Comparison of Generalized Entropy Functions

Purpose of this notebook is to compare the Coupled-Entropy function with Entropy functions, such as Tsallis, Normalized Tsallis, Renyi, Shannon, and Hanel-Thurner

Compare Average Uncertainty Functions

Care is needed in formulating these entropies and the constraints properly. The CoupledLogarithm is defined in terms of the source of coupling κ, and uses the separate definitions for the multiplicative coupling K = $\frac{-\alpha\kappa}{1+\kappa}$ and the additive coupling -$\alpha\kappa$. It is easiest to define the Tsallis Entropy in terms of the multiplicative coupling since q=1-K, but proper translation to the source of coupling will be needed. Alternatively using the Coupled Logarithm definition, the Tsallis Logarithm is CoupledLogarithm/(1+dκ). The -α term doesn't affect the translation since this factor is the same for both the multiplicative and additive coupling. Two equivalent equations for the Tsallis Entropy are then:

TsallisEntropy[p_,κ_,α_:1,d_:1]

$:= -\int_{-\infty}^{\infty} \text{PDF}[p, x] \times \text{CoupledLogarithm}[\text{PDF}[p, x], \kappa, -\alpha, d] / (1 + d\,\kappa)\, dx$

$:= \int_{-\infty}^{\infty} \text{PDF}[p, x] \times \text{CoupledLogarithm}[1 / \text{PDF}[p, x], \kappa, \alpha, d] / (1 + d\,\kappa)\, dx$

Tsallis Entropy

```
TsallisEntropy[p_, κ_, α_ : 1, d_ : 1, lowerlim_, upperlim_] :=
 ∫_lowerlim^upperlim FullSimplify[PDF[p, x] × CoupledLogarithm[1 / PDF[p, x], κ, α, d] (1 + d κ)] ⅆx //
  FullSimplify
```

```
Assuming[κ > 0, TsallisEntropy[CoupledNormalDistribution[κ, 0, σ], κ, 2, 1, -∞, ∞]]
```

$$\frac{1}{2}\left(1+\frac{1}{\kappa}-\pi^{-1+\frac{1}{1+\kappa}}\,\kappa^{-\frac{1}{1+\kappa}}\left(\frac{\sigma\,\text{Gamma}\left[\frac{1}{2\,\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\,\kappa}\right]}\right)^{-\frac{2\,\kappa}{1+\kappa}}\right)$$

```
Assuming[κ > 0,
 TsallisEntropy[CoupledExponentialDistribution[κ, 0, σ], κ, 1, 1, 0, ∞]]
```

$$\frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$$

$\frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa} = 1 - \frac{1}{1+\kappa}\left(\frac{1+\kappa}{\kappa}\right)\left(\sigma^{-1+\frac{1}{1+\kappa}} - 1\right) = 1 - \frac{1}{1+\kappa}\ln_{\frac{\kappa}{1+\kappa}}\sigma^{-1}.$

Verification of the two forms of Tsallis Entropy

```
Null
```

```
Null
```

```
Null
```

```
Null
```

Plot of Tsallis Entropy of Coupled Gaussian versus Coupling

The Tsallis Entropy of the Coupled Gaussian when σ=1 and $\kappa_{\text{entropy}} = \kappa_{\text{distribution}}$

$$\text{TECoupledGaussian}[\kappa\_, \sigma\_] := \frac{1}{2}\left(1 + \frac{1}{\kappa} - \pi^{-1+\frac{1}{1+\kappa}}\,\kappa^{-\frac{1}{1+\kappa}}\left(\frac{\sigma\,\text{Gamma}\left[\frac{1}{2\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{-\frac{2\kappa}{1+\kappa}}\right)$$

Tsallis Entropy - Numerical

```
TsallisEntropyNumerical[p_, κ_, α_ : 1, d_ : 0, lim_ : 100] :=
 -NIntegrate[PDF[p, x] × CoupledLogarithm[PDF[p, x], κ, - α, d], {x, - lim, lim}]
```

```
Thread[TsallisEntropyNumerical[CoupledNormalDistribution[
   {0.2, 0.4, 0.6, 0.8}, 0, 1], {0.2, 0.4, 0.6, 0.8}, 2, 0, 100]]
```

```
Null

Null

Null

Null

Null

Null
```

```
TsallisEntropyNumerical[CoupledNormalDistribution[0.1, 0, 1], 0.1, 2, 0]
```

```
Null
```

```
TsallisEntropyNumerical[CoupledNormalDistribution[0.2, 0, 1], 0.2, 2, 0]
```

```
Null
```

```
TsallisEntropyNumerical[CoupledNormalDistribution[0.4, 0, 1], 0.4, 2, 0]
```

```
Null
```

```
TsallisEntropyNumerical[CoupledNormalDistribution[0.6, 0, 1], 0.6, 2, 0]
```

```
Null

Null

Null
```

```
TsallisEntropyNumerical[CoupledNormalDistribution[0.8, 0, 1], 0.8, 2, 0]
Null
Null
Null
```

```
TsallisEntropyNumerical[CoupledNormalDistribution[1, 0, 1], 1, 2, 0]
Null
Null
Null
```

Normalized Tsallis Entropy

```
NormalizedTsallisEntropy[p_, κ_, α_ : 1, d_ : 1, lowerlim_, upperlim_] :=
```

$$-\int_{\text{lowerlim}}^{\text{upperlim}} \text{FullSimplify}\left[\text{CoupledProbability}\left[p, \frac{-\alpha\,\kappa}{1+\kappa}, x\right] \times \text{CoupledLogarithm}[\text{PDF}[p, x], \kappa, \alpha, d]\,(1 + d\,\kappa), \kappa > 0\right] dx \; // \; \text{FullSimplify}$$

```
Assuming[κ > 0, NormalizedTsallisEntropy[CoupledNormalDistribution[κ, 0, σ], κ, 2, 1]]
Null
```

$$\text{NTECoupledGaussian}[\kappa\_, \sigma\_] := \frac{(1+\kappa)\left(-\kappa + \pi^{\frac{\kappa}{1+\kappa}}\,\kappa^{\frac{1}{1+\kappa}}\,(1+\kappa)\left(\frac{\sigma\,\text{Gamma}\left[\frac{1}{2\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{\frac{2\kappa}{1+\kappa}}\right)}{2\,\kappa^2}$$

```
Assuming[κ > 0,
 NormalizedTsallisEntropy[CoupledExponentialDistribution[κ, 0, σ], κ, 1, 1, 0, ∞]]
```

$$\frac{(1+\kappa)\left(-1 + (1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}\right)}{\kappa}$$

