Information Relative Risk Aversion

Derivation of IRRA

Relative Risk Aversion (RRA) of the Coupled Logarithm utility function

First replicate Wikipedia page on Relative Risk Aversion

In[∘]:=

$$\texttt{RRA[c\_, q\_] := } -c\,\frac{\texttt{D[qln[c, q], \{c, 2\}]}}{\texttt{D[qln[c, q], c]}};$$

$$\texttt{qln[c\_, q\_] := } \frac{c^{1-q}-1}{1-q};$$

In[∘]:=

```
FullSimplify[RRA[c, q], 1 < q < 3]
```

Out[∘]= q

Review the Information Relative Risk from the perspective of utility to double check aversion/tolerance direction. With straight line as utility function, risk averse has negative second derivative and risk tolerant has positive second derivative. For utility, use ln p rather than - ln p.

In[∘]:=

$$\texttt{IRRA[p\_, }\kappa\texttt{\_, d\_, }\alpha\texttt{\_] :=}$$

$$-p\left(\frac{\texttt{D}\left[-\frac{1}{\alpha}\,\texttt{CoupledLogarithm[}p^{-\alpha}\texttt{, }\kappa\texttt{, d], \{p, 2\}}\right]}{\texttt{D}\left[-\frac{1}{\alpha}\,\texttt{CoupledLogarithm[}p^{-\alpha}\texttt{, }\kappa\texttt{, d], p}\right]}-\frac{\texttt{D[Log[p], \{p, 2\}]}}{\texttt{D[Log[p], p]}}\right);$$

In[∘]:=

```
FullSimplify[{IRRA[p, κ, d, α], IRRA[p, κ, d, α] /. {κ → qToCoupling[q, α, d]}},
 κ > 0 && α > 0 && d > 0 && 0 < p < 1]
```

Out[∘]=

$$\left\{\frac{\alpha\,\kappa}{1+d\,\kappa},\ \begin{cases}-1+q & \frac{1-q}{d-d\,q+\alpha}\neq 0\\ 0 & \text{True}\end{cases}\right\}$$

Solve for Informational Relative Risk Aversion (IRRA) using the coupled surprisal in terms of the loss rather than the utility function. The variable p is inverted to make it a loss function. The sign of the IRAA equation does not change because the sign of both the second and the first derivative changed. The ratio is unchanged.

In[∘]:=

$$\texttt{IRRA[p\_, }\kappa\texttt{\_, d\_, }\alpha\texttt{\_] :=}$$

$$-p\left(\frac{\texttt{D}\left[\frac{1}{\alpha}\,\texttt{CoupledLogarithm[}p^{-\alpha}\texttt{, }\kappa\texttt{, d], \{p, 2\}}\right]}{\texttt{D}\left[\frac{1}{\alpha}\,\texttt{CoupledLogarithm[}p^{-\alpha}\texttt{, }\kappa\texttt{, d], p}\right]}-\frac{\texttt{D[-Log[p], \{p, 2\}]}}{\texttt{D[-Log[p], p]}}\right);$$

In[∘]:=
```
FullSimplify[{IRRA[p, κ, d, α], IRRA[p, κ, d, α] /. {κ → qToCoupling[q, α, d]}},
 κ > 0 && α > 0 && d > 0 && 0 < p < 1]
```

Out[∘]= $\left\{\frac{\alpha\,\kappa}{1+d\,\kappa},\ \begin{cases} -1+q & \frac{1-q}{d-d\,q+\alpha}\neq 0 \\ 0 & \text{True}\end{cases}\right\}$

Thus it is the positive κ values that have positive values of IRAA. The domain is - 0 $< \kappa < \infty$.

Will use the expressions r = $r_a = -r_t$to label relative risk aversion and relative risk tolerance.

