Coupled Exponentials & Logarithms

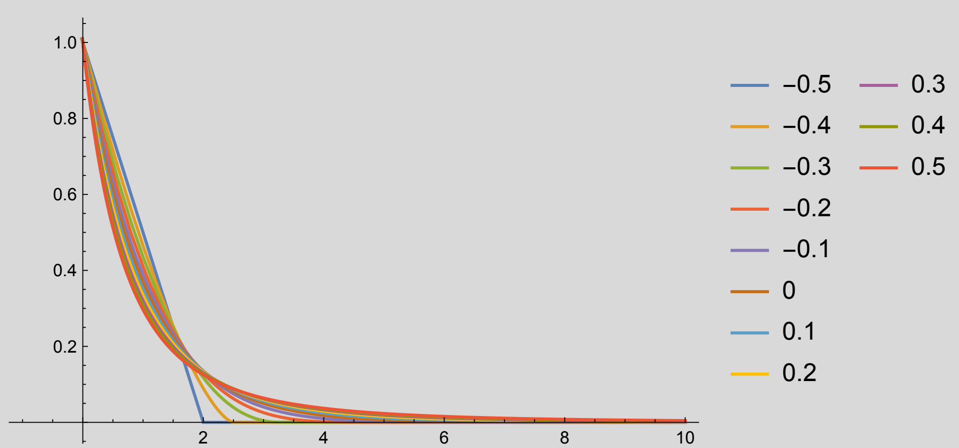
Graphic of Coupled Exponential

Graph shows curves from linear (κ = -0.5) to exponential (κ = 0)

In[ ]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Plot[CoupledExponential[x, #]^-1 & /@ CouplingValues // Evaluate,
 {x, -1, 10}, PlotLegends → CouplingValues, PlotRange → Automatic]
```

Out[ ]=



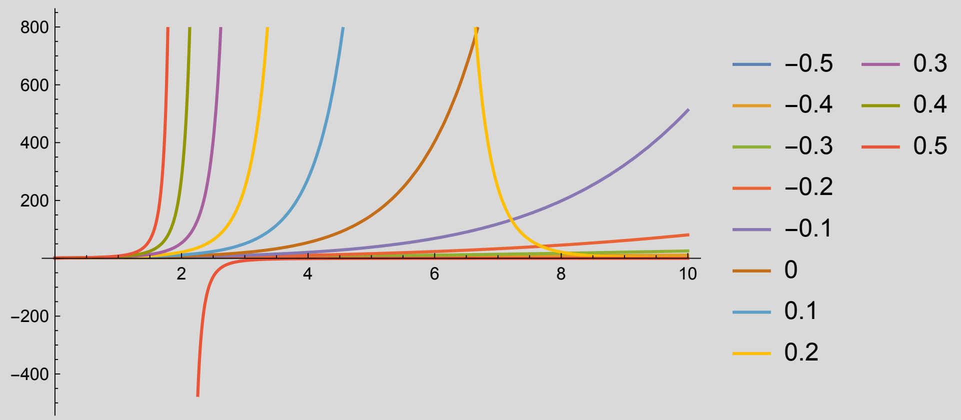
The curves are produced by the Coupled Exponential Function

$$(1 + \kappa\, x)^{-\frac{1+\kappa}{\kappa}}$$

In[∘]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Plot[CoupledExponential[-x, #]^-1 & /@ CouplingValues // Evaluate,
 {x, 0, 10}, PlotLegends → CouplingValues]
```