Shannon Entropy

Since a closed form could not be achieved for the Shannon Entropy of the Coupled Gaussian a numerical integration is completed. *NShannonEntropy* function was modified to specify lower and upper limits; the use of the *NShannonEntropy* with the Coupled Gaussian distribution needs to be updated.

```
Clear[NShannonEntropy, HTCoupledNormal, NSECoupledGaussian]
```

```
NShannonEntropy[p_, lowerlim_, upperlim_] :=
 -NIntegrate[PDF[p, x] Log[PDF[p, x]], {x, lowerlim, upperlim}]
```

Due to numerical issues, the calculation of Shannon Entropy is formed as a piecewise function with increasing limit on the range of the integration;
See plots below

```
HTCoupledNormal[κ_, 0, σ_] := Simplify[CoupledNormalDistribution[κ, 0, σ], κ > 0]
```

```
NSECoupledGaussian[κ_, σ_] :=
 Assuming[∞ > κ > 0,
  Piecewise[{
    {NShannonEntropy[HTCoupledNormal[κ, 0, σ], -100, 100], 0 < κ < 0.09},
    {NShannonEntropy[HTCoupledNormal[κ, 0, σ], -1000, 1000], 0.09 ≤ κ < 0.74},
    {NShannonEntropy[HTCoupledNormal[κ, 0, σ], -10 000, 10 000], 0.74 ≤ κ < 1.5},
    {NShannonEntropy[HTCoupledNormal[κ, 0, σ], -15 000, 15 000], 1.5 ≤ κ}
   }]
 ]

NSECoupledGaussian[#, 0.5] & /@ {0.1, 1, 2}

{0.828115, 1.83717, 2.901}
```

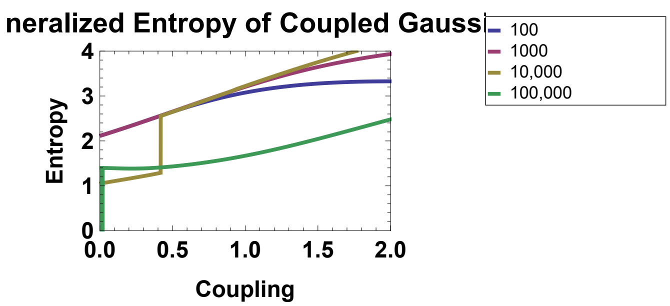
```
Plot[{
  NShannonEntropy[CoupledNormalDistribution[κ, 0, 2], 100],
  NShannonEntropy[CoupledNormalDistribution[κ, 0, 2], 1000],
  NShannonEntropy[CoupledNormalDistribution[κ, 0, 2], 10000],
  NShannonEntropy[CoupledNormalDistribution[κ, 0, 2], 100000]
 },
 {κ, 0.01, 2},
 PlotRange → {{0, 2}, {0, 4}},
 Frame → True,
 FrameLabel →
  {"Coupling", "Entropy"},
 PlotLabel → "Generalized Entropy of Coupled Gaussian",
 LabelStyle → Directive[12, Bold],
 PlotStyle → Thick,
 PlotLegend → {"100",
   "1000", "10,000", "100,000"},
 LegendShadow → None,
 LegendPosition → {1, 0.2},
 LegendSize → 0.75,
 LegendSpacing → 0.01,
 LegendTextSpace → 7.5]
```

NIntegrate ::slwcon : Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small. ≫

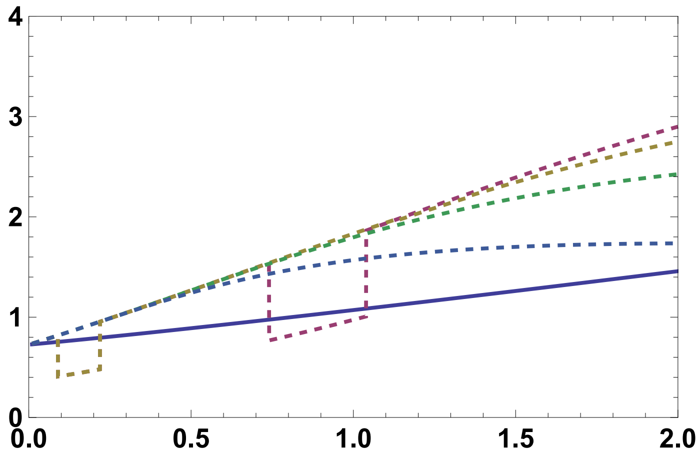
NIntegrate ::ncvb : NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in x near {x} = {−0.310833 }. NIntegrate obtained −1.05531 and 0.17599785130133294` for the integral and error estimates. ≫

NIntegrate ::slwcon : Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small. ≫

NIntegrate ::ncvb : NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in x near {x} = {387.517 }. NIntegrate obtained −1.39807 and 1.398071533952922` for the integral and error estimates. ≫



```
Plot[{
  Style[CECoupledGaussian[κ, 0.5], Green],
  Style[Piecewise[{
     {NShannonEntropy[CoupledNormalDistribution[κ, 0, 0.5], 100], 0 < κ < 0.09},
     {NShannonEntropy[CoupledNormalDistribution[κ, 0, 0.5], 1000], 0.09 ≤ κ < 0.74},
     {NShannonEntropy[CoupledNormalDistribution[κ, 0, 0.5], 10000], 0.74 ≤ κ}
    }], Dashed, Green],
  Style[Piecewise[{
     {NShannonEntropy[CoupledNormalDistribution[κ, 0, 0.5], 100], 0 < κ < 0.09},
     {NShannonEntropy[CoupledNormalDistribution[κ, 0, 0.5], 1000], 0.09 ≤ κ}
    }], Dashed, Green],
  Style[Piecewise[{
     {NShannonEntropy[CoupledNormalDistribution[κ, 0, 0.5], 100], 0 < κ}
    }], Dashed, Green],
  Style[NShannonEntropy[CoupledNormalDistribution[κ, 0, 0.5], 10], Green, Dashed ]
 },
 {κ, 0.01, 2},
 PlotRange → {{0, 2}, {0, 4}},
 Frame → True,
 (*FrameLabel→
  {"Coupling", "Entropy"},
 PlotLabel→"Generalized Entropy of Coupled Gaussian",*)
 LabelStyle → Directive[14, Bold],
 PlotStyle → Thick
 (*PlotLegend→{"Coupled Entropy",(*"Shannon 15,000",*)
   "Shannon 10,000", "Shannon 1,000", "Shannon 100", "Shannon 10"},
 LegendShadow→None,
 LegendPosition→{1,0.2},
 LegendSize → 0.75,
 LegendSpacing→0.01,
 LegendTextSpace→7.5*)
]
```



