

## Coupled Entropy NSP & Paper 2025

$In[*] := \$Assumptions = \kappa > 0 \ \&\& \ \kappa \in \text{Reals} \ \&\& \ x \in \text{Reals} \ \&\& \ \mu \in \text{Reals} \ \&\& \ \sigma \in \text{Reals} \ \&\& \ \sigma > 0 \ \&\& \ \alpha \in \text{Reals} \ \&\& \ \alpha > 0 \ \&\& \ d \in \text{Integers} \ \&\& \ d > 0 \ \&\& \ \gamma \in \text{Reals} \ \&\& \ 0 < \gamma < \infty$

$Out[*] =$

$\kappa > 0 \ \&\& \ \kappa \in \mathbb{R} \ \&\& \ x \in \mathbb{R} \ \&\& \ \mu \in \mathbb{R} \ \&\& \ \sigma \in \mathbb{R} \ \&\& \ \sigma > 0 \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ \alpha > 0 \ \&\& \ d \in \mathbb{Z} \ \&\& \ d > 0 \ \&\& \ \gamma \in \mathbb{R} \ \&\& \ 0 < \gamma < \infty$

## Compute Entropies of Gen. Pareto Distribution

**Clear[CECoupledExp]**

**Assuming** $[\kappa > 0 \ \&\& \ x > 0,$   
**CoupledExponentialDistribution** $[\kappa, 0, \sigma]$   
**]**

**ProbabilityDistribution** $\left[ \begin{cases} \frac{\text{If}[\kappa \neq 0, \text{If}[\text{Simplify}[1 - \frac{\kappa (-x)}{\sigma}] > 0, (1 - \frac{\kappa (-x)}{\sigma})^{-\frac{1+\kappa}{1\kappa}}, 0], \text{Exp}[-\frac{x}{\sigma}]]}{\sigma} & x \geq 0, \{x, -\infty, \infty\} \\ 0 & \text{True} \end{cases} \right]$

$Out[*] =$

$\kappa > 0 \ \&\& \ \kappa \in \mathbb{R} \ \&\& \ x \in \mathbb{R} \ \&\& \ \mu \in \mathbb{R} \ \&\& \ \sigma \in \mathbb{R} \ \&\& \ \sigma > 0 \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ \alpha > 0 \ \&\& \ d \in \mathbb{Z} \ \&\& \ d > 0$

This computes the Coupled Entropy of the Pareto Distribution. See below for computation of the coupled probability and the coupled logarithm of the distribution

$$- \int_0^{\infty} \left( (1 + \kappa) \sigma^{1 + \frac{1}{\kappa}} (x \kappa + \sigma)^{-2 - \frac{1}{\kappa}} \right) \left( \frac{1 - \left( \sigma^{\frac{1}{\kappa}} (x \kappa + \sigma)^{-\frac{1+\kappa}{\kappa}} \right)^{-1 + \frac{1}{1+\kappa}}}{\kappa} \right) dx // \text{FullSimplify}$$

$$\frac{-1 + (1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}}}{\kappa}$$

$$(1 + \kappa) \text{CoupledLogarithm}[\sigma^{-1}, \kappa, 1, 1] - \frac{1 + \kappa}{\kappa} // \text{FullSimplify}$$

$$- \frac{(1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}}}{\kappa}$$

$In[*] := \text{SECoupledExp}[\kappa_, \sigma_] := 1 + \kappa \text{Log}[\sigma];$

$In[*] := \text{CECoupledExp}[\kappa_, \sigma_] := \frac{-1 + (1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}}}{\kappa};$

$$\frac{-1 + (1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}}}{\kappa} = \frac{1 + \kappa}{\kappa} \left( \sigma^{\frac{\kappa}{1+\kappa}} - 1 \right) + \frac{1 + \kappa}{\kappa} - \frac{1}{\kappa} = 1 + \ln_{\frac{1+\kappa}{\kappa}} \sigma \text{ Confirms 2025 Solution}$$

$$\text{In}[*]:= \text{TCCoupledExp}[\kappa_, \sigma_] := \frac{1 + \kappa - \sigma^{-1 + \frac{1}{1+\kappa}}}{\kappa}$$

$$\text{In}[*]:= \frac{1 + \kappa - \sigma^{-1 + \frac{1}{1+\kappa}}}{\kappa} // \text{FullSimplify}$$

Out[\*]=

$$\frac{1 + \kappa - \sigma^{-1 + \frac{1}{1+\kappa}}}{\kappa}$$

$$\frac{1 + \kappa - \sigma^{-1 + \frac{1}{1+\kappa}}}{\kappa} = 1 - \frac{1}{\kappa} \left( \sigma^{-\frac{\kappa}{1+\kappa}} - 1 \right) \text{ Confirms 2025 Result}$$

$$\text{In}[*]:= \text{NTECoupledExp}[\kappa_, \sigma_] := \frac{(1 + \kappa) \left( -1 + (1 + \kappa) \sigma^{\frac{\kappa}{1+\kappa}} \right)}{\kappa}$$

CoupledProbability[

$$\text{ProbabilityDistribution} \left[ \frac{1}{\sigma} \left( 1 - \frac{\kappa (-x)}{\sigma} \right)^{-\frac{1+\kappa}{1+\kappa}}, \{x, 0, \infty\} \right],$$

$$\frac{-\kappa}{1 + \kappa}, x$$

]

$$\begin{cases} 0 & x \leq 0 \\ (1 + \kappa) \sigma^{1 + \frac{1}{\kappa}} (x \kappa + \sigma)^{-2 - \frac{1}{\kappa}} & \text{True} \end{cases}$$

CoupledLogarithm[

$$\frac{1}{\sigma} \left( 1 - \frac{\kappa (-x)}{\sigma} \right)^{-\frac{1+\kappa}{1+\kappa}},$$

$$\kappa, 1, 1] // \text{FullSimplify}$$

$$\text{ConditionalExpression} \left[ \frac{1 - \left( \sigma^{\frac{1}{\kappa}} (x \kappa + \sigma)^{-\frac{1+\kappa}{\kappa}} \right)^{-1 + \frac{1}{1+\kappa}}}{\kappa}, (x \kappa + \sigma)^{-1 - \frac{1}{\kappa}} \geq 0 \right]$$

Assuming[ $\kappa > 0$ , CoupledEntropy[

Assuming[ $\kappa > 0 \ \&\& \ x > 0$ ,

CoupledExponentialDistribution[ $\kappa, 0, \sigma$

],  $\kappa, 1, 1$ ] // Simplify

$$-\int_0^{\infty} \text{If}\left[\left\{\begin{array}{ll} \sigma^{\frac{1}{\kappa}} (x \kappa + \sigma)^{-1-\frac{1}{\kappa}} & x \geq 0 \\ 0 & \text{True} \end{array}\right\} \geq 0,\right.$$

$$\left. \text{If}\left[\kappa \neq 0, -\frac{1}{1 \kappa} \left( \left\{\begin{array}{ll} \frac{\text{If}[\kappa \neq 0, \text{If}[\text{Simplify}[1 - \frac{\kappa (-x)}{\sigma}] > 0, (1 - \frac{\kappa (-x)}{\sigma})^{-\frac{1+1 \kappa}{1 \kappa}}, 0], \text{Exp}[-\frac{x}{\sigma}]]}{\sigma} & x \geq 0 \\ 0 & \text{True} \end{array}\right\} \right)^{-\frac{\kappa}{1+1 \kappa}} - 1 \right], \right.$$

$$\left. \text{Log}\left[\left\{\begin{array}{ll} \frac{\text{If}[\kappa \neq 0, \text{If}[\text{Simplify}[1 - \frac{\kappa (-x)}{\sigma}] > 0, (1 - \frac{\kappa (-x)}{\sigma})^{-\frac{1+1 \kappa}{1 \kappa}}, 0], \text{Exp}[-\frac{x}{\sigma}]]}{\sigma} & x \geq 0 \\ 0 & \text{True} \end{array}\right\}\right], \right.$$

$$\left. \text{Undefined}\right] \left( \left\{\begin{array}{ll} 0 & x < 0 \\ (1 + \kappa) \sigma^{1+\frac{1}{\kappa}} (x \kappa + \sigma)^{-2-\frac{1}{\kappa}} & \text{True} \end{array}\right\} \right) dx$$

NSECoupledExp[ $\kappa_$ ,  $\sigma_$ ] :=

Assuming[ $\infty > \kappa > 0$ ,

Piecewise[{

{NShannonEntropy[CoupledExponentialDistribution[ $\kappa, 0, \sigma$ ], 0, 100], 0 <  $\kappa$  < 0.09},

{NShannonEntropy[

CoupledExponentialDistribution[ $\kappa, 0, \sigma$ ], 0, 1000], 0.09 ≤  $\kappa$  < 0.74},

{NShannonEntropy[

CoupledExponentialDistribution[ $\kappa, 0, \sigma$ ], 0, 10 000], 0.74 ≤  $\kappa$  < 1.5},

{NShannonEntropy[CoupledExponentialDistribution[ $\kappa, 0, \sigma$ ], 0, 15 000], 1.5 ≤  $\kappa$ }

}]

]

NSECoupledExp[#, 1] & /@ {0.25, 0.5, 0.75, 1, 1.25}

{1.25, 1.49992, 1.74985, 1.99796, 2.23985}

NSECoupledExp[0.01, 0.25]

-0.376294

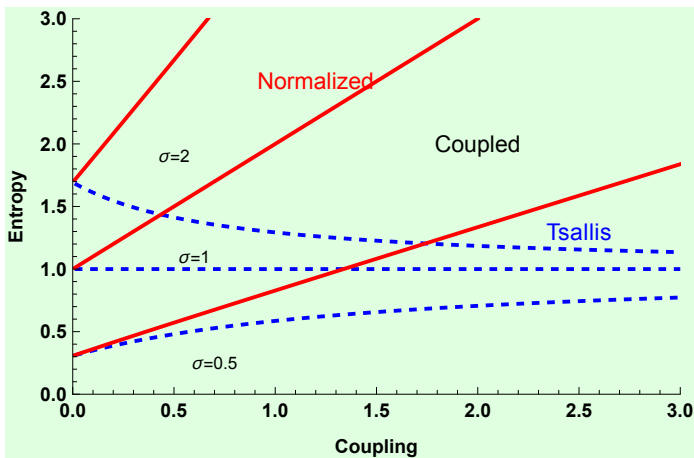
NSECoupledExp[2, 2]

3.54523

## Plot Comparison of GPD with TE, NTE, & Shannon

```
Assuming[0 <  $\kappa$  <  $\infty$ ,
Plot[
{
(*Table[SECoupledExp[ $\kappa$ ,  $\sigma$ ], { $\sigma$ , {0.5, 1, 2}}], *)
{Style[Table[TECoupledExp[ $\kappa$ ,  $\sigma$ ], { $\sigma$ , {0.5, 1, 2}}],
{Blue, Dashed}}],
{Style[Table[NTECoupledExp[ $\kappa$ ,  $\sigma$ ], { $\sigma$ , {0.5, 1, 2}}],
{Red, Dotted}}],
{Style[Table[CECoupledExp[ $\kappa$ ,  $\sigma$ ], { $\sigma$ , {0.5, 1, 2}}],
Black}}
},
{ $\kappa$ , 0.01, 3},
Background → LightGreen,
PlotRange → {{0, 3}, {0, 3}},
Frame → {{True, False}, {True, False}},
FrameLabel → {"Coupling", "Entropy"},
FrameStyle → Directive[10, Bold, Black],
Epilog → {
Inset[Style["Coupled", Black, Medium], {2, 2}],
Inset[Style["Tsallis", Blue, Medium], {2.5, 1.3}],
Inset[Style["Normalized", Red, Medium], {1.2, 2.5}],
Inset[Style[" $\sigma=2$ ", Black], {0.5, 1.9}],
Inset[Style[" $\sigma=1$ ", Black], {0.6, 1.1}],
Inset[Style[" $\sigma=0.5$ ", Black], {0.7, 0.25}]
}
]
]
```

Out[ ] =



## Compute Tsallis Entropy with different $q$ values

$$In[\bullet] := \$Assumptions =$$

$$\kappa \in \text{Reals} \ \&\& \ 0 < \kappa < \infty \ \&\& \ \alpha \in \text{Reals} \ \&\& \ 0 < \alpha < \infty \ \&\& \ d \in \text{Reals} \ \&\& \ 0 < d < \infty \ \&\& \ \sigma \in \text{Reals} \ \&\& \ 0 < \sigma < \infty$$

*Out*[•]=

$$\kappa \in \mathbb{R} \ \&\& \ 0 < \kappa < \infty \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ 0 < \alpha < \infty \ \&\& \ d \in \mathbb{R} \ \&\& \ 0 < d < \infty \ \&\& \ \sigma \in \mathbb{R} \ \&\& \ 0 < \sigma < \infty$$

$$qEnt = \frac{1}{qDist}$$

$$ln[*] := \int_0^\infty \text{FullSimplify}\left[\frac{1}{\sigma} \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}} \left(1 + \text{qToCoupling}\left[\frac{1+\kappa}{1+2\kappa}\right]\right)\right. \\ \left. \text{CoupledLogarithm}\left[\sigma \left(1 + \frac{\kappa x}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}, \text{qToCoupling}\left[\frac{1+\kappa}{1+2\kappa}\right], 1\right]\right] dx // \text{FullSimplify}$$

$$\int_0^\infty \frac{1}{(2+5\kappa)(\kappa\kappa+\sigma)} 2(1+2\kappa) \left(1+\frac{\kappa\kappa}{\sigma}\right)^{-1/\kappa} \text{If}\left[\left(1+\frac{\kappa\kappa}{\sigma}\right)^{\frac{1}{\kappa}}(\kappa\kappa+\sigma) \geq 0, \right.$$

$$\text{If}\left[-\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}\neq 0,-\frac{\left(\left(1+\frac{\kappa\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}\sigma\right)^{\left(3-\frac{1+\kappa}{1+2\kappa}\right)\left(1-\frac{1+\kappa}{3-\frac{1+\kappa}{1+2\kappa}}\right)}-1}{\frac{1-\frac{1+\kappa}{1+2\kappa}}{3-\frac{1+\kappa}{1+2\kappa}}},\text{Log}\left[\left(1+\frac{\kappa\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}\sigma\right],\text{Undefined}\right]\mathrm{d}x$$

$$\ln[*]:= \frac{2}{(2+5\kappa)(\kappa\kappa+\sigma)}(1+2\kappa)\left(1+\frac{\kappa\kappa}{\sigma}\right)^{-1/\kappa} - \frac{\left(\left(1+\frac{\kappa\kappa}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}\sigma\right)^{-\frac{1+\kappa}{1+2\kappa}} - 1}{3 - \frac{1+\kappa}{1+2\kappa}} // \text{FullSimplify}$$

