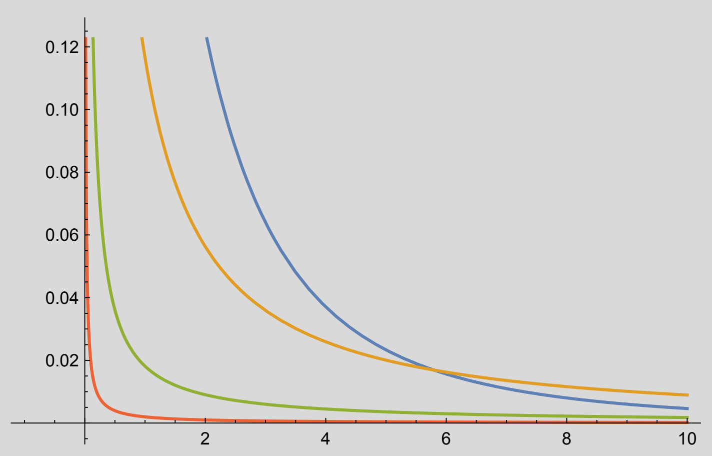
In[∘]:=
```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
LogLogPlot[CoupledExponential[x, #]^-1 & /@ CouplingValues // Evaluate,
 {x, 10^-3, 10^3}, PlotLegends → CouplingValues, PlotRange → Automatic]
```

Out[∘]=



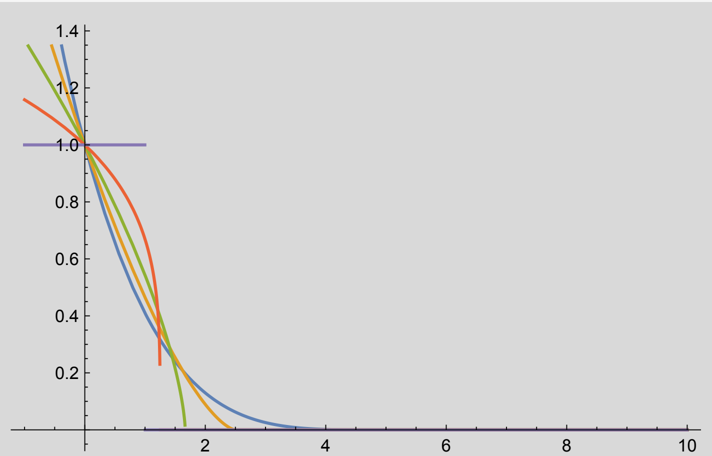
In[∘]:=
```
CouplingValues = {0.5, 5, 50, 500};
Plot[CoupledExponential[x, #]^-1 & /@ CouplingValues // Evaluate,
 {x, -1, 10}, PlotLegends → CouplingValues, PlotRange → Automatic]
```

Out[∘]=



In[∘]:=
```
CouplingValues = {-0.2, -0.4, -0.6, -0.8, -1.0};
Plot[CoupledExponential[x, #]^-1 & /@ CouplingValues // Evaluate,
 {x, -1, 10}, PlotLegends → CouplingValues, PlotRange → Automatic]
```

Out[∘]=



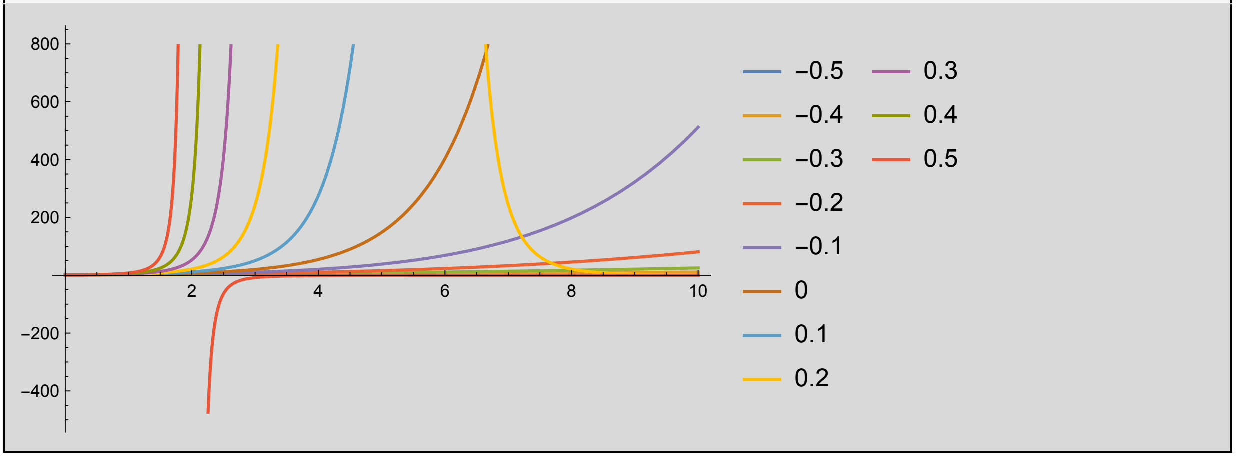
The curves are produced by the Coupled Exponential Function