Below are attempts to complete functional evaluation of the Shannon Entropy for the Coupled Gaussian distribution

```
ShannonEntropy[p_] := - ∫_{-∞}^{∞} PDF[p, x] Log[PDF[p, x]] ⅆx
```

```
ShannonEntropy[CoupledNormalDistribution[κ, 0, σ]]
```

$$
\begin{cases}
\text{Indeterminate} \\
\frac{1}{2}\left(-1-\text{Log}\left[2\,\pi\,\sigma^2\right]\right) \\
-\text{Integrate}\Bigg[\text{Log}\Bigg[\begin{cases}
\frac{1}{\sqrt{2\pi}\,\sigma}\begin{cases}\begin{cases}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} & 1+\frac{x^2\kappa}{\sigma^2}>0\\ 0 & \text{True}\end{cases} & \kappa\neq 0\\ e^{-\frac{x^2}{2\sigma^2}} & \text{True}\end{cases} \\
\left(\sqrt{-\kappa}\,\text{Gamma}\left[1-\frac{1}{2\kappa}\right]\left(\begin{cases}\begin{cases}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} & 1+\frac{x^2\kappa}{\sigma^2}>0\\ 0 & \text{True}\end{cases} & \kappa\neq 0\\ e^{-\frac{x^2}{2\sigma^2}} & \text{True}\end{cases}\right)\right)\Big/\left(\sqrt{\pi}\;\sigma\,\text{Gamm}\ldots\right. \\
\left(\sqrt{\kappa}\,\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]\left(\begin{cases}\begin{cases}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} & 1+\frac{x^2\kappa}{\sigma^2}>0\\ 0 & \text{True}\end{cases} & \kappa\neq 0\\ e^{-\frac{x^2}{2\sigma^2}} & \text{True}\end{cases}\right)\right)\Big/\left(\sqrt{\pi}\;\sigma\,\text{Gamma}\left[\frac{1}{2\ldots}\right.\right.
\end{cases} \\
\qquad\left(\begin{cases}
\frac{1}{\sqrt{2\pi}\,\sigma}\begin{cases}\begin{cases}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} & 1+\frac{x^2\kappa}{\sigma^2}>0\\ 0 & \text{True}\end{cases} & \kappa\neq 0\\ e^{-\frac{x^2}{2\sigma^2}} & \text{True}\end{cases} & \kappa =\ldots \\
\left(\sqrt{-\kappa}\,\text{Gamma}\left[1-\frac{1}{2\kappa}\right]\left(\begin{cases}\begin{cases}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} & 1+\frac{x^2\kappa}{\sigma^2}>0\\ 0 & \text{True}\end{cases} & \kappa\neq 0\\ e^{-\frac{x^2}{2\sigma^2}} & \text{True}\end{cases}\right)\right)\Big/\left(\sqrt{\pi}\;\sigma\,\text{Gamma}\left[\frac{-1+\kappa}{2\kappa}\right]\right) & \kappa <\ldots \\
\left(\sqrt{\kappa}\,\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]\left(\begin{cases}\begin{cases}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} & 1+\frac{x^2\kappa}{\sigma^2}>0\\ 0 & \text{True}\end{cases} & \kappa\neq 0\\ e^{-\frac{x^2}{2\sigma^2}} & \text{True}\end{cases}\right)\right)\Big/\left(\sqrt{\pi}\;\sigma\,\text{Gamma}\left[\frac{1}{2\kappa}\right]\right) & \text{Tr}\ldots
\end{cases}\right) \\
\qquad\text{Assumptions}\to d\in\text{Integers}\,\&\&\,x\in\text{Reals}\,\&\&\,\alpha\in\text{Reals}\,\&\&\,\kappa\in\text{Reals}\,\&\&\,\mu\in\text{Reals}\,\&\&\,\sigma\in\text{Reals}\ldots
\end{cases}
$$

```
ShannonEntropy[Simplify[CoupledNormalDistribution[κ, 0, σ], κ > 0]]
```

$$
-\int_{-\infty}^{\infty}\text{Log}\Bigg[\text{PDF}\left[\begin{cases}\text{ProbabilityDistribution}[0,\{x,-\infty,\infty\}] & x^2\kappa+\sigma^2\le 0\\ \text{ProbabilityDistribution}\left[\frac{\sqrt{\kappa}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}}\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\,\text{Gamma}\left[\frac{1}{2\kappa}\right]},\{x,-\infty,\infty\}\right] & \text{True}\end{cases},\;x\right]\Bigg]
$$

$$
\text{PDF}\left[\begin{cases}\text{ProbabilityDistribution}[0,\{x,-\infty,\infty\}] & x^2\kappa+\sigma^2\le 0\\ \text{ProbabilityDistribution}\left[\frac{\sqrt{\kappa}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}}\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\,\text{Gamma}\left[\frac{1}{2\kappa}\right]},\{x,-\infty,\infty\}\right] & \text{True}\end{cases},\;x\right]dx
$$

```
Simplify[ShannonEntropy[
  Simplify[CoupledNormalDistribution[κ, 0, σ], κ > 0 && σ > 0]], κ > 0 && σ > 0]
```

$$-\int_{-\infty}^{\infty} \mathrm{Log}\Big[\mathrm{PDF}\Big[\begin{cases} \mathrm{ProbabilityDistribution}[0, \{x, -\infty, \infty\}] & x^2 \kappa + \sigma^2 \le 0 \\ \mathrm{ProbabilityDistribution}\Big[\frac{\sqrt{\kappa}\left(1+\frac{x^2 \kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} \mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\,\mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}, \{x, -\infty, \infty\}\Big] & \mathrm{True} \end{cases}, x\Big]\Big]$$
$$\mathrm{PDF}\Big[\begin{cases} \mathrm{ProbabilityDistribution}[0, \{x, -\infty, \infty\}] & x^2 \kappa + \sigma^2 \le 0 \\ \mathrm{ProbabilityDistribution}\Big[\frac{\sqrt{\kappa}\left(1+\frac{x^2 \kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} \mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\,\mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}, \{x, -\infty, \infty\}\Big] & \mathrm{True} \end{cases}, x\Big]\, dx$$

$$\mathrm{Assuming}\Big[0.01 \le \kappa \le 0.49 \;\&\&\; \sigma > 0,$$
$$-\int_{-\infty}^{\infty} \mathrm{Log}\Big[\mathrm{PDF}\Big[\mathrm{ProbabilityDistribution}\Big[\frac{\sqrt{\kappa}\left(1+\frac{x^2 \kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} \mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\,\mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}, \{x, -\infty, \infty\}\Big], x\Big]\Big]$$
$$\mathrm{PDF}\Big[\mathrm{ProbabilityDistribution}\Big[\frac{\sqrt{\kappa}\left(1+\frac{x^2 \kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} \mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\,\mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}, \{x, -\infty, \infty\}\Big], x\Big]$$
$$dx \;//\; \mathrm{Simplify}\Big]$$

$$-\int_{-\infty}^{\infty} \frac{\sqrt{\kappa}\left(1+\frac{x^2 \kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} \mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right] \mathrm{Log}\Big[\frac{\sqrt{\kappa}\left(1+\frac{x^2 \kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} \mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\,\mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}\Big]}{\sqrt{\pi}\,\sigma\,\mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}\, dx$$

```
$Assumptions
```