Simplification of Terms

In[∘]:= $D\left[-\frac{1}{\alpha}\,\text{CoupledLogarithm}[p^{-\alpha}, \kappa, d], \{p, 2\}\right]$ // FullSimplify

Out[∘]= $\begin{cases} -\frac{1}{p^2} & p^{-\alpha}\geq 0\ \&\&\ \kappa == 0 \\ -\frac{(p^{-\alpha})^{\frac{\kappa}{1+d\,\kappa}}\,(1+(d+\alpha)\,\kappa)}{(p+d\,p\,\kappa)^2} & p^{-\alpha}\geq 0\ \&\&\ \kappa\neq 0 \\ 0 & \text{True}\end{cases}$

In[∘]:= $D\left[-\frac{1}{\alpha}\,\text{CoupledLogarithm}[p^{-\alpha}, \kappa, d], \{p, 1\}\right]$ // FullSimplify

Out[∘]= $\begin{cases} \frac{1}{p} & p^{-\alpha}\geq 0\ \&\&\ \kappa == 0 \\ \frac{(p^{-\alpha})^{\frac{\kappa}{1+d\,\kappa}}}{p+d\,p\,\kappa} & p^{-\alpha}\geq 0\ \&\&\ \kappa\neq 0 \\ 0 & \text{True}\end{cases}$

In[∘]:= $-\frac{1}{\alpha}\,\text{CoupledLogarithm}[p^{-\alpha}, 0, d]$ // FullSimplify

Out[∘]= $-\frac{\text{Log}[p^{-\alpha}]}{\alpha}\ \text{ if }\ p^{-\alpha}\geq 0$

Derivation of Decisive-Accuracy-Robustness Metric

From the Reduced Perplexity book chapter

$$P_r\left(\mathbf{P}^{(r)},\mathbf{p}\right)=\left(\sum_{i=1}^{N}\left(\frac{p_i^{1-r}}{\sum_{j=1}^{N}p_j^{1-r}}\right)p_i^r\right)^{\frac{1}{r}}=\left(\sum_{i=1}^{N}p_i^{1-r}\right)^{\frac{-1}{r}}=P_{-r}(\mathbf{p},\mathbf{p}). \qquad (12.9)$$

Perhaps, I haven't fully absorbed the significance of this equation. The effect of weighting the generalized mean by the coupled probability is to change the sign of the generalized mean. If I apply the generalized mean again, then the sign will reverse back.

However, the distinction with the assessment metric is that the equation is for the cross-entropy. For p - empirical distribution and q - quoted forecast, the equation is

In[ ]:=
$$\left(\sum_{i=1}^{N}\left(\frac{p_i^{1-r}}{\sum_{j=1}^{N} p_j^{1-r}}\right) q_i^r\right)^{\frac{1}{r}} = \left(\frac{1}{\sum_{j=1}^{N} p_j^{1-r}} \sum_{i=1}^{N} p_i \left(\frac{q_i}{p_i}\right)^r\right)^{\frac{1}{r}}$$

Set : Tag Power in $\left(\sum_{i=1}^{N}\frac{p_i^{1-r}\, q_i^r}{\sum_{j=1}^{N} p_j^{1 \ll 1\gg \ll 5\gg \ll 1\gg \ll 1\gg]}}\right)^{\frac{1}{r}}$ is Protected.

Out[ ]=
$$\left(\frac{\sum_{i=1}^{N} p_i \left(\frac{q_i}{p_i}\right)^r}{\sum_{j=1}^{N} p_j^{1-r}}\right)^{\frac{1}{r}}$$

The derivation of the DAR metric should focus on use of the inverse coupled logarithm.
Given $y = \frac{1}{\alpha}$ CoupledLogarithm[$p^{-\alpha}$, κ, d] as the coupled surprisal, the inverse of this to provide a probablity is (CoupledExponential[α y, κ, d])$^{\frac{-1}{\alpha}}$.

