Risk Assessment using Coupled Mean

Introduction

The weighted generalized mean has been used to evaluate probabilistic forecasted. The power of the mean is the degree of relative risk tolerance. The result *Risk Profile* has provided useful insights into the performance of machine learning and other probabilistic algorithms. Two issues need to be addressed with this approach to risk-biased assessment:

1) The risk profile is only 'proper' for $r = 0$. While the local nature of the assessment has been the priority, this does mean that optimization of an algorithm for $r \neq 0$ will be biased. This may not be a problem since the point is to account for relative risk; however, a better understand of the difference between the risk profile and related proper scores would be valuable.

2) A more serious concern is that the risk profile does not fulfil the generalized triangular relationship between cross-entropy (CE), entropy (E) and divergence (D). In the probability domain for $r = 0$, $P_{\text{CE}} = P_E\, P_D$. A related function should hold for $r \neq 0$, $P_{\text{CE}}^r = P_E^r \otimes_r P_D^r$. Unfortunately, the generalized mean using weights of $\frac{1}{N}$ results in the possibility that $\mathrm{P}_{\text{CE}}^r > \mathrm{P}_{\text{E}}^r$ which violates the necessity of $0 \leqslant P_D^r \leqslant 1$.

To address this issue this notebook will explore a definition of the 'Coupled Mean' which follows the theoretical development of the coupled algebra more closely. The derivation of the generalized mean from the generalized cross-entropy relied on the fact that probability of a test sample is $\frac{1}{N}$. Because all the samples have the same probability the role of the escort or coupled Probability cancelled out. If instead the probability of a sample is treated as a function of the outcome probability from the histogram analysis, then the coupled Probability would be an important factor.

For a set $\{1, 2, \ldots, i, \ldots, N\}$ of test samples with $\{1, 2, \ldots, j, \ldots, M\}$ classes arranged into $\{1, 2, \ldots, k, \ldots, N_{\text{bins}}\}$ bins, the outcome probability for the j^{th}class of the k^{th} bin is $p_{\text{jk}} = \frac{n_{\text{jk}}}{N_k}$, where n_{jk} is the number of j outcomes and $N_k = \sum_{j=1}^{M} n_{\text{jk}}$ is the total outcomes for the k^{th}bin. The weights for computing the coupled mean (also a probability) are determined by normalizing the outcome probabilities over all the samples, $w_{\text{jk}} = \frac{n_{\text{jk}}}{N_k} \frac{N_k}{N} = \frac{n_{\text{jk}}}{N}$.

From (Nelson, 2017) the following relationship is defined for the coupled average uncertainty for forecasts q, here in terms of $r = \frac{-\alpha\kappa}{1+\kappa}$ and referred to as the coupled mean

Frequency and Probability Computation per Bin

For a set $\{1, 2, \ldots, i, \ldots, N\}$ of test events with $\{1, 2, \ldots, j, \ldots, M\}$ classes per event, the probabilities are sorted and arranged into $\{1, 2, \ldots, k, \ldots, K\}$ bins. It is also possible, though not necessary for the algorithm to described in terms of separate bins for each class. For each true event class j^*, i.e. the event that actually happened, the outcome probability for the $j^{*\text{th}}$ class of the k^{th} bin is $p_{j^* k} = \frac{n_{j^* k}}{N_k} \frac{N_k}{N} = \frac{n_{j^* k}}{N}$, where $n_{j^* k}$ is the number of probabilities in the k^{th} bin for the true event j^* and $N_k = \sum_{j=1}^{M} n_{\text{jk}}$ is the total outcomes for the k^{th}bin. The outcome probability is also the weight for the coupled mean computations, $w_{j^* k} = p_{j^* k}$The average forecasted probability (or quoted (q) probability) for each bin and class is computed using the geometric of the forecasts for the true events, $q_{j^* k} = \prod_{i=1}^{N} q_{\text{ij}^* k}{}^{p_{j^* k}}$. (Can this notation be improved? It's an increment through the ij* probability samples in the k^{th} bin rather than all the test events). Use of the weight $p_{j^* k}$ rather than assuming independence of each event with $\frac{1}{N}$changes the composition of the metrics.

Equiprobability, Accuracy, Robustness and Decisiveness Metrics

From (Nelson, 2017) the following relationship is defined for the coupled average uncertainty for forecasts q and outcome frequencies p, in terms of the relative risk aversion $r = \frac{\alpha\kappa}{1+\kappa}$ and referred to as the coupled mean

$$\overline{Q}_r = \left(\frac{\sum_{k=1}^{K} \sum_{j^*=1}^{M} w_{j^* k}^{1+r} q_{j^* k}^{-r}}{\sum_{k=1}^{K} \sum_{j^*=1}^{M} w_{j^* k}^{1+r}}\right)^{-\frac{1}{r}} \text{ and } \overline{P}_r = \left(\frac{\sum_{k=1}^{K} \sum_{j^*=1}^{M} w_{j^* k}^{1+r} p_{j^* k}^{-r}}{\sum_{k=1}^{K} \sum_{j^*=1}^{M} w_{j^* k}^{1+r}}\right)^{-\frac{1}{r}}.$$

The original Risk Assessment was derived from a computation of the Coupled Average Uncertainty of a distribution and then substitute $\frac{1}{N}$for the weight. This derivation is shown in equations (32-33) of (Nelson, 2017). Advancing the metric by accounting for the frequency of outcomes, requires $w_{j^* k} = p_{j^* k}$,

$$\overline{Q}_r = \left(\frac{\sum_{k=1}^{K} \sum_{j^*=1}^{M} p_{j^* k}^{1+r} q_{j^* k}^{-r}}{\sum_{k=1}^{K} \sum_{j^*=1}^{M} p_{j^* k}^{1+r}}\right)^{-\frac{1}{r}} \text{ and } \overline{P}_r = \left(\frac{\sum_{k=1}^{K} \sum_{j^*=1}^{M} p_{j^* k}}{\sum_{k=1}^{K} \sum_{j^*=1}^{M} p_{j^* k}^{1+r}}\right)^{-\frac{1}{r}}$$

Note: This derivation is different than what is implemented below. For this reason, the results below are going to be reproduced using this more precise derivation. The difference arises from the fact that the implementation below started with equation 33 of (Nelson, 2017) and then substituted $p_{j^* k}$ for the weights; however, the more careful derivation started with the cross-entropy rather than the entropy function.

References

(Nelson, et. al. 2017) K. P. Nelson, S. R. Umarov, and M. A. Kon, "On the average uncertainty for systems with nonlinear coupling," Phys. A Stat. Mech. its Appl., vol. 468, pp. 30–43, 2017.

Example Using Precipitation Forecasts

The Import Media Forecast Data Section copied from NOAA Forecasting v6

Follows the functions from Election Assessment with some modification of the names
- prevBin -> outcomeProb; geoBin -> forecastProbs
- prevAll -> geoMeanOutcome; geoAll -> geoMeanForecast; divAll -> divergenceProb

The functions are self-contained within this file

Import Media Forecast Data

The dataset includes 321 forecasts for rain or no rain, one to seven days in advance. Each rain forecast is for a particular percentile {0,5,10,15,20,30,40,50,60,70,80,90,100}. To facilitate analysis the 0% and 100% need to be adjusted. By observation, the actual forecasts were on the order of 0.1% and 99.9% so these will be used initially. Also, the 5% and 15% rain forecasts require the corresponding 95% and 85% no rain percentiles, so these are added as categories.

Data is imported as a 39 by 14 array. Row spaces added to keep data in 10 row segments

Rows 2-9: Station 1 Forecasts
Rows 12-19: Station 1 Rain Days
Rows 22-29: Station 2 Forecasts
Rows 32-39: Station 2 Rain Days

In[91]:=
```
Folder = "/Users/kenricnelson/Documents/Photrek/Business
    Development/Pursuits/NOAA Tornado Forecasting/";
MediaForecastData =
  Import[Folder <> "Precipitation Forecasts.xlsx"]⟦1⟧ /. x_Real → Floor[x];
```

In[93]:=
```
Forecasts =
  {Riffle[MediaForecastData⟦#, 2 ;; 14⟧, 0, {-4, -2, 2}] & /@ Table[i, {i, 3, 9}],
    Riffle[MediaForecastData⟦#, 2 ;; 14⟧, 0, {-4, -2, 2}] & /@ Table[i, {i, 23, 29}]} /.
   x_ /; x == 0 → "NA";
DaysAhead = MediaForecastData⟦3 ;; 9, 1⟧;
Percentiles = Riffle[ReplacePart[MediaForecastData⟦2, 2 ;; 14⟧ // Floor,
    {1 → 1, 13 → 99}], {85, 95}, {12, 14, 2}];
(* Append Baseline as third list *)
AppendTo[Forecasts, Table["NA", 5, 15]];
Forecasts⟦3, 1, 4⟧ = Forecasts⟦3, 2, 5⟧ = Total[MediaForecastData⟦3, 2 ;; 14⟧];
Forecasts⟦3, {3, 4, 5}, 1 ;; 7⟧ = Reverse[Forecasts⟦1, {1, 4, 7}, 1 ;; 7⟧, 2];
Forecasts⟦3, {3, 4, 5}, 8⟧ = Forecasts⟦1, {1, 4, 7}, 8⟧;
Forecasts⟦3, {3, 4, 5}, 9 ;; 15⟧ = Reverse[Forecasts⟦1, {1, 4, 7}, 9 ;; 15⟧, 2];
BaselineLabel = {"B1", "B2", "U1", "U4", "U7"};

Print["Forecasts per Percentile for Station 1"];
TableForm[Forecasts⟦1⟧,
 TableHeadings → {DaysAhead, Percentiles}
]
Print["Forecasts per Percentile for Station 2"]
TableForm[Forecasts⟦2⟧,
 TableHeadings → {DaysAhead, Percentiles}
]
Print["Forecasts per Percentile for Baseline"]
TableForm[Forecasts⟦3⟧,
 TableHeadings → {BaselineLabel, Percentiles}
]
```

Forecasts per Percentile for Station 1

Out[103]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85	90	95
D1	162	1	10	15	37	36	16	12	18	4	4	NA	2	NA
D2	169	NA	4	20	32	42	25	10	14	4	1	NA	NA	NA
D3	176	NA	1	13	53	32	21	16	6	2	NA	NA	NA	NA
D4	172	NA	1	16	51	37	29	13	2	NA	NA	NA	NA	NA
D5	169	NA	2	10	78	39	15	7	1	NA	NA	NA	NA	NA
D6	181	NA	NA	13	71	43	9	3	1	NA	NA	NA	NA	NA
D7	140	NA	NA	16	117	40	8	NA	NA	NA	NA	NA	NA	NA

Forecasts per Percentile for Station 2

Out[105]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85	90	95
D1	170	NA	9	NA	58	28	24	9	10	6	5	NA	1	NA
D2	182	NA	8	NA	67	28	26	5	5	NA	NA	NA	NA	NA
D3	207	NA	7	NA	60	29	15	1	2	NA	NA	NA	NA	NA
D4	193	NA	8	NA	88	21	10	1	NA	NA	NA	NA	NA	NA
D5	202	NA	5	NA	82	24	8	NA	NA	NA	NA	NA	NA	NA
D6	212	NA	3	NA	92	12	2	NA	NA	NA	NA	NA	NA	NA
D7	233	NA	1	NA	80	7	NA	NA	NA	NA	NA	NA	NA	NA

Forecasts per Percentile for Baseline

Out[107]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85	90	95
B1	NA	NA	NA	321	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
B2	NA	NA	NA	NA	321	NA	NA	NA	NA	NA	NA	NA	NA	NA
U1	16	36	37	15	10	1	162	12	4	NA	2	NA	4	4
U4	29	37	51	16	1	NA	172	13	NA	NA	NA	NA	NA	NA
U7	8	40	117	16	NA	NA	140	NA	NA	NA	NA	NA	NA	NA

