

Coupled Entropy NSP & Paper 2025

```
In[1]:= $Assumptions = κ > 0 && κ ∈ Reals && x ∈ Reals && μ ∈ Reals && σ ∈ Reals && σ > 0 && α ∈ Reals && α > 0 && d ∈ Integers && d > 0 && γ ∈ Reals && 0 < γ < ∞
```

```
Out[1]=
κ > 0 && κ ∈ ℝ && x ∈ ℝ && μ ∈ ℝ && σ ∈ ℝ &&
σ > 0 && α ∈ ℝ && α > 0 && d ∈ ℤ && d > 0 && γ ∈ ℝ && 0 < γ < ∞
```

Compute Entropies of Gen. Pareto Distribution

```
Clear[CECoupledExp]
```

```
Assuming[κ > 0 && x > 0,
CoupledExponentialDistribution[κ, 0, σ]
]
```

```
ProbabilityDistribution[ $\begin{cases} \text{If}[\kappa \neq 0, \text{If}[\text{Simplify}\left[1 - \frac{\kappa(-x)}{\sigma}\right] > 0, \left(1 - \frac{\kappa(-x)}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}, 0], \text{Exp}\left[-\frac{x}{\sigma}\right]] & x \geq 0, \{x, -\infty, \infty\} \\ 0 & \text{True} \end{cases}]$ 
```

```
Out[2]=
κ > 0 && κ ∈ ℝ && x ∈ ℝ && μ ∈ ℝ && σ ∈ ℝ && σ > 0 && α ∈ ℝ && α > 0 && d ∈ ℤ && d > 0
```

This computes the Coupled Entropy of the Pareto Distribution. See below for computation of the coupled probability

and the coupled logarithm of the distribution

$$-\int_0^\infty \left((1 + \kappa) \sigma^{1+\frac{1}{\kappa}} (x \kappa + \sigma)^{-2-\frac{1}{\kappa}} \right) \left(\frac{1 - \left(\sigma^{\frac{1}{\kappa}} (x \kappa + \sigma)^{-\frac{1+\kappa}{\kappa}} \right)^{-1+\frac{1}{1+\kappa}}}{\kappa} \right) dx // \text{FullSimplify}$$

$$\frac{-1 + (1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}}}{\kappa}$$

$$(1 + \kappa) \text{CoupledLogarithm}[\sigma^{-1}, \kappa, 1, 1] - \frac{1 + \kappa}{\kappa} // \text{FullSimplify}$$

$$-\frac{(1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}}}{\kappa}$$

```
In[3]:= SECoupledExp[κ_, σ_] := 1 + κ Log[σ];
```

```
In[4]:= CECoupledExp[κ_, σ_] :=  $\frac{-1 + (1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}}}{\kappa};$ 
```

$$\frac{-1 + (1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}}}{\kappa} = \frac{1+\kappa}{\kappa} \left(\sigma^{\frac{\kappa}{1+\kappa}} - 1 \right) + \frac{1+\kappa}{\kappa} - \frac{1}{\kappa} = 1 + \ln_{\frac{1+\kappa}{\kappa}} \sigma$$

Confirms 2025 Solution

```

In[0]:= TECoupledExp[\kappa_, \sigma_] := 
$$\frac{1 + \kappa - \sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$$

In[0]:= 
$$\frac{1 + \kappa - \sigma^{-1+\frac{1}{1+\kappa}}}{\kappa} // \text{FullSimplify}$$

Out[0]= 
$$\frac{1 + \kappa - \sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$$


$$\frac{1 + \kappa - \sigma^{-1+\frac{1}{1+\kappa}}}{\kappa} = 1 - \frac{1}{\kappa} \left( \sigma^{-\frac{\kappa}{1+\kappa}} - 1 \right)$$
 Confirms 2025 Result
In[0]:= NTECoupledExp[\kappa_, \sigma_] := 
$$\frac{(1 + \kappa) \left( -1 + (1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}} \right)}{\kappa}$$


CoupledProbability[
  ProbabilityDistribution[ $\frac{1}{\sigma} \left( 1 - \frac{\kappa(-x)}{\sigma} \right)^{-\frac{1+1\kappa}{1\kappa}}$ , {x, 0, \infty}],
   $\frac{-\kappa}{1 + \kappa}, x$ 
]

$$\begin{cases} 0 & x \leq 0 \\ (1 + \kappa) \sigma^{1+\frac{1}{\kappa}} (x \kappa + \sigma)^{-2-\frac{1}{\kappa}} & \text{True} \end{cases}$$


CoupledLogarithm[
   $\frac{1}{\sigma} \left( 1 - \frac{\kappa(-x)}{\sigma} \right)^{-\frac{1+1\kappa}{1\kappa}}$ ,
  \kappa, 1, 1
] // FullSimplify
ConditionalExpression[ $\frac{1 - \left( \sigma^{\frac{1}{\kappa}} (x \kappa + \sigma)^{-\frac{1+\kappa}{\kappa}} \right)^{-1+\frac{1}{1+\kappa}}}{\kappa}$ ,  $(x \kappa + \sigma)^{-1-\frac{1}{\kappa}} \geq 0$ ]

```

```

Assuming[x > 0, CoupledEntropy[
  Assuming[x > 0 && x > 0,
    CoupledExponentialDistribution[x, 0, σ]
  ], κ, 1, 1]] // Simplify
-∫₀^∞ If[ $\left\{ \begin{array}{ll} \sigma^{\frac{1}{κ}} (xκ + σ)^{-1-\frac{1}{κ}} & x \geq 0 \\ 0 & \text{True} \end{array} \right\} \geq 0,$ 
If[κ ≠ 0, - $\frac{1}{1+κ} \left( \left( \begin{array}{ll} \frac{If[\kappa \neq 0, If[Simplify[1 - \frac{\kappa(-x)}{\sigma}] > 0, (1 - \frac{\kappa(-x)}{\sigma})^{-\frac{1+1/\kappa}{1/\kappa}}, 0], Exp[-\frac{x}{\sigma}]]}{\sigma} & x \geq 0 \\ 0 & \text{True} \end{array} \right)^{-\frac{\kappa}{1+1/\kappa}} - 1 \right)$ ,
Log[ $\left\{ \begin{array}{ll} \frac{If[\kappa \neq 0, If[Simplify[1 - \frac{\kappa(-x)}{\sigma}] > 0, (1 - \frac{\kappa(-x)}{\sigma})^{-\frac{1+1/\kappa}{1/\kappa}}, 0], Exp[-\frac{x}{\sigma}]]}{\sigma} & x \geq 0 \\ 0 & \text{True} \end{array} \right\}],$ 
Undefined]  $\left( \begin{array}{ll} 0 & x < 0 \\ (1+κ) \sigma^{1+\frac{1}{κ}} (xκ + σ)^{-2-\frac{1}{κ}} & \text{True} \end{array} \right) dx$ 

NSECoupledExp[κ_, σ_] :=
Assuming[∞ > x > 0,
Piecewise[{{
  {NShannonEntropy[CoupledExponentialDistribution[κ, 0, σ], 0, 100], 0 < κ < 0.09},
  {NShannonEntropy[
    CoupledExponentialDistribution[κ, 0, σ], 0, 1000], 0.09 ≤ κ < 0.74},
  {NShannonEntropy[
    CoupledExponentialDistribution[κ, 0, σ], 0, 10000], 0.74 ≤ κ < 1.5},
  {NShannonEntropy[CoupledExponentialDistribution[κ, 0, σ], 0, 15000], 1.5 ≤ κ}
  }]]
]

NSECoupledExp[#, 1] & /@ {0.25, 0.5, 0.75, 1, 1.25}
{1.25, 1.49992, 1.74985, 1.99796, 2.23985}

NSECoupledExp[0.01, 0.25]
-0.376294

NSECoupledExp[2, 2]
3.54523

```

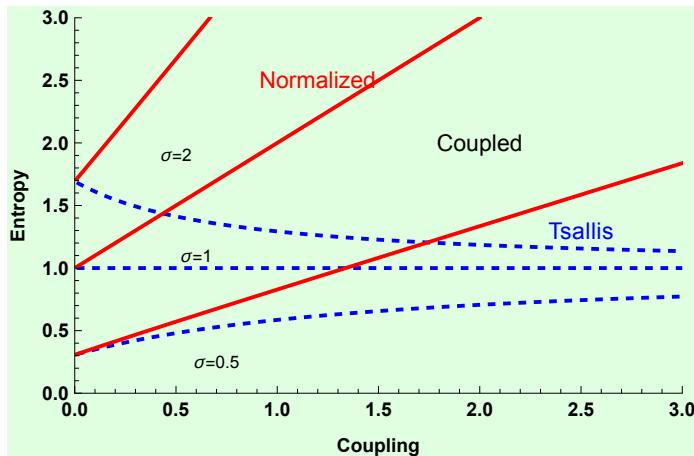
Plot Comparison of GPD with TE, NTE, & Shannon

```

Assuming[ $\kappa < \infty$ ,
Plot[
{
(*Table[SECoupledExp[ $\kappa, \sigma$ ], { $\sigma$ , {0.5, 1, 2}}], *)
{Style[Table[TECoupledExp[ $\kappa, \sigma$ ], { $\sigma$ , {0.5, 1, 2}}],
{Blue, Dashed}],
{Style[Table[NTECoupledExp[ $\kappa, \sigma$ ], { $\sigma$ , {0.5, 1, 2}}],
{Red, Dotted}],
{Style[Table[CECoupledExp[ $\kappa, \sigma$ ], { $\sigma$ , {0.5, 1, 2}}],
Black]}
},
{ $\kappa$ , 0.01, 3},
Background -> LightGreen,
PlotRange -> {{0, 3}, {0, 3}},
Frame -> {{True, False}, {True, False}},
FrameLabel -> {"Coupling", "Entropy"},
FrameStyle -> Directive[10, Bold, Black],
Epilog -> {
Inset[Style["Coupled", Black, Medium], {2, 2}],
Inset[Style["Tsallis", Blue, Medium], {2.5, 1.3}],
Inset[Style["Normalized", Red, Medium], {1.2, 2.5}],
Inset[Style[" $\sigma=2$ ", Black], {0.5, 1.9}],
Inset[Style[" $\sigma=1$ ", Black], {0.6, 1.1}],
Inset[Style[" $\sigma=0.5$ ", Black], {0.7, 0.25}]
}
}
]
]

```

Out[•] =



Compute Tsallis Entropy with different q values

In[1]:= \$Assumptions =

$\kappa \in \text{Reals} \& \& 0 < \kappa < \infty \&& \alpha \in \text{Reals} \& \& 0 < \alpha < \infty \&& d \in \text{Reals} \& \& 0 < d < \infty \&& \sigma \in \text{Reals} \& \& 0 < \sigma < \infty$

Out[1]=

$\kappa \in \mathbb{R} \& \& 0 < \kappa < \infty \&& \alpha \in \mathbb{R} \& \& 0 < \alpha < \infty \&& d \in \mathbb{R} \& \& 0 < d < \infty \&& \sigma \in \mathbb{R} \& \& 0 < \sigma < \infty$

$$\text{qEnt} = \frac{1}{\text{qDist}}$$

$$\begin{aligned} \text{In[2]:= } & \int_0^\infty \text{FullSimplify}\left[\frac{1}{\sigma} \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}} \left(1 + \text{qToCoupling}\left[\frac{1+\kappa}{1+2\kappa}\right]\right) \right. \\ & \quad \left. \text{CoupledLogarithm}\left[\sigma \left(1 + \frac{\kappa x}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}, \text{qToCoupling}\left[\frac{1+\kappa}{1+2\kappa}\right], 1\right]\right] dx // \text{FullSimplify} \\ & \int_0^\infty \frac{1}{(2+5\kappa)(x\kappa+\sigma)} 2(1+2\kappa) \left(1 + \frac{x\kappa}{\sigma}\right)^{-1/\kappa} \text{If}\left[\left(1 + \frac{x\kappa}{\sigma}\right)^{\frac{1}{\kappa}} (x\kappa+\sigma) \geq 0, \right. \\ & \quad \left. -\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}} \neq 0, -\frac{\left(\left(1 + \frac{x\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}} \sigma\right)^{\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}} - 1}{\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}}, \text{Log}\left[\left(1 + \frac{x\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}} \sigma\right], \text{Undefined}\right] dx \\ & \text{In[3]:= } \frac{2}{(2+5\kappa)(x\kappa+\sigma)} (1+2\kappa) \left(1 + \frac{x\kappa}{\sigma}\right)^{-1/\kappa} \left(-\frac{\left(\left(1 + \frac{x\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}} \sigma\right)^{\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}} - 1}{\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}} \right) // \text{FullSimplify} \end{aligned}$$

Out[3]=

$$-\frac{2(1+2\kappa) \left(1 + \frac{x\kappa}{\sigma}\right)^{-1/\kappa} \left(-1 + \left(\left(1 + \frac{x\kappa}{\sigma}\right)^{\frac{1}{\kappa}} (x\kappa+\sigma)\right)^{-\frac{\kappa}{2+4\kappa}}\right)}{x\kappa(x\kappa+\sigma)}$$