*Out[• ] =*

$$\frac{2^{1+2\kappa} \left(1 + \frac{\kappa}{\sigma}\right)^{-1/\kappa} \left(-1 + \left(1 + \frac{\kappa}{\sigma}\right)^{\frac{1}{\kappa}} (\kappa + \sigma)\right)^{-\frac{\kappa}{2+\kappa}}}{\kappa (\kappa + \sigma)}$$

$$\ln[ ] := \int_0^\infty - \frac{2 (1 + 2 \kappa) \left(1 + \frac{x \kappa}{\sigma}\right)^{-1/\kappa} \left(-1 + \left(1 + \frac{x \kappa}{\sigma}\right)^{\frac{1}{\kappa}} (x \kappa + \sigma)\right)^{-\frac{\kappa}{2 + 4 \kappa}}}{\kappa (x \kappa + \sigma)} dx // \text{FullSimplify}$$

*Out*[•] =

$$2(1+2\kappa)\sigma^{\frac{1}{\kappa}}\left(-\sigma^{-1/\kappa}+\frac{(2+4\kappa)\sigma^{-\frac{2+4\kappa}{2\kappa+4\kappa^2}}}{2+\kappa(5+\kappa)}\right)$$

$$q_{\text{Ent}} = 2 - q_{\text{Dist}}$$

```

In[ ]:= Integrate[FullSimplify[ $\frac{1}{\sigma} \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}} \left(1 + \text{qToCoupling}\left[1 - \frac{\kappa}{1+\kappa}\right]\right)$ 
    CoupledLogarithm[ $\sigma \left(1 + \frac{\kappa x}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}$ ,  $\text{qToCoupling}\left[1 - \frac{\kappa}{1+\kappa}\right]$ , 1]] dx // FullSimplify

Out[ ]:= 
$$\frac{2 (1 + \kappa) \left(1 - \frac{2 \sigma^{-\frac{\kappa}{2+2\kappa}}}{2+\kappa}\right)}{\kappa}$$


Dual kEnt = -  $\frac{\kappa}{1+\kappa}$ 

In[ ]:= Integrate[FullSimplify[ $\frac{1}{\sigma} \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}} \left(1 - \frac{\kappa}{1+\kappa}\right)$  CoupledLogarithm[ $\sigma \left(1 + \frac{\kappa x}{\sigma}\right)^{\frac{1+\kappa}{\kappa}}$ ,  $-\frac{\kappa}{1+\kappa}$ , 1]]
    dx // FullSimplify

Out[ ]:= 
$$\frac{1 - \frac{\sigma^{-\kappa}}{1+\kappa+\kappa^2}}{\kappa}$$


```

## Coupled Entropy of Coupled Stretched Exponential Distribution

Structure of solution is  $\frac{d}{\alpha} + \text{Log}_{\frac{\alpha\kappa}{1+d\kappa}}[Z_{\kappa}(\sigma, \alpha, d)] = \frac{d}{\alpha} + \text{Log}_{\frac{\alpha\kappa}{1+d\kappa}}[\sigma] \oplus \frac{\alpha\kappa}{1+d\kappa} \text{Log}_{\frac{\alpha\kappa}{1+d\kappa}}[Z'_{\kappa}(\alpha, d)]$

Computation of Coupled Entropy of Coupled Stretched Exponential

```

In[ ]:= Clear[CoupledEntropyCSE];
CoupledEntropyCSE[σ_, κ_, α_, d_] :=
  CoupledEntropyCSE[σ, κ, α, d] =
     $\frac{d}{\alpha} + \text{CoupledLogarithm}\left[
      \text{NormMultiCoupled}[\sigma, \kappa, \alpha, d],
      \frac{\alpha\kappa}{1+d\kappa}, 0\right];$ 

```

Plot of Coupled Entropies

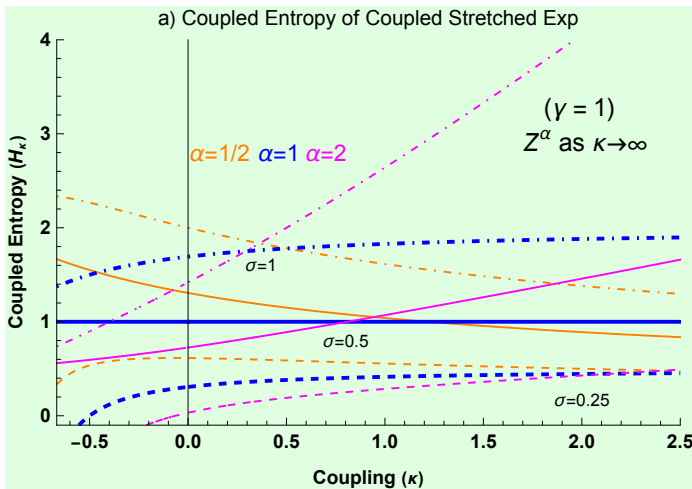
In[ ]:=

```

Plot[
  Evaluate@{
    Table[CoupledEntropyCSE[σ, κ, 1/2, 1]^1, {σ, {0.25, 0.5, 1}}],
    Table[CoupledEntropyCSE[σ, κ, 1, 1]^1, {σ, {0.25, 0.5, 1}}],
    Table[CoupledEntropyCSE[σ, κ, 2, 1]^1, {σ, {0.25, 0.5, 1}}],
  },
  {κ, -0.667, 2.5},
  Background → LightGreen,
  PlotRange → {{-0.667, 2.5}, {-0.1, 4}},
  Frame → {{True, False}, {True, False}},
  FrameLabel → {"Coupling (κ)", "Coupled Entropy (Hκ)"},
  FrameStyle → Directive[10, Bold, Black],
  PlotStyle → Flatten[Table[{αCol, σCol},
    {αCol, {{Orange, Thickness[Medium]}, {Blue, Thickness[Large]},
      {Magenta, Thickness[Medium]}}}, {σCol, {Dashed, , DotDashed}}], 1],
  PlotLabel →
    "a) Coupled Entropy of Coupled Stretched Exp",
  Epilog → {
    Inset[Style["α=1/2", Orange, Medium], {.16, 2.8}],
    Inset[Style["α=1", Blue, Medium], {.46, 2.8}],
    Inset[Style["α=2", Magenta, Medium], {.7, 2.8}],
    Inset[Style["σ=1", Black], {0.37, 1.6}],
    Inset[Style["σ=0.5", Black], {0.8, .8}],
    Inset[Style["σ=0.25", Black], {2.0, 0.17}],
    Inset[Style[DisplayForm[HoldForm[(γ = 1) "
Zα as κ→∞"]], Black, FontSize → 13], {2, 3.1}]
  }
]

```

Out[ ]:=



In[ ]:=

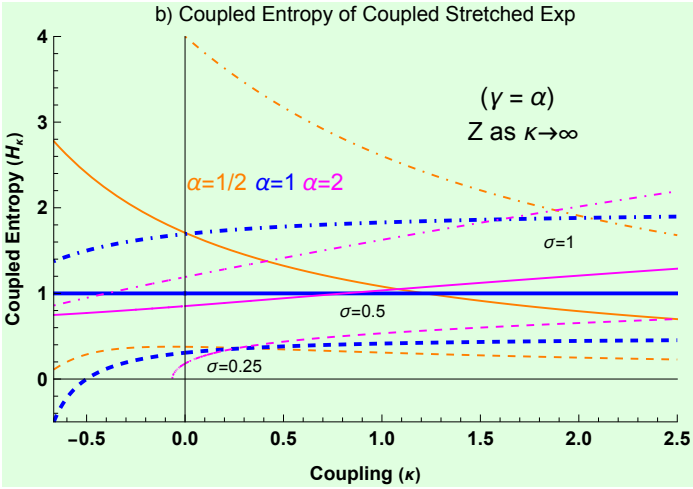
```

Plot[
  Evaluate@{
    Table[CoupledEntropyCSE[ $\sigma$ ,  $\kappa$ , 1/2, 1] $\frac{2}{1}$ , { $\sigma$ , {0.25, 0.5, 1}}],
    Table[CoupledEntropyCSE[ $\sigma$ ,  $\kappa$ , 1, 1], { $\sigma$ , {0.25, 0.5, 1}}],
    Table[CoupledEntropyCSE[ $\sigma$ ,  $\kappa$ , 2, 1] $\frac{1}{2}$ , { $\sigma$ , {0.25, 0.5, 1}}],
  },
  { $\kappa$ , -0.667, 2.5},
  Background → LightGreen,
  PlotRange → {{-0.667, 2.5}, {-0.5, 4}},
  Frame → {{True, False}, {True, False}},
  FrameLabel → {"Coupling ( $\kappa$ )", "Coupled Entropy ( $H_\kappa$ )"},
  FrameStyle → Directive[10, Bold, Black],
  PlotStyle → Flatten[Table[{ $\alpha$ Col,  $\sigma$ Col},
    { $\alpha$ Col, {{Orange, Thickness[Medium]}, {Blue, Thickness[Large]},
      {Magenta, Thickness[Medium]}}}, { $\sigma$ Col, {Dashed, , DotDashed}}], 1],
  PlotLabel →
    "b) Coupled Entropy of Coupled Stretched Exp",
  Epilog → {
    Inset[Style[" $\alpha=1/2$ ", Orange, Medium], {.16, 2.3}],
    Inset[Style[" $\alpha=1$ ", Blue, Medium], {.46, 2.3}],
    Inset[Style[" $\alpha=2$ ", Magenta, Medium], {.7, 2.3}],
    Inset[Style[" $\sigma=1$ ", Black], {1.9, 1.6}],
    Inset[Style[" $\sigma=0.5$ ", Black], {0.9, .8}],
    Inset[Style[" $\sigma=0.25$ ", Black], {0.25, 0.15}],
    Inset[Style[DisplayForm[HoldForm[( $\gamma = \alpha$ ) "
Z as  $\kappa \rightarrow \infty$ "]], Black, FontSize → 13], {1.7, 3.1}]
  }
]

```



Out[ ]=



Plot with variable dimensions

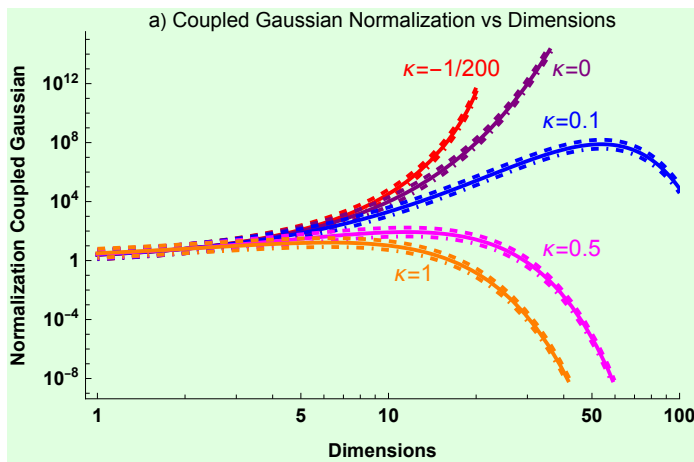
In[ ]:=

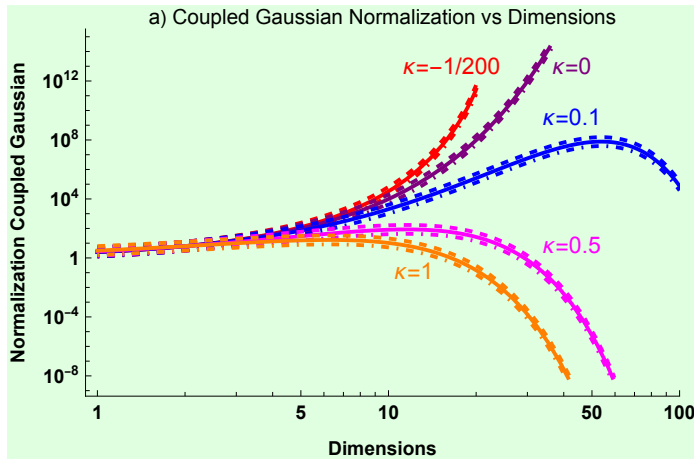
```

LogLogPlot[
  Evaluate@{
    Table[NormMultiCoupled[ $\sigma$ ,  $-0.5 \times 10^{-1}$ , 2, d], { $\sigma$ , {0.5, 1, 2}}],
    Table[NormMultiCoupled[ $\sigma$ , 0.0, 2, d], { $\sigma$ , {0.5, 1, 2}}],
    Table[NormMultiCoupled[ $\sigma$ , 0.1, 2, d], { $\sigma$ , {0.5, 1, 2}}],
    Table[NormMultiCoupled[ $\sigma$ , 0.5, 2, d], { $\sigma$ , {0.5, 1, 2}}],
    Table[NormMultiCoupled[ $\sigma$ , 1.0, 2, d], { $\sigma$ , {0.5, 1, 2}}],
  },
  {d, 1, 100},
  Background → LightGreen,
  PlotRange → {{0, 100}, {0, Automatic}},
  Frame → {{True, False}, {True, False}},
  FrameLabel → {"Dimensions", "Normalization Coupled Gaussian"},
  FrameStyle → Directive[10, Bold, Black],
  PlotStyle → Flatten[Table[{ $\kappa$ Col,  $\sigma$ Col},
    { $\kappa$ Col, {Red, Purple, Blue, Magenta, Orange}}, { $\sigma$ Col, {DotDashed, Dashed}}, 1],
  PlotLabel → "a) Coupled Gaussian Normalization vs Dimensions",
  Epilog → {
    Inset[Style[" $\kappa=-1/200$ ", Red, Medium], {2.8, 30}],
    Inset[Style[" $\kappa=0$ ", Purple, Medium], {3.75, 30}],
    Inset[Style[" $\kappa=0.1$ ", Blue, Medium], {3.75, 22}],
    Inset[Style[" $\kappa=0.5$ ", Magenta, Medium], {3.75, 2}],
    Inset[Style[" $\kappa=1$ ", Orange, Medium], {2.5, -2}],
    (*Inset[Style[" $\sigma=2$ ", Black], {1.45, 1.45}],
    Inset[Style[" $\sigma=1$ ", Black], {0.8, .8}],
    Inset[Style[" $\sigma=0.5$ ", Black], {0.5, 0.1}]*
  )
]

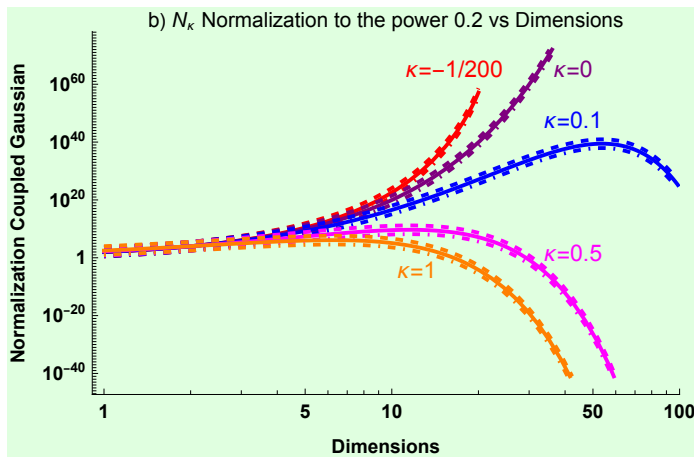
```