In[108]:=
```
ForecastRain = {
    Riffle[Quiet[MediaForecastData⟦#, 2 ;; 14⟧ /
          MediaForecastData⟦# - 10, 2 ;; 14⟧],
        "NA", {-4, -2, 2}] & /@ Table[i, {i, 13, 19}] /.
     {Indeterminate → "NA", 0 → 0.01, 1 → 0.99},
    Riffle[Quiet[MediaForecastData⟦#, 2 ;; 14⟧ /
          MediaForecastData⟦# - 10, 2 ;; 14⟧],
        "NA", {-4, -2, 2}] & /@ Table[i, {i, 33, 39}] /.
     {Indeterminate → "NA", 0 → 0.01, 1 → 0.99},
    Table["NA", 5, 15]
   };
ForecastRain⟦3, 1, 4⟧ =
  ForecastRain⟦3, 2, 5⟧ = Total[MediaForecastData⟦13, 2 ;; 14⟧] / Forecasts⟦3, 1, 4⟧;
ForecastRain⟦3, {3, 4, 5}, 1 ;; 7⟧ = Reverse[ForecastRain⟦1, {1, 4, 7}, 1 ;; 7⟧, 2];
ForecastRain⟦3, {3, 4, 5}, 8⟧ = ForecastRain⟦1, {1, 4, 7}, 8⟧;
ForecastRain⟦3, {3, 4, 5}, 9 ;; 15⟧ =
  Reverse[ForecastRain⟦1, {1, 4, 7}, 9 ;; 15⟧, 2];

Print["Fraction of Rain Days for Station 1"];
TableForm[ForecastRain⟦1⟧,
 TableHeadings → {DaysAhead, Percentiles}]
Print["Fraction of Rain Days for Station 2"];
TableForm[ForecastRain⟦2⟧,
 TableHeadings → {DaysAhead, Percentiles}
]
Print["Fraction of Rain Days for Baseline"];
TableForm[ForecastRain⟦3⟧,
 TableHeadings → {BaselineLabel, Percentiles}
]
```

Fraction of Rain Days for Station 1

Out[114]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85
D1	$\frac{5}{162}$	0.01	0.01	$\frac{2}{15}$	$\frac{7}{37}$	$\frac{13}{36}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{11}{18}$	0.99	0.99	NA
D2	$\frac{10}{169}$	NA	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{5}{32}$	$\frac{2}{7}$	$\frac{12}{25}$	$\frac{3}{5}$	$\frac{11}{14}$	$\frac{3}{4}$	0.99	NA
D3	$\frac{5}{44}$	NA	0.01	0.01	$\frac{11}{53}$	$\frac{13}{32}$	$\frac{11}{21}$	$\frac{3}{8}$	$\frac{5}{6}$	0.01	NA	NA
D4	$\frac{5}{43}$	NA	0.01	$\frac{1}{8}$	$\frac{14}{51}$	$\frac{11}{37}$	$\frac{16}{29}$	$\frac{3}{13}$	$\frac{1}{2}$	NA	NA	NA
D5	$\frac{19}{169}$	NA	0.01	$\frac{1}{5}$	$\frac{10}{39}$	$\frac{16}{39}$	$\frac{7}{15}$	$\frac{2}{7}$	0.99	NA	NA	NA
D6	$\frac{24}{181}$	NA	NA	$\frac{2}{13}$	$\frac{21}{71}$	$\frac{14}{43}$	$\frac{4}{9}$	$\frac{1}{3}$	0.99	NA	NA	NA
D7	$\frac{1}{5}$	NA	NA	$\frac{1}{8}$	$\frac{22}{117}$	$\frac{13}{40}$	$\frac{1}{4}$	NA	NA	NA	NA	NA

Fraction of Rain Days for Station 2

Out[116]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85	90
D1	$\frac{7}{170}$	NA	0.01	NA	$\frac{6}{29}$	$\frac{3}{7}$	$\frac{3}{8}$	$\frac{7}{9}$	$\frac{9}{10}$	$\frac{5}{6}$	$\frac{4}{5}$	NA	0.99
D2	$\frac{8}{91}$	NA	0.01	NA	$\frac{19}{67}$	$\frac{9}{28}$	$\frac{15}{26}$	$\frac{4}{5}$	$\frac{4}{5}$	NA	NA	NA	NA
D3	$\frac{25}{207}$	NA	$\frac{3}{7}$	NA	$\frac{7}{30}$	$\frac{18}{29}$	$\frac{1}{3}$	0.01	0.99	NA	NA	NA	NA
D4	$\frac{22}{193}$	NA	$\frac{3}{8}$	NA	$\frac{7}{22}$	$\frac{1}{3}$	$\frac{7}{10}$	0.01	NA	NA	NA	NA	NA
D5	$\frac{29}{202}$	NA	0.01	NA	$\frac{27}{82}$	$\frac{1}{3}$	$\frac{3}{8}$	NA	NA	NA	NA	NA	NA
D6	$\frac{19}{106}$	NA	$\frac{1}{3}$	NA	$\frac{21}{92}$	$\frac{1}{2}$	$\frac{1}{2}$	NA	NA	NA	NA	NA	NA
D7	$\frac{47}{233}$	NA	0.01	NA	$\frac{19}{80}$	$\frac{1}{7}$	NA	NA	NA	NA	NA	NA	NA

Fraction of Rain Days for Baseline

Out[118]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85	90
B1	NA	NA	NA	$\frac{67}{321}$	NA	NA	NA	NA	NA	NA	NA	NA	NA
B2	NA	NA	NA	NA	$\frac{67}{321}$	NA	NA	NA	NA	NA	NA	NA	NA
U1	$\frac{1}{2}$	$\frac{13}{36}$	$\frac{7}{37}$	$\frac{2}{15}$	0.01	0.01	$\frac{5}{162}$	$\frac{3}{4}$	$\frac{3}{4}$	NA	$\frac{1}{2}$	NA	0.99
U4	$\frac{16}{29}$	$\frac{11}{37}$	$\frac{14}{51}$	$\frac{1}{8}$	0.01	NA	$\frac{5}{43}$	$\frac{3}{13}$	NA	NA	NA	NA	NA
U7	$\frac{1}{4}$	$\frac{13}{40}$	$\frac{22}{117}$	$\frac{1}{8}$	NA	NA	$\frac{1}{5}$	NA	NA	NA	NA	NA	NA

In[119]:=
```
ForecastNonRain = {
    Reverse[ForecastRain⟦1⟧ /. x_ /; NumericQ[x] → 1 - x, 2],
    Reverse[ForecastRain⟦2⟧ /. x_ /; NumericQ[x] → 1 - x, 2],
    Table["NA", 5, 15]
   };
ForecastNonRain⟦3, 1, -4⟧ = ForecastNonRain⟦3, 2, -5⟧ = 1 - ForecastRain⟦3, 1, 4⟧;
ForecastNonRain⟦3, {3, 4, 5}, 1 ;; 7⟧ =
  Reverse[ForecastNonRain⟦1, {1, 4, 7}, 1 ;; 7⟧, 2];
ForecastNonRain⟦3, {3, 4, 5}, 8⟧ = ForecastNonRain⟦1, {1, 4, 7}, 8⟧;
ForecastNonRain⟦3, {3, 4, 5}, 9 ;; 15⟧ =
  Reverse[ForecastNonRain⟦1, {1, 4, 7}, 9 ;; 15⟧, 2];

Print["Fraction of non-Rain Days for Station 1"]
TableForm[ForecastNonRain⟦1⟧,
 TableHeadings → {DaysAhead, Percentiles}
]
Print["Fraction of non-Rain Days for Station 2"]
TableForm[ForecastNonRain⟦2⟧,
 TableHeadings → {DaysAhead, Percentiles}
]
Print["Fraction of non-Rain Days for Baseline"]
TableForm[ForecastNonRain⟦3⟧,
 TableHeadings → {BaselineLabel, Percentiles}
]
```

Fraction of non-Rain Days for Station 1

Out[125]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85	9
D1	$\frac{1}{4}$	NA	$\frac{1}{2}$	NA	0.01	0.01	$\frac{7}{18}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{23}{36}$	$\frac{30}{37}$	$\frac{13}{15}$	0
D2	NA	NA	NA	NA	0.01	$\frac{1}{4}$	$\frac{3}{14}$	$\frac{2}{5}$	$\frac{13}{25}$	$\frac{5}{7}$	$\frac{27}{32}$	$\frac{7}{10}$	[illegible]
D3	0.01	NA	NA	NA	NA	0.99	$\frac{1}{6}$	$\frac{5}{8}$	$\frac{10}{21}$	$\frac{19}{32}$	$\frac{42}{53}$	0.99	0
D4	NA	NA	NA	NA	NA	NA	$\frac{1}{2}$	$\frac{10}{13}$	$\frac{13}{29}$	$\frac{26}{37}$	$\frac{37}{51}$	$\frac{7}{8}$	0
D5	NA	NA	NA	NA	NA	NA	0.01	$\frac{5}{7}$	$\frac{8}{15}$	$\frac{23}{39}$	$\frac{29}{39}$	$\frac{4}{5}$	0
D6	NA	NA	NA	NA	NA	NA	0.01	$\frac{2}{3}$	$\frac{5}{9}$	$\frac{29}{43}$	$\frac{50}{71}$	$\frac{11}{13}$	N
D7	NA	NA	NA	NA	NA	NA	NA	NA	$\frac{3}{4}$	$\frac{27}{40}$	$\frac{95}{117}$	$\frac{7}{8}$	N

Fraction of non-Rain Days for Station 2

Out[127]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85	90
D1	0.01	NA	0.01	NA	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{2}{9}$	$\frac{5}{8}$	$\frac{4}{7}$	$\frac{23}{29}$	NA	0.99
D2	NA	NA	NA	NA	NA	NA	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{11}{26}$	$\frac{19}{28}$	$\frac{48}{67}$	NA	0.99
D3	NA	NA	NA	NA	NA	NA	0.01	0.99	$\frac{2}{3}$	$\frac{11}{29}$	$\frac{23}{30}$	NA	$\frac{4}{7}$
D4	NA	NA	NA	NA	NA	NA	NA	0.99	$\frac{3}{10}$	$\frac{2}{3}$	$\frac{15}{22}$	NA	$\frac{5}{8}$
D5	NA	NA	NA	NA	NA	NA	NA	NA	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{55}{82}$	NA	0.99
D6	NA	NA	NA	NA	NA	NA	NA	NA	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{71}{92}$	NA	$\frac{2}{3}$
D7	NA	NA	NA	NA	NA	NA	NA	NA	NA	$\frac{6}{7}$	$\frac{61}{80}$	NA	0.99

```
Fraction of non-Rain Days for Baseline
```

Out[129]//TableForm=

	1	5	10	15	20	30	40	50	60	70	80	85	90
B1	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	$\frac{254}{321}$	NA
B2	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	$\frac{254}{321}$	NA	NA
U1	$\frac{7}{18}$	0.01	0.01	NA	$\frac{1}{2}$	NA	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{157}{162}$	0.99	0.99	$\frac{13}{15}$	$\frac{30}{37}$
U4	$\frac{1}{2}$	NA	NA	NA	NA	NA	NA	$\frac{10}{13}$	$\frac{38}{43}$	NA	0.99	$\frac{7}{8}$	$\frac{37}{51}$
U7	NA	NA	NA	NA	NA	NA	NA	NA	$\frac{4}{5}$	NA	NA	$\frac{7}{8}$	$\frac{95}{11[illegible]}$

Rain Forecast Risk Assessment with Coupled Mean

In[220]:=
```
numMeans = 7; (* Assumed odd, so that zero included *)
Lim = 1;
positiveRiskValues = Range[0, Lim, 2 * Lim / (numMeans - 1)];
negativeRiskValues = (-2 Drop[positiveRiskValues, 1]) / (2 + Drop[positiveRiskValues, 1]);
(* riskValues are the relative risk aversion *)
riskValues = -Reverse[negativeRiskValues] ~Join~ positiveRiskValues
```

