Coupled Entropy NSP & Paper 2025

In[∘]:= $\$Assumptions = \kappa > 0 \,\&\&\, \kappa \in \text{Reals} \,\&\&\, x \in \text{Reals} \,\&\&\, \mu \in \text{Reals} \,\&\&\, \sigma \in \text{Reals} \,\&\&\, \sigma > 0 \,\&\&\, \alpha \in \text{Reals} \,\&\&\, \alpha > 0 \,\&\&\, d \in \text{Integers} \,\&\&\, d > 0 \,\&\&\, \gamma \in \text{Reals} \,\&\&\, 0 < \gamma < \infty$

Out[∘]=

$\kappa > 0 \,\&\&\, \kappa \in \mathbb{R} \,\&\&\, x \in \mathbb{R} \,\&\&\, \mu \in \mathbb{R} \,\&\&\, \sigma \in \mathbb{R} \,\&\&\, \sigma > 0 \,\&\&\, \alpha \in \mathbb{R} \,\&\&\, \alpha > 0 \,\&\&\, d \in \mathbb{Z} \,\&\&\, d > 0 \,\&\&\, \gamma \in \mathbb{R} \,\&\&\, 0 < \gamma < \infty$

Compute Entropies of Gen. Pareto Distribution

```
Clear[CECoupledExp]
```

```
Assuming[κ > 0 && x > 0,
 CoupledExponentialDistribution[κ, 0, σ]
]
```

$$\text{ProbabilityDistribution}\left[\begin{cases} \dfrac{\text{If}\left[\kappa\neq 0,\text{If}\left[\text{Simplify}\left[1-\frac{\kappa\,(-x)}{\sigma}\right]>0,\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\kappa}{1\kappa}},0\right],\text{Exp}\left[-\frac{x}{\sigma}\right]\right]}{\sigma} & x \geq 0 \\ 0 & \text{True} \end{cases}, \{x, -\infty, \infty\}\right]$$

Out[∘]=

$\kappa > 0 \,\&\&\, \kappa \in \mathbb{R} \,\&\&\, x \in \mathbb{R} \,\&\&\, \mu \in \mathbb{R} \,\&\&\, \sigma \in \mathbb{R} \,\&\&\, \sigma > 0 \,\&\&\, \alpha \in \mathbb{R} \,\&\&\, \alpha > 0 \,\&\&\, d \in \mathbb{Z} \,\&\&\, d > 0$

This computes the Coupled Entropy of the Pareto Distribution. See below for computation of the coupled probability
and the coupled logarithm of the distribution

$$-\int_0^\infty \left((1+\kappa)\,\sigma^{1+\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-2-\frac{1}{\kappa}}\right)\left(\frac{1-\left(\sigma^{\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-\frac{1+\kappa}{\kappa}}\right)^{-1+\frac{1}{1+\kappa}}}{\kappa}\right)dx \text{ // FullSimplify}$$

$$\frac{-1+(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}}{\kappa}$$

$$(1+\kappa)\,\text{CoupledLogarithm}\left[\sigma^{-1},\kappa,1,1\right]-\frac{1+\kappa}{\kappa} \text{ // FullSimplify}$$

$$-\frac{(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}}{\kappa}$$

In[∘]:= `SECoupledExp[κ_, σ_] := 1 + κ Log[σ];`

In[∘]:= $\text{CECoupledExp}[\kappa\_, \sigma\_] := \dfrac{-1+(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}}{\kappa};$

$\frac{-1+(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}}{\kappa} = \frac{1+\kappa}{\kappa}\left(\sigma^{\frac{\kappa}{1+\kappa}}-1\right)+\frac{1+\kappa}{\kappa}-\frac{1}{\kappa} = 1+\ln_{\frac{1+\kappa}{\kappa}}\sigma$ Confirms 2025 Solution

In[∘]:= $\text{TECoupledExp}[\kappa\_, \sigma\_] := \dfrac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$

In[∘]:= $\dfrac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$ // FullSimplify

Out[∘]=

$$\frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa}$$

$\frac{1+\kappa-\sigma^{-1+\frac{1}{1+\kappa}}}{\kappa} = 1 - \frac{1}{\kappa}\left(\sigma^{-\frac{\kappa}{1+\kappa}} - 1\right)$ Confirms 2025 Result

In[∘]:= $\text{NTECoupledExp}[\kappa\_, \sigma\_] := \dfrac{(1+\kappa)\left(-1+(1+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}\right)}{\kappa}$

$\text{CoupledProbability}\Big[$

$\quad \text{ProbabilityDistribution}\Big[\frac{1}{\sigma}\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\kappa}{1\kappa}}, \{x, 0, \infty\}\Big],$

$\quad \frac{-\kappa}{1+\kappa}, x$

$\Big]$

$$\begin{cases} 0 & x \le 0 \\ (1+\kappa)\,\sigma^{1+\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-2-\frac{1}{\kappa}} & \text{True} \end{cases}$$

$\text{CoupledLogarithm}\Big[$

$\quad \frac{1}{\sigma}\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\kappa}{1\kappa}},$

$\quad \kappa, 1, 1\Big]$ // FullSimplify

$$\text{ConditionalExpression}\left[\frac{1-\left(\sigma^{\frac{1}{\kappa}}\,(x\,\kappa+\sigma)^{-\frac{1+\kappa}{\kappa}}\right)^{-1+\frac{1}{1+\kappa}}}{\kappa},\ (x\,\kappa+\sigma)^{-1-\frac{1}{\kappa}} \ge 0\right]$$

```
Assuming[κ > 0, CoupledEntropy[
   Assuming[κ > 0 && x > 0,
    CoupledExponentialDistribution[κ, 0, σ]
   ], κ, 1, 1]] // Simplify
```

$$-\int_0^\infty \mathrm{If}\Big[\Big(\begin{cases}\sigma^{\frac{1}{\kappa}}\ (x\,\kappa+\sigma)^{-1-\frac{1}{\kappa}} & x\geq 0\\ 0 & \mathrm{True}\end{cases}\Big)\geq 0,$$
$$\mathrm{If}\Big[\kappa\neq 0,\ -\frac{1}{1\,\kappa}\Big(\Big(\begin{cases}\frac{\mathrm{If}\left[\kappa\neq0,\mathrm{If}\left[\mathrm{Simplify}\left[1-\frac{\kappa\,(-x)}{\sigma}\right]>0,\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\,\kappa}{1\,\kappa}},0\right],\mathrm{Exp}\left[-\frac{x}{\sigma}\right]\right]}{\sigma} & x\geq 0\\ 0 & \mathrm{True}\end{cases}\Big)^{-\frac{\kappa}{1+1\,\kappa}}-1\Big),$$
$$\mathrm{Log}\Big[\begin{cases}\frac{\mathrm{If}\left[\kappa\neq0,\mathrm{If}\left[\mathrm{Simplify}\left[1-\frac{\kappa\,(-x)}{\sigma}\right]>0,\left(1-\frac{\kappa\,(-x)}{\sigma}\right)^{-\frac{1+1\,\kappa}{1\,\kappa}},0\right],\mathrm{Exp}\left[-\frac{x}{\sigma}\right]\right]}{\sigma} & x\geq 0\\ 0 & \mathrm{True}\end{cases}\Big]\Big],$$
$$\mathrm{Undefined}\Big]\ \Big(\begin{cases}0 & x<0\\ (1+\kappa)\ \sigma^{1+\frac{1}{\kappa}}\ (x\,\kappa+\sigma)^{-2-\frac{1}{\kappa}} & \mathrm{True}\end{cases}\Big)\,dx$$

```
NSECoupledExp[κ_, σ_] :=
 Assuming[∞ > κ > 0,
  Piecewise[{
    {NShannonEntropy[CoupledExponentialDistribution[κ, 0, σ], 0, 100], 0 < κ < 0.09},
    {NShannonEntropy[
      CoupledExponentialDistribution[κ, 0, σ], 0, 1000], 0.09 ≤ κ < 0.74},
    {NShannonEntropy[
      CoupledExponentialDistribution[κ, 0, σ], 0, 10 000], 0.74 ≤ κ < 1.5},
    {NShannonEntropy[CoupledExponentialDistribution[κ, 0, σ], 0, 15 000], 1.5 ≤ κ}
   }]
 ]
```

```
NSECoupledExp[#, 1] & /@ {0.25, 0.5, 0.75, 1, 1.25}
```

{1.25, 1.49992, 1.74985, 1.99796, 2.23985}

```
NSECoupledExp[0.01, 0.25]
```

-0.376294

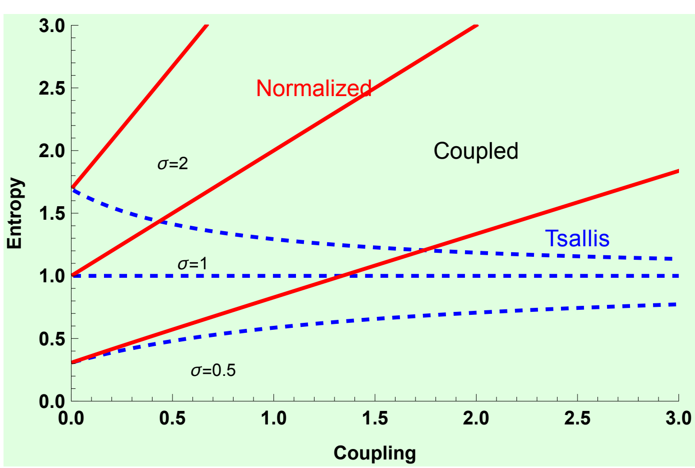
```
NSECoupledExp[2, 2]
```

3.54523

Plot Comparison of GPD with TE, NTE, & Shannon

```
Assuming[0 < κ < ∞,
 Plot[
  {
   (*Table[SECoupledExp[κ,σ],{σ,{0.5,1,2}}],*)
   {Style[Table[TECoupledExp[κ, σ], {σ, {0.5, 1, 2}}],
     {Blue, Dashed}]},
   {Style[Table[NTECoupledExp[κ, σ], {σ, {0.5, 1, 2}}],
     {Red, Dotted}]},
   {Style[Table[CECoupledExp[κ, σ], {σ, {0.5, 1, 2}}],
     Black]}
  },
  {κ, 0.01, 3},
  Background → LightGreen,
  PlotRange → {{0, 3}, {0, 3}} ,
  Frame → {{True, False}, {True, False}},
  FrameLabel → {"Coupling", "Entropy"},
  FrameStyle → Directive[10, Bold, Black],
  Epilog → {
    Inset[Style["Coupled", Black, Medium], {2, 2}],
    Inset[Style["Tsallis", Blue, Medium], {2.5, 1.3}],
    Inset[Style["Normalized", Red, Medium], {1.2, 2.5}],
    Inset[Style["σ=2", Black], {0.5, 1.9}],
    Inset[Style["σ=1", Black], {0.6, 1.1}],
    Inset[Style["σ=0.5", Black], {0.7, 0.25}]
   }
 ]
]
```

Out[∘]=



Compute Tsallis Entropy with different q values

In[∘]:=

$$\$\text{Assumptions} = \kappa \in \text{Reals} \;\&\&\; 0 < \kappa < \infty \;\&\&\; \alpha \in \text{Reals} \;\&\&\; 0 < \alpha < \infty \;\&\&\; d \in \text{Reals} \;\&\&\; 0 < d < \infty \;\&\&\; \sigma \in \text{Reals} \;\&\&\; 0 < \sigma < \infty$$

Out[∘]=

$$\kappa \in \mathbb{R} \;\&\&\; 0 < \kappa < \infty \;\&\&\; \alpha \in \mathbb{R} \;\&\&\; 0 < \alpha < \infty \;\&\&\; d \in \mathbb{R} \;\&\&\; 0 < d < \infty \;\&\&\; \sigma \in \mathbb{R} \;\&\&\; 0 < \sigma < \infty$$

$\text{qEnt} = \frac{1}{\text{qDist}}$

In[∘]:=

$$\int_0^\infty \text{FullSimplify}\left[\frac{1}{\sigma}\left(1+\frac{\kappa x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}\left(1+\text{qToCoupling}\left[\frac{1+\kappa}{1+2\kappa}\right]\right)\right.$$
$$\left.\text{CoupledLogarithm}\left[\sigma\left(1+\frac{\kappa x}{\sigma}\right)^{\frac{1+\kappa}{\kappa}},\ \text{qToCoupling}\left[\frac{1+\kappa}{1+2\kappa}\right],\ 1\right]\right] dx \;//\; \text{FullSimplify}$$

$$\int_0^\infty \frac{1}{(2+5\kappa)\,(x\kappa+\sigma)}\, 2\,(1+2\kappa)\left(1+\frac{x\kappa}{\sigma}\right)^{-1/\kappa} \text{If}\left[\left(1+\frac{x\kappa}{\sigma}\right)^{\frac{1}{\kappa}}(x\kappa+\sigma) \geq 0,\right.$$
$$\left.\text{If}\left[-\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}} \neq 0,\ -\frac{\left(\left(1+\frac{x\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}\sigma\right)^{-\frac{1-\frac{1+\kappa}{1+2\kappa}}{\left(3-\frac{1+\kappa}{1+2\kappa}\right)\left(1+-\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}\right)}}-1}{\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}},\ \text{Log}\left[\left(1+\frac{x\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}\sigma\right]\right],\ \text{Undefined}\right] dx$$

In[∘]:=

$$\frac{2}{(2+5\kappa)\,(x\kappa+\sigma)}\,(1+2\kappa)\left(1+\frac{x\kappa}{\sigma}\right)^{-1/\kappa}\left(-\frac{\left(\left(1+\frac{x\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}\sigma\right)^{-\frac{1-\frac{1+\kappa}{1+2\kappa}}{\left(3-\frac{1+\kappa}{1+2\kappa}\right)\left(1+-\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}\right)}}-1}{\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}}\right) \;//\; \text{FullSimplify}$$

Out[∘]=

$$-\frac{2\,(1+2\kappa)\left(1+\frac{x\kappa}{\sigma}\right)^{-1/\kappa}\left(-1+\left(\left(1+\frac{x\kappa}{\sigma}\right)^{\frac{1}{\kappa}}(x\kappa+\sigma)\right)^{-\frac{\kappa}{2+4\kappa}}\right)}{\kappa\,(x\kappa+\sigma)}$$

