Notes on Mathematics

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Contents

Part I

General Theorem

Chapter 1

Set Theory and Logic

1.1 Set Theory

Set theory is the foundation of modern Mathematics. Here, we discuss about axiomatic set theory.

Intuitively, sets are collections of objects. However, we have no way to give a further definition for collection and object. The only thing we can deal with sets is to give a axiomatic definition of sets. Most proofs in this section which are just direct applications of the axioms presented below is omitted.

First, we should specify hwo can be sets generated.

Axiom 1.1 (Sets are objects). *If A is a set, then A is an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.*

Axiom 1.2 (Equality of sets). *Two sets A and B are equal, denoted by $A = B$ if every element of A is an element of B and vice versa.*

Axiom 1.3 (Empty set). *There exists a set $\varnothing$ which contains no elements,* i.e., *for every object x we have $x \notin \varnothing$.*

Lemma 1.1. *If A is a nonempty set, then there exists an object x such that $x \in A$.*

Axiom 1.4 (Singleton sets). *If a is an object, then there exists a set $\{a\}$ whose only element is a.*

Axiom 1.5 (Pairwise union). *Given any two sets A and B, there exists a set $A \cup B$, called the union of A and B, whose elements of all the elements which belong to A or B or both, namely, for any object*

$$x \in A \cup B \iff x \in A \text{ or } x \in B.$$

Proposition 1.1. *(1) If a, b are objects, then $\{a, b\} = \{a\} \cup \{b\}$.*

(2) The union operation is commutative, i.e., *$A \cup B = B \cup A$ for any sets A, B.*

(3) $A \cup A = A \cup \varnothing = A$.

Definition 1.1 (Subsets). Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, if for any object x, $x \in B$ implies $x \in B$.

Proposition 1.2 (Sets are partially ordered by set inclusion). *Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. If $A \subsetneq B$ and $B \subsetneq C$, then $A \subsetneq C$.*

Axiom 1.6 (Specification)**.** *Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (*i.e.*, $P(x)$ is either a true statement or a false statement). Then there exists a set $\{x \in A : P(x)\}$ whose elements are precisely those elements in A for which $P(x)$ is true,* i.e.

$$y \in \{x \in A : P(x)\} \iff y \in A \textit{ and } P(y) \textit{ is true.}$$

Definition 1.2 (Intersection)**.** The intersection $A \cap B$ of two sets A, B is defined to be the set

$$A \cap B = \{x \in A : x \in B\}.$$

Definition 1.3 (Disjoint set)**.** Two sets A and B are said to be *disjoint* if $A \cap B = \varnothing$.

Definition 1.4 (Difference set)**.** Let A, B be sets. We define the set $A \backslash B$ (or $A - B$) to be the set

$$A \backslash B = \{x \in A : x \notin B\}.$$

Proposition 1.3 (Sets form a boolean algebra)**.** *Let A, B, C be sets and let X be a set containing A, B, C as subsets. Then we have (1) (Minimal element) $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$; (2) (Maximal element) $A \cup X = X$ and $A \cap X = A$; (3) (Identity) $A \cup A = A$ and $A \cap A = A$; (4) (Commutativity) $A \cup B = B \cup B$ and $A \cap B = B \cap A$; (5) (Associativity) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$; (6) (Distributivity) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; (7) (Partition) $A \cup (X \backslash A) = X$ and $A \cap (X \backslash) = \varnothing$; (8) (De Morgan laws) $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$ and $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.*

Axiom 1.7 (Replacement)**.** *Let A be a set. For any object $x \in A$ and any object y, suppose that we have a statement $P(x, y)$ pertaining to x and y such that for each $x \in A$ there exists at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y), x \in A\}$ such that for any object z,*

$$z \in \{y : P(x, y), x \in A\} \iff P(x, z) \textit{ is true for some } x \in A.$$

For instance, let $A = \{2, 4, 5\}$ and $P(x, y)$ be the statement $y = x{+}{+}$. Then $\{y : P(x, y), x \in A\} = \{3, 5, 6\}$. We often abbreviate a set of the form

$$\{y : y = f(x) \text{ for some } x \in A\}$$

as $\{f(x) : x \in A\}$.

Axiom 1.8 (Existence of infinity)**.** *There exists a set $\mathbb{N}$, whose elements are called natural numbers, as well as an object $0 \in \mathbb{N}$, and an object $n{+}{+}$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.*

To avoid Russell's Paradox, we need the following axiom that restrain the extent of sets we considered.

Axiom 1.9 (Regularity)**.** *If A is a nonempty set, then there is at least one element x of A which is either not a set, or is disjoint from A.*

This axiom filters out "too large" sets. This axiom is less intuitive and fortunately is never need in the study of analysis, in which the sets we consider are of very low hierarchy.

In analysis, we need a further axiom for our set theory.

Axiom 1.10 (Union)**.** *Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A,* i.e.,

$$x \in \bigcup A \iff x \in S \textit{ for some } S \in A.$$

If one has a set I to which we often refer as an *index set* and for every element $\alpha \in I$, we have a set A_α, then we can form the union set $\bigcup_{\alpha \in I} A_\alpha$ by defining

$$\bigcup_{\alpha \in I} A_\alpha = \bigcup \{A_\alpha : \alpha \in I\},$$

which is a set thanks to the Axiom of Replacement 1.7. Besides, the set $\{A_\alpha : \alpha \in I\}$ is called a *family of set.* However, given a family of sets $\{A_\alpha : \alpha \in I\}$ such that I is nonempty, notice that the existence intersection of defined as

$$\bigcap_{\alpha \in I} A_\alpha = \left\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\right\}$$

where β is an element of I (We can do this since I is nonempty), is guaranteed by the Axiom of Specification. One can show that result of $\left\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\right\}$ does not depend on the choice of β, whence $\bigcap_{\alpha \in I} A_\alpha$ is well-defined.

The collection of axioms of set theory we have given are known as the *Zermelo-Fraenkel axioms of set theory.* There is one further axiom we need, the famous axiom of choice, giving rise to the *Zermelo-Fraenkel-Choice (ZFC) axioms of set theory.*

Although a large portion of the foundation of analysis can be developped without it, the Axiom of Choice is a very powerful and even essential tool for further development.

Before giving the Axiom of Choice, we present the definition of Cartesian product:

Definition 1.5 (Cartesian product)**.** Let I be a set, and for each $\alpha \in I$, let X_α be a set. We then define the Cartesian product $\prod_{\alpha \in I} X_\alpha$ to be the set

$$\prod_{\alpha \in I} X_\alpha = \left\{ (x_\alpha)_{\alpha \in I} \in \left(\bigcup_{\beta \in I} X_\beta \right)^I : x_\alpha \in X_\alpha \quad \text{for all } \alpha \in I \right\}$$

Axiom 1.11 (Choice)**.** *Let I be a set, and for each $\alpha \in I$, let X_α be a non-empty set. Then $\prod_{\alpha \in I} X$ is also a non-empty set.*

There are some other forms of the Axiom of Choice.

Proposition 1.4. *Let X and Y be sets, and let $P(x, y)$ be a property pertaining to objects $x \in X, y \in Y$ such that for every $x \in X$ there exists at least one $y \in X$ such that $P(x, y)$ is true. Then there exists a function $f : X :\to Y$ such that $P(x, f(x))$ holds true for all $x \in X$.*

Proof. We will show that this proposition is equivalent to the Axiom of Choice. Let $Y_x = \{y \in Y : P(x, y) \text{ is true}\}$. Then by the Axiom of Choice, the set $\prod_{x \in X} Y_x$ is non-empty. Therefore, there exists a function

$$(f : X \to Y) \in \left(\bigcup_{x \in X} Y_x \right)^X$$

such that $P(x, f(x))$ is true.

On the other hand, suppose the proposition holds true. Let I be a set such that for every $\alpha \in I$ there is a corresponding set X_α, and let $P(\alpha, x)$ be a property that holds true if and only if $x \in X_\alpha$ for $\alpha \in I$ and $x \in \bigcup_{\alpha\in I} X_\alpha$. Then there exists a function $f : I \to \bigcup_{\alpha\in I} X_\alpha$ such that $P(\alpha, f(\alpha))$ holds true for all $\alpha \in I$. It is clear that $f \in \prod_{\alpha\in I} X_\alpha$, which completes the proof. $\square$

1.2 Functions

Definition 1.6 (Function)**.** Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (this is sometimes known as the vertical line test). Then we define the function $f : X \to Y$ defined by P on the domain X and range Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any xX and yY,

$$y = f(x) \iff P(x, y) \text{ is true.}$$

The set X is said to be the domain of function $f : X \to Y$. We refer the range of a function $f : X \to Y$ the subset $f(X) \coloneqq \{f(x) : x \in X\}$ of Y.

Remark 1. Sometimes, functions are referred to as maps, mappings and transformations. And on some context, the terminlogy "function" means functions that whose range is number filed.

Definition 1.7 (Equality of functions)**.** Two functions $f : X \to Y$, $g : X \to Y$ with the same domain and range are said to be equal, denoted as $f = g$ if $f(x) = g(x)$ for all $x \in X$.

Definition 1.8 (Composition)**.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions. We then define the *composition* $g \circ f$ of g and f to be the function defined explicitly by the formula

$$(g \circ f)(x) \coloneqq g(f(x)).$$

Proposition 1.5 (Composition is associative)**.** *Let $f : X \to W$, $g : Y \to Z$ and $h : XtoY$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Definition 1.9 (One-to-one function, Onto function, Bijective function)**.** A function $f : X \to Y$ is *one-to-one* (or injective) if different elements map to different elements:

$$x \neq x' \implies f(x) \neq f(x').$$

A function $f : X \to Y$ is *onto* (or *surjective*) if $f(X) = Y$, *i.e.*, for every element $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

A function $f : X \to Y$ is said to be *bijective* if it is both one- to-one and onto.

Remark 2. If a function $f : X \to Y$ is bijective, then we sometimes call f a *perfect function* or *one-to-one correspondence*. In this case, denote the action of f using the notation $x \leftrightarrow f(x)$ instead of $x \mapsto f(x)$.

If $f : X \to Y$ is a bijective function, then for each $y \in Y$, there exists exactly one element $x \in X$ such that $f(x) = y$. In this way, denoting x by $f^{-1}(y)$, we obtains a function f^{-1} from Y to X by $y \mapsto f^{-1}(y)$. We call f^{-1} the inverse of f.

To consider the set of all functions from X to Y, we need the following axiom of set theory.

Axiom 1.12 (Power set)**.** *Let X, Y be two sets. Then there exists a set, denoted Y^X, that consists of all the functions from X to Y.*

Proposition 1.6. *Let X be a set. Then $\{Y : Y \subseteq X\}$ is a set.*

Proof. By the axiom of power set, there exists a set $\{0,1\}^X$ consisting of all functions from $X \to \{0,1\}$. For each subset Y of X, we can construct a function $f_Y : X \to \{0,1\}$ such that $f(x) = 1$ if $x \in Y$, else $f(x) = 0$, whence

$$\left\{f(\{1\}) : f \in \{0,1\}^X\right\} \supseteq \{Y : Y \subset X\}.$$

Conversely, for each $f \in \{0,1\}^X$, $f(\{1\})$ is exactly one subset of X. Hence,

$$\left\{f(\{1\}) : f \in \{0,1\}^X\right\} \subseteq \{Y : Y \subset X\}.$$

Then we can draw a conclusion that

$$\left\{f(\{1\}) : f \in \{0,1\}^X\right\} = \{Y : Y \subset X\}$$

does exist. □

If X is a set, we denote the set $\{Y : Y \subset X\}$ by 2^X, which is conventionally called the power set of X.

1.3 Ordered Sets

Definition 1.10 (Partially ordered set)**.** A *partially ordered set* (or *poset*) is a pair $(X, \leq_X)$ where a set X and $\leq_X$ a relation on X (That is, for any two objects $x, y \in X$, the statement $x \leq y$ is either a true or a false one) that obeys the following three properties:

(1) (Reflexivity) For any $x \in X$, we have $x \leq_X x$.

(2) (Anti-symmetry) If $x, yinX$ are such that $x \leq_X y$ and $y \leq_X x$, then $x = y$.

(3) (Transitivity) If $x, y, z \in X$ are such that $x \leq_X y$ and $y \leq_X z$, then $x \leq_X z$.

We refer to $\leq_X$ as the *ordering relation* and sometimes write $\leq$ for short when there is no ambiguity. By $x <_X y$ for $x, y \in X$, we mean $x \leq_X y$ and $x \neq y$. Note that for a partially ordered set X, it may occur that two elements $x, y \in X$ are such that neither $x \leq_X y$ nor $y \leq_X x$ is true.

Definition 1.11 (Totally ordered set)**.** A subset Y in a partially ordered set $(X, \leq_X)$ is said to be a *totally ordered set* if for any $y, y' \in Y$, we either have $y \leq_X y'$ or $y' \leq_X y$ or both.

Example 1.1. Let X be a set. Then $(2^X, \subseteq)$ is a partially ordered set but is not a totally ordered set.

Definition 1.12 (Maximal and minimal element)**.** Let X be a partially ordered set and let Y be a subset of X. We say that $y \in Y$ is a *minimal element* of Y if there is no element $y' \in Y$ such that $y' < y$, *i.e.*,

$$y' \leq y \implies y' = y \quad \text{for any } y' \in Y.$$

Likewise, y is said to be a *maximal element* of Y if there is no element $y' \in Y$ such that $y < y'$.

Definition 1.13 (Well-ordered set)**.** Let X be a partially ordered set and let Y be a totally ordered subset of X. We say that Y is *well-ordered* if every non-empty subset of Y has a minimal element $\min(Y) \in Y$.

Example 1.2. The natural numbers $\mathbb{N}$ together with the canonical order is a totally ordered and well-ordered set, while the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$ and the rational numbers $\mathbb{R}$ are not well-ordered.

One can show that finite totally ordered sets are well-ordered and every subset of a well-ordered set is again well-ordered.

Theorem 1.1 (Principle of strong induction)**.** *Let X be a well-ordered set with an ordering relation $\leq$, and let $P(x)$ be a property pertaining to element $x \in X$ (*i.e.*, for each $x \in X$), $P(x)$ is either a true or false statement). If for every $x \in X$, we have the following implication:*

$$P(m) \text{ is true for all } x' < x \implies P(x) \text{ is true,}$$

then $P(x)$ is true for all $x \in X$.

Proof. Suppose the hypothesis is satisfied, and consider the set

$$Y = \{x \in X : P(x') \text{ is false for some } x' \leq x\}.$$

Then since X is well-ordered, Y has a minimal element x_0. Thus for every $x \in X$ with $x < x_0$, $P(x')$ holds true for every $x' \in X$ such that $x' \leq x$. This means that for every $x < x_0$, $P(x)$ holds true, which implies $P(x')$ holds true for all $x' \leq x_0$, which is a contradiction to the fact $x_0 \in Y$. □

Until now the Axiom of Choice hasn't played a role in our text.

Definition 1.14 (Upper bound and strict upper bound)**.** Let X be a partially ordered set and let Y be a totally ordered subset of X. If $x \in X$, we say x is a *upper bound* of Y if $y \leq x$ for all $y \in Y$. If in addition $x \neq Y$, then we say that x is a *strict upper bound* of Y. Equivalently, x is a strict upper bound of Y if and only if $y < x$ for every $y \in Y$

With Axiom of Choice, we can prove the following lemma:

Lemma 1.2. *Let X be a partially ordered set with ordering relation $\leq$, and let x_0 be an element of X. Then there exists a well-ordered subset Y of X which has x_0 as its minimal element and which has no strict upper bound.*

Proof. □

Now we are ready to demostrate the well-known Zorn's Lemma:

Lemma 1.3 (Zorn's Lemma)**.** *Let X be a non-empty partially ordered set, with the property that every totally ordered subset Y of X has an upper bound. Then X contains at least one maximal element.*

1.4 Cardinality

For sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$, we have a intuition that these two sets are the the same size. But now we have not defined what does "have the same size" mean. Besides, the natural number constructed on Peano axioms behaves much more like ordinals (first, second, third, ...) but not cardinality (one, two, three, ...). In this section, we are dealing with this problem.

Definition 1.15 (Equal cardinality)**.** Two sets X and Y are said to have equal cardinality if there is a bijective function from X to Y.

Example 1.3. The set $\mathbb{N}$ of all natural numbers and the set $Y = \{2n : n \in \mathbb{N}\}$ of all non-negative even numbers have equal cardinality since $f : \mathbb{N} \to Y, n \mapsto 2n$ is a bijective function.

From the example above, we find that two sets having equal cardinality does not preclude one of the sets from containing the other. In fact, this property can be viewed as the definition of infinite sets, which we will see below.

Proposition 1.7. *Having equal cardinality is a equivalence relation.*

Definition 1.16. Let n be a natural number. A set X is said to have n elements or have cardinality n if X have equal cardinality with the set $\{1, 2, \ldots, n\}$.

Proposition 1.8 (Uniqueness of cardinality)**.** *Let n, m be distinct natural numbers. If X is a set having cardinality n, then X cannot have cardinality m.*

Before we proceed with the proof, we need the following lemma.

Lemma 1.4. *Let X be a set having cardinality $n > 0$. Then X is non-empty if x is an element of X, then $X \backslash \{x\}$ has cardinality $n - 1$.*

Proof of Proposition 1.8. This can be done by inducting on n using lemma above. □

Definition 1.17 (Finite sets)**.** A set X is *finite* if it has cardinality $n \in \mathbb{N}$, and we write $\#(X)$ for the cardinality of X. Otherwise, X is said to be infinite.

Proposition 1.9. *The set $\mathbb{N}$ of all natural numbers is infinite.*

Proof. Suppose for the sake of contradiction that $\mathbb{N}$ has cardinality $n \in \mathbb{N}$. Then there exists a bijective function from $\{1, 2, \ldots, n\}$ to $\mathbb{N}$. However, $f(\{1, 2, \ldots, n\})$ is bounded while $\mathbb{N}$ is unbounded, which is absurd. □

Chapter 2

Inequality

2.1 Inequality in Integer Field

Theorem 2.1. *Let a, b be two positive real numbers, $a < b$ and n a positive integer, then*

$$b^n - a^n < n(b-a)b^{n-1}. \tag{2.1}$$

This formula can be easily obtained from Theorem 3.1.

2.2 Cauchy Schwarz's Inequality

Theorem 2.2 (Cauchy Schwarz's Inequality)**.** *Let $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n)$ be elements of $\mathbb{R}^n$, then*

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right). \tag{2.2}$$

Proof. Consider the following quadratic polynomial in x:

$$0 \leq (u_1 x + v_1)^2 + \cdots + (u_n x + v_n)^2 = \left(\sum_{i=1}^{n} u_i^2\right) x^2 + 2\left(\sum_{i=1}^{n} u_i v_i\right) x + \sum v_i^2.$$

Since it is nonnegative, it has at most one real root, whence its discriminant satisfying

$$\Delta = 4\left(\sum_{i=1}^{n} u_i v_i\right)^2 - 4\left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right) \leq 0,$$

which is exactly the inequality we want. □

2.3 Hölder's Inequality

Lemma 2.1 (Young's Inequality). *Let p, q be positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Then for all $a, b \in \mathbb{C}$,*

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}. \tag{2.3}$$

Proof. Define $\phi(t) = \frac{t^p}{p} + \frac{|b|^q}{q} - |b|\,t, t \geq 0$. It suffices to show that ϕ is non negative. Observing that

$$\phi'(t) = t^{p-1} - |b|$$

is negative when $t < |b|^{\frac{1}{p-1}}$ and positive when $t > |b|^{\frac{1}{p-1}}$, thus ϕ achieves its minium at $|b|^{\frac{1}{p-1}}$, which is

$$\frac{1}{p}|b|^{\frac{p}{p-1}} - |b|^{1+\frac{1}{p-1}} + \frac{1}{q}|b|^q = \frac{1}{p}|b|^q + \frac{1}{q}|b|^q - |b|^q = 0,$$

which is what we want. □

Theorem 2.3 (Hölder's inequality). *Let (X, Σ, μ) be a measure space and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all measurable real- or complex-valued function f and g on X,*

$$\|fg\| \leq \|f\|_p \|g\|_q \,,$$

where norm $\|\cdot\|_p$ is defined in Chapter 15 and Chapter 18.

Proof. If $\|f\|_p = 0$, then f vanishes almost anywhere thus fg does too. Hence Hölder's inequality holds true. The same is true for $\|g\|_q = 0$. Suppose that $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$ in the following.

Case 1. $p \in (1, \infty)$**.** Consider

$$a = \frac{|f(t)|}{\|f\|_p}, \quad b = \frac{|g(t)|}{\|g\|_q}.$$

By Young's inequality, we have

$$\frac{|f(t)||g(t)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p}\frac{|f(t)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(t)|^q}{\|g\|_q^q}.$$

Integrating both sides of the above inequality yields

$$\frac{1}{\|f\|_p \|g\|_q}\int_X |fg|\,\mathrm{d}t \leq \frac{1}{p\|f\|_p^p}\int_X |f|^p\,\mathrm{d}p + \frac{1}{q\|g\|_q^q}\int_X |g|^q\,\mathrm{d}q,$$

which is

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq 1.$$

Case 2. $p = 1$ **and** $q = \infty$**.** Inequality (2.3) follows directly from the monotonicity of Lebesgue integration. □

2.4 Minkowski's Inequality

Theorem 2.4. *Let (X, Σ, μ) be a measure space and let $p \in [1, \infty]$. Then for all measurable real- or complex-valued function f and g on X,*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \,.$$

Proof. The result is clearly true if $p = 1$.

Assume that $p \in (1, \infty)$, then let $q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Observing that

$$\begin{aligned}
\|f+g\|_p^p &= \int_X |f+g||f+g|^{p-1}\,\mathrm{d}t \\
&\leq \int_X \big(|f| + |g|\big)|f+g|^{p-1}\,\mathrm{d}t \\
&= \int_X |f||f+g|^{p-1}\,\mathrm{d}t + \int_X |g||f+g|^{p-1}\,\mathrm{d}t \\
&\leq \left\| f|f+g|^{p-1} \right\|_1 + \left\| g|f+g|^{p-1} \right\|_1 \\
\text{(Hölder's inequality)} &\leq \|f\|_p \left\| |f+g|^{p-1} \right\|_q + \|g\|_p \left\| |f+g|^{p-1} \right\|_q
\end{aligned}$$

Since $(p-1)q = p$, it follows that

$$\|f+g\|_p^p \leq \Big(\|f\|_p + \|g\|_p\Big) \|f+g\|_p^{\frac{p}{q}} \,,$$

whence

$$\|f+g\|_p^{p-\frac{p}{q}} = \left\|f_g\right\|_p \leq \|f\|_p + \|g\|_p \,.$$

$\square$

Chapter 3

Identity

3.1 Identity in Integer Field

Theorem 3.1. *Let a, b be two integers and n a positive integer, then*

$$b^n - a^n = (b - a) \sum_{i=1}^{n} b^{n-i} a^{i-1}. \tag{3.1}$$

We deduce an inequality (theorem 2.1) from his identity.

Chapter 4

Special Functions

4.1 Gamma Functions

Definition 4.1. The gamma function Γ defined on $\{z \in \mathbb{C} : \Re(z) > 0\}$ is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t. \tag{4.1}$$

4.2 Bessel Functions

Part II

Algebra

Chapter 5

Rings

5.1 Rings and Homomorphisms

Definition 5.1. A ring is a set $\mathbf{R}$ together with two binary operations (usually denoted as addition and multiplication) such that

(1) $\mathbf{R}$ is an *abelian group* under addition;

(2) $\mathbf{R}$ is a semigroup under multiplication;

(3) $a(b+c) = ab = ac$ and $(a+b)c = ac + bc$ for any $a, b, c \in \mathbf{R}$ (left and right distributive laws).

In addition, if $ab = ba$ for all $a, b \in \mathbf{R}$, then $\mathbf{R}$ is said to be commutative; if $\mathbf{R}$ has an element $1_{\mathbf{R}}$ such that

$$1_{\mathbf{R}} r = r 1_{\mathbf{R}} = r \text{ for any } r \in \mathbf{R},$$

then $\mathbf{R}$ is said to be a ring with identity.

Definition 5.2. A nonzero element a in a ring $\mathbf{R}$ is called a *left (resp. right) zero divisor* if $ab = 0$ (resp. $ba = 0$) for some nonzero element $b \in \mathbf{R}$. An element of $\mathbf{R}$ is a *zero divisor* if it is both a left and right zero divisor.

A commutative ring with identity $1_{\mathbf{R}} \neq 0$ is called an *integral domain* if it has no zero divisors.

Here is a direct and useful property of rings with no zero divisors.

Proposition 5.1. *A ring $\mathbf{R}$ has no zero divisors if and only if the left and right cancellation laws hold in $\mathbf{R}$, that is, for any $a, b, c \in \mathbf{R}$ with $a \neq 0$,*

$$ab = ac \textit{ or } ba = ca \implies a = c. \tag{5.1}$$

Definition 5.3. An element a in a ring $\mathbf{R}$ is said to be *left (resp. right) invertible* if there exists $b \in \mathbf{R}$ such that $ab = 1_{\mathbf{R}}$ (resp. $ba = 1_{\mathbf{R}}$). The element b of $\mathbf{R}$ is said to be a *left (resp. right) inverse* of a. An element $a \in \mathbf{R}$ is *invertible* or called a *unit* if it is both left and right invertible.

Definition 5.4. Let X be a subset of a ring $\mathbf{R}$. The intersection of all [left] ideals containing X is called the [left] *ideal generated by* X, denoted by (X). The elements of X are called the generators of the ideal (X).

An ideal (a) generated by a single element $a \in \mathbf{R}$ is called a *principal ideal.* A *principal ideal ring* is a ring in which every ideal is principal. A principal ideal ring which is an integral domian is called a *principal ideal domain.*

5.2 Ideals

In the theory of rings, ideals play an analogous role to normal subgroups in the theory of groups.

Definition 5.5. Let $\mathbf{R}$ be a ring and $\mathbf{S}$ be a nonempty subset of $\mathbf{R}$ that is closed under addition and multiplication of $\mathbf{R}$. If $\mathbf{S}$ is itself a ring under these operations, then $\mathbf{S}$ is called a *subring* of $\mathbf{R}$.