Out[224]= $\left\{\frac{2}{3}, \frac{1}{2}, \frac{2}{7}, 0, -\frac{1}{3}, -\frac{2}{3}, -1\right\}$

Compute a Table of Overall Statistics with dimension {numMeans, 3} where the three values are: outcomeProb, forecastProb, divergenceProb

In[270]:=
```
TVStation = 3; ForecastRows = 5;
TotalForecasts = Total[Forecasts, {3}] /. x_ /; x == "NA" → 0;
weatherForecastMetrics = ComputeCoupledMeanNOAA[
      Transpose[{ForecastRain⟦TVStation⟧, ForecastNonRain⟦TVStation⟧}, {3, 2, 1}],
      Transpose[Table[Percentiles / 100, ForecastRows, 2], {2, 3, 1}],
      Transpose[{ForecastRain⟦TVStation⟧ Forecasts⟦TVStation⟧ /
           TotalForecasts⟦TVStation⟧, ForecastNonRain⟦TVStation⟧
         Reverse[Forecasts⟦TVStation⟧, 2] / TotalForecasts⟦TVStation⟧},
        {3, 2, 1}],
      #
     ] & /@ riskValues;
(*
weatherForecastMetricsBasedonCoupledEntropy=ComputeCoupledMeanNOAA[
      Transpose[{ForecastRain⟦TVStation⟧,ForecastNonRain⟦TVStation⟧},{3,2,1}],
      Transpose[Table[Percentiles/100,ForecastRows,2],{2,3,1}],
      Transpose[{ForecastRain⟦TVStation⟧
         Forecasts⟦TVStation⟧/TotalForecasts⟦TVStation⟧,ForecastNonRain⟦TVStation⟧
         Reverse[Forecasts⟦TVStation⟧,2]/TotalForecasts⟦TVStation⟧},
        {3,2,1}],
      #
     ]&/@riskValues;
*)
```

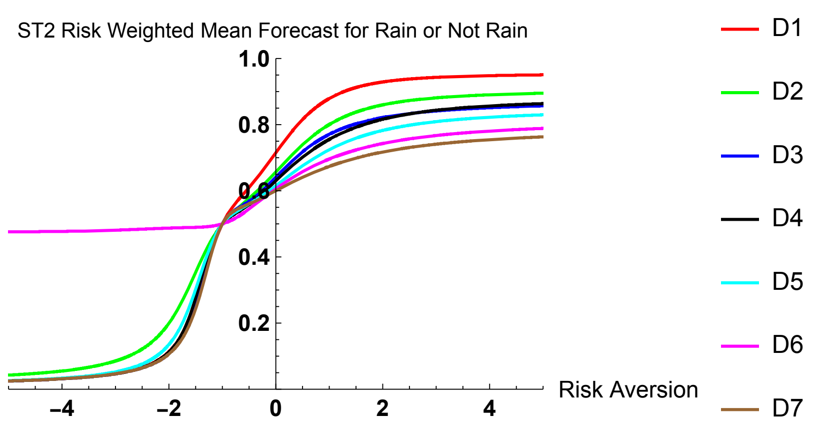
Extract the Robust (-2/3), Accuracy (0), and Decisive (1) metrics from the weatherForecastMetrics

In[228]:=
```
RobustMetrics = weatherForecastMetrics⟦1⟧;
AccuracyMetrics = weatherForecastMetrics⟦Ceiling[numMeans / 2]⟧;
DecisiveMetrics = weatherForecastMetrics⟦-1⟧;
```

Risk Assessment using Coupled Mean of Rain Forecasts

In[273]:=
```
TVStation = 2; numberDays = 7;
ForecastStats = 2; OutcomeStats = 1;
rMin = -5; rMax = 5;
TotalForecasts = Total[Forecasts, {3}] /. x_ /; x == "NA" → 0;
Colors = {Red, Green, Blue, Black, Cyan, Magenta, Brown};
Show @@ Table[
  Plot[Legended[ComputeCoupledMeanNOAA[
      Transpose[{ForecastRain〚TVStation〛, ForecastNonRain〚TVStation〛}, {3, 2, 1}],
      Transpose[Table[Percentiles / 100, numberDays, 2], {2, 3, 1}],
      Transpose[{ForecastRain〚TVStation〛 Forecasts〚TVStation〛 /
          TotalForecasts〚TVStation〛, ForecastNonRain〚TVStation〛
         Reverse[Forecasts〚TVStation〛, 2] / TotalForecasts〚TVStation〛},
       {3, 2, 1}],
      Risk
     ]〚OutcomeStats, DayAhead〛, DaysAhead〚DayAhead〛],
   {Risk, rMin, rMax},
   PlotRange → {{rMin, rMax}, {0, 1}},
   PlotStyle → Colors〚DayAhead〛,
   AxesStyle → {Medium, Bold},
   AxesLabel → {"Risk Aversion", None},
   PlotLabel → "ST" <> ToString[TVStation] <>
     " Risk Weighted Mean Forecast for Rain or Not Rain ",
   TicksStyle → {Bold, Medium}
  ],
  {DayAhead, numberDays}]
```

Out[278]=



Rain Forecast Assessment Results

Results are shown for Risk Bias values between -5 and 5; however, published results will focus on -1 to 1.

1 - Always equal to the inverse of the number of states

0 - Geometric Mean associated with Shannon Entropy

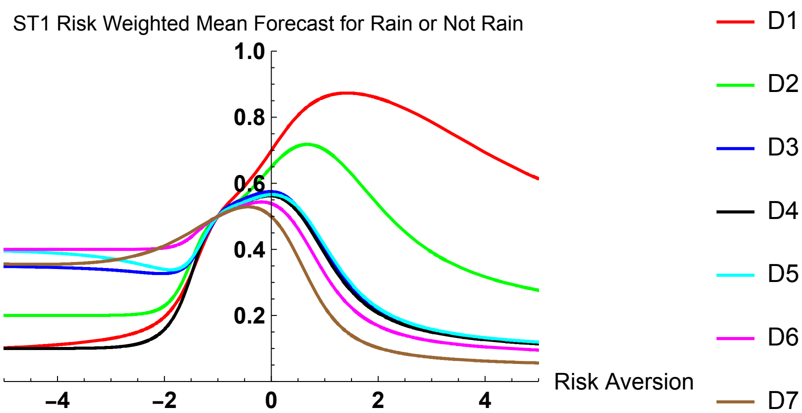
-2/3 - Metric used to define Robustness

1 - Use of the Coupled Mean extends the range of useful negative values.

TV Station 1

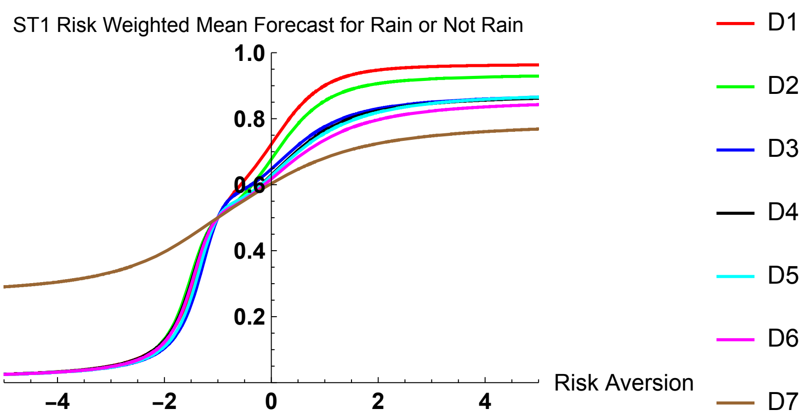
Forecast Statistics

Out[ ]=



Outcome Statistics

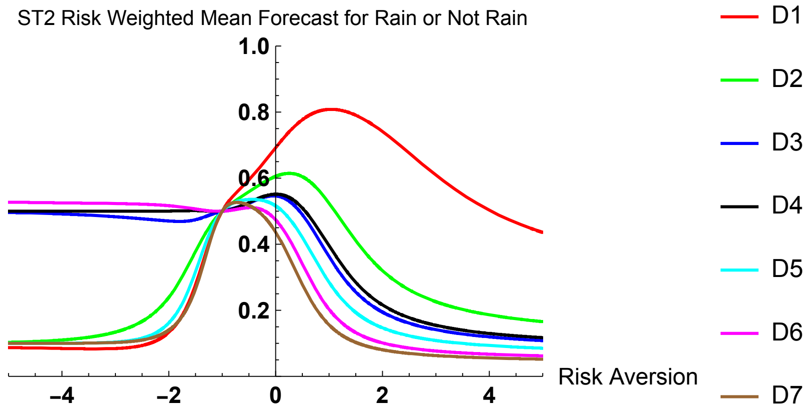
Out[ ]=



TV Station 2

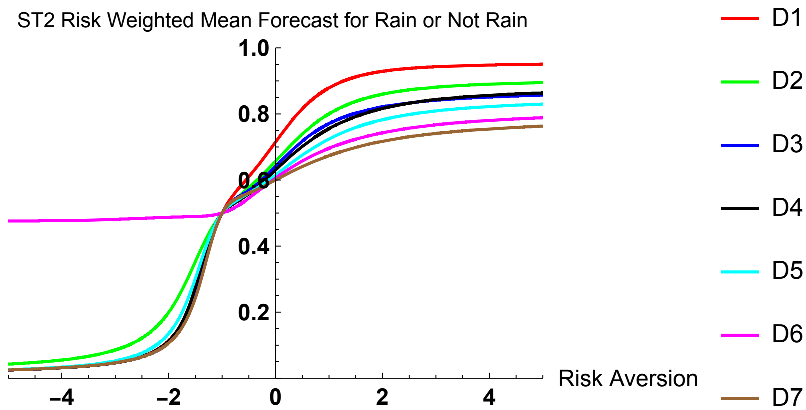
Forecast Statistics

Out[ ]=



Outcome Statistics

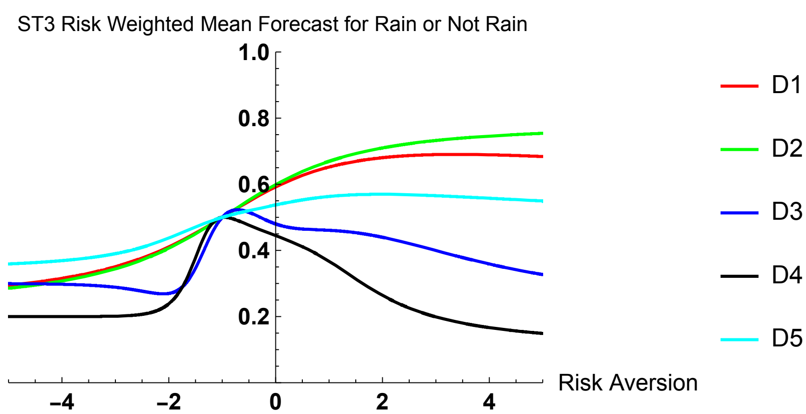
Out[ ]=



Baseline Forecasts

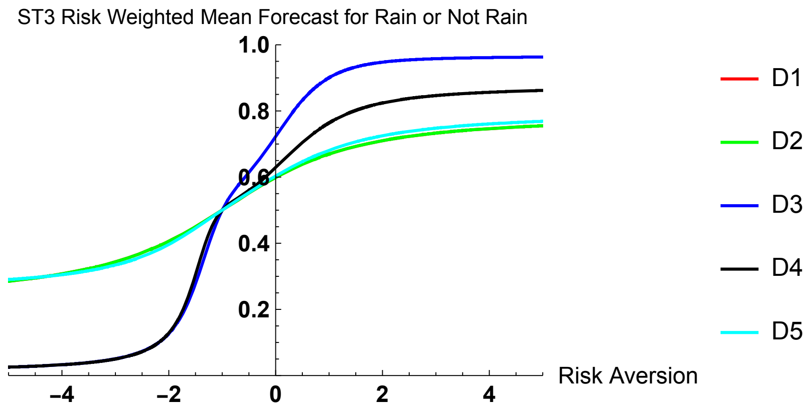
Forecast Statistics

Out[ ]=



Outcome Statistics

Out[ ]=



Forecast vs. Source Assessment of Rain Forecasts

Follows the functions from Election Assessment with some modification of the names

prevBin -> outcomeProb; geoBin -> forecastProbs

prevAll -> geoMeanOutcome; geoAll -> geoMeanForecast; divAll -> divergenceProb

Calls functions from “Assess Probabilities v7”