In[∘]:=

$$\int_0^\infty -\frac{2\,(1+2\kappa)\left(1+\frac{x\kappa}{\sigma}\right)^{-1/\kappa}\left(-1+\left(\left(1+\frac{x\kappa}{\sigma}\right)^{\frac{1}{\kappa}}(x\kappa+\sigma)\right)^{-\frac{\kappa}{2+4\kappa}}\right)}{\kappa\,(x\kappa+\sigma)}\, dx \;//\; \text{FullSimplify}$$

Out[∘]=

$$-\frac{2\,(1+2\kappa)\,\sigma^{\frac{1}{\kappa}}\left(-\sigma^{-1/\kappa}+\frac{(2+4\kappa)\,\sigma^{-\frac{2+4\kappa+\kappa^2}{2\kappa+4\kappa^2}}}{2+\kappa\,(5+\kappa)}\right)}{\kappa}$$

qEnt = 2-qDist

In[∘]:=
$$\int_0^\infty \text{FullSimplify}\left[\frac{1}{\sigma}\left(1+\frac{\kappa\,x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}\left(1+\text{qToCoupling}\left[1-\frac{\kappa}{1+\kappa}\right]\right)\text{CoupledLogarithm}\left[\sigma\left(1+\frac{\kappa\,x}{\sigma}\right)^{\frac{1+\kappa}{\kappa}},\text{qToCoupling}\left[1-\frac{\kappa}{1+\kappa}\right],1\right]\right]dx\ //\ \text{FullSimplify}$$

Out[∘]=
$$\frac{2\,(1+\kappa)\left(1-\frac{2\,\sigma^{-\frac{\kappa}{2+2\kappa}}}{2+\kappa}\right)}{\kappa}$$

Dual kEnt = $-\frac{\kappa}{1+\kappa}$

In[∘]:=
$$\int_0^\infty \text{FullSimplify}\left[\frac{1}{\sigma}\left(1+\frac{\kappa\,x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}\left(1-\frac{\kappa}{1+\kappa}\right)\text{CoupledLogarithm}\left[\sigma\left(1+\frac{\kappa\,x}{\sigma}\right)^{\frac{1+\kappa}{\kappa}},-\frac{\kappa}{1+\kappa},1\right]\right]dx\ //\ \text{FullSimplify}$$

Out[∘]=
$$\frac{1-\frac{\sigma^{-\kappa}}{1+\kappa+\kappa^2}}{\kappa}$$

Coupled Entropy of Coupled Stretched Exponential Distribution

Structure of solution is $\frac{d}{\alpha}+\text{Log}_{\frac{\alpha\kappa}{1+d\kappa}}[Z_\kappa(\sigma,\alpha,d)]=\frac{d}{\alpha}+\text{Log}_{\frac{\alpha\kappa}{1+d\kappa}}[\sigma]\oplus_{\frac{\alpha\kappa}{1+d\kappa}}\text{Log}_{\frac{\alpha\kappa}{1+d\kappa}}[Z'_\kappa(\alpha,d)]$

Computation of Coupled Entropy of Coupled Stretched Exponential

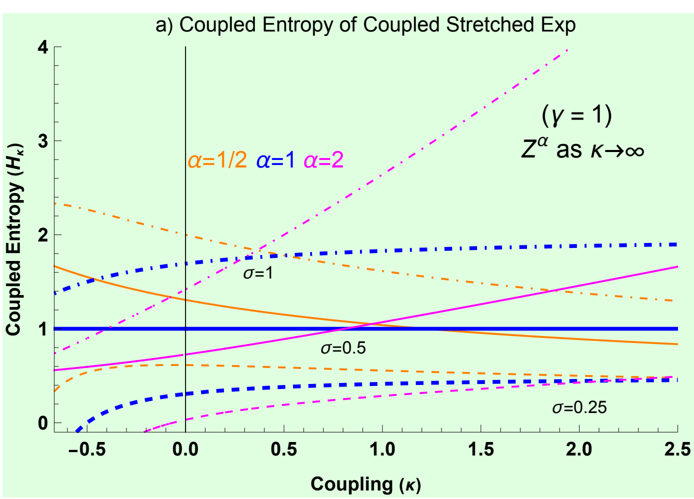
In[∘]:=
```
Clear[CoupledEntropyCSE];
CoupledEntropyCSE[σ_, κ_, α_, d_] :=
  CoupledEntropyCSE[σ, κ, α, d] =
   d/α + CoupledLogarithm[
     NormMultiCoupled[σ, κ, α, d],
     α κ/(1 + d κ), 0];
```

Plot of Coupled Entropies

In[•]:=

```
Plot[
 Evaluate@{
   Table[CoupledEntropyCSE[σ, κ, 1 / 2, 1]^1, {σ, {0.25, 0.5, 1}}],
   Table[CoupledEntropyCSE[σ, κ, 1, 1]^1, {σ, {0.25, 0.5, 1}}],
   Table[CoupledEntropyCSE[σ, κ, 2, 1]^1, {σ, {0.25, 0.5, 1}}],
  },
 {κ, -0.667, 2.5},
 Background → LightGreen,
 PlotRange → {{-0.667, 2.5}, {-0.1, 4}} ,
 Frame → {{True, False}, {True, False}},
 FrameLabel → {"Coupling (κ)", "Coupled Entropy (Hκ)"},
 FrameStyle → Directive[10, Bold, Black],
 PlotStyle → Flatten[Table[{αCol, σCol},
    {αCol, {{Orange, Thickness[Medium]}, {Blue, Thickness[Large]},
      {Magenta, Thickness[Medium]}}}, {σCol, {Dashed, , DotDashed}}], 1],
 PlotLabel →
  "a) Coupled Entropy of Coupled Stretched Exp",
 Epilog → {
   Inset[Style["α=1/2", Orange, Medium], {.16, 2.8}],
   Inset[Style["α=1", Blue, Medium], {.46, 2.8}],
   Inset[Style["α=2", Magenta, Medium], {.7, 2.8}],
   Inset[Style["σ=1", Black], {0.37, 1.6}],
   Inset[Style["σ=0.5", Black], {0.8, .8}],
   Inset[Style["σ=0.25", Black], {2.0, 0.17}],
   Inset[Style[DisplayForm[HoldForm[(γ = 1) "
 Z^α as κ→∞"]], Black, FontSize → 13], {2, 3.1}]
  }
]
```

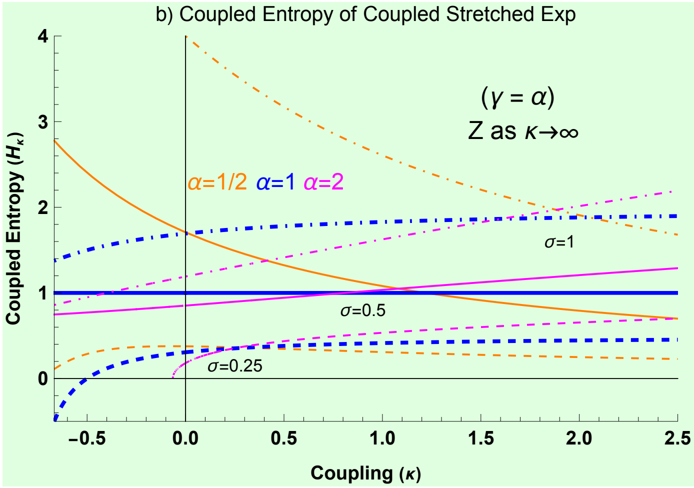
Out[•]=



In[ ]:=

```
Plot[
 Evaluate@{
   Table[CoupledEntropyCSE[σ, κ, 1 / 2, 1]^(2/1), {σ, {0.25, 0.5, 1}}],
   Table[CoupledEntropyCSE[σ, κ, 1, 1], {σ, {0.25, 0.5, 1}}],
   Table[CoupledEntropyCSE[σ, κ, 2, 1]^(1/2), {σ, {0.25, 0.5, 1}}],
  },
 {κ, -0.667, 2.5},
 Background → LightGreen,
 PlotRange → {{-0.667, 2.5}, {-.5, 4}} ,
 Frame → {{True, False}, {True, False}},
 FrameLabel → {"Coupling (κ)", "Coupled Entropy (Hκ)"},
 FrameStyle → Directive[10, Bold, Black],
 PlotStyle → Flatten[Table[{αCol, σCol},
    {αCol, {{Orange, Thickness[Medium]}, {Blue, Thickness[Large]},
      {Magenta, Thickness[Medium]}}}, {σCol, {Dashed, , DotDashed}}], 1],
 PlotLabel →
  "b) Coupled Entropy of Coupled Stretched Exp",
 Epilog → {
   Inset[Style["α=1/2", Orange, Medium], {.16, 2.3}],
   Inset[Style["α=1", Blue, Medium], {.46, 2.3}],
   Inset[Style["α=2", Magenta, Medium], {.7, 2.3}],
   Inset[Style["σ=1", Black], {1.9, 1.6}],
   Inset[Style["σ=0.5", Black], {0.9, .8}],
   Inset[Style["σ=0.25", Black], {0.25, 0.15}],
   Inset[Style[DisplayForm[HoldForm[(γ = α) "
 Z as κ→∞"]], Black, FontSize → 13], {1.7, 3.1}]
  }
]
```

Out[ ]=

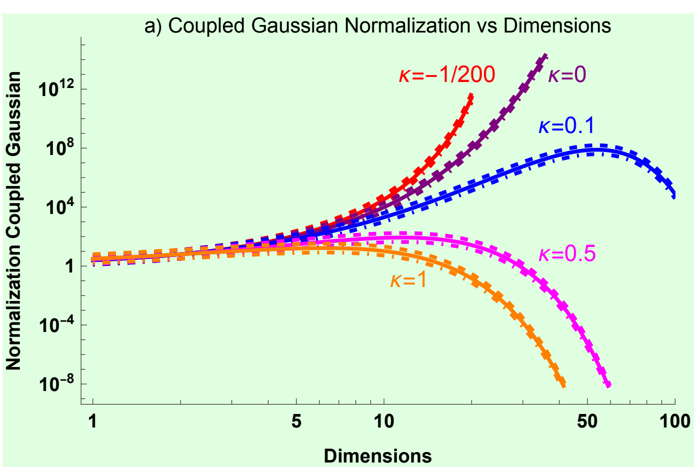


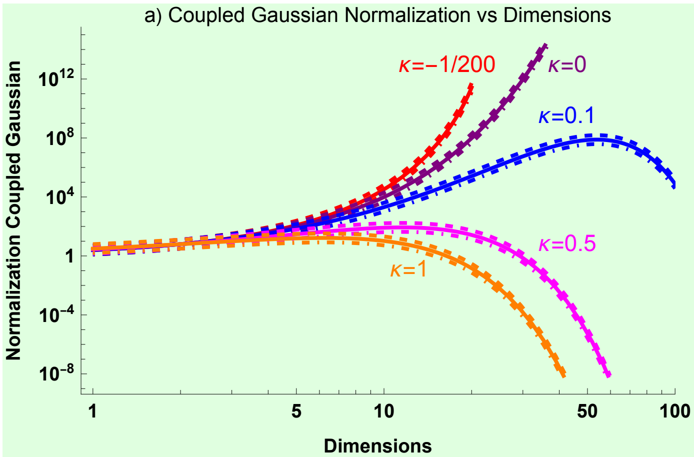
Plot with variable dimensions

In[ ]:=

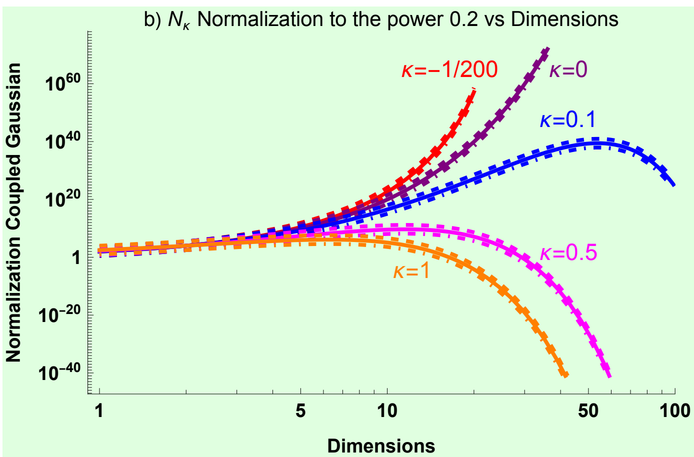
```
LogLogPlot[
 Evaluate@{
   Table[NormMultiCoupled[σ, -0.5×10^-1, 2, d], {σ, {0.5, 1, 2}}],
   Table[NormMultiCoupled[σ, 0.0, 2, d], {σ, {0.5, 1, 2}}],
   Table[NormMultiCoupled[σ, 0.1, 2, d], {σ, {0.5, 1, 2}}],
   Table[NormMultiCoupled[σ, 0.5, 2, d], {σ, {0.5, 1, 2}}],
   Table[NormMultiCoupled[σ, 1.0, 2, d], {σ, {0.5, 1, 2}}],
  },
 {d, 1, 100},
 Background → LightGreen,
 PlotRange → {{0, 100}, {0, Automatic}} ,
 Frame → {{True, False}, {True, False}},
 FrameLabel → {"Dimensions", "Normalization Coupled Gaussian"},
 FrameStyle → Directive[10, Bold, Black] ,
 PlotStyle → Flatten[Table[{κCol, σCol},
    {κCol, {Red, Purple, Blue, Magenta, Orange}}, {σCol, {DotDashed, , Dashed}}], 1],
 PlotLabel → "a) Coupled Gaussian Normalization vs Dimensions",
 Epilog → {
   Inset[Style["κ=-1/200", Red, Medium], {2.8, 30}],
   Inset[Style["κ=0", Purple, Medium], {3.75, 30}],
   Inset[Style["κ=0.1", Blue, Medium], {3.75, 22}],
   Inset[Style["κ=0.5", Magenta, Medium], {3.75, 2}],
   Inset[Style["κ=1", Orange, Medium], {2.5, -2}],
   (*Inset[Style["σ=2",Black],{1.45,1.45}],
   Inset[Style["σ=1",Black],{0.8,.8}],
   Inset[Style["σ=0.5",Black],{0.5,0.1}]*)
  }
]
```

Out[ ]=





With $\gamma = 2/10$



Score Function Plots

The score function of the GPD is $-\sigma^{-1}$, which is a powerful description of the scales unique properties. The score function is computed from the derivative of the log of the pdf.