A subring $\mathbf{I}$ of a ring $\mathbf{R}$ is a *left ideal* in $\mathbf{R}$ if

$$r \in \mathbf{R}, x \in \mathbf{I} \implies rx \in I; \tag{5.2}$$

$\mathbf{I}$ is a *right ideal* if

$$r \in \mathbf{R}, x \in \mathbf{I} \implies xr \in I. \tag{5.3}$$

$\mathbf{I}$ is an ideal if it is both a left and right ideal.

Example 5.1. The center of a ring $\mathbf{R}$ is the set $\mathbf{C} = \{c \in \mathbf{R} : cr = rc \text{ for any } r \in \mathbf{R}\}$. Then $\mathbf{C}$ is a subring of $\mathbf{R}$ but may not be an ideal. Notice that $fC = \mathbf{R}$ if and only if $\mathbf{R}$ is a commutative ring.

There is a very useful characterization of ideals:

Theorem 5.1. *A nonempty subset $\mathbf{I}$ of of a ring $\mathbf{R}$ is a left ideal if and only if for all $a, b \in \mathbf{I}$ and $r \in \mathbf{R}$, $a - b \in \mathbf{I}$ and $ra \in \mathbf{I}$.*

Proof. It suffices to prove that $\mathbf{I}$ is a subring of $\mathbf{R}$ if and only if $a - b \in \mathbf{I}$ and $ab \in \mathbf{I}$ for any $a, b \in \mathbf{I}, r \in \mathbf{R}$. Let $a \in \mathbf{I}$. We have $0 = a - a \in \mathbf{I}$ and $-a = 0 - a \in \mathbf{I}$. Thus $a + b = a - (-b) \in \mathbf{I}$ for $a, b \in \mathbf{I}$. Consequently, $\mathbf{I}$ is a subring of $\mathbf{R}$. □

Theorem 5.2. *Let $\mathbf{R}$ be a ring, $a \in \mathbf{R}$ and $X \subseteq \mathbf{R}$. Then*

(1) The principal ideal (a) consists of all elements of the form

$$ra + as + na + \sum_{i=1}^{m} r_i a s_i \quad r, s, r_i, s_i \in \mathbf{R}; m \in \mathbb{N}^* \text{ and } n \in \mathbb{Z}. \tag{5.4}$$

(2) If $\mathbb{R}$ has an identity, then $(a) = \left\{\sum_{i=1}^{m} r_i a s_i : r_i, s_i \in \mathbb{R}, m \in \mathbb{N}^*\right\}$.

(3) If a is in the center of $\mathbf{R}$ (in particular, $\mathbf{R}$ is a commutative ring), then $(a) = \{ra + na : r \in \mathbf{R}, n \in \mathbb{Z}\}$.

(4) If $\mathbf{R}$ has an identity and a is in the center of $\mathbf{R}$, then $\mathbf{R}a = (a) = a\mathbf{R}$.

Proof of Theorem 5.2. (1) Put

$$\mathbf{I} = \left\{ ra + as + na + \sum_{i=1}^{m} r_i a s_i : r, s, r_i, s_i \in \mathbf{R}, m \in \mathbb{N}^* \right\}. \tag{5.5}$$

Obviously, $\mathbf{I}$ is a subring of $\mathbf{R}$ and $a \in \mathbf{I}$. On the other hand, assume that $\mathbf{I}'$ is a ideal containing a, then by definition of ideals, elements of the form ra, as, ns are contained in $\mathbf{I}'$ whence $ras \in \mathbf{I}'$ for any $r, s \in \mathbf{I}'$. Thereafter,

$$x = ra + as + na + \sum_{i=1}^{m} r_i a s_i : r, s, r_i, s_i \in \mathbf{R}, m \in \mathbb{N}^* \tag{5.6}$$

is an element of $\mathbf{I}'$. Consequently, $\mathbf{I} \subseteq \mathbf{I}'$ and $\mathbf{I} = (a)$.

(2) (4) Trivial. $\square$

Remark 3. If $\mathbf{R}$ is a commutative ring and $a, b \in \mathbf{R}$, then $(a)(b) \subseteq (ab)$.

Let $A_1, A_2, \ldots, A_n$ be a family of subsets of ring $\mathbf{R}$. Denote by $A_1 + A_2 + \ldots + A_n$ the set

$$\left\{ \sum_{i=1}^{n} a_n : a_i \in A_i \text{ for all } i = 1, 2, \ldots, n \right\}.$$

Likewise, denote by $A_1 A_2 \cdots A_n$ the set

$$\{a_1 a_2 \cdots a_n : a_i \in A_i \text{ for all } i = 1, 2, \ldots, n\}.$$

As we have stated before, ideals is a basic tool to study the structure of rings as normal subgroups in the theory of groups. More specifically, ideals can be used to define quotient rings.

Definition 5.6.

5.2.1 Prime Ideals and Maximal Ideals

Definition 5.7. Ideal!Prime ~) An ideal P of a ring $\mathbf{R}$ is said to be *prime* if $P \neq \mathbf{R}$ and for any ideals A, B of $\mathbf{R}$

$$AB \subseteq P \implies A \subseteq P \text{ or } B \subseteq P. \tag{5.7}$$

Here is a useful characterization of prime ideals.

Theorem 5.3. *If P is an ideal in a ring $\mathbf{R}$ such that $P \neq \mathbf{R}$ and for any $a, b \in \mathbf{R}$*

$$ab \in P \implies a \in P \text{ or } b \in P, \tag{5.8}$$

then P is prime. Conversely, if $\mathbf{R}$ is commutative and P is prime, then P satisfies condition (5.8).

Proof of Theorem 5.3. We first prove the former statement. If P is not prime, then there exist ideals ideals A, B in $\mathbf{R}$ such that $AB \subseteq P$ and $A \not\subseteq P$ and $B \not\subseteq P$. Thus there exist elements $a \in A$ and $b \in B$ such that $a \notin P$ and $b \notin P$. Clearly, $ab \in AB \subseteq P$. However, according to condition (5.8), we deduce that $a \in P$ or $b \in P$, which is a contradiction. So P is prime.

Now we prove the second statement. Assume that elements $a, b \in fR$ satisfy $ab \in P$. Then $(a)(b) \subseteq (ab)$ since $\mathbf{R}$ is commutative. By definition, $(ab) \subseteq P$, whence $(a)(b) \subseteq P$. Because P is prime, we have $(a) \subseteq P$ or $(b) \subseteq P$, which implies $a \in b$ or $b \in P$. $\square$

The following theorem illustrates the functionality of prime ideals in the study of rings.

Theorem 5.4. *Let $\mathbf{R}$ be a commutative ring with identity $1_{\mathbf{R}} \neq 0$ and P is an ideal in $\mathbf{R}$. Then P is a prime ideal in $\mathbf{R}$ if and only if the quotient ring $\mathbf{R}/P$ is an integral domain.*

Definition 5.8. An ideal (resp. left ideal) M in a ring $\mathbf{R}$ is called *maximal* provided that $M \neq \mathbf{R}$ and for any ideal (resp. left ideal) N in $\mathbf{R}$

$$M \subseteq N \subseteq \mathbf{R} \implies N = M \text{ or } N = \mathbf{R}. \tag{5.9}$$

Remark 4. According to the definition above, M is a maximal element of the set of all ideals except $\mathbf{R}$. Sometimes we say an ideal I is maximal with respect to a property, meaning that I is a maximal element of the set of all ideals with the given property except $\mathbf{R}$.

Theorem 5.5. *If $\mathbf{R}$ is a commutative ring such that $\mathbf{R}^2 = \mathbf{R}$ (in particular, $\mathbf{R}$ has identity), then every maximal ideal M in $\mathbf{R}$ is prime.*

Theorem 5.6. *In a nonzero ring $\mathbf{R}$ with identity maximal [left] ideals always exist. In fact every [left] ideal $\mathbf{I} \neq \mathbf{R}$ is contained in a maximal [left] ideal.*

5.3 Factorization in Commutative Rings

5.3.1 Divisors

Let's extend the definitions of divisibility, greatest common divisor and prime in the ring of integers to arbitrary commutative rings.

Definition 5.9. A nonzero element a of a commutative ring R divides an element $b \in R$, denoted by $a \mid b$ if there exists an element $x \in R$ such that $b = ax$. If a divides b, then we say a is a factor of b. If a is a factor of b while b is not a factor of a, then a is said to be a *proper factor* of a. Elements a, b are said to be associates if $a \mid b$ and $b \mid a$, denoted as $a \sim b$.

Theorem 5.7. *Let a, b, u be elements of a commutative ring R. Then*

(1) $a \mid b$ if and only if $(b) \subseteq (a)$.

(2) a and b are associates if and only if $(a) = (b)$.

(3) u is a unit if and only if $(u) = R$.

Proof of Theorem 5.7. (1) By Theorem 5.2 (4), $(a) = \mathbf{R}a = a\mathbf{R}$ and $(b) = \mathbf{R}b = b\mathbf{R}$.

(2) Obvious.

(3) a is a unit $\iff$ a^{-1} exists $\implies$ $r = ra^{-1}a \in \mathbf{R}a = (a)$ for any $r \in \mathbf{R}$. Conversely, $(a) = \mathbf{R} \implies 1_{\mathbf{R}} = ra$ for some $r \in \mathbf{R}$. □

Definition 5.10. Let X be a subset of a commutative ring $\mathbf{R}$. An element $d \in \mathbf{R}$ is said to be a *greatest common divisor (GCD)* of X if

(1) $d \mid x$ for all $x \in X$;

(2) $c \mid x$ for all $x \in X \implies c \sim d$.

Greatest common divisor does not always exist.

Definition 5.11. Let $\mathbf{R}$ be a commutative ring with identity. An element $c \in \mathbf{R}$ is irreducible if

(1) c is a nonzero nonunit;

(2) $c = ab \implies a$ or b is a unit, for any $a, b \in \mathbf{R}$.

An element p of $\mathbf{R}$ is prime if

(1) p is a nonzero nonunit;

(2) $p \mid ab \implies p \mid a$ or $p \mid b$, for any $a, b \in \mathbf{R}$.

There is a close connection between prime (resp. irreducible) elements and prime (resp. maximal) principal ideals in $\mathbf{R}$.

Theorem 5.8. *Let p and c be nonzero elements in an integral domain $\mathbf{R}$.*

(1) p is prime if and only if (p) is a nonzero prime ideal.

(2) c is maximal if and only if (c) is maximal in the set S of all proper principal ideals of $\mathbf{R}$.

(3) Every prime element of $\mathbf{R}$ is irreducible.

(4) If $\mathbf{R}$ is a principal ideal domain, then p is prime if and only if p is irreducible.

(5) The only divisors of an irreducible element are its associates and units.

Proof of Theorem 5.8. (2) Assume that c is an irreducible element of $\mathbf{R}$. Then (c) is a proper principal ideal in $\mathbf{R}$ since (c) is a nonunit. Let a be an element in $\mathbf{R}$ such that (a) is a proper principal ideal in $\mathbf{R}$ such that $(c) \subseteq (a)$. a is a nonunit by Theorem 5.7 (3). By Theorem 5.2 (4), $c = ra$ for some $r \in \mathbf{R}$. Hence r is a unit because of the irreducibility of c. Thus $a = r^{-1}c$ and $r'a = r'r^{-1}c \in (c)$ for any $r' \in \mathbf{R}$, meaning that $(a) \subseteq (c)$. Conversely, if (c) is maximal in S, then c is a nonzero nonunit. Suppose that $c = ab$ and a is a nonunit, which means that $(c) \subseteq (a)$ and $(a) \neq \mathbf{R}$, whence $(a) = (c)$. Hence $a = rc$ for some $r \in \mathbf{R}$. Consequently, we have $c = ab = rbc$. Since $\mathbf{R}$ is an integral domain, $1_{\mathbf{R}} = rb$, whence b is a unit. [1]

[1] Why?

(3) p is prime and $p = ab \implies p \mid a$ or $p \mid b$; say $p \mid a$. THus $pr = a$ for some $r \in \mathbf{R}$, which implies $p = prb$. Since $\mathbf{R}$ is an integral domain, $rb = 1_{\mathbf{R}}$. Therefore, b is a unit and p is irreducible.

(4) It suffices to prove that if $\mathbf{R}$ is prime and p is irreducible, then p is prime. Before that, we prove that if $\mathbf{R}$ is a principal ideal domain, and (p) is a maximal ideal in the set of all proper principal ideals, then (p) is a maximal ideal in $\mathbf{R}$. Suppose that there exist a proper ideal X in $\mathbf{R}$ such that $(p) \subseteq X$. For any $x \in X$, $(x) \subseteq X$.

(5) Trivial. □

5.3.2 Unique Factorization Domains

Definition 5.12. An integral domain $\mathbf{R}$ is called a *unique factorization domain (UFD)* provided that

(1) every nonzero nonunit element $a \in \mathbf{R}$ can be written $c_1c_2\cdots c_n$ with $c_1, c_2, \ldots, c_n$ irreducible. (Existence of factorization.)

(2) If $a \in \mathbf{R}$ can be factorized as $a = c_1c_2\cdots c_n$ and $a = d_1d_2\cdots d_m$ with c_i and d_j irreducible, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, then $n = m$ and for some permutation σ of $\{1, 2, \ldots, n\}$, $c_i \sim d_{\sigma(i)}$. (Uniqueness of factorization.)

As mentioned in Theorem 5.8, a prime element in a integral domain is irreducible. For unique factorization domain, the converse is true. Consequently, irreducible and prime elements coincide in UFD.

Proposition 5.2. *Every irreducible element in a unique factorization domain is prime.*

Proof. Suppose that p is an irreducible element in a UFD $\mathbf{R}$ and $c \mid ab$, for nonunits $a, b \in \mathbf{R}$, which is equivalent to $pr = ab$ for some $r \in \mathbf{R}$. By definition of UFD, a, b and r can be factorized as $a = c_1c_2\cdots c_n$, $b = d_1d_2\cdots d_m$ and $r = q_1q_2\cdots c_l$. Then we have, $pq_1q_2\cdots q_l = c_1c_2\cdots c_nd_1d_2\cdots d_m$. Hence, with uniqueness of factorization, p associates with some c_i or d_j, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Thus $p \mid a$ or $p \mid b$. $\square$

From Theorem 5.8 we know for principal ideal domain, prime and irreducible elements do coincide as the case for UFD. It seems plausible that every principal ideal domain is a UFD. In order to prove that this is indeed the case, we first prove the following lemma.

Lemma 5.1. *If $\mathbf{R}$ is a principal ideal ring and $(a_1) \subseteq (a_2) \subseteq \cdots$ is a chain of ideals in $\mathbf{R}$, then for some positive integer n, $(a_i) = (a_n)$ for all $i > n$.*

Proof. Let $\mathbf{A} = \bigcup_{i=1}^{\infty}(a_i)$. We contend that $\mathbf{A}$ is an ideal. Indeed, for any $a, b \in \mathbf{A}$, $a \in (a_i), b \in (a_j)$ for some positive integers i, j. Assuming that $i \leq j$, $a \in (a_j)$ and $a - b \in (a_j) \subseteq \mathbf{A}$. Furthermore, for any $r \in \mathbf{R}$ we have $ra \in \mathbf{R}a \subseteq (a_i) \subseteq \mathbf{A}$.

Since $\mathbf{R}$ is a principal ideal domain, $\mathbf{I} = (d)$ for some $d \in \mathbf{R}$. Obviously, $d \in \mathbf{I}$ thus d belonging to some (a_n), $n \in \mathbb{Z}_+^*$. Hence, $(d) \subseteq (a_i)$ for all $i \geq n$. Therefore, $(a_i) = \mathbf{A}$ for all $i \geq n$. $\square$

Theorem 5.9. *Every principal ideal domain $\mathbf{R}$ is a unique factorization domain.*

Proof. Let S be the set of all nonzero nonunit elements of $\mathbf{R}$ that cannot be factored a finite product of irreducible elements. We shall first show that S is empty. Suppose that S is nonempty and $a \in S$. Then $(a) \subsetneqq \mathbf{R}$ by Theorem (3) and is contained in a maximal ideal (c) by Theorem 5.6. Hence, c is a irreducible element of $\mathbf{R}$ by Theorem 5.7 (2). Therefore, $(a) \subseteq (c)$ implies that c divides a, *i.e.*, $a = cx$ for some $x \in \mathbf{R}$. We contend that $x \in S$. Indeed, if x were a unit, then a, c are associates and thus a is irreducible, contradicting the hypothesis that $a \in S$. If x were a nonunit and $x \notin S$, then x has a finite factorization, whence a does, which is also a contradiction. Hence, $x \in S$. Furthermore, we claim that the ideal $(a) \subsetneqq (x)$. Indeed, since $x \mid a$, we have $(a) \subseteq (x)$. However, if $(a) = (x)$ were true, we deduce that $x = ay$ for some $y \in S$. Thus, $a = cx = cay$ and $cy = 1$ by cancellation law, which contradicts the irreducibility of c.

With Axiom of Choice [2], we can choose a $x_a \in \mathbf{R}$ for each $a \in S$ in the way given in the proceeding paragraph and denote the choice function by $f : S \to S, a \mapsto x_a$. By Recursion Theorem, we have a unique function $\phi : \mathbb{N} \to S$ such that $\phi(0) = a$ and $\phi(n+1) = f(\phi(n))$. We can denote $\phi(n)$ by a_n, and then we obtain a sequence $a_0, a_1, \ldots$ of elements of S such that

$$(a_0) \subsetneqq (a_1) \subsetneqq (a_2) \subsetneqq \cdots . \tag{5.10}$$

[2] To be completed.

But by Lemma 5.1, this is not allowed. Therefore, S must be empty, whence every nonzero nonunit element of $\mathbf{R}$ can be factored as af finite product of irreducible elements.

Finally, we need to prove the uniqueness of factorization. If $c_1c_2\cdots c_n = c = d_1d_2\cdots d_m$ where c_i, d_j are irreducible, then c_1 divides d_j for some $j = 1, 2, \ldots, m$. Without loss of generality, assume $j = 1$. Since the factors of an irreducible element are either units or its associates (Theorem 5.8 (5)), we have c_1 and d_1 are associates. The proof of uniqueness is now completed by a routine inductive argument. □

Although greatest common divisor may not exist for some commutative ring $\mathbf{R}$, in the case of uniqueness factorization domain, greatest common divisor does exists for any finite subset of $\mathbf{R}$.

Theorem 5.10. *Let* $\mathbf{R}$ *is a unique factorization domain. If* $X = \{a_1, a_2, \cdots, a_n\} \subseteq \mathbf{R}$*, then there exits a greatest common divisor of* X*.*

Proof. a_i can be factored as $a_i = c_1^{s_{i,1}} c_2^{s_{i,2}} \cdots c_m^{s_{i,m}}$, $i = 1, 2, \ldots, n$. Set $t_j = \min\{s_{1,j}, s_{2,j}, \ldots, s_{n,j}\}$ for $j = 1, 2, \ldots, m$. We can prove that $c = c_1^{t_1} c_2^{t_2} \cdots c_m^{t_m}$ is a greatest common divisor of X. It is trivial to prove that c is a common divisor of X. Suppose that d is a common divisor of X, and now we prove that d is a divisor of c, which can be done by establishing the fact that any common divisor of X is of the form $c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}$ with $k_j \leq t_j, j = 1, 2, \ldots, m$. □

Apart from the condition of uniqueness of factorization, we have more conditions that characterize unique factorization domain:

Theorem 5.11. *Let* $\mathbf{R}$ *be an integral domain. Then the following statements are equivalent:*

(1) $\mathbf{R}$ *is a uniqueness factorization domain.*

(2) $\mathbf{R}$ *satisfies the condition of existence of factorization and the hypothesis of Proposition 5.2.*

(3) $\mathbf{R}$ *satisfies the condition of existence of factorization and the hypothesis of Theorem 5.10.*

5.3.3 Euclidean Domains

Definition 5.13. Let $\mathbf{R}$ be a commutative ring. $\mathbf{R}$ is called a *Euclidean ring* if there exists a function $\phi : \mathbf{R}^* = \mathbf{R} - \{0\} \to \mathbb{N}$ such that:

(1) if $a, b \in \mathbf{R}$ and $ab \neq 0$, then $\phi(a) \leq \phi(ab)$;

(2) if $a, b \in \mathbf{R}$ and $b \neq 0$, then there exist $q, r \in \mathbf{R}$ such that $a = qb + r$ with $r = 0$, or $r \neq 0$ and $\phi(r) \leq \phi(b)$.

A Euclidean ring is called a *Euclidean domain* if it is an integral domain.

Example 5.2. The ring $\mathbb{Z}$ of integers is a Euclidean domain with $\phi(x) = |x|$.

Example 5.3. The ring $\mathbf{F}[x]$ of polynomials over a field $\mathbf{F}$ is a Euclidean domain $\phi(f) = \deg f$ for $f \neq 0$.

Example 5.4. The ring $\mathbb{Z}[\sqrt{-1}] = \{a + b\sqrt{-1} : a, b \in \mathbb{Z}\}$, known as *Gaussian integers* is a Euclidean domain with $\phi(a + b\sqrt{-1}) = a^2 + b^2$. It is easy to verify that $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ for any $\alpha, \beta \in \mathbb{Q}[\sqrt{-1}]$. Suppose $\beta \neq 0, \beta \in \mathbb{Z}[\sqrt{-1}]$ and $\alpha \in \mathbb{Z}[\sqrt{-1}]$. Then $\beta^{-1} \in \mathbb{Q}[\sqrt{-1}]$ and assume that $\alpha\beta^{-1}$

$$\alpha\beta^{-1} = u + v\sqrt{-1} \in \mathbb{Q}[\sqrt{-1}].$$

There exist $c, d \in \mathbb{Z}$ such that $|c - u| \leq \frac{1}{2}$ and $|d - v| \leq \frac{1}{2}$. Setting $x = u - c$ and $y = v - d$, we have $\alpha = \beta\left((x + c) + (y + d)\sqrt{-1}\right)$. Then we have $\alpha = \beta\theta + \gamma$ with

$$\theta = c + d\sqrt{-1} \text{ and } \gamma = \beta(x + y\sqrt{-1}) = \alpha - \beta(c + d\sqrt{-1}) = \alpha - \beta\theta. \tag{5.11}$$

If $\gamma \neq 0$, then $\pi(r) = \phi(\beta(x + y\sqrt{-1})) = \phi(\beta)\phi(x + y\sqrt{-1}) = \phi(\beta)\left(x^2 + y^2\right) \leq \frac{1}{2}\phi(\beta) \leq \phi(\beta)$.

Theorem 5.12. *Every Euclidean domain* $\mathbf{R}$ *is a principal ideal domain.*

Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction

Definition 6.1. Suppose A is a n by n matrix. A complex number λ is an eigenvalue of A if there exists a nonzero vector $v \in \setminus^n$ such that

$$Av = \lambda v. \tag{6.1}$$

And v is called an eigenvector of A corresponding to λ.

Transforming the equation 6.1 to $(A-\lambda I)v = 0$, we know that v is in the null space of $(A - \lambda I)$. In the meanwhile, any vector in the null space of $A - \lambda I$ is a eigenvector of eigenvalue *lambda*. Since v is nonzero, we have $A - \lambda I$ is singular, that is,

$$\det(A - \lambda I) = 0. \tag{6.2}$$

Equation 6.2 is named the characteristic equation for eigenvalues to satisfy. Conversely, any λ satisfying the characteristic equation, $A - \lambda I$ is singular and has nontrivial null space, leading to $Av = \lambda v$ for some v. Thus we have the following theorem

Theorem 6.1. *Let A be an n by n matrix. Then λ is an eigenvalue of A if and only if λ is a root of the characteristic equation 6.2.*

Obviously, equation 6.2 is a polynomial of order n with respect to λ. By the fundamental theorem of algebra, it has n roots in , and hence n eigenvalues in .

6.2 Diagonalization of Matrices

Now we assume that a n by n matrix A has n linearly independent eigenvectors $x_1, x_2, \dots, x_n$, and denote $X = (x_1, x_2, \dots, x_n)$. By definition, we have $Ax_i = \lambda_i x_i$, $i = 1, 2, \dots, n$. Thus

$$AX = A(x_1, \dots, x_n) = (Ax_1, \dots, Ax_n) = (x_1, \dots, x_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = A\Lambda. \tag{6.3}$$

By linearly independence of the column vectors of X, X is invertible. Hence, we can write $AX = X\Lambda$ in two good ways:

$$X^{-1}AX = \Lambda \text{ or } A = X\Lambda X^{-1}. \tag{6.4}$$

That is, we have diagonalize A thanks to X. The kth power of A then can be computed simply by

$$A^k = (X\Lambda X^{-1})^k = X\Lambda^k X^{-1}. \tag{6.5}$$

A^k shares the eigenvectors with A and its eigenvalues are the kth power of that of A, which is quite straightforward geometrically. An interesting result is that A^k tends to zero matrix if the absolute values of all eigenvalues are less than 1.

Remark 5. Note that there is no connection with invertibility and diagonalizability. Invertibility is concerned with eigenvalues (Whether the matrix has a zero eigenvalue or not) while diagonalizability is concerned with the number of linearly independent eigenvectors.

Theorem 6.2. *A matrix A with n distinct eigenvalues is diagonalizable.*

6.2.1 Application of Diagonalization

Example 6.1 (Fibonacci Number)**.** The Fibonacci number is given iteratively by

$$F_{n+2} = F_n + F_{n+1} \tag{6.6}$$

and $F_0 = 0$, $F_1 = 1$. Our goal is to obtain a formula for F_n.

Let

$$u_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}. \tag{6.7}$$

Then we can rewrite 6.6 as $u_{n+1} = Au_n$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.8}$$

It is easy to find that A has eigenvalues

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{5}}{2} \tag{6.9}$$

with the corresponding eigenvectors

$$x_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}. \tag{6.10}$$

Then u_0 can be decomposed to $u_0 = \frac{x_1 - x_2}{\lambda_1 - \lambda_2}$, and $u_n = A^n u_0 = X\Lambda^n X^{-1} u_0$.