$$(1 + \kappa\, x)^{-\frac{1+\kappa}{\kappa}}$$

In[∘]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Plot[CoupledExponential[-x, #]^-1 & /@ CouplingValues // Evaluate,
 {x, 0, 10}, PlotLegends → CouplingValues]
```

Out[∘]=



The curves are produced by the Coupled Exponential Function

$$(1-\kappa\,x)^{\frac{1+\kappa}{-\kappa}}$$

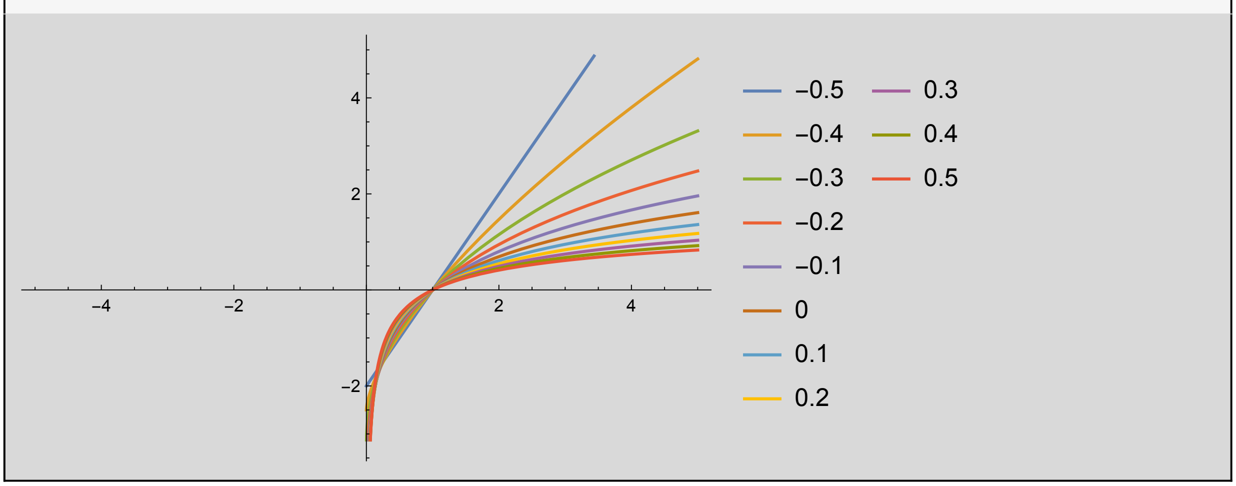
Graphic of Coupled Logarithm

Graph shows curves from linear to logarithmic

In[∘]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Quiet[Plot[
  -CoupledLogarithm[x^-1, #] & /@ CouplingValues // Evaluate, {x, -5, 5},
  PlotLegends → CouplingValues],
 {Power::infy}]
```

Out[∘]=



The curves are produced by the Coupled Logarithmic Function

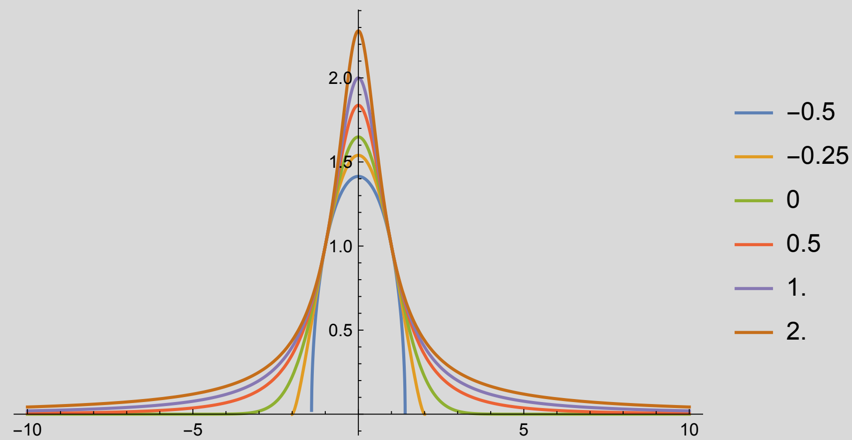
$$\frac{1}{-\kappa}\left(x^{\frac{-\kappa}{1+\kappa}}-1\right)$$

Coupled Normal Distribution

In[∘]:=

```
CouplingValues = {-0.5, -0.25, 0, 0.5, 1.0, 2.0};
Quiet[Plot[
  PDF[CoupledNormalDistribution[0, 1, #], {x}] /
     PDF[CoupledNormalDistribution[0, 1, #], {1}] & /@
   CouplingValues // Evaluate, {x, -10, 10},
  PlotLegends → CouplingValues,
  PlotRange → Full],
 {Power::infy}]
```