In[ ]:=
```
FullSimplify[(CoupledExponential[
    FullSimplify[Sum[α/(α N) CoupledLogarithm[p^-α, κ, d], {i, 1, N}],
     0 < p < 1 && 0 < α < ∞ && 0 < κ < ∞ && 0 < d < ∞ && N ∈ Integers],
    κ, d])^(-1/α), 0 < p < 1 && 0 < α < ∞ && 0 < κ < ∞ && 0 < d < ∞ && N ∈ Integers]
```

Out[ ]= p

In[ ]:=

$$\text{FullSimplify}\left[\left(\text{CoupledExponential}\left[\text{FullSimplify}\left[\sum_{i=1}^{n}\frac{\alpha}{\alpha\, n}\,\text{CoupledLogarithm}[p[i]^{-\alpha},\kappa,d],\ 0<p<1\ \&\&\ 0<\alpha<\infty\ \&\&\ 0<\kappa<\infty\ \&\&\ 0<d<\infty\ \&\&\ N\in\text{Integers}\right],\ \kappa,d\right]\right)^{\frac{-1}{\alpha}},\ 0<p<1\ \&\&\ 0<\alpha<\infty\ \&\&\ 0<\kappa<\infty\ \&\&\ 0<d<\infty\ \&\&\ N\in\text{Integers}\right]$$

Out[ ]=

$$\text{If}\left[1+\kappa\sum_{i=1}^{n}\frac{\text{If}\left[p[i]^{-\alpha}\geq 0,\ \text{If}\left[\kappa\neq 0,\ \frac{(p[i]^{-\alpha})^{\frac{\kappa}{1+d\kappa}}-1}{\kappa},\ \text{Log}[p[i]^{-\alpha}]\right],\ \text{Undefined}\right]}{n}>0,\right.$$

$$\text{If}\left[\kappa\neq 0,\ \left(1+\kappa\sum_{i=1}^{n}\frac{\text{If}\left[p[i]^{-\alpha}\geq 0,\ \text{If}\left[\kappa\neq 0,\ \frac{(p[i]^{-\alpha})^{\frac{\kappa}{1+d\kappa}}-1}{\kappa},\ \text{Log}[p[i]^{-\alpha}]\right],\ \text{Undefined}\right]}{n}\right)^{\frac{1+d\kappa}{\kappa}},\right.$$

$$\left.\text{Exp}\left[\sum_{i=1}^{n}\frac{\text{If}\left[p[i]^{-\alpha}\geq 0,\ \text{If}\left[\kappa\neq 0,\ \frac{(p[i]^{-\alpha})^{\frac{\kappa}{1+d\kappa}}-1}{\kappa},\ \text{Log}[p[i]^{-\alpha}]\right],\ \text{Undefined}\right]}{n}\right]\right],$$

$$\left.\text{If}\left[\frac{1+d\kappa}{\kappa}>0,\ 0,\ \infty\right]\right]^{-1/\alpha}$$

In[ ]:=
$$\text{FullSimplify}\left[\left(1+\kappa\sum_{i=1}^{n}\frac{\frac{(p[i]^{-\alpha})^{\frac{\kappa}{1+d\kappa}}-1}{\kappa}}{n}\right)^{\frac{1+d\kappa}{-\alpha\kappa}}\right]$$

Out[ ]=
$$\left(1+\kappa\sum_{i=1}^{n}\frac{-1+(p[i]^{-\alpha})^{\frac{\kappa}{1+d\kappa}}}{n\,\kappa}\right)^{-\frac{1+d\kappa}{\alpha\kappa}}$$

$$\left(\frac{1}{n}\sum_{i=1}^{n}\left(p[i]^{\frac{-\alpha\kappa}{1+d\kappa}}\right)\right)^{-\frac{1+d\kappa}{\alpha\kappa}}$$