Uses inputs from the “Import Media Forecast Data” Section

Does not use the “Compute Generalized Mean of Forecasts” Section which will be superseded by this section

Generate complementary values of risk for the generalized mean computation

In[307]:=

```
numMeans = 11; (* Assumed odd, so that zero included *)
Lim = 1;
positiveRiskValues = Range[0, Lim, 2 * Lim / (numMeans - 1)];
(*negativeRiskValues =-Drop[positiveRiskValues,1];*)
negativeRiskValues = (-2 Drop[positiveRiskValues, 1]) / (2 + Drop[positiveRiskValues, 1]);
(* riskValues are the relative risk aversion *)
riskValues = -Reverse[negativeRiskValues]~Join~positiveRiskValues
```

Out[311]= $\left\{\frac{2}{3}, \frac{4}{7}, \frac{6}{13}, \frac{1}{3}, \frac{2}{11}, 0, -\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{4}{5}, -1\right\}$

Compute a Table of Overall Statistics with dimension {numMeans, 3} where the three values are: outcomeProb, forecastProb, divergenceProb

```
In[336]:= TVStation = 3; ForecastRows = 5;
TotalForecasts = Total[Forecasts, {3}] /. x_ /; x == "NA" → 0;
(* weatherForecastMetrics Dimensions: 7 days; 3 metrics; 5 riskValues *)
weatherForecastMetrics = ComputeCoupledMeanNOAA[
     Transpose[{ForecastRain⟦TVStation⟧, ForecastNonRain⟦TVStation⟧}, {3, 2, 1}],
     Transpose[Table[Percentiles / 100, ForecastRows, 2], {2, 3, 1}],
     Transpose[{ForecastRain⟦TVStation⟧ Forecasts⟦TVStation⟧ /
         TotalForecasts⟦TVStation⟧, ForecastNonRain⟦TVStation⟧
        Reverse[Forecasts⟦TVStation⟧, 2] / TotalForecasts⟦TVStation⟧},
      {3, 2, 1}],
     #
    ] & /@ riskValues;
```

Extract the Robust (-2/3), Accuracy (0), and Decisive (1) metrics from the weatherForecastMetrics

```
In[339]:= RobustMetrics = weatherForecastMetrics⟦1⟧;
AccuracyMetrics = weatherForecastMetrics⟦Ceiling[numMeans / 2]⟧;
DecisiveMetrics = weatherForecastMetrics⟦-1⟧;
```

```
In[344]:= Show[
  (* Plot the raw forecasts and outcomes for rain and not rain *)
  DayOut = 2;
  ListPlot[{
    Flatten[
      {ForecastRain⟦TVStation⟧⟦#⟧, Percentiles / 100} & /@ {DayOut}, 1] // Transpose,
    Flatten[
      {ForecastNonRain⟦TVStation⟧⟦#⟧, Percentiles / 100} & /@ {DayOut}, 1] // Transpose
   },
   PlotRange → {{-0.01, 1.01}, {-0.01, 1.01}},
   (*PlotLegends→{"Rain","Not Rain"},*)
   (*PlotLabel→
    Style["ST"<>ToString[TVStation]<>" Rain Outcomes vs. Forecasts - "<>
      ToString[DayOut]<>" Day"<>If[DayOut≠1,"s",""]<>" Out",
     16, Black],*)
   PlotLabel →
    Style[" Forecasts vs. Outcomes - Baseline" <> If[DayOut == 1, " 15%", " 20%"],
     16, Black],
   Frame → {{True, False}, {True, False}},
   FrameLabel → {"Forecast Probabilities", "Outcome Frequencies"},
   LabelStyle → Directive[12],
   PlotStyle → PointSize[Medium],
   GridLines → Automatic,
   Epilog → Inset[Style[
      NumberForm[TableForm[
        Flatten[Permute[
```

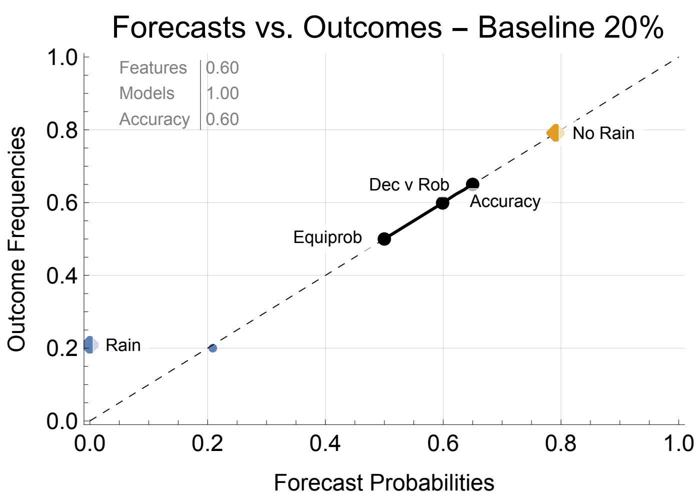
```
          weatherForecastMetrics⟦
           Ceiling[Length[weatherForecastMetrics] / 2], All, #⟧,
          {1, 3, 2}] & /@ {DayOut}],
      TableHeadings → {{"Features ", "Models   ", "Accuracy "}, None}], {2, 2}],
    Gray],
   {0.154, 0.895}]
 ],
 Graphics[{Dashed, Line[{{0, 0}, {1, 1}}]}], (* Dashed line of equality *)
 (* Plot the Robust, Accurate & Decisive Metrics *)
 ListPlot[{
     Labeled[{RobustMetrics⟦2, #⟧, RobustMetrics⟦1, #⟧},
      "Dec v Rob ", Left],
     Labeled[{AccuracyMetrics⟦2, #⟧, AccuracyMetrics⟦1, #⟧},
      "  Accuracy", Right],
     Labeled[{DecisiveMetrics⟦2, #⟧, DecisiveMetrics⟦1, #⟧},
      "Equiprob ", Left]},
    PlotRange → {{-0.01, 1.01}, {-0.01, 1.01}},
    PlotStyle → {PointSize[Large], Black}
   ] & /@ {DayOut},
 (* Plot the Generalized Means for Forecast and Outcome *)
 ListPlot[weatherForecastMetrics⟦All, {2, 1}, #⟧,
    Joined → True, InterpolationOrder → 4,
    PlotStyle → Black
   ] & /@ {DayOut},

 (* Plot the Classification for Rain or NonRain *)
 numRainForecasts =
  Round[Total[(ForecastRain⟦TVStation⟧⟦#, All⟧ * Forecasts⟦TVStation⟧⟦#, All⟧)]] & /@
   {DayOut};
 numNonRainForecats =
  Round[Total[Reverse[Reverse[ForecastNonRain⟦TVStation⟧⟦#, All⟧] *
        Forecasts⟦TVStation⟧⟦#, All⟧]]] & /@ {DayOut};
 numForecasts = Total[Forecasts⟦TVStation⟧⟦#, All⟧] & /@ {DayOut};

 ListPlot[
  {{
    Labeled[
     {((0.5 Total[Take[(ForecastRain⟦TVStation⟧⟦#, All⟧ * Forecasts⟦TVStation⟧⟦
                  #, All⟧), {8}]] +
              Total[Take[(ForecastRain⟦TVStation⟧⟦#, All⟧ *
                   Forecasts⟦TVStation⟧⟦#, All⟧), -7]]) /
            numForecasts & /@ {DayOut})⟦1, 1⟧,
       (Total[ForecastRain⟦TVStation⟧⟦#, All⟧ Forecasts⟦TVStation⟧⟦#, All⟧] /
            numForecasts & /@ {DayOut})⟦1, 1⟧
```

```
      } /. x_ /; x == "NA" → 0, " Rain", {Right, Left}]
   },
   {
    Labeled[{
       ((0.5 Total[Take[Reverse[Reverse[ForecastNonRain⟦TVStation⟧⟦#, All⟧] *
                 Forecasts⟦TVStation⟧⟦#, All⟧], {8}]] +
              Total[Take[Reverse[Reverse[ForecastNonRain⟦TVStation⟧⟦#, All⟧] *
                  Forecasts⟦TVStation⟧⟦#, All⟧], -7]]) /
            numForecasts & /@ {DayOut})⟦1, 1⟧,
       (Total[Reverse[Reverse[ForecastNonRain⟦TVStation⟧⟦#, All⟧] *
                Forecasts⟦TVStation⟧⟦#, All⟧]]] /
            numForecasts & /@ {DayOut})⟦1, 1⟧
      } /. x_ /; x == "NA" → 0, " No Rain", {Right, Left}]
   }
  },
  ngon[p_, q_] := Polygon[Table[{Cos[2 Pi k q / p], Sin[2 Pi k q / p]}, {k, p}]];
  PlotMarkers → {{Graphics[ngon[4, 1]], 9}, {Graphics[ngon[4, 1]], 9}}
 ]
]
```

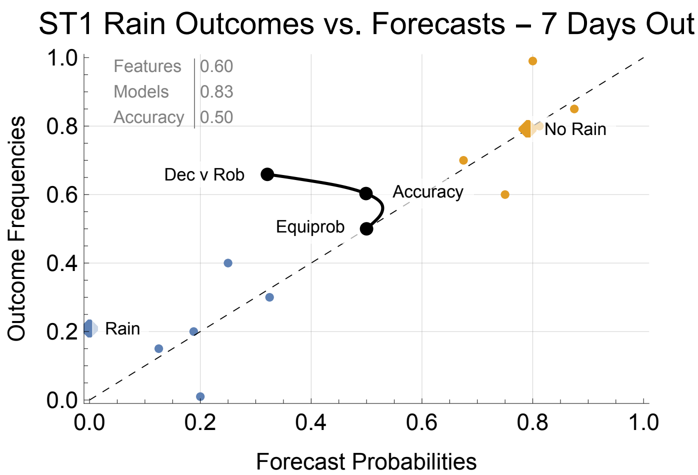
Out[344]=



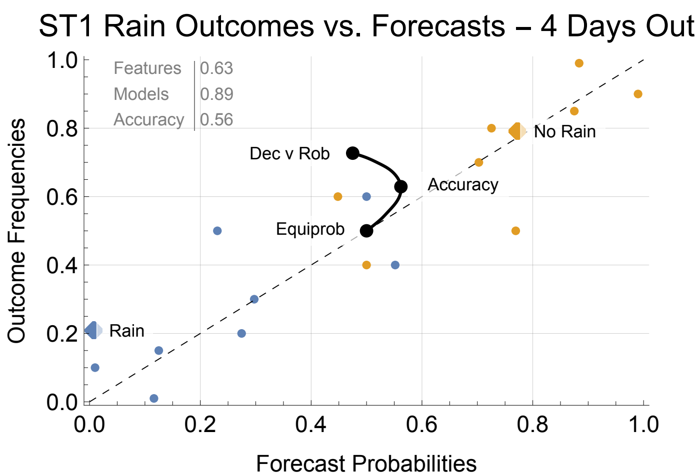
Source vs. Forecast Assessment of Rain Forecast Results with negative Risk limit of -2/3