$\kappa \in \mathrm{Reals} \;\&\&\; x \in \mathrm{Reals} \;\&\&\; \mu \in \mathrm{Reals} \;\&\&\; \sigma \in \mathrm{Reals} \;\&\&\; \sigma > 0 \;\&\&\; \alpha \in \mathrm{Reals} \;\&\&\; \alpha > 0 \;\&\&\; d \in \mathrm{Integers} \;\&\&\; d > 0$

Renyi Entropy

$$\mathrm{NRenyiEntropy}[p\_, K\_, \mathrm{lowlim}\_, \mathrm{upperlim}\_] := \frac{1}{K} \mathrm{Log}\Big[\mathrm{NIntegrate}\Big[\left(\mathrm{PDF}[p, x]^{1-K}\right), \{x, \mathrm{lowlim}, \mathrm{upperlim}\}\Big]\Big]$$

$$\mathrm{NRenyiEntropy15}[p\_, K\_] := \frac{1}{K} \mathrm{Log}\Big[\mathrm{NIntegrate}\Big[\left(\mathrm{PDF}[p, x]^{1-K}\right), \{x, -2000, 2000\}\Big]\Big]$$

```
NRECoupledGaussian[κ_, σ_] :=
 NRenyiEntropy[CoupledNormalDistribution[κ, 0, σ], -2 κ / (1 + κ), -1000, 1000]
```

```
NRECoupledGaussian[#, 2] & /@ {0.1, 1, 2}
```

{2.16125, 2.53102, 2.82764}

```
NRenyiEntropy[CoupledNormalDistribution[#, 0, 1], -2 # / (1 + #), -1000, 1000] & /@
 {0.1, 1, 2}
```

{1.4681, 1.83788, 2.13449}

```
NRenyiEntropy15[CoupledNormalDistribution[#, 0, 1], -2 # / (1 + #)] & /@ {0.1, 1, 2}
```

{1.4681, 1.83788, 2.13449}

RenyiEntropy[p_, K_] := $\frac{1}{K}$ Log$\left[\int_{-\infty}^{\infty}\left(\text{PDF}[p, x]^{1-K}\right) dx\right]$

```
Simplify[RenyiEntropy[
  Simplify[CoupledNormalDistribution[κ, 0, σ], κ > 0], -2 κ / (1 + κ)], κ > 0]
```

$$\frac{1}{2\kappa}(1+\kappa)\,\text{Log}\left[\int_{-\infty}^{\infty}\text{PDF}\left[\begin{cases}\text{ProbabilityDistribution}[0, \{x, -\infty, \infty\}] & x^2\kappa+\sigma^2\le 0\\ \text{ProbabilityDistribution}\left[\frac{\sqrt{\kappa}\left(1+\frac{x^2\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}}\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\,\text{Gamma}\left[\frac{1}{2\kappa}\right]}, \{x, -\infty, \infty\}\right] & \text{True}\end{cases}, x\right]^{1+\frac{2\kappa}{1+\kappa}} dx\right]$$

Coupled Entropy

See the file Coupled Exponentials for the definition of the Coupled Entropy function

CoupledEntropy[p_, κ_, α_ : 1, d_ : 1] :=
 $-\int_{-\infty}^{\infty}$ CoupledProbability$\left[p, \frac{-\alpha\kappa}{1+\kappa}, x\right]$ × CoupledLogarithm[PDF[p, x], κ, α, d] dx // FullSimplify

```
Assuming[κ > 0,
  CoupledEntropy[CoupledNormalDistribution[κ, 0, σ], κ, 2, 1]] // Simplify
```

$$\frac{-\kappa+\pi^{\frac{\kappa}{1+\kappa}}\kappa^{\frac{1}{1+\kappa}}(1+\kappa)\left(\frac{\sigma\,\text{Gamma}\left[\frac{1}{2\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{\frac{2\kappa}{1+\kappa}}}{2\kappa^2}$$

```
Clear[CECoupledGaussian]
```

CECoupledGaussian[κ_] := $\frac{-\kappa+\pi^{\frac{\kappa}{1+\kappa}}\kappa^{\frac{1}{1+\kappa}}(1+\kappa)\left(\frac{\text{Gamma}\left[\frac{1}{2\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{\frac{2\kappa}{1+\kappa}}}{2\kappa^2}$

CECoupledGaussian[κ_, σ_] :=
$$\frac{-\kappa + \pi^{\frac{\kappa}{1+\kappa}}\, \kappa^{\frac{1}{1+\kappa}}\, (1+\kappa) \left(\frac{\sigma\, \mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}{\mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{\frac{2\kappa}{1+\kappa}}}{2\,\kappa^2}$$

CECoupledGaussian[#, 2] & /@ {0.25, 1, 1.75}

$$\left\{2.88359,\ \frac{1}{2}\,(-1+4\,\pi),\ 9.31623\right\}$$

Conjugate Coupled Entropy

First attempt encumbered by inability to resolve if statements; will need to complete computation in pieces to check these statements

First Attempt

Null

Null

Null

Null

Null

Second Attempt - Use average uncertainty

AvgUncertCG = FullSimplify[
$$\left(\int_{-\infty}^{\infty} \mathrm{HTCoupledGaussian}^{1-m}\, dx\right)^{\frac{-1}{m}},$$
0 < κ < ∞]

$$\frac{2^{-\frac{1+\kappa}{2\kappa}}\, \kappa^{-\frac{1}{2}/\kappa}\, \mathrm{Gamma}\left[\frac{1}{2}\left(3+\frac{1}{\kappa}\right)\right]^{-\frac{1+\kappa}{2\kappa}}\, \mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]^{\frac{1}{2}\left(3+\frac{1}{\kappa}\right)}}{\sqrt{\pi}\,\sigma\, \mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}$$