In[∘]:=

$$\text{NegDerGPD}[\sigma\_, \kappa\_, x\_] := \frac{1+\kappa}{x\,\kappa+\sigma};$$

$$\text{NegDerGPDq}[\beta\_, q\_, x\_] := \frac{\beta}{1+(-1+q)\,x\,\beta};$$

In[∘]:=

```
Clear[NegDerGPD]
```

In[∘]:=

$$\text{Assuming}\left[0<\kappa<\infty,\ -D\left[\text{Log}\left[\frac{1}{\sigma}\left(1+\frac{\kappa\,x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}\right], x\right]\right]\ //\ \text{FullSimplify}$$

Out[∘]=

$$\frac{1+\kappa}{x\,\kappa+\sigma}$$

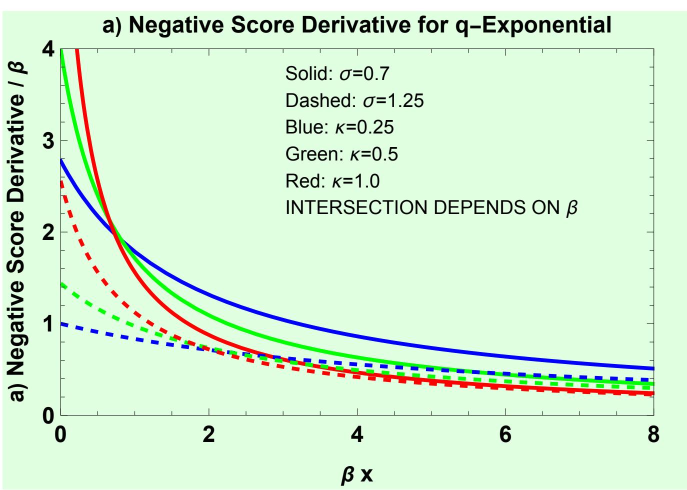
In[∘]:=
$$\frac{1+\kappa}{x\,\kappa+\sigma}\;/.\;\left\{\kappa\to\frac{-(1-q)}{2-q},\;\sigma\to\frac{1}{(2-q)\,\beta}\right\}\;//\;\text{FullSimplify}$$

Out[∘]=
$$\frac{\beta}{1+(-1+q)\,x\,\beta}$$

Plot -β Score versus $\frac{x}{\beta}$

In[∘]:=
```
Plot[MapThread[scaleShapeToBeta[#1, #2, 1] ×
      NegDerGPD[#1, #2, scaleShapeToBeta[#1, #2, 1] x] &,
    {{0.75, 0.75, 0.75, 1.25, 1.25, 1.25}, {0.25, 0.5, 1, 0.25, 0.5, 1}}] //
   Evaluate, {x, 0.0001, 10},
  Background → LightGreen,
  PlotRange → {{0, 8}, {0, 4}},
  PlotStyle → {Blue, Green, Red, {Blue, Dashed}, {Green, Dashed}, {Red, Dashed}},
  Epilog → Inset[Style[Text["Solid: σ=0.7
Dashed: σ=1.25
Blue: κ=0.25
Green: κ=0.5
Red: κ=1.0
INTERSECTION DEPENDS ON β"], Larger], {5, 3}],
  LabelStyle → Directive[Bold, Medium],
  Frame → {{True, True}, {True, True}},
  FrameLabel → {{"a) Negative Score Derivative / β", ""}, {"β x",
     "a) Negative Score Derivative for q-Exponential"}}
]
```

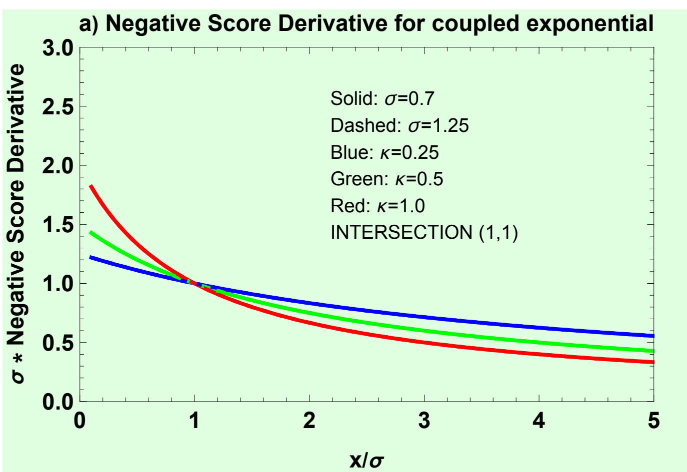
Out[∘]=



Translation of sigma and kappa

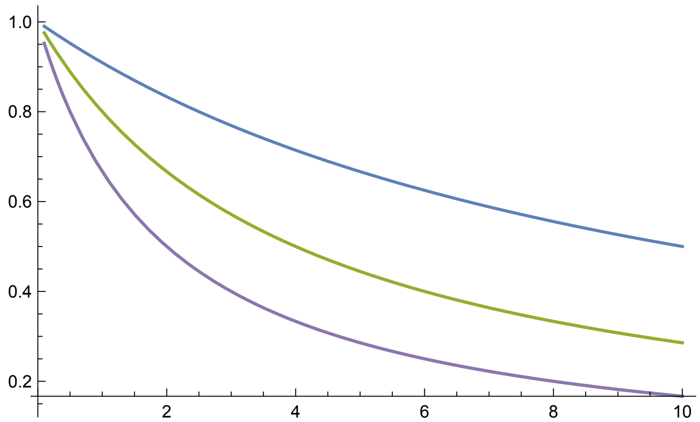
```
Plot[MapThread[
   #1 NegDerGPD[#1, #2, #1 x] &, {{0.75, 0.75, 0.75, 1.25, 1.25, 1.25},
    {0.25, 0.5, 1, 0.25, 0.5, 1}}] // Evaluate, {x, 0.1, 10},
 Background → LightGreen,
 PlotRange → {{0, 5}, {0, 3}},
 PlotStyle → {Blue, Green, Red, {Blue, Dashed}, {Green, Dashed}, {Red, Dashed}},
 Epilog → Inset[Style[Text["Solid: σ=0.7
Dashed: σ=1.25
Blue: κ=0.25
Green: κ=0.5
Red: κ=1.0
INTERSECTION (1,1)"], Larger], {3, 2}],
 LabelStyle → Directive[Bold, Medium],
 Frame → {{True, True}, {True, True}},
 FrameLabel → {{"σ * Negative Score Derivative", ""},
   {"x/σ", "a) Negative Score Derivative for Coupled Exponential"}}
]
```

Out[ ]=



In[∘]:=
```
Plot[MapThread[#1^-1 NegDerGPDq[#1, #2, #1^-1 x] &,
    {{0.5, 0.5, .2, .2, .5}, {1.1, 1.25, 1.25, 1.5, 1.5}}] // Evaluate, {x, 0.1, 10}]
```

Out[∘]=



Generalized Weibull Distribution

In[∘]:= GWDSF[σ_, κ_, α_, x_] := $\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}}$;

In[∘]:= GWDPDF[σ_, κ_, α_, x_] := $\frac{x^{\alpha-1}}{\sigma^{\alpha}}\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}-1}$;

In[∘]:=
```
D[-Log[GWDPDF[σ, κ, α, x]], x] \\ FullSimplify
```

Syntax: "D[−Log[GWDPDF[σ, κ, α, x]], x] \\ FullSimplify" is incomplete; more input is needed.

In[∘]:= D$\left[-\text{Log}\left[\frac{x^{\alpha-1}}{\sigma^{\alpha}}\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}-1}\right], x\right]$ \\ FullSimplify

Syntax: "D$\left[-\text{Log}\left[\frac{x^{\alpha-1}}{\sigma^{\alpha}}\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}-1}\right], x\right]$ \\ FullSimplify" is incomplete; more input is needed.

In[∘]:= $-\partial_x \text{Log}\left[x^{-1+\alpha}\,\sigma^{-\alpha}\,(1+x^{\alpha}\,\kappa\,\sigma^{-\alpha})^{-1-\frac{1}{\alpha\kappa}}\right]$ // FullSimplify

Out[∘]=

$$\frac{x^{\alpha}\,(1+\kappa)-(-1+\alpha)\,\sigma^{\alpha}}{x\,(x^{\alpha}\,\kappa+\sigma^{\alpha})}$$

In[∘]:= $\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}}+(1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}$ /. x → σ // FullSimplify

Out[∘]=

$$\frac{2-\alpha+\kappa}{\sigma+\kappa\,\sigma}$$

So the use of the Generalized Weibull will require care regarding the definition for the scale of the distribution

If $\alpha = 2$, then

$$\frac{1}{\sigma}\ \frac{\kappa}{1+\kappa}$$

So if this is defined as σW let’s see what happens. Could also do this generally for alpha.

In[ ]:= $\frac{1}{\frac{2-\alpha+\kappa}{\sigma+\kappa\,\sigma}}$

Out[ ]= $\frac{\sigma+\kappa\,\sigma}{2-\alpha+\kappa}$

In[ ]:= $\sigma W = \frac{\sigma+\kappa\,\sigma}{2-\alpha+\kappa};$

Recompute the derivative using the expression - $\frac{f'(x)}{f(x)}$

In[ ]:= $D\left[x^{-1+\alpha}\,\sigma^{-\alpha}\,(1+x^{\alpha}\,\kappa\,\sigma^{-\alpha})^{-1-\frac{1}{\alpha\kappa}},\ x\right]$ // FullSimplify

Out[ ]= $x^{-2+\alpha}\,\sigma^{\frac{1}{\kappa}}\,(x^{\alpha}\,\kappa+\sigma^{\alpha})^{-2-\frac{1}{\alpha\kappa}}\,(-x^{\alpha}\,(1+\kappa)+(-1+\alpha)\,\sigma^{\alpha})$

In[ ]:= $-\frac{x^{-2+\alpha}\,\sigma^{\frac{1}{\kappa}}\,(x^{\alpha}\,\kappa+\sigma^{\alpha})^{-2-\frac{1}{\alpha\kappa}}\,(-x^{\alpha}\,(1+\kappa)+(-1+\alpha)\,\sigma^{\alpha})}{x^{-1+\alpha}\,\sigma^{-\alpha}\,(1+x^{\alpha}\,\kappa\,\sigma^{-\alpha})^{-1-\frac{1}{\alpha\kappa}}}$ // FullSimplify

Out[ ]= $\frac{x^{\alpha}\,(1+\kappa)-(-1+\alpha)\,\sigma^{\alpha}}{x\,(x^{\alpha}\,\kappa+\sigma^{\alpha})}$

Confirmed that it is the same expression

Consider solving for x, such that the score is equal to its inverse.

In[ ]:= Solve$\left[\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}}+(1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1} == \left(\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}}+(1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}\right)^{-1},\ x\right]$

Solve$\left[\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}}+(1-\alpha)\,x^{-1}\right)^{2}\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-2} == 1,\ x\right]$

$\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}}+(1-\alpha)\,x^{-1}\right)^{2}\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-2}$ // FullySimplify

Considering possible values of the scale

In[ ]:= $\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}}+(1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}$ /. $x\to\sigma\,\frac{\kappa}{\alpha}$ // FullSimplify

Out[ ]= $\frac{\alpha\left(1+\kappa-\frac{(1+\alpha\,\kappa)\,\sigma^{\alpha}}{\sigma^{\alpha}+\kappa\left(\frac{\kappa\,\sigma}{\alpha}\right)^{\alpha}}\right)}{\kappa^{2}\,\sigma}$

Examine the log-log derivative

From the following, this is x times derivative of the Log[

Therefore:

$$\frac{d}{d(\log x)}\log(f(x)) = \frac{f'(x)}{f(x)}\cdot x = x\cdot\frac{f'(x)}{f(x)}$$

In[∘]:= $\text{FullSimplify}\left[x\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}} + (1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}\right]$

Out[∘]=

$$1-\alpha+\frac{x^{\alpha}\,(1+\alpha\,\kappa)}{x^{\alpha}\,\kappa+\sigma^{\alpha}}$$

$\text{Solve}\left[1-\alpha+\frac{x^{\alpha}\,(1+\alpha\,\kappa)}{x^{\alpha}\,\kappa+\sigma^{\alpha}} == 1,\ x\right]$

Out[∘]=

$Aborted

$\text{Solve}\left[\frac{x^{\alpha}}{\sigma^{\alpha}}\ \frac{(1+\alpha\,\kappa)}{1+\frac{x^{\alpha}}{\sigma^{\alpha}}\,\kappa} == \alpha,\ x\right]$

If $x = \alpha^{1/\alpha}\,\sigma$

$\text{Solve}\left[\frac{\alpha\,\sigma^{\alpha}}{\sigma^{\alpha}} == \alpha,\ x\right]$

In[∘]:= $1-\alpha+\frac{x^{\alpha}\,(1+\alpha\,\kappa)}{x^{\alpha}\,\kappa+\sigma^{\alpha}}$ /. $x \to \alpha^{1/\alpha}\,\sigma$ // FullSimplify

Out[∘]=

1

In[∘]:= $\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}} + (1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}$ /. $x \to \alpha^{1/\alpha}\,\sigma$ // FullSimplify

Out[∘]=

$$\frac{\alpha^{-1/\alpha}}{\sigma}$$

In[∘]:= $x\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}} + (1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}$ /. $x \to \alpha^{1/\alpha}\,\sigma$ // FullSimplify