Out[4]=

$$\int_0^\infty -\frac{2(1+2\kappa) \left(1 + \frac{x\kappa}{\sigma}\right)^{-1/\kappa} \left(-1 + \left(\left(1 + \frac{x\kappa}{\sigma}\right)^{\frac{1}{\kappa}} (x\kappa+\sigma)\right)^{-\frac{\kappa}{2+4\kappa}}\right)}{x\kappa(x\kappa+\sigma)} dx // \text{FullSimplify}$$

Out[4]=

$$-\frac{2(1+2\kappa) \sigma^{\frac{1}{\kappa}} \left(-\sigma^{-1/\kappa} + \frac{(2+4\kappa) \sigma^{-\frac{2+4\kappa^2}{2+4\kappa}}}{2+\kappa(5+\kappa)}\right)}{\kappa}$$

$$\text{qEnt} = 2 - \text{qDist}$$

```
In[6]:= Integrate[FullSimplify[(1/σ)^(1+κ) (1 + κ x/σ)^(-1/κ) (1 + qToCoupling[1 - κ/(1 + κ)]) CoupledLogarithm[(σ (1 + κ x/σ))^(1/κ), qToCoupling[1 - κ/(1 + κ)], 1]], {x, 0, ∞}] // FullSimplify
Out[6]= 2 (1 + κ) (1 - 2 σ^(2+2κ)/(2+κ)) / κ
Dual kEnt = -κ/(1+κ)

In[7]:= Integrate[FullSimplify[(1/σ)^(1+κ) (1 + κ x/σ)^(-1/κ) (1 - κ/(1 + κ)) CoupledLogarithm[(σ (1 + κ x/σ))^(1/κ), -κ/(1 + κ), 1]], {x, 0, ∞}] // FullSimplify
Out[7]= 1 - σ^κ / (1+κ+κ^2) / κ
```

Coupled Entropy of Coupled Stretched Exponential Distribution

Structure of solution is $\frac{d}{\alpha} + \text{Log}_{\frac{\alpha\kappa}{1+d\kappa}} [Z_\kappa(\sigma, \alpha, d)] = \frac{d}{\alpha} + \text{Log}_{\frac{\alpha\kappa}{1+d\kappa}} [\sigma] \oplus_{\frac{\alpha\kappa}{1+d\kappa}} \text{Log}_{\frac{\alpha\kappa}{1+d\kappa}} [Z'_{\kappa}(\alpha, d)]$

Computation of Coupled Entropy of Coupled Stretched Exponential

```
In[8]:= Clear[CoupledEntropyCSE];
CoupledEntropyCSE[σ_, κ_, α_, d_] :=
  CoupledEntropyCSE[σ, κ, α, d] =
    d/α + CoupledLogarithm[
      NormMultiCoupled[σ, κ, α, d],
      α κ / (1 + d κ), 0];
```

Plot of Coupled Entropies

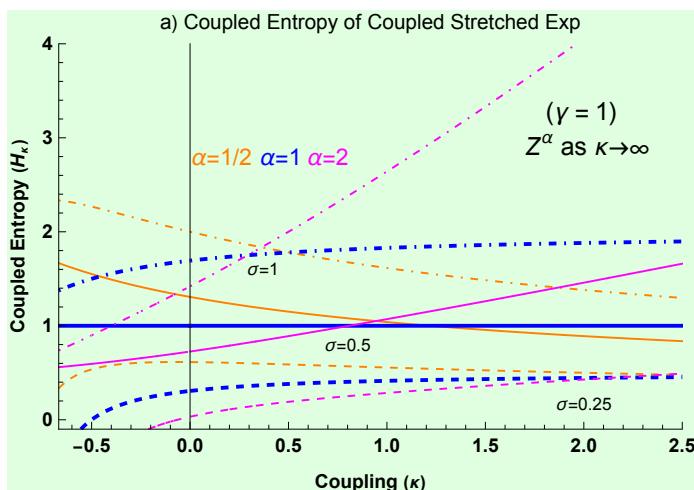
In[6]:=

```

Plot[
 Evaluate@{
  Table[CoupledEntropyCSE[\sigma, \kappa, 1/2, 1]^1, {\sigma, {0.25, 0.5, 1}}],
  Table[CoupledEntropyCSE[\sigma, \kappa, 1, 1]^1, {\sigma, {0.25, 0.5, 1}}],
  Table[CoupledEntropyCSE[\sigma, \kappa, 2, 1]^1, {\sigma, {0.25, 0.5, 1}}],
 },
 {\kappa, -0.667, 2.5},
 Background \rightarrow LightGreen,
 PlotRange \rightarrow {{-0.667, 2.5}, {-0.1, 4}},
 Frame \rightarrow {{True, False}, {True, False}},
 FrameLabel \rightarrow {"Coupling (\kappa)", "Coupled Entropy (H_\kappa)" },
 FrameStyle \rightarrow Directive[10, Bold, Black],
 PlotStyle \rightarrow Flatten[Table[{ \alpha Col, \sigma Col},
   {\alpha Col, {{Orange, Thickness[Medium]}, {Blue, Thickness[Large]},
     {Magenta, Thickness[Medium]}}}, {\sigma Col, {Dashed, , DotDashed}}]], 1],
 PlotLabel \rightarrow
 "a) Coupled Entropy of Coupled Stretched Exp",
 Epilog \rightarrow {
  Inset[Style["\alpha=1/2", Orange, Medium], {.16, 2.8}],
  Inset[Style["\alpha=1", Blue, Medium], {.46, 2.8}],
  Inset[Style["\alpha=2", Magenta, Medium], {.7, 2.8}],
  Inset[Style["\sigma=1", Black], {0.37, 1.6}],
  Inset[Style["\sigma=0.5", Black], {0.8, .8}],
  Inset[Style["\sigma=0.25", Black], {2.0, 0.17}],
  Inset[Style[DisplayForm[HoldForm[(\gamma = 1) " Z^\alpha as \kappa \rightarrow \infty"]], Black, FontSize \rightarrow 13], {2, 3.1}]
 }
]

```

Out[6]=

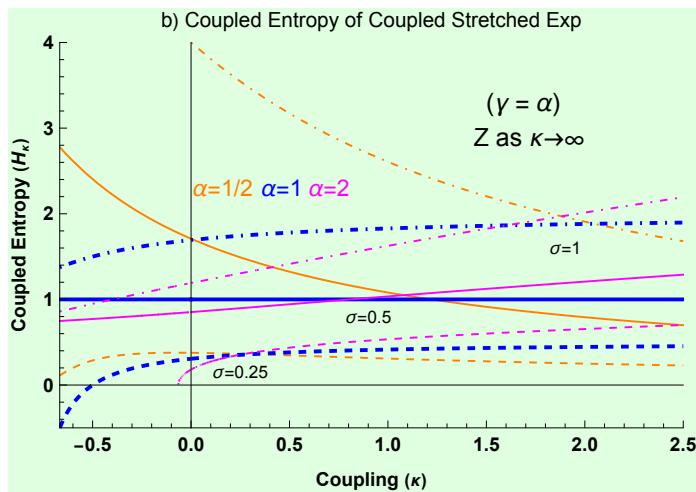


```

In[6]:= Plot[
  Evaluate@{
    Table[CoupledEntropyCSE[\sigma, \kappa, 1/2, 1]^2, {\sigma, {0.25, 0.5, 1}}],
    Table[CoupledEntropyCSE[\sigma, \kappa, 1, 1], {\sigma, {0.25, 0.5, 1}}],
    Table[CoupledEntropyCSE[\sigma, \kappa, 2, 1]^1, {\sigma, {0.25, 0.5, 1}}]
  },
  {\kappa, -0.667, 2.5},
  Background → LightGreen,
  PlotRange → {{-0.667, 2.5}, {-0.5, 4}},
  Frame → {{True, False}, {True, False}},
  FrameLabel → {"Coupling (\kappa)", "Coupled Entropy (H_\kappa)" },
  FrameStyle → Directive[10, Bold, Black],
  PlotStyle → Flatten[Table[{\alphaCol, \sigmaCol},
    {\alphaCol, {{Orange, Thickness[Medium]}, {Blue, Thickness[Large]},
      {Magenta, Thickness[Medium]}}}, {\sigmaCol, {Dashed, , DotDashed}}]], 1],
  PlotLabel →
    "b) Coupled Entropy of Coupled Stretched Exp",
  Epilog → {
    Inset[Style["\alpha=1/2", Orange, Medium], {0.16, 2.3}],
    Inset[Style["\alpha=1", Blue, Medium], {0.46, 2.3}],
    Inset[Style["\alpha=2", Magenta, Medium], {0.7, 2.3}],
    Inset[Style["\sigma=1", Black], {1.9, 1.6}],
    Inset[Style["\sigma=0.5", Black], {0.9, 0.8}],
    Inset[Style["\sigma=0.25", Black], {0.25, 0.15}],
    Inset[Style[DisplayForm[HoldForm[(\gamma = \alpha) "Z as \kappa \rightarrow \infty]]], Black, FontSize → 13], {1.7, 3.1}]
  }
]

```

Out[\circ] =



Plot with variable dimensions

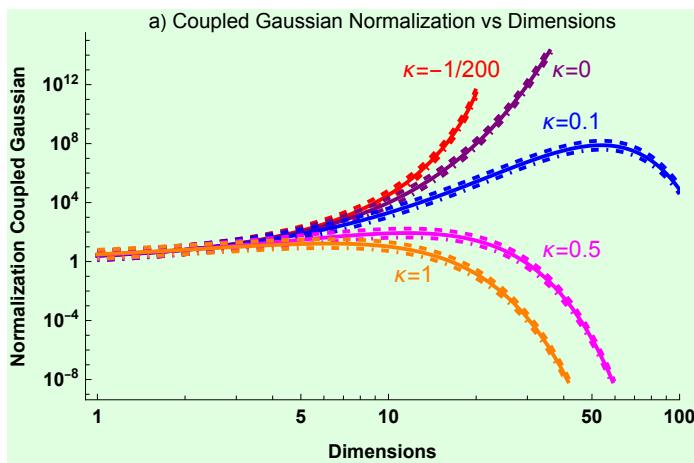
In[6]:=

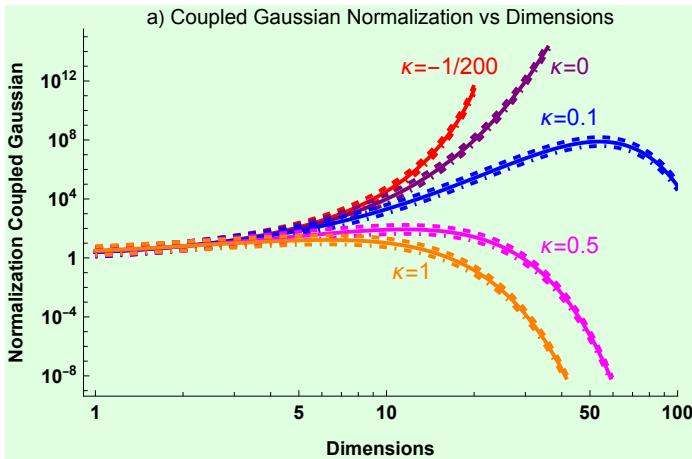
```

LogLogPlot[
 Evaluate@{
  Table[NormMultiCoupled[ $\sigma$ , -0.5×10-1, 2, d], { $\sigma$ , {0.5, 1, 2}}],
  Table[NormMultiCoupled[ $\sigma$ , 0.0, 2, d], { $\sigma$ , {0.5, 1, 2}}],
  Table[NormMultiCoupled[ $\sigma$ , 0.1, 2, d], { $\sigma$ , {0.5, 1, 2}}],
  Table[NormMultiCoupled[ $\sigma$ , 0.5, 2, d], { $\sigma$ , {0.5, 1, 2}}],
  Table[NormMultiCoupled[ $\sigma$ , 1.0, 2, d], { $\sigma$ , {0.5, 1, 2}}],
 },
 {d, 1, 100},
 Background → LightGreen,
 PlotRange → {{0, 100}, {0, Automatic}},
 Frame → {{True, False}, {True, False}},
 FrameLabel → {"Dimensions", "Normalization Coupled Gaussian"},
 FrameStyle → Directive[10, Bold, Black],
 PlotStyle → Flatten[Table[{xCol,  $\sigma$ Col},
   {xCol, {Red, Purple, Blue, Magenta, Orange}}, { $\sigma$ Col, {DotDashed, , Dashed}}], 1],
 PlotLabel → "a) Coupled Gaussian Normalization vs Dimensions",
 Epilog → {
  Inset[Style[" $\kappa=-1/200$ ", Red, Medium], {2.8, 30}],
  Inset[Style[" $\kappa=0$ ", Purple, Medium], {3.75, 30}],
  Inset[Style[" $\kappa=0.1$ ", Blue, Medium], {3.75, 22}],
  Inset[Style[" $\kappa=0.5$ ", Magenta, Medium], {3.75, 2}],
  Inset[Style[" $\kappa=1$ ", Orange, Medium], {2.5, -2}],
  (*Inset[Style[" $\sigma=2$ ", Black], {1.45, 1.45}],
  Inset[Style[" $\sigma=1$ ", Black], {0.8, .8}],
  Inset[Style[" $\sigma=0.5$ ", Black], {0.5, 0.1}]*)
 }
]

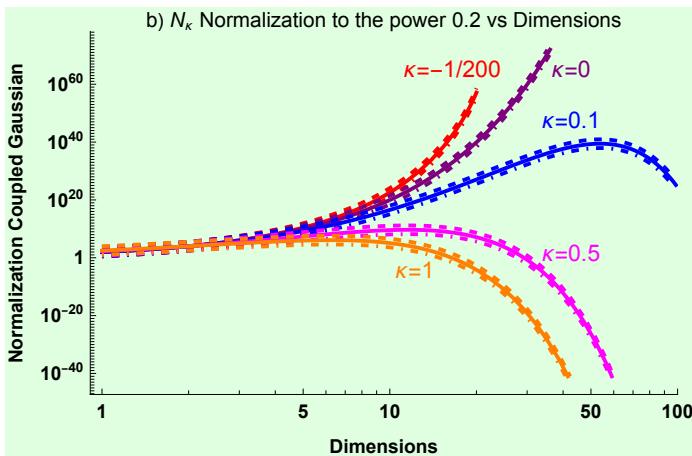
```