Station 1

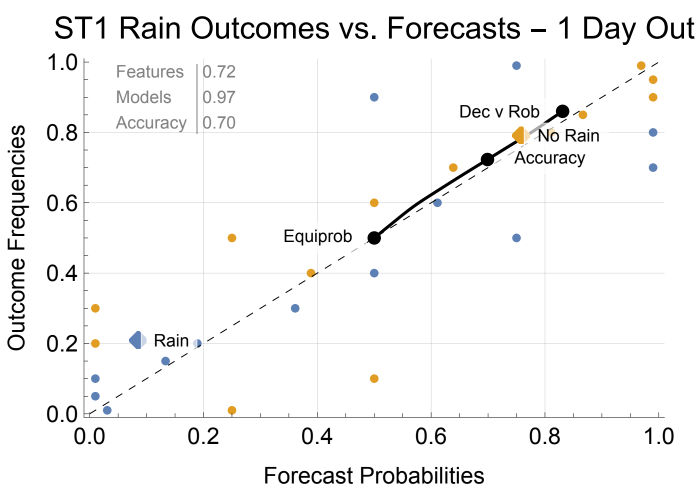
Out[ ]=



Out[ ]=

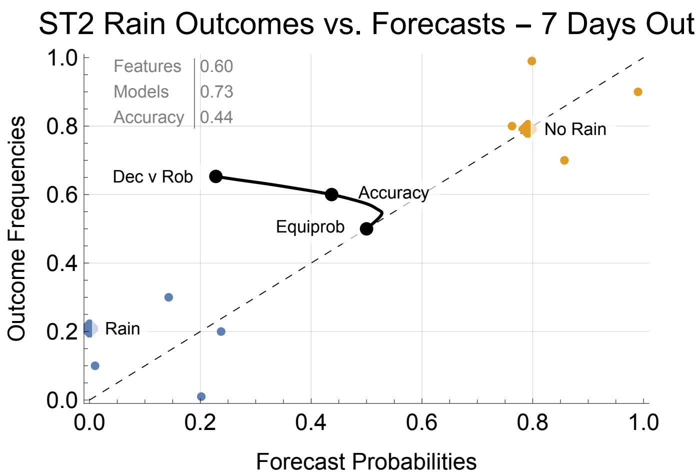


Out[ ]=

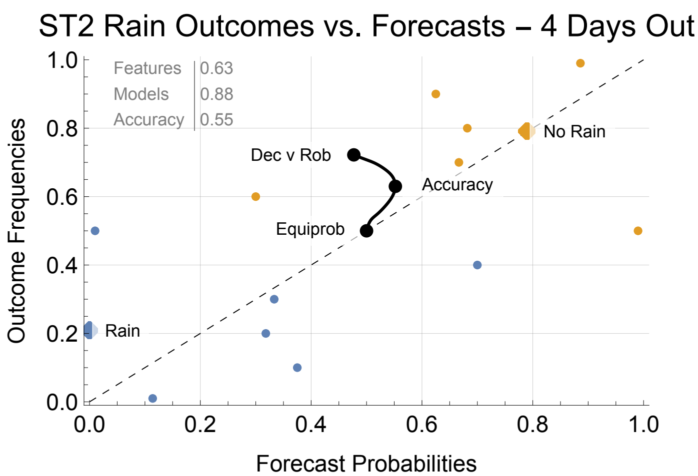


Station 2

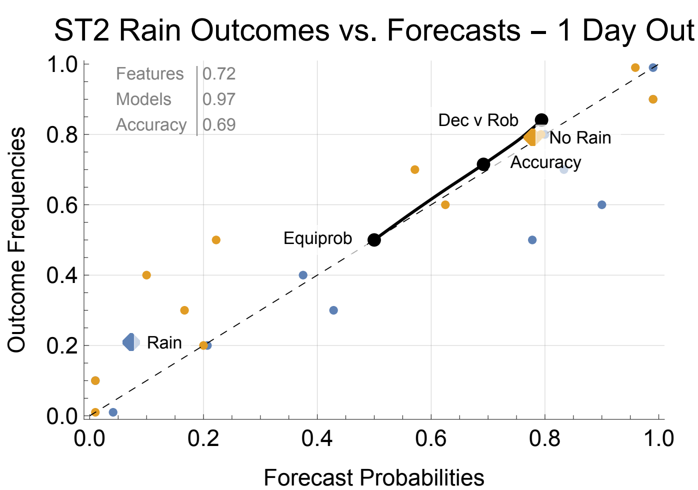
Out[ ]=



Out[ ]=

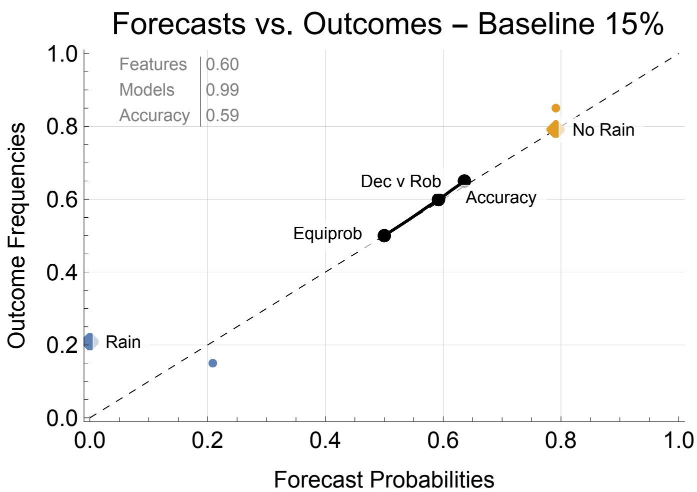


Out[ ]=

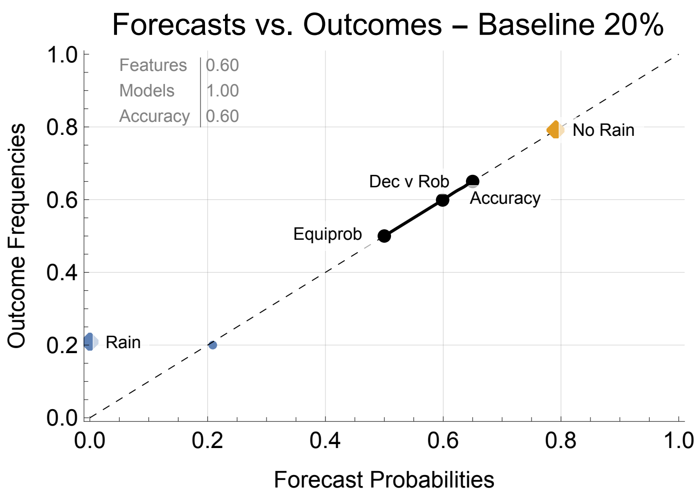


Baseline Forecasts

Out[ ]=

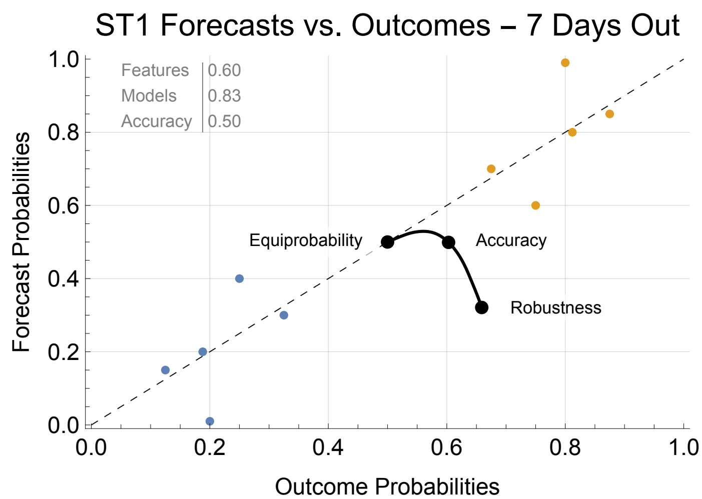

Forecasts vs. Outcomes – Baseline 15%
Features 0.60
Models 0.99
Accuracy 0.59
No Rain
Dec v Rob
Accuracy
Equiprob
Rain
Outcome Frequencies
Forecast Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

Out[ ]=

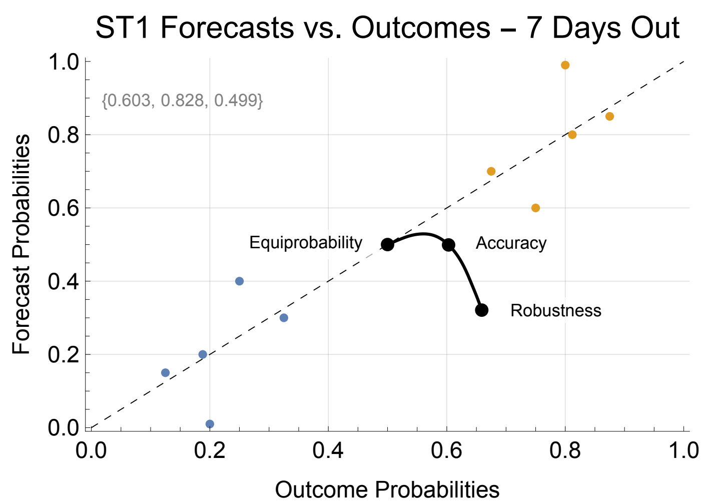

Forecasts vs. Outcomes – Baseline 20%
Features 0.60
Models 1.00
Accuracy 0.60
No Rain
Dec v Rob
Accuracy
Equiprob
Rain
Outcome Frequencies
Forecast Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

Forecast vs. Source Assessment of Rain Forecasts Results Using Modification of Coupled Entropy rather than Coupled Cross Entropy

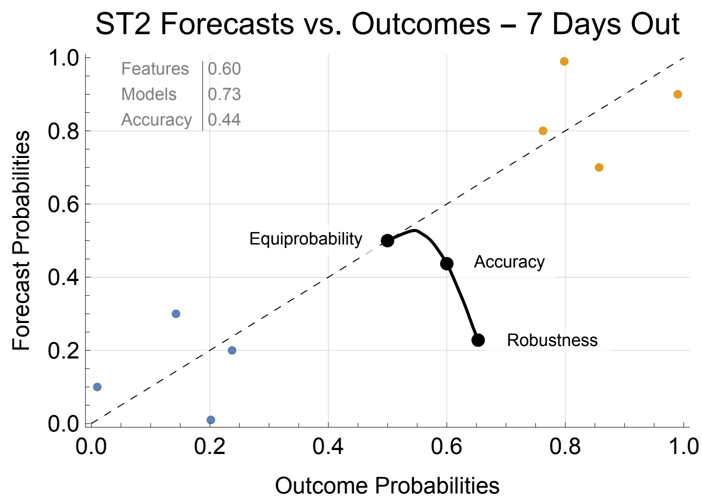
In[ ]:=



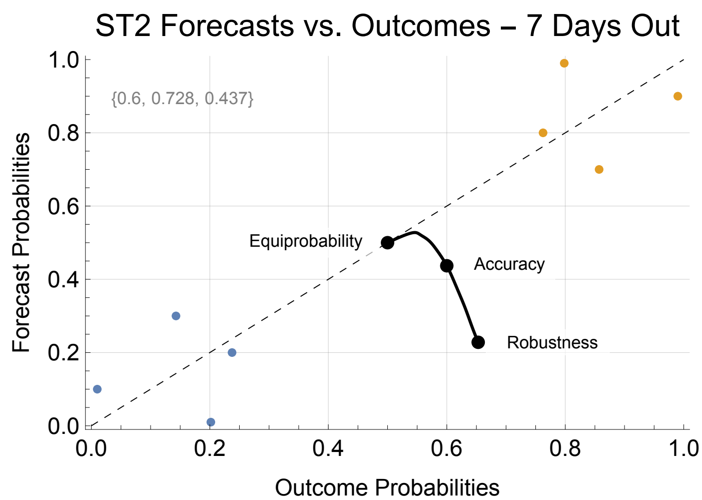
Out[ ]=



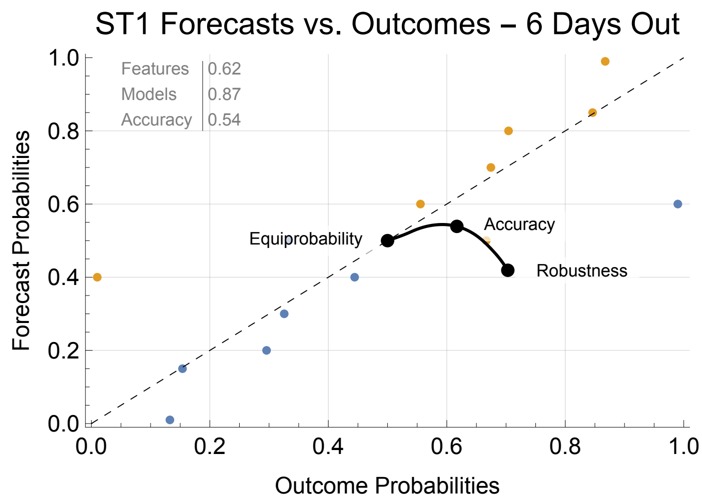
In[ ]:=


ST2 Forecasts vs. Outcomes – 7 Days Out
Features 0.60
Models 0.73
Accuracy 0.44
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

