Linear Algebra

Phree

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Chapter 1

Eigenvalues and Eigenvectors

1.1 Introduction

Definition 1.1. Suppose A is a n by n matrix. A complex number λ is an eigenvalue of A if there exists a nonzero vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v. \tag{1.1}$$

And v is called an eigenvector of A corresponding to λ.

Transforming the equation 1.1 to $(A - \lambda I)v = 0$, we know that v is in the null space of $(A - \lambda I)$. In the meanwhile, any vector in the null space of $A - \lambda I$ is a eigenvector of eigenvalue *lambda*. Since v is nonzero, we have $A - \lambda I$ is singular, that is,

$$\det(A - \lambda I) = 0. \tag{1.2}$$

Equation 1.2 is named the characteristic equation for eigenvalues to satisfy. Conversely, any λ satisfying the characteristic equation, $A - \lambda I$ is singular and has nontrivial null space, leading to $Av = \lambda v$ for some v. Thus we have the following theorem

Theorem 1.1. *Let A be an n by n matrix. Then λ is an eigenvalue of A if and only if λ is a root of the characteristic equation 1.2.*

Obviously, equation 1.2 is a polynomial of order n with respect to λ. By the fundamental theorem of algebra, it has n roots in $\mathbb{C}$, and hence n eigenvalues in $\mathbb{C}$.

1.2 Diagonalization of Matrices

Now we assume that a n by n matrix A has n linearly independent eigenvectors $x_1, x_2, \ldots, x_n$, and denote $X = (x_1, x_2, \ldots, x_n)$. By definition, we have $Ax_i =$

$\lambda_i x_i$, $i = 1, 2, \ldots, n$. Thus

$$AX = A(x_1, \ldots, x_n) = (Ax_1, \ldots, Ax_n) = (x_1, \ldots, x_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = A\Lambda. \tag{1.3}$$

By linearly independence of the column vectors of X, X is invertible. Hence, we can write $AX = X\Lambda$ in two good ways:

$$X^{-1}AX = \Lambda \text{ or } A = X\Lambda X^{-1}. \tag{1.4}$$

That is, we have diagonalize A thanks to X. The kth power of A then can be computed simply by

$$A^k = (X\Lambda X^{-1})^k = X\Lambda^k X^{-1}. \tag{1.5}$$

A^k shares the eigenvectors with A and its eigenvalues are the kth power of that of A, which is quite straightforward geometrically. An interesting result is that A^k tends to zero matrix if the absolute values of all eigenvalues are less than 1.

Remark 1. Note that there is no connection with invertibility and diagonalizability. Invertibility is concerned with eigenvalues (Whether the matrix has a zero eigenvalue or not) while diagonalizability is concerned with the number of linearly independent eigenvectors.

Theorem 1.2. *A matrix A with n distinct eigenvalues is diagonalizable.*

1.2.1 Application of Diagonalization

Example 1.1 (Fibonacci Number)**.** The Fibonacci number is given iteratively by

$$F_{n+2} = F_n + F_{n+1} \tag{1.6}$$

and $F_0 = 0$, $F_1 = 1$. Our goal is to obtain a formula for F_n.

Let

$$u_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}. \tag{1.7}$$

Then we can rewrite 1.6 as $u_{n+1} = Au_n$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.8}$$

It is easy to find that A has eigenvalues

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{5}}{2} \tag{1.9}$$

with the corresponding eigenvectors

$$x_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}. \tag{1.10}$$

Then u_0 can be decomposed to $u_0 = \frac{x_1 - x_2}{\lambda_1 - \lambda_2}$, and $u_n = A^n u_0 = X\Lambda^n X^{-1} u_0$.

Example 1.2 (Linear Constant Coefficient Differential Equations)**.**

[Linear Constant Coefficient Differential Equations]

1.3 Square Root of Positive Definite Matrix

For positive definite matrix, it is reasonable to give the definition of square root of a matrix.

Definition 1.2. Let G be a positive definite matrix. The square root of G is the positive definite matrix A that $A^2 = G$.

This definition is well-defined by the next theorem.

Theorem 1.3. *Every positive definite matrix has one and only one square root.*

Proof. Existence. G can be reformulated to $G = V^T \Lambda V$ where $\Sigma = (\lambda_1, \cdots, \lambda_n)$ is the matrix of eigenvalues. Denoting $\Lambda^{\frac{1}{2}} = (\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$ and $B = V^T \Lambda^{\frac{1}{2}} V$, we have

$$G = V^T \Lambda^{\frac{1}{2}} V V^T \Lambda^{\frac{1}{2}} V = (V^T \Lambda^{\frac{1}{2}} V)^2 = B^2. \tag{1.11}$$

Clearly, B has positive eigenvalues $\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}$.

Uniqueness. Suppose there exists another positive definite matrix A such that $A^2 = G$. Then A can be decompose as

$$A = P^T D P, \tag{1.12}$$

where $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$. Then $PGP^T = PA^2P^T = P(P^TDP)^2P^T = D^2$. That is, P diagonalize G, and it follows that up to permutation $d_1^2, d_2^2, \cdots, d_n^2$ are equal to $\lambda_1, \lambda_2, \cdots, \lambda_n$. Without loss of generality, we assume $\lambda_i = d_i$, $i = 1, 2, \cdots, n$. Thus $PCP^T = \Lambda^{\frac{1}{2}}$, or equivalently,

$$A = P^T \Lambda^{\frac{1}{2}} P. \tag{1.13}$$

Since $G = B^2 = A^2$, we obtain

$$\begin{aligned} B^2 &= A^2 \\ \Leftrightarrow V^T \Lambda V &= P^T \Lambda P \\ \Leftrightarrow (PV^T)\Lambda &= \Lambda(PV^T). \end{aligned} \tag{1.14}$$

Let $W = PV^T$. Then W is commutative with Λ.

Suppose

$$\Lambda = \begin{pmatrix} \lambda_{i_1} I_1 & & & \\ & \lambda_{i_2} I_2 & & \\ & & \ddots & \\ & & & \lambda_{i_k} I_k \end{pmatrix},$$

where λ_{i_j} are distinct eigenvalues of A and I_j are identity matrix. Partition W with the same manner, we have

$$W = \begin{pmatrix} W_{1,1} & \cdots & W_{1,k} \\ W_{2,1} & \cdots & W_{2,k} \\ \vdots & \ddots & \vdots \\ W_{k,1} & \cdots & W_{k,k} \end{pmatrix}.$$

It follows from $W\Lambda = \Lambda W$ that $\lambda_j W_{i,j} = \lambda W_{i,j}$. Since $\lambda_i \neq \lambda_j$ if $i \neq j$, we have $W_{i,j} = 0$ for $i \neq j$. Therefore, it yields that

$$W\Lambda^{\frac{1}{2}} = \Lambda\frac{1}{2}W, \tag{1.15}$$

since Λ and $\Lambda\frac{1}{2}$ share the same structure. Finally, Noting that W is orthogonal, $\Lambda\frac{1}{2} = W^T\Lambda\frac{1}{2}W$.

Hence, $B = V^T\Lambda^{\frac{1}{2}}V = V^T(VP^T\Lambda^{\frac{1}{2}}PV^T)V = P^T\Lambda^{\frac{1}{2}}P = A$. □