Out[6]=





With $\gamma = 2/10$



Score Function Plots

The score function of the GPD is $-\sigma^{-1}$, which is a powerful description of the scale's unique properties. The score function is computed from the derivative of the log of the pdf.

```
In[6]:= NegDerGPD[\sigma_, \kappa_, x_]:= \frac{1+\kappa}{x\kappa+\sigma};  
NegDerGPDq[\beta_, q_, x_]:= \frac{\beta}{1+(-1+q)x\beta};  
  
In[7]:= Clear[NegDerGPD]  
  
In[8]:= Assuming[0 < \kappa < \infty, -D[Log[\frac{1}{\sigma} \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}], x]] // FullSimplify  
  
Out[8]= \frac{1+\kappa}{x\kappa+\sigma}
```

```

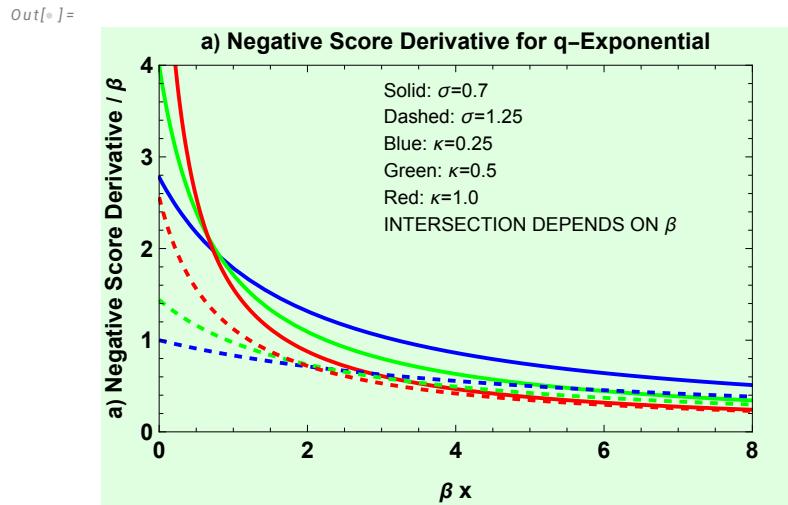
In[8]:= 
$$\frac{1 + \kappa}{x \kappa + \sigma} / . \left\{ \kappa \rightarrow \frac{-(1 - q)}{2 - q}, \sigma \rightarrow \frac{1}{(2 - q) \beta} \right\} // FullSimplify$$

Out[8]= 
$$\frac{\beta}{1 + (-1 + q) \times \beta}$$


Plot - $\beta$  Score versus  $\frac{x}{\beta}$ 

In[9]:= Plot[MapThread[scaleShapeToBeta[#1, #2, 1] x
  NegDerGPD[#1, #2, scaleShapeToBeta[#1, #2, 1] x] &,
  {{0.75, 0.75, 0.75, 1.25, 1.25, 1.25}, {0.25, 0.5, 1, 0.25, 0.5, 1}}] //
  Evaluate, {x, 0.0001, 10},
  Background → LightGreen,
  PlotRange → {{0, 8}, {0, 4}},
  PlotStyle → {Blue, Green, Red, {Blue, Dashed}, {Green, Dashed}, {Red, Dashed}},
  Epilog → Inset[Style[Text["Solid:  $\sigma=0.7$ "], "Solid:  $\sigma=1.25$ "], "Blue:  $\kappa=0.25$ ", "Green:  $\kappa=0.5$ ", "Red:  $\kappa=1.0$ "],
  "INTERSECTION DEPENDS ON  $\beta$ "], Larger], {5, 3}],
  LabelStyle → Directive[Bold, Medium],
  Frame → {{True, True}, {True, True}},
  FrameLabel → {"a) Negative Score Derivative /  $\beta$ ", " $\beta x$ "}
]

```



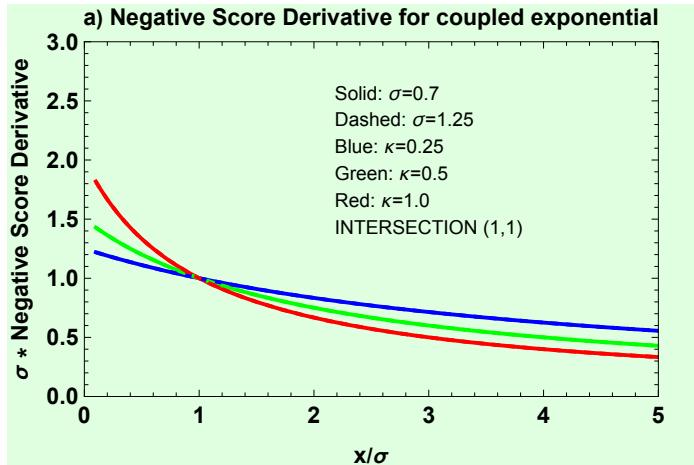
Translation of sigma and kappa

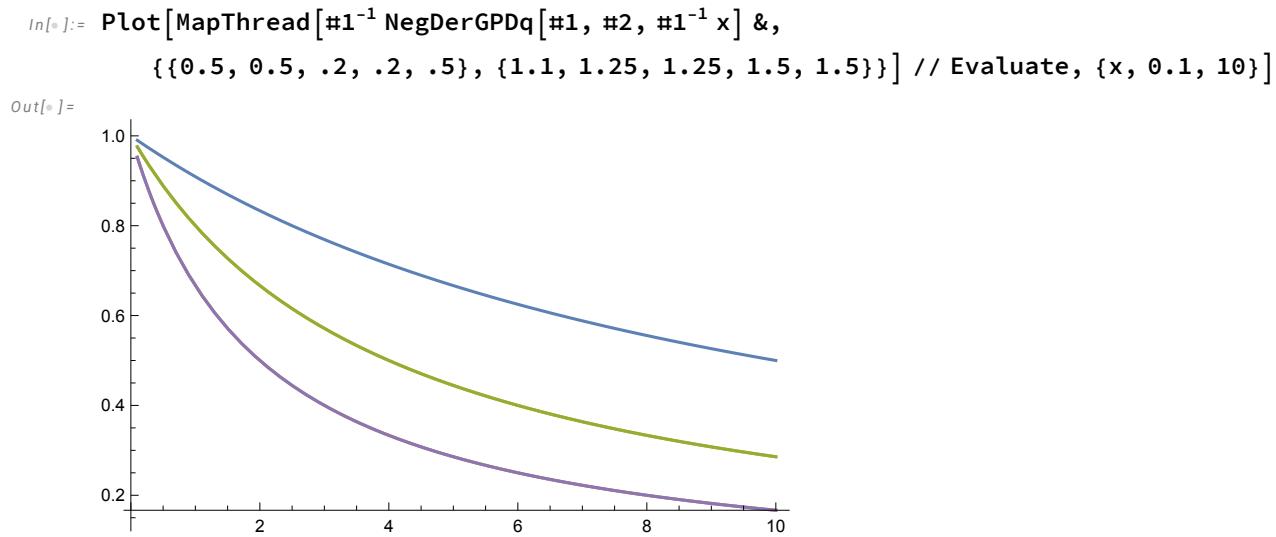
```

Plot[MapThread[
  #1 NegDerGPD[#1, #2, #1 x] &, {{0.75, 0.75, 0.75, 1.25, 1.25, 1.25},
   {0.25, 0.5, 1, 0.25, 0.5, 1}}] // Evaluate, {x, 0.1, 10},
  Background -> LightGreen,
  PlotRange -> {{0, 5}, {0, 3}},
  PlotStyle -> {Blue, Green, Red, {Blue, Dashed}, {Green, Dashed}, {Red, Dashed}},
  Epilog -> Inset[Style[Text["Solid:  $\sigma=0.7$ "],
  Dashed:  $\sigma=1.25$ ,
  Blue:  $\kappa=0.25$ ,
  Green:  $\kappa=0.5$ ,
  Red:  $\kappa=1.0$ ,
  INTERSECTION (1,1)"], Larger], {3, 2}],
  LabelStyle -> Directive[Bold, Medium],
  Frame -> {{True, True}, {True, True}},
  FrameLabel -> {" $\sigma * \text{Negative Score Derivative}$ ", ""},
  {" $x/\sigma$ ", "a) Negative Score Derivative for Coupled Exponential"}]
]

```

Out[\circ] =





Generalized Weibull Distribution

$$\text{In[1]:= GWDSF}[\sigma, \kappa, \alpha, x] := \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-\frac{1}{\alpha\kappa}};$$

$$\text{In[2]:= GWDPDF}[\sigma, \kappa, \alpha, x] := \frac{x^{\alpha-1}}{\sigma^\alpha} \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-\frac{1}{\alpha\kappa}-1};$$

In[3]:= D[-Log[GWDPDF[\sigma, \kappa, \alpha, x]], x] // FullSimplify

Syntax: "D[-Log[GWDPDF [\sigma, \kappa, \alpha, x]], x] // FullSimplify" is incomplete; more input is needed.

$$\text{In[4]:= D}\left[-\text{Log}\left[\frac{x^{\alpha-1}}{\sigma^\alpha} \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-\frac{1}{\alpha\kappa}-1}\right], x\right] // \text{FullSimplify}$$

Syntax: "D[-Log[\frac{x^{\alpha-1}}{\sigma^\alpha} \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-\frac{1}{\alpha\kappa}-1}], x] // FullSimplify" is incomplete; more input is needed.

$$\text{In[5]:= } -\partial_x \text{Log}\left[x^{-1+\alpha} \sigma^{-\alpha} (1 + x^\alpha \kappa \sigma^{-\alpha})^{-1-\frac{1}{\alpha\kappa}}\right] // \text{FullSimplify}$$

Out[5]=

$$\frac{x^\alpha (1 + \kappa) - (-1 + \alpha) \sigma^\alpha}{x (x^\alpha \kappa + \sigma^\alpha)}$$

$$\text{In[6]:= } \left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-1} / . x \rightarrow \sigma // \text{FullSimplify}$$

Out[6]=

$$\frac{2 - \alpha + \kappa}{\sigma + \kappa \sigma}$$

So the use of the Generalized Weibull will require care regarding the definition for the scale of the distribution

If $\alpha=2$, then

$$\frac{1}{\sigma} \frac{\kappa}{1 + \kappa}$$

So if this is defined as σW let's see what happens. Could also do this generally for alpha.

$$\begin{aligned} In[1]:= & \frac{1}{\frac{2-\alpha+\kappa}{\sigma+\kappa \sigma}} \\ Out[1]:= & \frac{\sigma + \kappa \sigma}{2 - \alpha + \kappa} \\ In[2]:= & \sigma W = \frac{\sigma + \kappa \sigma}{2 - \alpha + \kappa}; \end{aligned}$$

Recompute the derivative using the expression - $\frac{f'(x)}{f(x)}$

$$\begin{aligned} In[3]:= & D[x^{-1+\alpha} \sigma^{-\alpha} (1 + x^\alpha \kappa \sigma^{-\alpha})^{-1-\frac{1}{\alpha}}, x] // FullSimplify \\ Out[3]:= & x^{-2+\alpha} \frac{1}{\sigma^\kappa} (x^\alpha \kappa + \sigma^\alpha)^{-2-\frac{1}{\alpha}} (-x^\alpha (1 + \kappa) + (-1 + \alpha) \sigma^\alpha) \\ In[4]:= & -\frac{x^{-2+\alpha} \frac{1}{\sigma^\kappa} (x^\alpha \kappa + \sigma^\alpha)^{-2-\frac{1}{\alpha}} (-x^\alpha (1 + \kappa) + (-1 + \alpha) \sigma^\alpha)}{x^{-1+\alpha} \sigma^{-\alpha} (1 + x^\alpha \kappa \sigma^{-\alpha})^{-1-\frac{1}{\alpha}}} // FullSimplify \\ Out[4]:= & \frac{x^\alpha (1 + \kappa) - (-1 + \alpha) \sigma^\alpha}{x (x^\alpha \kappa + \sigma^\alpha)} \end{aligned}$$