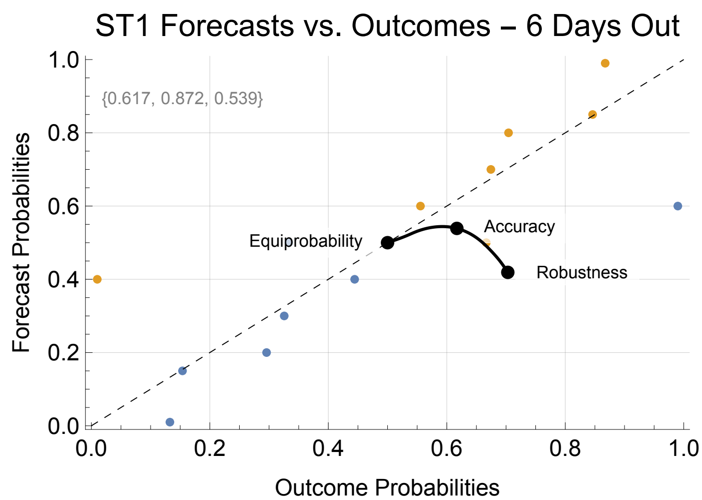
Out[ ]=


ST2 Forecasts vs. Outcomes – 7 Days Out
{0.6, 0.728, 0.437}
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

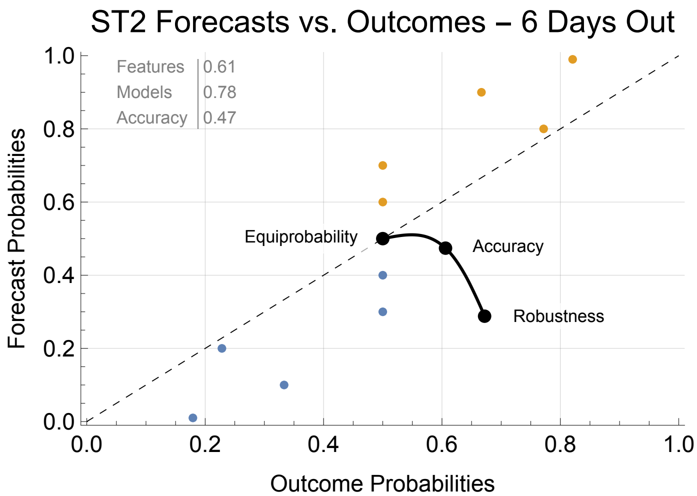
In[∘]:=


ST1 Forecasts vs. Outcomes – 6 Days Out
Features 0.62
Models 0.87
Accuracy 0.54
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

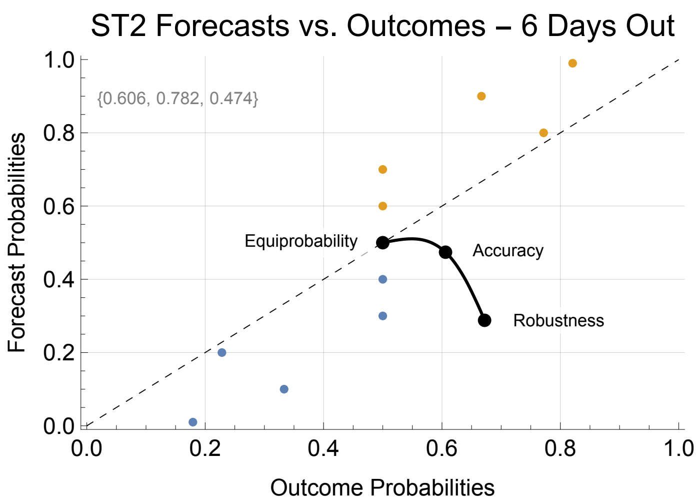
Out[∘]=


ST1 Forecasts vs. Outcomes – 6 Days Out
{0.617, 0.872, 0.539}
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

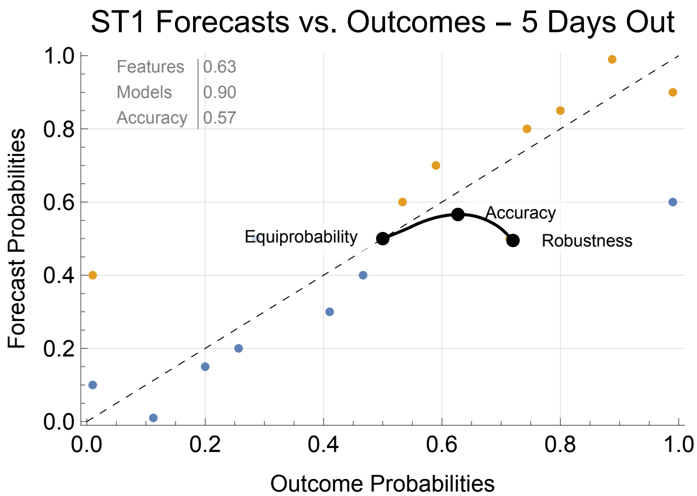
In[∘]:=


ST2 Forecasts vs. Outcomes – 6 Days Out
Features 0.61
Models 0.78
Accuracy 0.47
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

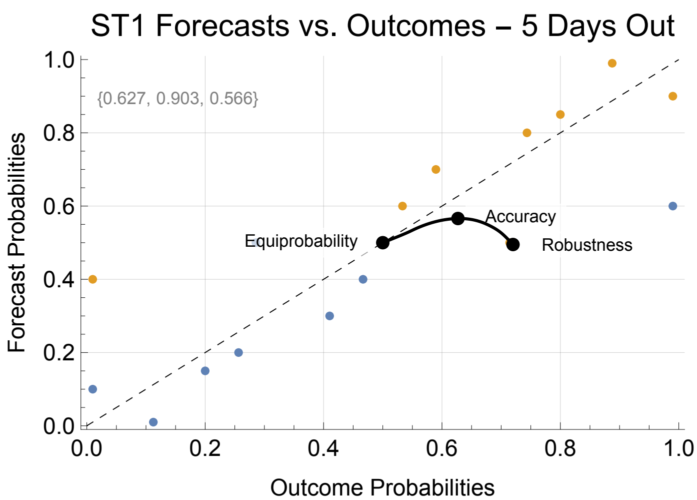
Out[∘]=


ST2 Forecasts vs. Outcomes – 6 Days Out
{0.606, 0.782, 0.474}
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

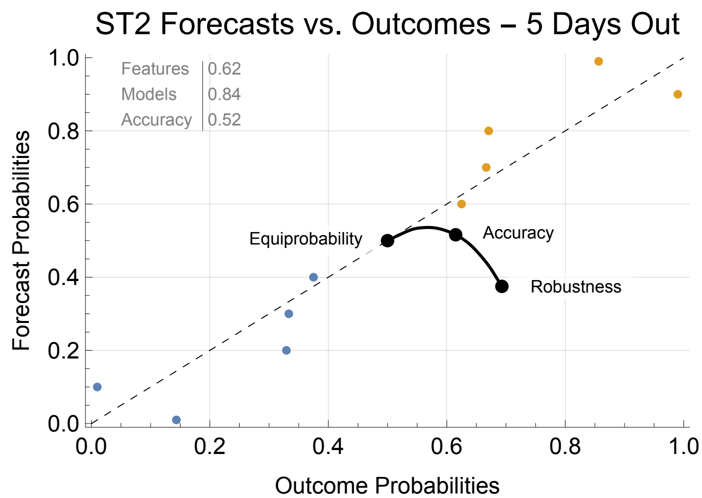
In[∘]:=


ST1 Forecasts vs. Outcomes – 5 Days Out
Features 0.63
Models 0.90
Accuracy 0.57
Accuracy
Equiprobability
Robustness
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

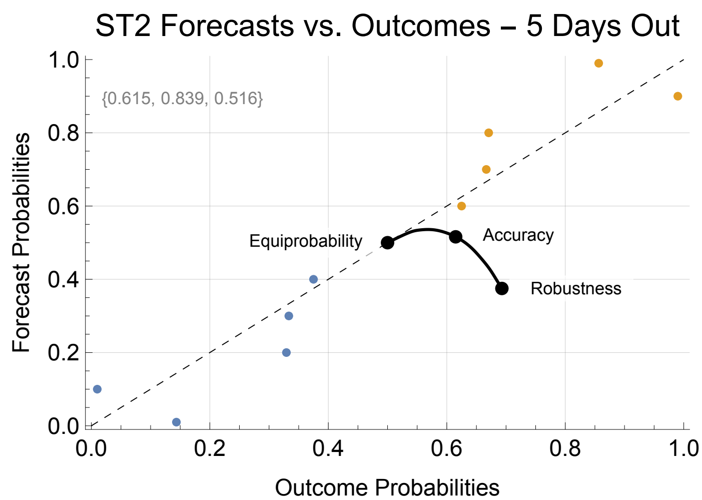
Out[∘]=


ST1 Forecasts vs. Outcomes – 5 Days Out
{0.627, 0.903, 0.566}
Accuracy
Equiprobability
Robustness
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

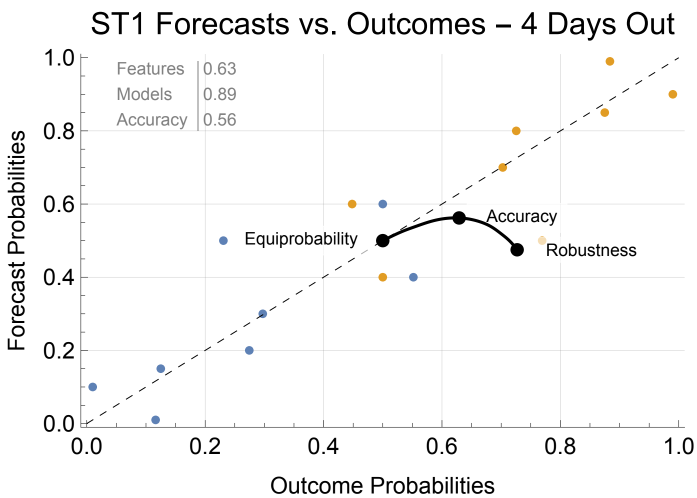
In[•]:=


ST2 Forecasts vs. Outcomes – 5 Days Out
Features 0.62
Models 0.84
Accuracy 0.52
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