Out[∘]=



The curves are produced by the Coupled Exponential Function

$$(1-\kappa\, x)^{\frac{1+\kappa}{-\kappa}}$$

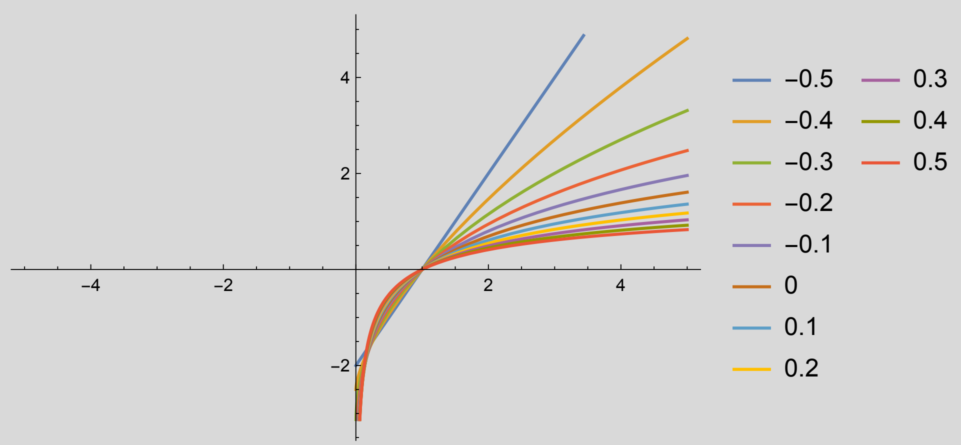
Graphic of Coupled Logarithm

Graph shows curves from linear to logarithmic

In[∘]:=

```
CouplingValues = {-0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5};
Quiet[Plot[
  -CoupledLogarithm[x^-1, #] & /@ CouplingValues // Evaluate, {x, -5, 5},
  PlotLegends → CouplingValues],
 {Power::infy}]
```

Out[∘]=



The curves are produced by the Coupled Logarithmic Function

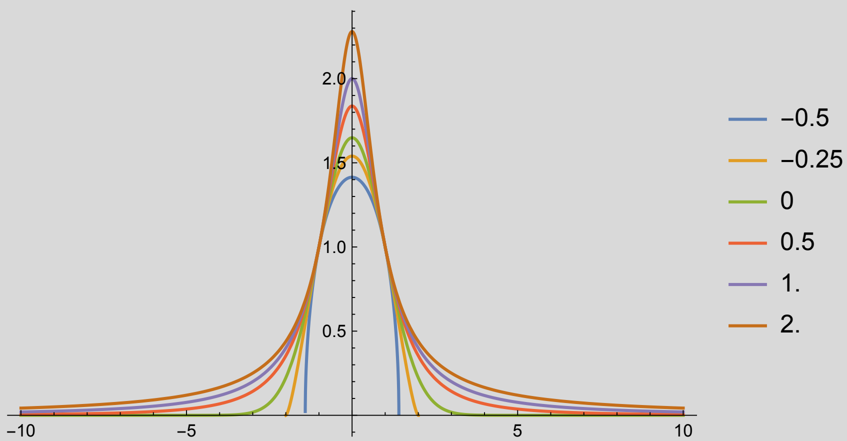
$$\frac{1}{-\kappa}\left(x^{\frac{-\kappa}{1+\kappa}}-1\right)$$

Coupled Normal Distribution

In[ ]:=

```
CouplingValues = {-0.5, -0.25, 0, 0.5, 1.0, 2.0};
Quiet[Plot[
  PDF[CoupledNormalDistribution[0, 1, #], {x}] /
      PDF[CoupledNormalDistribution[0, 1, #], {1}] & /@
    CouplingValues // Evaluate, {x, -10, 10},
  PlotLegends → CouplingValues,
  PlotRange → Full],
 {Power::infy}]
```