Out[∘]=

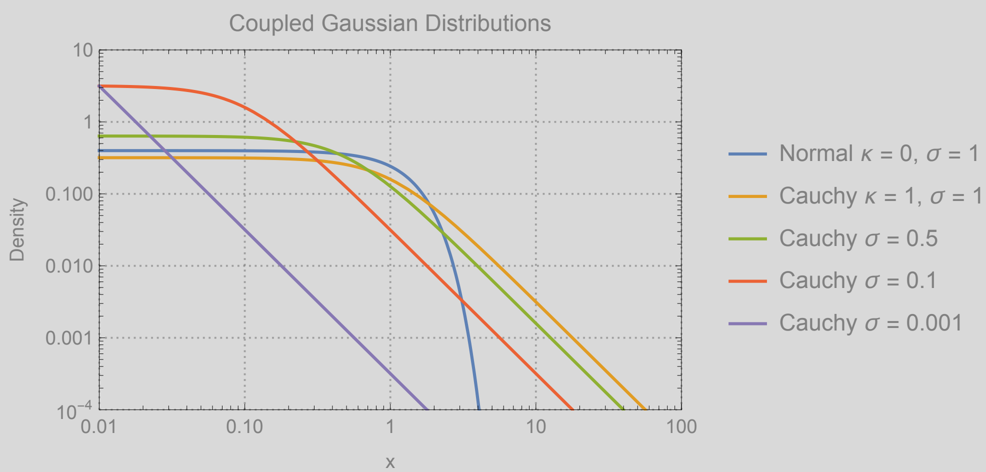


Coupled Gaussian is Scale-Free as $\sigma \to 0$

In[∘]:=

```
Parameters = {{1, 1, 0.5, 0.1, 0.001}, {0, 1, 1, 1, 1}};
Quiet[LogLogPlot[MapThread[
    PDF[CoupledNormalDistribution[0, #1, #2], {x}] &, Parameters] // Evaluate,
  {x, 0.01, 100},
  PlotLegends → {"Normal κ = 0, σ = 1",
    "Cauchy κ = 1, σ = 1", "Cauchy σ = 0.5",
    "Cauchy σ = 0.1", "Cauchy σ = 0.001"},
  LabelStyle → Directive[Gray, Smaller],
  PlotRange → {{0.01, 100}, {10^-4, 10}},
  PlotTheme → {"Detailed"},
  FrameLabel → {"x", "Density"},
  PlotLabel → "Coupled Gaussian Distributions"],
 {Power::infy}]
```

Out[∘]=



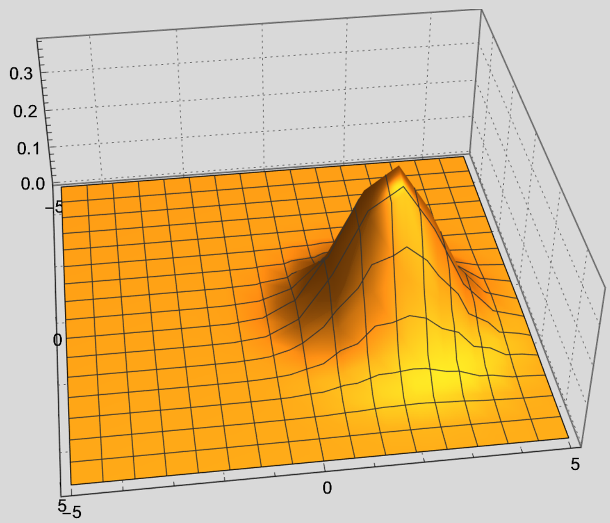
Multivariate Coupled Distribution

Multivariate Coupled Exponential

Multivariate Coupled Gaussian

In[•]:=
```
Plot3D[
 PDF[MultivariateCoupledDistribution[{1, 2}, {{1, -0.01}, {0.01, 1}}, 0.01, 2],
  {x, y}],
 {x, -5, 5}, {y, -5, 5},
 PlotLegends → None,
 PlotTheme → "Detailed",
 PlotRange → Full
]
```

Out[•]=



Test Normalization of Coupled Multivariate Gaussian

In[•]:=
```
Assuming[-1 / 2 < κ < ∞,
  Integrate[PDF[MultivariateCoupledDistribution[{0, 0}, {{1, 0}, {0, 1}}, κ, 2],
    {x, y}],
   {x, -∞, ∞}, {y, -∞, ∞}
  ]] // FullSimplify
```

Out[•]=
```
1
```

In[ ]:=
```
Assuming[-1 / 3 < κ < ∞, Integrate[PDF[MultivariateCoupledDistribution[
     {0, 0, 0}, {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}, κ, 2],
    {x, y, z}],
   {x, -∞, ∞}, {y, -∞, ∞}, {z, -∞, ∞}
  ]] // FullSimplify
```