Confirmed that it is the same expression

Consider solving for x , such that the score is equal to its inverse.

$$\begin{aligned} In[5]:= & \text{Solve}\left[\left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-1} = \left(\left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-1}\right)^{-1}, x\right] \\ & \text{Solve}\left[\left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right)^2 \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-2} = 1, x\right] \\ & \left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right)^2 \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-2} // FullySimplify \end{aligned}$$

Considering possible values of the scale

$$\begin{aligned} In[6]:= & \left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-1} /. x \rightarrow \sigma \frac{\kappa}{\alpha} // FullSimplify \\ Out[6]:= & \alpha \left(1 + \kappa - \frac{(1+\alpha \kappa) \sigma^\alpha}{\sigma^\alpha + \kappa \left(\frac{\kappa \sigma}{\alpha}\right)^\alpha}\right) \\ & \kappa^2 \sigma \end{aligned}$$

Examine the log-log derivative

From the following, this is x times derivative of the Log[

Therefore:

$$\frac{d}{d(\log x)} \log(f(x)) = \frac{f'(x)}{f(x)} \cdot x = x \cdot \frac{f'(x)}{f(x)}$$

```
In[1]:= FullSimplify[x \left( \frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1} \right) \left( 1 + \frac{\kappa x^\alpha}{\sigma^\alpha} \right)^{-1}]
```

```
Out[1]=
```

$$1 - \alpha + \frac{x^\alpha (1 + \alpha \kappa)}{x^\alpha \kappa + \sigma^\alpha}$$

```
Solve[1 - \alpha + \frac{x^\alpha (1 + \alpha \kappa)}{x^\alpha \kappa + \sigma^\alpha} == 1, x]
```

```
Out[2]= $Aborted
```

```
Solve[\frac{x^\alpha}{\sigma^\alpha} \frac{(1 + \alpha \kappa)}{1 + \frac{x^\alpha}{\sigma^\alpha} \kappa} == \alpha, x]
```

If $x = \alpha^{1/\alpha} \sigma$

```
Solve[\frac{\alpha \sigma^\alpha}{\sigma^\alpha} == \alpha, x]
```

```
In[3]:= 1 - \alpha + \frac{x^\alpha (1 + \alpha \kappa)}{x^\alpha \kappa + \sigma^\alpha} /. x \rightarrow \alpha^{1/\alpha} \sigma // FullSimplify
```

```
Out[3]=
```

$$1$$

```
In[4]:= \left( \frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1} \right) \left( 1 + \frac{\kappa x^\alpha}{\sigma^\alpha} \right)^{-1} /. x \rightarrow \alpha^{1/\alpha} \sigma // FullSimplify
```

```
Out[4]=
```

$$\frac{\alpha^{-1/\alpha}}{\sigma}$$

```
In[5]:= x \left( \frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1} \right) \left( 1 + \frac{\kappa x^\alpha}{\sigma^\alpha} \right)^{-1} /. x \rightarrow
```

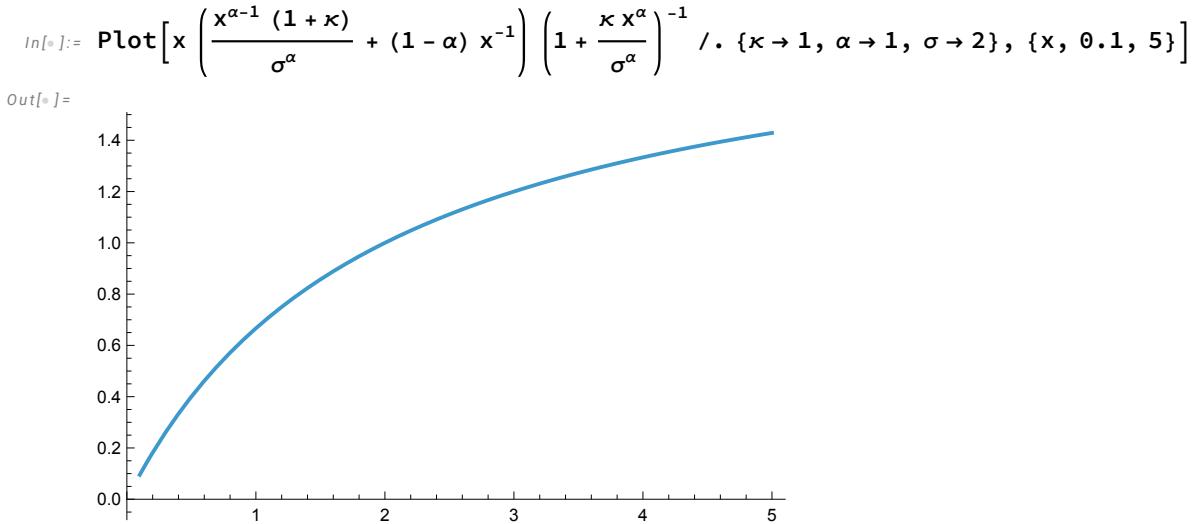
```
Out[5]=
```

$$1$$

$$x \left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1} \right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha} \right)^{-1} / . x \rightarrow$$

This is equivalent to modifying the definition of the Coupled Weibull Distribution to be

$$\frac{x^{\alpha-1}}{\sigma^\alpha} \left(1 + \alpha \kappa \frac{x^\alpha}{\alpha \sigma^\alpha} \right)^{-\frac{1}{\alpha \kappa} - 1}$$



Redefined Coupled Weibull with σ equal to the “Informational Scale”

In[2]:= GWDSF[\sigma_, \kappa_, \alpha_, x_] := \left(1 + \alpha \kappa \frac{x^\alpha}{\sigma^\alpha} \right)^{-\frac{1}{\alpha \kappa}};

In[3]:= GWDPDF[\sigma_, \kappa_, \alpha_, x_] := \frac{\alpha x^{\alpha-1}}{\sigma^\alpha} \left(1 + \alpha \kappa \frac{x^\alpha}{\sigma^\alpha} \right)^{-\frac{1}{\alpha \kappa}-1};

In[4]:= -x \frac{D[GWDPDF[\sigma, \kappa, \alpha, x], x]}{GWDPDF[\sigma, \kappa, \alpha, x]} // FullSimplify

Out[4]=

$$\frac{\alpha (x^\alpha - \sigma^\alpha)}{1 + \frac{\alpha (x^\alpha - \sigma^\alpha)}{x^\alpha \alpha \kappa + \sigma^\alpha}}$$

In[5]:= 1 + \frac{\alpha (x^\alpha - \sigma^\alpha)}{x^\alpha \alpha \kappa + \sigma^\alpha} /. x \rightarrow \sigma // FullSimplify

Out[5]=

$$1$$

Score of Coupled Stretched Exponential Distribution

Computing $-d \log[f(x)]/dx$ where $f(x)$ is the Coupled Stretched Exponential Distribution. The normalization, Z , can be ignored since the expression is equivalent to $-d f'(x)/(f(x) dx)$ and constants cancel.

In[6]:= \partial_x \log \left[\left(1 + \kappa \left(\frac{x}{\sigma} \right)^\alpha \right)^{-\frac{1}{\alpha} \left(\frac{1}{\kappa} + 1 \right)} \right] // FullSimplify

Out[6]=

$$-\frac{x^{-1+\alpha} (1 + \kappa)}{x^\alpha \kappa + \sigma^\alpha}$$

```

In[8]:= 
$$\frac{x^{-1+\alpha} (1+\kappa)}{x^\alpha \kappa + \sigma^\alpha} / . x \rightarrow \sigma // \text{FullSimplify}$$

Out[8]= 
$$\frac{1}{\sigma}$$


In[9]:= -x \partial_x \text{Log}\left[\left(1+\kappa \left(\frac{x}{\sigma}\right)^\alpha\right)^{-\frac{1}{\alpha} \left(\frac{1}{\kappa}+1\right)}\right] // \text{FullSimplify}
Out[9]= 
$$\frac{(1+\kappa) \left(\frac{x}{\sigma}\right)^\alpha}{1+\kappa \left(\frac{x}{\sigma}\right)^\alpha}$$


In[10]:= 
$$\frac{(1+\kappa) \left(\frac{x}{\sigma}\right)^\alpha}{1+\kappa \left(\frac{x}{\sigma}\right)^\alpha} / . x \rightarrow \sigma // \text{FullSimplify}
Out[10]= 1$$


In[11]:= (1+\kappa) \left(1+\kappa \left(\frac{x}{\sigma}\right)^\alpha\right)^{-1+\frac{1}{\kappa}} \left(\frac{x}{\sigma}\right)^\alpha / . x \rightarrow \sigma // \text{FullSimplify}
Out[11]= (1+\kappa)^{\frac{1}{\kappa}}

```

Multiplicative Noise Simulation

Lemma (Multiplicative Process With Coupled Gaussian Limit)

Let a stochastic process X_t be defined by the Stratonovich differential equation,

$$dX_t = \underbrace{f(X_t)dt}_{\text{Drift}} + \underbrace{A \circ dW_t^{(a)}}_{\text{Additive Noise}} + \underbrace{g(X_t)M \circ dW_t^{(m)}}_{\text{Multiplicative Noise}}$$

where $dW_t^{(a)}$ and $W_t^{(m)}$ are independent Wiener processes which define the additive (a) and multiplicative (m) noise, and A and M are the amplitudes of each noise source. Let the drift, $J(x) = f(X_t)$, be related to the diffusion, $D(x) = \frac{1}{2}(A^2 + M^2 g^2(x))$ by a restorative potential, $V(x)$, in which $f(x) = -\tau g(x)g'(x) = -V'(x)$ and τ is a time-domain scaling. Then the probability density $p_X(x, t)$ for this system has a coupled Gaussian limit distribution of $p_X(x) = \lim_{t \rightarrow \infty} p_X(x, t) \propto \exp^{-\frac{1+\kappa}{2} \left(\frac{g^2(x)}{\sigma^2}\right)}$, with $\sigma^2 = \frac{A^2}{2\tau}$ and $\kappa = \frac{D'(x)}{V'(x)} = \frac{M^2}{2\tau}$.

Define Stratonovich Process for Velocity with Multiplicative Noise and a Coupled Gaussian Limit Distribution

```

In[12]:= MultVelProc = StratonovichProcess[dv[t] == -v[t] dt + A dWAdd[t] + M v[t] \times dWMult[t],
                                         {v[t], {v, 1}}, t, {WAdd \approx WienerProcess[], WMult \approx WienerProcess[]}]

Out[12]= StratonovichProcess[{{-v[t]}, {{A, M v[t]}}, v[t], {{v}, {1}}, {t, 0}}]

```

```

In[1]:= MultVelProc["KolmogorovForwardEquation"] // TraditionalForm
Out[1]//TraditionalForm=

$$p^{(0,1)}(v, t) = \frac{1}{2} \frac{\partial^2 ((A^2 + M^2 v^2) p(v, t))}{\partial v \partial v} - \frac{\partial}{\partial v} \left( \frac{1}{2} (M^2 v - 2 v) p(v, t) \right)$$


In[2]:= Limit[PDF[MultVelProc[t] /. A → Sqrt[2] σ, v] // FullSimplify, t → Infinity]
Out[2]=

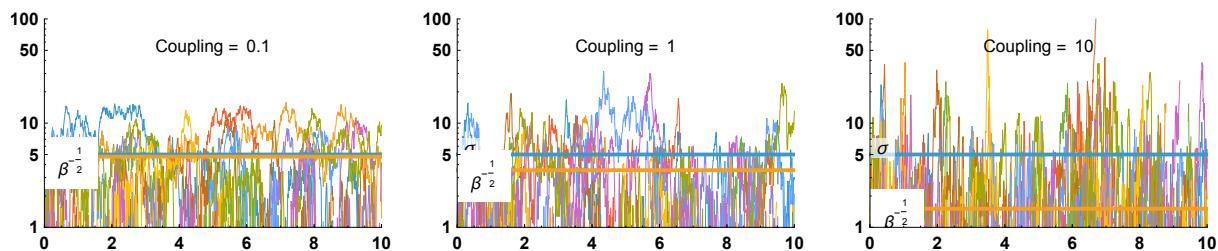
$$\lim_{t \rightarrow \infty} \text{PDF}[\text{StratonovichProcess}[\{-v[t]\}, \{\{\sqrt{2} \sigma, M v[t]\}\}, v[t]^2], \{v\}, \{1\}], \{t, 0\}][t], v]$$


In[3]:= AddSample[σ_, κ_] :=
  RandomFunction[AddVelProc /. {A → σ Sqrt[2], M → Sqrt[2 * κ]}, {0, 10, .01}, 50];
In[4]:= MultSample[σ_, κ_] :=
  RandomFunction[MultVelProc /. {A → σ Sqrt[2], M → Sqrt[2 * κ]}, {0, 10, .01}, 10];
σ=2, κ=0.1