Out[ ]:=





With  $\gamma = 2/10$



## Score Function Plots

The score function of the GPD is  $-\sigma^{-1}$ , which is a powerful description of the scales unique properties. The score function is computed from the derivative of the log of the pdf.

$$\text{In}[*]:= \text{NegDerGPD}[\sigma\_ , \kappa\_ , x\_ ] := \frac{1 + \kappa}{x \kappa + \sigma};$$

$$\text{NegDerGPDq}[\beta\_ , q\_ , x\_ ] := \frac{\beta}{1 + (-1 + q) x \beta};$$

$$\text{In}[*]:= \text{Clear}[\text{NegDerGPD}]$$

$$\text{In}[*]:= \text{Assuming}\left[0 < \kappa < \infty, -\text{D}\left[\text{Log}\left[\frac{1}{\sigma} \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}\right], x\right]\right] // \text{FullSimplify}$$

$$\text{Out}[*]= \frac{1 + \kappa}{x \kappa + \sigma}$$

```
In[*]:= 
$$\frac{1+\kappa}{x\kappa+\sigma} /. \left\{ \kappa \rightarrow \frac{-(1-q)}{2-q}, \sigma \rightarrow \frac{1}{(2-q)\beta} \right\} // \text{FullSimplify}$$

```

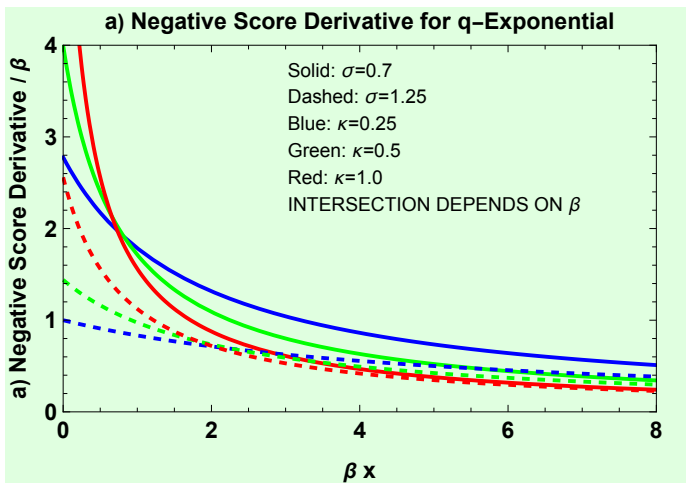
```
Out[*]= 
$$\frac{\beta}{1+(-1+q)x\beta}$$

```

Plot - $\beta$  Score versus  $\frac{x}{\beta}$

```
In[*]:= Plot[MapThread[scaleShapeToBeta[#1, #2, 1] x &,  
  NegDerGPD[#1, #2, scaleShapeToBeta[#1, #2, 1] x] &,  
  {{0.75, 0.75, 0.75, 1.25, 1.25, 1.25}, {0.25, 0.5, 1, 0.25, 0.5, 1}}] //  
  Evaluate, {x, 0.0001, 10},  
  Background -> LightGreen,  
  PlotRange -> {{0, 8}, {0, 4}},  
  PlotStyle -> {Blue, Green, Red, {Blue, Dashed}, {Green, Dashed}, {Red, Dashed}},  
  Epilog -> Inset[Style[Text["Solid:  $\sigma=0.7$   
Dashed:  $\sigma=1.25$   
Blue:  $\kappa=0.25$   
Green:  $\kappa=0.5$   
Red:  $\kappa=1.0$   
INTERSECTION DEPENDS ON  $\beta$ "], Larger], {5, 3}],  
  LabelStyle -> Directive[Bold, Medium],  
  Frame -> {{True, True}, {True, True}},  
  FrameLabel -> {{ "a) Negative Score Derivative /  $\beta$ ", ""}, {" $\beta x$ ",  
  "a) Negative Score Derivative for q-Exponential"}},  
  ]
```

```
Out[*]=
```



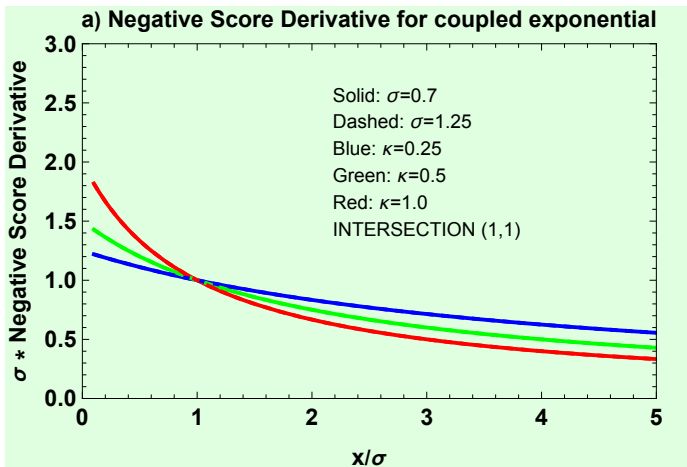
Translation of sigma and kappa

```

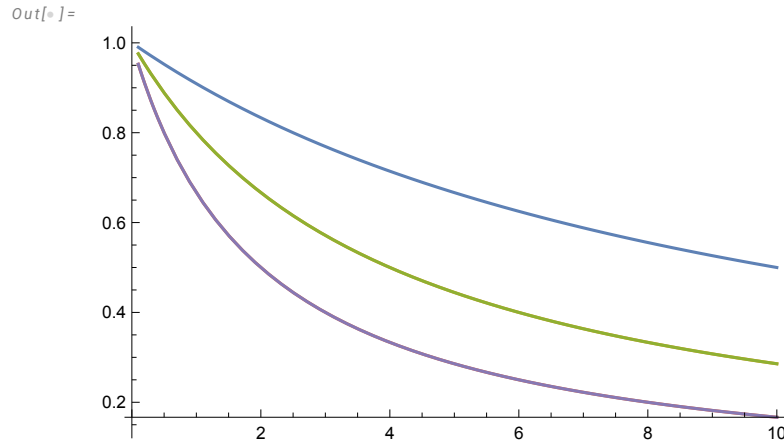
Plot[MapThread[
  #1 NegDerGPD[#1, #2, #1 x] &, {{0.75, 0.75, 0.75, 1.25, 1.25, 1.25},
    {0.25, 0.5, 1, 0.25, 0.5, 1}}] // Evaluate, {x, 0.1, 10},
  Background → LightGreen,
  PlotRange → {{0, 5}, {0, 3}},
  PlotStyle → {Blue, Green, Red, {Blue, Dashed}, {Green, Dashed}, {Red, Dashed}},
  Epilog → Inset[Style[Text["Solid:  $\sigma=0.7$ 
Dashed:  $\sigma=1.25$ 
Blue:  $\kappa=0.25$ 
Green:  $\kappa=0.5$ 
Red:  $\kappa=1.0$ 
INTERSECTION (1,1)"], Larger], {3, 2}],
  LabelStyle → Directive[Bold, Medium],
  Frame → {{True, True}, {True, True}},
  FrameLabel → {{ $\sigma * \text{Negative Score Derivative}$ , ""},
    { $x/\sigma$ , "a) Negative Score Derivative for Coupled Exponential"}}
]

```

Out[ ] =



```
In[ ]:= Plot[MapThread[#1^-1 NegDerGPDq[#1, #2, #1^-1 x] &,
  {{0.5, 0.5, .2, .2, .5}, {1.1, 1.25, 1.25, 1.5, 1.5}}] // Evaluate, {x, 0.1, 10}]
```



## Generalized Weibull Distribution

```
In[ ]:= GWDSF[σ_, κ_, α_, x_] :=  $\left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-\frac{1}{\alpha\kappa}};$ 
```

```
In[ ]:= GWDPDF[σ_, κ_, α_, x_] :=  $\frac{x^{\alpha-1}}{\sigma^\alpha} \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-\frac{1}{\alpha\kappa}-1};$ 
```

```
In[ ]:= D[-Log[GWDPDF[σ, κ, α, x]], x] \ FullSimplify
```

⋯ Syntax: "D[-Log[GWDPDF[σ, κ, α, x]], x] \ FullSimplify" is incomplete; more input is needed.

```
In[ ]:= D[-Log[ $\frac{x^{\alpha-1}}{\sigma^\alpha} \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-\frac{1}{\alpha\kappa}-1}$ ], x] \ FullSimplify
```

⋯ Syntax: "D[-Log[ $\frac{x^{\alpha-1}}{\sigma^\alpha} \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-\frac{1}{\alpha\kappa}-1}$ ], x] \ FullSimplify" is incomplete; more input is needed.

```
In[ ]:= -∂x Log[x-1+α σ-α (1 + xα κ σ-α)-1- $\frac{1}{\alpha\kappa}$ ] // FullSimplify
```

Out[ ]:=

$$\frac{x^\alpha (1 + \kappa) - (-1 + \alpha) \sigma^\alpha}{x (x^\alpha \kappa + \sigma^\alpha)}$$

```
In[ ]:=  $\left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-1} /. x \rightarrow \sigma // FullSimplify$ 
```

Out[ ]:=

$$\frac{2 - \alpha + \kappa}{\sigma + \kappa \sigma}$$

So the use of the Generalized Weibull will require care regarding the definition for the scale of the distribution

If  $\alpha = 2$ , then

$$\frac{1}{\sigma} \frac{\kappa}{1 + \kappa}$$

So if this is defined as  $\sigma W$  let's see what happens. Could also do this generally for alpha.

$$\begin{aligned} In[*] &:= \frac{1}{\frac{2-\alpha+\kappa}{\sigma+\kappa}\sigma} \\ Out[*] &= \frac{\sigma + \kappa \sigma}{2 - \alpha + \kappa} \end{aligned}$$

$$In[*] := \sigma W = \frac{\sigma + \kappa \sigma}{2 - \alpha + \kappa};$$

Recompute the derivative using the expression  $-\frac{f'(x)}{f(x)}$

$$\begin{aligned} In[*] &:= D\left[x^{-1+\alpha} \sigma^{-\alpha} (1 + x^\alpha \kappa \sigma^{-\alpha})^{-1-\frac{1}{\alpha\kappa}}, x\right] // FullSimplify \\ Out[*] &= x^{-2+\alpha} \sigma^{\frac{1}{\kappa}} (\kappa^\alpha \kappa + \sigma^\alpha)^{-2-\frac{1}{\alpha\kappa}} (-x^\alpha (1 + \kappa) + (-1 + \alpha) \sigma^\alpha) \\ In[*] &:= -\frac{x^{-2+\alpha} \sigma^{\frac{1}{\kappa}} (\kappa^\alpha \kappa + \sigma^\alpha)^{-2-\frac{1}{\alpha\kappa}} (-x^\alpha (1 + \kappa) + (-1 + \alpha) \sigma^\alpha)}{x^{-1+\alpha} \sigma^{-\alpha} (1 + x^\alpha \kappa \sigma^{-\alpha})^{-1-\frac{1}{\alpha\kappa}}} // FullSimplify \\ Out[*] &= \frac{x^\alpha (1 + \kappa) - (-1 + \alpha) \sigma^\alpha}{x (\kappa^\alpha \kappa + \sigma^\alpha)} \end{aligned}$$

Confirmed that it is the same expression

Consider solving for x, such that the score is equal to its inverse.

$$\begin{aligned} In[*] &:= Solve\left[\left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-1} == \left(\left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-1}\right)^{-1}, x\right] \\ &Solve\left[\left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right)^2 \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-2} == 1, x\right] \\ &\left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right)^2 \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-2} // FullySimplify \end{aligned}$$

Considering possible values of the scale

$$\begin{aligned} In[*] &:= \left(\frac{x^{\alpha-1} (1 + \kappa)}{\sigma^\alpha} + (1 - \alpha) x^{-1}\right) \left(1 + \frac{\kappa x^\alpha}{\sigma^\alpha}\right)^{-1} /. x \rightarrow \sigma \frac{\kappa}{\alpha} // FullSimplify \\ Out[*] &= \frac{\alpha \left(1 + \kappa - \frac{(1+\alpha\kappa)\sigma^\alpha}{\sigma^\alpha + \kappa \left(\frac{\kappa\sigma}{\alpha}\right)^\alpha}\right)}{\kappa^2 \sigma} \end{aligned}$$

Examine the log-log derivative

From the following, this is x times derivative of the Log[

Therefore:

$$\frac{d}{d(\log x)} \log(f(x)) = \frac{f'(x)}{f(x)} \cdot x = x \cdot \frac{f'(x)}{f(x)}$$

$$\text{In[*]} := \text{FullSimplify}\left[x \left( \frac{x^{\alpha-1} (1+\kappa)}{\sigma^\alpha} + (1-\alpha) x^{-1} \right) \left( 1 + \frac{\kappa x^\alpha}{\sigma^\alpha} \right)^{-1}\right]$$