Out[ ]=

$$\begin{cases} 1 & \kappa \geq 0 \\ -\frac{1}{2\,\pi\,\mathrm{Beta}\left[-\frac{1+\kappa}{2\kappa},\frac{3}{2}\right]}\sqrt{-\kappa}\;\kappa\;\mathrm{Integrate}\left[\frac{1}{\sqrt{\begin{cases}\left(1+x^2\kappa+y^2\kappa+z^2\kappa\right)^{3+\frac{1}{\kappa}} & \left(x^2+y^2+z^2\right)\kappa\geq -1 \\ \infty & \mathrm{True}\end{cases}}},\{x,-\infty,\infty\},\right. & \mathrm{True} \\ \quad \left.\{y,-\infty,\infty\},\{z,-\infty,\infty\},\mathrm{Assumptions}\to -\frac{1}{3}<\kappa<\infty \,\&\&\, \left(-\frac{1}{3}<\kappa<0 \,||\, \kappa\leq -\frac{1}{3}\right)\right] & \end{cases}$$

In[ ]:=
```
Assuming[-1 / 4 < κ < ∞,
  Integrate[PDF[MultivariateCoupledDistribution[{0, 0, 0, 0},
     {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}, κ, 2],
    {w, x, y, z}],
   {w, -∞, ∞}, {x, -∞, ∞}, {y, -∞, ∞}, {z, -∞, ∞}
  ]] // FullSimplify
```

Out[ ]=

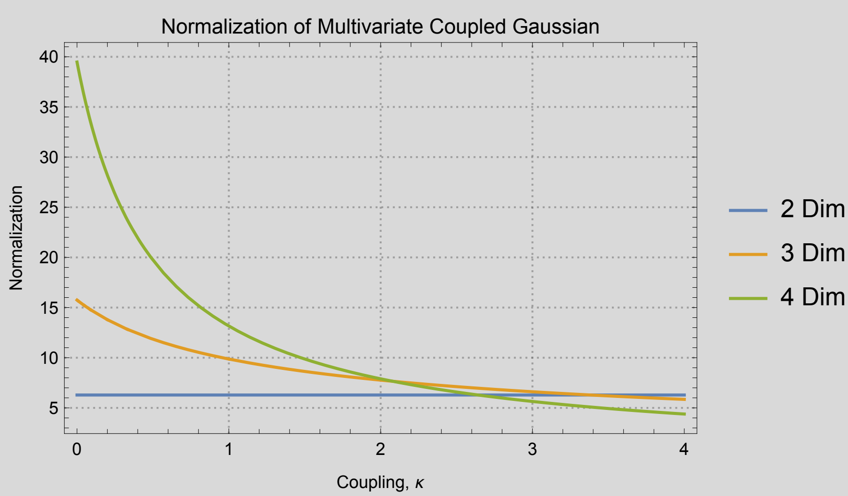
$$\begin{cases} 1 & \kappa \geq 0 \\ \frac{1}{\pi^2\,\mathrm{Beta}\left[-1-\frac{1}{2\kappa},2\right]} & \mathrm{True} \\ \quad \kappa^2\;\mathrm{Integrate}\left[\frac{1}{\sqrt{\begin{cases}\left(1+w^2\kappa+x^2\kappa+y^2\kappa+z^2\kappa\right)^{4+\frac{1}{\kappa}} & \left(w^2+x^2+y^2+z^2\right)\kappa\geq -1 \\ \infty & \mathrm{True}\end{cases}}},\{w,-\infty,\infty\},\{x,-\infty,\infty\},\right. & \\ \quad \left.\{y,-\infty,\infty\},\{z,-\infty,\infty\},\mathrm{Assumptions}\to -\frac{1}{4}<\kappa<\infty \,\&\&\, \left(-\frac{1}{4}<\kappa<0 \,||\, \kappa\leq -\frac{1}{4}\right)\right] & \end{cases}$$

Normalization of Multivariate Coupled Gaussian

In[∘]:=

```
Plot[Evaluate@MapThread[NormMultiCoupled[
     #1, κ, 2, #2] &, {{
     {{1, 0}, {0, 1}},
     {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}},
     {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
    },
    {2, 3, 4}
   }],
 {κ, 0, 4},
 PlotRange → Full,
 PlotTheme → "Detailed",
 PlotLegends → {"2 Dim", "3 Dim", "4 Dim"},
 FrameLabel → {"Coupling, κ", "Normalization"},
 PlotLabel → "Normalization of Multivariate Coupled Gaussian"
]
```