Example 6.2 (Linear Constant Coefficient Differential Equations)**.**

[Linear Constant Coefficient Differential Equations]

6.3 Square Root of Positive Definite Matrix

For positive definite matrix, it is reasonable to give the definition of square root of a matrix.

Definition 6.2. Let G be a positive definite matrix. The square root of G is the positive definite matrix A that $A^2 = G$.

This definition is well-defined by the next theorem.

Theorem 6.3. *Every positive definite matrix has one and only one square root.*

Proof. Existence. G can be reformulated to $G = V^T \Lambda V$ where $\Sigma = (\lambda_1, \cdots, \lambda_n)$ is the matrix of eigenvalues. Denoting $\Lambda^{\frac{1}{2}} = (\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$ and $B = V^T \Lambda^{\frac{1}{2}} V$, we have

$$G = V^T \Lambda^{\frac{1}{2}} V V^T \Lambda^{\frac{1}{2}} V = (V^T \Lambda^{\frac{1}{2}} V)^2 = B^2. \tag{6.11}$$

Clearly, B has positive eigenvalues $\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}$.

Uniqueness. Suppose there exists another positive definite matrix A such that $A^2 = G$. Then A can be decompose as

$$A = P^T D P, \tag{6.12}$$

where $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$. Then $PGP^T = PA^2P^T = P(P^TDP)^2P^T = D^2$. That is, P diagonalize G, and it follows that up to permutation $d_1^2, d_2^2, \cdots, d_n^2$ are equal to $\lambda_1, \lambda_2, \cdots, \lambda_n$. Without loss of generality, we assume $\lambda_i = d_i$, $i = 1, 2, \cdots, n$. Thus $PCP^T = \Lambda^{\frac{1}{2}}$, or equivalently,

$$A = P^T \Lambda^{\frac{1}{2}} P. \tag{6.13}$$

Since $G = B^2 = A^2$, we obtain

$$\begin{aligned} & B^2 = A^2 \\ \Leftrightarrow\ & V^T \Lambda V = P^T \Lambda P \\ \Leftrightarrow\ & (PV^T)\Lambda = \Lambda(PV^T). \end{aligned} \tag{6.14}$$

Let $W = PV^T$. Then W is commutative with Λ.

Suppose

$$\Lambda = \begin{pmatrix} \lambda_{i_1} I_1 & & & \\ & \lambda_{i_2} I_2 & & \\ & & \ddots & \\ & & & \lambda_{i_k} I_k \end{pmatrix},$$

where λ_{i_j} are distinct eigenvalues of A and I_j are identity matrix. Partition W with the same manner, we have

$$W = \begin{pmatrix} W_{1,1} & \cdots & W_{1,k} \\ W_{2,1} & \cdots & W_{2,k} \\ \vdots & \ddots & \vdots \\ W_{k,1} & \cdots & W_{k,k} \end{pmatrix}.$$

It follows from $W\Lambda = \Lambda W$ that $\lambda_j W_{i,j} = \lambda W_{i,j}$. Since $\lambda_i \neq \lambda_j$ if $i \neq j$, we have $W_{i,j} = 0$ for $i \neq j$. Therefore, it yields that

$$W\Lambda^{\frac{1}{2}} = \Lambda\frac{1}{2}W, \tag{6.15}$$

since Λ and $\Lambda\frac{1}{2}$ share the same structure. Finally, Noting that W is orthogonal, $\Lambda\frac{1}{2} = W^T\Lambda\frac{1}{2}W$.

Hence, $B = V^T\Lambda^{\frac{1}{2}}V = V^T(VP^T\Lambda^{\frac{1}{2}}PV^T)V = P^T\Lambda^{\frac{1}{2}}P = A$. $\square$

Part III

Topology

Chapter 7

Topological Spaces and Continuous Functions

7.1 Topological Spaces

Definition 7.1. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

(1) $\varnothing$ and X are in $\mathcal{T}$;

(2) the union of the elements of any subcollection of $\mathcal{T}$ is contained in $\mathcal{T}$;

(3) the intersection of elements of any finite subcollection of $\mathcal{T}$ is contained in $\mathcal{T}$.

A set X on which a topology $\mathcal{T}$ is specified is called a topological space, denoted by $(X, \mathcal{T})$. An element of $\mathcal{T}$ is called an open set of topological space $(X, \mathcal{T})$.

Sometimes, we denote topological space $(X, \mathcal{T})$ by X for convenience if the topology is not emphasized.

Open set in topological space plays an essential role in the study of topology. By a neighborhood of x in a topological space X we mean a open set containing x.

7.2 Construction of Topological Spaces

Sometimes we are given a collection $\mathcal{B}$ of subsets of a set X and we want to construct a topological space that contains $\mathcal{B}$. Here we restrict collection $\mathcal{B}$ with some requirements. Or often it is too tremendous for a topology for us to specify that we hope to characterize a topology by some "basic elements".

Definition 7.2 (Basis for a topology)**.** Let X be a set.A collection $\mathcal{B}$ of subsets of X (called basis elements) is called a basis for a topology if

(1) for each $x \in X$, there is at least one basis element B such that $x \in B$;

(2) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there exists a basis elements B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ obeys these two conditions, then we define a topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is an element of $\mathcal{T}$ if for any $x \in U$, there is a basis element B such that $x \in B$ and $B \subseteq U$. Obviously, $\mathcal{B} \subseteq \mathcal{T}$.

A typical and useful topological space is the metric space, induced by distance function. More concepts in topological spaces can be successively applied to metric spaces, see Chapter 10.

Theorem 7.1. *Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{B}$ is a basis of $\mathcal{T}$, then $\mathcal{T}$ is the class of unions of any elements of X,* i.e.,

$$\mathcal{T} = \left\{ \bigcup_{B \in \mathcal{B}'} B : \mathcal{B}' \subset \mathcal{B} \right\}.$$

Proposition 7.1. *For any subset $\mathcal{B}' \subseteq \mathcal{B}$, every element of $\mathcal{B}'$ belongs to $\mathcal{T}$, and therefore the intersection of all elements of $\mathcal{B}'$ belongs to $\mathcal{T}$. Conversely, suppose $U \in \mathcal{T}$. Then for any $x \in U$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Consequently, $U = \bigcup_{x \in U} B_x$.*

The preceding theorem tells us that given a basis, one can construct a topology from it by taking all intersections of subsets.

Definition 7.3 (Subbasis for a topology)**.** A *subbasis* $\mathcal{S}$ of a topological space X is a class of subsets in X if the union of all elements of $\mathcal{S}$ is X. The topology $\mathcal{T}$ generated by $\mathcal{S}$ is the class of all unions of any finite intersections of the elements of $\mathcal{S}$, *i.e.*,

$$\mathcal{T} = \left(\bigcup_{B \in \mathcal{B}'} B : \mathcal{B}' \subset \mathcal{B} \right),$$

where $\mathcal{B}$ is the basis generated by $\mathcal{S}$ in the following way:

$$\mathcal{B} = \{ \bigcap_{k=1}^{n} S_k : n \in \mathbb{N}^*; S_k \in \mathcal{S}, \text{ for all } k = 1, 2, \dots, n \}. \tag{7.1}$$

One can verify without much effort that $\mathcal{B}$ defined in Equation (7.1) is exactly a basis.

7.3 Closed Sets and Limit Points

Definition 7.4. A subset A of a topological space X is said to be closed if X^C is open.

Theorem 7.2. *Let X be a topological space. Then*

(1) $\varnothing$ and X are closed;

(2) arbitrary intersections of closed sets are closed;

(3) finite unions of closed sets are closed.

Proof. These are direct results from definition of closed set and DeMorgan's law. □

Definition 7.5. Let X be a topological space and A is a subset of X. The interior of A, denoted by $\mathring{A}$, is defined as the intersection of all open sets contained in A. The elements of $\mathring{A}$ are called interior points of A. The closure of A, denoted by $\overline{A}$, is defined as the union of all closed sets containing A.

A set in a topological space is open if and only if all the points of A are interior points, *i.e.*, $A = \mathring{A}$. Clearly, the interior of a set A is the maximal open set contained in A and the closure of A is the mimimal closed set containing A. Thus the following property holds:

$$\mathring{A} \subseteq A \subseteq \overline{A}.$$

If A is closed, then $A = \overline{A}$; if A is open, then $A = \mathring{A}$. The interior of A can also be denoted as $\operatorname{int} A$ and the closure $\operatorname{cl} A$.

Here are two ways to describe the closure of a set.

Theorem 7.3. *Let A be a subset of a topological space X. Then*

(1) $\operatorname{cl}(A)^C = \operatorname{int}(A^C)$.

(2) $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

(3) Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof. (1) $x \in \overline{A} \Leftrightarrow$ for any closed set C containing A, we have $x \in C \Leftrightarrow$ for any open set O such that $O \cap A = \varnothing$, we have $x \notin O \Leftrightarrow$ for any open set $O \subseteq A^C$, we have $x \notin O \Leftrightarrow x \notin \operatorname{int}(A^C)$.

Here is an alternative proof. $x \in \operatorname{cl}(A)^C \Leftrightarrow$ there exists a closed set C such that $A \subseteq C$ and $x \notin C \Leftrightarrow$ there exists an open set O such that $O \cap A = \varnothing$ and $x \in O \Leftrightarrow$ there exists an open set $O \subseteq \operatorname{int}(A^C)$ such that $x \in O \Leftrightarrow x \in \operatorname{int}(A^C)$.

(2) It would be convenient to prove the following equivalent problem statement: $x \notin \overline{A} \Leftrightarrow$ there exists a neighborhood U of x such that $U \cap A = \varnothing$.

A direct proof is as follows. $x \in \overline{A} \Leftrightarrow$ for any closed set $C \supset A$, $x \in C \Leftrightarrow$ for any closed set C such that $x \notin C$, $A \cap C^C \neq \varnothing \Leftrightarrow$ for any neighborhood U of x, $A \cap U \neq \varnothing$. □

With closed sets, we can now talk about limit points.

Definition 7.6. Let A be a subset of a topological space bX. We say $x \in X$ is a limit point (or cluster point, or point of accumulation) of A if every neighborhood of x intersets A in some point other than x itself. We denote the set of all limit points of A by A'. In other words, x is a limit point of A if x is a element of the closure of A x.

Note that a limit point of A needs not to be an element of A and the criterion of limit point requires a point other than x.

Proposition 7.2. *Let X be a topological space. If A and B are subsets of X, then*

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Proof. If $x \in \overline{A \cup B}$, then any neighborhood V of x intersects A or B. Assume that $x \notin \overline{A}$ and $x \notin \overline{B}$. Then there exist neighborhoods V_1 and V_2 such that $V_1 \cap A = \varnothing$ and $V_2 \cap B = \varnothing$. Therefore, the neighborhood $V_1 \cap V_2$ does not intersect $A \cup B$, is a contradiction. Thus $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

On the other hand, since $A \subseteq A \cup B$, we have $\overline{A} \subseteq \overline{A \cup B}$. Likewise, $\overline{B} \subseteq \overline{A \cup B}$. Hence, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. □

Theorem 7.4. *Let A be a subset of a topological space X. Then*

$$\overline{A} = A \cup A'.$$

Proof. Supposing $x \in \overline{A}$ and $x \notin A$. According to Theorem 7.3, every neighborhood of x intersects A. Thus $x \in A'$.

Conversely, if $x \in A'$, then the intersection of any neighborhood of x and A contains a point other than x itself. Thus, $x \in \overline{A}$. □

From the Theorem above, we have a very important characterization of closed sets, which is often used as an alternative definition of closed set.

Corollary 7.1. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. A is closed $\Leftrightarrow A = \overline{A} \Leftrightarrow A' \subseteq A$. □

7.3.1 Hausdorff Spaces

Definition 7.7. Let $\{x_1, x_2, \ldots\}$ be a sequence in a topological space. If for any neighborhood of $x \in X$, there is a positive integer N such that for any $n > N$, $x_n \in U$, then $\{x_1, x_2, \ldots\}$ is said to converge to x. In this case, we say $\{x_1, x_2, \ldots\}$ is convergent and x is a limit of $\{x_1, x_2, \ldots\}$.

In the case of metric space, there is a well-known conclusion: if a sequence is convergent, then it converges to a unique point. However, for general topological spaces, this is not true. For instance, we consider the tree points set $X = \{a, b, c\}$ equipped with the topology $\mathcal{T} = \big\{\{a, b, c\}, \{a, b\}, \{b, c\}, \{b\}, \varnothing\big\}$. Then the sequence $x_n = b$ converges to a, b and c! What's more, in this space, not all of the sets of a single point are closed.

This kind of topological spaces is abnormal and less important. Therefore, Hausdorff gave a condition to make limit unique, which is known as Hausdorff's axiom.

Definition 7.8. Let X be a topological space. X is said to be a Hausdorff space if for any two distinct points x_1 and x_2, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 7.5. *Every finite set in a Hausdorff space X is closed.*

Proof. Let $A = \{x_1, x_2, \ldots, x_n\}$ be a finite set in X and $x \notin A$. Then for all $m = 1, 2, \ldots, n$, there exists neighborhood U_m of x such that $x_m \notin U_m$. Thus the union $\bigcap_{m=1}^{n} U_m$ is a subset of A^C, whence A^C is open and A is closed. □

The converse, *i.e.*, a topological space whose finite subsets are closed is a Hausdorff space, is not true. In fact, this condition every finite subset is closed is named T_1 axiom and topological spaces satisfying T_1 axiom are named T_1 spaces. But we have little interst in T_1 spaces here, since only Hausdorff gives us the following property:

Theorem 7.6. *If $\{x_1, x_2, \ldots\}$ be a sequence in a topological space X, then $\{x_1, x_2, \ldots\}$ converges to a most one point.*

Proof. Supposing $\{x_1, x_2, \ldots\}$ converges to two distinct points x and x', then there exist two neighborhoods U_1 and U_2 of x and x', respectively, that are disjoint. This means that there exist two positive intergers N_1 and N_2 such that for any $n > N_1$, $x_n \in U_1$ and for any $n > N_2$, $x_n \in U_2$, which is impossible. □

For Hausdorff spaces, we write $\lim_{n\to\infty} x_n = x$ if the sequence $\{x_1, x_2, \ldots\}$ converges to x.

7.4 Continuous Functions

Definition 7.9. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Theorem 7.7. *Let X and Y be topological spaces. If $f : X \to Y$, then the following statements are equivalent*

(1) f is continuous.

(2) for every subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$.

(3) for every closed subset B of Y, $f^{-1}(B)$ is a closed subset in X.

(4) for every $x \in X$ and neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.

Definition 7.10. Let X and Y be topological spaces. A one to one function f that maps X onto Y is called a homeomorphism if both f and f^{-1} are continuous.

7.5 Metric Topology

Metric spaces is a very important kind of spaces in the study of analysis. Facts that matter in analysis are presented in Chapter 10. Here we mainly focus on the topological properties of metric spaces.

Chapter 8

Contability and Seperation Axioms

8.1 Countability Axioms

Definition 8.1. Let A, B be subsets in a topological space X. If $\operatorname{cl} A \supset B$, then A is said to be *dense* in B.

Definition 8.2. A topological space X is called separable if there exists a countable dense subset of X.

8.2 The Seperation Axioms

Definition 8.3. Let X be a topological space.

(1) (T_1 axiom) X is said to satisfy T_1 axiom if every set of single point in X is closed.

(2) (T_2 axiom) X is said to be a *Hausdorff space* if for any $x_1, x_2 \in X$, there exist disjoint neighborhoods U_1, U_2 of $x_{1,2}$ respectively.

The following proposition illustrates that T_2 axiom is stronger than T_1 axiom.

Proposition 8.1. *Every set of single point is closed in a Hausdorff space.*

Part IV

Mathematical Analysis

Chapter 9

Real Number Theory

9.1 Field

Since rational numbers constitute a set showing some arithmetic properties, we abstract these properties as the definition of the so-called field.

Definition 9.1. A field is a set F with two operations, called addition and multiplication, which satisfy the following field axioms:

A *Axioms for addition*

(1) *Close.* If $x, y \in F$, then $x + y \in F$;

(2) *Commutativity.* If $x, y \in F$, then $x + y = y + x$;

(3) *Associativity.* If $x, yandz \in F$, then $(x + y) + z = x + (y + z)$.

(4) There is an element 0 in F, named zero element, which satisfies that $x + 0 = x$ for all $x \in F$;

(5) For any $x \in F$, there is corresponding element $-x$, named negative element, which satisfies that $x + (-x) = 0$.

M *Axioms for multiplication*

(1) *Close.* If $x, y \in F$, then $xy \in F$;

(2) *Commutativity.* If $x, y \in F$, then $xy = yx$;

(3) *Associativity.* If $x, yandz \in F$, then $(xy)z = x(yz)$.

(4) There is an element 1 in F, named unit element, which satisfies that $1x = x$ for all $x \in F$;

(5) For any $x \in F$, there is corresponding element $\frac{1}{x}$, named inverse element, which satisfies that $x\frac{1}{x} = 1$.

D *The distributive law.* If $x, y, z \in F$, then $x * (y + z) = xy + xz$.

9.2 The Construction of Real Field

The main theory of this chapter is given as follows.

Theorem 9.1. *There exists an ordered field referred as $\mathbb{R}$, which has the least-upper-bound property and $\mathbb{R}$ contains $\mathbb{Q}$.*

Proof. We will complete our proof by constructing such a $\mathbb{R}$. We make $\mathbb{R}$ contains exactly some subsets of $\mathbb{Q}$, called cuts (denoted by Greek letters). If α is a cut in $\mathbb{R}$, then by definition, it has the following tree properties:

(1) α is not empty and $\alpha \neq \mathbb{Q}$;

(2) If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$;

(3) If $p \in \alpha$, then there exists some $r \in \alpha$ such that $p < r$.

From the definition of cuts, we can make some observations:

- Every cut of $\mathbb{Q}$ is bounded above;
- If $p \in \alpha$ and $p \notin \alpha$, then $p < q$;
- If $r \notin \alpha$ and $r < s$, then $s \notin \alpha$, *i.e.*, there is no maximum element of α.

In order to make $\mathbb{R}$ an ordered set, we define $\alpha < \beta$ as $\alpha \in \beta$. The addition of $\mathbb{R}$ is defined by

$$\gamma = \alpha + \beta = \{r + s : r \in \alpha, s \in \beta\}. \tag{9.1}$$

Making $\mathbb{R}$ fits the axioms of addition, we have to find a zero element by letting 0^* be the set of all negative number in $\mathbb{Q}$. Clearly, 0^* is a cut, thus in $\mathbb{R}$.

It would be harder to construct multiplication for $\mathbb{R}$. First, let us confine us on $\mathbb{R}^+ = \{\alpha \in \mathbb{R} : \alpha > 0^*\}$. Likewise, define $\alpha\beta$ as all the number less than rs for some $r \in \alpha$, $s \in \beta$, $r > 0$ and $s > 0$. Futhermore, we complete the definition of multiplication by letting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^* \text{ and } \beta < 0^*, \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^* \text{ and } \beta > 0^*, \\ -[\alpha(-\beta)] & \text{if } \alpha > 0^* \text{ and } \beta < 0^*. \end{cases} \tag{9.2}$$

And we define $1^* = \{p \in \mathbb{Q} : p < 1\}$.

Finally we define a mapping $\phi : \mathbb{Q} \to \mathbb{R}$ as $\phi(p) = \{r \in \mathbb{Q} : r < p\}$. Obviously, ϕ is a injection and is a homomorphism, *i.e.*, the following conditions is satisfied

(1) $\phi(rs) = \phi(r)\phi(s)$;

(2) $\phi(r + s) = \phi(r) + \phi(s)$;

(3) $r < s$ if and only if $\phi(r) < \phi(s)$.

In fact, these conditions can be expressed as that $\phi(\mathbb{Q})$ is isomorphic to $\mathbb{Q}$. So, in concerns of the arithmetic and ordering properties they are exactly the same.

Then by proving the following lemmas, we show that such $\mathbb{R}$ satisfies our requirements. The proof is rather tedious and omitted right now, may be presented someday. □

Lemma 9.1. $\mathbb{R}$ *is an ordered set and has least-upper-bound property.*

Proof. Let A be an nonempty bounded above subset of $\mathbb{R}$, β a upper bound of A and $\gamma = \prod_{\alpha \in A} \alpha$. Then it is trivial to prove that $\gamma = \sup A$. □

Lemma 9.2. *R with addition, zero element, multiplication and unit element defined above is a field.*

Chapter 10

Metric Spaces

10.1 The Induction of Metric Spaces

Definition 10.1. A distance on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following conditions:

(1) (definitive positiveness) $d(x, y) > 0$ if $x \neq y; d(x, x) = 0$

(2) (symmetry) $d(x, y) = d(y, x)$

(3) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$

for any x, y and $z \in X$. We call $d(x, y)$ is the distance of x and y.

Proposition 10.1. *Let (X, d) be a metric space. Then for any $x, x_1, y, y_1 \in X$,*

$$\left|d(x, y) - d(x_1, y_1)\right| \leq d(x, x_1) + d(y, y_1). \tag{10.1}$$

Proof. From triangle inequality, we have

$$\begin{aligned}\left|d(x, y) - d(x_1, y_1)\right| &= \left|d(x, y) - d(x_1, y) + d(x_1, y) - d(x_1, y_1)\right| \\ &\leq \left|d(x, y) - d(x_1, y)\right| + \left|d(x_1, y) - d(x_1, y_1)\right| \\ &\leq d(x, x_1) + d(y, y_1).\end{aligned}$$

□

Let (X, d) be a metric space. The ϵ-ball centered at $x \in X$ is denoted by

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

With the definition of distance, a topology on X can be induced by d with basis being the collection $\{B_d(x, \epsilon) : x \in X, \epsilon > 0, \epsilon \in \mathbb{R}\}$.

Theorem 10.1. *The collection $\{B_d(x, \epsilon) : x \in X, \epsilon > 0, \epsilon \in \mathbb{R}\}$ is a basis for a topology.*

Proof. $x \in B_d(x, \epsilon)$ for any $\epsilon > 0$, whence the first condition of the definition of basis. Before verifying the second condition, we first prove the facts: if $y \in B_d(x, \epsilon)$, then there is a basis element $B_d(y, \delta)$ such that $B_d(y, \delta) \subseteq B_d(x, \epsilon)$. Indeed, setting

$\delta < \epsilon - d(x, y)$, then for any $z \in B_d(y, \delta)$, $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta < \epsilon$. Hence $B_d(y, \delta) \subseteq B_d(x, \epsilon)$.

Now we start to prove the second condition. Let $B_1 = B_d(x_1, \epsilon_1)$ and $B_2 = B_d(x_2, \epsilon_2)$ be two basis elements, and $x \in B_1 \cap B_2$. Then there exist basis elements $B_d(x, \delta_1)$ and $B_d(x, \delta_2)$ such that $B_d(x, \delta_1) \in B_1$ and $B_d(x, \delta_2) \subseteq B_2$ respectively. Thus basis element $B_d(x, \min(\delta_1, \delta_2)) \in B_1 \cap B_2$ and the second condition holds. □

Definition 10.2. We call the topology induced by distance d the metric topology.

A topological space $(X, \mathcal{T})$ is called a metrizable space if $\mathcal{T}$ is induced by a distance d of X. A metric space (or distance space) is a metrizable space X together with a specific metric d that gives the topology of X, denoted by (X, d).

With the metric topology, we further introduce the concepts of open/closed set, limit point, neighborhood and convergence from topological space. Note that the following concepts are consistent with those in topological spaces, see Chapter 7.

Definition 10.3. Let X be a metric space. The points and sets mentioned below are understood to be elements and subsets of X.

(1) A *neighborhood* of p is set $N_r(p)$ consisting of all points in $B(p, r) = \{q \in X : d(p, q) < r\}$.

(2) A point p is a *limit point* of a set E if every neighborhood of p consists a point $q \in E$ such that $q \neq p$.

(3) A set E is *closed* if every limit point of E is a point of E.

(4) A point p is an *interior point* of a set E there exists a neighborhood of p contained in E.

(5) A set E is *open* if every point of E is an interior point of E.

Every metric space is a Hausdorff space. Indeed, supposing x_1 and x_2 are two distinct points of a metric space (X, d), then the neighborhoods $B_d(x, \epsilon)$ and $B_d(y, \epsilon)$ are disjoint with $\epsilon = \frac{d(x,y)}{2}$ by triangle inequality. Therefore, convergence in metric spaces makes sense.

Definition 10.4. Let (X, d) be a metric space. A sequence $\{x_1, x_2, \ldots\}$ of X is said to converge to $x \in X$ if $lim_{n \to \infty} d(x_n, x) = 0$. In this case, we say that sequence $\{x_1, x_2, \ldots\}$ is convergent and x is called a limit of $\{x_1, x_2, \ldots\}$.

We have some basic facts about convergence in metric spaces.

Proposition 10.2. *Suppose that sequences $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ of a metric space (X, d) converges to $x, y \in X$ respectively. Then*

(1) the limit of $\{x_1, x_2, \ldots\}$ is unique.

(2) any subsequence of $\{x_1, x_2, \ldots\}$ converges to x.

(3) $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$.

Proof. The proofs of (1) and (2) is trivial and have been omitted. (3) is straitforward result of Equation 10.1. □

Example 10.1. The sets $\mathbb{R}^d$ and $\mathbb{C}^d$ equipped with the Euclidean distance $d(x, y) = \sqrt{\sum_{i=1}^{d} (\xi_i - \zeta_i)^2}$ for $x = (\xi_1, \xi_2, \ldots, \xi_d)$ and $y = (\zeta_1, \zeta_2, \ldots, \zeta_d)$ is a metric space and convergence in them is equivalent to convergence by coordinates.

Example 10.2 (Discreate space D)**.** Let X be a nonempty set. The distance d is defined as

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Then $\lim_{n\to\infty} x_n = x$ if and only if there exists a positive interger N such that for any $n > N$, $x_n = x$.

Example 10.3. Let $C([a,b])$ be the set of all continuous functions defined on $[a,b]$, and the distance defind as $d(f,g) = max_{t\in[a,b]}\big|f(t) - g(t)\big|$. The convergence in $C([a,b])$ is equivalent to uniform convergence.