Out[∘]=

1

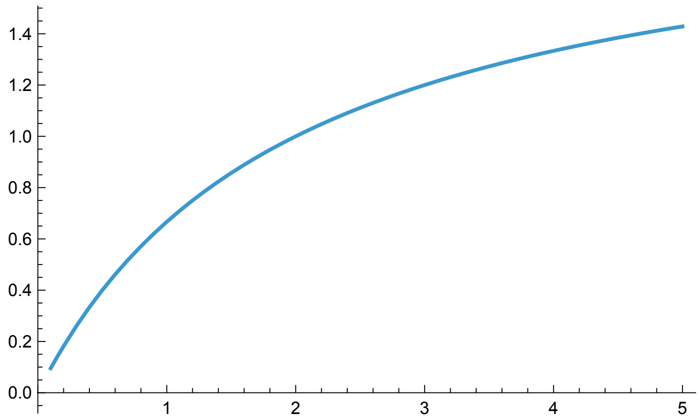
$x\left(\frac{x^{\alpha-1}\,(1+\kappa)}{\sigma^{\alpha}} + (1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}$ /. $x \to$

This is equivalent to modifying the definition of the Coupled Weibull Distribution to be

$$\frac{x^{\alpha-1}}{\sigma^{\alpha}}\left(1+\alpha\kappa\,\frac{x^{\alpha}}{\alpha\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}-1}$$

In[•]:= $\text{Plot}\left[x\left(\frac{x^{\alpha-1}(1+\kappa)}{\sigma^{\alpha}}+(1-\alpha)\,x^{-1}\right)\left(1+\frac{\kappa\,x^{\alpha}}{\sigma^{\alpha}}\right)^{-1}\text{/. }\{\kappa\to1,\alpha\to1,\sigma\to2\},\{x,0.1,5\}\right]$

Out[•]=



Redefined Coupled Weibull with σ equal to the “Informational Scale”

In[•]:= $\text{GWDSF}[\sigma\_,\kappa\_,\alpha\_,x\_]:=\left(1+\alpha\,\kappa\frac{x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}};$

In[•]:= $\text{GWDPDF}[\sigma\_,\kappa\_,\alpha\_,x\_]:=\frac{\alpha\,x^{\alpha-1}}{\sigma^{\alpha}}\left(1+\alpha\,\kappa\frac{x^{\alpha}}{\sigma^{\alpha}}\right)^{-\frac{1}{\alpha\kappa}-1};$

In[•]:= $-x\frac{\text{D}[\text{GWDPDF}[\sigma,\kappa,\alpha,x],x]}{\text{GWDPDF}[\sigma,\kappa,\alpha,x]}\text{ // FullSimplify}$

Out[•]=

$$1+\frac{\alpha\,(x^{\alpha}-\sigma^{\alpha})}{x^{\alpha}\,\alpha\,\kappa+\sigma^{\alpha}}$$

In[•]:= $1+\frac{\alpha\,(x^{\alpha}-\sigma^{\alpha})}{x^{\alpha}\,\alpha\,\kappa+\sigma^{\alpha}}\text{ /. }x\to\sigma\text{ // FullSimplify}$

Out[•]=

1

Score of Coupled Stretched Exponential Distribution

Computing - d Log[f(x)]/dx where f(x) is the Coupled Stretched Exponential Distribution. The normalization, Z, can be ignored since the expression is equivalent to -d f’(x)/(f(x) dx) and constants cancel.

In[•]:= $\partial_x\text{Log}\left[\left(1+\kappa\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{-\frac{1}{\alpha}\left(\frac{1}{\kappa}+1\right)}\right]\text{ // FullSimplify}$

Out[•]=

$$-\frac{x^{-1+\alpha}\,(1+\kappa)}{x^{\alpha}\,\kappa+\sigma^{\alpha}}$$

In[∘]:=
```
x^(-1+α) (1+κ) / (x^α κ + σ^α) /. x → σ // FullSimplify
```
$$\frac{x^{-1+\alpha}\,(1+\kappa)}{x^{\alpha}\kappa+\sigma^{\alpha}}\ /.\ x\to\sigma\ //\ \text{FullSimplify}$$

Out[∘]=
$$\frac{1}{\sigma}$$

In[∘]:=
$$-x\,\partial_x \text{Log}\left[\left(1+\kappa\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{-\frac{1}{\alpha}\left(\frac{1}{\kappa}+1\right)}\right]\ //\ \text{FullSimplify}$$

Out[∘]=
$$\frac{(1+\kappa)\left(\frac{x}{\sigma}\right)^{\alpha}}{1+\kappa\left(\frac{x}{\sigma}\right)^{\alpha}}$$

In[∘]:=
$$\frac{(1+\kappa)\left(\frac{x}{\sigma}\right)^{\alpha}}{1+\kappa\left(\frac{x}{\sigma}\right)^{\alpha}}\ /.\ x\to\sigma\ //\ \text{FullSimplify}$$

Out[∘]=
$$1$$

In[∘]:=
$$(1+\kappa)\left(1+\kappa\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{-1+\frac{1+\frac{1}{\kappa}}{\alpha}}\left(\frac{x}{\sigma}\right)^{\alpha}\ /.\ x\to\sigma\ //\ \text{FullSimplify}$$

Out[∘]=
$$(1+\kappa)^{\frac{1+\frac{1}{\kappa}}{\alpha}}$$

Multiplicative Noise Simulation

Lemma (Multiplicative Process With Coupled Gaussian Limit)
Let a stochastic process X_t be defined by the Stratonovich differential equation,

$$dX_t = \underbrace{f(X_t)dt}_{\text{Drift}} + \underbrace{A\circ dW_t^{(a)}}_{\text{Additive Noise}} + \underbrace{g(X_t)M\circ dW_t^{(m)}}_{\text{Multiplicative Noise}}$$

where $dW_t^{(a)}$ and $W_t^{(m)}$ are independent Wiener processes which define the additive (a) and multiplicative (m) noise, and A and M are the amplitudes of each noise source. Let the drift, $J(x)=f(X_t)$, be related to the diffusion, $D(x)=\frac{1}{2}(A^2+M^2g^2(x))$ by a restorative potential, $V(x)$, in which $f(x)=-\tau g(x)g'(x)=-V'(x)$ and τ is a time-domain scaling. Then the probability density $p_X(x,t)$ for this system has a coupled Gaussian limit distribution of $p_X(x)=\lim_{t\to\infty}p_X(x,t)\propto \exp_\kappa^{-\frac{1+\kappa}{2}}\left(\frac{g^2(x)}{\sigma^2}\right)$, with $\sigma^2=\frac{A^2}{2\tau}$ and $\kappa=\frac{D'(x)}{V'(x)}=\frac{M^2}{2\tau}$.

Define Stratonovich Process for Velocity with Multiplicative Noise and a Coupled Gaussian Limit Distribution

In[∘]:=
```
MultVelProc = StratonovichProcess[dv[t] == -v[t] dt + A dWAdd[t] + M v[t] × dWMult[t],
   v[t], {v, 1}, t, {WAdd ≈ WienerProcess[], WMult ≈ WienerProcess[]}]
```

Out[∘]=
```
StratonovichProcess[{{-v[t]}, {{A, M v[t]}}, v[t]}, {{v}, {1}}, {t, 0}]
```

In[ ]:=
```
MultVelProc["KolmogorovForwardEquation"] // TraditionalForm
```

Out[ ]//TraditionalForm=

$$p^{(0,1)}(v,t) = \frac{1}{2}\,\frac{\partial^2\left(\left(A^2 + M^2 v^2\right) p(v,t)\right)}{\partial v\,\partial v} - \frac{\partial}{\partial v}\left(\frac{1}{2}\left(M^2 v - 2v\right) p(v,t)\right)$$

In[ ]:=
```
Limit[PDF[MultVelProc[t] /. A → √2 σ, v] // FullSimplify, t → Infinity]
```

Out[ ]=
```
lim PDF[
t→∞
  StratonovichProcess[{{-v[t]}, {{√2 σ, M v[t]}}, v[t]^2}, {{v}, {1}}, {t, 0}][t], v]
```

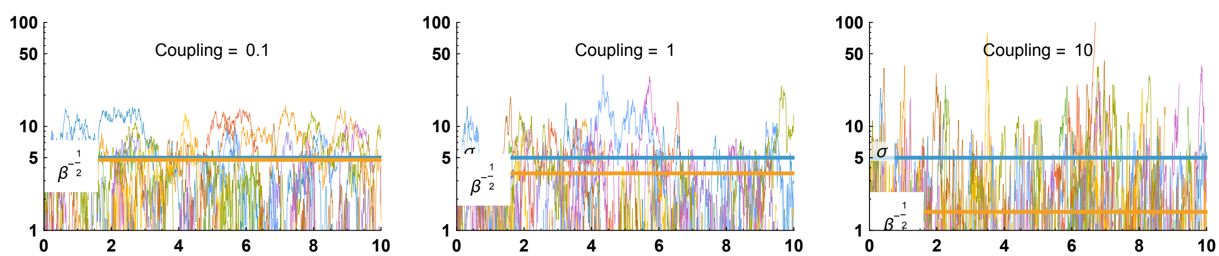
In[ ]:=
```
AddSample[σ_, κ_] :=
  RandomFunction[AddVelProc /. {A → σ Sqrt[2], M → Sqrt[2 * κ]}, {0, 10, .01}, 50];
```

In[ ]:=
```
MultSample[σ_, κ_] :=
  RandomFunction[MultVelProc /. {A → σ Sqrt[2], M → Sqrt[2 * κ]}, {0, 10, .01}, 10];
```

$\sigma = 2$, $\kappa = 0.1$

In[∘]:=
```
Module[{σ = 5, κ = {0.1, 1, 10}, yLimit = {1, 100}},
 GraphicsRow[Table[
   Show[
    ListLinePlot[MultSample[σ, κ[[i]]],
     PlotRange → {{0, 10}, {yLimit[[1]], yLimit[[2]]}},
     PlotStyle → Thin,
     ScalingFunctions → {"Linear", "Log"},
     LabelStyle → Bold,
     Epilog → Inset["Coupling = " Text[κ[[i]]], {5, 4}]
    ],
    Plot[{Labeled[σ, "σ", {Bottom, Left}],
      Labeled[σ/Sqrt[1 + κ[[i]]], "β^(-1/2)", {Top, Left}]}, {x, 0, 10},
     PlotRange → {{0, 10}, {yLimit[[1]], yLimit[[2]]}},
     PlotStyle → Thick,
     ScalingFunctions → {"Linear", "Log"}
    ]
   ]
   , {i, 3}]]]
```

Out[∘]=



Checking DeepSeek Solutions

In[∘]:= $\text{Solve}\left[1 + \frac{\tau}{\alpha M} == \frac{1+\kappa}{\alpha\kappa}, \kappa\right]$

Out[∘]=

$$\left\{\left\{\kappa \to \frac{M}{-M + M\alpha + \tau}\right\}\right\}$$

In[∘]:= $\frac{M}{-M + M\,\alpha + \tau}\;\frac{A}{M} = \frac{A}{-M + M\,\alpha + \tau}$

$\text{Solve}\left[\frac{\tau + \frac{3}{4}\,\alpha\,M}{\alpha\,M} == \frac{1+\kappa}{\kappa},\ \kappa\right]$ \\ FullSimplify

••• Set: Tag Times in $\frac{A\,M}{M\,(-M + M\,\alpha + \tau)}$ is Protected.

Out[∘]=

$\frac{A}{-M + M\,\alpha + \tau}$

In[∘]:= $\text{Solve}\left[\frac{1 + \frac{3}{4}\,\alpha\,\frac{M}{\tau}}{\alpha\,\frac{M}{\tau}} == \frac{1+\kappa}{\kappa},\ \kappa\right]$ \\ FullSimplify

••• Syntax: "Solve $\left[\frac{\tau + \frac{3}{4}\,\alpha\,M}{\alpha\,M} == \frac{1+\kappa}{\kappa},\ \kappa\right]$ \\ FullSimplify" is incomplete; more input is needed.

In[∘]:= `Clear[A, M, α, τ]`

Check independent equals transformation

In[∘]:= $\text{Solve}\left[\frac{1 + (1+m)\,\kappa}{\kappa} == \frac{1 + \kappa M}{\kappa M},\ \kappa M\right]$

Out[∘]=

$\left\{\left\{\kappa M \to \frac{\kappa}{1 + m\,\kappa}\right\}\right\}$

In[∘]:= $\text{Solve}\left[1 + \frac{1}{M^2} == \frac{1}{2}\left(1 + \frac{1}{\kappa}\right),\ \kappa\right]$

Out[∘]=

{ }

$$2\left(\frac{1}{2} + \frac{1}{M^2}\right) == \frac{1}{\kappa}$$

$$\kappa == \frac{M^2}{1 + M^2}$$

Uncertainty at the Scale: Definition for Gen. Entropy

Since the generalized entropy must compute a solution that is equal to the generalized logarithm of the coupled stretched exponential distribution at the scale, it is illustrative to compute this value for a variety of generalized entropies. For simplicity the case of d=1, $\alpha = 1$, and $\mu = 0$ is computed. The entropy transformed to the domain of the distribution is $\exp_\kappa{}^{-(1+\kappa)}[H]$. This value must be set equal to f(x) and then one must solve for x, which requires taking ln_$\kappa\, f(x)^{-1}$, reversing the coupled exponential. So the equation setting H equal to the argument of the density.