Out[ ]=

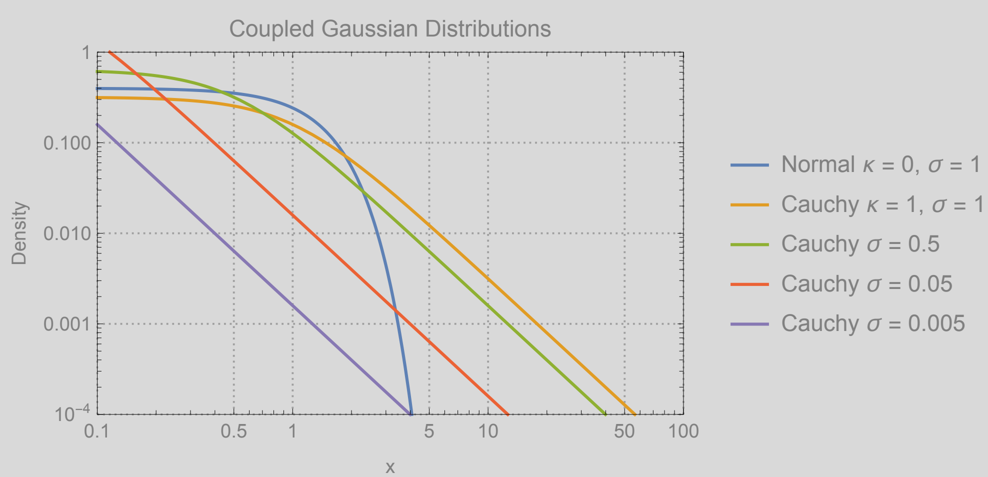


Coupled Gaussian is Scale-Free as $\sigma \to 0$

In[∘]:=

```
Parameters = {{1, 1, 0.5, 0.05, 0.005}, {0, 1, 1, 1, 1}};
Quiet[LogLogPlot[MapThread[
    PDF[CoupledNormalDistribution[0, #1, #2], {x}] &, Parameters] // Evaluate,
  {x, 0.1, 100},
  PlotLegends → {"Normal κ = 0, σ = 1",
    "Cauchy κ = 1, σ = 1", "Cauchy σ = 0.5",
    "Cauchy σ = 0.05", "Cauchy σ = 0.005"},
  LabelStyle → Directive[Gray, Smaller],
  PlotRange → {{0.1, 100}, {10^-4, 1}},
  PlotTheme → {"Detailed"},
  FrameLabel → {"x", "Density"},
  PlotLabel → "Coupled Gaussian Distributions"],
 {Power::infy}]
```

Out[∘]=



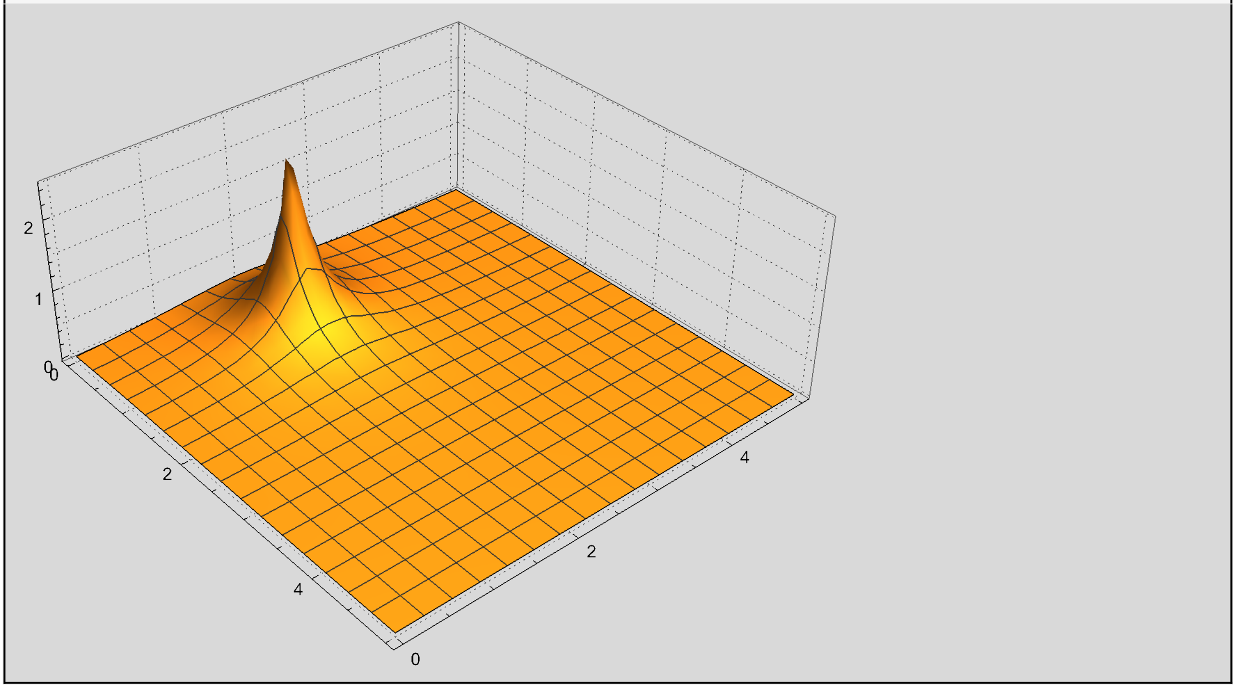
Multivariate Coupled Distribution

Multivariate Coupled Exponential

In[•]:=

```
Plot3D[PDF[MultivariateCoupledDistribution[{1, 2}, {{1, 0}, {0, 1}}, 2, 1],
   {x, y}],
 {x, 0, 5}, {y, 0, 5},
 PlotLegends → None,
 PlotTheme → "Detailed",
 PlotRange → Full
]
```