Example 10.4 (The space S of Lebesgue mesurable functions)**.** Let S be the set of all lebesgue measurable functions on a measurable set set E such that $0 < m(E) < \infty$ where m is the Lebesgue measure. The distance is $d(f,g) = \int_E \frac{\big|f(t)-g(t)\big|}{1+\big|f(t)-g(t)\big|}\,\mathrm{d}t$. We contend that convergence in space S is equivalent to convergence in measure.

Indeed, Let $\{f_1, f_2, \ldots\}$ be a sequence in S and $lim_{n\to\infty} = f$. Supposing $0 < \epsilon < 1$, then

$$\begin{aligned} \int_E \frac{|f_n - f|}{1+|f_n - f|}\,\mathrm{d}t &\geq \int_{E[\![f_n - f|>\epsilon]} \frac{|f_n - f|}{1+|f_n - f|}\,\mathrm{d}t \\ &\geq \int_{E[\![f_n - f|>\epsilon]} \frac{\epsilon}{1+\epsilon}\,\mathrm{d}t \geq \frac{\epsilon}{2} m\left(m[\![f_n - f| > \epsilon]\right). \end{aligned}$$

Thus $m\left(m[\![f_n - f| > \epsilon]\right) \leq \frac{2d(f_n,f)}{\epsilon} \to 0$ as $n \to \infty$.

Conversely, suppose that $\{f_1, f_2, \ldots\}$ converges to $f \in S$ such in measure. Setting $\epsilon > 0$, by assumption, $\lim_{n\to\infty} m\left(E[\![f_n - f| > \epsilon]\right) = 0$. That is, there exists a positive interger N such that for any $n > N$, $m\left(E[\![f_n - f| > \epsilon]\right) < \epsilon$. Thus, for $n > N$,

$$\begin{aligned} d(f_n, f) &= \int_E \frac{|f_n - f|}{1+|f_n - f|}\,\mathrm{d}t \\ &= \int_{E[\![f_n - f|>\epsilon]} \frac{|f_n - f|}{1+|f_n - f|}\,\mathrm{d}t + \int_{E[\![f_n - f|<\epsilon]} \frac{|f_n - f|}{1+|f_n - f|}\,\mathrm{d}t \\ &\leq \epsilon + \epsilon m(E) = \epsilon(1 + m(E)). \end{aligned}$$

Therefore, $\lim_{n\to\infty} f_n = f$.

For metric space, we have two equivalent definitions of continuous function.

Theorem 10.2. *Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A function $f : X_1 \to X_2$ is continuous if and only if for any $x \in X$ and $\epsilon > 0$, there exists a $\delta > 0$ such that for any $y \in X$ satisfying $d_1(x,y) < \delta$,*

$$d_2(f(x), f(y)) < \epsilon.$$

Proof. $\Leftarrow$ Let V be a open subset Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exits a open ball $B_{d_Y}(f(x), \epsilon) \in V$. Hence, by assumption, there exits a open ball $B_{d_X}(x, \delta)$ such that for any $y \in B_{d_X}(x,\delta)$, $y \in B_{d_Y}(f(x), \epsilon) \subseteq V$. That is, for any $B_{d_X}(x, \epsilon) \subseteq f^{-1}(V)$, whence f is continuous.

$\Rightarrow$ Let f be a continuous function and ϵ be an arbitrary positive real number. By definition, $f^{-1}(B_{d_Y}(f(x), \epsilon))$ is an open set for each $x \in X$. Thus there exists open ball $B_{d_X}(x, \delta) \subseteq f^{-1}(B_{d_Y}(f(x), \epsilon))$. $\square$

Theorem 10.3. *Let X and Y be topological spaces and f be a function from X to Y. If X is a metric space, then f is continuous if and only if for every convergent sequence $\{x_1, x_2, \ldots\} \subseteq X$ such that $\lim_{n\to\infty} x_n = x$, the sequence $\{f(x_1), f(x_2), \ldots\} \subseteq Y$ converges to $f(x)$.*

Before we start to prove this theorem, we introduce the following lemma:

Lemma 10.1 (Sequence Lemma)**.** *Let X be a topological space and $A \subseteq X$. If sequence $\{x_1, x_2, \ldots\} \subseteq X$ converges to $x \in X$, then $x \in \overline{A}$. If X is a metric space, then the converse is true.*

Proof. The first statement is trivial. Now we prove the second statement. Suppose that X is a metric space and $x \in \overline{A}$. Then for every $n \in \mathbb{N}^*$, the neighborhood $B(x, \frac{1}{n})$ intersects A. Thus, for each n, we choose a element from $B(x, \frac{1}{n}) \cap A$ as x_n. Obviously, the sequence $\{x_1, x_2, \ldots\}$ converges to x. □

Proof of Theorem 10.3. $\Leftarrow$ Suppose that f is continuous and sequence $\{x_1, x_2, \ldots\} \subseteq X$ converging to x. Then for every neighborhood V of $f(x)$, $U = f^{-1}(V)$ is a neighborhood of x. Thus there exists $N \in \mathbb{N}^*$ such that for $n > N$, $x_n \in V$. That is, for $n > N$, $f(x_n) \in V$, whence $\{f(x_1), f(x_2), \ldots\}$ converges to $f(x)$.

$\Rightarrow$ Suppose that the convergent sequence condition is satisfied. It suffices to prove $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq X$. Let $y \in f(\overline{A})$. There exists $x \in \overline{A}$ such that $f(x) = y$. Thus, there exists a sequence $\{x_1, x_2, \ldots\}$ converging to x, by by Sequence Lemma 5.1. Therefore, $\{f(x_1), f(x_2), \ldots\}$ converges to $f(x) = y$, whence $y \in \overline{f(A)}$. □

As homeomorphism in topological space preserves the topological properties, we define isometry in metric space that preserves the metric properties.

Definition 10.5. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A function $f : X_1 \to X_2$ is said to be an isometry (or isometric mapping) if for any $x_1, x_2 \in X_1$, $d_1(x_1, x_2) = d_2(f(x_1), f(x_2))$. If there exists an isometry f maps X_1 onto X_2, then we say X_1 is isometric to X_2.

Clearly, isometry is a equivalent relation and a isometric mapping is injective.

10.2 Complete Metric Spaces

10.2.1 Cauchy Sequences

Definition 10.6. A sequence $\{x_1, x_2, \ldots\}$ in a metric space (X, d) is a *Cauchy sequence* if for any $\epsilon > 0$, there exists an $N \in \mathbb{N}^*$ such that

$$d(x_m, x_n) < \epsilon, \text{ for any } n, m \geq N.$$

There is a geometry perspective of Cauchy sequences.

Definition 10.7. The diameter of a nonempty set E is defined as

$$\operatorname{diam} E = \sup_{p,q \in E} d(p, q). \tag{10.2}$$

By definition, a sequence $\{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{N\to\infty} \operatorname{diam} E_N = 0,$$

where $E_N = \{p_i : i \geq N\}$.

Proposition 10.3. *Every convergent sequence is a Cauchy sequence.*

Proof. The proof is trivial. □

Definition 10.8. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Theorem 10.4. *If every Cauchy sequence in a metric space X has an convergent subsequence, then X is complete.*

Proof. Supposing that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X, we contend that if $\{x_n\}_{n=1}^{\infty}$ has subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to some $x \in X$, then $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$. Indeed, for any $\epsilon > 0$, there exist $N_1, N_2 > 0$ such that

$$\begin{aligned} d(x_n, x_m) &< \frac{\epsilon}{2}, \quad \text{for every } n, m \geq N_1, \\ d(x_{n_k}, x) &< \frac{\epsilon}{2}, \quad \text{for every } k > N_2. \end{aligned}$$

Therefore, it yields that for given $m > \max\{N_1, N_2\}$,

$$d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) < \epsilon, \quad \text{for every } n > \max\{N_1, N_2\}.$$

This has completed the proof. □

10.2.2 Completion of Metric Spaces

There is a very interesting fact about metric space: every metric space can is isometric to a dense subspace of a complete.

Theorem 10.5 (Completion of Metric Spaces)**.** *For every metric space (X, d), there exists a complete metric space $\tilde{X}, \tilde{d}$ such that (X, d) is isometric to a dense subspace in $(\tilde{X}, \tilde{d})$. The subspace $(\tilde{X}, \tilde{d})$ is unique up to an isometric mapping.*

Proof. Let $\tilde{X}$ be the set of all Cauchy sequences in X and the distance of $\tilde{x} = \{x_n\}$ and $\tilde{y} = \{y_n\}$ is defined as

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n \to \infty} d(x_n, y_n). \tag{10.3}$$

Step 1. The limit in (10.3) exists. Assuming that $\epsilon > 0$, by Equation (10.1), we have

$$\big|(x_n, y_n) - d(x_m, y_m)\big| \leq d(x_n, x_m) + d(y_n, y_m) \leq \epsilon.$$

Thus $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{N}$ and thus has a limit. For two Cauchy sequences $\tilde{x} = \{x_n\}$ and $\tilde{y} = \{y_n\}$, we regard $\tilde{x}$ and $\tilde{y}$ as the same element of $\tilde{X}$ if $d(\tilde{x}, \tilde{y})$.

Step 2. The metric d is well-defined. That is, for elements $\tilde{x} = \tilde{x}'$ and $\tilde{y} = \tilde{y}'$ of $\tilde{X}$, $\lim_{n \to \infty} \tilde{d}(\tilde{x}, \tilde{y}) = d(\tilde{x}', \tilde{y}')$. Indeed, $\big|d(x_n, y_n) - d(x_n', y_n')\big| \leq d(x_n, x_n') + d(y_n, y_n') \to 0$ when $n \to \infty$.

Step 3. The metric space $(\tilde{X}, \tilde{d})$ is complete. Suppose that $\{\tilde{x}_n\}$ is a Cauchy sequence in $\tilde{X}$. For any $n \in \mathbb{N}^*$, there exists $N_n \in \mathbb{N}^*$ such that

$$d(x_{nk} - x_{nk'}) < \frac{1}{n} \quad \text{for } k, k' > N.$$

Let $y_n = x_{n(N_n+1)}$, and set $\tilde{y} = \{y_n\}$. Then $\tilde{y}$ is an element of $\tilde{X}$ and $\lim_{n\to\infty} \tilde{x}_n = \tilde{y}$. Indeed, for any $m, n \in \mathbb{N}$ and $k > \max(N_n, N_m)$,

$$\begin{aligned} d(y_n, y_m) &= d(x_{n(N_n+1)}, x_{m(N_m+1)}) \\ &< d(x_{n(N_n} + 1), x_{nk}) + d(x_{nk}, x_{mk}) + d(x_{m(N_m+1)}, x_k) \\ &< \frac{1}{n} + d(x_{nk}, x_{mk}) + \frac{1}{m}. \end{aligned}$$

Thus $d(y_n, y_m) \to 0$ as $n, m \to \infty$. And

$$\begin{aligned} \tilde{d}(\tilde{y}, \tilde{x}_n) &\le \tilde{d}(\tilde{y}, \tilde{y}_n) + \tilde{d}(\tilde{y}_n, \tilde{x}_n) \\ &< \lim_{k\to\infty} d(y_k, y_n) + \frac{1}{n}. \end{aligned}$$

tends to 0 as $n \to \infty$.

Step 4. (X, d) is isometric to a dense subspace in $(\tilde{X}, \tilde{d})$. Let G be the set of all constant sequences in $\tilde{X}$ and define

$$\begin{aligned} T : X &\to \tilde{X}, \\ x &\mapsto \tilde{x} = \{x, x, \ldots\}. \end{aligned}$$

Then obviously $d(x, y) = \tilde{d}(\tilde{x}, \tilde{y})$, whence (X, d) is isometric to $(T(X) = G, \tilde{d})$. Supposing that $\tilde{y} = \{y_n\}$ is a Cauchy sequence in $\tilde{X}$, then $\tilde{y}_n = \{y_n, y_n, \ldots\}$ is a constant sequence. Observing that

$$\tilde{d}(\tilde{y}, \tilde{y}_n) = \lim_{m\to\infty} d(y_m, n),$$

we have $\tilde{y}_n \to \tilde{y}$ as $n \to \infty$, which means $G = T(X)$ is dense in $\tilde{X}$.

Step 5. Suppose that $(\tilde{X}, \tilde{d})$ and $(\tilde{X}', \tilde{d}')$ are two complete metric spaces and there exist two isometric mappings T and T' such that $T : X \to \tilde{X}$ and $T' : X \to \tilde{X}'$ are dense in $\tilde{X}$ and $\tilde{X}'$ respectively. We contend that $(\tilde{X}, \tilde{d})$ is isometric to $(\tilde{X}', \tilde{d}')$. Indeed, for any $\tilde{x} \in \tilde{X}$, there exists a sequence $\{x_n\}$ such that $\{Tx_n\}$ converges to $\tilde{x}$. Define $T_1; \tilde{X}_1 \to \tilde{X}_2, \tilde{x} \mapsto \lim_{n\to\infty} T'x_n$. For any $\tilde{x}$ and $\tilde{y}$ of $\tilde{X}$,

$$\begin{aligned} \tilde{d}'(T_1\tilde{x}, T_1\tilde{y}) &= \lim_{n\to\infty} \tilde{d}'(T'\tilde{x}_n, T'\tilde{y}_n) = \lim_{n\to\infty} d(x_n, y_n) \\ &= \lim_{n\to\infty} \tilde{d}(Tx_n, Ty_n) = \tilde{d}(\lim_{n\to\infty} Tx_n, \lim_{n\to\infty} Ty_n) \\ &= \tilde{d}(\tilde{x}, \tilde{y}). \end{aligned}$$

Thus T_1 is an isometry.

The conclusion now has been established. □

10.3 Compact Sets

Definition 10.9. Let $\{G_\alpha\}$, $\alpha \in I$ be a collection of open subsets of metric space X and E a set in X. Then $\{G_\alpha\}$ is a open cover of E if $E \in \bigcup_{\alpha\in I} G_\alpha$.

Definition 10.10. A subset K in a metric space X is a compact if every open cover of K contains a finite subcover. In other words, for every open cover $\{G_\alpha : \alpha \in I\}$ of K there exist a finite subcover $\{G_{\alpha_n} : 0 \le n \le N, \alpha_n \in I\}$ such that

$$A \subseteq \bigcup_{n=1}^{N} G_{\alpha_n}.$$

While the concepts of closed set and open set is relative to the metric space we considered, compactness is not. We have the following theorem.

10.3.1 Basic Properties of Compact Sets

Theorem 10.6. *Suppose $K \subseteq Y \subseteq X$. Then K is compact relative to X if and only if K is compact relative to Y. Here, compact relative to set Y means that the open cover is considered in Y.*

Proof. $\Rightarrow$ Suppose K is compact relative to X and $\{U_\alpha\}$ is an open cover of K relative to Y, *i.e.*, U_α is open relative to Y and $K \subseteq \bigcup_\alpha U_\alpha$. From the previously presented theorem, we know for any U_α, there exists a corresponding open set $V_\alpha \in X$ which makes $U_\alpha = V_\alpha \cap Y$. So we can choose a finite open cover from $\{V_\alpha\}$, say $V_1, V_2, \ldots, V_n$. Evidently, the corresponding $\{U_1, U_2, \ldots, U_n\}$ where $U_i = V_i \cap Y$ is an open cover of K.

$\Leftarrow$ Conversely, suppose K is compact relative to Y and $\{V_\alpha\}$ is an open cover relative to X. Then the collection of all $U_\alpha = V_\alpha \cap Y$ is an open cover of K. Thus we can draw a finite open subcover from U_α, say $U_1, U_2, \ldots, U_n$, which indicates that $V_1, V_2, \ldots, V_n$ where $U_i = V_i \cap Y$ is also a finite open subcover of K. $\square$

Compactness is a stronger requirement to a set than the closedness.

Theorem 10.7. *Compact sets in a metric space are closed and bounded.*

Proof. Assume X is a metric space and K is a compact set in X. In order to go to a contradiction, suppose K is not closed, *i.e.*, there exists a limit point p of K such that $p \notin K$. Then $\left\{B(q, \frac{1}{2}d(q,p)) : q \in k\right\}$ is an open cover of K. Note that $p \notin \bigcup_{p\in K} B\left(p, \frac{1}{2}d(p,q)\right)$. Since K is compact, we can draw a finite subcover from $\left\{B(q, \frac{1}{2}d(q,p)) : q \in K\right\}$, say $\{B(q_i, \frac{1}{2}d(q_i,p)) : q_i \in K, i = 1, 2, \ldots, n\}$. Let $\delta = \min_i \frac{1}{2}d(p, q_i)$ and $N = B(p, \delta)$. Then we obtain $N \cap K = \varnothing$, which is a contradiction to the condition that p is a limit point of K.

Since $\left\{B(q, \frac{1}{2}) : q \in K\right\}$ is a open cover of K, there exists a subcover

$$\left\{B(q_i, \frac{1}{2}) : 0 \leq i \leq N, q_i \in K\right\}$$

of K, which is obviously bounded. $\square$

Remark 6. Note the theorem above is quite general, illustaring that compactness is a stronger condition than closedness. The converse holds true for finite-dimensional metric space. However, for infinite-dimensional space, the converse is not true.

Theorem 10.8. *Closed subset of a compact set is compact.*

Proof. Suppose $A \subseteq X$ be compact and $B \subseteq A$ be closed. Let $G = \{G_\alpha : \alpha \in I\}$ be an open cover of B. As B is compact, B is closed, whence B^C is open. Thus $G \cup \left\{B^C\right\}$ is an open cover of A, and then it contains a finite subcover G' of A. Hence $G' \setminus \left\{B^C\right\} \subseteq G$ is a finite subcover of B. $\square$

Theorem 10.9. *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of any finite subcollection of $\{K_\alpha\}$ is nonempty, then*

$\bigcap_\alpha K_\alpha$ is nonempty.

Proof. Fix a subset K_1 from $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^C$. With no loss of generality, suppose that for any point $p \in K_1$, there exists a subset K_α such that $p \notin K_\alpha$. Then G_α is an open cover of K_1; thus we can choose a finite number of indices such that $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ is an open subcover of K_1, *i.e.*, $K_1 \subseteq \bigcup_{i=1}^n G_{\alpha_i}$, which means $K_1 \cap (\bigcap_{i=1}^n K_{\alpha_i})$ is empty, contradicting the condition that the intersection of any finite subcollection of K_α is nonempty. □

Remark 7. This theorem is the key to prove the so-called nested interval theorem when we set the metric space to $\mathbb{R}$.

The next theorem is an extension to the classical nested closed ball theorem.

Theorem 10.10. *Let K_n is a sequence of compact sets in a metric space X such that $K_n \supset K_{n+1}, n = 1, 2, \ldots$. If*

$$\lim_{n\to\infty} \operatorname{diam} K_n = 0, \tag{10.4}$$

then $K = \bigcap_{n=1}^\infty K_n$ contains exactly one point.

Proof. By Theorem 10.9, $\bigcap_{n=1}^\infty K_n$ is nonempty. If K consists of more than one point, then $\operatorname{diam} K = a > 0$. Since $K \subseteq K_n$ for all positive integer n, we have $0 < a \le \operatorname{diam} K < \operatorname{diam} K_n$, which is a contradiction to the condition. □

Theorem 10.11. *Let X and Y be two metric spaces and $f : X \to Y$ be a continuous mapping. If $A \subseteq X$ is compact in X, then $f(A)$ is compact in Y.*

Proof. This is trivial with the definition 7.9 of continuous function. □

Corollary 10.1. *Let $A \subseteq X$ be compact and $f : A \to \mathbb{R}$ be continuous. Then f achieves its maximum and minimum on A.*

10.3.2 Compactness in Metric Spaces

Definition 10.11 (sequentially compact)**.** Let X be a metric space. A subset A in X is said to be *sequentially compact* if every sequence in A has a convergent subsequence.

Definition 10.12 (ϵ-net, totally bounded)**.** Let A, B be subsets in X and $\epsilon > 0$. If

$$A \subseteq \bigcup_{b\in B} B(b, \epsilon),$$

then we call B an *ϵ-net* of A. If for any $\epsilon > 0$, A has a finite ϵ-net, then A is said to be *totally bounded*.

Clearly, a totally bounded set is bounded, while the converse may not be true. For instance, consider the open ball $B(0, 2)$ in the infinite-dimensional metric space l^∞. Let $e_i = (0, 0, \ldots, 1, \ldots)$ where the the nonzero element appear in the i-th entry. Then $e_i \in B(0, 1)$ for $i \in \mathbb{N}^*$ and $d(e_i, e_j) = 1$ if $i \ne j$. Thus for all $\epsilon < 1$, there is no ϵ-net of A.

Theorem 10.12. *Let X be a complete metric space and $A \subseteq X$ be closed. Then the following are equivalent*

(1) A is compact.

(2) A is totally bounded.

(3) A is sequentially compact.

Proof. (1) $\implies$ (2) Supposing that A is compact and $\epsilon > 0$, then

$$\{B(a, \epsilon) : a \in A\}$$

is an open cover of A, and thus has a finite subcover, which is a ϵ-net of A. It follows that A is totally bounded.

(1) $\implies$ (3) [1] Supposing A is compact and $\{x_n\}_{n=1}^{\infty}$ is a sequence in A consisting of infinitely many distinct elements of X. We want to prove that A has a limit point of $\{x_n\}_{n=1}^{\infty}$. Assume to the contrary that every $a \in A$ is not a limit point of $\{x_n\}_{n=1}^{\infty}$. Then for every $a \in A$, there exists an $\epsilon_a > 0$ such that

$$B(a, \epsilon_a) \cap \{x_n\}_{n=1}^{\infty} \text{ is finite.}$$

Thus $\{B(a, \epsilon_a) : a \in A\}$ is an open cover of A, which has no finite subcover, contradicting the assumption.

(3) $\implies$ (2) Suppose that A is sequentially compact and $\epsilon > 0$. Assume that A does not have a finite ϵ-net. Choose an arbitrary $x_1 \in A$ and consider $B(x_1, \epsilon)$. Then $A \not\subseteq B(x_1, \epsilon)$. Choose $x_2 \in A$ such that $x_2 \notin B(x_1, \epsilon)$. Repeatedly, we obtain a sequence $\{x_n\}$ in A that has no limit point, which is a contradiction.

(2) $\implies$ (1) [2] Let A be totally bounded and closed. Assume A is not compact. Then there exists an open cover $\{G_\alpha : \alpha \in I\}$ that does not have a finite subcover of A. Firstly, as A is totally bounded, there exists a finite 1-net covering A, *i.e.*, $A \subseteq \bigcup_{i=1}^{n} B(x_i, 1)$. Thus $A = \bigcup_{i=1}^{n} \left[A \cap \overline{B(x_i, 1)}\right]$. Hence, there exists some x_i such that $A_1 = A \cap \overline{B(x_i, 1)}$ cannot be covered by any finitely many G_α. Secondly, as $A_1 \subseteq A$ is totally bounded, there exists a finite $\frac{1}{2}$-net $\{B(y_i, \frac{1}{2}) : i = 1, 2, \dots, n_1\}$ covering A_1. Then $A_1 = \bigcup_{i=1}^{n} \left[A_1 \cap \overline{B(y_i, \frac{1}{2})}\right]$. Likewise, there exists y_i such that $A_2 = A_1 \cap \overline{B(y_i, \frac{1}{2})}$ cannot be covered by finitely many G_α. Repeating this process, we get a collection $\{A_i\}$ such that for all $n \in \mathbb{N}$

$$\begin{cases} A_1 \supset A_2 \supset A_3 \supset \cdots, \\ \operatorname{diam}(A_n) \leq \dfrac{2}{n}, \\ A_n \text{ cannot be covered by any finitely many } G_\alpha. \end{cases}$$

Choosing $x_n \in A_n$, then for $n > m$, $x_n, x_m \in A_m$, which implies $d(x_m, x_n)\frac{2}{m}$ and thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, we assume that $\{x_n\}$ converges to $x \in X$. Since A is closed and $x_n \in A$, it follows that $x \in A$. Hence $x \in G_\alpha$ for some $\alpha \in I$. That G_α is open yields that there exists $B(x, \delta) \in G_\alpha$ and there exists $N \in \mathbb{N}$ such that for every $n > N$,

$$d(x_n, a) < \frac{\delta}{2}.$$

[1] Requires only compactness of A. Compactness is stronger than sequentially compactness.

[2] Requires completeness of X and Closedness of A.

Therefore, for $n > \max\left\{\frac{4}{\delta}, N\right\}$ and $x \in A_n$

$$d(x,a) \le d(x,x_n) + d(x_n,a) < \frac{2}{n} + \frac{\delta}{2} < \delta.$$

i.e., $A_n \subseteq B(a,\delta) \subseteq G_\alpha$. which contradicts the hypothesis. □

Corollary 10.2. *A bounded subset F in $\mathbb{R}^n$ is totally bounded.*

Proof. F is bounded in $\mathbb{R}^n \implies \bar{F}$ is a compact set in $\mathbb{R}^n \implies \bar{F}$ is totally bounded $\implies F$ is totally bounded. □

10.3.3 Arzelà–Ascoli Theorem

As metric space $C[a,b]$ is of fundamental importance in analysis, there is a theorem describing the compactness of $C[a,b]$, which is the well- known Arzelà–Ascoli Theorem. Before stating the theorem, we need the following definitions:

Definition 10.13 (uniformly bounded, equicontinuous)**.** Let F be a subset of $C[a,b]$ where $-\infty < a < b < \infty$. We call functions in F are *uniformly bounded* if there exists a constant $M \ge 0$ such that for every $f \in F$,

$$\left|f(x)\right| \le M, \quad \text{for any } x \in [a,b].$$

F is called *equicontinuous* if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $t_1, t_2 \in [a,b]$ such that $|t_1 - t_2| < \epsilon$ and $f \in F$,

$$\left|f(t_1) - f(t_2)\right| < \epsilon.$$

Theorem 10.13 (Arzelà–Ascoli Theorem)**.** *Let $F \subseteq C[a,b]$ be closed where $-\infty < a < b < \infty$. Then F is compact if and only if F is uniformly bounded and equicontinuous.*

Proof. By Theorem 10.12, it suffices to prove that F is totally compact if and only if F is uniformly bounded and equicontinuous.