Out[\*] =

$$1 - \alpha + \frac{x^\alpha (1 + \alpha \kappa)}{x^\alpha \kappa + \sigma^\alpha}$$

$$\text{Solve}\left[1 - \alpha + \frac{x^\alpha (1 + \alpha \kappa)}{x^\alpha \kappa + \sigma^\alpha} == 1, x\right]$$

Out[\*] =

\$Aborted

$$\text{Solve}\left[\frac{x^\alpha}{\sigma^\alpha} \frac{(1 + \alpha \kappa)}{1 + \frac{x^\alpha}{\sigma^\alpha} \kappa} == \alpha, x\right]$$

$$\text{If } x = \alpha^{1/\alpha} \sigma$$

$$\text{Solve}\left[\frac{\alpha \sigma^\alpha}{\sigma^\alpha} == \alpha, x\right]$$

$$\text{In[*]} := 1 - \alpha + \frac{x^\alpha (1 + \alpha \kappa)}{x^\alpha \kappa + \sigma^\alpha} /. x \rightarrow \alpha^{1/\alpha} \sigma // \text{FullSimplify}$$

Out[\*] =

1

$$\text{In[*]} := \left( \frac{x^{\alpha-1} (1+\kappa)}{\sigma^\alpha} + (1-\alpha) x^{-1} \right) \left( 1 + \frac{\kappa x^\alpha}{\sigma^\alpha} \right)^{-1} /. x \rightarrow \alpha^{1/\alpha} \sigma // \text{FullSimplify}$$

Out[\*] =

$$\frac{\alpha^{-1/\alpha}}{\sigma}$$

$$\text{In[*]} := x \left( \frac{x^{\alpha-1} (1+\kappa)}{\sigma^\alpha} + (1-\alpha) x^{-1} \right) \left( 1 + \frac{\kappa x^\alpha}{\sigma^\alpha} \right)^{-1} /. x \rightarrow \alpha^{1/\alpha} \sigma // \text{FullSimplify}$$

Out[\*] =

1

$$x \left( \frac{x^{\alpha-1} (1+\kappa)}{\sigma^\alpha} + (1-\alpha) x^{-1} \right) \left( 1 + \frac{\kappa x^\alpha}{\sigma^\alpha} \right)^{-1} /. x \rightarrow$$

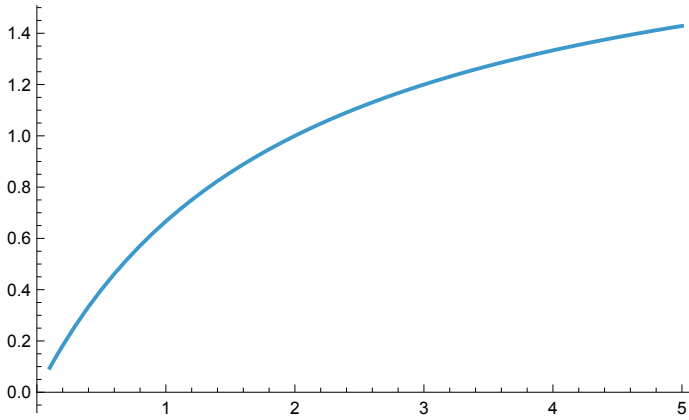
This is equivalent to modifying the definition of the Coupled Weibull Distribution to be

$$\frac{x^{\alpha-1}}{\sigma^\alpha} \left( 1 + \alpha \kappa \frac{x^\alpha}{\alpha \sigma^\alpha} \right)^{-\frac{1}{\alpha \kappa} - 1}$$



```
In[ ]:= Plot[x ( (x^(alpha-1) (1+κ) / σ^α) + (1-α) x^-1 ) (1 + (κ x^α / σ^α))^-1 /. {κ→1, α→1, σ→2}, {x, 0.1, 5}]
```

```
Out[ ]:=
```



## Redefined Coupled Weibull with $\sigma$ equal to the “Informational Scale”

```
In[ ]:= GWDSF[σ_, κ_, α_, x_] := (1 + α κ (x^α / σ^α))^-1 / (α κ);
```

```
In[ ]:= GWDPDF[σ_, κ_, α_, x_] := (α x^(α-1) / σ^α) (1 + α κ (x^α / σ^α))^-1 / (α κ) - 1;
```

```
In[ ]:= -x (D[GWDPDF[σ, κ, α, x], x] / GWDPDF[σ, κ, α, x]) // FullSimplify
```

```
Out[ ]:=
```

$$1 + \frac{\alpha (x^\alpha - \sigma^\alpha)}{x^\alpha \alpha \kappa + \sigma^\alpha}$$

```
In[ ]:= 1 + (α (x^α - σ^α) / (x^α α κ + σ^α)) /. x→σ // FullSimplify
```

```
Out[ ]:=
```

$$1$$

## Score of Coupled Stretched Exponential Distribution

Computing  $-d \text{Log}[f(x)]/dx$  where  $f(x)$  is the Coupled Stretched Exponential Distribution. The normalization,  $Z$ , can be ignored since the expression is equivalent to  $-d f'(x)/(f(x) dx)$  and constants cancel.

```
In[ ]:= ∂_x Log[ (1 + κ (x/σ)^α)^-1 / (α (1/x + 1)) ] // FullSimplify
```

```
Out[ ]:=
```

$$-\frac{x^{-1+\alpha} (1+\kappa)}{x^\alpha \kappa + \sigma^\alpha}$$

```

In[*]:= 
$$\frac{x^{-1+\alpha} (1 + \kappa)}{x^\alpha \kappa + \sigma^\alpha} /. x \rightarrow \sigma // \text{FullSimplify}$$

Out[*]= 
$$\frac{1}{\sigma}$$


In[*]:= 
$$-x \partial_x \text{Log} \left[ \left( 1 + \kappa \left( \frac{x}{\sigma} \right)^\alpha \right)^{-\frac{1}{\alpha} \left( \frac{1}{\kappa} + 1 \right)} \right] // \text{FullSimplify}$$

Out[*]= 
$$\frac{(1 + \kappa) \left( \frac{x}{\sigma} \right)^\alpha}{1 + \kappa \left( \frac{x}{\sigma} \right)^\alpha}$$


In[*]:= 
$$\frac{(1 + \kappa) \left( \frac{x}{\sigma} \right)^\alpha}{1 + \kappa \left( \frac{x}{\sigma} \right)^\alpha} /. x \rightarrow \sigma // \text{FullSimplify}$$

Out[*]= 1

In[*]:= 
$$(1 + \kappa) \left( 1 + \kappa \left( \frac{x}{\sigma} \right)^\alpha \right)^{-1 + \frac{1+\frac{1}{\kappa}}{\alpha}} \left( \frac{x}{\sigma} \right)^\alpha /. x \rightarrow \sigma // \text{FullSimplify}$$

Out[*]= 
$$(1 + \kappa)^{\frac{1+\frac{1}{\kappa}}{\alpha}}$$


```

## Multiplicative Noise Simulation

**Lemma (Multiplicative Process With Coupled Gaussian Limit)**

Let a stochastic process  $X_t$  be defined by the Stratonovich differential equation,

$$dX_t = \underbrace{f(X_t)dt}_{\text{Drift}} + \underbrace{A \circ dW_t^{(a)}}_{\text{Additive Noise}} + \underbrace{g(X_t)M \circ dW_t^{(m)}}_{\text{Multiplicative Noise}}$$

where  $dW_t^{(a)}$  and  $W_t^{(m)}$  are independent Wiener processes which define the additive (a) and multiplicative (m) noise, and  $A$  and  $M$  are the amplitudes of each noise source. Let the drift,  $J(x) = f(X_t)$ , be related to the diffusion,  $D(x) = \frac{1}{2}(A^2 + M^2 g^2(x))$  by a restorative potential,  $V(x)$ , in which  $f(x) = -\tau g(x)g'(x) = -V'(x)$  and  $\tau$  is a time-domain scaling. Then the probability density  $p_X(x, t)$  for this system has a coupled Gaussian limit distribution of  $p_X(x) = \lim_{t \rightarrow \infty} p_X(x, t) \propto \exp_{\kappa}^{-\frac{1+\kappa}{2}} \left( \frac{g^2(x)}{\sigma^2} \right)$ , with  $\sigma^2 = \frac{A^2}{2\tau}$  and  $\kappa = \frac{D'(x)}{V'(x)} = \frac{M^2}{2\tau}$ .

Define Stratonovich Process for Velocity with Multiplicative Noise and a Coupled Gaussian Limit Distribution

```

In[*]:= MultVelProc = StratonovichProcess[dv[t] == -v[t] d[t] + A dWAdd[t] + M v[t] x dWMult[t],
      v[t], {v, 1}, t, {WAdd ≈ WienerProcess[], WMult ≈ WienerProcess[]}]
Out[*]= StratonovichProcess[{{-v[t]}, {{A, M v[t]}}, v[t]}, {{v}, {1}}, {t, 0}]

```

```
In[*]:= MultVelProc["KolmogorovForwardEquation"] // TraditionalForm
```

```
Out[*]//TraditionalForm=
```

$$p^{(0,1)}(v, t) = \frac{1}{2} \frac{\partial^2 ((A^2 + M^2 v^2) p(v, t))}{\partial v \partial v} - \frac{\partial}{\partial v} \left( \frac{1}{2} (M^2 v - 2 v) p(v, t) \right)$$

```
In[*]:= Limit[PDF[MultVelProc[t] /. A -> Sqrt[2] σ, v] // FullSimplify, t -> Infinity]
```

```
Out[*]=
```

$$\lim_{t \rightarrow \infty} \text{PDF} \left[ \text{StratonovichProcess} \left[ \{ \{-v[t]\}, \{ \{ \sqrt{2} \sigma, M v[t] \} \}, v[t]^2 \}, \{ \{v\}, \{1\} \}, \{t, 0\} \right] [t], v \right]$$

```
In[*]:= AddSample[σ_, κ_] :=
```

```
RandomFunction[AddVelProc /. {A -> σ Sqrt[2], M -> Sqrt[2 * κ]}, {0, 10, .01}, 50];
```

```
In[*]:= MultSample[σ_, κ_] :=
```

```
RandomFunction[MultVelProc /. {A -> σ Sqrt[2], M -> Sqrt[2 * κ]}, {0, 10, .01}, 10];
```

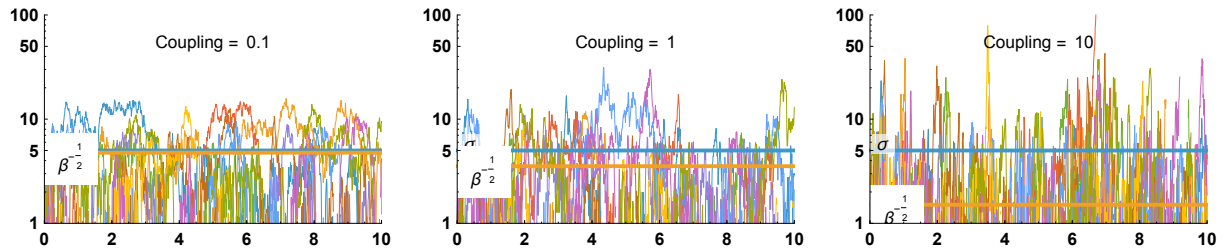
```
σ=2, κ=0.1
```

```

In[ ]:= Module[{σ = 5, κ = {0.1, 1, 10}, yLimit = {1, 100}},
  GraphicsRow[Table[
    Show[
      ListLinePlot[MultSample[σ, κ[[i]],
        PlotRange → {{0, 10}, {yLimit[[1], yLimit[[2]]}},
        PlotStyle → Thin,
        ScalingFunctions → {"Linear", "Log"},
        LabelStyle → Bold,
        Epilog → Inset["Coupling = " Text[κ[[i]], {5, 4}]
      ],
      Plot[{Labeled[σ, "σ", {Bottom, Left}],
        Labeled[ $\frac{\sigma}{\text{Sqrt}[1 + \kappa[[i]]]}$ , " $\beta^{-\frac{1}{2}}$ ", {Top, Left}]}], {x, 0, 10},
        PlotRange → {{0, 10}, {yLimit[[1], yLimit[[2]]}},
        PlotStyle → Thick,
        ScalingFunctions → {"Linear", "Log"}
    ]
  ], {i, 3}]]]

```

Out[ ]=



## Checking DeepSeek Solutions

```

In[ ]:= Solve[1 +  $\frac{\tau}{\alpha M}$  ==  $\frac{1 + \kappa}{\alpha \kappa}$ , κ]

```

Out[ ]=

$$\left\{ \left\{ \kappa \rightarrow \frac{M}{-M + M \alpha + \tau} \right\} \right\}$$

$$\text{In}[*]:= \frac{M}{-M + M \alpha + \tau} \frac{A}{M} = \frac{A}{-M + M \alpha + \tau}$$

$$\text{Solve}\left[\frac{\tau + \frac{3}{4} \alpha M}{\alpha M} = \frac{1 + \kappa}{\kappa}, \kappa\right] \backslash \backslash \text{FullSimplify}$$

Set: Tag Times in  $\frac{A M}{M (-M + M \alpha + \tau)}$  is Protected. ?

$$\text{Out}[*]= \frac{A}{-M + M \alpha + \tau}$$

$$\text{In}[*]:= \text{Solve}\left[\frac{1 + \frac{3}{4} \alpha \frac{M}{\tau}}{\alpha \frac{M}{\tau}} = \frac{1 + \kappa}{\kappa}, \kappa\right] \backslash \backslash \text{FullSimplify}$$

Syntax: "Solve  $\left[\frac{\tau + \frac{3}{4} \alpha M}{\alpha M} = \frac{1 + \kappa}{\kappa}, \kappa\right] \backslash \backslash \text{FullSimplify}$ " is incomplete; more input is needed.