Out[•]=

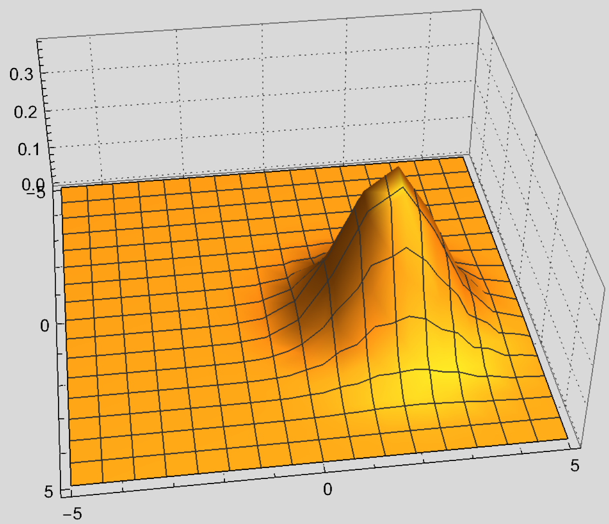


Multivariate Coupled Gaussian

In[•]:=

```
Plot3D[
 PDF[MultivariateCoupledDistribution[{1, 2}, {{1, -0.01}, {0.01, 1}}, 0.01, 2],
  {x, y}],
 {x, -5, 5}, {y, -5, 5},
 PlotLegends → None,
 PlotTheme → "Detailed",
 PlotRange → Full
]
```

Out[•]=



Test Normalization of Coupled Multivariate Gaussian

In[•]:=

```
Assuming[-1/2 < κ < ∞,
  Integrate[PDF[MultivariateCoupledDistribution[{0, 0}, {{1, 0}, {0, 1}}, κ, 2],
    {x, y}],
   {x, -∞, ∞}, {y, -∞, ∞}
  ]] // FullSimplify
```

Out[•]=

```
1
```

In[∘]:=

```
Assuming[-1/3 < κ < ∞, Integrate[PDF[MultivariateCoupledDistribution[
     {0, 0, 0}, {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}, κ, 2],
    {x, y, z}],
   {x, -∞, ∞}, {y, -∞, ∞}, {z, -∞, ∞}
  ]] // FullSimplify
```

Out[∘]=

$$\begin{cases} 1 & \kappa \geq 0 \\ -\frac{1}{2\pi\,\text{Beta}\left[-\frac{1+\kappa}{2\kappa},\frac{3}{2}\right]}\sqrt{-\kappa}\;\kappa\;\text{Integrate}\left[\frac{1}{\sqrt{\begin{cases}\left(1+x^2\kappa+y^2\kappa+z^2\kappa\right)^{3+\frac{1}{\kappa}} & \left(x^2+y^2+z^2\right)\kappa\geq -1 \\ \infty & \text{True}\end{cases}}},\{x,-\infty,\infty\},\right. \\ \quad \left.\{y,-\infty,\infty\},\{z,-\infty,\infty\},\text{Assumptions}\to -\frac{1}{3}<\kappa<\infty\;\&\&\;\left(-\frac{1}{3}<\kappa<0\;||\;\kappa\leq -\frac{1}{3}\right)\right] & \text{True} \end{cases}$$

In[∘]:=

```
Assuming[-1/4 < κ < ∞,
  Integrate[PDF[MultivariateCoupledDistribution[{0, 0, 0, 0},
     {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}, κ, 2],
    {w, x, y, z}],
   {w, -∞, ∞}, {x, -∞, ∞}, {y, -∞, ∞}, {z, -∞, ∞}
  ]] // FullSimplify
```

Out[∘]=

$$\begin{cases} 1 & \kappa \geq 0 \\ \frac{1}{\pi^2\,\text{Beta}\left[-1-\frac{1}{2\kappa},2\right]} & \text{True} \\ \quad \kappa^2\;\text{Integrate}\left[\frac{1}{\sqrt{\begin{cases}\left(1+w^2\kappa+x^2\kappa+y^2\kappa+z^2\kappa\right)^{4+\frac{1}{\kappa}} & \left(w^2+x^2+y^2+z^2\right)\kappa\geq -1 \\ \infty & \text{True}\end{cases}}},\{w,-\infty,\infty\},\{x,-\infty,\infty\},\right. & \\ \quad \left.\{y,-\infty,\infty\},\{z,-\infty,\infty\},\text{Assumptions}\to -\frac{1}{4}<\kappa<\infty\;\&\&\;\left(-\frac{1}{4}<\kappa<0\;||\;\kappa\leq -\frac{1}{4}\right)\right] & \end{cases}$$