$\exp_\kappa{}^{-(1+\kappa)}[H] = \exp_\kappa{}^{-(1+\kappa)}\left[\frac{x}{\sigma} + \ln_\kappa \sigma^{\frac{1}{1+\kappa}} + \kappa\,\frac{x}{\sigma}\,\ln_\kappa \sigma^{\frac{1}{1+\kappa}}\right]$

$$H = \frac{x}{\sigma}\left(1 + \kappa \ln_\kappa \sigma^{\frac{1}{1+\kappa}}\right) + \ln_\kappa \sigma^{\frac{1}{1+\kappa}}$$

$$\frac{x}{\sigma} = \frac{\sigma\left(H - \ln_\kappa \sigma^{\frac{1}{1+\kappa}}\right)}{\left(1 + \kappa \ln_\kappa \sigma^{\frac{1}{1+\kappa}}\right)}$$

In[∘]:= entValue[H_, σ_, κ_] :=

$$\sigma \frac{\left(\text{H} - \text{CoupledLogarithm}\left[\sigma^{\frac{1}{1+\kappa}}, \kappa, 0\right]\right)}{1 + \kappa\, \text{CoupledLogarithm}\left[\sigma^{\frac{1}{1+\kappa}}, \kappa, 0\right]} \text{ // FullSimplify;}$$

In[∘]:= entDensity[H_, κ_] :=

CoupledExponential[H, κ]$^{-1}$ // FullSimplify;

In[∘]:= HCoupled[σ_, κ_] := 1 + CoupledLogarithm$\left[\sigma, \frac{\kappa}{1+\kappa}, 0\right]$;

In[∘]:= entValue[HCoupled[σ, κ], σ, κ]

Out[∘]=

$$\sigma$$

In[∘]:= entDensity[HCoupled[σ, κ], κ]

Out[∘]=

$$\frac{(1+\kappa)^{-\frac{1+\kappa}{\kappa}}}{\sigma}$$

In[∘]:= HBGS[σ_, κ_] := 1 + Log[σ] + κ;

In[∘]:= entValue[HBGS[σ, κ], σ, κ] // FullSimplify

Out[∘]=

$$\frac{\sigma^{\frac{1}{1+\kappa}}\left(1 + \kappa + \kappa^2 - \sigma^{\frac{\kappa}{1+\kappa}} + \kappa \,\text{Log}[\sigma]\right)}{\kappa}$$

In[∘]:= entDensity[HBGS[σ, κ], κ]

Out[∘]=

$$1 \Big/ \text{If}\Big[1 + \kappa\,(1 + \kappa + \text{Log}[\sigma]) > 0,\; \text{If}\Big[\kappa \neq 0,\; (1 + \kappa\,(1 + \kappa + \text{Log}[\sigma]))^{\frac{1+1\kappa}{\kappa}},\; \text{Exp}[1 + \kappa + \text{Log}[\sigma]]\Big],\; \text{If}\Big[\frac{1 + 1\kappa}{\kappa} > 0,\; 0,\; \infty\Big]\Big]$$

In[∘]:= HRenyi[σ_, κ_] := Log[σ] + $\left(1 + \frac{1}{\kappa}\right)$ Log[1 + κ];

In[∘]:= entValue[HRenyi[σ, κ], σ, κ] // FullSimplify

Out[∘]=

$$\frac{\sigma^{\frac{1}{1+\kappa}}\left(1 - \sigma^{\frac{\kappa}{1+\kappa}} + (1+\kappa)\,\text{Log}[1+\kappa] + \kappa\,\text{Log}[\sigma]\right)}{\kappa}$$

In[∘]:=
```
entDensity[HRenyi[σ, κ], κ]
```

Out[∘]=

$$1\Big/\mathrm{If}\Big[1+\mathrm{Log}[1+\kappa]+\kappa\,\mathrm{Log}[(1+\kappa)\,\sigma]>0,\ \mathrm{If}\Big[\kappa\neq 0,\ \left(1+\kappa\left(\left(1+\frac{1}{\kappa}\right)\mathrm{Log}[1+\kappa]+\mathrm{Log}[\sigma]\right)\right)^{\frac{1+1\kappa}{\kappa}},\ \mathrm{Exp}\Big[\left(1+\frac{1}{\kappa}\right)\mathrm{Log}[1+\kappa]+\mathrm{Log}[\sigma]\Big]\Big],\ \mathrm{If}\Big[\frac{1+1\kappa}{\kappa}>0,\ 0,\ \infty\Big]\Big]$$

In[∘]:=

$$\mathrm{HTsallis}[\sigma\_,\kappa\_] := 1-\frac{1}{1+\kappa}\,\mathrm{CoupledLogarithm}\Big[\sigma^{-1},\ \frac{\kappa}{1+\kappa},\ 0\Big];$$

In[∘]:=
```
entValue[HTsallis[σ, κ], σ, κ] // FullSimplify
```

Out[∘]=

$$\frac{\sigma^{\frac{1}{1+\kappa}}\left(2+\kappa-2\,\mathrm{Cosh}\left[\frac{\kappa\,\mathrm{Log}[\sigma]}{1+\kappa}\right]\right)}{\kappa}$$

In[∘]:=
```
entDensity[HTsallis[σ, κ], κ]
```

Out[∘]=

$$1\Big/\mathrm{If}\Big[(2+\kappa)\,\sigma^{\frac{\kappa}{1+\kappa}}>1,\ \mathrm{If}\Big[\kappa\neq 0,\ \left(1+\kappa\left(1-\frac{1}{1+\kappa}\,\mathrm{If}\Big[\frac{1}{\sigma}\geq 0,\ \mathrm{If}\Big[\frac{\kappa}{1+\kappa}\neq 0,\ \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1+\kappa)\left(1+\frac{0\kappa}{1+\kappa}\right)}}-1}{\frac{\kappa}{1+\kappa}},\ \mathrm{Log}\Big[\frac{1}{\sigma}\Big]\Big],\ \mathrm{Undefined}\Big]\right)\right)^{\frac{1+1\kappa}{\kappa}},$$
$$\mathrm{Exp}\Big[1-\frac{1}{1+\kappa}\,\mathrm{If}\Big[\frac{1}{\sigma}\geq 0,\ \mathrm{If}\Big[\frac{\kappa}{1+\kappa}\neq 0,\ \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1+\kappa)\left(1+\frac{0\kappa}{1+\kappa}\right)}}-1}{\frac{\kappa}{1+\kappa}},\ \mathrm{Log}\Big[\frac{1}{\sigma}\Big]\Big],\ \mathrm{Undefined}\Big]\Big]\Big],$$
$$\mathrm{If}\Big[\frac{1+1\kappa}{\kappa}>0,\ 0,\ \infty\Big]\Big]$$

In[∘]:=

$$\mathrm{HNormTsallis}[\sigma\_,\kappa\_] := 1+(1+\kappa)\,\mathrm{CoupledLogarithm}\Big[\sigma,\ \frac{\kappa}{1+\kappa},\ 0\Big]+\kappa;$$

In[∘]:=
```
entValue[HNormTsallis[σ, κ], σ, κ]
```

Out[∘]=

$$\frac{\sigma^{-1+\frac{2}{1+\kappa}}\left((1+\kappa)^2-\kappa\,\sigma^{\frac{\kappa}{1+\kappa}}-\sigma^{\frac{2\kappa}{1+\kappa}}\right)}{\kappa}$$

In[∘]:=
```
entDensity[HNormTsallis[σ, κ], κ]
```

Out[∘]=

$$1\Big/\,\text{If}\Big[(1+\kappa)^2>\kappa\,\sigma^{\frac{\kappa}{1+\kappa}},\ \text{If}\Big[\kappa\neq 0,\ \left(1+\kappa\left(1+\kappa+(1+\kappa)\ \text{If}\left[\frac{1}{\sigma}\geq 0,\ \text{If}\left[\frac{\kappa}{1+\kappa}\neq 0,\ \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1+\kappa)\left(1+\frac{0\kappa}{1+\kappa}\right)}}-1}{\frac{\kappa}{1+\kappa}},\ \text{Log}\left[\frac{1}{\sigma}\right]\right],\ \text{Undefined}\right]\right)\right)^{\frac{1+1\kappa}{\kappa}},$$

$$\text{Exp}\left[1+\kappa+(1+\kappa)\ \text{If}\left[\frac{1}{\sigma}\geq 0,\ \text{If}\left[\frac{\kappa}{1+\kappa}\neq 0,\ \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1+\kappa)\left(1+\frac{0\kappa}{1+\kappa}\right)}}-1}{\frac{\kappa}{1+\kappa}},\ \text{Log}\left[\frac{1}{\sigma}\right]\right],\ \text{Undefined}\right]\right]\Big],$$

$$\text{If}\left[\frac{1+1\kappa}{\kappa}>0,\ 0,\ \infty\right]\Big]$$

In[∘]:=
```
Module[{EntData, σPlotBottom = 0.5, σPlotTop = 2,
  κValues = {1 / 2, 1, 2}},
 EntData = Table[
   Table[{entValue[entName[σPlot, κPlot], σPlot, κPlot],
     entDensity[entName[σPlot, κPlot], κPlot]},
    {σPlot, {σPlotBottom, σPlotTop}}, {κPlot, κValues}],
   {entName, {HCoupled, HBGS, HRenyi, HTsallis, HNormTsallis}}
  ];
 ResourceFunction["PlotGrid"][
  {{Plot[
     Table[PDF[CoupledExponentialDistribution[σPlotTop, κPlot], x],
      {κPlot, κValues}]
     , {x, 0, 7.5},
     PlotRange → {{0, 5}, {0, 0.55}},
     Frame → {{True, True}, {True, True}},
     FrameStyle → Directive[Bold, Medium],
     Epilog → {
       {Dashed, Gray, Line[{{σPlotTop, 0}, {σPlotTop, 0.6}}]},
       PointSize[Large],
       {Black, Point[EntData⟦1, 2⟧]},
       {Blue, Point[EntData⟦2, 2⟧]},
       {Brown, Point[EntData⟦3, 2⟧]},
       {Magenta, Point[EntData⟦4, 2⟧]},
       {Orange, Point[EntData⟦5, 2⟧]},
       Inset[Style["Required Solution",
         Medium], {2.8, 0.32}],
       Inset[Style["= Coupled Entropy",
         Medium], {3.0, 0.28}],
       Inset[Style["BGS", Blue, Medium], {3.0, 0.12}],
```

```
    Inset[Style["Renyi", Brown, Medium], {2.5, 0.16}],
    Inset[Style["Tsallis", Magenta, Medium], {1, 0.12}],
    Inset[Style["Norm Tsallis", Orange, Medium], {4.0, 0.1}] ,
    Inset[Style["σ", Black, Medium], {1.9, 0.03}],
    Inset[Style["Coupled Exponential", Black, Medium, Bold],
     Scaled[{0.77, 0.92}]],
    Inset[Style["κ = 0.5, 1, 2, σ = 2", Black, Medium, Bold],
     Scaled[{0.77, 0.84}]],
    Inset[Style["Higher κ → Lower f(x(H))", Black, Medium, Bold],
     Scaled[{0.77, 0.76}]],
    Inset[Style["a)", Black, Medium],
     Scaled[{0.03, 0.95}]]
   }
 ]},
{Plot[
  Table[PDF[CoupledExponentialDistribution[σPlotBottom, κPlot], x],
   {κPlot, κValues}]
  , {x, 0, 5},
  PlotRange → {{0, 2.5}, {0, 1.05}} ,
  Frame → {{True, True}, {True, True}},
  FrameStyle → Directive[Bold, Medium],
  Epilog → {
    {Dashed, Gray, Line[{{σPlotBottom, 0}, {σPlotBottom, 1.0}}]},
    PointSize[Large],
    {Black, Point[EntData〚1, 1〛]},
    {Blue, Point[EntData〚2, 1〛]},
    {Brown, Point[EntData〚3, 1〛]},
    {Magenta, Point[EntData〚4, 1〛]},
    {Orange, Point[EntData〚5, 1〛]},
    Inset[Style["Required Solution",
      Medium], {0.9, 0.77}],
    Inset[Style["= Coupled Entropy",
      Medium], {0.92, 0.69}],
    Inset[Style["BGS", Blue, Medium], {1.3, 0.22}],
    Inset[Style["Renyi", Brown, Medium], {0.8, 0.5}],
    Inset[Style["Tsallis", Magenta, Medium], {0.68, 0.2}],
    Inset[Style["Norm Tsallis", Orange, Medium], {1.15, 0.35}],
    Inset[Style["σ", Black, Medium], {0.55, 0.05}],
    Inset[Style["Coupled Exponential", Black, Medium, Bold],
     Scaled[{0.77, 0.92}]],
    Inset[Style["κ = 0.5, 1, 2, σ = 0.5", Black, Medium, Bold],
     Scaled[{0.77, 0.84}]],
    Inset[Style["Higher κ → Lower f(x(H))", Black, Medium, Bold],
     Scaled[{0.77, 0.76}]],
```

```
      Inset[Style["b)", Black, Medium],
       Scaled[{0.03, 0.95}]]
     }
   ]}},
 Spacings → 20,
 FrameLabel → {"x", "Density, f(x)"},
 PlotLabel → "Entropies Mapped to Density"
]]
```

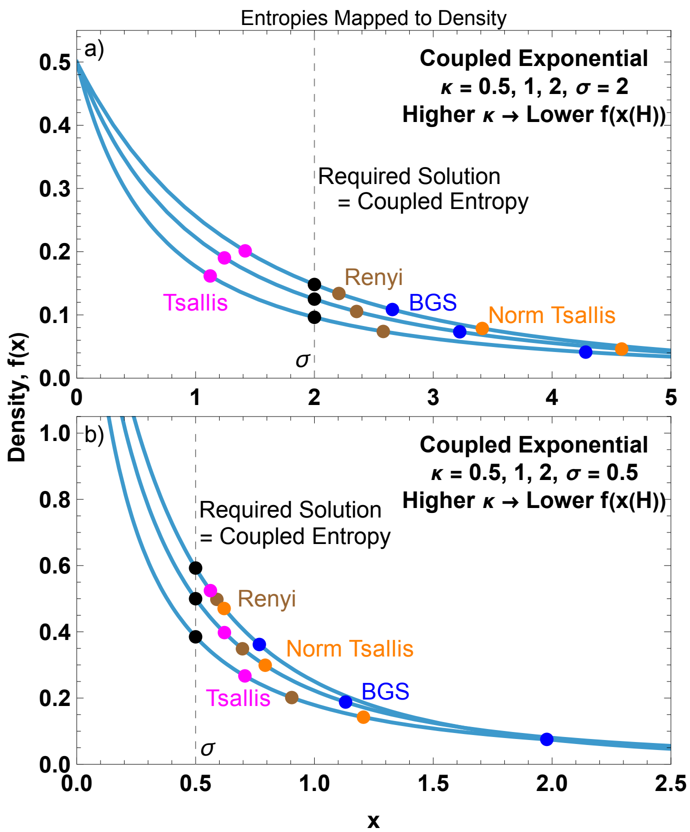
••• OptionValue: Unknown option DefaultBaseStyle for Graphics.

••• OptionValue: Unknown option DefaultBaseStyle for Graphics.