```

```
In[6]:= Module[{σ = 5, κ = {0.1, 1, 10}, yLimit = {1, 100}}, 
GraphicsRow[Table[
Show[
ListLinePlot[MultSample[σ, κ[[i]]], 
PlotRange → {{0, 10}, {yLimit[[1]], yLimit[[2]]}}, 
PlotStyle → Thin, 
ScalingFunctions → {"Linear", "Log"}, 
LabelStyle → Bold, 
Epilog → Inset["Coupling = " Text[κ[[i]]], {5, 4}]],
],
Plot[{Labeled[σ, "σ", {Bottom, Left}], 
Labeled[ $\frac{\sigma}{\sqrt{1+\kappa[i]}}$ , " $\beta^{-\frac{1}{2}}$ ", {Top, Left}]}, {x, 0, 10}, 
PlotRange → {{0, 10}, {yLimit[[1]], yLimit[[2]]}}, 
PlotStyle → Thick, 
ScalingFunctions → {"Linear", "Log"}]
]
]
], {i, 3}]]]
```

Out[6]=



Checking DeepSeek Solutions

```
In[7]:= Solve[1 +  $\frac{\tau}{\alpha M} = \frac{1 + \kappa}{\alpha \kappa}$ , κ]
```

Out[7]=

$$\left\{ \kappa \rightarrow \frac{M}{-M + M \alpha + \tau} \right\}$$

```
In[1]:= 
$$\frac{M}{-M + M \alpha + \tau} \frac{A}{M} = -\frac{A}{-M + M \alpha + \tau}$$


Solve[ $\frac{\tau + \frac{3}{4} \alpha M}{\alpha M} = \frac{1 + \kappa}{\kappa}, \kappa$ ] // FullSimplify

Set: Tag Times in  $\frac{A M}{M (-M + M \alpha + \tau)}$  is Protected. ⓘ

Out[1]= 
$$\frac{A}{-M + M \alpha + \tau}$$


In[2]:= Solve[ $\frac{1 + \frac{3}{4} \alpha \frac{M}{\tau}}{\alpha \frac{M}{\tau}} = \frac{1 + \kappa}{\kappa}, \kappa$ ] // FullSimplify

Syntax: "Solve [ $\frac{\tau + \frac{3}{4} \alpha M}{\alpha M} == \frac{1 + \kappa}{\kappa}, \kappa$ ] // FullSimplify" is incomplete; more input is needed.

In[3]:= Clear[A, M, \alpha, \tau]
```

Check independent equals transformation

```
In[1]:= Solve[ $\frac{1 + (1 + m) \kappa}{\kappa} = \frac{1 + \kappa M}{\kappa M}, \kappa M$ ]
```

```
Out[1]=  $\left\{ \kappa M \rightarrow \frac{\kappa}{1 + m \kappa} \right\}$ 
```

```
In[2]:= Solve[ $1 + \frac{1}{M^2} = \frac{1}{2} \left( 1 + \frac{1}{\kappa} \right), \kappa$ ]
```

```
Out[2]= {}
```

$$2 \left(\frac{1}{2} + \frac{1}{M^2} \right) = \frac{1}{\kappa}$$

$$\kappa = \frac{M^2}{1 + M^2}$$

Uncertainty at the Scale: Definition for Gen. Entropy

Since the generalized entropy must compute a solution that is equal to the generalized logarithm of the coupled stretched exponential distribution at the scale, it is illustrative to compute this value for a variety of generalized entropies. For simplicity the case of $d=1$, $\alpha=1$, and $\mu=0$ is computed. The entropy transformed to the domain of the distribution is $\exp_{\kappa}^{-(1+\kappa)}[H]$. This value must be set equal to $f(x)$ and then one must solve for x , which requires taking $\ln_{\kappa} f(x)^{-1}$, reversing the coupled exponential. So the equation setting H equal to the argument of the density.

$$\exp_{\kappa}^{-(1+\kappa)}[H] = \exp_{\kappa}^{-(1+\kappa)} \left[\frac{x}{\sigma} + \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} + \kappa \frac{x}{\sigma} \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} \right]$$

$$H = \frac{\kappa}{\sigma} \left(1 + \kappa \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} \right) + \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}}$$

$$\frac{\kappa}{\sigma} = \frac{\sigma \left(H - \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} \right)}{\left(1 + \kappa \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} \right)}$$

$$\text{In[0]:= entValue[H_, \sigma_, \kappa_] :=}$$

$$\sigma \frac{\left(H - \text{CoupledLogarithm}\left[\sigma^{\frac{1}{1+\kappa}}, \kappa, 0\right] \right)}{1 + \kappa \text{CoupledLogarithm}\left[\sigma^{\frac{1}{1+\kappa}}, \kappa, 0\right]} // \text{FullSimplify};$$

$$\text{In[0]:= entDensity[H_, \kappa_] :=}$$

$$\text{CoupledExponential}[H, \kappa]^{-1} // \text{FullSimplify};$$

$$\text{In[0]:= HCoupled}[\sigma_, \kappa_] := 1 + \text{CoupledLogarithm}\left[\sigma, \frac{\kappa}{1 + \kappa}, 0\right];$$

$$\text{In[0]:= entValue[HCoupled}[\sigma, \kappa], \sigma, \kappa]$$

Out[0]=

$$\sigma$$

$$\text{In[0]:= entDensity[HCoupled}[\sigma, \kappa], \kappa]$$

Out[0]=

$$\frac{(1 + \kappa)^{-\frac{1+\kappa}{\kappa}}}{\sigma}$$

$$\text{In[0]:= HBGS}[\sigma_, \kappa_] := 1 + \text{Log}[\sigma] + \kappa;$$

$$\text{In[0]:= entValue[HBGS}[\sigma, \kappa], \sigma, \kappa] // \text{FullSimplify}$$

Out[0]=

$$\frac{\sigma^{\frac{1}{1+\kappa}} \left(1 + \kappa + \kappa^2 - \sigma^{\frac{\kappa}{1+\kappa}} + \kappa \text{Log}[\sigma] \right)}{\kappa}$$

$$\text{In[0]:= entDensity[HBGS}[\sigma, \kappa], \kappa]$$

Out[0]=

$$1 / \text{If}\left[1 + \kappa (1 + \kappa + \text{Log}[\sigma]) > 0, \text{If}\left[\kappa \neq 0, (1 + \kappa (1 + \kappa + \text{Log}[\sigma]))^{\frac{1+1\kappa}{\kappa}}, \text{Exp}[1 + \kappa + \text{Log}[\sigma]] \right], \text{If}\left[\frac{1 + 1\kappa}{\kappa} > 0, 0, \infty \right] \right]$$

$$\text{In[0]:= HRenyi}[\sigma_, \kappa_] := \text{Log}[\sigma] + \left(1 + \frac{1}{\kappa} \right) \text{Log}[1 + \kappa];$$

$$\text{In[0]:= entValue[HRenyi}[\sigma, \kappa], \sigma, \kappa] // \text{FullSimplify}$$

Out[0]=

$$\frac{\sigma^{\frac{1}{1+\kappa}} \left(1 - \sigma^{\frac{\kappa}{1+\kappa}} + (1 + \kappa) \text{Log}[1 + \kappa] + \kappa \text{Log}[\sigma] \right)}{\kappa}$$

In[1]:= **entDensity**[**HRenyi**[σ , κ], κ]

Out[1]=

$$\frac{1}{\text{If}\left[1 + \text{Log}[1 + \kappa] + \kappa \text{Log}[(1 + \kappa) \sigma] > 0, \text{If}\left[\kappa \neq 0, \left(1 + \kappa \left(\left(1 + \frac{1}{\kappa}\right) \text{Log}[1 + \kappa] + \text{Log}[\sigma]\right)\right)^{\frac{1+1 \kappa}{\kappa}}, \text{Exp}\left[\left(1 + \frac{1}{\kappa}\right) \text{Log}[1 + \kappa] + \text{Log}[\sigma]\right]\right], \text{If}\left[\frac{1+1 \kappa}{\kappa} > 0, 0, \infty\right]\right]},$$

$$\text{Exp}\left[\left(1 + \frac{1}{\kappa}\right) \text{Log}[1 + \kappa] + \text{Log}[\sigma]\right], \text{If}\left[\frac{1+1 \kappa}{\kappa} > 0, 0, \infty\right]$$

$$\text{In[2]:= HTsallis}[\sigma, \kappa] := 1 - \frac{1}{1+\kappa} \text{CoupledLogarithm}\left[\sigma^{-1}, \frac{\kappa}{1+\kappa}, 0\right];$$

In[3]:= **entValue**[**HTsallis**[σ , κ], σ , κ] // FullSimplify

Out[3]=

$$\frac{\sigma^{\frac{1}{1+\kappa}} \left(2 + \kappa - 2 \cosh\left[\frac{\kappa \text{Log}[\sigma]}{1+\kappa}\right]\right)}{\kappa}$$

In[4]:= **entDensity**[**HTsallis**[σ , κ], κ]

Out[4]=

$$\frac{1}{\text{If}\left[(2 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}} > 1, \text{If}\left[\kappa \neq 0, \left(1 + \kappa \left(1 - \frac{1}{1+\kappa} \text{If}\left[\frac{1}{\sigma} \geq 0, \text{If}\left[\frac{\kappa}{1+\kappa} \neq 0, \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1+\kappa)(1+\frac{0 \kappa}{1+\kappa})}} - 1}{\frac{\kappa}{1+\kappa}}, \text{Log}\left[\frac{1}{\sigma}\right]\right], \text{Undefined}\right]\right)\right]^{\frac{1+1 \kappa}{\kappa}}, \text{Exp}\left[1 - \frac{1}{1+\kappa} \text{If}\left[\frac{1}{\sigma} \geq 0, \text{If}\left[\frac{\kappa}{1+\kappa} \neq 0, \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1+\kappa)(1+\frac{0 \kappa}{1+\kappa})}} - 1}{\frac{\kappa}{1+\kappa}}, \text{Log}\left[\frac{1}{\sigma}\right]\right], \text{Undefined}\right]\right], \text{If}\left[\frac{1+1 \kappa}{\kappa} > 0, 0, \infty\right]\right]}},$$

$$\text{In[5]:= HNormTsallis}[\sigma, \kappa] := 1 + (1 + \kappa) \text{CoupledLogarithm}\left[\sigma, \frac{\kappa}{1+\kappa}, 0\right] + \kappa;$$

In[6]:= **entValue**[**HNormTsallis**[σ , κ], σ , κ]

Out[6]=

$$\frac{\sigma^{-1+\frac{2}{1+\kappa}} \left((1 + \kappa)^2 - \kappa \sigma^{\frac{\kappa}{1+\kappa}} - \sigma^{\frac{2 \kappa}{1+\kappa}}\right)}{\kappa}$$

```

In[8]:= entDensity[HNormTsallis[\sigma, \kappa], \kappa]
Out[8]=

$$\frac{1}{\kappa} \text{If}\left[(1 + \kappa)^2 > \kappa \sigma^{\frac{\kappa}{1+\kappa}}, \text{If}\left[\kappa \neq 0, \right.\right.$$


$$\left.\left.1 + \kappa \left(1 + \kappa + (1 + \kappa) \text{If}\left[\frac{1}{\sigma} \geq 0, \text{If}\left[\frac{\kappa}{1 + \kappa} \neq 0, \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1+\kappa)(1+\frac{0}{1+\kappa})}} - 1}{\frac{\kappa}{1+\kappa}}, \text{Log}\left[\frac{1}{\sigma}\right]\right], \text{Undefined}\right]\right)\right]^{\frac{1+1/\kappa}{\kappa}},$$


$$\text{Exp}\left[1 + \kappa + (1 + \kappa) \text{If}\left[\frac{1}{\sigma} \geq 0, \text{If}\left[\frac{\kappa}{1 + \kappa} \neq 0, \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1+\kappa)(1+\frac{0}{1+\kappa})}} - 1}{\frac{\kappa}{1+\kappa}}, \text{Log}\left[\frac{1}{\sigma}\right]\right], \text{Undefined}\right]\right],$$


$$\text{If}\left[\frac{1 + 1/\kappa}{\kappa} > 0, 0, \infty\right]$$


In[9]:= Module[{EntData, \sigmaPlotBottom = 0.5, \sigmaPlotTop = 2,
  \kappaValues = {1/2, 1, 2}},
  EntData = Table[
    Table[{entValue[entName[\sigmaPlot], \kappaPlot], \sigmaPlot, \kappaPlot},
      entDensity[entName[\sigmaPlot], \kappaPlot], \kappaPlot}],
    {\sigmaPlot, {\sigmaPlotBottom, \sigmaPlotTop}}, {\kappaPlot, \kappaValues}],
  {entName, {HCoupled, HBGS, HRenyi, HTsallis, HNormTsallis}}];
ResourceFunction["PlotGrid"][
{{Plot[
  Table[PDF[CoupledExponentialDistribution[\sigmaPlotTop, \kappaPlot], x],
    {\kappaPlot, \kappaValues}]
  , {x, 0, 7.5},
  PlotRange \rightarrow {{0, 5}, {0, 0.55}},
  Frame \rightarrow {{True, True}, {True, True}},
  FrameStyle \rightarrow Directive[Bold, Medium],
  Epilog \rightarrow {
    {Dashed, Gray, Line[{{\sigmaPlotTop, 0}, {\sigmaPlotTop, 0.6}}]},
    PointSize[Large],
    {Black, Point[EntData[[1, 2]]]},
    {Blue, Point[EntData[[2, 2]]]},
    {Brown, Point[EntData[[3, 2]]]},
    {Magenta, Point[EntData[[4, 2]]]},
    {Orange, Point[EntData[[5, 2]]]},
    Inset[Style["Required Solution",
      Medium], {2.8, 0.32}],
    Inset[Style["= Coupled Entropy",
      Medium], {3.0, 0.28}],
    Inset[Style["BGS", Blue, Medium], {3.0, 0.12}]}]}]}

```



```

        Inset[Style["b)", Black, Medium],
        Scaled[{0.03, 0.95}]]
    }
]}}},
Spacings → 20,
FrameLabel → {"x", "Density, f(x)" },
PlotLabel → "Entropies Mapped to Density"
]
]

```

•• OptionValue: Unknown option DefaultBaseStyle for Graphics.