This solution still doesn't seem correct; see below for density at scale

```
HTCoupledGaussian = FullSimplify[
  PDF[CoupledNormalDistribution[κ, 0, σ], x],
  0 < κ < ∞
 ]
```

$$\frac{\sqrt{\kappa}\left(1+\frac{x^2\,\kappa}{\sigma^2}\right)^{-\frac{1+\kappa}{2\kappa}} \mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\,\sigma\, \mathrm{Gamma}\left[\frac{1}{2\kappa}\right]}$$

m = -2 κ / (1 + κ);

```
AvgUncertCG2 = HTCoupledGaussian /. x → σ
```

$$\frac{\sqrt{\kappa}\ (1+\kappa)^{-\frac{1+\kappa}{2\kappa}}\ \text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}{\sqrt{\pi}\ \sigma\ \text{Gamma}\left[\frac{1}{2\kappa}\right]}$$

Check whether the Coupled Entropy is correct; yes this matches

```
FullSimplify[-CoupledLogarithm[AvgUncertCG2, κ, 2, 1],
 0 < κ < ∞
]
```

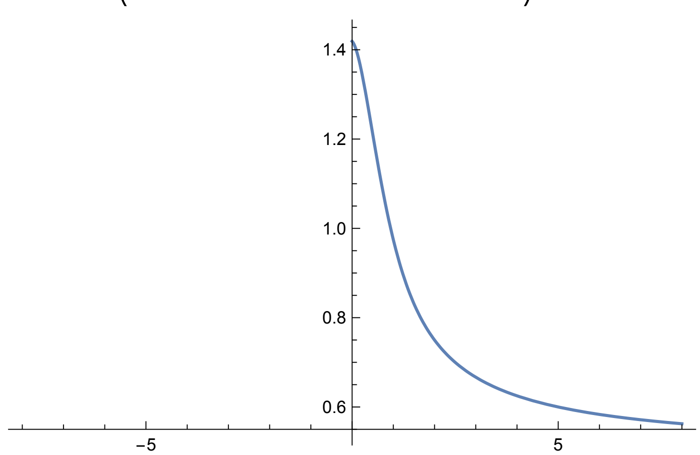
$$\frac{-\kappa+\pi^{\frac{\kappa}{1+\kappa}}\ \kappa^{\frac{1}{1+\kappa}}\ (1+\kappa)\left(\frac{\sigma\ \text{Gamma}\left[\frac{1}{2\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{\frac{2\kappa}{1+\kappa}}}{2\ \kappa^2}$$

```
CoupledConjugateEntCpldGauss =
 FullSimplify[-CoupledLogarithm[AvgUncertCG2, -κ / (1 + κ), 2, 1],
  0 < κ < ∞
 ]
```

$$\frac{1}{2}\left(1+\frac{1-\left(\pi+\frac{\pi}{\kappa}\right)^{-\kappa}\left(\frac{\sigma\ \text{Gamma}\left[\frac{1}{2\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{-2\kappa}}{\kappa}\right)$$

$$\text{ConjCECoupledGaussian}[\kappa\_,\ \sigma\_] := \frac{1}{2}\left(1+\frac{1-\left(\pi+\frac{\pi}{\kappa}\right)^{-\kappa}\left(\frac{\sigma\ \text{Gamma}\left[\frac{1}{2\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{-2\kappa}}{\kappa}\right)$$

$$\text{Plot}\left[\frac{1}{2}\left(1+\frac{1-\left(\pi+\frac{\pi}{\kappa}\right)^{-\kappa}\left(\frac{\text{Gamma}\left[\frac{1}{2\kappa}\right]}{\text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]}\right)^{-2\kappa}}{\kappa}\right),\ \{\kappa,\ -8,\ 8\}\right]$$



Plot Entropy Comparison

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Compare Entropy for Pareto Distribution

Compute Coupled, Shannon, Renyi, Tsallis & Normalized Tsallis Entropy

```
Clear[CECoupledExp]
```

```
Assuming[κ > 0 && x > 0,
 CoupledExponentialDistribution[κ, 0, σ]
]
```

$$\text{ProbabilityDistribution}\left[\begin{cases} \dfrac{\text{If}\left[\kappa\neq 0,\text{If}\left[\text{Simplify}\left[1-\frac{\kappa\,(-x)}{\sigma}\right]>0,\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\,\kappa}{1\,\kappa}},0\right],\text{Exp}\left[-\frac{x}{\sigma}\right]\right]}{\sigma} & x\geq 0 \\ 0 & \text{True}\end{cases},\ \{x,-\infty,\infty\}\right]$$

```
$Assumptions = κ > 0 && κ ∈ Reals && x ∈ Reals &&
   μ ∈ Reals && σ ∈ Reals && σ > 0 && α ∈ Reals && α > 0 && d ∈ Integers && d > 0
```

```
κ > 0 && κ ∈ Reals && x ∈ Reals && μ ∈ Reals &&
 σ ∈ Reals && σ > 0 && α ∈ Reals && α > 0 && d ∈ Integers && d > 0
```

This computes the Coupled Entropy of the Pareto Distribution. See below for computation of the coupled probability
and the coupled logarithm of the distribution

$$-\int_0^\infty \left((1+\kappa)\,\sigma^{1+\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-2-\frac{1}{\kappa}}\right)\left(\frac{1-\left(\sigma^{\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-\frac{1+\kappa}{\kappa}}\right)^{-1+\frac{1}{1+\kappa}}}{\kappa}\right)dx\ \text{// FullSimplify}$$

$$\frac{-1+(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}}{\kappa}$$

$$(1+\kappa)\,\text{CoupledLogarithm}\left[\sigma^{-1},\kappa,1,1\right]-\frac{1+\kappa}{\kappa}\ \text{// FullSimplify}$$

$$-\frac{(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}}{\kappa}$$

```
SECoupledExp[κ_, σ_] := 1 + κ + Log[σ];
```

In[ ]:= $$\text{CECoupledExp}[\kappa\_,\sigma\_] := \frac{-1+(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}}{\kappa};$$

$\frac{-1+(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}}{\kappa} = \frac{1+\kappa}{\kappa}\left(\sigma^{\frac{\kappa}{1+\kappa}}-1\right)+\frac{1+\kappa}{\kappa}-\frac{1}{\kappa} = 1+\ln_{\frac{1+\kappa}{\kappa}}\sigma$ Confirms 2025 Solution

In[ ]:= $$\text{TECoupledExp}[\kappa\_,\sigma\_] := \frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$$

In[ ]:= $$\frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}\ \text{// FullSimplify}$$

Out[ ]=

$$\frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$$

$\frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa} = 1 - \frac{1}{\kappa}\left(\sigma^{-\frac{\kappa}{1+\kappa}}-1\right)$ Confirms 2025 Result