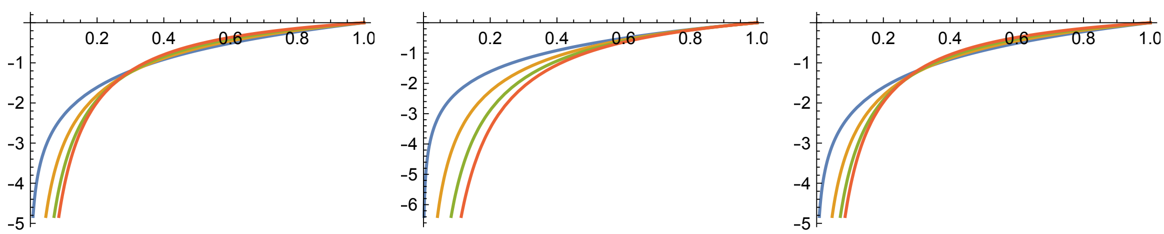
So, indeed the RAD metric uses rt = $\frac{-\alpha\,\kappa}{1+d\,\kappa}$ as the metric; thus it is based on the relative risk tolerance.

Plots of q-log and coupled log

Plot of the coupled logarithm as a utility function shows more negative curvature as κ is more positive.

```
GraphicsGrid[{{Plot[Evaluate@
     (-1/α CoupledLogarithm[p^-α, #, d] & /@ {0, 0.25, .5, 0.75} //. {α → 2, d → 1}),
    {p, 0, 1}],
   (* Not the exact same function *)
   Plot[Evaluate@
     (qln[p, #] & /@ couplingToq[{10^-6, 0.25, .5, 0.75}, α, d] //. {α → 2, d → 1}),
    {p, 0, 1}],
   Plot[Evaluate@
     (1/α CoupledLogarithm[p^α, -#, -d] & /@ {0, 0.25, .5, 0.75} //. {α → 2, d → 1}),
    {p, 0, 1}]
  }}]
```

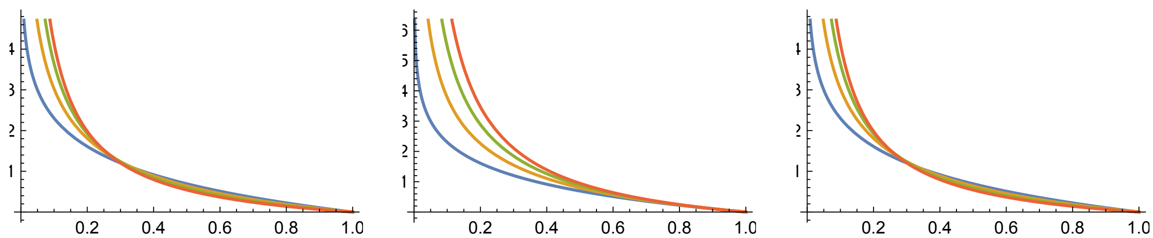
Out[ ]=



Plot the coupled-surprisal as the information loss function

In[ ]:=
```
GraphicsGrid[{{Plot[Evaluate@
      (1/α CoupledLogarithm[p^-α, #, d] &/@ {0, 0.25, .5, 0.75} //. {α → 2, d → 1}),
    {p, 0, 1}],
   Plot[Evaluate@
     (-qln[p, #] &/@ couplingToq[{10^-6, 0.25, .5, 0.75}, α, d] //. {α → 2, d → 1}),
    {p, 0, 1}],
   Plot[Evaluate@
     (-1/α CoupledLogarithm[p^α, -#, -d] &/@ {0, 0.25, .5, 0.75} //. {α → 2, d → 1}),
    {p, 0, 1}]
  }}]
```

Out[ ]=



Curious what the curves look like if alpha = 1

In[ ]:=
```
GraphicsGrid[{{Plot[Evaluate@
      (1/α CoupledLogarithm[p^-α, #, d] &/@ {0, 0.25, .5, 0.75} //. {α → 1, d → 1}),
    {p, 0, 1}],
   Plot[Evaluate@
     (-qln[p, #] &/@ couplingToq[{10^-6, 0.25, .5, 0.75}, α, d] //. {α → 1, d → 1}),
    {p, 0, 1}],
   Plot[Evaluate@
     (-1/α CoupledLogarithm[p^α, -#, -d] &/@ {0, 0.25, .5, 0.75} //. {α → 1, d → 1}),
    {p, 0, 1}]
  }}]
```

Out[ ]=