$\Rightarrow$ Suppose F is totally bounded. Then F is bounded, implying uniform boundedness of F. Setting $\epsilon > 0$, there exist $f_1, f_2 \ldots, f_m \in C[a,b]$ such that

$$F \subseteq \bigcup_{i=1}^{m} B(f_i, \frac{\epsilon}{3}).$$

Noting that for every $i \in \{1,2,\ldots,m\}$, f_i is uniformly continuous, thus there exists $\delta_i > 0$ such that for $t_1, t_2 \in [a,b]$ satisfying $|t_1 - t_2| < \delta_i$,

$$\left|f_i(t_1) - f_i(t_2)\right| < \epsilon.$$

Take $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Thus for every $f \in F$, there exists $i \in \{1,2,\ldots,m\}$ such that $f \in B(f_i, \frac{\epsilon}{3})$. This is, for $t_1, t_2 \in [a,b]$ satisfying $|t_1 - t_2| < \delta$,

$$\left|f(t_1) - f(t_2)\right| \le \left|f(t_1) - f_i(t_1)\right| + \left|f_i(t_1) - f_i(t_2)\right| + \left|f_i(t_2) - f(t_2)\right| < \epsilon.$$

Hence, F is equicontinuous.

$\Leftarrow$ Suppose that F is uniformly bounded and equicontinuous. Letting $\epsilon > 0$, as F is equicontinuous, there exists $\delta > 0$ such that for any $|t_1 - t_2| < \delta$ and $f \in F$,

$$\big|f(t_1) - f(t_2)\big| < \frac{\epsilon}{5}.$$

Uniformly divide the closed interval $[a, b]$ into $a = t_0 < t_1 < t_2 < \cdots < t_m = b$ such that

$$|t_{i+1} - t_i| = \frac{b-a}{m} < \delta.$$

Hence, for every $i \in \{1, 2, \dots, m\}$, $t \in [t_i, t_{i+1}]$ implies $\big|f(t) - f(t_i)\big| < \frac{\epsilon}{5}$. Consider the set

$$E = \big\{(f(t_0), f(t_1), \dots, f(t_m)) : f \in \mathcal{F}\big\} \subseteq \mathbb{C}^m.$$

By uniform boundedness of F, E is bounded and thus totally bounded with Corollary 10.2. Therefore, there exists $f_1, f_2, \dots, f_p \in F$ such that for any $f \in F$ and $i \in \{1, 2, \dots, m\}$, there exists $j \in \{1, 2, \dots, p\}$ such that

$$\big|f(t_i) - f_j(t_i)\big| < \frac{\epsilon}{5}.$$

We claim that

$$F \subseteq \bigcup_{i=1}^{p} B(f_i, \epsilon).$$

Indeed, for every $f \in F$, there exists $f_i \in F$ such that $\big|f(t_j) - f_i(t_j)\big| < \frac{\epsilon}{5}$ for any $j \in \{1, 2, \dots, m\}$. Supposing $t \in [a, b]$ and $t \in [t_j, t_{j+1}]$, then

$$\big|f(t) - f_i(t)\big| \leq \big|f(t) - f(t_j)\big| + \big|f(t_j) - f_i(t_j)\big| + \big|f_i(t_j) - f_i(t)\big| \leq \frac{3}{5}\epsilon < \epsilon.$$

Thus we have proven that F is totally bounded. □

10.4 Cauchy Sequences

Definition 10.14. By the upper limit of a sequence $\{x_k\}$, we mean

$$l = \lim_{n\to\infty} \sup_{k\geq n} x_k, \tag{10.5}$$

denoted by $\limsup_{n\to\infty} x_n$. If $\{x_k\}$ is unbounded above, then $\limsup_{n\to\infty} x_n = \infty$. Likewise, the lower limit of $\{x_k\}$ is

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k. \tag{10.6}$$

Let $b_n = \sup_{k>n} x_k, \dots\}$. We have $b_n > b_{n+1}$ for any n; so $\lim_{n\to\infty} b_n = \inf b_n$ is either a finite number or ∞.

Chapter 11

Fourier Analysis

11.1 Fejér's Theorem

In this section, we consider continuous function f on circle.

Definition 11.1. Suppose $\sum c_n$ is a complex valued series and s_n the partial sum of c_n. The Cesàro mean of sequence $\{s_n\}$ is defined as

$$\sigma_N = \frac{s_1 + s_2 + \ldots + s_N}{N}. \tag{11.1}$$

Definition 11.2. The Fejér sum of f is the Cesàro mean of the Fourier series of f, *i.e.*,

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + S_1(f)(x) + \ldots + S_{N-1}(f)(x)}{N}. \tag{11.2}$$

Fejér's theorem show us the completeness of trigonometric polynomial baasis.

Theorem 11.1. *Suppose f is a continuous function on circle, then the Fejér sum of f converges uniformly to f.*

Remark 8. This theorem is stronger than the Wierstrass's theorem; the latter one only prove the existence of trigonometric sequence that converges uniformly to any continuous function but the former one exactly specify such a sequence.

Part V

Real Analysis

Chapter 12

Measurable Sets

12.1 Classes of Sets

To establish measure on a set X, we should give a set of subsets of X (also call a set of sets a class) that are measurable. For the class of measurable sets, we should impose some constraints on them.

Definition 12.1. A π-system on a set X is a nonempty class $\mathcal{P}$ of subsets in X that are closed under intersections, *i.e.*,

$$A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}.$$

Example 12.1. The class $\mathcal{P}_{\mathbb{R}} = \{(-\infty, a] : a \in \mathbb{R}\}$ is a π-system. In fact, class that consists of all open intervals is a π-system. So do left open right closed intervals and left closed right open intervals.

Definition 12.2. A *semiring of sets* is a π-system $\mathcal{D}$ on a set X such that for any $B \subseteq A \in \mathcal{D}$, there exists a finite disjoint subclass $\{C_k \in \mathcal{D} : k = 1, 2, \ldots, n\}$ of $\mathcal{D}$ such that

$$A \backslash B = \bigcup_{k=1}^{n} C_k.$$

Since semirings are π-systems, for any A, B in a semiring $\mathcal{D}$, $A \backslash B = A \backslash (A \cap B) = \bigcup_{k=1}^{n} C_k$, $C_k \in \mathcal{D}, k = 1, 2, \ldots, n$.

Example 12.2. The class $\mathcal{D}_{\mathbb{R}} = \{(a, b] : a, b \in \mathbb{R}\}$ is a semiring on $\mathbb{R}$.

Example 12.3. Let X be a finite set. Then the class of all sets of single point is a semiring on X.

Definition 12.3. A *ring of sets* on a set X is a nonempty class $\mathcal{R}$ of sets of X that is closed under the union and difference of pairs of sets, *i.e.*,

$$A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}, A \backslash B \in \mathcal{R}.$$

Example 12.4. The class $\mathcal{R}_{\mathbb{R}} = \bigcup_{n=1}^{\infty} \left\{\bigcup_{k=1}^{n} (a_k, b_k] : a_k, b_k \in \mathbb{R}\right\}$ is a ring on $\mathbb{R}$.

Proposition 12.1. *A ring is a semiring.*

Proof. Let A, B be elements of a ring $\mathcal{R}$. Then $A \backslash B \in \mathcal{R}$, $B \backslash A \in \mathcal{R}$ and $A \cap B = A \backslash (A \backslash B)$. $\square$

Definition 12.4. An algebra (or Boolean algebra) over a set X is a nonempty π-system $\mathcal{A}$ on X that is closed under the complement, *i.e.*, $E\ in\mathcal{A} \implies A^C \in \mathcal{A}$.

Proposition 12.2. *A ring $\mathcal{R}$ on a set X is an algebra if and only if $X \in \mathcal{R}$.*

Proof. Let A and B be elements of an algebra $\mathcal{A}$. Then $A \backslash B = A \cap B^C \in \mathcal{A}$ and $A \cup B = \left(A^C \cap B^C\right)^C \in \mathcal{A}$. Since $\mathcal{A}$ is nonempty, there is an element $E \in \mathcal{A}$, whence $X = E \cup E^C \in \mathcal{A}$.

Obviously, let A and B be elements of a ring $\mathcal{R}$ containing X. Then $A^C = X \backslash A \in \mathcal{R}$. By Proposition 12.1 and definition of algebra, $\mathcal{R}$ is an algebra. □

The classes defined above have the property of closedness under finite operations, which is insufficient for measurable sets.

Definition 12.5. A nonempty class $\mathcal{M}$ of sets is *monotone* if, for every monotone sequence $\{E_n\}$ of sets in $\mathcal{M}$, we have

$$\lim_{n\to\infty} E_n \in \mathcal{M}.$$

Definition 12.6. A λ-*system* on a set X is a class $\mathcal{L}$ of sets of X such that

(1) $X \in \mathcal{L}$;

(2) $A, B \in \mathcal{L}$ and $B \subseteq A \implies A \backslash B \in \mathcal{L}$;

(3) $\{A_n\}$ is an increasing sequence of sets $\mathcal{L}$ of $\implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$.

Proposition 12.3. *A λ-system is a monotone class.*

Proof. It is trivial. □

Definition 12.7. A σ-*system* on a set X is a class of sets $\mathcal{F}$ of X such that

(1) $X \in \mathcal{F}$;

(2) $A \in \mathcal{F} \implies A^C \in \mathcal{F}$;

(3) $A_n \in \mathcal{F}, n = 1, 2, \ldots \implies \bigcup_{i=1}^{\infty} A_n \in \mathcal{F}$.

If $\{B_n\}$ is a sequence in a σ-algebra, then

$$\bigcap_{n=1}^{\infty} B_n = \left(\bigcup_{n=1}^{\infty} B_n^C\right)^C \in \mathcal{F}.$$

Example 12.5. Let X be a set. The class $\{\varnothing, X\}$ is the minimal *sigma*- algebra on X while the power set of X is the maximal σ- algebra on X.

Proposition 12.4. *A σ-algebra is a λ-system.*

Proof. Let A and B belong to a σ-algebra $\mathcal{F}$. Then $A \backslash B = A \cap B^C \in \mathcal{F}$. □

Proposition 12.5. *A σ-algebra is an algebra.*

Proof. Trivial. □

In conclusion, we have the following relations:

π-systems $\supset$ semirings $\supset$ rings $\supset$ algebras $\supset$ σ-algebras.

monotone classes $\supset$ λ-systems $\supset$ σ-algebras.

Amongst these classes, the most essential one is σ-algebra, which is used to establish measure. The elements of σ-algebra on which the *measure* is defined are what we call measurable sets.

Proposition 12.6. *A class is a σ-algebra if*

(1) it's a monotone algebra;

(2) it's both a π-system and a λ-system.

Definition 12.8. A *σ-ring* on a set X is a ring $\mathcal{S}$ on X that is closed under the formation of countable unions.

If $\mathcal{S}$ is a σ-ring on X, there is an element in $\mathcal{S}$ and thus $\varnothing \in \mathcal{S}$. Furthermore, $X \in \mathcal{S}$ and $A \in \mathcal{S} \implies A^C \in \mathcal{S}$. It is easy to verify that a σ-ring containing X is a σ-algebra.

Proposition 12.7. *A σ-ring is a monotone class; a monotone ring is a σ-ring.*

Proof. The first statement is obvious. To prove the second assertion, we must show that a monotone ring is closed under the formation of countable union. If $\mathcal{M}$ is a monotone ring and if $\{A_n\}$ is a sequence of sets of $\mathcal{M}$, then the sequence

$$B_n = \bigcup_{k=1}^{n} A_n$$

is a monotone sequence of sets whose limit is $\bigcup_{k=1}^{\infty} A_n$. Since $\mathcal{M}$ is monotone, $\bigcup_{k=1}^{\infty} A_k = \lim_{n\to\infty} B_n \in \mathcal{M}$. □

12.2 Generated Rings and Algebras

Proposition 12.8. *Let X be a set. If $\mathcal{E}$ is a class of sets of X, then there is a unique ring $\mathcal{R}$ (resp., monotone class, λ-system, or σ-algebra) such that $\mathcal{R} \supset \mathcal{E}$ and if $\mathcal{R}'$ is another ring (resp., monotone class, λ-system, or σ-algebra) on X containing $\mathcal{E}$, then $\mathcal{R} \subseteq \mathcal{R}'$.*

Proof. Since the class of all subsets of X is a ring, the collection of rings containing $\mathcal{E}$ is nonempty. It is easy to verify that the intersection any collection of rings is also a ring. Thus the the intersection of all rings containing $\mathcal{E}$ is the ring with disired property. The proofs of other cases are similar. □

Definition 12.9. The ring (resp., monotone class, λ-system, σ-algebra) is called the ring (resp., monotone class, λ-system, σ-algebra) generated by $\mathcal{E}$; it will be denoted by $\mathcal{R}(\mathcal{E})$ (resp., $\mathcal{M}(\mathcal{E})$), $\mathcal{A}(\mathcal{E})$).

The tree core theorems of this section are as follows.

Theorem 12.1. *If $\mathcal{D}$ is a semiring, then*

$$\mathcal{R}(\mathcal{D}) = \left\{ \bigcup_{k=1}^{n} U_k : n \in \mathbb{N}^*, U_k \in \mathcal{D} \text{ are disjoint} \right\}. \tag{12.1}$$

Proof. Denote by $\mathcal{B}$ the set given by Equation (12.1). Clearly, $\mathcal{B} \subseteq \mathcal{R}(\mathcal{D})$. Supposint that $A = \bigcup_{k=1}^{n_1} A_k \in \mathcal{D}$ and $B = \bigcup_{k=1}^{n_2} B_k \in \mathcal{D}$, then

$$\begin{aligned} A \backslash B &= \left(\bigcup_{k=1}^{n_1} A_k\right) \backslash \left(\bigcup_{l=1}^{n_2} B_k\right) = \bigcup_{k=1}^{n} \left(A_k \backslash \bigcup_{l=1}^{n_2} B_l\right) \\ &= \bigcup_{k=1}^{n_1} \bigcap_{l=1}^{n_2} (A_k \backslash B_l) = \bigcup_{k=1}^{n_1} \bigcap_{l=1}^{n_2} (A_k \backslash (A_k \cap B_l)) \\ &= \bigcup_{k=1}^{n_1} \bigcap_{l=1}^{n_2} \left(\bigcup_{i=1}^{n_{kl}} C_i^{k,l}\right) \in \mathcal{B} \end{aligned}$$

where $\{C_i^{k,l} \in \mathcal{D} : i = 1, 2, \ldots, n_{kl}\}$ are disjoint for each pair (k, l).

Then $A \cup B = A \cup (B \backslash A)$ since A and $B \backslash A$ are disjoint and are elements of $\mathcal{B}$ and thus can be written as unions of finite disjoint sets of $\mathcal{D}$. Thus, $\mathcal{B}$ is a ring.

Now we have to show that $\mathcal{B}$ is a minimal ring containing $\mathcal{D}$. This is true because of closedness of the formation of finite unions. □

Theorem 12.2. *If $\mathcal{A}$ is a algebra on a set X, then $\mathcal{M}(A) = \mathcal{F}(A)$.*

Proof. On the one hand, $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{F}(\mathcal{A})$, since $\mathcal{F}(\mathcal{A})$ is monotone. On the other hand, to prove $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$, it suffices to prove that $\mathcal{M}(\mathcal{A})$ is a σ-algebra, by Proposition 12.6. However, to show this it suffices to show that $\mathcal{M}(\mathcal{A})$ is a ring since $X \in \mathcal{A} \subseteq \mathcal{M}(\mathcal{A})$ by Proposition 12.2.

First, we show that for any $A \in \mathcal{A}$ and $B \in \mathcal{M}(\mathcal{A})$,

$$A \cup B \in \mathcal{M}(\mathcal{A}) \text{ and } A \backslash B \in \mathcal{M}(\mathcal{A}). \tag{12.2}$$

Setting

$$\mathcal{H}_A = \{B \in \mathcal{M}(\mathcal{A}) : B \cup A \in \mathcal{M}(\mathcal{A}), A \backslash B \in \mathcal{M}(\mathcal{A})\},$$

it is not difficult to verify that $\mathcal{H}_A$ is monotone and $\mathcal{A} \subseteq \mathcal{H}_A$, whence $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{H}_A$ and Equation (12.2) follows.

Second, setting

$$\mathcal{G}_B = \{A \in \mathcal{M}(\mathcal{A}) : A \cup B \in \mathcal{M}(\mathcal{A}), A \backslash B \in \mathcal{M}(\mathcal{A})\},$$

likewise, $\mathcal{G}_B$ is monotone and by Equation (12.2), $\mathcal{G}_B \supset \mathcal{A}$, whence $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{G}_B$. Therefore, for any $A, B \in \mathcal{M}(\mathcal{A})$,

$$A \cup B \in \mathcal{M}(\mathcal{A}) \text{ and } A \backslash B \in \mathcal{M}(\mathcal{A}).$$

The conclusion is established. □

In practice, an equivalent form of Theorem 12.2 is frequently used.

Corollary 12.1. *If $\mathcal{A}$ is an algebra and $\mathcal{M}$ is a monotone class, then*

$$\mathcal{A} \subseteq \mathcal{M} \implies \mathcal{F}(\mathcal{A}) \subseteq \mathcal{M}.$$

Theorem 12.3. *If $\mathcal{P}$ is a π-system on a set X, then $\mathcal{L}(\mathcal{P}) = \mathcal{F}(\mathcal{P})$.*

Proof. Since σ-algebras are π-systems, we have $\mathcal{F}(\mathcal{P}) \subseteq \mathcal{L}(\mathcal{P})$. Now we prove $\mathcal{F}(\mathcal{P}) \supset \mathcal{L}(\mathcal{P})$ and it suffices to prove that $\mathcal{L}(\mathcal{P})$ is a π-system thus a σ-algebra.

Putting $A \in \mathcal{P}$ and $B \in \mathcal{L}(\mathcal{P})$, let

$$\mathcal{H}_A = \{B \in \mathcal{L}(\mathcal{P}) : B \cap A \in \mathcal{L}(\mathcal{P})\},$$

which is a λ-system containing $\mathcal{P}$, whence $\mathcal{L}(\mathcal{P}) \subseteq \mathcal{H}_A$.

Further, set

$$\mathcal{G}_B = \{A \in \mathcal{L}(\mathcal{P}) : A \cap B \in \mathcal{L}(\mathcal{P})\},$$

which is also a λ-system containing $\mathcal{P}$, whence $\mathcal{L}(\mathcal{P}) \subseteq \mathcal{G}_B$, *i.e.*, for $A, B \in \mathcal{L}(\mathcal{P})$

$$A \cap B \in \mathcal{L}(\mathcal{P}).$$

The conclusion is established. □

This theorem has an equivalent form:

Corollary 12.2. *If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system, then*

$$\mathcal{P} \subseteq \mathcal{L} \implies \mathcal{F}(\mathcal{P}) \subseteq \mathcal{L}.$$

Chapter 13

Measure Spaces

13.1 Measures

Let $\mathcal{E}$ be a class on a set X. A function defined on $\mathcal{E}$ is called a *set function*, usually denoted by Greek letters μ, ν, ...

An extended real valued set function is *(finitely) additive* if for any disjoint $A, B \in \mathcal{E}$ such that $A \cup B \in \mathcal{E}$, we have $\mu(A \cup B) = \mu(A) + \mu(B)$.

An extended real valued set function is *countably additive (or σ-additive)* if for every disjoint sequence $\{E_n\}_{n=1}^{\infty}$ of sets in $\mathcal{E}$ whose union is also in $\mathcal{E}$, we have $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} E_n$.

Definition 13.1. Let $\mathcal{E}$ be a class on a set X and $\varnothing \in \mathcal{E}$. A non negative set function $\mu : \mathcal{E} \to \overline{\mathbb{R}}$ is called a *measure* on $\mathcal{E}$ if μ is countably additive and $\mu(\varnothing) = 0$.

A measure μ is called *finite* if $\mu(A) < \infty$ for all $A \in \mathcal{E}$; A measure μ called *σ-finite* if for every $A \in \mathcal{E}$, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{E}$ such that $\mu(A_i) < \infty$ and $A \in \bigcup_{n=1}^{\infty}$.

13.2 Properties of Measures

An extended real valued set function μ on a class $\mathcal{E}$ is monotonic if, whenever $E \in \mathcal{E}$, $F \in \mathcal{E}$, $E \subseteq F$, then $\mu(E) < \mu(F)$.

An extended real valued set function μ on a class $\mathcal{E}$ is subtractive if, whenever $E \in \mathcal{E}$, $F \in \mathcal{E}$, $E \subseteq F$, $F \backslash E \in \mathcal{E}$ and $\left|\mu(E)\right| < \infty$, then $\mu(F \backslash E) = \mu(F) - \mu(E)$.

Theorem 13.1. *If μ is a measure on a ring $\mathcal{R}$, then μ is monotonic and subtractive.*

Proof. The proof is trivial. $\square$

Theorem 13.2. *Let μ be a measure on a ring $\mathcal{R}$. If $E \in \mathcal{R}$ and $\{E_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{R}$ such that $E \in \bigcup_{n=1}^{\infty} E_n$, then*

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Proof. Let $F_n = E_n \cap E$ and $G_n = F_n \backslash \left(\bigcup_{i<n} F_i\right)$. Notice that $G_n \subseteq F_n \subseteq E_n$. Then

$$E = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} G_n.$$

Since $G_n, n = 1, 2, \ldots$ are disjoint, by countably additivity and monotonicity,

$$\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} G_n\right) = \sum_{n=1}^{\infty} \mu(G_n) \le \sum_{n=1}^{\infty} \mu(E_n).$$

This is the desired result. □

Theorem 13.3. *Let μ be a measure on a ring $\mathcal{R}$. If $E \in \mathcal{R}$ and $\{E_n\}_{n=1}^{\infty}$ is a disjoint sequence of sets in $\mathcal{R}$ such that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \subseteq E$, then*

$$\sum_{n=1}^{\infty} \mu(E_n) \le \mu(E).$$

Proof. For a given $N \in \mathbb{N}^*$,

$$\bigcup_{n=1}^{N} E_n \subseteq E.$$

The monotonicity of μ yields that

$$\mu\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} \mu(E_n) \le \subseteq E.$$

Letting $N \to \infty$, the validity for infinite sequence follows. □

Theorem 13.4. *Let μ is a measure on a ring $\mathcal{R}$. The following hold true:*

(1) If $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of sets in $\mathcal{R}$ for which $\lim_{n\to\infty} E_n = E \in \mathcal{R}$, then $\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$.

(2) If $\{E_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathcal{R}$ of which at least one has finite measure and for which $\lim_{n\to\infty} E_n = E \in \mathcal{R}$, then $\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$.

Proof. (1) If we write $E_0 = 0$, then

$$\begin{aligned}
\mu(\lim_{n\to\infty} E_n) &= \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} (E_i - E_{i-1})\right) \\
&= \sum_{i=1}^{\infty} \mu\left(E_i - E_{i-1}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu\left(E_i - E_{i-1}\right) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right) = \lim_{n\to\infty} \mu(E_n).
\end{aligned}$$

(2) If $\mu(E_m) < \infty$, then for $n \geq m$, $\mu(E_n) < \infty$ and therefore $\mu\left(\lim_{n\to\infty} E_n\right) < \infty$. It follows from

$$\begin{aligned}\mu(E_m) - \mu(\lim_{n\to\infty} E_n) &= \mu\left(E_m - \lim_{n\to\infty} E_n\right) = \mu\left(\lim_{n\to\infty}(E_m - E_n)\right)\\ &= \lim_{n\to\infty}\mu(E_m - E_n) = \lim_{n\to\infty}\big(\mu(E_m) - \mu(E_n)\big)\\ &= \mu(E_m) - \lim_{n\to\infty}\mu(E_m).\end{aligned}$$

□

We shall say that an extended real valued set function μ defined on class $\mathcal{E}$ is *continuous from below at a set E in $\mathcal{E}$* if, for every increasing sequence $\{E_n\}_{n=1}^{\infty}$ of sets in $\mathcal{E}$ for which $\lim_{n\to\infty}\mu(E_n) = \mu(E)$. Similarly, μ is *continuous from above* at E if for every decreasing sequence $\{E_n\}_{n=1}^{\infty}$ for which at $\left|\mu(E_m)\right| < \infty$ for at least one value of m and for which $\lim_{n\to\infty} E_n = E$, we have $\lim_{n\to\infty}\mu(E_n) = E_n$.

Theorem 13.5. *Let μ be a finite, non negative, and additive set function on a ring $\mathcal{R}$. If μ is either continuous from below at every set E in $\mathcal{R}$, or continuous from above at $\varnothing$, then μ is a measure on $\mathcal{R}$.*

Chapter 14

Lebesgue Measurability

This chapter is mainly focused on Lebesgue measurability. First we need to clarify some concepts. Suppose $\{f_n\}$ is a sequence of extended-real-valued functions on X. If

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{14.1}$$

exists for every point in X, then we call f the point wise limit of sequence $\{f_n\}$. $\sup_n f_n$ and $\limsup_n f_n$ are functions defined on X that

$$\left(\sup_n f_n\right)(x) = \sup_n f_n(x), \tag{14.2}$$

$$\left(\limsup_{n\to\infty} f_n\right)(x) = \limsup_{n\to\infty} f_n(x). \tag{14.3}$$

Theorem 14.1. *If f_n is a sequence of measurable functions $X \to [-\infty, \infty]$ and $h = \sup_n f_n$, $g = \limsup_{n\to\infty} f_n$, then h and g are measurable functions.*

Definition 14.1. A complex function s on a measurable set X is called a simple function if its range consists of only finitely many values. Among these are nonnegative simple functions, whose range is a finite subset of $[0, \infty)$. Note that we explicitly exclude ∞ from the values of simple functions.

A simple function can be represented as

$$s = \sum_{i=1}^{n} s_i \chi_{A_i}. \tag{14.4}$$

Simple functions play an essential role in real analysis, since they are very good approximation to measurable functions. In fact ....