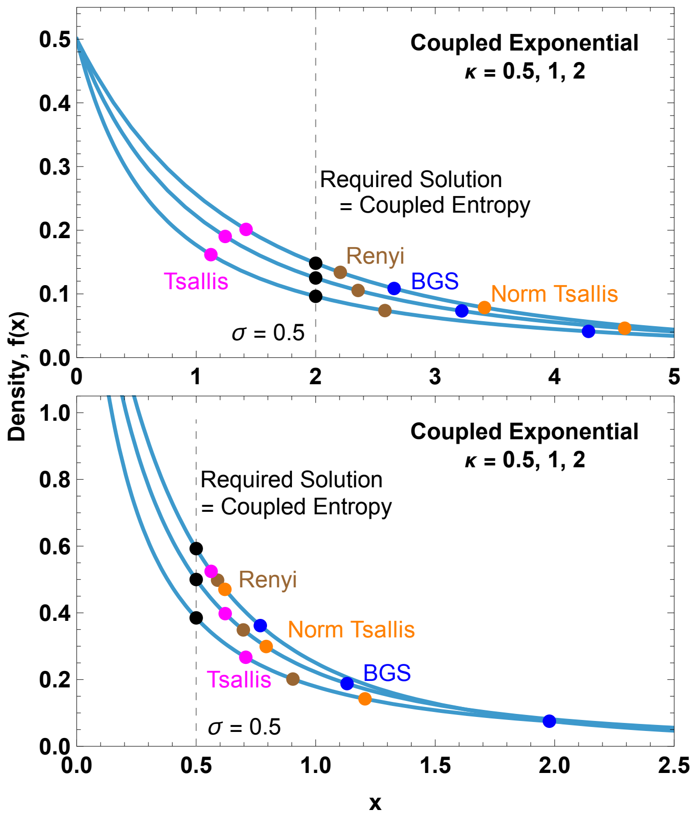
••• OptionValue: Unknown option DefaultBaseStyle for Graphics.

••• General: Further output of OptionValue::nodef will be suppressed during this calculation.

Out[ ]=



Saved Plots



Moments of Coupled Weibull distribution

Keeping 1+$\alpha\,\kappa$ term

In[∘]:= $\int x^{\alpha}\,\frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{\alpha-1}\,\mathrm{Exp}\left[-(1+\alpha\,\kappa)\left(\frac{x}{\sigma}\right)^{\alpha}\right]dx$ // FullSimplify

Out[∘]=

$$-\frac{\sigma^{\alpha}\,\mathrm{Gamma}\left[2,\ (1+\alpha\,\kappa)\left(\frac{x}{\sigma}\right)^{\alpha}\right]}{(1+\alpha\,\kappa)^{2}}$$

In[∘]:= $-\frac{\sigma^{\alpha}\,\mathrm{Gamma}\left[2,\ (1+\alpha\,\kappa)\left(\frac{x}{\sigma}\right)^{\alpha}\right]}{(1+\alpha\,\kappa)^{2}}$ /. x → 0 // FullSimplify

Out[∘]=

$$-\frac{\sigma^{\alpha}}{(1+\alpha\,\kappa)^{2}}$$

In[∘]:= $-\frac{\sigma^{\alpha}}{(1+\alpha\,\kappa)^{2}}$ /. $\left\{\sigma^{\alpha}\to\frac{\sigma^{\alpha}}{1+\alpha\,\kappa},\ \kappa\to\frac{\kappa}{1+\alpha\,\kappa}\right\}$ // FullSimplify

Out[∘]=

$$-\frac{(1+\alpha\,\kappa)\,\sigma^{\alpha}}{(1+2\,\alpha\,\kappa)^{2}}$$

Removing 1+$\alpha\,\kappa$ term

In[∘]:=

$$\int x^{\alpha}\,\frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{\alpha-1}\,\text{Exp}\left[-\left(\frac{x}{\sigma}\right)^{\alpha}\right]dx\ \text{// FullSimplify}$$

Out[∘]=

$$-\sigma^{\alpha}\,\text{Gamma}\left[2,\ \left(\frac{x}{\sigma}\right)^{\alpha}\right]$$

In[∘]:=

$$-\sigma^{\alpha}\,\text{Gamma}\left[2,\ \left(\frac{x}{\sigma}\right)^{\alpha}\right]\ \text{/. } x\to 0\ \text{// FullSimplify}$$

Out[∘]=

$$-\sigma^{\alpha}$$

In[∘]:=

$$-\frac{\sigma^{\alpha}}{1+\alpha\,\kappa}\,\text{Gamma}\left[2,\ (1+\alpha\,\kappa)\left(\frac{x}{\sigma}\right)^{\alpha}\right]\ \text{/. } x\to 0\ \text{// FullSimplify}$$

Out[∘]=

$$-\frac{\sigma^{\alpha}}{1+\alpha\,\kappa}$$

In[∘]:=

$$\int x^{\frac{-1}{\kappa}-1}\,dx$$

Out[∘]=

$$-x^{-1/\kappa}\,\kappa$$

In[∘]:=

$$\text{Limit}\left[-x^{-1/\kappa}\,\kappa,\ x\to\infty\right]$$

Limit: Warning: Assumptions that involve the limit variable are ignored.

Out[∘]=

$$0$$

Fluctuation Model

Syntax: Incomplete expression; more input is needed.

$$\int_0^1 \text{Exp}\left[-\frac{t}{\sigma}\right]\,\text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa},\ \frac{\kappa}{\sigma}\right],\ \frac{1}{t}\right]dt$$

Out[∘]=

$$\int_0^1 \left(e^{-\frac{t}{\sigma}}\right)^{x}\left(\begin{cases}\dfrac{e^{-\frac{\sigma}{t\kappa}}\left(\frac{1}{t}\right)^{-1+\frac{1}{\kappa}}\left(\frac{\kappa}{\sigma}\right)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & \frac{1}{t}>0\\ 0 & \text{True}\end{cases}\right)dt$$

In[∘]:=

$$\int_0^{\infty} c\,\text{Exp}[-c\,t]\,\text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa},\ \kappa\,\beta\right],\ c\right]dc$$

Out[∘]=

$$\begin{cases}\dfrac{\beta^{1-\frac{1}{\kappa}}\left(\frac{1}{\beta}+t\,\kappa\right)^{-1/\kappa}}{1+t\,\beta\,\kappa} & \text{Re}\left[t+\frac{1}{\beta\,\kappa}\right]>0\\ \text{Integrate}\left[c\,e^{-c\,t}\left(\begin{cases}\dfrac{c^{-1+\frac{1}{\kappa}}\,e^{-\frac{c}{\beta\kappa}}\,(\beta\,\kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c>0\\ 0 & \text{True}\end{cases}\right),\ \{c,\ 0,\ \infty\},\right. & \text{True}\\ \quad \text{Assumptions}\to d\in\mathbb{Z}\ \&\&\ x\in\mathbb{R}\ \&\&\ \alpha\in\mathbb{R}\ \&\&\ \kappa\in\mathbb{R}\ \&\&\ \mu\in\mathbb{R}\ \&\& & \\ \quad\quad \left.\sigma\in\mathbb{R}\ \&\&\ \kappa>0\ \&\&\ \sigma>0\ \&\&\ \alpha>0\ \&\&\ d>0\ \&\&\ \text{Re}\left[t+\frac{1}{\beta\,\kappa}\right]\le 0\right] & \end{cases}$$

In[•]:=

$$\begin{cases} \dfrac{\beta^{1-\frac{1}{\kappa}}\left(\frac{1}{\beta}+t\,\kappa\right)^{-1/\kappa}}{1+t\,\beta\,\kappa} & \text{Re}\left[t+\frac{1}{\beta\,\kappa}\right]>0 \\ \text{Integrate}\left[c\,e^{-c\,t}\left(\begin{cases}\dfrac{c^{-1+\frac{1}{\kappa}}\,e^{-\frac{c}{\beta\kappa}}\,(\beta\,\kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c>0 \\ 0 & \text{True}\end{cases}\right),\ \{c,0,\infty\},\right. & \text{True} \\ \quad \text{Assumptions}\to d\in\mathbb{Z}\ \&\&\ x\in\mathbb{R}\ \&\&\ \alpha\in\mathbb{R}\ \&\&\ \kappa\in\mathbb{R}\ \&\&\ \mu\in\mathbb{R}\ \&\& & \\ \qquad \left.\sigma\in\mathbb{R}\ \&\&\ \kappa>0\ \&\&\ \sigma>0\ \&\&\ \alpha>0\ \&\&\ d>0\ \&\&\ \text{Re}\left[t+\frac{1}{\beta\,\kappa}\right]\le 0\right] & \end{cases}$$

/ FullSimplify

Out[•]=

$$\begin{cases} \dfrac{\beta^{1-\frac{1}{\kappa}}\left(\frac{1}{\beta}+t\,\kappa\right)^{-1/\kappa}}{1+t\,\beta\,\kappa} & \text{Re}\left[t+\frac{1}{\beta\,\kappa}\right]>0 \\ \text{Integrate}\left[c\,e^{-c\,t}\left(\begin{cases}\dfrac{c^{-1+\frac{1}{\kappa}}\,e^{-\frac{c}{\beta\kappa}}\,(\beta\,\kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c>0 \\ 0 & \text{True}\end{cases}\right),\ \{c,0,\infty\},\right. & \text{True} \\ \quad \text{Assumptions}\to d\in\mathbb{Z}\ \&\&\ x\in\mathbb{R}\ \&\&\ \alpha\in\mathbb{R}\ \&\&\ \kappa\in\mathbb{R}\ \&\&\ \mu\in\mathbb{R}\ \&\& & \\ \qquad \left.\sigma\in\mathbb{R}\ \&\&\ \kappa>0\ \&\&\ \sigma>0\ \&\&\ \alpha>0\ \&\&\ d>0\ \&\&\ \text{Re}\left[t+\frac{1}{\beta\,\kappa}\right]\le 0\right] & \end{cases}$$

$$\beta^1\ (1+t\,\beta\,\kappa)^{-1/\kappa-1}$$

In[•]:=

$$\int_0^\infty c\ \text{Exp}\left[-c\,t^2\right]\ \text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa},\ \kappa\,\beta\right],\ c\right]\,dc$$

Out[•]=

$$\begin{cases} \dfrac{\beta^{1-\frac{1}{\kappa}}\left(\frac{1}{\beta}+t^2\,\kappa\right)^{-1/\kappa}}{1+t^2\,\beta\,\kappa} & \text{Re}\left[t^2+\frac{1}{\beta\,\kappa}\right]>0 \\ \text{Integrate}\left[c\,e^{-c\,t^2}\left(\begin{cases}\dfrac{c^{-1+\frac{1}{\kappa}}\,e^{-\frac{c}{\beta\kappa}}\,(\beta\,\kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c>0 \\ 0 & \text{True}\end{cases}\right),\ \{c,0,\infty\},\right. & \text{True} \\ \quad \text{Assumptions}\to d\in\mathbb{Z}\ \&\&\ x\in\mathbb{R}\ \&\&\ \alpha\in\mathbb{R}\ \&\&\ \kappa\in\mathbb{R}\ \&\&\ \mu\in\mathbb{R}\ \&\& & \\ \qquad \left.\sigma\in\mathbb{R}\ \&\&\ \kappa>0\ \&\&\ \sigma>0\ \&\&\ \alpha>0\ \&\&\ d>0\ \&\&\ \text{Re}\left[t^2+\frac{1}{\beta\,\kappa}\right]\le 0\right] & \end{cases}$$

In[∘]:= $\int_0^\infty c\,\mathrm{Exp}\left[-c^2\,t^2\right]\,\mathrm{PDF}\left[\mathrm{GammaDistribution}\left[\frac{1}{\kappa},\,\kappa\,\beta^2\right],\,c\right]\,dc$

Out[∘]=

$$\begin{cases} -\frac{1}{4\,\mathrm{Gamma}\left[\frac{1}{\kappa}\right]}\left(t^2\right)^{-\frac{3}{2}-\frac{1}{2\kappa}}\left(\beta^2\right)^{-1-\frac{1}{\kappa}}\kappa^{-2-\frac{1}{\kappa}}\left(\sqrt{t^2}\,\mathrm{Gamma}\left[\frac{1}{2\kappa}\right]\,\mathrm{Hypergeometric1F1}\left[1+\frac{1}{2\kappa},\,\frac{3}{2},\,\frac{1}{4\,t^2\,\beta^4\,\kappa^2}\right]-2\,t^2\,\beta^2\,\kappa^2\,\mathrm{Gamma}\left[\frac{1+\kappa}{2\kappa}\right]\,\mathrm{Hypergeometric1F1}\left[\frac{1+\kappa}{2\kappa},\,\frac{1}{2},\,\frac{1}{4\,t^2\,\beta^4\,\kappa^2}\right]\right) & \left(\mathrm{Re}\left[t^2\right]\geq 0\;\&\&\;\mathrm{Re}\left[\beta^2\right]>0\right)\;||\;\mathrm{Re}\left[t^2\right]>0 \\ \mathrm{Integrate}\left[c\,e^{-c^2\,t^2}\left(\begin{cases}\frac{c^{-1+\frac{1}{\kappa}}\,e^{-\frac{c}{\beta^2\kappa}}\left(\beta^2\,\kappa\right)^{-1/\kappa}}{\mathrm{Gamma}\left[\frac{1}{\kappa}\right]} & c>0 \\ 0 & \mathrm{True}\end{cases}\right),\,\{c,\,0,\,\infty\},\,\mathrm{Assumptions}\to\left(d\in\mathbb{Z}\;\&\&\;x\in\mathbb{R}\;\&\&\;\alpha\in\mathbb{R}\;\&\&\;\kappa\in\mathbb{R}\;\&\&\;\mu\in\mathbb{R}\;\&\&\;\sigma\in\mathbb{R}\;\&\&\;\mathrm{Re}\left[t^2\right]\leq 0\;\&\&\;d>0\;\&\&\;\alpha>0\;\&\&\;\kappa>0\;\&\&\;\sigma>0\;\&\&\;\mathrm{Re}\left[\beta^2\right]\leq 0\right)\;||\;\left(d\in\mathbb{Z}\;\&\&\;x\in\mathbb{R}\;\&\&\;\alpha\in\mathbb{R}\;\&\&\;\kappa\in\mathbb{R}\;\&\&\;\mu\in\mathbb{R}\;\&\&\;\sigma\in\mathbb{R}\;\&\&\;\mathrm{Re}\left[t^2\right]<0\;\&\&\;d>0\;\&\&\;\alpha>0\;\&\&\;\kappa>0\;\&\&\;\sigma>0\right)\right] & \mathrm{True}\end{cases}$$

In[∘]:= $\int_\infty^0 \frac{1}{c}\,\mathrm{Exp}\left[\frac{-1}{c}\,t^2\right]\,\mathrm{PDF}\left[\mathrm{GammaDistribution}\left[\frac{1}{\kappa},\,\frac{\kappa}{\sigma^2}\right],\,\frac{1}{c}\right]\,d\left(\frac{1}{c}\right)$

Integrate: Invalid integration variable or limit (s) in $\left\{\frac{1}{c},\,\infty,\,0\right\}$.