In[\*]:= Clear[A, M, α, τ]

## Check independent equals transformation

$$\text{In}[*]:= \text{Solve}\left[\frac{1 + (1 + m) \kappa}{\kappa} = \frac{1 + \kappa M}{\kappa M}, \kappa M\right]$$

$$\text{Out}[*]= \left\{ \left\{ \kappa M \rightarrow \frac{\kappa}{1 + m \kappa} \right\} \right\}$$

$$\text{In}[*]:= \text{Solve}\left[1 + \frac{1}{M^2} = \frac{1}{2} \left(1 + \frac{1}{\kappa}\right), \kappa\right]$$

$$\text{Out}[*]= \left\{ \right\}$$

$$2 \left( \frac{1}{2} + \frac{1}{M^2} \right) = \frac{1}{\kappa}$$

$$\kappa = \frac{M^2}{1 + M^2}$$

## Uncertainty at the Scale: Definition for Gen. Entropy

Since the generalized entropy must compute a solution that is equal to the generalized logarithm of the coupled stretched exponential distribution at the scale, it is illustrative to compute this value for a variety of generalized entropies. For simplicity the case of  $d=1$ ,  $\alpha=1$ , and  $\mu=0$  is computed. The entropy transformed to the domain of the distribution is  $\exp_{\kappa}^{-(1+\kappa)}[H]$ . This value must be set equal to  $f(x)$  and then one must solve for  $x$ , which requires taking  $\ln_{\kappa} f(x)^{-1}$ , reversing the coupled exponential. So the equation setting  $H$  equal to the argument of the density.

$$\exp_{\kappa}^{-(1+\kappa)}[H] = \exp_{\kappa}^{-(1+\kappa)}\left[\frac{x}{\sigma} + \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} + \kappa \frac{x}{\sigma} \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}}\right]$$

$$H = \frac{x}{\sigma} \left( 1 + \kappa \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} \right) + \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}}$$

$$\frac{x}{\sigma} = \frac{\sigma \left( H - \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} \right)}{\left( 1 + \kappa \ln_{\kappa} \sigma^{\frac{1}{1+\kappa}} \right)}$$

In[\*]:= entValue[H\_, σ\_, κ\_] :=

$$\sigma \frac{\left( H - \text{CoupledLogarithm}\left[\sigma^{\frac{1}{1+\kappa}}, \kappa, \theta\right] \right)}{1 + \kappa \text{CoupledLogarithm}\left[\sigma^{\frac{1}{1+\kappa}}, \kappa, \theta\right]} // \text{FullSimplify};$$

In[\*]:= entDensity[H\_, κ\_] :=

$$\text{CoupledExponential}[H, \kappa]^{-1} // \text{FullSimplify};$$

In[\*]:= HCoupled[σ\_, κ\_] := 1 + CoupledLogarithm[σ,  $\frac{\kappa}{1+\kappa}$ , θ];

In[\*]:= entValue[HCoupled[σ, κ], σ, κ]

Out[\*] =

$$\sigma$$

In[\*]:= entDensity[HCoupled[σ, κ], κ]

Out[\*] =

$$\frac{(1 + \kappa)^{-\frac{1+\kappa}{\kappa}}}{\sigma}$$

In[\*]:= HBGS[σ\_, κ\_] := 1 + Log[σ] + κ;

In[\*]:= entValue[HBGS[σ, κ], σ, κ] // FullSimplify

Out[\*] =

$$\frac{\sigma^{\frac{1}{1+\kappa}} \left( 1 + \kappa + \kappa^2 - \sigma^{\frac{\kappa}{1+\kappa}} + \kappa \text{Log}[\sigma] \right)}{\kappa}$$

In[\*]:= entDensity[HBGS[σ, κ], κ]

Out[\*] =

$$1 / \text{If}\left[1 + \kappa (1 + \kappa + \text{Log}[\sigma]) > 0, \right. \\ \left. \text{If}\left[\kappa \neq 0, (1 + \kappa (1 + \kappa + \text{Log}[\sigma]))^{\frac{1+\kappa}{\kappa}}, \text{Exp}[1 + \kappa + \text{Log}[\sigma]]\right], \text{If}\left[\frac{1 + 1 \kappa}{\kappa} > 0, 0, \infty\right]\right]$$

In[\*]:= HREnyi[σ\_, κ\_] := Log[σ] +  $\left(1 + \frac{1}{\kappa}\right) \text{Log}[1 + \kappa]$ ;

In[\*]:= entValue[HREnyi[σ, κ], σ, κ] // FullSimplify

Out[\*] =

$$\frac{\sigma^{\frac{1}{1+\kappa}} \left( 1 - \sigma^{\frac{\kappa}{1+\kappa}} + (1 + \kappa) \text{Log}[1 + \kappa] + \kappa \text{Log}[\sigma] \right)}{\kappa}$$

In[\*]:= **entDensity**[HRenyi[ $\sigma$ ,  $\kappa$ ],  $\kappa$ ]

Out[\*]=

$$\frac{1}{\text{If}\left[1 + \text{Log}[1 + \kappa] + \kappa \text{Log}[(1 + \kappa) \sigma] > 0, \text{If}\left[\kappa \neq 0, \left(1 + \kappa \left(1 + \frac{1}{\kappa}\right) \text{Log}[1 + \kappa] + \text{Log}[\sigma]\right)\right]^{\frac{1 + \kappa}{\kappa}}, \text{Exp}\left[\left(1 + \frac{1}{\kappa}\right) \text{Log}[1 + \kappa] + \text{Log}[\sigma]\right], \text{If}\left[\frac{1 + \kappa}{\kappa} > 0, 0, \infty\right]\right]}$$

In[\*]:= **HTsallis**[ $\sigma$ \_,  $\kappa$ \_] :=  $1 - \frac{1}{1 + \kappa} \text{CoupledLogarithm}\left[\sigma^{-1}, \frac{\kappa}{1 + \kappa}, 0\right]$ ;

In[\*]:= **entValue**[HTsallis[ $\sigma$ ,  $\kappa$ ],  $\sigma$ ,  $\kappa$ ] // FullSimplify

Out[\*]=

$$\frac{\frac{1}{\sigma^{1 + \kappa}} \left(2 + \kappa - 2 \text{Cosh}\left[\frac{\kappa \text{Log}[\sigma]}{1 + \kappa}\right]\right)}{\kappa}$$

In[\*]:= **entDensity**[HTsallis[ $\sigma$ ,  $\kappa$ ],  $\kappa$ ]

Out[\*]=

$$\frac{1}{\text{If}\left[(2 + \kappa) \sigma^{\frac{\kappa}{1 + \kappa}} > 1, \text{If}\left[\kappa \neq 0, \left(1 + \kappa \left(1 - \frac{1}{1 + \kappa} \text{If}\left[\frac{1}{\sigma} \geq 0, \text{If}\left[\frac{\kappa}{1 + \kappa} \neq 0, \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1 + \kappa) \left(1 + \frac{\kappa}{1 + \kappa}\right)} - 1}}{\frac{\kappa}{1 + \kappa}}, \text{Log}\left[\frac{1}{\sigma}\right]\right], \text{Undefined}\right]\right)^{\frac{1 + \kappa}{\kappa}}, \text{Exp}\left[1 - \frac{1}{1 + \kappa} \text{If}\left[\frac{1}{\sigma} \geq 0, \text{If}\left[\frac{\kappa}{1 + \kappa} \neq 0, \frac{\left(\frac{1}{\sigma}\right)^{\frac{\kappa}{(1 + \kappa) \left(1 + \frac{\kappa}{1 + \kappa}\right)} - 1}}{\frac{\kappa}{1 + \kappa}}, \text{Log}\left[\frac{1}{\sigma}\right]\right], \text{Undefined}\right]\right], \text{If}\left[\frac{1 + \kappa}{\kappa} > 0, 0, \infty\right]\right]}$$

In[\*]:= **HNormTsallis**[ $\sigma$ \_,  $\kappa$ \_] :=  $1 + (1 + \kappa) \text{CoupledLogarithm}\left[\sigma, \frac{\kappa}{1 + \kappa}, 0\right] + \kappa$ ;

In[\*]:= **entValue**[HNormTsallis[ $\sigma$ ,  $\kappa$ ],  $\sigma$ ,  $\kappa$ ]

Out[\*]=

$$\frac{\sigma^{-1 + \frac{2}{1 + \kappa}} \left((1 + \kappa)^2 - \kappa \sigma^{\frac{\kappa}{1 + \kappa}} - \sigma^{\frac{2\kappa}{1 + \kappa}}\right)}{\kappa}$$

```

In[*]:= entDensity[HNormTsallis[σ, κ], κ]
Out[*]:=

$$1 / \text{If} \left[ (1 + \kappa)^2 > \kappa \frac{\kappa}{\sigma^{1+\kappa}}, \text{If} \left[ \kappa \neq 0, \left( 1 + \kappa \left( 1 + \kappa + (1 + \kappa) \text{If} \left[ \frac{1}{\sigma} \geq 0, \text{If} \left[ \frac{\kappa}{1 + \kappa} \neq 0, \frac{\left( \frac{1}{\sigma} \right)^{\frac{\kappa}{(1+\kappa) \left( 1 + \frac{\kappa}{1+\kappa} \right)} - 1}}{\frac{\kappa}{1+\kappa}}, \text{Log} \left[ \frac{1}{\sigma} \right] \right], \text{Undefined} \right] \right)^{\frac{1+\kappa}{\kappa}}, \right. \right. \\ \left. \left. \text{Exp} \left[ 1 + \kappa + (1 + \kappa) \text{If} \left[ \frac{1}{\sigma} \geq 0, \text{If} \left[ \frac{\kappa}{1 + \kappa} \neq 0, \frac{\left( \frac{1}{\sigma} \right)^{\frac{\kappa}{(1+\kappa) \left( 1 + \frac{\kappa}{1+\kappa} \right)} - 1}}{\frac{\kappa}{1+\kappa}}, \text{Log} \left[ \frac{1}{\sigma} \right] \right], \text{Undefined} \right] \right] \right], \right. \\ \left. \text{If} \left[ \frac{1 + 1 \kappa}{\kappa} > 0, 0, \infty \right] \right]$$


In[*]:= Module[{EntData, σPlotBottom = 0.5, σPlotTop = 2,
  κValues = {1/2, 1, 2}},
  EntData = Table[
    Table[{entValue[entName[σPlot, κPlot], σPlot, κPlot],
      entDensity[entName[σPlot, κPlot], κPlot]},
      {σPlot, {σPlotBottom, σPlotTop}}, {κPlot, κValues}],
    {entName, {HCoupled, HBGS, HRenyi, HTsallis, HNormTsallis}}
  ];
  ResourceFunction["PlotGrid"][
    {{Plot[
      Table[PDF[CoupledExponentialDistribution[σPlotTop, κPlot], x],
        {κPlot, κValues}]
      , {x, 0, 7.5},
      PlotRange → {{0, 5}, {0, 0.55}},
      Frame → {{True, True}, {True, True}},
      FrameStyle → Directive[Bold, Medium],
      Epilog → {
        {Dashed, Gray, Line[{σPlotTop, 0}, {σPlotTop, 0.6}]},
        PointSize[Large],
        {Black, Point[EntData[[1, 2]]]},
        {Blue, Point[EntData[[2, 2]]]},
        {Brown, Point[EntData[[3, 2]]]},
        {Magenta, Point[EntData[[4, 2]]]},
        {Orange, Point[EntData[[5, 2]]]},
        Inset[Style["Required Solution",
          Medium], {2.8, 0.32}],
        Inset[Style["= Coupled Entropy",
          Medium], {3.0, 0.28}],
        Inset[Style["BGS", Blue, Medium], {3.0, 0.12}],

```



```

Inset[Style["Renyi", Brown, Medium], {2.5, 0.16}],
Inset[Style["Tsallis", Magenta, Medium], {1, 0.12}],
Inset[Style["Norm Tsallis", Orange, Medium], {4.0, 0.1}],
Inset[Style[" $\sigma$ ", Black, Medium], {1.9, 0.03}],
Inset[Style["Coupled Exponential", Black, Medium, Bold],
  Scaled[{0.77, 0.92}]],
Inset[Style[" $\kappa = 0.5, 1, 2, \sigma = 2$ ", Black, Medium, Bold],
  Scaled[{0.77, 0.84}]],
Inset[Style["Higher  $\kappa \rightarrow$  Lower  $f(x(H))$ ", Black, Medium, Bold],
  Scaled[{0.77, 0.76}]],
Inset[Style["a)", Black, Medium],
  Scaled[{0.03, 0.95}]]
}
]],
{Plot[
  Table[PDF[CoupledExponentialDistribution[ $\sigma$ PlotBottom,  $\kappa$ Plot], x],
    { $\kappa$ Plot,  $\kappa$ Values}]
  , {x, 0, 5},
  PlotRange  $\rightarrow$  {{0, 2.5}, {0, 1.05}},
  Frame  $\rightarrow$  {{True, True}, {True, True}},
  FrameStyle  $\rightarrow$  Directive[Bold, Medium],
  Epilog  $\rightarrow$  {
    {Dashed, Gray, Line[{ $\sigma$ PlotBottom, 0}, { $\sigma$ PlotBottom, 1.0}]},
    PointSize[Large],
    {Black, Point[EntData[[1, 1]]]},
    {Blue, Point[EntData[[2, 1]]]},
    {Brown, Point[EntData[[3, 1]]]},
    {Magenta, Point[EntData[[4, 1]]]},
    {Orange, Point[EntData[[5, 1]]]},
    Inset[Style["Required Solution",
      Medium], {0.9, 0.77}],
    Inset[Style["= Coupled Entropy",
      Medium], {0.92, 0.69}],
    Inset[Style["BGS", Blue, Medium], {1.3, 0.22}],
    Inset[Style["Renyi", Brown, Medium], {0.8, 0.5}],
    Inset[Style["Tsallis", Magenta, Medium], {0.68, 0.2}],
    Inset[Style["Norm Tsallis", Orange, Medium], {1.15, 0.35}],
    Inset[Style[" $\sigma$ ", Black, Medium], {0.55, 0.05}],
    Inset[Style["Coupled Exponential", Black, Medium, Bold],
      Scaled[{0.77, 0.92}]],
    Inset[Style[" $\kappa = 0.5, 1, 2, \sigma = 0.5$ ", Black, Medium, Bold],
      Scaled[{0.77, 0.84}]],
    Inset[Style["Higher  $\kappa \rightarrow$  Lower  $f(x(H))$ ", Black, Medium, Bold],
      Scaled[{0.77, 0.76}]],

```

```

Inset[Style["b"), Black, Medium],
  Scaled[{0.03, 0.95}]]
}
]]},
Spacings → 20,
FrameLabel → {"x", "Density, f(x)"},
PlotLabel → "Entropies Mapped to Density"
]]

```

OptionValue: Unknown option DefaultBaseStyle for Graphics.