Theorem 14.2 (Lebesgue's Monotone Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that*

(1) $0 \le f_1 \le f_2 \le \cdots \le \infty$;

(2) $f_n(x) \to f(x)$ *as* $n \to \infty$ *for any* $x \in X$.

Then f is measurable, and

$$\int_X f_n(x)\,\mathrm{d}\mu \to \int_X f(x)\,\mathrm{d}\mu. \tag{14.5}$$

Proof. Since $\int f_n \le \int f_{n+1}$, there exists an $\alpha \in [0,\infty]$ such that

$$\int f_n \to \alpha. \tag{14.6}$$

Since $f = \sup f_n$, with theorem 14.1, we know that f is measurable. Besides, $\int f_n \le \int f$ for every n, so $\alpha \le \int f$.

Let s be any simple measurable function satisfying $0 \le s \le f$, c be a constant in $(0,1)$ and denote

$$E_n = \{x : f_n(x) \le cs(x)\} \quad (n = 1, 2, \cdots). \tag{14.7}$$

Then we make two observations

(1) For any n, E_n is measurable, $E_n \subseteq E_{n+1}$, and $X = \bigcup_n E_n$. To see the last identity, consider some $x \in X$. If $f(x) = 0$, then clearly $x \in E_1$; else $f(x) > 0$, then since $0 < s \le f$, $\lim_{n\to\infty} f_n = f$ and $c < 1$, $x \in E_n$ for some sufficiently large n.

(2)

□

Theorem 14.3 (Fatou's Lemma). *If $\{f_n\}$ is a sequence of measurable nonnegative functions. then*

$$\int_X \left(\liminf_{n\to\infty} f_n(x)\right) \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu. \tag{14.8}$$

Proof. Let $g_k(x) = \inf_{i\ge k} f_i(x)$. Then $g_k(x) \le f_k(x)$ and g_k is a measurable function by theorem 14.1. Thus

$$\int_X g_k(x)\,\mathrm{d}\mu \le \int_X f_k(x)\,\mathrm{d}\mu, \forall k \in \mathbb{N}^*. \tag{14.9}$$

Noticing that $0 \le g_1(x) \le g_2(x) \le \cdots$ and $\lim_{k\to\infty} g_k = \limsup_{n\to\infty} f_n$, by Lebesgue's monotone convergence theorem, $\int_X g_k(x)\,\mathrm{d}\mu$ converges to left side of equation 14.8 as $k \to \infty$. Hence

$$\liminf_{k\to\infty} \int_X g\,\mathrm{d}\mu = \lim_{k\to\infty} g_k(x)\,\mathrm{d}\mu = \int_X \liminf_{n\to\infty} f_n\,\mathrm{d}\mu \le \liminf_{k\to\infty} \int_X f_n(x)\,\mathrm{d}\mu. \tag{14.10}$$

□

Chapter 15

L^p Spaces

15.1 L^2 Spaces

Definition 15.1 (L^p space)**.** Suppose a, $b \in \mathbb{R}$, $p \in [1, \infty)$ and $-\infty < a < b < \infty$. Let $L^p([a, b])$ denote all the Lebesgue measurable functions $u : [a, b] \to \mathbb{R}$ such that

$$\int_a^b \left|u(x)\right|^p \, \mathrm{d}x < \infty. \tag{15.1}$$

When not leading to ambiguousness, we may omit the domain where the functions are defined, just writing L^p.

Addition and scalar multiplication can be introduced as the ordinary pointwise addition and scalar multiplication of functions. Naturally, the domain of L^p can be easily extend to $\mathbb{R}^n$, $n \geq 1$.

Theorem 15.1. *Square-integrable functions are absolutely integrable,* i.e., $L^2 \in L^1$.

Proof. Proof is easily obtained by the inequality

$$|u| \leq \frac{1 + u^2}{2}. \tag{15.2}$$

□

Theorem 15.2. L^p *space is a linear space.*

Proof. Suppose $u, v \in L_2$. From the basic inequality, we have

$$|uv| \leq \frac{u^2 + v^2}{2}. \tag{15.3}$$

Along with theorem 15.1, uv is square-integrable.

$$(u + v)^2 = u^2 + 2uv + v^2, \tag{15.4}$$

which means $u + v$ is a square-integrable function, *i.e.*, $u + v \in L_2$.

Obviously, αu is square-integrable for any $alpha \in \mathbb{R}$. □

By introducing inner product to L_2, we make L_2 a Hilbert space. Define

$$\langle u, v\rangle = \int_a^b u(x)v(x)\,\mathrm{d}x. \tag{15.5}$$

By induction, norm on L_2 is given as

$$\|u\| = \sqrt{\langle u, u\rangle}. \tag{15.6}$$

Theorem 15.3. *With inner product defined in 15.5, L_2 is a Hilbert space.*

Proof. Clearly, formula 15.5 is well-defined inner product of L_2. Then the only thing left is to prove L_2 is complete. □

Chapter 16

Functions of Bounded Variation

Definition 16.1 (total variation)**.** The *total variation* of a continuous real-valued function f, defined on an interval $[a, b] \subseteq \mathbb{R}$ is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} \left| f(x_{i+1}) - f(x_i) \right|$$

where the supremum is taken over the set

$$\mathcal{P} = \left\{ P = \left\{ \left(x_0, x_1, \ldots, x_{n_P} \right) \right\} : a = x_0 < x_1 < \cdots < x_{n_P} = b \right\}$$

of all partitions of the interval $[a, b]$. Denote by $V_P(f)$ the variation of f over partion P, *i.e.*,

$$V_P(f) = \sum_{i=0}^{n_P - 1} \left| f(x_{i+1}) - f(x_i) \right|.$$

Definition 16.2 (BV function)**.** A function $f \in C[a, b]$ is of bounded variation (BV function) if its total variation is finite.

Part VI

Functional Analysis

Chapter 17

Metric Spaces

This chapter is based on metric space, and the basic definition of metric space is presented in chapter 10.1.

17.1 Fixed Point Theorem of Banach

The fixed point theorem of Banach, also known as contraction principle, plays a fundamental role in the analysis of iterative algorithm.

Theorem 17.1 (Fixed Point Theorem of Banach)**.** *Suppose that X is a nonempty complete metric space and operator $A : X \to X$ is k-contractive,* i.e.*,*

$$d(Ax, Ay) \leq kd(x, y) \text{ for any } x, y \in X \tag{17.1}$$

where $k \in [0, 1)$. Then we have the following results:

(1) Existence and Uniqueness of the fixed-point*. The equation*

$$Ax = x \tag{17.2}$$

has a unique solution in X.

(2) Convergence of the Iterative Method*. For a given $x_0 \in X$, the sequence $\{x_n\}$ generated by the iterative equation*

$$x_{n+1} = Ax_n$$

converges to the unique solution to equation $Ax = x$ (fixed point).

(3) Error estimates*. For all $n \in \mathbb{N}$, we have the so-called* a priori*error estimate*

$$d(x_n, x) \leq \frac{k^n}{1-k} d(x_1, x_0), \tag{17.3}$$

and the so-called a posterior*error estimate*

$$d(x_{n+1}, x) \leq \frac{k}{1-k} d(x_{n+1}, x_n). \tag{17.4}$$

(4) Rate of Convergence. *For all $n \in \ltimes$, we have*

$$d(x_{n+1}, u) \leq kd(x_n, x).$$

Proof. (1) (2) **Step 1. $\{x_n\}$ is a Cauchy sequence in X.** Since A is a k-contractive operator, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Ax_n, Ax_{n-1}) \leq kd(x_n, x_{n-1}) \\ &\leq k^2 d(x_{n-1}, x_{n-2}) \leq k^n d(x_1, x_0). \end{aligned}$$

Now let $n, m \in \mathbb{N}$, by triangle inequality and the sum formula for geometric series, it yields that

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{m-n-1} k^{n+i} d(x_1, x_0) \\ &\leq \frac{k^n}{1-k} d(x_1, x_0). \end{aligned}$$

It follows that $d(x_n, x_m) \to 0$ as $n \to \infty$ since $0 \leq k < 1$. Hence $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to some point $u \in X$ since X is complete.

Step 2. x is a solution to fixed-point of A. The definition of contractive operator yields that A is a continuous operator. Thus $x = \lim_{n\to\infty} x_n = \lim_{N\to\infty} Ax_{n-1} = A\left(\lim_{n\to\infty} x_{n-1}\right) = Ax$.

Step 3. Uniqueness of fixed-point. Supposing that $x, x' \in X$ such that $x = Ax$ and $x' = Ax'$, it follows that $d(x, x') = d(Ax, Ax') \leq kd(x, x') < d(x, x')$, whence $d(x, x') = 0$.

(3) Letting $m \to \infty$, it follows from

$$d(x_n, x_{n+m}) \leq \frac{k^n}{1-k} d(x_1, x_0)$$

that for every $n \in \ltimes$,

$$d(x_n, x) \leq \frac{k^n}{1-k} d(x_1, x_0).$$

This is the estimate (17.3).

To prove estimate (17.4), observe that

$$d(x_{n+1}, x_{n+m+1}) \leq \sum_{i=1}^{m} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=1}^{m} k^i d(x_n, x_{n+1}).$$

Letting $m \to \infty$, we get

$$d(x_{n+1}, x) \leq \frac{k}{1-k} d(x_n, x_{n+1}).$$

(4) Observing that

$$d(x_{n+1}, x) = d(Ax_n, Ax) \leq kd(x_n, x).$$

□

Remark 9. The condition that $A : X \to X$ can be replaced with that $A : M \to M$, where M is a closed subset of X.

Corollary 17.1. *Supposing that (X,d) is a complete metric space and $T : X \to X$ is an operator such that T^{n_0} is a contractive operator for given $n_0 \in \mathbb{N}$, then T has a unique fixed-point.*

Proof. As T^{n_0} is a contractive operator, it has a unique fixed point x_0. Observe that

$$T^{n_0}(Tx_0) = T(T^{n_0}x_0) = T(x_0),$$

which implies that Tx_0 is a fixed-point of T^{n_0}. Thus $Tx_0 = x_0$.

Assume that x_0 and x_0' are fixed-points of T. It follows that $T^{n_0}(x_0) = x_0$ and $T^{n_0}(x_0') = x_0'$, whence $x_0 = x_0'$. □

Example 17.1 (Existence and Uniqueness of Solution to ODE)**.** For a given point $(t_0, x_0$ in $\mathbb{R}^2$ consider the initial value problem of ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [t_0 - a, t_0 + a] \\ x(t_0) = x_0 \end{cases} \tag{17.5}$$

where f is continuous and satisfies a Lipschitz condition with respect to x, *i.e.*,

$$\left|f(t,x) - f(t,x')\right| \le K\left|x - x'\right|.$$

If $F(x,y,u)$ is of the form

$$F(x,y,u) = K(x,y)u,$$

then it is called an linear integral equation. We are looking for a solution $x = x(t)$ to Equation (17.5) such that $x(t)$ is differentiable and $(t, x(t)) \in S$ with the retangular area $S = \left\{(t,x) \in \mathbb{R}^2 : |t - t_0| \le a, |x - x_0| \le b\right\}$ for given $a, b > 0$. Then we have the following result:

Theorem 17.2 (The Picard-Lindelof Theorem)**.** *The Equation (17.5) has a unique solution defined on the interval $[t_0 - h, t_0 + h]$ where $h \in (0, b]$ satisfying $hK < 1$ and $hM \le b$ where $M = \max_{(t,x)\in S}\left|f(t,x)\right|$.*

Proof. **Step 1.** The solution of Equation (17.5) is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(\tau), \tau)\,\mathrm{d}\tau, \quad t \in [t_0 - a, t_0 - b] \tag{17.6}$$

and vice versa.

Step 2. Let E be the subspace of $X = C[t_0 - h, t_0 + h]$ such that $d(x - x_0) \le b$ and consider the integral operator $A : E \to E$ defined as

$$Ax = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\,\mathrm{d}\tau, \quad t \in [t_0 - h, t_0 + h]$$

where $x_0(t) = x_0$. We then show that $A(M) \subseteq M$. Indeed, letting $x \in M$, then

$$\left|\int_{t_0}^{t} f(\tau, x(\tau))\,\mathrm{d}\tau\right| \le |t - t_0| \max_{(t,x)\in S} f(x,t) \le hM \le b,$$

and hence

$$d(Ax, x) = \max_{t\in[t_0-h,t_0+h]} \left| \int_{t_0}^{t} f(\tau, x(\tau))\,\mathrm{d}\tau \right| \leq b.$$

Step 3. We now prove that A is a contractive operator. By the assumption of Lipschitz continuity,

$$\begin{aligned} d(Ax_1, Ax_2) &= \max_{t\in[t_0-h,t_0+h]} \left| \int_{t_0}^{t} f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\,\mathrm{d}\tau \right| \\ &\leq hKd(x_1, x_2) = kd(x_1, x_2) \end{aligned}$$

where $k = hK < 1$. Hence A is a k-contractive operator and there is a unique solution to Equation (17.5). □

Example 17.2 (Freholm Integral Equations)**.** Consider the integral equation

$$u(x) = f(t) + \lambda \int_a^b F(x, y, u(y))u(y)\,\mathrm{d}y, \quad x \in [a, b]. \tag{17.7}$$

along with the iteration method

$$u_{n+1} = f(t) + \lambda \int_a^b F(x, y, u_n(y))\,\mathrm{d}y, \quad x \in [a, b], n = 1, 2, \ldots \tag{17.8}$$

where $u_0 \equiv 0$ and $-\infty < a < b < \infty$.

Theorem 17.3. *Assume the following:*

(1) The function $f : [a, b] \to \mathbb{R}$ is continuous.

(2) The function $F : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and the partial derivative $F_u : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous.

(3) There is a number L such that $\left|F_u(x, y, u)\right| \leq L$ for all $x, y \in [a, b]$ and $u \in \mathbb{R}$.

(4) Let the real number λ be given such that $(b - a)|\lambda|\, L < 1$.

(5) Set $X = C[a, b]$.

Then the sequence $\{u_n\}$ constructed by (17.8) converges to the unique solution of (17.7).

Proof. Define the operator

$$A(u)(x) \coloneqq f(x) + \lambda \int_a^b F(x, y, u(y))\,\mathrm{d}y, \quad \text{for all } x \in [a, b].$$

Then the original equation (17.7) corresponds to the fixed-point problem $u = Ax$. By the classical mean value theorem,

$$\left|F(x, y, u) - F(x, y, v)\right| \leq L|u - v|\,,$$

which implies

$$\begin{aligned} d(Au, Av) &= \max_{a\leq x\leq b} \left|(Au)(x) - (Av)(x)\right| \\ &= \max_{a\leq x\leq b} \left| \lambda \int_a^b F(x, y, u(y)) - F(x, y, v(y))\,\mathrm{d}y \right| \\ &\leq |\lambda|\,(b - a)L \max_{a\leq x\leq b} \left|u(x) - v(x)\right| = kd(x, u). \end{aligned}$$

where $k \coloneqq |\lambda| (b-a)L < 1$. Hence $A : X \to X$ is a k-contractive operator ad has a unique solution. ☐

Example 17.3 (Volterra Integral Equations)**.** Consider the Equation

$$x(t) = f(t) + \lambda \int_a^t K(t,s)x(s)\,\mathrm{d}s, \quad a \le t \le b \tag{17.9}$$

where K is continous on $[a,b] \times [a,b]$. Letting $M = \max_{a\le s,t\le b}\left|K(t,s)\right|$, then we have following:

Theorem 17.4. *Let $X = C[a,b]$ and define*

$$(Tx)(t) = f(t) + \lambda \int_a^b K(s,t)x(s)\,\mathrm{d}s, \quad a \le t \le b. \tag{17.10}$$

Then T has a unique fixed-point in X.

Proof. We shall prove that there exists some $n > 0$ such that T^n is a contractive mapping. This is proved by showing

$$\left|(T^n x_1)(t) - (T^n x_2)(t)\right| \le |\lambda|^n M^n \frac{(t-a)^n}{n!} d(x_1,x_2). \tag{17.11}$$

We prove the above equation by induction as follows. For $n = 1$,

$$\begin{aligned}\left|(Tx_1)(t) - Tx_2(t)\right| &= |\lambda| \left| \int_a^t K(t,s)(x_1(s) - x_2(s))\,\mathrm{d}s \right| \\ &\le |\lambda| M(t-a)d(x_1,x_2).\end{aligned}$$

Supposing Equation (17.11) holds true for n, then for $n+1$,

$$\begin{aligned}\left|(T^{n+1})(x_1) - (T^{n+1})(x_2)\right| &= \left|T\left(T^n x_1\right)(t) - T\left(T^n x_2\right)(t)\right| \\ &= |\lambda| \left| \int_a^t K(t,s)\left((T^n x_1)(t) - (T^n x_2)(t)\right)\mathrm{d}s \right| \\ &\le |\lambda| M \int_a^t \left|(T^n x_1)(t) - (T^n x_2)(t)\right| \mathrm{d}s \\ &\le |\lambda| M \int_a^t \frac{|\lambda|^n M^n}{n!} d(x_1,x_2)\,\mathrm{d}s \\ &= \frac{|\lambda|^{n+1} M^{n+1}}{(n+1)!}(t-a)^{n+1} d(x_1,x_2).\end{aligned}$$

Therefore,

$$d(T^n x_1, T^n x_2) \le \frac{|\lambda|^n M^n}{n!}(b-a)^n d(x_1,x_2) = k_n d(x_1,x_2)$$

where $k_n \coloneqq \frac{|\lambda|^n M^n}{n!}(b-a)^n$. Since $k_n \to \infty$ as $n \to \infty$, T^n is a contractive mapping for n big enough. ☐

Chapter 18

Banach Spaces

18.1 Norms

Definition 18.1. A real-valued function p defined on a linear space X is called a semi-norm on X, if the following conditions are satisfied

(1) (Subadditivity) $p(x+y) \leq p(x) + p(y)$;

(2) $p(\alpha x) = |\alpha|\, p(x)$,

for any $x, y \in X$ and $\alpha \in \mathbb{K}$.

Definition 18.2 (norm)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X be a linear space over K. A norm $\|\cdot\| : X \to \mathbb{R}$ is a function such that

(1) (positive definiteness) $\|x\| \geq 0$ for every $x \in X$ and $\|x\| = 0 \iff x = 0$.

(2) (absolute homogeneousness) $\|\alpha x\| = |\alpha| \|x\|$ for any $x \in X$ and $\alpha \in \mathbb{K}$.

(3) (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

Definition 18.3 (normed space)**.** A linear space X equipped with a norm $\|\cdot\|$ is called a *normed (linear) space*, denoted by $(X, \|\cdot\|)$.

For the sake of conciseness, we simply write X for a normed space when it is unambiguous.

Obviously, a normed space $(X, \|\cdot\|)$ is a metric space together with the distance defined by $d(x, y) = \|x - y\|$, called the induced metric from the norm. From now on, normed spaces mentioned will be implicitly equipped with the distances induced from the norm. Therefore, normed spaces are finer than metric spaces. All definitions and theorems for metric spaces can be transferred to normed spaces at once. Specifically, a complete normed space is called a *Banach space*.

We have some basic facts about normed spaces, whose proofs has been omitted:

Proposition 18.1. *The norm in a normed space is continuous.*

Example 18.1. Consider the set $\mathbb{K}^n$. Define

$$\|x\|_2 = \sqrt{\sum_{k=1}^{\infty} x_k^2},$$
$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k|,$$
$$\|x\|_\infty = \max_{1\le k\le n} |x_k|.$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{K}^n$. It can be verified that all of them above are norms. They are known as 2-norm, 1-norm and ∞-norm respectively.

Example 18.2. Consider the set $C[a,b]$. For $f \in C[a,b]$, define

$$\|f\| = \max_{t\in[a,b]} \left|f(t)\right|.$$

The induced distance coincides with that given in Example 10.3. Under this norm, $C[a,b]$ is a Banach space.

Example 18.3. The linear space $l^\infty = \left\{x = (x_1, x_2, \dots) : x \text{ is bounded.}\right\}$ equipped with the norm

$$\|x\| = \sup_{n\in\mathbb{N}^*} |x_n|$$

is an inseparable Banach space.

Example 18.4. The set $V[a,b]$ of all BV function (Definition 16.2) along with the norm

$$\|x\| = \left|f(a)\right| + V_a^b(f) \tag{18.1}$$

is a Banach space.

Proof. **Step 1.** Obviously, $\|f\| \ge 0$ for every $f \in V[a,b]$ and $\|f\| = 0$ if $f = 0$. Assuming that $\|f\| = 0$, then for any $t \in [a,b]$,

$$\left|f(t)\right| \le \left|f(t) - f(a)\right| + f(a) \le V_a^b(f) + f(a) = 0.$$

The absolute homogeneousness is easy to verified.

Step 2. Let $f, g \in V[a,b]$. We need to show $\|f+g\| = \left|f(a)+g(a)\right| + V_a^b(f+g) \le \|f\| + \|g\| = \left|f(a)\right| + \left|g(a)\right| + V_a^b(f) + V_a^b(g)$. For any partition $T : a = x_0 < x_1 < \cdots < x_n = b$,

$$\begin{aligned} V_T(f+g) &= \sum_{k=1}^{n} \left|f(x_k) + g(x_k) - f(x_{k-1} - g(x_{k-1}))\right| \\ &\le \sum_{k=1}^{n} \left[\left|f(x_k) - f(x_{k-1})\right| + \left|g(x_k) - g(x_{k-1})\right|\right] \\ &= V_T(f) + V_T(g). \end{aligned}$$

Therefore, $V_a^b(f+g) \le V_a^b(f) + V_a^b(g)$. Consequently, the function defined by Equation (18.1) is a norm.

Step 3. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $V[a,b]$. Then for any $t \in [a,b]$, there exists $n, m \in \mathbb{N}$ such that

$$\begin{aligned} \left|f_n(t) - f(t)\right| &= \left|f_n(t) - f_m(t) - f_n(a) - f_m(a) + f_m(a) + f_n(a)\right| \\ &\le V_a^b(f_n - f_m) + \left|f_m(a) - f_n(a)\right| = \|f_n - f_m\|. \end{aligned}$$

Therefore, for any $t \in [a,b]$, $\{f_n(t)_n\}_{n=1}^{\infty}$ converges, to $f(t)$ say. We claim that $\{f_n\}_{n=1}^{\infty}$ converges to f, which is a BV function. Indeed, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \epsilon$ for any $n, m > N$, whence for any partition $T : a = x_0 < x_1 < \cdots < x_p = b$, we have

$$\left|f_n(a) - f_m(a)\right| + V_T(f_n - f_m) < \epsilon.$$

Letting $n \to \infty$ yields

$$\left|f(a) - f_m(a)\right| + V_T(f - f_m) \leq \epsilon.$$

It follows that for $m > N$,

$$\left|f(a) - f_m(a)\right| + V_a^b(f - f_m) \leq \epsilon,$$

which means $f \in V[a,b]$ (sum of BV functions is a BV function) $\lim_{n\to\infty} f_n = f$. $\square$

Definition 18.4. The linear space $X = C[a,b]$ along with the norm

$$\|x\| = \int_a^b \left|x(t)\right| \mathrm{d}t$$

is not complete.

Proposition 18.2. *Suppose that $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ are sequences in a normed space $(X, \|\cdot\|)$ and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{K}$. If $x_n \to x \in X$, $y_n \to y \in X$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $x_n + y_n \to x + y$ and $\alpha_n x_n \to \alpha x$.*

Theorem 18.1. *Let X be a normed space and $\{x_n\}_{n=1}^{\infty}$ is a sequence in X. If X is complete and series $\sum_{n=1}^{\infty}\|x_n\|$ converges, then the series $\sum_{n=1}^{\infty} x_n$ converges and*

$$\left\|\sum_{n=1}^{\infty} x_n\right\| \leq \sum_{n}^{\infty}\|x_n\|.$$

Conversely, if $\sum_{n=1}^{\infty} x_n$ converges always implies that $\sum_{n=1}^{\infty} x_n$ converges, then X is complete.

Proof. If X is complete and $\sum_{n=1}^{\infty}\|x_n\|$ is convergent, setting $S_N = \sum_{n=1}^{N}$, then it suffices to prove that $\{S_n\}$ is a Cauchy sequence, which is shown by

$$\begin{aligned}\|S_m - S_n\| &= \|x_{n+1} + x_{n+2} + \ldots + x_m\| \\ &\leq \|x_{n+1}\| + \|x_{n+2}\| + \ldots + \|x_m\| \to 0, \quad \text{as } n, m \to \infty.\end{aligned}$$

Suppose conversely, for every $\{x_n\}_{n=1}^{\infty}$, convergence of $\sum_{n=1}^{\infty}|x_n|$ implies $\sum_{n=1}^{\infty} x_n$. Assume that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Then for every $i \in \mathbb{N}$, there exists $N_i \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \frac{1}{2^i}, \quad \text{for any } n, m > N_i.$$

Choosing $\{n_k\}_{k=1}^{\infty}$ such that $n_k > N_k$ and $n_{k+1} > n_k$, then

$$\left\|x_{n_{k+1}} - x_{n_k}\right\| < \frac{1}{2^k}.$$

Consider the series $x_{n_1} + \sum_{k=1}^{\infty}(x_{n_{k+1}} - x_{n_k})$, which is absolutely convergent by the above Equation. By our assumption, $x_{n_1} + \sum_{k=1}^{\infty}(x_{n_{k+1}} - x_{n_k})$ is convergent. As $x_{n_{m+1}} = x_{n_1} + \sum_{k=1}^{m}(x_{n_{k+1}} - x_{n_k})$, $\{x_{n_k}\}_{k=1}^{\infty}$ converges to some $x \in X$. Therefore, by Theorem 10.4, X is complete. $\square$

Remark 10. In general, absolute convergence in an arbitrary normed space does not imply convergence.

18.2 Convex Sets

Definition 18.5. Let X be a linear space and $A \subseteq X$. A is a *convex set* if for all $x, y \in A$ and for all $t \in [0,1]$,

$$tx + (1-t)y \in A.$$

About convex sets, we have a simple fact.

Proposition 18.3. *The intersection of any class of convex sets remains convex.*

Henceforce, it is fine to define the *convex hull* of arbitrary set A as the intersection of all convex sets containing A, denoted by $\mathrm{co}(A)$. Clearly, the convex hull of a set A is the smallest set that contains A.