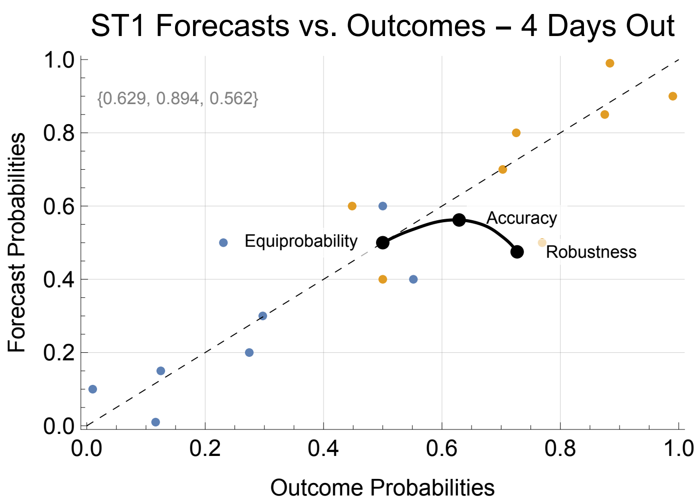
Out[•]=


ST2 Forecasts vs. Outcomes – 5 Days Out
{0.615, 0.839, 0.516}
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

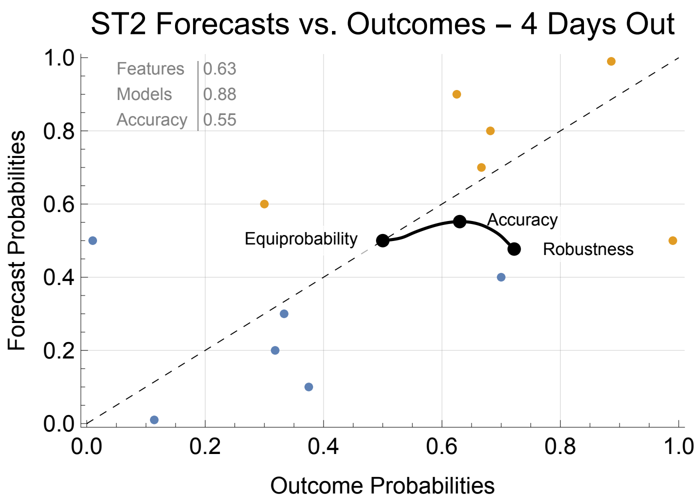
In[∘]:=


ST1 Forecasts vs. Outcomes – 4 Days Out
Features 0.63
Models 0.89
Accuracy 0.56
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

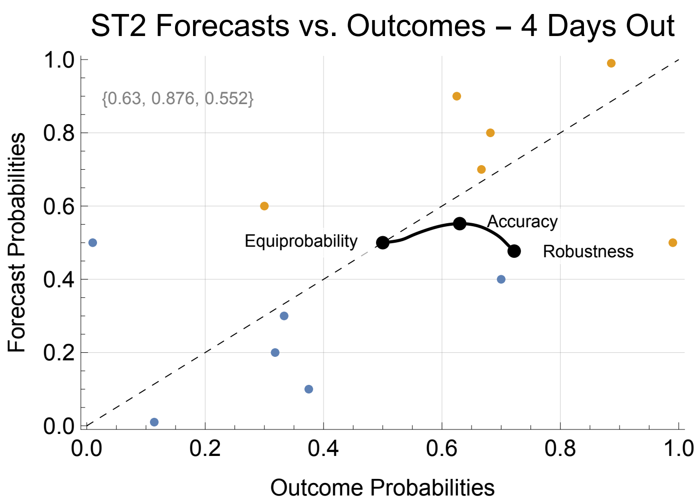
Out[∘]=


ST1 Forecasts vs. Outcomes – 4 Days Out
{0.629, 0.894, 0.562}
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

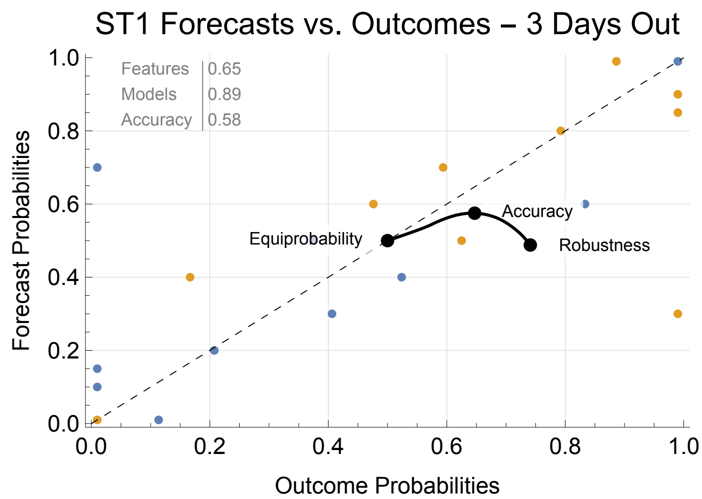
In[•]:=


ST2 Forecasts vs. Outcomes – 4 Days Out
Features 0.63
Models 0.88
Accuracy 0.55
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

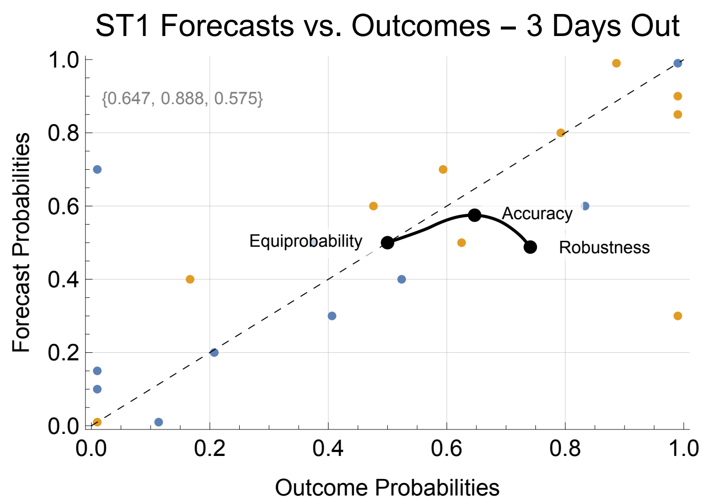
Out[•]=


ST2 Forecasts vs. Outcomes – 4 Days Out
{0.63, 0.876, 0.552}
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

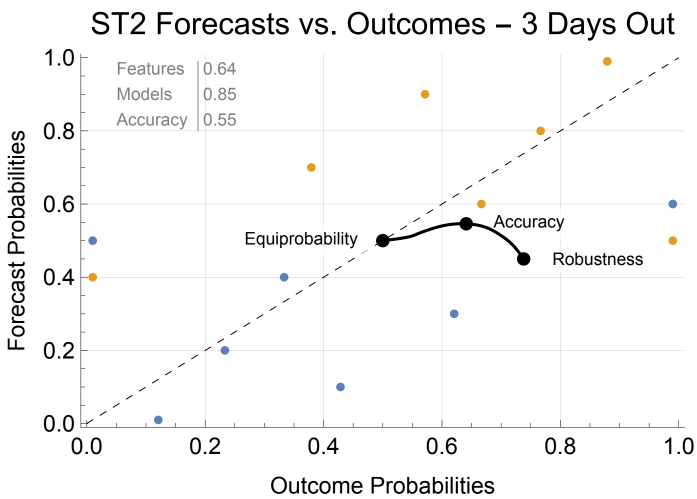
In[∘]:=


ST1 Forecasts vs. Outcomes – 3 Days Out
Features 0.65
Models 0.89
Accuracy 0.58
Accuracy
Equiprobability
Robustness
Forecast Probabilities
Outcome Probabilities

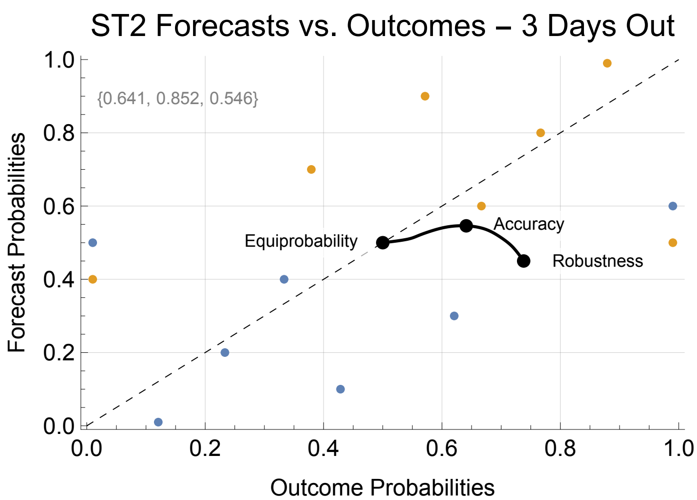
Out[∘]=


ST1 Forecasts vs. Outcomes – 3 Days Out
{0.647, 0.888, 0.575}
Accuracy
Equiprobability
Robustness
Forecast Probabilities
Outcome Probabilities

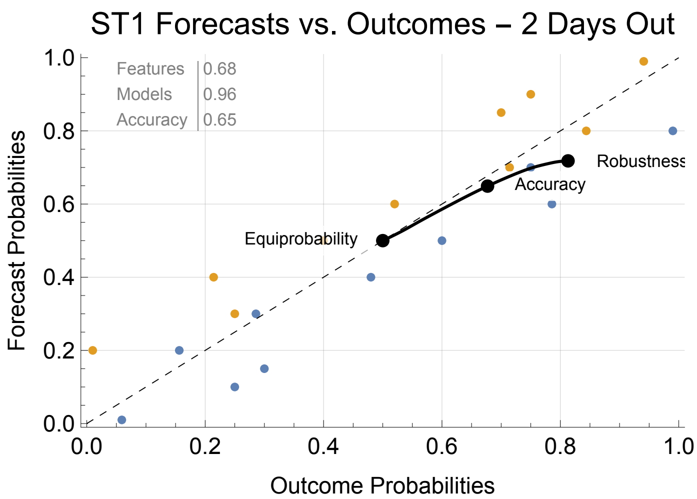
In[◦]:=


ST2 Forecasts vs. Outcomes – 3 Days Out
Features 0.64
Models 0.85
Accuracy 0.55
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

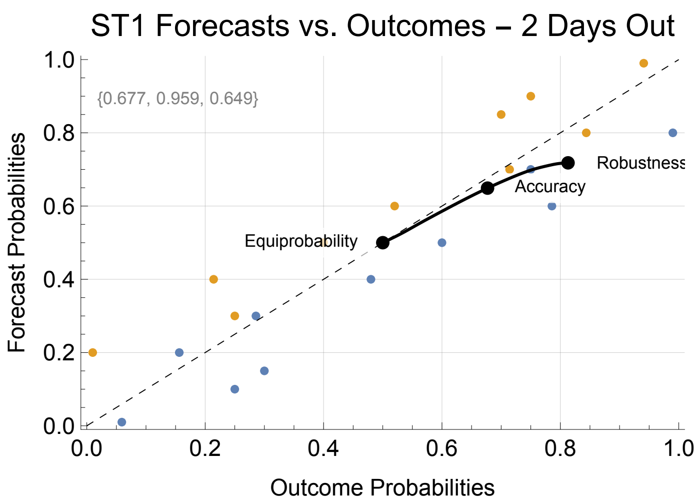
Out[◦]=


ST2 Forecasts vs. Outcomes – 3 Days Out
{0.641, 0.852, 0.546}
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

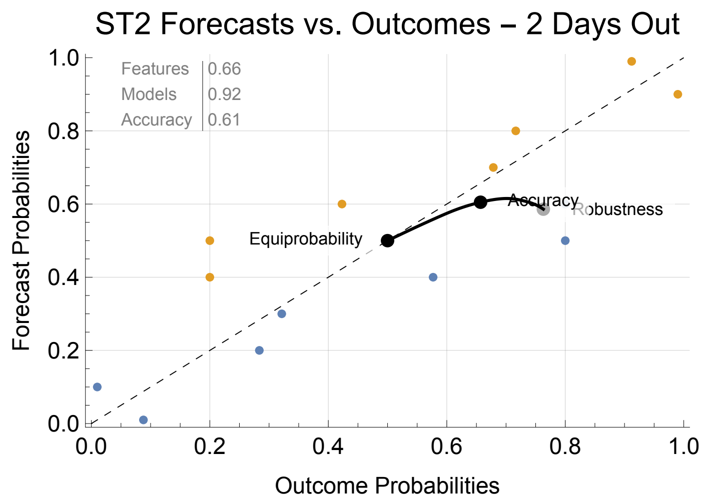
In[ ]:=


ST1 Forecasts vs. Outcomes – 2 Days Out
Features | 0.68
Models | 0.96
Accuracy | 0.65
Robustness
Accuracy
Equiprobability
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