Out[∘]=

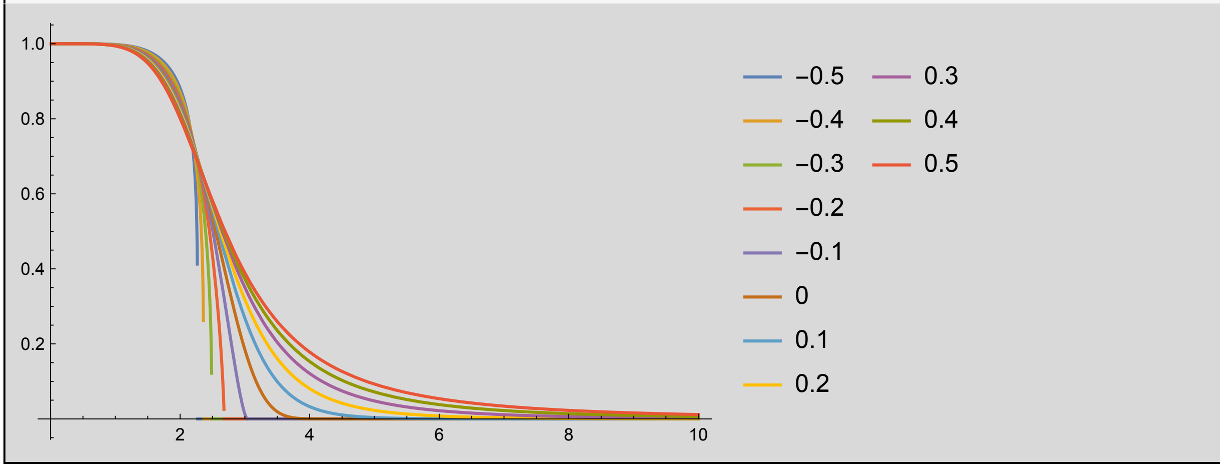


Coupled Exponential with variable power α

In[ ]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Plot[CoupledExponential[(x / σ)^α, #]^(-1/α) /. {α → 5.5, σ → 2} & /@ CouplingValues //
  Evaluate, {x, 0.01, 10}, PlotLegends → CouplingValues, PlotRange → Full]
```

Out[ ]=



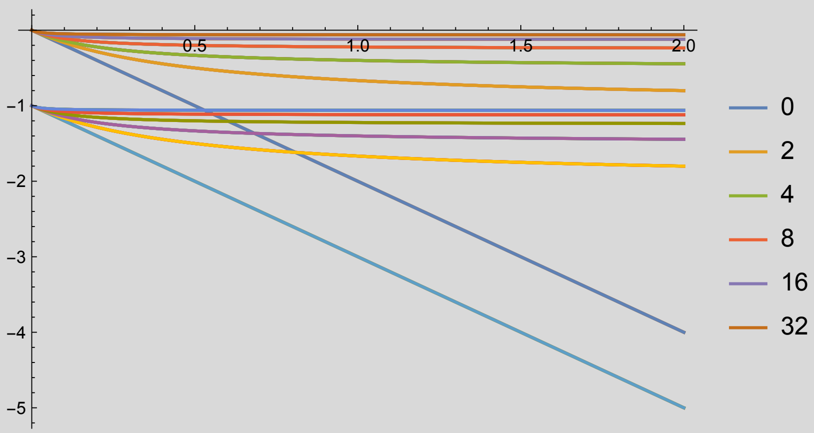
Coupled Logarithm Power as function of d

The coupled logarithm when applied to $x^{-\alpha}$ has a power of $\frac{-\alpha\,\kappa}{1+d\,\kappa}$. The power of the derivative is $\frac{-\alpha\,\kappa}{1+d\,\kappa} - 1 = \frac{-1-(\alpha+d)\,\kappa}{1+d\,\kappa}$

In[72]:=

```
d = {0, 2, 4, 8, 16, 32};
Quiet[Plot[
  {(-2 κ)/(1 + d κ), (-1 - (2 + d) κ)/(1 + d κ)} & /@ d // Evaluate, {κ, 10^-6, 2},
  PlotLegends → d],
 {Power::infy}]
```

Out[73]=



In[65]:=