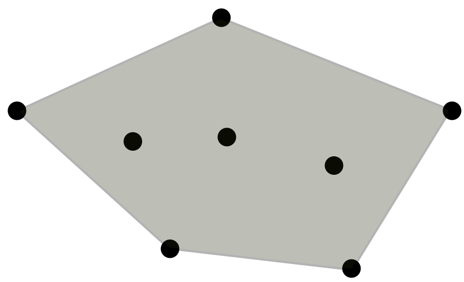

Figure 18.1: Illustration of convex hull.

Example 18.5. The unit ball $B(x,1)$ in a normed space X is a convex set for any given $x \in X$.

18.3 L^p Spaces

Recall that given $p \in [1,\infty]$ and measure space $(X, \mathcal{F}, \mu)$, space $L^p(X, \mathcal{F}, \mu)$ consisting of all real valued measurable functions f such that $|f|^p$ is integrable $(1 \le p < \infty)$ or that is bounded a.e. (essentially bounded) $(p = \infty)$, along with the norm

$$\|f\|_p = \left(\int_X |f|^p\right)^{\frac{1}{p}}, \quad p \in [0,1)$$

or

$$\|f\|_\infty = \operatorname{ess\,sup} \|f\| := \inf\left\{a \in \mathbb{R}_+ : \mu(|f| > a) = 0\right\}$$

is a normed space. See Chapter 15. For the sake of conciseness, we simply write L^p when it is unambiguous.

Theorem 18.2. *The $L^p(1 \le p \le \infty)$ space is a Banach space.*

Proof. We first deal with the case $p \in [0, \infty)$. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in L^p. Then by definition of Cauchy sequence we can find a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of it such that

$$\left\|f_{n_{i+1}} - f_{n_i}\right\| < \frac{1}{2^i}, \quad i \in \mathbb{N}^*.$$

For $k \in \mathbb{N}^*$, put

$$g_k = |f_{n_1}| + \sum_{i=1}^{k} \left|f_{n_{i+1}} - f_{n+i}\right|$$

and

$$g = |f_{n_1}| + \sum_{i=1}^{\infty} \left|f_{n_{i+1}} - f_{n+i}\right|.$$

It follows from triangle inequality (Minkowski's inequality) that

$$\|g_k\|_p \leq \|f_{n_1}\|_p + \sum_{i=1}^{k} \left\|f_{n_{k+1}} - f_{n_k}\right\|_p \leq \|f_{n_1}\|_p + 1.$$

Since g_k is monotonic and non negative, by Monotone Convergence Theorem we have

$$\int_X g^p \,\mathrm{d}x = \lim_{k\to\infty} \int_X g_k^p \,\mathrm{d}x \leq \left(\|f_{n_1}\| + 1\right)^p < \infty,$$

which implies $g \in L^p$ and thus $g < \infty$ a.e.. Therefore, $\lim_{i\to\infty} f_{n_i} = f_{n_1} + \sum_{i=1}^{\infty}$ converges a.e.. Setting $f = \lim_{i\to\infty} f_{n_i}$, it follows from

$$|f| = \lim_{k\to\infty} \left| f_{n_1} + \sum_{i=1}^{k} \left(f_{n_{i+1}} - f_{n_i}\right) \right| \leq \lim_{k\to\infty} g_k = g, \quad \text{a.e.}$$

that $f \in L^p$. Furhermore,

$$\left\|f - f_{n_k}\right\|_p = \left\| \sum_{i=k}^{\infty} (f_{n_{i+1}} - f_{n_k}) \right\|_p \leq \sum_{i=k}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_p \leq \frac{1}{2^{k-1}} \to 0$$

as $k \to \infty$. In this way, we have found a convergent subsequence of $\{f_n\}_{n=1}^{\infty}$. Hence $L^p (1 \leq p < \infty)$ is complete.

□

Theorem 18.3. *(1) For $p \in [0, \infty$, L^p is separable.*
(2) L^∞ is inseparable.

Proof. □

18.4 Properties of Normed Spaces

Theorem 18.4. *Every normed space is isometric to a dense subspace of a Banach space.*

Proof. The process of completion of normed space is the same as of metric space, except that we have to make the set of all elements a vector space, by introducing addition and scalar multiplication to it. $\square$

Suppose that X is a linear space and M is a subspace of X. Define a relation $\sim$ in X as $x \sim y \iff x - y \in M$. It is easy to see that $\sim$ is a equivalence relationship.

Definition 18.6 (Equivalent norms)**.** Let X be a linear space. Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ are said to be *equivalent* if there exists c_1, $c_2 \in \mathbb{R}_+$ such that

$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1\,, \quad \text{for any } x \in X.$$

It is easy to see that equivalence of norms is a equivalence relation.

18.5 Finite-dimensional Normed Spaces

Theorem 18.5. *The norms on a finite-dimensional normed space are equivalent.*

Proof. Let $\|\cdot\|$ be a norm on a n-dimensional linear space X. There exists n linearly independent vectors e_i, $1 \le n \le n$ in X such that $X = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. Then it suffices to prove that $\|\cdot\|$ on X is equivalent to a specified norm $\|\cdot\|_\infty$, which is defined as

$$\|x\|_\infty = \max_{1\le i\le n} |\alpha_i|$$

for any $x = \sum_{i=1}^n \alpha_i e_i \in X$, $\alpha_i \in \mathbb{K}$, $1 \le i \le n$. It is easy to show that $\|\cdot\|_\infty$ is a norm on X.

For $x = \sum_{i=1}^n \alpha_i e_i$,

$$\begin{aligned}\|x\| &= \left\|\sum_{i=1}^n \alpha_i e_i\right\| \le \sum_{i=1}^n \|\alpha_i e_i\| \\ &= \sum_{i=1}^n |\alpha_i|\|e_i\| \le \max_{1\le i\le n} \sum_{i=1}^n \|e_i\| \;\; = \|x\|_\infty c_1,\end{aligned} \tag{18.2}$$

where $c_1 := \sum_{i=1}^n \|e_i\|$. Next, consider $f : \mathbb{K}^n \to \mathbb{R}$, $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\|\sum_{i=1}^n \alpha_i e_i\right\|$. We claim that f is continuous, since for for $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{K}^n$,

$$\begin{aligned}\left|f(\alpha_1, \alpha_2, \ldots, \alpha_n) - f(\beta_1, \beta_2, \ldots, \beta_n)\right| &= \left|\left\|\sum_{i=1}^n \alpha_i e_i\right\| - \left\|\sum_{i=1}^n \beta_i e_i\right\|\right| \\ &\le \left\|\sum_{i=1}^n \alpha_i e_i - \sum_{i=1}^n \beta_i e_i\right\| = \left\|\sum_{i=1}^n (\alpha_i - \beta_i) e_i\right\| \\ &\le \sum_{i=1}^n |\alpha_i - \beta_i|\|e_i\| \le \left(\max_{1\le i\le n}\right) \sum_{i=1}^n \|e_i\|\,.\end{aligned}$$

Let $\Omega = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \max_{1\leq i\leq n}|\alpha_i| = 1\}$, which is obviously is a bounded closed subset of $\mathbb{K}^n$ and thus compact in $\mathbb{K}^n$. This implies that f attains its minimum on Ω. Let $c_2 = \min_{(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\Omega}\left\|\sum_{i=1}^n e_i\right\|$. Then we have for any $x = \sum_{i=1}^n \alpha_i e_i$,

$$\|x\| = \max_{1\leq i\leq n}|\alpha_i|\left\|\frac{x}{\max_{1\leq i\leq n}|\alpha_i|}\right\| \geq c_2\|x\|_\infty . \tag{18.3}$$

Combining, Equation (18.2) and (18.3), it follows that $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$. $\square$

Let X be an n-dimensional normed space and define

$$T : X \to \mathbb{K}^n,$$
$$x = \sum_{i=1}^n \alpha_i e_i \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of X. And we further equip $\mathbb{K}^n$ with the standard Euclidean norm. Then by the theorem above, we know that T is a topological homeomorphism from $X \to \mathbb{K}^n$. In this sense, call n-dimensional normed spaces are topologically the same as $\mathbb{K}^n$.

We have already known that bounded closed sets in $\mathbb{K}^n$ are compact, and by the theorem above, so does in finite-dimensional normed space. We can further show that this property is also a characterization of finite-dimensional normed space. Before stating the miraculous result, we need the following lemma:

Lemma 18.1 (F.Riesz's Lemma)**.** *Let X_9 be a proper closed subspace in a normed space X. Then for any $\epsilon > 0$, there exists $y_0 \in X$ such that $\|y_0\| = 1$ and $\|y_0 - x\| > 1 - \epsilon$ for every $x \in X_0$.*

Proof. As X_0 is a proper closed subsequence of X, we can find some point $y \in X\ X_0$ and thus $d(y, X_0) = \inf_{x\in X_0}\|y - x\| > 0$. Setting $\epsilon > 0$, there exists $x_1 \in X_0$ such that $d(y, X_0) \leq \|y - x_1\| < \frac{d(y,X_0)}{1-\epsilon}$. Set $y_0 = \frac{y-x_1}{\|y-x_1\|}$. Then $\|y_0\| = 1$ and for any $x \in X_0$,

$$\begin{aligned}\|y_0 - x\| &= \left\|\frac{y - x_1}{\|y - x_1\|} - x\right\| = \frac{1}{\|y - x_1\|}\left\|y - \left(x_1 + \|y - x_1\|\, x\right)\right\| \\ &\geq \frac{d}{\|y - x_1\|} > \frac{d}{\frac{d}{1-\epsilon}} = 1 - \epsilon.\end{aligned}$$

This completes the proof. $\square$

Theorem 18.6. *A normed space is finite-dimensional if and only if every bounded closed set in the space is compact.*

Proof. The necessity is the famous Heine-Borel Theorem.

For sufficiency, assume that every bounded closed subset in X is compact while X is infinite-dimensional. Let $S = \{x \in X : \|x\| = 1\}$. Then S is bounded and closed (Why?). Firstly, choose some $x_1 \in S$ and let $X_0 = \operatorname{span} x_1$. Then X_0 is a proper closed subspace of X, whence there exists $x_2 \in S$ such that $\|x_2 - x_1\| > \frac{1}{2}$ by Lemma 18.1. Next, set $X_1 = \operatorname{span}\{x_1, x_2\}$. X_1 is a proper closed subspace of X, whence there exists $x_3 \in S$ such that $\|x_2 - x_1\| > \frac{1}{2}$, $\|x_3 - x_1\| > \frac{1}{2}$ by Lemma 18.1. Repeating this process, we obtain a sequence $\{x_n\}_{n=1}^\infty$ such that $\|x_n\| = 1$ and $\|x_n - x_m\| > \frac{1}{2}$ for any $n, m \in \mathbb{N}$, $n \neq m$. Therefore, $\{x_n\}_{n=1}^\infty$ is not compact because any subsequence of $\{x_n\}_{n=1}^\infty$ does not converges. However, $\{x_n\}_{n=1}^\infty$ is a bounded and closed subset in X, which contradicting our hypothesis. $\square$

Chapter 19

Bounded Linear Operators

The operator theory is one of the most essential topics in functional analysis. In this chapter, we will encounter the four "great theorems" in functional analysis, *i.e.*, (1) Banach-Steinhaus Theorem (Uniform Boundedness Theorem); (2) Banach Open Mapping Theorem; (3) Inverse Mapping Theorem; (4) Hahn-Banach Theorem.

19.1 Bounded Linear Operators and Bounded Linear Functionals

Definition 19.1 (Linear operator, linear functional)**.** Let X and Y be two normed spaces and $D(T)$ be a subspace of X. A *linear operator* T from $D(T)$ to Y is a mapping that is linear, *i.e.*,

(1) $T(x+y) = Tx + Ty$ for any $x, y \in D(T)$;

(2) $T(\alpha x) = \alpha Tx$ for any $\alpha \in \mathbb{K}$ and $x \in D(T)$.

Denote by $R(T)$ the range of T.

A linear operator $f : X \to Y$ is said to be a *linear functional* If $Y = \mathbb{K}$.

Definition 19.2 (bounded linear operator)**.** Let X and Y be two normed spaces and $T : D(T) \subseteq X \to Y$. T is said to be bounded if there exists a constant $M \in \mathbb{R}_+$ such that $\|Tx\| \leq M\|x\|$ for every $x \in D(T)$.

An linear operator $T : X \to Y$ is bounded if and only if there exists $M \in \mathbb{R}_+$ such that $Tx \leq M$ for every $x \in X$ satisfying $\|x\| \leq 1$.

Proposition 19.1. *If linear operator $T : X \to Y$ is continuous at some $x_0 \in X$, then T is continuous on X.*

Proof. This is trivial. □

Proposition 19.2. *A linear operator $T : X \to Y$ is injective if and only if for every $x \in X$ such that $x \neq 0$, $Tx \neq 0$.*

Theorem 19.1. *Let X be a finite dimensional linear space and $T : X \to X$ be a linear operator. T is a injective if and only if T is surjective.*

Proof. To be completed. □

Remark 11. However, an injective operator on infinite dimensional space may not be surjective. For example, consider $X = l^2$ and $T : X \to X, x = (x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$. Then obviously T is injective but not surjective.

Theorem 19.2. *A linear operator $T : X \to Y$ is continuous if and only if T is bounded.*

Proof. Supposing that T is bounded, then $\|Tx\| \leq M\|x\|$ implies the continuity of T.

Conversely, supposing T is continuous, then there exists sequence $\{x_n\}_{n=1}^{\infty}$ such that $\|Tx_n\| > n\|x_n\|$ for $n \in \mathbb{N}^*$. Setting $y_n = \frac{x_n}{n\|x_n\|}$, $n \in \mathbb{N}^*$, then $\|y_n\| = \frac{1}{n}$. Thus $y_n \to 0$ as $n \to \infty$. However, $\|Ty_n\| = \frac{\|Tx_n\|}{n\|x_n\|} > 1$, whence Ty_n does not converges to $T0$, contradicting the assumption that T is continuous. $\square$

Definition 19.3 (operator norm)**.** Let T be a bounded linear operator from X to Y. Define

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

We call $\|T\|$ the operator norm of T.

It can be proven that

$$\|T\| = \inf \left\{ M \in \mathbb{R}_+ : \|Tx\| \leq M\|x\| \text{ for every } x \in X \right\}.$$

Moreover,

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\| \leq 1} \|Tx\|.$$

Indeed, on the one hand, by definition

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \geq \sup_{\|x\| \leq 1} \frac{\|Tx\|}{\|x\|} \geq \sup_{\|x\| \leq 1} \|Tx\| \geq \sup_{\|x\|=1} \|Tx\|.$$

On the other hand, for $x \neq 0$,

$$\frac{\|Tx\|}{\|x\|} = \left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \sup_{\|x\|=1} \|Tx\|,$$

implying that $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \leq \sup_{\|x\|=1} \|Tx\|$.

Example 19.1. Consider the n-dimensional Euclidean space $\mathbb{R}^n$. Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix and define $T_A : X \to X, x \mapsto y$ where y is obtained by matrix multiplication Ax. Then A is a bounded as by conventional Cauchy-Schwarz's Inequality (2.2),

$$\|T_A x\| = \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \sum_{j=1}^{n} |x_j|^2 \right) \right) \frac{1}{2}$$

$$= \|x\|_2 \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Example 19.2. Let $g \in C[a,b]$ and define $T : C[a,b] \to \mathbb{K}, f \mapsto \int_a^b f(t)g(t)\,\mathrm{d}t$. Then T is a linear functional on $C[a,b]$ and

$$|Tf| \le \int_a^b |fg|\,\mathrm{d}t \le \max_{a\le t\le b}\left|f(t)\right| \int_a^b |g|\,\mathrm{d}t = \|f\| \int_a^b |g|\,\mathrm{d}t.$$

Thus T is bounded and $\|T\| \le \int_a^b |g|\,\mathrm{d}t$. We next show that $\|T\| = \int_a^b |g|\,\mathrm{d}t$.

Proof. Letting $h(t) = \operatorname{sgn} g(t) = \frac{g(t)}{|g(t)|}$ if $g(t) \neq 0$ else 0, then we have $\left|h(t)\right| \le 1$ and $\overline{h(t)}g(t) = \left|g(t)\right|$ for every $t \in [a,b]$. By Luzin's Theorem, for any $\epsilon > 0$, there exists $f \in C[a,b]$ such that $\|f\| \le 1$ and $m(\Omega) \le \epsilon$ where m is the Lebesgue measure and

$$\Omega = \{t \in [a,b] : f(t) \neq \overline{h(t)}\}.$$

Then

$$\begin{aligned}\|T\| \ge |Tf| &= \left|\int_a^b fg\,\mathrm{d}t\right| = \left|\int_a^b \overline{h}g\,\mathrm{d}t + \int_a^b (f-\overline{h})g\,\mathrm{d}t\right| \\ &\ge \int_a^b |g|\,\mathrm{d}t - \int_a^b \left|f-\overline{h}\right||g|\,\mathrm{d}t \ge \int_a^b |g|\,\mathrm{d}t - 2\epsilon\|g\|.\end{aligned}$$

Since ϵ is arbitrary, we arrive at $\|T\| \ge \int_a^b g\,\mathrm{d}t$, which completes the proof. □

Example 19.3. Let $X = l^p$ and $Y = l^q$ where p, q are conjugate indices. Define $T : X \to Y, x = (x_1, x_2, \ldots) \mapsto y = (y_1, y_2, \ldots)$, where $y_i = \sum_{j=1}^{\infty} a_{ij}x_j$ and the infinite matrix $A = (a_{ij})_{1\le i,j<\infty}$ satisfies

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|^q < \infty.$$

We shall prove $\|T\| \ge \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|^q\right)^{\frac{1}{q}}$.

Proof. By Hölder's Inequality (2.3),

$$\begin{aligned}\|Tx\|^q = \sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty} a_{ij}x_j\right|^q &\le \sum_{i=1}^{\infty}\left(\left(\sum_{j=1}^{\infty}\left|a_{ij}\right|^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{\infty}\left|x_j\right|^p\right)^{\frac{1}{p}}\right)^q \\ &= \sum_{i=1}^{\infty}\left(\left(\sum_{j=1}^{\infty}\left|a_{ij}\right|^q\right)^{\frac{1}{q}}\right)^q \|x\|_p^q.\end{aligned}$$

This implies $\|Tx\| \le \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|^q\right)^{\frac{1}{q}}\|x\|_p$. □

Example 19.4. Let $T : C^1[a,b] \to C[a,b], f \mapsto \frac{\mathrm{d}f}{\mathrm{d}t}$. We point out that T is unbounded. Indeed, letting $g_n = \sin(nt)$, then $\|g_n\| \le 1$ but

$$\left\|\frac{\mathrm{d}g_n}{\mathrm{d}t}\right\| = \|n\cos nt\| = n \to \infty$$

as $n \to \infty$.

Example 19.5. Let $K(s,t)$ be a continuous function on $[a,b]\times[a,b]$ and define $T: C[a,b]\to C[a,b], (Tf)(t)=\int_a^b K(t,s)f(s)\,\mathrm{d}s$. We claim that

$$\|T\| = \max_{a\le t\le b}\int_a^b \big|K(t,s)\big|\,\mathrm{d}s.$$

Firstly, for $f\in C[a,b]$,

$$\begin{aligned}\big|(Tf)(t)\big| &= \left|\int_a^b K(t,s)f(s)\,\mathrm{d}s\right| \le \int_a^b \big|K(t,s)f(s)\big|\,\mathrm{d}s\\ &\le \|f\|\int_a^b \big|K(t,s)\big|\,\mathrm{d}s.\end{aligned}$$

Thus, $\|Tf\|\le\|f\|\max_{t\in[a,b]}\int_a^b\big|K(t,s)\big|\,\mathrm{d}s$ and therefore $\|T\|\le\max_{t\in[a,b]}\int_a^b\big|K(t,s)\big|\,\mathrm{d}s$. Letting $\beta=\max_{t\in[a,b]}\int_a^b\big|K(t,s)\big|\,\mathrm{d}s$, As $\int_a^b\big|K(t,s)\big|\,\mathrm{d}s$ is continuous on $[a,b]$, $t_0=\arg\max_{t\in[a,b]}\int_a^b\big|K(t,s)\big|\,\mathrm{d}s$ exists. Hence,

$$\|Tf\|\ge\big|(Tf)(t_0)\big| = \int_a^b K(t_0,s)f(s)\,\mathrm{d}s = |Af|$$

where $A:=\int_a^b K(t_0,s)f(s)\,\mathrm{d}s$ is a continuous linear functional. And by Example 19.2,

$$\|A\| = \int_a^b \big|K(t_0,s)\big|\,\mathrm{d}s.$$

Therefore, $\|T\|\ge\|A\|=\int_a^b\big|K(t_0,s)\big|\,\mathrm{d}s=\max_{t\in[a,b]}\int_a^b\big|K(t,s)\big|\,\mathrm{d}s$.

Now we present some classical results on finite dimensional normed space. Let $A=(a_{ij})_{m\times n}$ be an $m\times n$ matrix and $X=\mathbb{K}^n$, $Y=\mathbb{K}^m$ equipped with p-norm. Define $T: X\to Y, x\mapsto Ax$. We want to study $\|A\|_p:=\sup_{x\ne0}\frac{\|Ax\|}{\|x\|}$ for $p=1,2,\infty$.

Theorem 19.3. *It holds true that*

(1) $\|A\|_2=\big(\rho_1(A^*A)\big)^{\frac12}=\sigma_1(A)$ *where* $\rho_i(A)$ *(resp.,* $\sigma(A)$*) is the i-th largest eigenvalue (resp., singular value) of matrix* A.

(2) $\|A\|_1=\|A\|_{1,\infty}:=\max_{1\le j\le n}\sum_{i=1}^n\big|a_{ij}\big|$.

(3) $\|A\|_\infty=\max_{1\le i\le m}\sum_{j=1}^n\big|a_{ij}\big|$.

Proof. Assume that A is not a zero matrix.

(1) As we know A^*A is a hermitian and positive semi-definite matrix, there exists $\{e_1,e_2,\dots,e_3\}\subseteq\mathbb{K}^n$ and $\lambda_1\ge\lambda_2\ge\dots\ge\lambda_n\ge0$ such that $A^*Ae_i=\lambda e_i,\quad 1\le i\le n$ and $e_1,e_2,\dots,e_n$ are orthogonal and normalized.

For $x\in\mathbb{K}^n$, we have the decomposition $x=\sum_{i=1}^n c_ie_i$ for some $c_1,c_2,\dots,c_n\in\mathbb{K}$ and $\|x\|_2=\left(\sum_{i=1}^n|c_i|^2\right)^{\frac12}$. Then $\|Ax\|_2=(Ax)^*(Ax)=x^*A^*Ax=\sum_{i=1}^n\lambda_i|c_i|^2$. It follows that

$$\frac{\|Ax\|_2}{\|x\|_2}=\left(\frac{\sum_{i=1}^n\lambda_i|c_i|^2}{\sum_{i=1}^n|c_i|^2}\right)^{\frac12}\le\sqrt{\lambda_1}.$$

In the meantime, the above inequality can be achieved if $c_i = 0$ for all $i \in \{2, 2, \ldots, n\}$, this completes the proof of ((1)).

(2) For $x \in \mathbb{K}^n$,

$$\begin{aligned}\|Ax\|_1 &= \sum_{i=1}^m \left|(Ax)_i\right| = \sum_{i=1}^m \left|\sum_{j=1}^n a_{ij}x_j\right| \\ &\leq \sum_{i=1}^m \sum_{i=j}^n |a_{ij}||x_j| = \sum_{j=1}^n |x_j| \sum_{i=1}^m |a_{ij}| \leq \|A\|_{1,\infty} \|x\|_1 ,\end{aligned}$$

and this achieves equality for $x = e_{j_0}$ whose i-th entry is equal to 1 while others are equal to 0. This implies $\|A\|_1 \leq \|A\|_{1,\infty}$.

(3) Firstly, for $x \in \mathbb{K}^n$,

$$\|Ax\|_\infty = \max_{1\leq i\leq m}\left|\sum_{j=1}^n a_{ij}x_j\right| \leq \max_{1\leq i\leq m}\sum_{j=1}^n |a_{ij}||x_j| \leq \left(\max_{1\leq i\leq m}\sum_{j=1}^n |a_{ij}|\right)\|x\|_\infty ,$$

which implies $\|A\|_\infty \leq \max_{1\leq i\leq m}\sum_{j=1}^n |a_{ij}|$. And the inequality above achieves equality for $x = (x_1, x_2, \ldots, x_n)$ such that

$$x_j = \overline{(\operatorname{sgn} a_{i_0 j})}$$

where $i_0 = \arg\max_{1\leq i\leq m}\sum_{j=1}^n |a_i j|$. Indeed,

$$\|Ax\|_\infty = \max_{1\leq i\leq m}\left|\sum_{j=1}^n a_{ij}x_j\right| \geq \sum_{j=1}^n |a_{i_0 j}x_j| = \sum_{j=1}^n |a_{i_0 j}| \geq \sum_{j=1}^n |a_{i_0 j}|\|x\|_\infty .$$

This completes our proof. □

Let X, Y be two normed space. Denote by $\mathscr{B}(X,Y)$ the set of all bounded linear operators from X to Y. Define for $A, B \in \mathscr{B}(X,Y)$, $x \in X$ and $\alpha \in \mathbb{K}$,

$$(A+B)(x) = Ax + Bx, \quad (\alpha A)(x) = \alpha(Ax).$$

Then $\mathscr{B}(X,Y)$ is a linear space. For the sake of simplicity, if $X = Y$, we write $\mathscr{B}(X)$ for $\mathscr{B}(X,X)$.