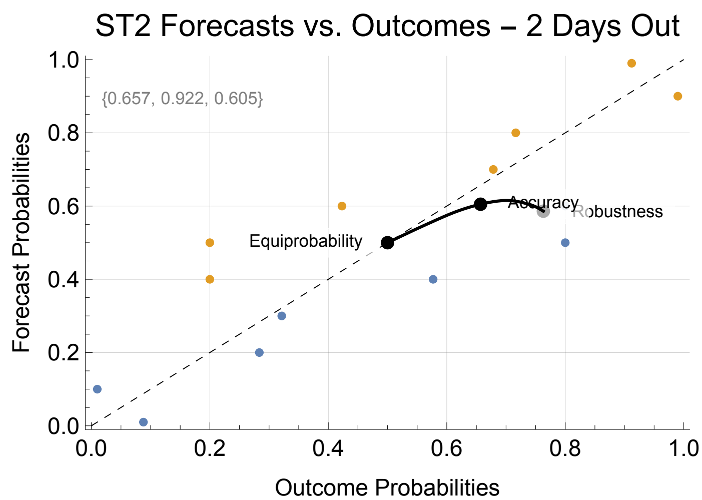
Out[ ]=


ST1 Forecasts vs. Outcomes – 2 Days Out
{0.677, 0.959, 0.649}
Robustness
Accuracy
Equiprobability
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

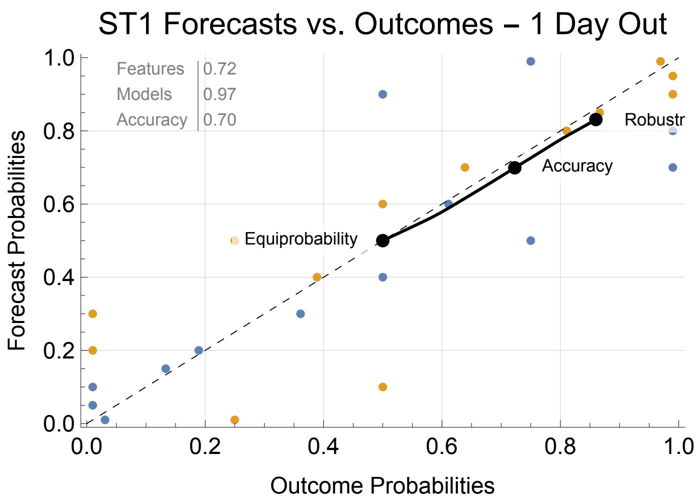
In[∘]:=


ST2 Forecasts vs. Outcomes – 2 Days Out
Features | 0.66
Models | 0.92
Accuracy | 0.61
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

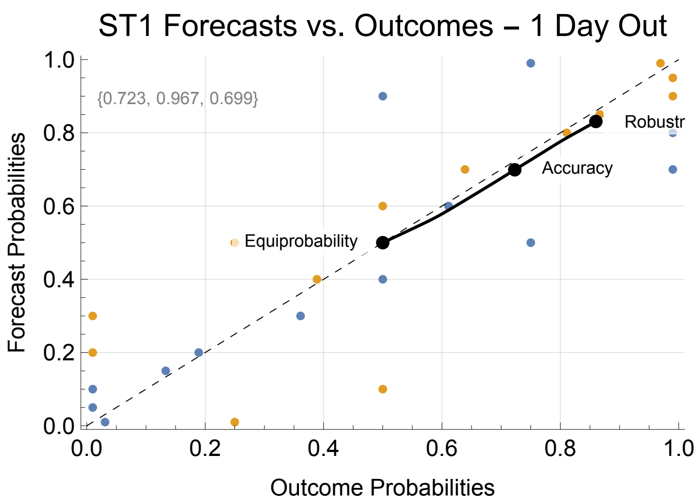
Out[∘]=


ST2 Forecasts vs. Outcomes – 2 Days Out
{0.657, 0.922, 0.605}
Equiprobability
Accuracy
Robustness
Forecast Probabilities
Outcome Probabilities

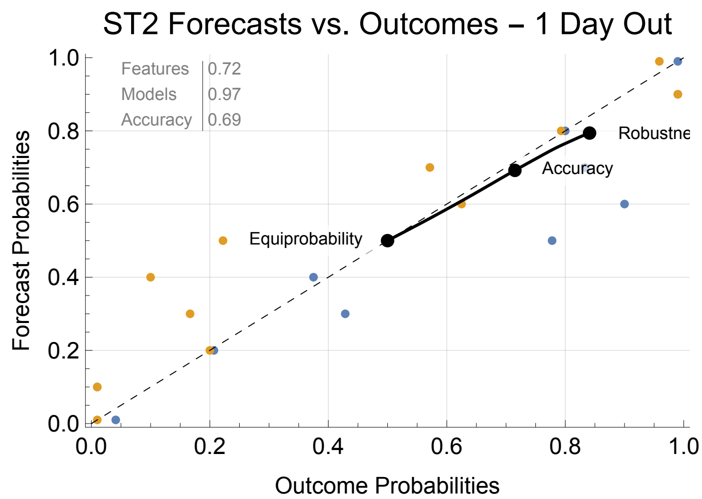
In[∘]:=


ST1 Forecasts vs. Outcomes – 1 Day Out
Features 0.72
Models 0.97
Accuracy 0.70
Robustr
Accuracy
Equiprobability
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

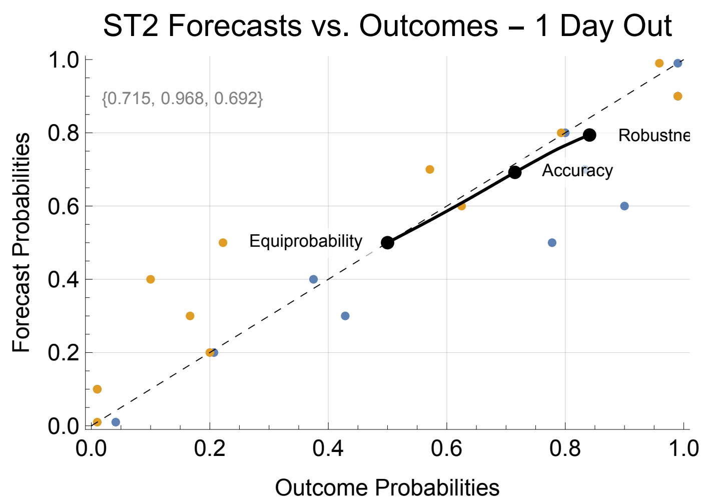
Out[∘]=


ST1 Forecasts vs. Outcomes – 1 Day Out
{0.723, 0.967, 0.699}
Robustr
Accuracy
Equiprobability
Forecast Probabilities
Outcome Probabilities
0.0
0.2
0.4
0.6
0.8
1.0

In[ ]:=

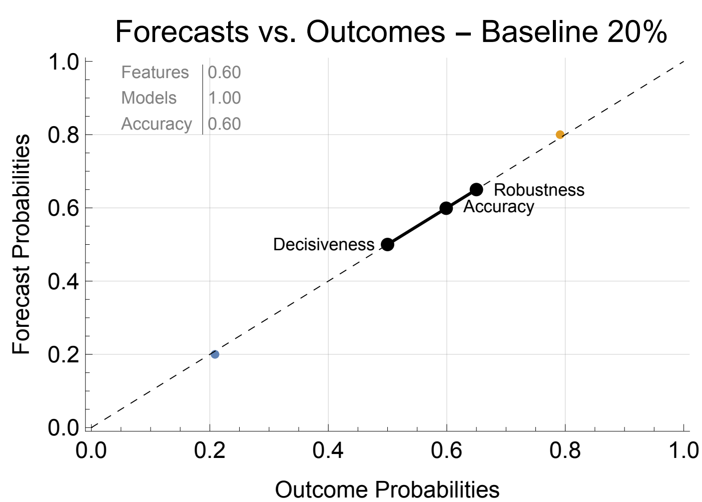


Out[ ]=



Baseline 20%

Out[ ]=



Functions for Coupled Mean Computations

Approach will be to modify the ComputeOverallStats NOAA function

In[149]:=
```
ComputeCoupledMeanNOAA[prevBin_, geoBin_, weightBin_, metricPower_] :=
 Module[{overallStats, prev, geo, weight, weightedPrev,
   numBins, numForecasters, numHypotheses},
  {numBins, numForecasters, numHypotheses} = Dimensions[prevBin];

  {prev, geo, weight} = ArrayReshape[
       Transpose[#, {2, 1, 3}], {numForecasters, numBins * numHypotheses}] & /@
    {prevBin, geoBin, weightBin};
  overallStats = ConstantArray[0, {2, numForecasters}];

  Assert[True]; (* Debug Break Point *)
  overallStats = {
    MapThread[CoupledCrossProb[ (* Frequencies, Forecasts, risk aversion *)
       MapThread[Pick[#2, NumericQ[#1]] &, {#1, #2}],
       MapThread[Pick[#, NumericQ[#]] &, {#1}],
       metricPower
      ] &, {prev, weight}],
    MapThread[CoupledCrossProb[ (* Frequencies, Forecasts, risk aversion *)
       MapThread[Pick[#2, NumericQ[#1]] &, {#1, #2}],
       MapThread[Pick[#2, NumericQ[#1]] &, {#1, #3}],
       metricPower
      ] &, {prev, weight, geo}]
   };
  Round[
   AppendTo[overallStats, overallStats[[2]] / overallStats[[1]]],
   0.001
  ]
 ];
```

In[211]:=
```
(* The CoupledMean takes as inputs:
            1) discrete variable - X,
          2) discrete probability weights - P,
           3) relative risk aversion - r,
and outputs the Coupled Mean
```

$$\text{CoupledMean}[X,r,P] = \left(\sum_{i=1}^{N} \frac{p_i^{1+r}}{\sum_{j=1}^{N} p_i^{1+r}} x_i^r\right)^{\frac{1}{r}}.$$

This is the weighted generalized mean with weights $\frac{p_i^{1+r}}{\sum_{j=1}^{N} p_i^{1+r}}$.

The CoupledCrossProbs[P,Q,r] is the Coupled Mean applied to probabilites Q.
$\text{CoupledCrossProb}[P,Q,r]=\text{CoupledMean}\left[Q^{-1},r,P\right]^{-1}$. This function is also equal to CoupledExponential[-CoupledCrossEntropy[P,Q,κ,d,α],κ,d,α]

The $\text{CoupledDivergenceProbs}[P,Q,r]=\text{CoupledMean}\left[P*Q^{-1},r,P\right]^{-1}$

The $\text{CoupledEntropyProbs}[P,r]= \text{CoupledMean}\left[P^{-1},r,P\right]^{-1}$

The definitions of the Coupled Divergence will need to be reexamined so that there is consistency with the probability expression. Since P* Q^{-1} is being used the divergence will also need to use this instead of CoupledLog[P]- CoupledLog[Q]. Also it is worthwhile to experiment with the two forms of divergence to see if either has more merit.
*)

```
CoupledMean[X_, r_, P_] :=
  WeightedGeneralizedMean[X, r, P^(1+r) / Table[Total[P], Dimensions[P][[1]]]];
CoupledCrossProb[P_, Q_, r_] :=
  CoupledMean[Q^-1, r, P]^-1;
CoupledEntropyProb[P_, r_] := CoupledCrossProb[P, P, r];
(* Add CoupledDivergenceProb as a function of CoupledCrossProb and
 CoupledEntropyProb once analysis on relationship is completed *)
```

Coupled Mean of Coupled Exponential Distributions