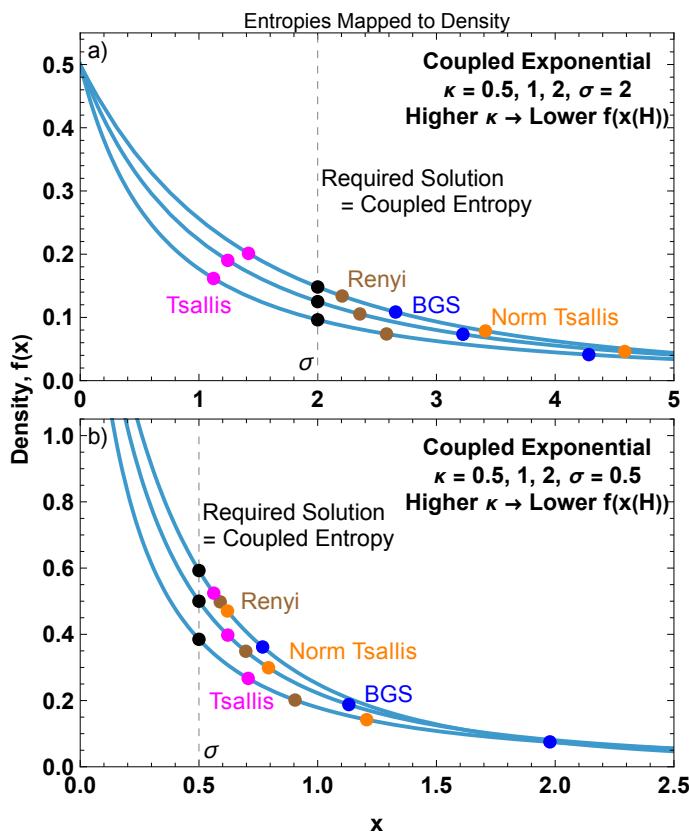
•• OptionValue: Unknown option DefaultBaseStyle for Graphics.

•• OptionValue: Unknown option DefaultBaseStyle for Graphics.

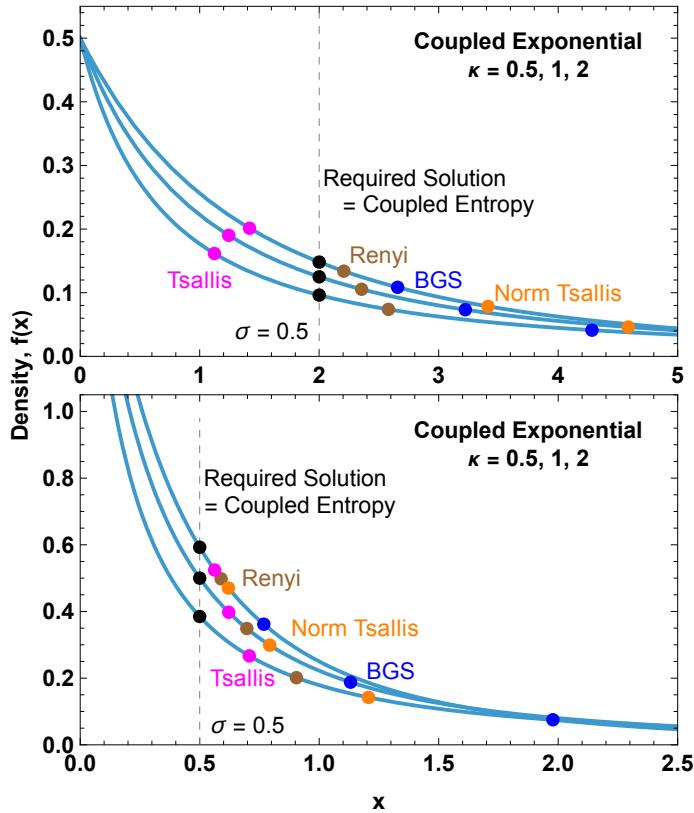
•• General: Further output of OptionValue::nodef will be suppressed during this calculation.



Out[•]=



Saved Plots



Moments of Coupled Weibull distribution

Keeping $1+\alpha \kappa$ term

$$\text{In[}]:= \int x^\alpha \frac{\alpha}{\sigma} \left(\frac{x}{\sigma}\right)^{\alpha-1} \text{Exp}\left[-(1 + \alpha \kappa) \left(\frac{x}{\sigma}\right)^\alpha\right] dx // \text{FullSimplify}$$

$$\text{Out[}]:= -\frac{\sigma^\alpha \text{Gamma}[2, (1 + \alpha \kappa) \left(\frac{x}{\sigma}\right)^\alpha]}{(1 + \alpha \kappa)^2}$$

$$\text{In[}]:= -\frac{\sigma^\alpha \text{Gamma}[2, (1 + \alpha \kappa) \left(\frac{x}{\sigma}\right)^\alpha]}{(1 + \alpha \kappa)^2} / . x \rightarrow 0 // \text{FullSimplify}$$

$$\text{Out[}]:= -\frac{\sigma^\alpha}{(1 + \alpha \kappa)^2}$$

$$\text{In[}]:= -\frac{\sigma^\alpha}{(1 + \alpha \kappa)^2} / . \left\{ \sigma^\alpha \rightarrow \frac{\sigma^\alpha}{1 + \alpha \kappa}, \kappa \rightarrow \frac{\kappa}{1 + \alpha \kappa} \right\} // \text{FullSimplify}$$

$$\text{Out[}]:= -\frac{(1 + \alpha \kappa) \sigma^\alpha}{(1 + 2 \alpha \kappa)^2}$$

Removing $1+\alpha \kappa$ term

```

In[5]:= Integrate[x^α/(x/σ)^(α-1) Exp[-(x/σ)^α], x] // FullSimplify
Out[5]= -σ^α Gamma[2, (x/σ)^α]

In[6]:= -σ^α Gamma[2, (x/σ)^α] /. x → 0 // FullSimplify
Out[6]= -σ^α

In[7]:= -σ^α/(1 + α κ) Gamma[2, (1 + α κ) (x/σ)^α] /. x → 0 // FullSimplify
Out[7]= -σ^α/(1 + α κ)

In[8]:= Integrate[x^{-1/κ}-1, x]
Out[8]= -x^{-1/κ} κ

In[9]:= Limit[-x^{-1/κ} κ, x → ∞]
Out[9]= Limit[-x^{-1/κ} κ, x → ∞]

••• Limit: Warning: Assumptions that involve the limit variable are ignored.

```

Fluctuation Model

 **Syntax:** Incomplete expression; more input is needed.

```

Out[=] = Integrate[Exp[-t/σ] PDF[GammaDistribution[1/κ, σ], t], {t, 0, ∞}]

In[=] := Integrate[c Exp[-c t] PDF[GammaDistribution[1/κ, β], c], {c, 0, ∞}]

Out[=] = 
$$\frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta} + t \kappa\right)^{-1/\kappa}}{1+t \beta \kappa}$$

Integrate[c e^{-c t} \left( \begin{array}{ll} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c > 0 \\ 0 & \text{True} \end{array} \right), \{c, 0, \infty\}, \text{Assumptions} \rightarrow d \in \mathbb{Z} \& \& x \in \mathbb{R} \& \& \alpha \in \mathbb{R} \& \& \kappa \in \mathbb{R} \& \& \mu \in \mathbb{R} \& \& \sigma \in \mathbb{R} \& \& \kappa > 0 \& \& \sigma > 0 \& \& \alpha > 0 \& \& d > 0 \& \& \text{Re}\left[t + \frac{1}{\beta \kappa}\right] \leq 0]

```

$$\text{In}[0]:= \frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta} + t \kappa\right)^{-1/\kappa}}{1+t \beta \kappa}$$

$$\text{Integrate}\left[c e^{-c t} \left(\begin{array}{ll} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c > 0 \\ 0 & \text{True} \end{array} \right), \{c, 0, \infty\}, \text{Assumptions} \rightarrow d \in \mathbb{Z} \& x \in \mathbb{R} \& \alpha \in \mathbb{R} \& \kappa \in \mathbb{R} \& \mu \in \mathbb{R} \& \sigma \in \mathbb{R} \& \kappa > 0 \& \sigma > 0 \& \alpha > 0 \& d > 0 \& \text{Re}\left[t + \frac{1}{\beta \kappa}\right] \leq 0\right]$$

$\text{Re}\left[t + \frac{1}{\beta \kappa}\right] > 0$
True

/ FullSimplify

$$\text{Out}[0]=$$

$$\frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta} + t \kappa\right)^{-1/\kappa}}{1+t \beta \kappa}$$

$$\text{Integrate}\left[c e^{-c t} \left(\begin{array}{ll} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c > 0 \\ 0 & \text{True} \end{array} \right), \{c, 0, \infty\}, \text{Assumptions} \rightarrow d \in \mathbb{Z} \& x \in \mathbb{R} \& \alpha \in \mathbb{R} \& \kappa \in \mathbb{R} \& \mu \in \mathbb{R} \& \sigma \in \mathbb{R} \& \kappa > 0 \& \sigma > 0 \& \alpha > 0 \& d > 0 \& \text{Re}\left[t + \frac{1}{\beta \kappa}\right] \leq 0\right]$$

$\text{Re}\left[t + \frac{1}{\beta \kappa}\right] > 0$
True

$$\beta^1 (1 + t \beta \kappa)^{-1/\kappa-1}$$

$$\text{In}[0]:= \int_0^\infty c \text{Exp}[-c t^2] \text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa}, \kappa \beta\right], c\right] dc$$

$$\text{Out}[0]=$$

$$\frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta} + t^2 \kappa\right)^{-1/\kappa}}{1+t^2 \beta \kappa}$$

$$\text{Integrate}\left[c e^{-c t^2} \left(\begin{array}{ll} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c > 0 \\ 0 & \text{True} \end{array} \right), \{c, 0, \infty\}, \text{Assumptions} \rightarrow d \in \mathbb{Z} \& x \in \mathbb{R} \& \alpha \in \mathbb{R} \& \kappa \in \mathbb{R} \& \mu \in \mathbb{R} \& \sigma \in \mathbb{R} \& \kappa > 0 \& \sigma > 0 \& \alpha > 0 \& d > 0 \& \text{Re}\left[t^2 + \frac{1}{\beta \kappa}\right] \leq 0\right]$$

$\text{Re}\left[t^2 + \frac{1}{\beta \kappa}\right] > 0$
True

$$\text{In}[1]:= \int_0^\infty c \text{Exp}[-c^2 t^2] \text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa}, \kappa \beta^2\right], c\right] dc$$

$$\text{Out}[1]= -\frac{1}{4 \text{Gamma}\left[\frac{1}{\kappa}\right]} \left(t^2\right)^{-\frac{3}{2}-\frac{1}{2 \kappa}} \left(\beta^2\right)^{-1-\frac{1}{\kappa}} \kappa^{-2-\frac{1}{\kappa}} \left(\text{Re}\left[t^2\right] \geq 0 \& \& \text{Re}\left[\beta^2\right] > 0\right) \mid\mid \left(\text{Re}\left[t^2\right] > 0\right)$$

$$\left(\sqrt{t^2} \text{Gamma}\left[\frac{1}{2 \kappa}\right] \text{Hypergeometric1F1}\left[1+\frac{1}{2 \kappa}, \frac{3}{2}, \frac{1}{4 t^2 \beta^4 \kappa^2}\right]-2 t^2 \beta^2 \kappa^2 \text{Gamma}\left[\frac{1+\kappa}{2 \kappa}\right] \text{Hypergeometric1F1}\left[\frac{1+\kappa}{2 \kappa}, \frac{1}{2}, \frac{1}{4 t^2 \beta^4 \kappa^2}\right]\right)$$

$$\begin{cases} \text{Integrate}\left[c e^{-c^2 t^2} \left(\frac{\frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta^2 \kappa}} (\beta^2 \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} \quad c > 0}{0} \right), \{c, 0, \infty\}, \text{True}\right] & \text{True} \\ \left(c e^{-c^2 t^2} \left(\frac{\frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta^2 \kappa}} (\beta^2 \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} \quad c > 0}{0} \right), \{c, 0, \infty\}, \text{True}\right) & \text{True} \end{cases}$$