In[∘]:= **NTECoupledExp[κ_, σ_] :=** $$\frac{(1+\kappa)\left(-1+(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}\right)}{\kappa}$$

CoupledProbability[

$$\texttt{ProbabilityDistribution}\left[\frac{1}{\sigma}\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\kappa}{1\kappa}},\ \{x,0,\infty\}\right],$$

$$\frac{-\kappa}{1+\kappa},\ x$$

]

$$\begin{cases} 0 & x \le 0 \\ (1+\kappa)\,\sigma^{1+\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-2-\frac{1}{\kappa}} & \text{True} \end{cases}$$

CoupledLogarithm[

$$\frac{1}{\sigma}\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\kappa}{1\kappa}},$$

κ, 1, 1] // FullSimplify

$$\texttt{ConditionalExpression}\left[\frac{1-\left(\sigma^{\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-\frac{1+\kappa}{\kappa}}\right)^{-1+\frac{1}{1+\kappa}}}{\kappa},\ (x\,\kappa+\sigma)^{-1-\frac{1}{\kappa}} \ge 0\right]$$

```
Assuming[κ > 0, CoupledEntropy[
   Assuming[κ > 0 && x > 0,
    CoupledExponentialDistribution[κ, 0, σ]
   ], κ, 1, 1]] // Simplify
```

$$-\int_0^\infty \texttt{If}\Bigg[\left(\begin{cases} \sigma^{\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-1-\frac{1}{\kappa}} & x \ge 0 \\ 0 & \text{True} \end{cases}\right) \ge 0,$$

$$\texttt{If}\left[\kappa \ne 0,\ -\frac{1}{1\,\kappa}\left(\left(\begin{cases} \frac{\texttt{If}\left[\kappa\neq 0,\texttt{If}\left[\texttt{Simplify}\left[1-\frac{\kappa\,(-x)}{\sigma}\right]>0,\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\kappa}{1\kappa}},0\right],\texttt{Exp}\left[-\frac{x}{\sigma}\right]\right]}{\sigma} & x \ge 0 \\ 0 & \text{True} \end{cases}\right)^{-\frac{\kappa}{1+1\kappa}}-1\right),\right.$$

$$\texttt{Log}\left[\begin{cases} \frac{\texttt{If}\left[\kappa\neq 0,\texttt{If}\left[\texttt{Simplify}\left[1-\frac{\kappa\,(-x)}{\sigma}\right]>0,\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\kappa}{1\kappa}},0\right],\texttt{Exp}\left[-\frac{x}{\sigma}\right]\right]}{\sigma} & x \ge 0 \\ 0 & \text{True} \end{cases}\right]\Bigg],$$

$$\texttt{Undefined}\Bigg]\left(\begin{cases} 0 & x < 0 \\ (1+\kappa)\,\sigma^{1+\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-2-\frac{1}{\kappa}} & \text{True} \end{cases}\right) dx$$

```
NSECoupledExp[κ_, σ_] :=
 Assuming[∞ > κ > 0,
  Piecewise[{
    {NShannonEntropy[CoupledExponentialDistribution[κ, 0, σ], 0, 100], 0 < κ < 0.09},
    {NShannonEntropy[
      CoupledExponentialDistribution[κ, 0, σ], 0, 1000], 0.09 ≤ κ < 0.74},
    {NShannonEntropy[
      CoupledExponentialDistribution[κ, 0, σ], 0, 10000], 0.74 ≤ κ < 1.5},
    {NShannonEntropy[CoupledExponentialDistribution[κ, 0, σ], 0, 15000], 1.5 ≤ κ}
   }]
 ]
```

```
NSECoupledExp[#, 1] & /@ {0.25, 0.5, 0.75, 1, 1.25}
```

{1.25, 1.49992, 1.74985, 1.99796, 2.23985}

```
NSECoupledExp[0.01, 0.25]
```

-0.376294

```
NSECoupledExp[2, 2]
```

3.54523

Compute the 2-q Tsallis entropy of the coupled exponential distribution

In[295]:=

```
(-PDF[CoupledExponentialDistribution[σ, κ], x] ×
   CoupledLogarithm[PDF[CoupledExponentialDistribution[σ, κ], x],
   -κ/(1 + κ), 0]) // FullSimplify
```

Out[295]=

$$\begin{cases} \frac{(1+\kappa)\,\sigma^{\frac{1}{\kappa+\kappa^2}}\,(x\,\kappa+\sigma)^{-\frac{1+\kappa}{\kappa}}\left(x\,\kappa+\sigma-\sigma^{\frac{1}{1+\kappa}}\right)}{\kappa} & x>0 \\ 0 & \text{True} \end{cases}$$

In[296]:=

$$\int_0^\infty \frac{(1+\kappa)\,\sigma^{\frac{1}{\kappa+\kappa^2}}\,(x\,\kappa+\sigma)^{-\frac{1+\kappa}{\kappa}}\left(x\,\kappa+\sigma-\sigma^{\frac{1}{1+\kappa}}\right)}{\kappa}\,dx$$

Out[296]=

$$\frac{(1+\kappa)\,\sigma^{-\frac{1}{\kappa}+\frac{1}{\kappa+\kappa^2}}\left(-\sigma-(-1+\kappa)\,\sigma^{\frac{1}{1+\kappa}}\right)}{(-1+\kappa)\,\kappa} \quad \text{if } \kappa<1$$

Compute the q Tsallis entropy of the coupled exponential distribution

In[297]:=

```
(-PDF[CoupledExponentialDistribution[σ, κ], x] ×
   CoupledLogarithm[PDF[CoupledExponentialDistribution[σ, κ], x],
    κ/(1 + κ), 0]) // FullSimplify
```

Out[297]=

$$\begin{cases} \frac{(1+\kappa)\,\sigma^{\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-2-\frac{1}{\kappa}}\left(x\,\kappa+\sigma-\sigma^{\frac{1}{1+\kappa}}\right)}{\kappa} & x>0 \\ 0 & \text{True} \end{cases}$$

In[298]:=

$$\int_0^\infty \frac{(1+\kappa)\,\sigma^{\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-2-\frac{1}{\kappa}}\left(x\,\kappa+\sigma-\sigma^{\frac{1}{1+\kappa}}\right)}{\kappa}\,dx$$