Out[∘]=

$$\int_\infty^0 \frac{e^{-\frac{t^2}{c}}\left(\begin{cases}\frac{\left(\frac{1}{c}\right)^{-1+\frac{1}{\kappa}}\,e^{-\frac{\sigma^2}{c\,\kappa}}\left(\frac{\kappa}{\sigma^2}\right)^{-1/\kappa}}{\mathrm{Gamma}\left[\frac{1}{\kappa}\right]} & \frac{1}{c}>0 \\ 0 & \mathrm{True}\end{cases}\right)}{c}\,d\frac{1}{c}$$

In[∘]:= $\int_0^\infty c\,\mathrm{Exp}\left[-c\,t^\alpha\right]\,\mathrm{PDF}\left[\mathrm{GammaDistribution}\left[\frac{1}{\kappa},\,\kappa\,\beta\right],\,c\right]\,dc$

Out[∘]=

$$\begin{cases}\frac{\beta^{1-\frac{1}{\kappa}}\left(\frac{1}{\beta}+t^\alpha\,\kappa\right)^{-1/\kappa}}{1+t^\alpha\,\beta\,\kappa} & \mathrm{Re}\left[t^\alpha+\frac{1}{\beta\,\kappa}\right]>0 \\ \mathrm{Integrate}\left[c\,e^{-c\,t^\alpha}\left(\begin{cases}\frac{c^{-1+\frac{1}{\kappa}}\,e^{-\frac{c}{\beta\kappa}}\left(\beta\,\kappa\right)^{-1/\kappa}}{\mathrm{Gamma}\left[\frac{1}{\kappa}\right]} & c>0 \\ 0 & \mathrm{True}\end{cases}\right),\,\{c,\,0,\,\infty\},\,\mathrm{Assumptions}\to d\in\mathbb{Z}\;\&\&\;x\in\mathbb{R}\;\&\&\;\alpha\in\mathbb{R}\;\&\&\;\kappa\in\mathbb{R}\;\&\&\;\mu\in\mathbb{R}\;\&\&\;\sigma\in\mathbb{R}\;\&\&\;\kappa>0\;\&\&\;\sigma>0\;\&\&\;\alpha>0\;\&\&\;d>0\;\&\&\;\mathrm{Re}\left[t^\alpha+\frac{1}{\beta\,\kappa}\right]\leq 0\right] & \mathrm{True}\end{cases}$$

In[∘]:= $\int_0^\infty \mathrm{Exp}\left[-c\,t^\alpha\right]\,\mathrm{PDF}\left[\mathrm{GammaDistribution}\left[\frac{1}{\kappa}+1,\,\frac{\kappa}{1+\kappa}\,\beta^\alpha\right],\,c\right]\,dc$

Out[∘]=

$Aborted

Complexity Measurement

Determine the Taylor Series expansion of the result

In[ ]:= $\text{EntRatioExp} = \dfrac{1+\text{Log}[\sigma\,(1-\kappa)]}{1+\text{Log}[\sigma]}$

Out[ ]=

$$\frac{1+\text{Log}[(1-\kappa)\,\sigma]}{1+\text{Log}[\sigma]}$$

In[ ]:= Series[EntRatioExp (1 - EntRatioExp), {κ, 0, 3}, Assumptions → (0 < κ < 1)]

Out[ ]=

$$\frac{\kappa}{1+\text{Log}[\sigma]}+\frac{(-1+\text{Log}[\sigma])\,\kappa^2}{2\,(1+\text{Log}[\sigma])^2}+\frac{(-2+\text{Log}[\sigma])\,\kappa^3}{3\,(1+\text{Log}[\sigma])^2}+O[\kappa]^4$$

In[ ]:= $\text{Series}\left[-\left(1+\dfrac{\text{Log}[1-\kappa]}{\text{Log}[\sigma]}\right)\left(\dfrac{\text{Log}[1-\kappa]}{\text{Log}[\sigma]}\right),\ \{\kappa,0,3\},\ \text{Assumptions}\to\left(0<\kappa<\dfrac{1}{2}\right)\right]$

Out[ ]=

$$\frac{\kappa}{\text{Log}[\sigma]}+\frac{(-2+\text{Log}[\sigma])\,\kappa^2}{2\,\text{Log}[\sigma]^2}+\frac{(-3+\text{Log}[\sigma])\,\kappa^3}{3\,\text{Log}[\sigma]^2}+O[\kappa]^4$$

First order approximation of the entropy of the coupled Gaussian with σ=1

In[ ]:= $\text{Series}\left[\dfrac{1+\kappa}{2\,\kappa}\left(\text{PolyGamma}\left[\dfrac{1+\kappa}{2\,\kappa}\right]-\text{PolyGamma}\left[\dfrac{1}{2\,\kappa}\right]\right)+\text{Log}\left[\dfrac{1}{\sqrt{\kappa}}\,\text{Beta}\left[\dfrac{1}{2\,\kappa},\dfrac{1}{2}\right]\right],\ \{\kappa,0,3\}\right]$ // FullSimplify

Out[ ]=

$$\frac{1}{2}\,(1+\text{Log}[2]+\text{Log}[\pi])+\kappa+\frac{\kappa^2}{4}-\frac{\kappa^3}{6}+O[\kappa]^{7/2}$$

Taylor Series of Complexity of Coupled Gaussian

In[ ]:= $\text{EntRatioGauss} = \dfrac{\frac{1}{2}+\frac{1}{2}\,\text{Log}\left[2\,\pi\,\sigma\,\sqrt{1-2\,\kappa}\right]}{\frac{1}{2}+\frac{1}{2}\,\text{Log}[2\,\pi\,\sigma]}$

Out[ ]=

$$\frac{\frac{1}{2}+\frac{1}{2}\,\text{Log}\left[2\,\pi\,\sqrt{1-2\,\kappa}\,\sigma\right]}{\frac{1}{2}+\frac{1}{2}\,\text{Log}[2\,\pi\,\sigma]}$$

In[ ]:= Series[EntRatio (1 - EntRatioGauss), {κ, 0, 3}]

Out[ ]=

$$\frac{\kappa}{1+\text{Log}[2\,\pi\,\sigma]}+\frac{\text{Log}[2\,\pi\,\sigma]\,\kappa^2}{(1+\text{Log}[2\,\pi\,\sigma])^2}+\frac{2\,(-1+2\,\text{Log}[2\,\pi\,\sigma])\,\kappa^3}{3\,(1+\text{Log}[2\,\pi\,\sigma])^2}+O[\kappa]^4$$

Relation to c,d-entropy

Asymptotic Classification

J. M. Amigó, S. G. Balogh, and S. Hernández, “A Brief Review of Generalized Entropies,” Entropy, vol. 20, no. 11, p. 813, Nov. 2018, doi: 10.3390/e20110813.

4. Hanel–Thurner Exponents

All generalized entropies $F_{G,g}$ group in classes labeled by two exponents (c, d) introduced by Hanel and Thurner [16], which are determined by the limits

$$\lim_{W\to\infty} \frac{F_{G,g}(p_1, \ldots, p_{\lambda W})}{F_{G,g}(p_1, \ldots, p_W)} = \lambda^{1-c} \tag{25}$$

(W being as before the cardinality of the probability distribution or the total number of microstates in the system, $\lambda > 1$) and

$$\lim_{W\to\infty} \frac{F_{G,g}(p_1, \ldots, p_{W^{1+a}})}{F_{G,g}(p_1, \ldots, p_W)} W^{a(c-1)} = (1+a)^d \tag{26}$$

($a > 0$). Note that the limit in Equation (26) does not depend actually on c. The limits in Equations (25) and (26) can be computed via the asymptotic equipartition property [26]. Thus,

$$F_{G,g}(p_1, \ldots, p_{\lambda W}) \approx G\left(\lambda W g\left(\frac{1}{\lambda W}\right)\right)$$

and

$$F_{G,g}(p_1, \ldots, p_{W^{1+a}}) \approx G\left(W^{1+a} g\left(\frac{1}{W^{1+a}}\right)\right)$$

(E1) For the BGS entropy, $g(x) = -x\ln x$ (see Equation (11)), so

$$\frac{g(zx)}{g(x)} = \frac{zx\ln(zx)}{x\ln x} = \frac{z\ln z + z\ln x}{\ln x} \to z$$

as $x \to 0+$. Therefore, $c = 1$. Furthermore,

$$\frac{g(x^{1+a})}{x^{ac}g(x)} = \frac{x^{1+a}\ln x^{1+a}}{x^{a+1}\ln x} = \frac{(1+a)\ln x}{\ln x} = 1+a$$

for all $x > 0$, so $d = 1$.

(E2) For the Tsallis entropy, see Equation (12),

$$g(x) = \begin{cases} \frac{1}{1-q}x^q + \mathcal{O}(x) & \text{if } 0 < q < 1, \\ -\frac{1}{1-q}x + \mathcal{O}(x) & \text{if } q > 1. \end{cases}$$

It follows readily that $(c,d) = (q,0)$ if $0 < q < 1$, and $(c,d) = (1,0)$ if $q > 1$. Hence, although $\lim_{q\to 1} T_q = S_{BGS}$, there is no parallel convergence concerning the HT exponents.

(E3) For the Rényi entropy, $g(x) = x^q$ and $G(u) = \frac{1}{1-q}e^u$ (see Equation (15)), so

$$\frac{G(\frac{\lambda}{x}g(\frac{x}{\lambda}))}{G(\frac{1}{x}g(x))} = \frac{\ln\left(\frac{\lambda}{x}(\frac{x}{\lambda})^q\right)}{\ln\left(\frac{1}{x}x^q\right)} = \frac{\ln x^{q-1} - \ln\lambda^{q-1}}{\ln x^{q-1}} \to 1$$

For Coupled Entropy, consider that for $p_i = \frac{1}{W}$, the independent equals distribution also equals p_i. Then $g(x) = x \ln_{\alpha\kappa} x^{-\frac{1}{1+\kappa}} = \frac{x}{\alpha\kappa}\, x^{-\alpha\frac{\kappa}{1+\kappa}} + O(x) = \frac{1}{\alpha\kappa}\, x^{1-\alpha\frac{\kappa}{1+\kappa}} + O(x) = \frac{1}{\alpha\kappa}\, x^{2-q} + O(x)$

and $G(u) = u^{\frac{1}{\gamma}}$.

```
In[ ]:= g[r_, κ_, α_, d_] := 1/(α κ) r^(1 - α κ/(1 + d κ));
        G[u_, γ_] := u^(1/γ);
In[ ]:= Limit[G[λ W g[1/(λ W), κ, α, d], γ] / G[W g[1/W, κ, α, d], γ], W → ∞]
Out[ ]=
```

$$\left(\left(\frac{1}{\lambda}\right)^{-\frac{\alpha\kappa}{1+d\kappa}}\right)^{\frac{1}{\gamma}} \quad \text{if condition}$$

$$\left(\lambda^{\frac{\alpha\kappa/\gamma}{1+d\kappa}}\right) = \lambda^{1-c}$$

$$c = 1 - \frac{\alpha\kappa/\gamma}{1+d\kappa}$$

In[∘]:= $\text{Limit}\left[\frac{G\left[W^{1+a}\, g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W\, g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a \frac{\alpha\kappa/\gamma}{1+d\kappa}}, W \to \infty\right]$

Out[∘]=

$1 \text{ if } \frac{1}{\gamma} \in \mathbb{R} \;\&\&\; a > -1 \;\&\&\; d\,\kappa > -1 \;\&\&\; \alpha\,\kappa > 0$

Thus for $(1+a)^d = 1$, d = 0.

In[∘]:= $\frac{G\left[W^{1+a}\, g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W\, g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a \frac{\alpha\kappa/\gamma}{1+d\kappa}}$ // FullSimplify

Out[∘]=

$\left(\left(\frac{1}{W}\right)^{-\frac{\alpha\kappa}{1+d\kappa}}\right)^{-1/\gamma} W^{-\frac{a\,\alpha\,\kappa}{\gamma+d\,\gamma\,\kappa}} \left(\left(W^{-1-a}\right)^{-\frac{\alpha\kappa}{1+d\kappa}}\right)^{\frac{1}{\gamma}}$

$\left(W^{-\frac{\alpha\kappa}{(1+d\kappa)\gamma}}\right) W^{-\frac{a\,\alpha\,\kappa}{\gamma+d\,\gamma\,\kappa}} W^{\frac{(1+a)\,\alpha\,\kappa}{(1+d\kappa)\,\gamma}}$

In[∘]:= $-\frac{\alpha\,\kappa}{(1+d\,\kappa)\,\gamma} - \frac{a\,\alpha\,\kappa}{\gamma+d\,\gamma\,\kappa} + \frac{(1+a)\,\alpha\,\kappa}{(1+d\,\kappa)\,\gamma}$ // FullSimplify

Out[∘]=

0

In[∘]:= $\text{Limit}\left[\frac{G\left[W^{1+a}\, g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W\, g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a\,(1-c)}, W \to \infty\right]$

Out[∘]=

∞ if *condition* +

In[∘]:= $\text{Limit}\left[\frac{G\left[W^{1+a}\, g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W\, g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a\,(1-1)}, W \to \infty\right]$

Out[∘]=

$\infty \text{ if } a > 0$

In[∘]:= $\text{Limit}\left[\frac{G\left[W^{1+a}\, g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W\, g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a\,(1-0)}, W \to \infty\right]$

Out[∘]=

$\infty \text{ if } a > -1 \;\&\&\; a \neq 0 \;\&\&\; a\,\gamma\,(1+d\,\kappa) < a\,\alpha\,\kappa$

What if the full form of the generalized logarithm is used?