Assumptions \rightarrow $(d \in \mathbb{Z} \&& x \in \mathbb{R} \&& \alpha \in \mathbb{R} \&& \kappa \in \mathbb{R} \&&$
 $\mu \in \mathbb{R} \&& \sigma \in \mathbb{R} \&& \text{Re}[t^2] \leq 0 \&& d > 0 \&&$
 $\alpha > 0 \&& \kappa > 0 \&& \sigma > 0 \&& \text{Re}[\beta^2] \leq 0) \mid\mid$
 $(d \in \mathbb{Z} \&& x \in \mathbb{R} \&& \alpha \in \mathbb{R} \&& \kappa \in \mathbb{R} \&& \mu \in \mathbb{R} \&& \sigma \in \mathbb{R} \&&$
 $\text{Re}[t^2] < 0 \&& d > 0 \&& \alpha > 0 \&& \kappa > 0 \&& \sigma > 0)$

$$\text{In}[2]:= \int_\infty^0 \frac{1}{c} \text{Exp}\left[\frac{-1}{c} t^2\right] \text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa}, \frac{\kappa}{\sigma^2}\right], \frac{1}{c}\right] dl\left(\frac{1}{c}\right)$$

••• Integrate: Invalid integration variable or limit (s) in $\left\{\frac{1}{c}, \infty, 0\right\}$. [i](#)

$$\text{Out}[2]= \int_\infty^0 \frac{e^{-\frac{t^2}{c}} \left(\frac{\left(\frac{1}{c}\right)^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\kappa}} \left(\frac{\kappa}{\sigma^2}\right)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} \quad \frac{1}{c} > 0}{c} dl\left(\frac{1}{c}\right)$$

$$\text{In}[3]:= \int_0^\infty c \text{Exp}[-c t^\alpha] \text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa}, \kappa \beta\right], c\right] dc$$

$$\text{Out}[3]= \frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta}+t^\alpha \kappa\right)^{-1/\kappa}}{1+t^\alpha \beta \kappa} \begin{cases} \text{Integrate}\left[c e^{-c t^\alpha} \left(\frac{\frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} \quad c > 0}{0} \right), \{c, 0, \infty\}, \text{True}\right] & \text{True} \\ \text{Assumptions} \rightarrow d \in \mathbb{Z} \&& x \in \mathbb{R} \&& \alpha \in \mathbb{R} \&& \kappa \in \mathbb{R} \&& \mu \in \mathbb{R} \&& \sigma \in \mathbb{R} \&& \kappa > 0 \&& \sigma > 0 \&& \alpha > 0 \&& d > 0 \&& \text{Re}\left[t^\alpha + \frac{1}{\beta \kappa}\right] \leq 0 & \text{True} \end{cases}$$

$$\text{In}[4]:= \int_0^\infty \text{Exp}[-c t^\alpha] \text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa}+1, \frac{\kappa}{1+\kappa} \beta^\alpha\right], c\right] dc$$

Out[4]= \$Aborted

Complexity Measurement

Determine the Taylor Series expansion of the result

```
In[1]:= EntRatioExp =  $\frac{1 + \text{Log}[\sigma(1 - \kappa)]}{1 + \text{Log}[\sigma]}$ 
Out[1]=  $\frac{1 + \text{Log}[(1 - \kappa)\sigma]}{1 + \text{Log}[\sigma]}$ 
In[2]:= Series[EntRatioExp (1 - EntRatioExp), {\kappa, 0, 3},
Assumptions → (0 < \kappa < 1)]
Out[2]=  $\frac{\kappa}{1 + \text{Log}[\sigma]} + \frac{(-1 + \text{Log}[\sigma])\kappa^2}{2(1 + \text{Log}[\sigma])^2} + \frac{(-2 + \text{Log}[\sigma])\kappa^3}{3(1 + \text{Log}[\sigma])^2} + O[\kappa]^4$ 
In[3]:= Series[- $\left(1 + \frac{\text{Log}[1 - \kappa]}{\text{Log}[\sigma]}\right) \left(\frac{\text{Log}[1 - \kappa]}{\text{Log}[\sigma]}\right)$ , {\kappa, 0, 3},
Assumptions →  $\left(0 < \kappa < \frac{1}{2}\right)$ ]
Out[3]=  $\frac{\kappa}{\text{Log}[\sigma]} + \frac{(-2 + \text{Log}[\sigma])\kappa^2}{2\text{Log}[\sigma]^2} + \frac{(-3 + \text{Log}[\sigma])\kappa^3}{3\text{Log}[\sigma]^2} + O[\kappa]^4$ 
```

First order approximation of the entropy of the coupled Gaussian with $\sigma=1$

```
In[4]:= Series[ $\frac{1 + \kappa}{2\kappa} \left(\text{PolyGamma}\left[\frac{1 + \kappa}{2\kappa}\right] - \text{PolyGamma}\left[\frac{1}{2\kappa}\right]\right) +$ 
 $\text{Log}\left[\frac{1}{\sqrt{\kappa}} \text{Beta}\left[\frac{1}{2\kappa}, \frac{1}{2}\right]\right]$ , {\kappa, 0, 3}] // FullSimplify
Out[4]=  $\frac{1}{2} (1 + \text{Log}[2] + \text{Log}[\pi]) + \kappa + \frac{\kappa^2}{4} - \frac{\kappa^3}{6} + O[\kappa]^{7/2}$ 
```

Taylor Series of Complexity of Coupled Gaussian

```
In[5]:= EntRatioGauss =  $\frac{\frac{1}{2} + \frac{1}{2} \text{Log}[2\pi\sigma\sqrt{1 - 2\kappa}]}{\frac{1}{2} + \frac{1}{2} \text{Log}[2\pi\sigma]}$ 
Out[5]=  $\frac{\frac{1}{2} + \frac{1}{2} \text{Log}[2\pi\sqrt{1 - 2\kappa}\sigma]}{\frac{1}{2} + \frac{1}{2} \text{Log}[2\pi\sigma]}$ 
In[6]:= Series[EntRatio (1 - EntRatioGauss), {\kappa, 0, 3}]
Out[6]=  $\frac{\kappa}{1 + \text{Log}[2\pi\sigma]} + \frac{\text{Log}[2\pi\sigma]\kappa^2}{(1 + \text{Log}[2\pi\sigma])^2} + \frac{2(-1 + 2\text{Log}[2\pi\sigma])\kappa^3}{3(1 + \text{Log}[2\pi\sigma])^2} + O[\kappa]^4$ 
```

Relation to c,d-entropy

Asymptotic Classification

J. M. Amigó, S. G. Balogh, and S. Hernández, “A Brief Review of Generalized Entropies,” Entropy, vol. 20, no. 11, p. 813, Nov. 2018, doi: 10.3390/e20110813.

4. Hanel–Thurner Exponents

All generalized entropies $F_{G,g}$ group in classes labeled by two exponents (c, d) introduced by Hanel and Thurner [16], which are determined by the limits

$$\lim_{W \rightarrow \infty} \frac{F_{G,g}(p_1, \dots, p_{\lambda W})}{F_{G,g}(p_1, \dots, p_W)} = \lambda^{1-c} \quad (25)$$

(W being as before the cardinality of the probability distribution or the total number of microstates in the system, $\lambda > 1$) and

$$\lim_{W \rightarrow \infty} \frac{F_{G,g}(p_1, \dots, p_{W^{1+a}})}{F_{G,g}(p_1, \dots, p_W)} W^{a(c-1)} = (1+a)^d \quad (26)$$

($a > 0$). Note that the limit in Equation (26) does not depend actually on c . The limits in Equations (25) and (26) can be computed via the asymptotic equipartition property [26]. Thus,

$$F_{G,g}(p_1, \dots, p_{\lambda W}) \approx G\left(\lambda W g\left(\frac{1}{\lambda W}\right)\right)$$

and

$$F_{G,g}(p_1, \dots, p_{W^{1+a}}) \approx G\left(W^{1+a} g\left(\frac{1}{W^{1+a}}\right)\right)$$

(E1) For the BGS entropy, $g(x) = -x \ln x$ (see Equation (11)), so

$$\frac{g(zx)}{g(x)} = \frac{zx \ln(zx)}{x \ln x} = \frac{z \ln z + z \ln x}{\ln x} \rightarrow z$$

as $x \rightarrow 0+$. Therefore, $c = 1$. Furthermore,

$$\frac{g(x^{1+a})}{x^a g(x)} = \frac{x^{1+a} \ln x^{1+a}}{x^{a+1} \ln x} = \frac{(1+a) \ln x}{\ln x} = 1+a$$

for all $x > 0$, so $d = 1$.

(E2) For the Tsallis entropy, see Equation (12),

$$g(x) = \begin{cases} \frac{1}{1-q} x^q + \mathcal{O}(x) & \text{if } 0 < q < 1, \\ -\frac{1}{1-q} x + \mathcal{O}(x) & \text{if } q > 1. \end{cases}$$

It follows readily that $(c, d) = (q, 0)$ if $0 < q < 1$, and $(c, d) = (1, 0)$ if $q > 1$. Hence, although $\lim_{q \rightarrow 1} T_q = S_{BGS}$, there is no parallel convergence concerning the HT exponents.

(E3) For the Rényi entropy, $g(x) = x^q$ and $G(u) = \frac{1}{1-q} e^u$ (see Equation (15)), so

$$\frac{G(\frac{\lambda}{x} g(\frac{x}{\lambda}))}{G(\frac{1}{x} g(x))} = \frac{\ln \left(\frac{\lambda}{x} (\frac{x}{\lambda})^q \right)}{\ln \left(\frac{1}{x} x^q \right)} = \frac{\ln x^{q-1} - \ln \lambda^{q-1}}{\ln x^{q-1}} \rightarrow 1$$

For Coupled Entropy, consider that for $p_i = \frac{1}{W}$, the independent equals distribution also equals p_i . Then $g(x) = x \ln_{\alpha \kappa} x^{-\frac{1}{1+\kappa}} = \frac{x}{\alpha \kappa} x^{-\alpha \frac{\kappa}{1+\kappa}} + \mathcal{O}(x) = \frac{1}{\alpha \kappa} x^{1-\alpha \frac{\kappa}{1+\kappa}} + \mathcal{O}(x) = \frac{1}{\alpha \kappa} x^{2-\kappa} + \mathcal{O}(x)$ and $G(u) = u^{\frac{1}{\kappa}}$.

$$\begin{aligned} \text{In[} &:= g[r_-, \kappa_-, \alpha_-, d_-] := \frac{1}{\alpha \kappa} r^{1-\frac{\alpha \kappa}{1+d \kappa}}; \\ &G[u_-, \gamma_-] := u^{\frac{1}{\kappa}}; \\ \text{In[} &:= \text{Limit}\left[\frac{G[\lambda W g[\frac{1}{\lambda W}, \kappa, \alpha, d], \gamma]}{G[W g[\frac{1}{W}, \kappa, \alpha, d], \gamma]}, W \rightarrow \infty\right] \end{aligned}$$

Out[] =

$$\left(\left(\frac{1}{\lambda} \right)^{-\frac{\alpha \kappa}{1+d \kappa}} \right)^{\frac{1}{\gamma}} \quad \text{if } \text{condition} \oplus$$

$$\left(\lambda^{\frac{\alpha \kappa / \gamma}{1+d \kappa}} \right) = \lambda^{1-c}$$

$$c = 1 - \frac{\alpha \kappa / \gamma}{1+d \kappa}$$

$$\text{In}[0]:= \text{Limit}\left[\frac{\text{G}\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{\text{G}\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a \frac{\alpha \kappa / \gamma}{1+d \kappa}}, W \rightarrow \infty\right]$$

Out[0]=

$$1 \text{ if } \frac{1}{\gamma} \in \mathbb{R} \text{ && } a > -1 \text{ && } d \kappa > -1 \text{ && } \alpha \kappa > 0$$

Thus for $(1+a)^d = 1$, $d=0$.

$$\text{In}[0]:= \frac{\text{G}\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{\text{G}\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a \frac{\alpha \kappa / \gamma}{1+d \kappa}} // \text{FullSimplify}$$

Out[0]=

$$\left(\left(\frac{1}{W}\right)^{-\frac{\alpha \kappa}{1+d \kappa}}\right)^{-1/\gamma} W^{-\frac{a \alpha \kappa}{\gamma+d \gamma \kappa}} \left(\left(W^{-1-a}\right)^{-\frac{\alpha \kappa}{1+d \kappa}}\right)^{\frac{1}{\gamma}}$$

$$\left(W^{-\frac{\alpha \kappa}{(1+d \kappa) \gamma}}\right) W^{-\frac{a \alpha \kappa}{\gamma+d \gamma \kappa}} W^{\frac{(1+a) \alpha \kappa}{(1+d \kappa) \gamma}}$$

$$\text{In}[0]:= -\frac{\alpha \kappa}{(1+d \kappa) \gamma} - \frac{a \alpha \kappa}{\gamma+d \gamma \kappa} + \frac{(1+a) \alpha \kappa}{(1+d \kappa) \gamma} // \text{FullSimplify}$$

Out[0]=

$$0$$

$$\text{In}[0]:= \text{Limit}\left[\frac{\text{G}\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{\text{G}\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a (1-c)}, W \rightarrow \infty\right]$$

Out[0]=

$$\infty \text{ if } \text{condition} +$$

$$\text{In}[0]:= \text{Limit}\left[\frac{\text{G}\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{\text{G}\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a (1-1)}, W \rightarrow \infty\right]$$

Out[0]=

$$\infty \text{ if } a > 0$$

$$\text{In}[0]:= \text{Limit}\left[\frac{\text{G}\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{\text{G}\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a (1-0)}, W \rightarrow \infty\right]$$

Out[0]=

$$\infty \text{ if } a > -1 \text{ && } a \neq 0 \text{ && } a \gamma (1+d \kappa) < a \alpha \kappa$$

What if the full form of the generalized logarithm is used?