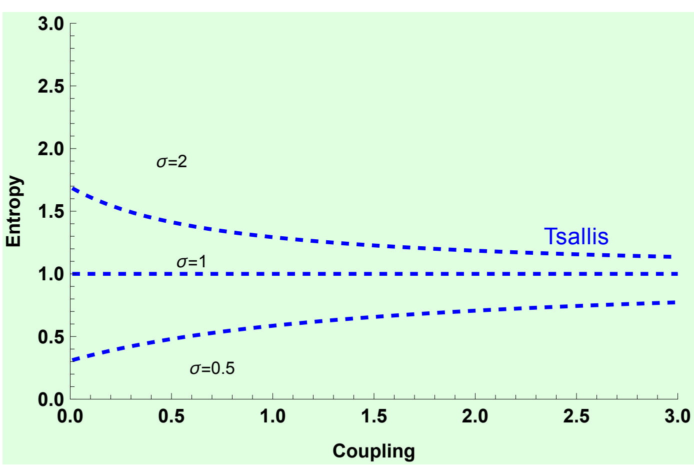
Out[298]=

$$\frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$$

Plot Comparison of GPD with TE, NTE, & Shannon

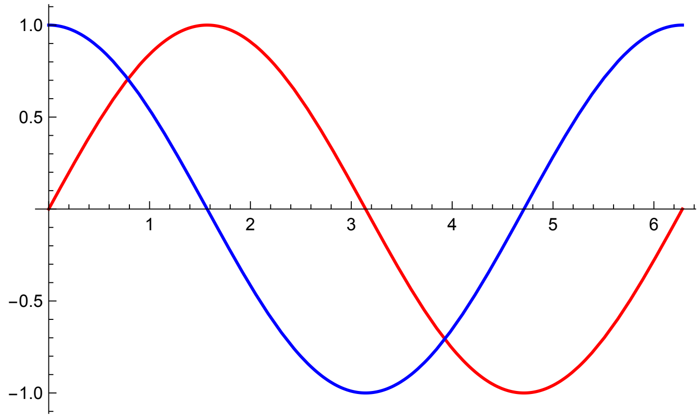
```
Assuming[0 < κ < ∞,
 Plot[
  {
   (*Table[SECoupledExp[κ,σ],{σ,{0.5,1,2}}],*)
   {Style[Table[TECoupledExp[κ, σ], {σ, {0.5, 1, 2}}],
     {Blue, Dashed}]},
   {Style[Table[NTECoupledExp[κ, σ], {σ, {0.5, 1, 2}}],
     {Red, Thick}]},
   {Style[Table[CECoupledExp[κ, σ], {σ, {0.5, 1, 2}}],
     Black]}
  },
  {κ, 0.01, 3},
  Background → LightGreen,
  PlotRange → {{0, 3}, {0, 3}} ,
  Frame → {{True, False}, {True, False}},
  FrameLabel → {"Coupling", "Entropy"},
  FrameStyle → Directive[10, Bold, Black],
  Epilog → {
    Inset[Style["Coupled", Black, Medium], {2, 2}],
    Inset[Style["Tsallis", Blue, Medium], {2.5, 1.3}],
    Inset[Style["Normalized", Red, Medium], {1.2, 2.5}],
    Inset[Style["σ=2", Black], {0.5, 1.9}],
    Inset[Style["σ=1", Black], {0.6, 1.1}],
    Inset[Style["σ=0.5", Black], {0.7, 0.25}]
   }
 ]
]
```

Out[ ]=



In[ ]:=
```
Plot[{Sin[x], Cos[x]}, {x, 0, 2 Pi},
 PlotStyle → {{PointSize[Large], Red, Point[{π / 2, Sin[π / 2]}]},
   {PointSize[Medium], Blue, Point[{3 π / 2, Sin[3 π / 2]}]},
   {PointSize[Small], Green, Point[{π / 4, Cos[π / 4]}]}}]
```

Out[ ]=



Plot Comparison of Generalized Pareto with Shannon Entropy

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Score Function Plots

The score function of the GPD is $-\sigma^{-1}$, which is a powerful description of the scales unique properties. The score function is computed from the derivative of the log of the pdf.

In[∘]:=

$$\text{NegDerGPD}[\sigma\_, \kappa\_, x\_] := \frac{1+\kappa}{x\,\kappa+\sigma};$$

$$\text{NegDerGPDq}[\beta\_, q\_, x\_] := \frac{\beta}{1+(-1+q)\,x\,\beta};$$

In[∘]:=

```
Clear[NegDerGPD]
```

In[∘]:=

$$\text{Assuming}\left[0<\kappa<\infty, -\text{D}\left[\text{Log}\left[\frac{1}{\sigma}\left(1+\frac{\kappa\,x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}\right], x\right]\right] \text{ // FullSimplify}$$

Out[∘]=

$$\frac{1+\kappa}{x\,\kappa+\sigma}$$

In[∘]:=

$$\frac{1+\kappa}{x\,\kappa+\sigma} \text{ /. } \left\{\kappa \to \frac{-(1-q)}{2-q}, \sigma \to \frac{1}{(2-q)\,\beta}\right\} \text{ // FullSimplify}$$