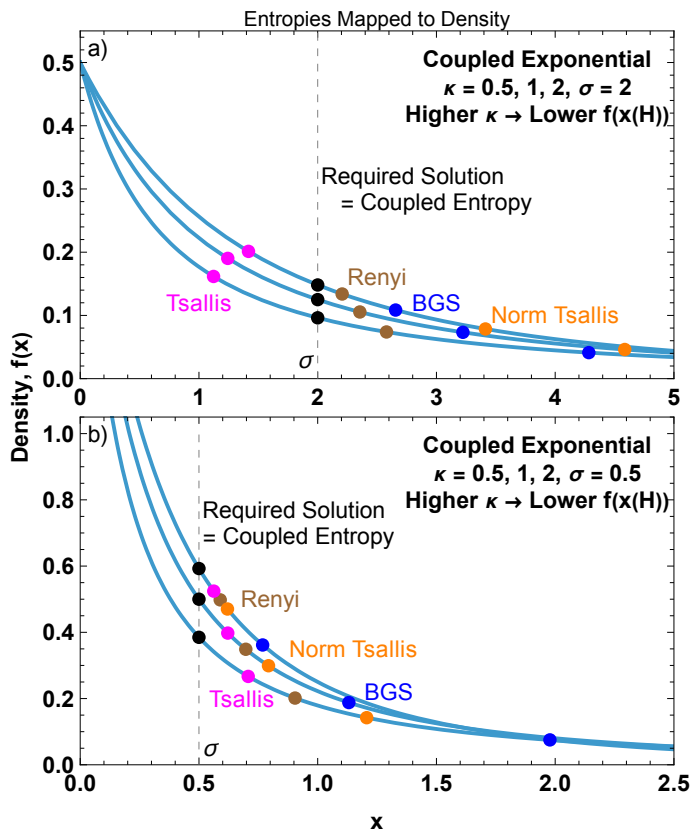
OptionValue: Unknown option DefaultBaseStyle for Graphics.

OptionValue: Unknown option DefaultBaseStyle for Graphics.

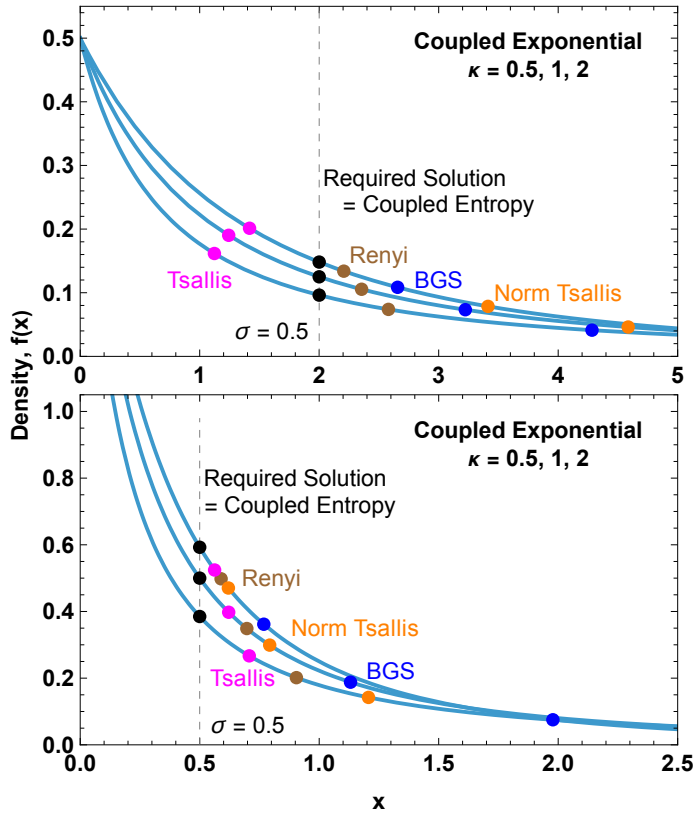
General: Further output of OptionValue::nofdef will be suppressed during this calculation.



Out[ ] =



Saved Plots



## Moments of Coupled Weibull distribution

Keeping  $1+\alpha \kappa$  term

$$\begin{aligned}
 In[*] &:= \int x^\alpha \frac{\alpha}{\sigma} \left(\frac{x}{\sigma}\right)^{\alpha-1} \text{Exp}\left[-(1+\alpha \kappa) \left(\frac{x}{\sigma}\right)^\alpha\right] dx // \text{FullSimplify} \\
 Out[*] &= -\frac{\sigma^\alpha \text{Gamma}\left[2, (1+\alpha \kappa) \left(\frac{x}{\sigma}\right)^\alpha\right]}{(1+\alpha \kappa)^2} \\
 In[*] &:= -\frac{\sigma^\alpha \text{Gamma}\left[2, (1+\alpha \kappa) \left(\frac{x}{\sigma}\right)^\alpha\right]}{(1+\alpha \kappa)^2} /. x \rightarrow 0 // \text{FullSimplify} \\
 Out[*] &= -\frac{\sigma^\alpha}{(1+\alpha \kappa)^2} \\
 In[*] &:= -\frac{\sigma^\alpha}{(1+\alpha \kappa)^2} /. \left\{\sigma^\alpha \rightarrow \frac{\sigma^\alpha}{1+\alpha \kappa}, \kappa \rightarrow \frac{\kappa}{1+\alpha \kappa}\right\} // \text{FullSimplify} \\
 Out[*] &= -\frac{(1+\alpha \kappa) \sigma^\alpha}{(1+2 \alpha \kappa)^2}
 \end{aligned}$$

Removing  $1+\alpha \kappa$  term

```

In[*]:= Integrate[x^alpha * (x/sigma)^(alpha-1) * Exp[-(x/sigma)^alpha] dx // FullSimplify
Out[*]= -sigma^alpha Gamma[2, (x/sigma)^alpha]

In[*]:= -sigma^alpha Gamma[2, (x/sigma)^alpha] /. x -> 0 // FullSimplify
Out[*]= -sigma^alpha

In[*]:= -sigma^alpha / (1 + alpha kappa) Gamma[2, (1 + alpha kappa) (x/sigma)^alpha] /. x -> 0 // FullSimplify
Out[*]= -sigma^alpha / (1 + alpha kappa)

In[*]:= Integrate[x^(1/kappa - 1) dx
Out[*]= -x^(1/kappa) kappa

In[*]:= Limit[-x^(1/kappa) kappa, x -> Infinity]
Limit: Warning: Assumptions that involve the limit variable are ignored.
Out[*]= 0

```

## Fluctuation Model

**Syntax:** Incomplete expression; more input is needed.

```

Out[*]= Integrate[Exp[-t/sigma] PDF[GammaDistribution[1/kappa, kappa/sigma], 1/t] dt

In[*]:= Integrate[c Exp[-c t] PDF[GammaDistribution[1/kappa, kappa beta], c] dc
Out[*]= 
  beta^(1-1/kappa) (1/beta + t kappa)^(-1/kappa)
  1 + t beta kappa
  {
    Integrate[c e^(-c t) (
      {
        c^(-1+1/kappa) e^(-c/beta kappa) (beta kappa)^(-1/kappa) / Gamma[1/kappa] c > 0
        0 True
      }
    ), {c, 0, Infinity},
    Assumptions -> d in Z && x in R && alpha in R && kappa in R && mu in R &&
      sigma in R && kappa > 0 && sigma > 0 && alpha > 0 && d > 0 && Re[t + 1/beta kappa] <= 0
  ]

```

$\text{Re}\left[t + \frac{1}{\beta \kappa}\right] > 0$   
 True

$$\begin{aligned}
& \frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta} + t \kappa\right)^{-1/\kappa}}{1+t \beta \kappa} \\
\text{In}[*]:= & \left\{ \begin{array}{l} \text{Integrate}\left[c e^{-c t} \left( \left\{ \begin{array}{ll} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c > 0 \\ 0 & \text{True} \end{array} \right\}, \{c, 0, \infty\}, \right. \\ \left. \text{Assumptions} \rightarrow d \in \mathbb{Z} \ \&\& \ x \in \mathbb{R} \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ \kappa \in \mathbb{R} \ \&\& \ \mu \in \mathbb{R} \ \&\& \right. \\ \left. \sigma \in \mathbb{R} \ \&\& \ \kappa > 0 \ \&\& \ \sigma > 0 \ \&\& \ \alpha > 0 \ \&\& \ d > 0 \ \&\& \ \text{Re}\left[t + \frac{1}{\beta \kappa}\right] \leq 0 \right] \end{array} \right. \\
& \text{True}
\end{aligned}$$

/ FullSimplify

$$\begin{aligned}
& \frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta} + t \kappa\right)^{-1/\kappa}}{1+t \beta \kappa} \\
\text{Out}[*]= & \left\{ \begin{array}{l} \text{Integrate}\left[c e^{-c t} \left( \left\{ \begin{array}{ll} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c > 0 \\ 0 & \text{True} \end{array} \right\}, \{c, 0, \infty\}, \right. \\ \left. \text{Assumptions} \rightarrow d \in \mathbb{Z} \ \&\& \ x \in \mathbb{R} \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ \kappa \in \mathbb{R} \ \&\& \ \mu \in \mathbb{R} \ \&\& \right. \\ \left. \sigma \in \mathbb{R} \ \&\& \ \kappa > 0 \ \&\& \ \sigma > 0 \ \&\& \ \alpha > 0 \ \&\& \ d > 0 \ \&\& \ \text{Re}\left[t + \frac{1}{\beta \kappa}\right] \leq 0 \right] \end{array} \right. \\
& \text{True}
\end{aligned}$$

$$\beta^1 (1+t \beta \kappa)^{-1/\kappa-1}$$

$$\begin{aligned}
& \int_0^\infty c \text{Exp}[-c t^2] \text{PDF}\left[\text{GammaDistribution}\left[\frac{1}{\kappa}, \kappa \beta\right], c\right] dc \\
\text{In}[*]:= & \\
& \frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta} + t^2 \kappa\right)^{-1/\kappa}}{1+t^2 \beta \kappa} \\
\text{Out}[*]= & \left\{ \begin{array}{l} \text{Integrate}\left[c e^{-c t^2} \left( \left\{ \begin{array}{ll} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} & c > 0 \\ 0 & \text{True} \end{array} \right\}, \{c, 0, \infty\}, \right. \\ \left. \text{Assumptions} \rightarrow d \in \mathbb{Z} \ \&\& \ x \in \mathbb{R} \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ \kappa \in \mathbb{R} \ \&\& \ \mu \in \mathbb{R} \ \&\& \right. \\ \left. \sigma \in \mathbb{R} \ \&\& \ \kappa > 0 \ \&\& \ \sigma > 0 \ \&\& \ \alpha > 0 \ \&\& \ d > 0 \ \&\& \ \text{Re}\left[t^2 + \frac{1}{\beta \kappa}\right] \leq 0 \right] \end{array} \right. \\
& \text{True}
\end{aligned}$$

$$In[*]:= \int_0^{\infty} c \text{Exp}[-c^2 t^2] \text{PDF}[\text{GammaDistribution}\left[\frac{1}{\kappa}, \kappa \beta^2\right], c] \, dc$$

$$Out[*]= -\frac{1}{4 \text{Gamma}\left[\frac{1}{\kappa}\right]} (t^2)^{-\frac{3}{2}-\frac{1}{2\kappa}} (\beta^2)^{-1-\frac{1}{\kappa}} \kappa^{-2-\frac{1}{\kappa}} \left( \sqrt{t^2} \text{Gamma}\left[\frac{1}{2\kappa}\right] \text{Hypergeometric1F1}\left[1+\frac{1}{2\kappa}, \frac{3}{2}, \frac{1}{4 t^2 \beta^4 \kappa^2}\right] - 2 t^2 \beta^2 \kappa^2 \text{Gamma}\left[\frac{1+\kappa}{2\kappa}\right] \text{Hypergeometric1F1}\left[\frac{1+\kappa}{2\kappa}, \frac{1}{2}, \frac{1}{4 t^2 \beta^4 \kappa^2}\right] \right)$$

$$\left\{ \begin{array}{l} \text{Integrate}\left[ \right. \quad \text{True} \\ \left. c e^{-c^2 t^2} \left( \left\{ \begin{array}{l} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta^2 \kappa}} (\beta^2 \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} \quad c > 0 \\ 0 \quad \text{True} \end{array} \right\}, \{c, 0, \infty\}, \right. \end{array} \right.$$

$$\text{Assumptions} \rightarrow (d \in \mathbb{Z} \ \&\& \ x \in \mathbb{R} \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ \kappa \in \mathbb{R} \ \&\& \ \mu \in \mathbb{R} \ \&\& \ \sigma \in \mathbb{R} \ \&\& \ \text{Re}[t^2] \leq 0 \ \&\& \ d > 0 \ \&\& \ \alpha > 0 \ \&\& \ \kappa > 0 \ \&\& \ \sigma > 0 \ \&\& \ \text{Re}[\beta^2] \leq 0) \ ||$$

$$(d \in \mathbb{Z} \ \&\& \ x \in \mathbb{R} \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ \kappa \in \mathbb{R} \ \&\& \ \mu \in \mathbb{R} \ \&\& \ \sigma \in \mathbb{R} \ \&\& \ \text{Re}[t^2] < 0 \ \&\& \ d > 0 \ \&\& \ \alpha > 0 \ \&\& \ \kappa > 0 \ \&\& \ \sigma > 0)$$

$$In[*]:= \int_{-\infty}^0 \frac{1}{c} \text{Exp}\left[\frac{-1}{c} t^2\right] \text{PDF}[\text{GammaDistribution}\left[\frac{1}{\kappa}, \frac{\kappa}{\sigma^2}\right], \frac{1}{c}] \, d\left(\frac{1}{c}\right)$$