For Coupled Entropy, consider that for $p_i = \frac{1}{W}$, the independent equals distribution also equals p_i . Then

$$g(x) = x \ln_{\alpha \kappa} x^{-\frac{1}{1+\kappa}} = \frac{x}{\alpha \kappa} \left(x^{-\alpha \frac{\kappa}{1+\kappa}} - 1\right)$$

$$\text{In}[0]:= g[r_, \kappa_, \alpha_, d_]:= \frac{r}{\alpha \kappa} \left(r^{-\frac{\alpha \kappa}{1+d \kappa}} - 1\right);$$

$$G[u_, \gamma_]:= u^{\frac{1}{\gamma}};$$

```

In[]:= Limit[ $\frac{G[\lambda W g[\frac{1}{\lambda W}, \kappa, \alpha, d], \gamma]}{G[W g[\frac{1}{W}, \kappa, \alpha, d], \gamma]}$  // FullSimplify, W → ∞]

Out[]=  $\left( \left( \frac{1}{\lambda} \right)^{-\frac{\alpha \kappa}{1+d \kappa}} \right)^{\frac{1}{\gamma}}$ 

In[]:= Limit[ $\frac{G[W^{1+a} g[\frac{1}{W^{1+a}}, \kappa, \alpha, d], \gamma]}{G[W g[\frac{1}{W}, \kappa, \alpha, d], \gamma]}$  W-a  $^{\frac{\alpha \kappa / \gamma}{1+d \kappa}}$  // FullSimplify, W → ∞]

Out[=] 1 if a > 1

In[]:=  $\frac{G[W^{1+a} g[\frac{1}{W^{1+a}}, \kappa, \alpha, d], \gamma]}{G[W g[\frac{1}{W}, \kappa, \alpha, d], \gamma]} W^{-a}^{\frac{\alpha \kappa / \gamma}{1+d \kappa}}$  // FullSimplify

Out[=]  $\left( -1 + \left( \frac{1}{W} \right)^{-\frac{\alpha \kappa}{1+d \kappa}} \right)^{-1/\gamma} W^{-\frac{a \alpha \kappa}{\gamma(1+d \kappa)}} \left( -1 + (W^{-1-a})^{-\frac{\alpha \kappa}{1+d \kappa}} \right)^{\frac{1}{\gamma}}$ 

With W → ∞ only the terms with W are significant and all the terms contain  $\frac{\alpha \kappa}{\gamma(1+d \kappa)}$ . Thus exponent multiplies this by (1 - a - 1 - a) = 0, which is why the expression converges to 1. But if κ → 0 first, then the -1 term is of significance in the terms converging to log Wa. Set r =  $\frac{\alpha \kappa}{(1+d \kappa)}$ , and factor out the 1+a from the coupled logarithm, then the ratio is  $\frac{(1+a) \ln_{-r}(1+a) W}{\ln_{-r} W}$ . The question is under what circumstance does this ratio converge to 1 + a? Certainly when r → 0 first. What about when r → 1, which is its other limit? Then LimW→∞  $\frac{(1+a)(W^{-(1+a)}-1)}{(1+a)(W^{-1}-1)} = 1$ .

```

Try using the form $e^{W^{1+a}}$ and see what the scaling is

```

In[]:= Limit[ $\frac{G[\text{Exp}[W^{1+a}] g[\frac{1}{\text{Exp}[W^{1+a}]}, \kappa, \alpha, d], \gamma]}{G[\text{Exp}[W] g[\frac{1}{\text{Exp}[W]}, \kappa, \alpha, d], \gamma]} \text{Exp}[W^{-a}^{\frac{\alpha \kappa / \gamma}{1+d \kappa}}]$ , W → ∞]

Out[=]  $\lim_{W \rightarrow \infty} \frac{e^{W^{-\frac{a \alpha \kappa}{\gamma(1+d \kappa)}}} G[e^{W^{1+a}} g[e^{-W^{1+a}}, \kappa, \alpha, d], \gamma]}{G[e^W g[e^{-W}, \kappa, \alpha, d], \gamma]}$ 

In[]:= Limit[ $\frac{G[\text{Exp}[W^a] g[\frac{1}{\text{Exp}[W^a]}, \kappa, \alpha, d], \gamma]}{G[\text{Exp}[W] g[\frac{1}{\text{Exp}[W]}, \kappa, \alpha, d], \gamma]}$ , W → ∞]

Out[=]  $\lim_{W \rightarrow \infty} \frac{G[e^{W^a} g[e^{-W^a}, \kappa, \alpha, d], \gamma]}{G[e^W g[e^{-W}, \kappa, \alpha, d], \gamma]}$ 

```

Taylor series of coupled exponential function

```
In[6]:= Series[CoupledExponential[x, κ, 1], {x, x0, 4}]
Out[6]=
∞
(x ∈ ℝ && -1 ≤ κ < 0 && x0 κ < -1) || 
(x0 ∈ ℝ && κ ≠ 0 && x0 κ == -1 &&
κ ≤ -1 && -1 ≤ κ < 0)
x0 ∈ ℝ && κ ≠ 0 && x0 κ == -1 &&
(κ > 0 && x κ > -1) || (κ < 0 && x κ > -1))
(x | x0) ∈ ℝ && κ == 0
((κ > 0 && x κ > -1) || (κ < 0 && x κ > -1))
x ∈ ℝ &&
((κ > 0 && x0 κ > -1) || (κ < 0 && x0 κ > -1))
0
True
```

Taylor series of cdFunction

Starting with the 0th branch which corresponds to d>0

```
In[7]:= cdFunction[x_, c_, d_, r_] :=
Exp[-d/(1-c) ProductLog[0, B[c, r] (1-x/r)^1/d] -
ProductLog[0, B[c, r]]] /; d > 0;

In[8]:= B[c_, r_] := (1-c)r/(1-(1-c)r) Exp[(1-c)r/(1-(1-c)r)];
Series[ProductLog[0, z], {z, z0, 4}]
Out[8]=
ProductLog[z0] + ProductLog[z0] (z - z0)/z0 (1 + ProductLog[z0]) -
((ProductLog[z0]^2 (2 + ProductLog[z0])) (z - z0)^2)/
2 (z0^2 (1 + ProductLog[z0])^3) +
ProductLog[z0]^3 (9 + 8 ProductLog[z0] + 2 ProductLog[z0]^2) (z - z0)^3/
6 z0^3 (1 + ProductLog[z0])^5 +
(ProductLog[z0]^4 (-64 - 79 ProductLog[z0] - 36 ProductLog[z0]^2 - 6 ProductLog[z0]^3) (z - z0)^4)/
(24 z0^4 (1 + ProductLog[z0])^7) + O[z - z0]^5
```

In[6]:= Series[cdfFunction[x, c, d, r], {x, x0, 4}]

Out[6]=

$$\begin{aligned}
 & \frac{d}{e} \left(-\text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r}{1+(-1+c) r} \right] + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right) - \\
 & \left(\frac{d}{e} \left(-\text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r}{1+(-1+c) r} \right] + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right) \right. \\
 & \quad \left. \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right) \\
 & \left((x - x_0) \right) / \left((-1+c) (r - x_0) \left(1 + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right) \right) - \\
 & \left(\frac{d}{e} \left(-\text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r}{1+(-1+c) r} \right] + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right) \right. \\
 & \quad \left. \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right) \\
 & \left(1 - c - d + c d - 3 d \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + 2 c d \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right. \\
 & \quad \left. - \frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right) - 2 d \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \\
 & \quad c d \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 \right) \left(x - x_0 \right)^2 \Bigg) / \\
 & \left(2 \left((-1+c)^2 d (r - x_0)^2 \left(1 + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right)^3 \right) \right)
 \end{aligned}$$

$$\begin{aligned}
& \left(\frac{\left(\frac{d \left[-\text{ProductLog} \left[\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r}{1+(-1+c) r} \right] + \text{ProductLog} \left[\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]} \right]}{-1+c} \right. \\
& \left. \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right) \left(1 - 2 c + c^2 - 3 d + 6 c d - 3 c^2 d + \right. \\
& \left. 2 d^2 - 4 c d^2 + 2 c^2 d^2 - 2 \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + \right. \\
& \left. 4 c \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - 2 c^2 \text{ProductLog} \left[\right. \right. \\
& \left. \left. -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - 9 d \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + \right. \\
& \left. 15 c d \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - 6 c^2 d \text{ProductLog} \left[\right. \right. \\
& \left. \left. -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + 11 d^2 \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - \right. \\
& \left. 19 c d^2 \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + 8 c^2 d^2 \text{ProductLog} \left[\right. \right. \\
& \left. \left. -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - 6 d \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \right. \\
& \left. 9 c d \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 - \right. \\
& \left. 3 c^2 d \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \right. \\
& \left. 22 d^2 \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 - \right. \\
& \left. 33 c d^2 \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \right. \\
& \left. 12 c^2 d^2 \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \right)
\end{aligned}$$

$$\begin{aligned}
& 19 d^2 \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^3 - \\
& 25 c d^2 \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^3 + \\
& 8 c^2 d^2 \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^3 + \\
& 6 d^2 \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^4 - \\
& 7 c d^2 \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^4 + \\
& 2 c^2 d^2 \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^4 \Bigg) (x-x \theta)^3 \Bigg) / \\
& \left(6 \left((-1+c)^3 d^2 (r-x \theta)^3 \left(1 + \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right)^5 \right) \right) + \\
& \left(\frac{d \left[-\operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r}{1+(-1+c) r} \right] + \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right]}{-1+c} \right. \\
& \left. \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right. \\
& \left. \left(-(-1+c)^3 (-1+6 d-11 d^2+6 d^3) - (-1+c)^2 (-1+d) \right. \right. \\
& \left. \left. (8+15 d-47 d^2+4 c (-2-2 d+9 d^2)) \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - \right. \right. \\
& \left. \left. (-1+c) (-1+d) (6+25 d+151 d^2+6 c^2 (1+4 d+15 d^2)) - c (12+49 d+235 d^2) \right) \right. \\
& \left. \operatorname{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x \theta}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \right. \\
& \left. d (c (52+252 d-604 d^2)+c^3 (12+44 d-120 d^2)+5 (-4-22 d+51 d^2)) + \right.
\end{aligned}$$

$$\begin{aligned}
& 2 c^2 (-22 - 93 d + 235 d^2) \left(\text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}} {1 + (-1+c) r} \right]^3 + \right. \\
& d^2 (-35 + c (75 - 526 d) + c^3 (11 - 90 d) + 239 d + c^2 (-51 + 380 d)) \\
& \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}} {1 + (-1+c) r} \right]^4 + \\
& (118 - 242 c + 163 c^2 - 36 c^3) d^3 \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}} {1 + (-1+c) r} \right]^5 + \\
& \left. \left(24 - 46 c + 29 c^2 - 6 c^3 \right) d^3 \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}} {1 + (-1+c) r} \right]^6 \right) (x - x_0)^4 \Bigg) \\
& 24 (-1+c)^4 d^3 (r - x_0)^4 \left(1 + \text{ProductLog} \left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1 - \frac{x_0}{r}\right)^{\frac{1}{d}}} {1 + (-1+c) r} \right]^7 \right) + \\
& 0 [x - x_0]^5
\end{aligned}$$