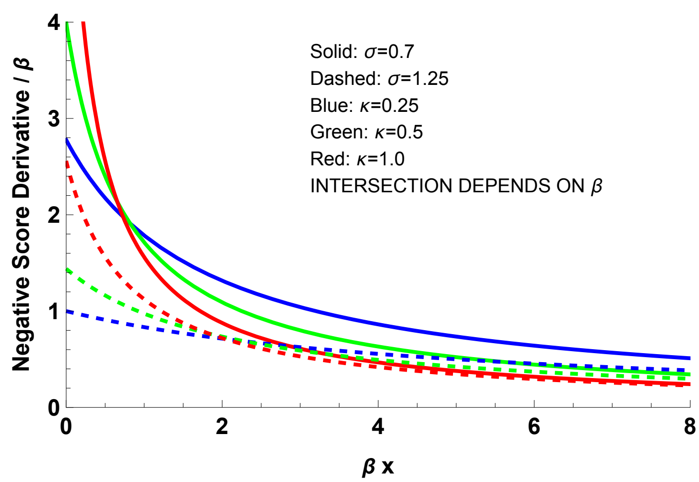
Out[∘]=

$$\frac{\beta}{1+(-1+q)\,x\,\beta}$$

Plot -β Score versus $\frac{x}{\beta}$

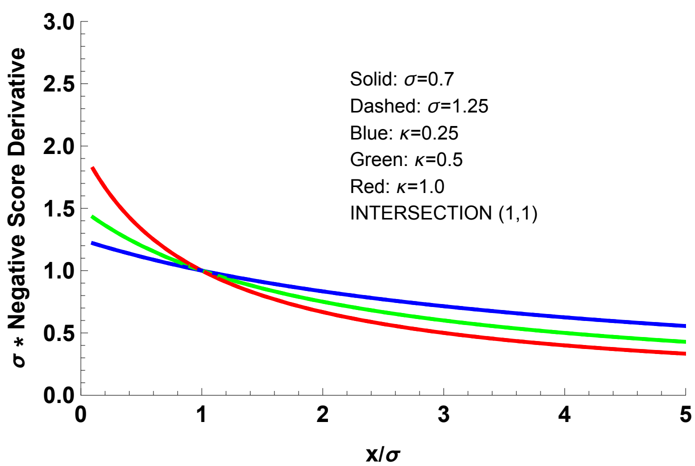
```
Plot[MapThread[scaleShapeToBeta[#1, #2, 1] ×
      NegDerGPD[#1, #2, scaleShapeToBeta[#1, #2, 1] x] &,
    {{0.75, 0.75, 0.75, 1.25, 1.25, 1.25}, {0.25, 0.5, 1, 0.25, 0.5, 1}}] //
   Evaluate, {x, 0.0001, 10},
 PlotRange → {{0, 8}, {0, 4}},
 PlotStyle → {Blue, Green, Red, {Blue, Dashed}, {Green, Dashed}, {Red, Dashed}},
 Epilog → Inset[Style[Text["Solid: σ=0.7
Dashed: σ=1.25
Blue: κ=0.25
Green: κ=0.5
Red: κ=1.0
INTERSECTION DEPENDS ON β"], Larger], {5, 3}],
 LabelStyle → Directive[Bold, Medium],
 Frame → {{True, False}, {True, False}},
 FrameLabel → {"β x", "Negative Score Derivative / β"}
]
```

Out[ ]=



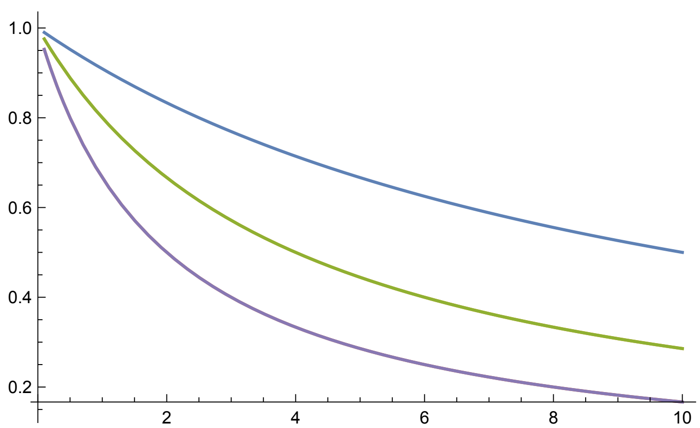
In[◦]:=
```
Plot[MapThread[
   #1 NegDerGPD[#1, #2, #1 x] &, {{0.75, 0.75, 0.75, 1.25, 1.25, 1.25},
    {0.25, 0.5, 1, 0.25, 0.5, 1}}] // Evaluate, {x, 0.1, 10},
 PlotRange → {{0, 5}, {0, 3}},
 PlotStyle → {Blue, Green, Red, {Blue, Dashed}, {Green, Dashed}, {Red, Dashed}},
 Epilog → Inset[Style[Text["Solid: σ=0.7
Dashed: σ=1.25
Blue: κ=0.25
Green: κ=0.5
Red: κ=1.0
INTERSECTION (1,1)"], Larger], {3, 2}],
 LabelStyle → Directive[Bold, Medium],
 Frame → {{True, False}, {True, False}},
 FrameLabel → {"x/σ", "σ * Negative Score Derivative"}
]
```

Out[◦]=



In[◦]:=
```
Plot[MapThread[#1^-1 NegDerGPDq[#1, #2, #1^-1 x] &,
   {{0.5, 0.5, .2, .2, .5}, {1.1, 1.25, 1.25, 1.5, 1.5}}] // Evaluate, {x, 0.1, 10}]
```

Out[◦]=



Generalized Weibull Distribution

In[∘]:= $\texttt{GWDSF}[\sigma\_, \kappa\_, \alpha\_, x\_] := \left(1 + \frac{\kappa\, x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}};$

In[∘]:= $\texttt{GWDPDF}[\sigma\_, \kappa\_, \alpha\_, x\_] := \frac{x^{\alpha-1}}{\sigma^{\alpha}} \left(1 + \frac{\kappa\, x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}-1};$

In[∘]:= `D[-Log[GWDPDF[σ, κ, α, x]], x] \\ FullSimplify`

••• Syntax: "D[−Log[GWDPDF[σ, κ, α, x]], x] \\ FullSimplify" is incomplete; more input is needed.

In[∘]:= $\texttt{D}\left[-\texttt{Log}\left[\frac{x^{\alpha-1}}{\sigma^{\alpha}} \left(1 + \frac{\kappa\, x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}-1}\right], x\right]$ `\\ FullSimplify`

••• Syntax: "$\texttt{D}\left[-\texttt{Log}\left[\frac{x^{\alpha-1}}{\sigma^{\alpha}} \left(1 + \frac{\kappa\, x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}-1}\right], x\right]$ \\ FullSimplify" is incomplete; more input is needed.

In[∘]:= $-\partial_x \texttt{Log}\left[x^{-1+\alpha}\, \sigma^{-\alpha}\, (1 + x^{\alpha}\, \kappa\, \sigma^{-\alpha})^{-1-\frac{1}{\alpha\kappa}}\right]$ `// FullSimplify`

Out[∘]=

$$\frac{x^{\alpha}\,(1+\kappa) - (-1+\alpha)\,\sigma^{\alpha}}{x\,(x^{\alpha}\,\kappa + \sigma^{\alpha})}$$

In[∘]:= $\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}} + (1-\alpha)\, x^{-1}\right) \left(1 + \frac{\kappa\, x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}$ `/. x → σ // FullSimplify`

Out[∘]=

$$\frac{2-\alpha+\kappa}{\sigma+\kappa\,\sigma}$$

So the use of the Generalized Weibull will require care regarding the definition for the scale of the distribution

If $\alpha = 2$, then

$$\frac{1}{\sigma}\,\frac{\kappa}{1+\kappa}$$

So if this is defined as σW let’s see what happens. Could also do this generally for alpha.

In[∘]:= $\frac{1}{\frac{2-\alpha+\kappa}{\sigma+\kappa\,\sigma}}$

Out[∘]=

$$\frac{\sigma+\kappa\,\sigma}{2-\alpha+\kappa}$$

In[∘]:= $\sigma W = \frac{\sigma + \kappa\,\sigma}{2 - \alpha + \kappa};$