$$\cdots \text{Integrate: Invalid integration variable or limit (s) in } \left\{\frac{1}{c}, \infty, 0\right\}. \text{ ?}$$

$$Out[*]= \int_{-\infty}^0 \frac{e^{-\frac{t^2}{c}} \left( \left\{ \begin{array}{l} \frac{\left(\frac{1}{c}\right)^{-1+\frac{1}{\kappa}} e^{-\frac{\kappa}{c \kappa}} \left(\frac{\kappa}{\sigma^2}\right)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} \quad \frac{1}{c} > 0 \\ 0 \quad \text{True} \end{array} \right\}}{c} \right) d\frac{1}{c}}$$

$$In[*]:= \int_0^{\infty} c \text{Exp}[-c t^{\alpha}] \text{PDF}[\text{GammaDistribution}\left[\frac{1}{\kappa}, \kappa \beta\right], c] \, dc$$

$$Out[*]= \frac{\beta^{1-\frac{1}{\kappa}} \left(\frac{1}{\beta} + t^{\alpha} \kappa\right)^{-1/\kappa}}{1+t^{\alpha} \beta \kappa} \left\{ \begin{array}{l} \text{Integrate}\left[c e^{-c t^{\alpha}} \left( \left\{ \begin{array}{l} \frac{c^{-1+\frac{1}{\kappa}} e^{-\frac{c}{\beta \kappa}} (\beta \kappa)^{-1/\kappa}}{\text{Gamma}\left[\frac{1}{\kappa}\right]} \quad c > 0 \\ 0 \quad \text{True} \end{array} \right\}, \{c, 0, \infty\}, \right. \right. \\ \text{Assumptions} \rightarrow d \in \mathbb{Z} \ \&\& \ x \in \mathbb{R} \ \&\& \ \alpha \in \mathbb{R} \ \&\& \ \kappa \in \mathbb{R} \ \&\& \ \mu \in \mathbb{R} \ \&\& \ \sigma \in \mathbb{R} \ \&\& \ \kappa > 0 \ \&\& \ \sigma > 0 \ \&\& \ \alpha > 0 \ \&\& \ d > 0 \ \&\& \ \text{Re}\left[t^{\alpha} + \frac{1}{\beta \kappa}\right] \leq 0 \end{array} \right.$$

$$\text{Re}\left[t^{\alpha} + \frac{1}{\beta \kappa}\right] > 0$$

$$\text{True}$$

$$In[*]:= \int_0^{\infty} \text{Exp}[-c t^{\alpha}] \text{PDF}[\text{GammaDistribution}\left[\frac{1}{\kappa} + 1, \frac{\kappa}{1+\kappa} \beta^{\alpha}\right], c] \, dc$$

$$Out[*]= \text{\$Aborted}$$

## Complexity Measurement

Determine the Taylor Series expansion of the result

$$\begin{aligned} \text{In}[*]:= & \text{EntRatioExp} = \frac{1 + \text{Log}[\sigma (1 - \kappa)]}{1 + \text{Log}[\sigma]} \\ \text{Out}[*]= & \frac{1 + \text{Log}[(1 - \kappa) \sigma]}{1 + \text{Log}[\sigma]} \end{aligned}$$

$$\begin{aligned} \text{In}[*]:= & \text{Series}[\text{EntRatioExp} (1 - \text{EntRatioExp}), \{\kappa, 0, 3\}, \\ & \text{Assumptions} \rightarrow (0 < \kappa < 1)] \\ \text{Out}[*]= & \frac{\kappa}{1 + \text{Log}[\sigma]} + \frac{(-1 + \text{Log}[\sigma]) \kappa^2}{2 (1 + \text{Log}[\sigma])^2} + \frac{(-2 + \text{Log}[\sigma]) \kappa^3}{3 (1 + \text{Log}[\sigma])^2} + O[\kappa]^4 \end{aligned}$$

$$\begin{aligned} \text{In}[*]:= & \text{Series}\left[-\left(1 + \frac{\text{Log}[1 - \kappa]}{\text{Log}[\sigma]}\right) \left(\frac{\text{Log}[1 - \kappa]}{\text{Log}[\sigma]}\right), \{\kappa, 0, 3\}, \right. \\ & \left. \text{Assumptions} \rightarrow \left(0 < \kappa < \frac{1}{2}\right)\right] \\ \text{Out}[*]= & \frac{\kappa}{\text{Log}[\sigma]} + \frac{(-2 + \text{Log}[\sigma]) \kappa^2}{2 \text{Log}[\sigma]^2} + \frac{(-3 + \text{Log}[\sigma]) \kappa^3}{3 \text{Log}[\sigma]^2} + O[\kappa]^4 \end{aligned}$$

First order approximation of the entropy of the coupled Gaussian with  $\sigma=1$

$$\begin{aligned} \text{In}[*]:= & \text{Series}\left[\frac{1 + \kappa}{2 \kappa} \left(\text{PolyGamma}\left[\frac{1 + \kappa}{2 \kappa}\right] - \text{PolyGamma}\left[\frac{1}{2 \kappa}\right]\right) + \right. \\ & \left. \text{Log}\left[\frac{1}{\sqrt{\kappa}} \text{Beta}\left[\frac{1}{2 \kappa}, \frac{1}{2}\right]\right], \{\kappa, 0, 3\}\right] // \text{FullSimplify} \\ \text{Out}[*]= & \frac{1}{2} (1 + \text{Log}[2] + \text{Log}[\pi]) + \kappa + \frac{\kappa^2}{4} - \frac{\kappa^3}{6} + O[\kappa]^{7/2} \end{aligned}$$

Taylor Series of Complexity of Coupled Gaussian

$$\begin{aligned} \text{In}[*]:= & \text{EntRatioGauss} = \frac{\frac{1}{2} + \frac{1}{2} \text{Log}[2 \pi \sigma \sqrt{1 - 2 \kappa}]}{\frac{1}{2} + \frac{1}{2} \text{Log}[2 \pi \sigma]} \\ \text{Out}[*]= & \frac{\frac{1}{2} + \frac{1}{2} \text{Log}[2 \pi \sqrt{1 - 2 \kappa} \sigma]}{\frac{1}{2} + \frac{1}{2} \text{Log}[2 \pi \sigma]} \\ \text{In}[*]:= & \text{Series}[\text{EntRatio} (1 - \text{EntRatioGauss}), \{\kappa, 0, 3\}] \\ \text{Out}[*]= & \frac{\kappa}{1 + \text{Log}[2 \pi \sigma]} + \frac{\text{Log}[2 \pi \sigma] \kappa^2}{(1 + \text{Log}[2 \pi \sigma])^2} + \frac{2 (-1 + 2 \text{Log}[2 \pi \sigma]) \kappa^3}{3 (1 + \text{Log}[2 \pi \sigma])^2} + O[\kappa]^4 \end{aligned}$$

## Relation to c,d-entropy

### Asymptotic Classification

J. M. Amigó, S. G. Balogh, and S. Hernández, “A Brief Review of Generalized Entropies,” *Entropy*, vol. 20, no. 11, p. 813, Nov. 2018, doi: 10.3390/e20110813.

#### 4. Hanel–Thurner Exponents

All generalized entropies  $F_{G,g}$  group in classes labeled by two exponents  $(c, d)$  introduced by Hanel and Thurner [16], which are determined by the limits

$$\lim_{W \rightarrow \infty} \frac{F_{G,g}(p_1, \dots, p_{\lambda W})}{F_{G,g}(p_1, \dots, p_W)} = \lambda^{1-c} \quad (25)$$

( $W$  being as before the cardinality of the probability distribution or the total number of microstates in the system,  $\lambda > 1$ ) and

$$\lim_{W \rightarrow \infty} \frac{F_{G,g}(p_1, \dots, p_{W^{1+a}})}{F_{G,g}(p_1, \dots, p_W)} W^{a(c-1)} = (1+a)^d \quad (26)$$

( $a > 0$ ). Note that the limit in Equation (26) does not depend actually on  $c$ . The limits in Equations (25) and (26) can be computed via the asymptotic equipartition property [26]. Thus,

$$F_{G,g}(p_1, \dots, p_{\lambda W}) \approx G\left(\lambda W g\left(\frac{1}{\lambda W}\right)\right)$$

and

$$F_{G,g}(p_1, \dots, p_{W^{1+a}}) \approx G\left(W^{1+a} g\left(\frac{1}{W^{1+a}}\right)\right)$$



(E1) For the BGS entropy,  $g(x) = -x \ln x$  (see Equation (11)), so

$$\frac{g(zx)}{g(x)} = \frac{zx \ln(zx)}{x \ln x} = \frac{z \ln z + z \ln x}{\ln x} \rightarrow z$$

as  $x \rightarrow 0+$ . Therefore,  $c = 1$ . Furthermore,

$$\frac{g(x^{1+a})}{x^a g(x)} = \frac{x^{1+a} \ln x^{1+a}}{x^a \ln x} = \frac{(1+a) \ln x}{\ln x} = 1+a$$

for all  $x > 0$ , so  $d = 1$ .

(E2) For the Tsallis entropy, see Equation (12),

$$g(x) = \begin{cases} \frac{1}{1-q} x^q + \mathcal{O}(x) & \text{if } 0 < q < 1, \\ -\frac{1}{1-q} x + \mathcal{O}(x) & \text{if } q > 1. \end{cases}$$

It follows readily that  $(c, d) = (q, 0)$  if  $0 < q < 1$ , and  $(c, d) = (1, 0)$  if  $q > 1$ . Hence, although  $\lim_{q \rightarrow 1} T_q = S_{BGS}$ , there is no parallel convergence concerning the HT exponents.

(E3) For the Rényi entropy,  $g(x) = x^q$  and  $G(u) = \frac{1}{1-q} e^u$  (see Equation (15)), so

$$\frac{G\left(\frac{1}{x} g\left(\frac{x}{\lambda}\right)\right)}{G\left(\frac{1}{x} g(x)\right)} = \frac{\ln\left(\frac{1}{x} \left(\frac{x}{\lambda}\right)^q\right)}{\ln\left(\frac{1}{x} x^q\right)} = \frac{\ln x^{q-1} - \ln \lambda^{q-1}}{\ln x^{q-1}} \rightarrow 1$$

For Coupled Entropy, consider that for  $p_i = \frac{1}{W}$ , the independent equals distribution also equals  $p_i$ . Then

$$g(x) = x \ln_{\alpha \kappa} x^{-\frac{1}{1+\kappa}} = \frac{x}{\alpha \kappa} x^{-\alpha \frac{\kappa}{1+\kappa}} + \mathcal{O}(x) = \frac{1}{\alpha \kappa} x^{1-\alpha \frac{\kappa}{1+\kappa}} + \mathcal{O}(x) = \frac{1}{\alpha \kappa} x^{2-q} + \mathcal{O}(x)$$

and  $G(u) = u^{\frac{1}{\gamma}}$ .

$$\ln[*] := g[r_-, \kappa_-, \alpha_-, d_-] := \frac{1}{\alpha \kappa} r^{1-\frac{\alpha \kappa}{1+d \kappa}};$$

$$G[u_-, \gamma_-] := u^{\frac{1}{\gamma}};$$

$$\ln[*] := \text{Limit}\left[\frac{G\left[\lambda W g\left[\frac{1}{\lambda W}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]}, W \rightarrow \infty\right]$$

Out[\*] =

$$\left(\left(\frac{1}{\lambda}\right)^{-\frac{\alpha \kappa}{1+d \kappa}}\right)^{\frac{1}{\gamma}} \text{ if } \text{condition} \oplus$$

$$\left(\lambda^{\frac{\alpha \kappa / \gamma}{1+d \kappa}}\right) = \lambda^{1-c}$$

$$c = 1 - \frac{\alpha \kappa / \gamma}{1+d \kappa}$$

$$\ln[\ast] := \text{Limit}\left[\frac{G\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a \frac{\alpha \kappa / \gamma}{1+d \kappa}}, W \rightarrow \infty\right]$$

Out[ ] =

$$1 \text{ if } \frac{1}{\gamma} \in \mathbb{R} \ \&\& \ a > -1 \ \&\& \ d \kappa > -1 \ \&\& \ \alpha \kappa > 0$$

Thus for  $(1+a)^d = 1, d = 0$ .

$$\ln[\ast] := \frac{G\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a \frac{\alpha \kappa / \gamma}{1+d \kappa}} // \text{FullSimplify}$$

Out[ ] =

$$\left(\left(\frac{1}{W}\right)^{-\frac{\alpha \kappa}{1+d \kappa}}\right)^{-1/\gamma} W^{-\frac{a \alpha \kappa}{\gamma+d \gamma \kappa}} \left(\left(W^{-1-a}\right)^{-\frac{\alpha \kappa}{1+d \kappa}}\right)^{\frac{1}{\gamma}}$$

$$\left(W^{-\frac{\alpha \kappa}{(1+d \kappa) \gamma}}\right) W^{-\frac{a \alpha \kappa}{\gamma+d \gamma \kappa}} W^{\frac{(1+a) \alpha \kappa}{(1+d \kappa) \gamma}}$$

$$\ln[\ast] := -\frac{\alpha \kappa}{(1+d \kappa) \gamma} - \frac{a \alpha \kappa}{\gamma+d \gamma \kappa} + \frac{(1+a) \alpha \kappa}{(1+d \kappa) \gamma} // \text{FullSimplify}$$

Out[ ] =

$$0$$

$$\ln[\ast] := \text{Limit}\left[\frac{G\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a (1-c)}, W \rightarrow \infty\right]$$

Out[ ] =

$$\infty \text{ if } \text{condition} \rightarrow$$

$$\ln[\ast] := \text{Limit}\left[\frac{G\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a (1-1)}, W \rightarrow \infty\right]$$

Out[ ] =

$$\infty \text{ if } a > 0$$

$$\ln[\ast] := \text{Limit}\left[\frac{G\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a (1-0)}, W \rightarrow \infty\right]$$

Out[ ] =

$$\infty \text{ if } a > -1 \ \&\& \ a \neq 0 \ \&\& \ a \gamma (1+d \kappa) < a \alpha \kappa$$

What if the full form of the generalized logarithm is used?