```
{0, 2, 4, 8, 16, 32}
```

Out[65]=

```
{0, 2, 4, 8, 16, 32}
```

The coupled logarithm for very low probabilities. Setting $\alpha = 2$, d = 16, and $\kappa = 0.1$

In[108]:=
```
CouplingValues = {0, 0.01, 0.05, 0.1};
Quiet[LogLogPlot[
  CoupledLogarithm[x^-2, #, 16] & /@ CouplingValues // Evaluate, {x, 10^-250, 10^-50},
  PlotLegends → CouplingValues,
  PlotRange → Full],
 {Power::infy}]
```

General: 1.00945×10^{-250} 1.00945×10^{-250} is too small to represent as a normalized machine number; precision may be lost.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

GreaterEqual: Invalid comparison with ComplexInfinity attempted.

GreaterEqual: Invalid comparison with ComplexInfinity attempted.

General: 1.00945×10^{-250} 1.00945×10^{-250} is too small to represent as a normalized machine number; precision may be lost.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: 1.00945×10^{-250} 1.00945×10^{-250} is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

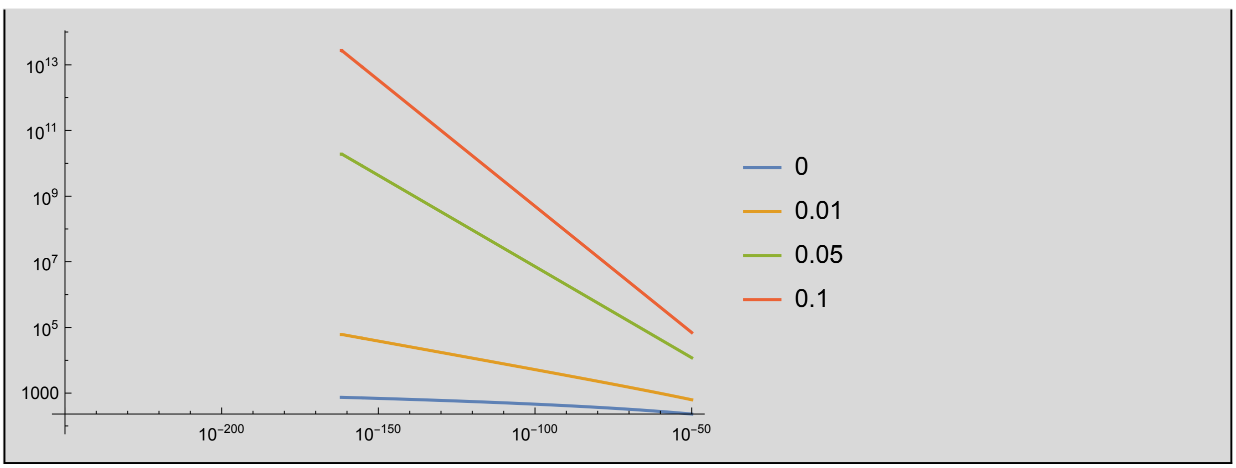
Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: Further output of Divide::infy will be suppressed during this calculation.

GreaterEqual: Invalid comparison with ComplexInfinity attempted.

General: Further output of GreaterEqual::nord will be suppressed during this calculation.

Out[109]=



The issue is due to the squaring of the probability. If the square term is moved into the exponent, then this problem is no longer an issue.

In[132]:=

$$\texttt{CouplingValues = \{0, 0.01, 0.05, 0.1\};}$$
$$\texttt{Quiet}\Big[\texttt{LogLogPlot}\Big[$$
$$\texttt{If}\left[\# \neq 0, \frac{\left(\frac{1}{x}\right)^{\left(\frac{2\#}{1+16\#}\right)} - 1}{\#}, \texttt{Log}\left[\frac{1}{x^2}\right]\right] \texttt{ \& /@ CouplingValues // Evaluate,}$$
$$\{x, 10^{-250}, 10^{-50}\},$$
$$\texttt{PlotLegends} \to \texttt{CouplingValues,}$$
$$\texttt{PlotRange} \to \texttt{Full}\Big],$$
$$\{\texttt{Power::infy}\}\Big]$$

General: 1.00945×10^{-250} 1.00945×10^{-250} is too small to represent as a normalized machine number; precision may be lost.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: 1.2182×10^{-246} 1.2182×10^{-246} is too small to represent as a normalized machine number; precision may be lost.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

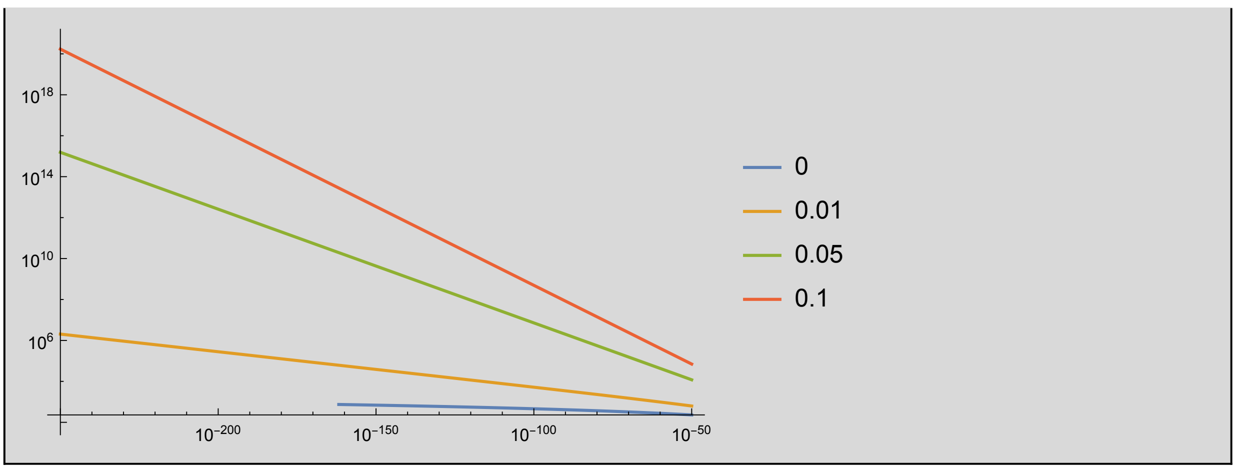
General: 1.47011×10^{-242} 1.47011×10^{-242} is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: Further output of Divide::infy will be suppressed during this calculation.

Out[133]=

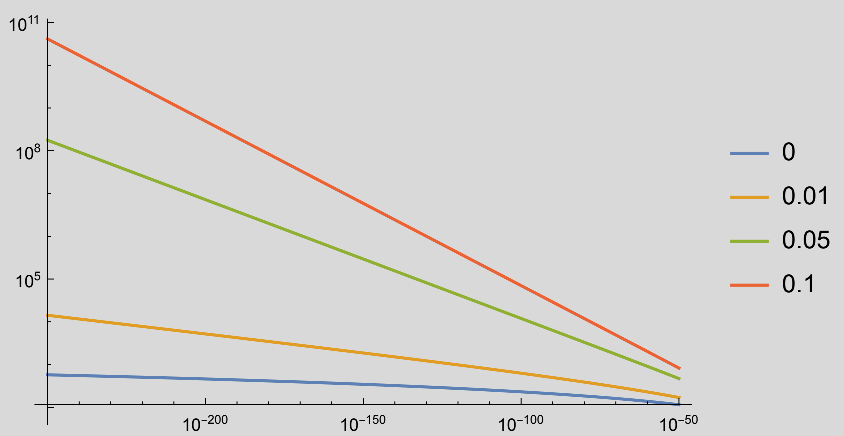


An alternative work around is to set alpha = 1.

In[134]:=