For Coupled Entropy, consider that for $p_i = \frac{1}{W}$, the independent equals distribution also equals p_i. Then $g(x) = x \ln_{\alpha\kappa} x^{-\frac{1}{1+\kappa}} = \frac{x}{\alpha\kappa}\left(x^{-\alpha\frac{\kappa}{1+\kappa}} - 1\right)$

In[∘]:= $g[r\_, \kappa\_, \alpha\_, d\_] := \frac{r}{\alpha\,\kappa}\left(r^{-\frac{\alpha\kappa}{1+d\kappa}} - 1\right);$

$G[u\_, \gamma\_] := u^{\frac{1}{\gamma}};$

In[ ]:= Limit[$\frac{G[\lambda W g[\frac{1}{\lambda W}, \kappa, \alpha, d], \gamma]}{G[W g[\frac{1}{W}, \kappa, \alpha, d], \gamma]}$ // FullSimplify, W → ∞]

Out[ ]=

$$\left(\left(\frac{1}{\lambda}\right)^{-\frac{\alpha\kappa}{1+d\kappa}}\right)^{\frac{1}{\gamma}}$$

In[ ]:= Limit[$\frac{G[W^{1+a} g[\frac{1}{W^{1+a}}, \kappa, \alpha, d], \gamma]}{G[W g[\frac{1}{W}, \kappa, \alpha, d], \gamma]} W^{-a \frac{\alpha\kappa/\gamma}{1+d\kappa}}$ // FullSimplify, W → ∞]

Out[ ]=

1 if a > 1

In[ ]:= $\frac{G[W^{1+a} g[\frac{1}{W^{1+a}}, \kappa, \alpha, d], \gamma]}{G[W g[\frac{1}{W}, \kappa, \alpha, d], \gamma]} W^{-a \frac{\alpha\kappa/\gamma}{1+d\kappa}}$ // FullSimplify

Out[ ]=

$$\left(-1+\left(\frac{1}{W}\right)^{-\frac{\alpha\kappa}{1+d\kappa}}\right)^{-1/\gamma} W^{-\frac{a\alpha\kappa}{\gamma+d\gamma\kappa}} \left(-1+\left(W^{-1-a}\right)^{-\frac{\alpha\kappa}{1+d\kappa}}\right)^{\frac{1}{\gamma}}$$

With W -> ∞ only the terms with W are significant and all the terms contain $\frac{\alpha\kappa}{\gamma(1+d\kappa)}$. Thus exponent multiplies this by (1 - a -1 -a) = 0, which is why the expression converges to 1. But if κ -> 0 first, then the -1 term is of significance in the terms converging to log W^z. Set r = $\frac{\alpha\kappa}{(1+d\kappa)}$, and factor out the 1+a from the coupled logarithm, then the ratio is $\frac{(1+a)\ln_{-r(1+a)} W}{\ln_{-r} W}$. The question is under what circumstance does this ratio converge to 1 + a? Certainly when r ->0 first. What about when r -> 1, which is its other limit? Then $\text{Lim}_{W\to\infty} \frac{(1+a)(W^{-(1+a)}-1)}{(1+a)(W^{-1}-1)} = 1$.

Try using the form $e^{W^{1+a}}$ and see what the scaling is

In[ ]:= Limit[$\frac{G[\text{Exp}[W^{1+a}]\, g[\frac{1}{\text{Exp}[W^{1+a}]}, \kappa, \alpha, d], \gamma]}{G[\text{Exp}[W]\, g[\frac{1}{\text{Exp}[W]}, \kappa, \alpha, d], \gamma]}$ Exp[$W^{-a \frac{\alpha\kappa/\gamma}{1+d\kappa}}$], W → ∞]

Out[ ]=

$$\lim_{W\to\infty} \frac{e^{W^{-\frac{a\alpha\kappa}{\gamma(1+d\kappa)}}} G[e^{W^{1+a}} g[e^{-W^{1+a}}, \kappa, \alpha, d], \gamma]}{G[e^{W} g[e^{-W}, \kappa, \alpha, d], \gamma]}$$

In[ ]:= Limit[$\frac{G[\text{Exp}[W^{a}]\, g[\frac{1}{\text{Exp}[W^{a}]}, \kappa, \alpha, d], \gamma]}{G[\text{Exp}[W]\, g[\frac{1}{\text{Exp}[W]}, \kappa, \alpha, d], \gamma]}$, W → ∞]

Out[ ]=

$$\lim_{W\to\infty} \frac{G[e^{W^{a}} g[e^{-W^{a}}, \kappa, \alpha, d], \gamma]}{G[e^{W} g[e^{-W}, \kappa, \alpha, d], \gamma]}$$

Taylor series of coupled exponential function

In[∘]:=
```
Series[CoupledExponential[x, κ, 1], {x, x0, 4}]
```

Out[∘]=

$$
\begin{cases}
\infty & (x \in \mathbb{R} \,\&\&\, -1 \le \kappa < 0 \,\&\&\, x0\,\kappa < -1) \,||\, (x0 \in \mathbb{R} \,\&\&\, \kappa \ne 0 \,\&\&\, x0\,\kappa == -1 \,\&\&\, x\,\kappa \le -1 \,\&\&\, -1 \le \kappa < 0) \\
((x-x0)\,\kappa)^{\frac{1}{\kappa}} \left(\kappa\,(x-x0) + O[x-x0]^5\right) & x0 \in \mathbb{R} \,\&\&\, \kappa \ne 0 \,\&\&\, x0\,\kappa == -1 \,\&\&\, ((\kappa > 0 \,\&\&\, x\,\kappa > -1) \,||\, (\kappa < 0 \,\&\&\, x\,\kappa > -1)) \\
e^{x0} + e^{x0}(x-x0) + \frac{1}{2} e^{x0}(x-x0)^2 + \frac{1}{6} e^{x0}(x-x0)^3 + \frac{1}{24} e^{x0}(x-x0)^4 + O[x-x0]^5 & (x \mid x0) \in \mathbb{R} \,\&\&\, \kappa == 0 \\
(1+x0\,\kappa)^{1+\frac{1}{\kappa}} + (1+\kappa)(1+x0\,\kappa)^{\frac{1}{\kappa}}(x-x0) + \frac{1}{2}(1+\kappa)(1+x0\,\kappa)^{-1+\frac{1}{\kappa}}(x-x0)^2 - \frac{1}{6}\left((-1+\kappa)(1+\kappa)(1+x0\,\kappa)^{-2+\frac{1}{\kappa}}\right)(x-x0)^3 + \frac{1}{24}(1+\kappa)(1+x0\,\kappa)^{-3+\frac{1}{\kappa}}\left(1-3\kappa+2\kappa^2\right)(x-x0)^4 + O[x-x0]^5 & x \in \mathbb{R} \,\&\&\, ((\kappa > 0 \,\&\&\, x0\,\kappa > -1) \,||\, (\kappa < 0 \,\&\&\, x0\,\kappa > -1)) \\
0 & \text{True}
\end{cases}
$$

Taylor series of cdFunction

Starting with the 0th branch which corresponds to d>0

In[∘]:=
```
cdFuntion[x_, c_, d_, r_] :=
  Exp[-d/(1 - c) (ProductLog[0, B[c, r] (1 - x/r)^(1/d)]
      - ProductLog[0, B[c, r]])] /; d > 0;
```

In[∘]:=
```
B[c_, r_] := (1 - c) r/(1 - (1 - c) r) Exp[(1 - c) r/(1 - (1 - c) r)];
```

In[∘]:=
```
Series[ProductLog[0, z], {z, z0, 4}]
```

Out[∘]=

$$
\mathrm{ProductLog}[z0] + \frac{\mathrm{ProductLog}[z0]\,(z-z0)}{z0\,(1+\mathrm{ProductLog}[z0])} - \frac{\left(\mathrm{ProductLog}[z0]^2\,(2+\mathrm{ProductLog}[z0])\right)(z-z0)^2}{2\left(z0^2\,(1+\mathrm{ProductLog}[z0])^3\right)} + \frac{\mathrm{ProductLog}[z0]^3\left(9+8\,\mathrm{ProductLog}[z0]+2\,\mathrm{ProductLog}[z0]^2\right)(z-z0)^3}{6\,z0^3\,(1+\mathrm{ProductLog}[z0])^5} + \left(\mathrm{ProductLog}[z0]^4\left(-64-79\,\mathrm{ProductLog}[z0]-36\,\mathrm{ProductLog}[z0]^2-6\,\mathrm{ProductLog}[z0]^3\right)(z-z0)^4\right) \Big/ \left(24\,z0^4\,(1+\mathrm{ProductLog}[z0])^7\right) + O[z-z0]^5
$$

In[ ]:=
```
Series[cdFuntion[x, c, d, r], {x, x0, 4}]
```

Out[ ]=

$$
e^{\frac{d\left(-\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r}{1+(-1+c) r}\right]+\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]\right)}{-1+c}}-
$$

$$
\left(e^{\frac{d\left(-\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r}{1+(-1+c) r}\right]+\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]\right)}{-1+c}}\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]\right.
$$

$$
\left.(x-x0)\right) \Big/ \left((-1+c)\,(r-x0)\left(1+\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]\right)\right)-
$$

$$
\left(\left(e^{\frac{d\left(-\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r}{1+(-1+c) r}\right]+\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]\right)}{-1+c}}\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]\right.\right.
$$

$$
\left(1-c-d+c\,d-3\,d\,\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]+2\,c\,d\,\text{ProductLog}\left[\right.\right.
$$

$$
\left.-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]-2\,d\,\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2+
$$

$$
\left.\left.c\,d\,\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2\right)\,(x-x0)^2\right) \Big/
$$

$$
\left(2\left((-1+c)^2\, d\,(r-x0)^2\left(1+\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]\right)^3\right)\right)-
$$

$$\left(\left(e^{\frac{d\left(-\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r}{1+(-1+c) r}\right]+\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]\right)}{-1+c}}\right.\right.$$

$$\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] \left(1-2 c+c^2-3 d+6 c d-3 c^2 d+\right.$$

$$2 d^2-4 c d^2+2 c^2 d^2-2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]+$$

$$4 c\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]-2 c^2\, \text{ProductLog}\left[\right.$$

$$\left.-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]-9 d\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]+$$

$$15 c d\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]-6 c^2 d\, \text{ProductLog}\left[\right.$$

$$\left.-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]+11 d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]-$$

$$19 c d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]+8 c^2 d^2\, \text{ProductLog}\left[\right.$$

$$\left.-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]-6 d\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2+$$

$$9 c d\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2-$$

$$3 c^2 d\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2+$$

$$22 d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2-$$

$$33 c d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2+$$

$$12 c^2 d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c r}{1+(-1+c) r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2+$$

$$19\, d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]^3 -$$

$$25\, c\, d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]^3 +$$

$$8\, c^2\, d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]^3 +$$

$$6\, d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]^4 -$$

$$7\, c\, d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]^4 +$$

$$\left.\left. 2\, c^2\, d^2\, \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]^4 \right)\right) (x-x0)^3 \Bigg) \Bigg/$$

$$\left(6 \left((-1+c)^3\, d^2\, (r-x0)^3 \left(1+\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]\right)^5\right)\right) +$$

$$\left(e^{\frac{d \left(-\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r}{1+(-1+c)\, r}\right]+\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]\right)}{-1+c}} \right.$$

$$\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]$$

$$\left(-(-1+c)^3 \left(-1+6\, d-11\, d^2+6\, d^3\right) - (-1+c)^2\, (-1+d)\right.$$

$$\left(8+15\, d-47\, d^2+4\, c \left(-2-2\, d+9\, d^2\right)\right) \text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right] -$$

$$(-1+c)\, (-1+d) \left(6+25\, d+151\, d^2+6\, c^2 \left(1+4\, d+15\, d^2\right) - c \left(12+49\, d+235\, d^2\right)\right)$$

$$\text{ProductLog}\left[-\frac{(-1+c)\, e^{\frac{r-c\, r}{1+(-1+c)\, r}}\, r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\, r}\right]^2 +$$

$$d \left(c \left(52+252\, d-604\, d^2\right) + c^3 \left(12+44\, d-120\, d^2\right) + 5 \left(-4-22\, d+51\, d^2\right) + \right.$$

$$2\,c^2\,\left(-22-93\,d+235\,d^2\right)\Big)\ \text{ProductLog}\left[-\frac{(-1+c)\,e^{\frac{r-c\,r}{1+(-1+c)\,r}}\,r\,\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\,r}\right]^3+$$

$$d^2\,\left(-35+c\,(75-526\,d)+c^3\,(11-90\,d)+239\,d+c^2\,(-51+380\,d)\right)$$

$$\text{ProductLog}\left[-\frac{(-1+c)\,e^{\frac{r-c\,r}{1+(-1+c)\,r}}\,r\,\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\,r}\right]^4+$$

$$\left(118-242\,c+163\,c^2-36\,c^3\right)\,d^3\,\text{ProductLog}\left[-\frac{(-1+c)\,e^{\frac{r-c\,r}{1+(-1+c)\,r}}\,r\,\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\,r}\right]^5+$$

$$\left(24-46\,c+29\,c^2-6\,c^3\right)\,d^3\,\text{ProductLog}\left[-\frac{(-1+c)\,e^{\frac{r-c\,r}{1+(-1+c)\,r}}\,r\,\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\,r}\right]^6\Bigg)\ (x-x0)^4\Bigg)\Bigg/$$

$$\left(24\,(-1+c)^4\,d^3\,(r-x0)^4\,\left(1+\text{ProductLog}\left[-\frac{(-1+c)\,e^{\frac{r-c\,r}{1+(-1+c)\,r}}\,r\,\left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c)\,r}\right]\right)^7\right)+$$

$$O[x-x0]^5$$