For Coupled Entropy, consider that for  $p_i = \frac{1}{W}$ , the independent equals distribution also equals  $p_i$ . Then

$$g(x) = x \ln_{\alpha \kappa} x^{-\frac{1}{1+\kappa}} = \frac{x}{\alpha \kappa} \left( x^{-\alpha \frac{\kappa}{1+\kappa}} - 1 \right)$$

$$\ln[\ast] := g[r_, \kappa_, \alpha_, d_] := \frac{r}{\alpha \kappa} \left( r^{-\frac{\alpha \kappa}{1+d \kappa}} - 1 \right);$$

$$G[u_, \gamma_] := u^{\frac{1}{\gamma}};$$

$$In[*]:= \text{Limit}\left[\frac{G\left[\lambda W g\left[\frac{1}{\lambda W}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} // \text{FullSimplify}, W \rightarrow \infty\right]$$

$$Out[*]= \left(\left(\frac{1}{\lambda}\right)^{-\frac{\alpha \kappa}{1+d \kappa}}\right)^{\frac{1}{\gamma}}$$

$$In[*]:= \text{Limit}\left[\frac{G\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a \frac{\alpha \kappa / \gamma}{1+d \kappa}} // \text{FullSimplify}, W \rightarrow \infty\right]$$

$$Out[*]= 1 \text{ if } a > 1$$

$$In[*]:= \frac{G\left[W^{1+a} g\left[\frac{1}{W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[W g\left[\frac{1}{W}, \kappa, \alpha, d\right], \gamma\right]} W^{-a \frac{\alpha \kappa / \gamma}{1+d \kappa}} // \text{FullSimplify}$$

$$Out[*]= \left(-1 + \left(\frac{1}{W}\right)^{-\frac{\alpha \kappa}{1+d \kappa}}\right)^{-1/\gamma} W^{-\frac{a \alpha \kappa}{\gamma+d \gamma \kappa}} \left(-1 + \left(W^{-1-a}\right)^{-\frac{\alpha \kappa}{1+d \kappa}}\right)^{\frac{1}{\gamma}}$$

With  $W \rightarrow \infty$  only the terms with  $W$  are significant and all the terms contain  $\frac{\alpha \kappa}{\gamma(1+d \kappa)}$ . Thus exponent multiplies this by  $(1 - a - 1 - a) = 0$ , which is why the expression converges to 1. But if  $\kappa \rightarrow 0$  first, then the  $-1$  term is of significance in the terms converging to  $\log W^a$ . Set  $r = \frac{\alpha \kappa}{(1+d \kappa)}$ , and factor out the  $1+a$  from the coupled logarithm, then the ratio is  $\frac{(1+a) \ln_{-r}(1+a) W}{\ln_{-r} W}$ . The question is under what circumstance does this ratio converge to  $1+a$ ? Certainly when  $r \rightarrow 0$  first. What about when  $r \rightarrow 1$ , which is its other limit?

$$\text{Then } \lim_{W \rightarrow \infty} \frac{(1+a)(W^{-(1+a)}-1)}{(1+a)(W^{-1}-1)} = 1.$$

Try using the form  $e^{W^{1+a}}$  and see what the scaling is

$$In[*]:= \text{Limit}\left[\frac{G\left[\text{Exp}\left[W^{1+a}\right] g\left[\frac{1}{\text{Exp}\left[W^{1+a}\right]}, \kappa, \alpha, d\right], \gamma\right]}{G\left[\text{Exp}\left[W\right] g\left[\frac{1}{\text{Exp}\left[W\right]}, \kappa, \alpha, d\right], \gamma\right]} \text{Exp}\left[W^{-a \frac{\alpha \kappa / \gamma}{1+d \kappa}}\right], W \rightarrow \infty\right]$$

$$Out[*]= \lim_{W \rightarrow \infty} \frac{e^{W^{\frac{a \alpha \kappa}{\gamma(1+d \kappa)}}} G\left[e^{W^{1+a}} g\left[e^{-W^{1+a}}, \kappa, \alpha, d\right], \gamma\right]}{G\left[e^W g\left[e^{-W}, \kappa, \alpha, d\right], \gamma\right]}$$

$$In[*]:= \text{Limit}\left[\frac{G\left[\text{Exp}\left[W^a\right] g\left[\frac{1}{\text{Exp}\left[W^a\right]}, \kappa, \alpha, d\right], \gamma\right]}{G\left[\text{Exp}\left[W\right] g\left[\frac{1}{\text{Exp}\left[W\right]}, \kappa, \alpha, d\right], \gamma\right]}, W \rightarrow \infty\right]$$

$$Out[*]= \lim_{W \rightarrow \infty} \frac{G\left[e^{W^a} g\left[e^{-W^a}, \kappa, \alpha, d\right], \gamma\right]}{G\left[e^W g\left[e^{-W}, \kappa, \alpha, d\right], \gamma\right]}$$

## Taylor series of coupled exponential function

```
In[*]:= Series[CoupledExponential[x, κ, 1], {x, x0, 4}]
Out[*]=
```

$$\begin{aligned} & \infty \quad (x \in \mathbb{R} \ \&\& \ -1 \leq \kappa < 0 \ \&\& \ x_0 \ \kappa < -1) \ || \\ & \quad (x_0 \in \mathbb{R} \ \&\& \ \kappa \neq 0 \ \&\& \ x_0 \ \kappa == -1 \ \&\& \\ & \quad \quad x \ \kappa \leq -1 \ \&\& \ -1 \leq \kappa < 0) \\ & \quad ( (x - x_0) \ \kappa )^{\frac{1}{\kappa}} \left( \kappa (x - x_0) + O[x - x_0]^5 \right) \quad x_0 \in \mathbb{R} \ \&\& \ \kappa \neq 0 \ \&\& \ x_0 \ \kappa == -1 \ \&\& \\ & \quad \quad ( (\kappa > 0 \ \&\& \ x \ \kappa > -1) \ || \ (\kappa < 0 \ \&\& \ x \ \kappa > -1) ) \\ & \quad \left[ \begin{aligned} & e^{x_0} + e^{x_0} (x - x_0) + \frac{1}{2} e^{x_0} (x - x_0)^2 + \\ & \quad \frac{1}{6} e^{x_0} (x - x_0)^3 + \frac{1}{24} e^{x_0} (x - x_0)^4 + O[x - x_0]^5 \\ & (1 + x_0 \ \kappa)^{1+\frac{1}{\kappa}} + (1 + \kappa) (1 + x_0 \ \kappa)^{\frac{1}{\kappa}} (x - x_0) + \\ & \quad \frac{1}{2} (1 + \kappa) (1 + x_0 \ \kappa)^{-1+\frac{1}{\kappa}} (x - x_0)^2 - \\ & \quad \frac{1}{6} \left( (-1 + \kappa) (1 + \kappa) (1 + x_0 \ \kappa)^{-2+\frac{1}{\kappa}} \right) (x - x_0)^3 + \\ & \quad \frac{1}{24} (1 + \kappa) (1 + x_0 \ \kappa)^{-3+\frac{1}{\kappa}} \\ & \quad (1 - 3 \ \kappa + 2 \ \kappa^2) (x - x_0)^4 + O[x - x_0]^5 \end{aligned} \right. \quad x \in \mathbb{R} \ \&\& \\ & \quad \quad ( (\kappa > 0 \ \&\& \ x_0 \ \kappa > -1) \ || \ (\kappa < 0 \ \&\& \ x_0 \ \kappa > -1) ) \\ & 0 \quad \quad \quad \text{True} \end{aligned}$$

## Taylor series of cdFunction

Starting with the 0th branch which corresponds to d>0

```
In[*]:= cdFunction[x_, c_, d_, r_] :=
  Exp[
$$\frac{-d}{1-c} \left( \text{ProductLog}[0, B[c, r] \left( 1 - \frac{x}{r} \right)^{\frac{1}{d}} \right. \right. \\ \left. \left. - \text{ProductLog}[0, B[c, r]] \right) \right] /; d > 0;$$

In[*]:= B[c_, r_] := 
$$\frac{(1-c) r}{1 - (1-c) r} \text{Exp} \left[ \frac{(1-c) r}{1 - (1-c) r} \right];$$

In[*]:= Series[ProductLog[0, z], {z, z0, 4}]
Out[*]=
```

$$\begin{aligned} & \text{ProductLog}[z_0] + \frac{\text{ProductLog}[z_0] (z - z_0)}{z_0 (1 + \text{ProductLog}[z_0])} - \\ & \quad \frac{(\text{ProductLog}[z_0]^2 (2 + \text{ProductLog}[z_0])) (z - z_0)^2}{2 (z_0^2 (1 + \text{ProductLog}[z_0])^3)} + \\ & \quad \frac{\text{ProductLog}[z_0]^3 (9 + 8 \text{ProductLog}[z_0] + 2 \text{ProductLog}[z_0]^2) (z - z_0)^3}{6 z_0^3 (1 + \text{ProductLog}[z_0])^5} + \\ & \quad \frac{(\text{ProductLog}[z_0]^4 (-64 - 79 \text{ProductLog}[z_0] - 36 \text{ProductLog}[z_0]^2 - 6 \text{ProductLog}[z_0]^3) \\ & \quad (z - z_0)^4)}{(24 z_0^4 (1 + \text{ProductLog}[z_0])^7)} + O[z - z_0]^5 \end{aligned}$$

In[\*]:= Series[cdFuntion[x, c, d, r], {x, x0, 4}]

Out[\*]=

$$\begin{aligned}
 & e^{\frac{d \left( -\text{ProductLog}\left[-\frac{r-c r}{1+(-1+c) r}\right] + \text{ProductLog}\left[-\frac{r-c r}{1+(-1+c) r} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}\right] \right)}{-1+c}} - \\
 & \left( e^{\frac{d \left( -\text{ProductLog}\left[-\frac{r-c r}{1+(-1+c) r}\right] + \text{ProductLog}\left[-\frac{r-c r}{1+(-1+c) r} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}\right] \right)}{-1+c}} \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] \right. \\
 & \left. (x-x0) \right) / \left( (-1+c) (r-x0) \left( 1 + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] \right) \right) - \\
 & \left( e^{\frac{d \left( -\text{ProductLog}\left[-\frac{r-c r}{1+(-1+c) r}\right] + \text{ProductLog}\left[-\frac{r-c r}{1+(-1+c) r} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}\right] \right)}{-1+c}} \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] \right. \\
 & \left( 1-c-d+c d-3 d \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] + 2 c d \text{ProductLog}\left[ \right. \right. \\
 & \left. \left. -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] - 2 d \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2 + \right. \\
 & \left. \left. c d \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2 \right) \right) (x-x0)^2 / \\
 & \left( 2 \left( (-1+c)^2 d (r-x0)^2 \left( 1 + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] \right)^3 \right) \right) -
 \end{aligned}$$

$$\begin{aligned}
& \left( \frac{d \left[ -\text{ProductLog} \left[ -\frac{\frac{r-c r}{1+(-1+c) r}}{1+(-1+c) r} \right] + \text{ProductLog} \left[ -\frac{\frac{r-c r}{1+(-1+c) r}}{1+(-1+c) r} \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}} \right] \right]}{e^{-1+c}} \right) \\
& \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \left( 1 - 2 c + c^2 - 3 d + 6 c d - 3 c^2 d + \right. \\
& 2 d^2 - 4 c d^2 + 2 c^2 d^2 - 2 \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + \\
& 4 c \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - 2 c^2 \text{ProductLog} \left[ \right. \\
& \left. -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - 9 d \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + \\
& 15 c d \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - 6 c^2 d \text{ProductLog} \left[ \right. \\
& \left. -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + 11 d^2 \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - \\
& 19 c d^2 \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] + 8 c^2 d^2 \text{ProductLog} \left[ \right. \\
& \left. -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] - 6 d \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \\
& 9 c d \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 - \\
& 3 c^2 d \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \\
& 22 d^2 \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 - \\
& 33 c d^2 \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 + \\
& 12 c^2 d^2 \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^2 +
\end{aligned}$$

$$\begin{aligned}
& 19 d^2 \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^3 - \\
& 25 c d^2 \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^3 + \\
& 8 c^2 d^2 \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^3 + \\
& 6 d^2 \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^4 - \\
& 7 c d^2 \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^4 + \\
& 2 c^2 d^2 \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^4 \right) (x-x0)^3 \Bigg/ \\
& \left( 6 \left( (-1+c)^3 d^2 (r-x0)^3 \left( 1 + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] \right)^5 \right) \right) + \\
& \left( \frac{d \left( -\text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] + \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] \right)}{e^{-1+c}} \right) \\
& \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] \\
& \left( -(-1+c)^3 (-1+6 d-11 d^2+6 d^3) - (-1+c)^2 (-1+d) \right. \\
& \left. (8+15 d-47 d^2+4 c (-2-2 d+9 d^2)) \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right] - \right. \\
& \left. (-1+c) (-1+d) (6+25 d+151 d^2+6 c^2 (1+4 d+15 d^2) - c (12+49 d+235 d^2)) \right. \\
& \left. \text{ProductLog}\left[-\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left(1-\frac{x0}{r}\right)^{\frac{1}{d}}}{1+(-1+c) r}\right]^2 + \right. \\
& \left. d (c (52+252 d-604 d^2) + c^3 (12+44 d-120 d^2) + 5 (-4-22 d+51 d^2) + \right.
\end{aligned}$$

$$\begin{aligned}
& 2 c^2 \left( -22 - 93 d + 235 d^2 \right) \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^3 + \\
& d^2 \left( -35 + c (75 - 526 d) + c^3 (11 - 90 d) + 239 d + c^2 (-51 + 380 d) \right) \\
& \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^4 + \\
& (118 - 242 c + 163 c^2 - 36 c^3) d^3 \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^5 + \\
& (24 - 46 c + 29 c^2 - 6 c^3) d^3 \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right]^6 \left( x - x0 \right)^4 \Bigg/ \\
& \left( 24 (-1+c)^4 d^3 (r - x0)^4 \left( 1 + \text{ProductLog} \left[ -\frac{(-1+c) e^{\frac{r-c r}{1+(-1+c) r}} r \left( 1 - \frac{x0}{r} \right)^{\frac{1}{d}}}{1+(-1+c) r} \right] \right)^7 \right) + \\
& 0 [x - x0]^5
\end{aligned}$$