```
CouplingValues = {0, 0.01, 0.05, 0.1};
Quiet[LogLogPlot[
  CoupledLogarithm[x^-1, #, 16] & /@ CouplingValues // Evaluate, {x, 10^-250, 10^-50},
  PlotLegends → CouplingValues,
  PlotRange → Full],
 {Power::infy}]
```

Out[135]=



In[110]:=

```
Quiet[LogLogPlot[
  CoupledLogarithm[x^-2, #, 0] & /@ CouplingValues // Evaluate, {x, 10^-250, 10^-50},
  PlotLegends → CouplingValues,
  PlotRange → Full],
 {Power::infy}]
```

General: 1.00945×10^{-250} 1.00945×10^{-250} is too small to represent as a normalized machine number; precision may be lost.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

GreaterEqual: Invalid comparison with ComplexInfinity attempted.

GreaterEqual: Invalid comparison with ComplexInfinity attempted.

General: 1.00945×10^{-250} 1.00945×10^{-250} is too small to represent as a normalized machine number; precision may be lost.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: 1.00945×10^{-250} 1.00945×10^{-250} is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

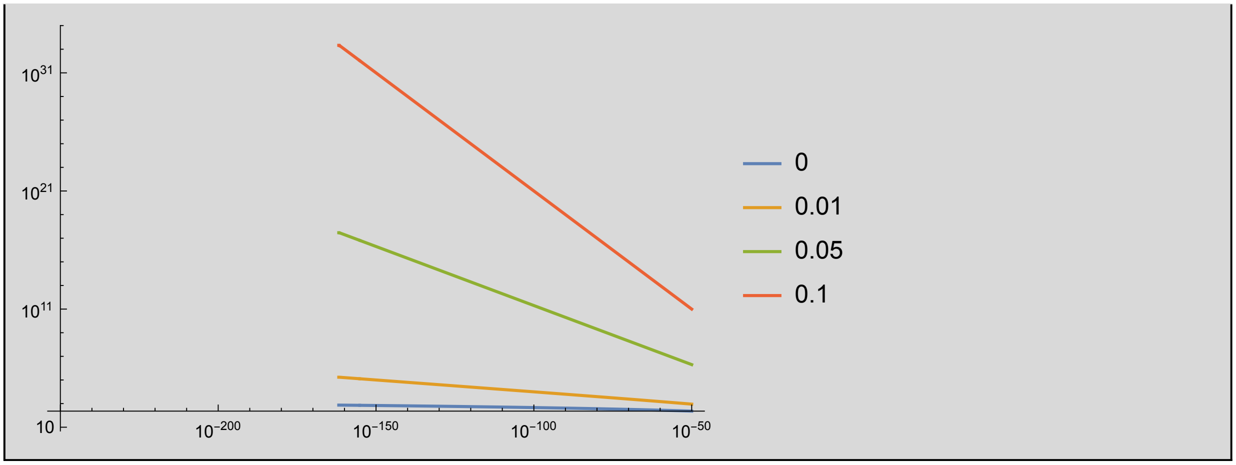
Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: Further output of Divide::infy will be suppressed during this calculation.

GreaterEqual: Invalid comparison with ComplexInfinity attempted.

General: Further output of GreaterEqual::nord will be suppressed during this calculation.

Out[110]=



In[115]:=

```
Clear[d]
```

In[116]:=

```
Assuming[0 < x < 1 && 0 < κ < ∞, FullSimplify[D[CoupledLogarithm[x^-2, κ, d], x]]]
```

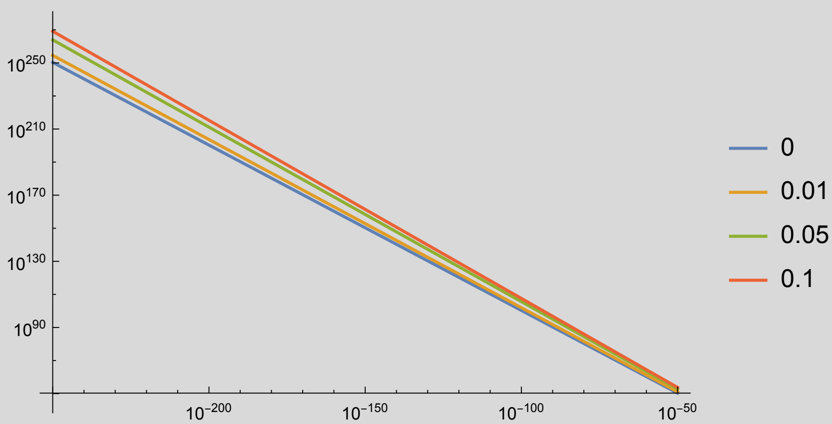
Out[116]=

$$-\frac{2\,x^{-\frac{2\kappa}{1+d\kappa}}}{x+d\,x\,\kappa}$$

In[119]:=

```
CouplingValues = {0, 0.01, 0.05, 0.1};
Quiet[LogLogPlot[
  (2 x^(-(2 #)/(1+16 #)))/(x + 16 x #) & /@ CouplingValues // Evaluate, {x, 10^-250, 10^-50},
  PlotLegends → CouplingValues,
  PlotRange → Full],
 {Power::infy}]
```

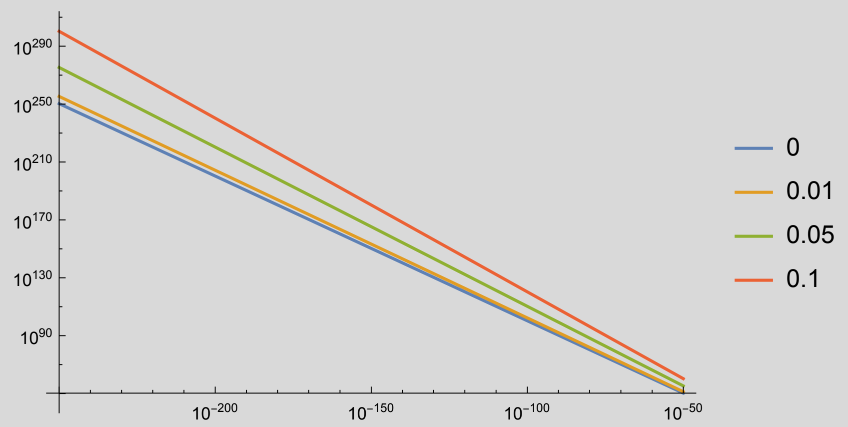
Out[120]=



In[121]:=

```
CouplingValues = {0, 0.01, 0.05, 0.1};
Quiet[LogLogPlot[
  2 x^(-2 #)/x & /@ CouplingValues // Evaluate, {x, 10^-250, 10^-50},
  PlotLegends → CouplingValues,
  PlotRange → Full],
 {Power::infy}]
```

Out[122]=



In[128]:=

```
CouplingValues = {0, 0.01, 0.05, 0.1};
Quiet[LogLogPlot[
   ((1/x^2)^(#/(1+16 #)) - 1)/# & /@ CouplingValues // Evaluate, {x, 10^-250, 10^-50},
   PlotLegends → CouplingValues,
   PlotRange → Full],
  {Power::infy}]
```

Infinity: Indeterminate expression 0 ComplexInfinity encountered.

General: 1.00945×10^{-250} 1.00945×10^{-250} is too small to represent as a normalized machine number; precision may be lost.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: 1.2182×10^{-246} 1.2182×10^{-246} is too small to represent as a normalized machine number; precision may be lost.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: 1.47011×10^{-242} 1.47011×10^{-242} is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

Divide: Infinite expression $\frac{1}{0.}$ encountered.

General: Further output of Divide::infy will be suppressed during this calculation.

Out[129]=