Theorem 19.4. *Equipped with the operator norm $\|\cdot\|$, $\mathscr{B}(X,Y)$ is a normed space.*

Proof. It suffices to prove that $\|\cdot\|$ is a norm on $\mathscr{B}(X,Y)$. Here we only verify the triangle inequality since the other two condition is obvious. For $A, B \in \mathscr{B}(X,Y)$,

$$\begin{aligned}\|A+B\| &= \sup_{\|x\|=1}\|Ax+Bx\| \leq \sup_{\|x\|=1}\left(\|Ax\| + \|Bx\|\right) \\ &\leq \sup_{\|x\|=1}\|Ax\| + \sup_{\|x\|=1}\|Bx\| = \|A\| + \|B\| .\end{aligned}$$

□

Theorem 19.5. *Let X, Y be two normed space. If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.*

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $B(X, Y)$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}^*$ such that for $n, m > N$,

$$\|A_n - A_m\| < \epsilon \tag{19.1}$$

which implies

$$\begin{aligned}\|A_n x - A_m x\| &= \big\|(A_n - A_m)x\big\| \le \|A_n - A_m\|\|x\| \\ &< \epsilon\|x\|, \quad \text{for any } x \in X.\end{aligned}$$

Thus $\{A_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in Y for every $x \in X$. Since Y is a Banach space, these Cauchy sequences converges. Denote $Ax = \lim_{n\to\infty} A_n x$ for every $x \in X$. Clearly, A is a linear operator from X to Y. By Equation (19.1), there for $\|x\| = 1$,

$$\|A_n x - A_m x\| < \epsilon, \quad \text{for any } n, m > N.$$

Letting $m \to \infty$ yields for $\|x\| = 1$,

$$\|A_n x - Ax\| \le \epsilon, \quad \text{for any } n > N,$$

which means $\|A_n - A\| \to 0$ as $n \to \infty$, $A_n - A \in B(X, Y)$ and thus $A \in \mathscr{B}(X, Y)$. $\square$

In particular, the set $\mathscr{B}(X, \mathbb{K})$ of all bounded linear functionals on X is denoted by X^*, which is also known as the *dual space* or *conjugate space* of X. Since $\mathbb{K}$ is complete for the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the dual space of X is always a Banach space.

Definition 19.4. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{B}(X, Y)$ and $A \in \mathscr{B}(X, Y)$. If $\lim_{n\to\infty}\|A_n - A\| = 0$, then we say that $\{A_n\}_{n=1}^{\infty}$ *converges uniformly* to A. If for every $x \in X$,

$$\lim_{n\to\infty} A_n x = Ax,$$

then we say that $\{A_n\}_{n=1}^{\infty}$, then we say $\{A_n\}_{n=1}^{\infty}$ *converges strongly* to A.

Remark 12. It is not hard to verify that uniform convergence implies strong convergence. The converse holds true in finite dimensional space, but false in infinite dimensional space. For instance, let $X = Y = l^p (p \ge 1)$ and for all $n \in \mathbb{N}^*$,

$$A_n x = (x_n, x_{n+1}, \ldots), \quad \text{for any } x = (x_1, x_2, \ldots) \in X.$$

For every $x \in X$ and $\epsilon > 0$, there exists $N \in \mathbb{N}^*$ such that

$$\sum_{k=n}^{\infty} |x_k|^p < \epsilon, \quad \text{for any } n > N.$$

Therefore, for any $n > N$ and $x \in X$,

$$\|A_n x\| = \left(\sum_{k=n}^{\infty} |x_k|\right)^{\frac{1}{p}} < \epsilon^{\frac{1}{p}},$$

implying that $\{A_n x\}_{n=1}^{\infty}$ converges strongly to zero. However, since $\|A_n\| = 1$ for all $n \in \mathbb{N}^*$, sequence $\{A_n\}_{n=1}^{\infty}$ would never converges to zero operator.

19.2 Banach-Steinhaus Theorem

Theorem 19.6 (Banach-Steinhaus Theorem, Uniform Bounded Principle, The Resonance Theorem)**.** *Let X be a Banach space and Y be a normed space. Suppose that I is a index set and $\{T_\alpha : \alpha \in I\}$ is a subset of $\mathscr{B}(X, Y)$. If for every $x \in X$,*

$$\sup_{\alpha \in I} \|T_\alpha x\| < \infty,$$

then $\{T_n\}_{n=1}^\infty$ is uniformly bounded, i.e.,

$$\sup_{\alpha \in I} \|T_\alpha\| < \infty.$$

Proof. Define

$$p(x) = \sup_{\alpha \in I} \|T_\alpha x\|, \quad x \in X.$$

It is not difficult to verify that for every $x, y \in X$ and $\lambda \in \mathbb{K}$,

(1) $p(x) < \infty$;
(2) $p(\lambda x) = |\lambda|\, p(x)$;
(3) $p(x+y) \leq p(x) + p(y)$.

For every $k \in \mathbb{N}$, let

$$M_k = \{x \in X : p(x) \leq k\} = \bigcap_{\alpha \in I} \{x \in X : \|T_\alpha x\| \leq k\}.$$

As T_α is continuous on X,

$$\{x \in X : \|T_\alpha\| \leq k\}$$

is closed in X and consequently, M_k is closed in X.

Noticing that

$$\bigcup_{k \in \mathbb{N}} M_k = X,$$

by Baire's Category Theorem, X is the second category set, since X is a Banach space. Thus there exists some k_0 and nonempty closed ball $\overline{B} = \overline{B(x_0, r)}$ such that $\overline{B} \subseteq M_{k_0}$.

For any $x \in X$, $x_0 \pm \frac{x}{\|x\|} r_0 \in \overline{B}$. Hence,

$$\begin{aligned} p(\frac{2r_0 x}{\|x\|}) &= p\left(\left(x_0 + \frac{x}{\|x\|} r_0\right) - \left(x_0 - \frac{x}{\|x\|} r_0\right)\right) \\ &\leq p\left(x_0 + \frac{x}{\|x\|} r_0\right) + p\left(x_0 - \frac{x}{\|x\|} r_0\right) \leq 2k_0, \end{aligned}$$

which implies

$$\begin{aligned} p(x) &\leq \frac{2k_0}{2r_0} \|x\| = \frac{k_0}{r_0} \|x\| \\ &\Longrightarrow \|T_\alpha x\| \leq \frac{k_0}{r_0} \|x\|, \quad \text{for any } \alpha \in I \\ &\Longrightarrow \|T_\alpha\| \leq \frac{k_0}{r_0}, \quad \text{for any } \alpha \in I \\ &\Longrightarrow \sup_{\alpha \in I} \|T_\alpha\| < \infty. \end{aligned}$$

This completes the proof. □

Banach-Steinhaus Theorem is one of the most essential theorems in functional analysis. Here are some applications of it.

Theorem 19.7. *Let X be a normed space and Y be a Banach space. Suppose that $\{T_n\}_{n=1}^{\infty}$ is a sequence in $\mathscr{B}(X,Y)$ which satisfies*

(1) $\{T_n\}_{n=1}^{\infty}$ is uniformly bounded, i.e., $\sup_{n\in\mathbb{N}^*}\|T_n\| < \infty$;

*(2) there exists a dense subset G of X such that $\{T_n x\}_{n=1}^{\infty}$ converges on G (*i.e., *for all $x \in G$).*

Then $\{T_n\}_{n=1}^{\infty}$ converges strongly to some $T \in \mathscr{B}(X,Y)$ and

$$\|T\| \le \liminf_{n\to\infty}\|T_n\| .$$

Proof. Let $\epsilon > 0$ and

$$M = \sup_{n\in\mathbb{N}^*}\|T_n\| < \infty.$$

Since G is dense, for every $x \in X$, there exists $y \in G$ such that $\|y - x\| < \frac{\epsilon}{3M}$. By assumption, there exists $N \in \mathbb{N}^*$ such that for every $n, m > N$,

$$\|T_n y - T_m y\| < \frac{\epsilon}{3}.$$

It follows from

$$\begin{aligned}\|T_n x - T_m x\| &\le \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m x - T_m y\| \\ &< \|T_n\|\|x - y\| + \frac{\epsilon}{3} + \|T_m\|\|x - y\| \le 2M\|x - y\| + \frac{\epsilon}{3} < \epsilon\end{aligned}$$

that $\{T_n x\}_{n=1}^{\infty}$ is a Cauchy sequence, whence it converges to some element of X, since Y is a Banach space. Denoting

$$Tx = \lim_{n\to\infty} T_n x,$$

it is clear that T is a linear operator. Also,

$$\|Tx\| = \lim_{n\to\infty}\|T_n x\| \le \liminf_{n\to\infty}\|T_n\|\|x\| ,$$

implying that $\|T\| \le \liminf_{n\to\infty}\|Tn\|$. ☐

In Theorem 19.5, we have illustrated that if Y is a Banach space, then $\mathscr{B}(X,Y)$ is complete in the sense of uniform convergence. As an application of the Banach-Steinhaus Theorem, if we further assume that X is also a Banach space, then $\mathscr{B}(X,Y)$ is complete in the sense of strong convergence.

Theorem 19.8. *Let X, Y be two Banach spaces. Then $\mathscr{B}(X,Y)$ is complete in the sense of strong convergence.*

Proof. Let $\{T_n\}$ be such that

$$\lim_{n\to\infty} T_n x$$

exists for every $x \in X$. Then $\sup_{n\in\mathbb{N}^*}\|T_n x\| < \infty$ for every $x \in X$. By the Banach-Steinhaus Theorem 19.6, $\sup_{n\in\mathbb{N}^*}\|T_n\| < \infty$. And by the theorem above, there $\{T_n\}$ converges strongly to some linear bounded operator from X to Y. ☐

Now we apply the Banach-Steinhaus Theorem to prove the convergence of quadrature formula of integral. In numerical analysis, given a function on $[a, b] \subseteq \mathbb{R}$ and $\alpha_{nk} \in \mathbb{K}, t_{nk} \in \mathbb{R}, k \in \{1, 2, \ldots, n\}, n \in \mathbb{N}^*$ such that $a \leq t_{n1} \leq t_{n2} \leq \cdots \leq t_{nn} \leq b$, the quadrature formula

$$\sum_{k=1}^{n} \alpha_{nk} f(t_{nk})$$

is widely used to approximate the integral $\int_a^b f(t)\,\mathrm{d}$. It remains a problem that how can we choose coefficients α_{nk} such that

$$A_n(f) = \sum_{k=1}^{n} \alpha_{nk} f(t_{nk})$$

converges to $\int_a^b f(t)\,\mathrm{d}t$ as $n \to \infty$.

Theorem 19.9. *The quadrature formula converges for every every $f \in C[a, b]$, i.e.,*

$$\lim_{n \to \infty} A_n(f) = \int_a^b f(t)\,\mathrm{d}t$$

if and only if the following conditions hold true:

(1) there exists $M > 0$ such that $\sum_{k=1}^{k} |\alpha_{nk}| \leq M$;

(2) for every polynomial function p, $\{A_n(p)\}_{n=1}^{\infty}$ converges to $\int_a^b p(t)\,\mathrm{d}t$.

Proof. ($\Leftarrow$) Let $X = C[a, b]$, $Y = \mathbb{K}$ and define

$$Tf = \int_a^b f(t)\,\mathrm{d}t, \quad T_n f = \sum_{k=1}^{n} \alpha_{nk} f(t_{nk})$$

for every $n \in \mathbb{N}^*$. Observing that $\|T_n\| = \sum_{k=1}^{n} |\alpha_{nk}| < M$ for every $n \in \mathbb{N}$ and polynomial functions are dense in $C[a, b]$, it follows from Theorem 19.7 that $\{T_n\}_{n=1}^{\infty}$ converges strongly to T.

($\Rightarrow$) Assuming that $\{T_n\}_{n=1}^{\infty}$ converges strongly on $C[a, b]$, $\{T_n x\}_{n=1}^{\infty}$converges bounded for every $x \in X$. Thus $\{\|T_n x\| = \sum_{k=1}^{n} |\alpha_{nk}|\}_{n=1}^{\infty}$ is bounded for every $x \in C[a, b]$. By the Banach-Steinhaus Theorem, condition (1) holds true. Condition (2) is obvious. □

Another well-known result of the Banach-Steinhaus Theorem is the divergence of the Fourier series. Denote by $C_{2\pi}$ the linear subspace of $C[-\infty, \infty]$ consisting of all 2π-periodic functions. We make $C_{2\pi}$ a normed space by equipping it with norm

$$\|f\|_{\infty} = \max_{t \in \mathbb{R}} \left| f(t) \right|, \quad \text{for every } f \in C_{2\pi}.$$

One can show that $C_{2\pi}$ is a Banach space. Recall that for $f \in C_{2\pi}$, its Fourier series is

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$$

where

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,\mathrm{d}t, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,\mathrm{d}t. \end{aligned}$$

We are concerned with the convergence of

$$\begin{aligned} (S_n f)(t) &= \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,\mathrm{d}s + \sum_{k=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\,(\cos ks \cos kt + \sin ks \sin kt)\,\mathrm{d}s \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left(\frac{1}{2} + \sum_{k=1}^{n} \cos k(s-t)\right)\mathrm{d}s \\ &= \int_{-\pi}^{\pi} f(s) K_n(s,t)\,\mathrm{d}s \end{aligned}$$

where $K_n(s,t) = \frac{\sin(n+\frac{1}{2})(s-t)}{2\pi \sin\frac{1}{2}(s-t)}$ is the Dirichlet kernel. We show that for every $t \in [\pi, 2\pi]$, there exists $f \in C_{2\pi}$ such that $\{S_n(f)\}_{n=1}^{\infty}$ is divergent. Without loss of generality, assume that $t = 0$. Let

$$T_n f = (S_n f)(0) = \int_{-\pi}^{\pi} f(s) K_n(s,0)\,\mathrm{d}s.$$

By Example 19.5,

$$\begin{aligned} \|T_n\| &= \frac{1}{2\pi} K_n(s,0)\,\mathrm{d}s \\ &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|\sin(n+\frac{1}{2}s)\right|}{\left|\frac{s}{2}\right|}\,\mathrm{d}s \\ &= \frac{2}{\pi}\int_{0}^{\pi} \frac{\left|\sin(n+\frac{1}{2}s)\right|}{s}\,\mathrm{d}s \\ &= \int_{0}^{(2n+1)\pi} \frac{|\sin u|}{u}\,\mathrm{d}u \to \infty \end{aligned}$$

as $n \to \infty$. Here,

$$\begin{aligned} \int_0^{\infty} &\geq \sum_{n=1}^{\infty} \int_{2(n-1)\pi}^{2n\pi} \frac{\sin u}{u}\,\mathrm{d}u \\ &\geq \sum_{n=1}^{\infty} \int_{2(n-1)\pi+\frac{4}{\pi}}^{2n\pi+\frac{4}{\pi}} \frac{\sin u}{u}\,\mathrm{d}u \\ &\geq \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty} \frac{1}{2n\pi}\frac{\pi}{4} \to \infty. \end{aligned}$$

Thus $\{\|T_n\|\}_{n=1}^{\infty}$ is unbounded. By the Banach-Steinhaus Theorem, there exists $f \in C_{2\pi}$ such that $\|T_n f\|$ is unbounded. Hence for such f, $T_n f$ does not converge.

19.3 Open Mapping Theorem and Closed Graph Theorem

Let X, Y, Z be normed space, $T_1 \in \mathscr{B}(X,Y)$ and $T_2 \in \mathscr{B}(Y,Z)$. Then we define the product of operators as $T = T_2T_1 = T_2 \circ T_1$. Clearly, $T \in \mathscr{B}(X,Z)$. Also,

$$\|Tx\| = \left\|T_2(T_1x)\right\| \le \|T_2\|\|T_1x\| \le \|T_2\|\|T_1\|\|x\|,$$

which implies that $\|T\| = \|T_2T_1\| \le \|T_2\|\|T_1\|$.

Definition 19.5 (inverse operator)**.** Let X, Y be linear operator and T is a bounded linear operator from $D(T) \subseteq X$ to Y. We say T is *invertible* if there exists a linear operator $T^{-1} : R(T) \to D(T)$ such that

$$T^{-1} \circ T = I_{D(T)}, \quad T \circ T^{-1} = I_{R(T)}$$

where $I_{D(T)} : D(T) \to D(T), I_R(T) : R(T) \to R(T)$ are the identity operators. We call T^{-1} is the *inverse (operator)* of T.

Clearly, T admits the inverse T^{-1} if and only if T is one-to-one. Besides, the inverse operator of a invertible operator is unique.

Theorem 19.10. *Let X, Y be two normed spaces and $T : X \to Y$ be a surjective linear operator. If there exists a positive real number m such that for every $x \in X$.*

$$\|Tx\| \ge m\|x\|,$$

then T has a bounded inverse operator T^{-1}.

Proof. Obviously, T is injective and thus has an inverse operator T^{-1}. It remains to show that T^{-1} is bounded. For $y \in Y$,

$$\|y\| = \left\|TT^{-1}y\right\| \ge m\left\|T^{-1}y\right\|,$$

which implies that

$$\left\|T^{-1}y\right\| \le \frac{1}{m}\|y\|.$$

It follows that $T^{-1} \in \mathscr{B}(X,Y)$. ☐

For $T \in \mathscr{B}(X)$, denote $T^n = T \circ T^{n-1}$ for $n = \mathbb{N}^*$ and $T^0 = I_X$. Then $\|T^n\| \le \|T\|^n$ and $T^n \in \mathscr{B}(X)$.

Theorem 19.11. *Let X be a Banach space and $T \in \mathscr{B}(X)$. If $\|T\| < 1$, then $I - T$ has a bounded inverse operator and $\left\|(I-T)^{-1}\right\| \le \frac{1}{1-\|T\|}$.*

Proof. Consider the series $S = \sum_{k=1}^{\infty} T^k$ and let $S_n = \sum_{k=1}^{n} T^k$. For $n, m \in \mathbb{N}^*$, $m > n$,

$$\|S_m - S_n\| = \left\|\sum_{k=n}^{m-1} T^k\right\| \le \sum_{k=n}^{m-1} \|T\|^k$$

is a Cauchy sequence as $\|T\| < 1$. Since $\mathscr{B}(X,Y)$ is a Banach space, $\{S_n\}_{n=1}^{\infty}$ has a limit, say S in $\mathscr{B}(X,Y)$. Observe that

$$\begin{aligned}(I-T)S &= \lim_{n\to\infty}(I-T)\sum_{k=1}^{n}T^k \\ &= \lim_{n\to\infty}(I+T+\cdots+T^{n-1})-(T+T^2+\ldots+T^n) \\ &= \lim_{n\to\infty}(I-T^n) = I.\end{aligned}$$

Likewise, $S(I-T)=I$. Therefore, $I-T)^{-1}=S$ and

$$\|S\| = \left\|\sum_{n=0}^{\infty}T^n\right\| \le \sum_{n=0}^{\infty}\|T\|^n = \frac{1}{1-\|T\|}.$$

This completes the proof. □

Corollary 19.1. *Let X be a Banach space and $T \in \mathscr{B}(X)$ have a bounded inverse operator. Then for any operator $\Delta T \in \mathscr{B}(X)$ with*

$$\|\Delta T\| < \frac{1}{\|T^{-1}\|},$$

$T+\Delta T$ has a bounded inverse operator and

$$(T+\Delta T)^{-1} = \sum_{k=0}^{\infty}(-1)^k\left(T^{-1}\Delta T\right)^k T^{-1}.$$

Proof. Let $S = T+\Delta T = T(I+T^{-1}\Delta T)$. As $\left\|T^{-1}\Delta T\right\| \le \left\|T^{-1}\right\|\|\Delta T\| < 1$, $I + T^{-1}\Delta T$ has the bounded inverse

$$\left(I+T^{-1}\Delta T\right)^{-1} = \sum_{n=0}^{\infty}(-1)^n\left(T^{-1}\Delta T\right)^n.$$

Then

$$S^{-1} = \left(I+T^{-1}\Delta T\right)^{-1}T^{-1} = \sum_{k=0}^{\infty}(-1)^k\left(T^{-1}\Delta T\right)^k T^{-1}$$

is bounded. □

Now we proceed with the main theorems in this section: the Open Mapping Theorem and the Closed Graph Theorem. Note that most theorems requires completeness of the space.

Definition 19.6 (Open mapping)**.** A mapping that maps every open set to an open set is called an open mapping.

Theorem 19.12 (Banach Open Mapping Theorem)**.** *Let X, Y be Banach spaces and $T : X \to Y \in \mathscr{B}(X,Y)$. If T is surjective, then T is an open mapping.*

Proof. **Step 1.** As $X = \bigcup_{k=1}^{\infty} \overline{B(0,k)}$, $Y = TX = \bigcup_{k=1}^{\infty} T\overline{B(0,k)}$ and Y is complete, by Baire's Category Theorem, there exists some $k_0 \in \mathbb{N}^*$ such that $T\overline{B(0,k_0)}$ is dense in some open ball $B_Y(y_0, r_0)$ in Y. Here, we use B and B_Y to denote the open balls in X and Y respectively.

Step 2. We claim that there exists $\delta > 0$ such that for any $\epsilon > 0$, $T\overline{B(0,\epsilon)}$ is dense in $\overline{B_Y(0,\delta\epsilon)}$. Setting $\delta = \frac{r_0}{k_0}$, $\epsilon > 0$ and letting $y \in \overline{B_Y(0,\epsilon\delta)}$, then $y_0 \pm \frac{k_0}{\epsilon} y$ satisfies

$$\left\| y_0 \pm \frac{k_0}{\epsilon} y - y_0 \right\| = \frac{k_0}{\epsilon} \|y\| < \frac{k_0}{\epsilon} \epsilon\delta = r_0,$$

which is $y_0 \pm \frac{k_0}{\epsilon} y \in B_Y(y_0, r_0)$. Hence, we can find a sequence $\{x_n\}_{n=1}^{\infty}$, $\{x'_n\}_{n=1}^{\infty}$ in $\overline{B(0,k_0)}$ such that

$$Tx_n \to y_0 + \frac{k_0}{\epsilon} y, \quad Tx'_n \to y_0 - \frac{k_0}{\epsilon} y \quad (n \to \infty)$$

since $T\overline{B(0,k_0)}$ is dense in $\overline{B(0,k_0)}$. Then

$$T\left(\frac{\epsilon}{2k_0} (x_n - x'_n) \right) \to y \quad (n \to \infty)$$

Note that

$$\frac{\epsilon}{2k_0}(x_n - x'_n) \subseteq \overline{B(0,\epsilon)}.$$

Thus we succeed to prove our claim that $T\overline{B(0,\epsilon\delta)}$ is dense in $B_Y(0,\epsilon\delta)$ for any $\epsilon > 0$.

Step 3. Now we try to prove the key claim: for the δ given above and any $r > 0$, it holds that $T\overline{B(0,r)} \supseteq B_Y(0, \frac{1}{2} r\delta)$. By the linearity of T, it suffices to prove the case $r = 1$. Choosing $y_0 \in B_Y(0, \frac{1}{2}\delta)$, by previous claim, there exists $x_1 \in \overline{B(0, \frac{1}{2})}$ such that

$$\|Tx_1 - y_0\| < \frac{1}{2^2}\delta.$$

Further, let $y_1 = y_0 - Tx_1 \in B_Y(0, \frac{1}{2^2}\delta)$, there exists $x_2 \in \overline{B(0,k_0)}$ such that

$$\|y_1 - Tx_2\| < \frac{1}{2^3}\delta.$$

Repeating the process, we obtain sequences $\{x_n\}_{n=1}^{\infty} \subseteq X$ and $\{y_n\}_{n=1}^{\infty} \subseteq Y$ such that $\|x_n\| \le \frac{1}{2^n}$, $y_n = y_{n-1} - Tx_n = y_0 - T\left(\sum_{i=1}^{n} x_n\right)$ and $\|y_n\| < \frac{1}{2^{n+1}}\delta$. Tt follows from the completeness of X and the fact that $\sum_{n=1}^{\infty} \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ that $\sum_{n=1}^{\infty} x_n$ is convergent to some element x_0 say, of X such that $\|x_0\| < 1$. Letting $n \to \infty$ yields that $y_0 = Tx_0$.

Step 4. Now we are prepared to show that T is a open mapping. Let G be an open subset in X and $x \in G$. Then there exists $r_0 > 0$ such that $\overline{B(x, \frac{r_0}{2})} \subseteq B(x, r_0) \subseteq G$. By Step 3, $T\overline{B(0,r)} \supseteq B_Y(0, \frac{1}{2} r\delta)$. Hence,

$$\begin{aligned} T\overline{B(x, \frac{r_0}{2})} &= T\left(x + \overline{B(0, \frac{r_0}{2})} \right) \\ &= Tx + T\overline{B(0,r)} \supseteq Tx + S_1(0, \frac{1}{4} r_0\delta), \end{aligned}$$

which implies that Tx is a inner point of TG. Thus TG is an open subset in Y and we establish the conclusion. ☐

The famous Inverse Mapping Theorem is an application of the Open Mapping Theorem.

Theorem 19.13 (Inverse Mapping Theorem)**.** *Let T be a bijective linear operator from a Banach space X to a Banach space Y. Then $T^{-1} \subset \mathscr{B}(Y, X)$.*

Proof. It suffices to show that for every open subset $G \subseteq X$, $\left(T^{-1}\right)^{-1}(G)$ is open in Y. Note that $\left(T^{-1}\right)^{-1}(G) = T(G)$. By the Open Mapping Theorem, TG is open in Y, which proves the result. □

Corollary 19.2. *Let $\|\cdot\|$ and $\|\cdot\|$ be two norms on a linear space X that is complete under each of these two norms. If there exists a constant $C > 0$ such that for every $x \in X$,*

$$\|\cdot\|_2 \leq C\|\cdot\|_1 ,$$

then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. Letting $X = (X, \|\cdot\|_1), X_2 = (X, \|\cdot\|_2)$, then by assumption, both X_1, X_2 are Banach space. Define

$$\begin{aligned} T : X_1 &\to X_2 \\ x &\mapsto x. \end{aligned}$$

It follows that for every $x \in X$, $\|Tx\|_2 \leq C\|x\|_1$, namely, $T \in \mathscr{B}(X_1, X_2)$. Since T is bijective, by the Inverse Mapping Theorem, T^{-1} is a bounded linear operator from X_2 to X_1. This means there exists a constant $D > 0$ such that for every $x \in X$,

$$\|x\|_1 \leq \left\|T^{-1}x\right\| \leq D\|x\|_2 .$$

Thus the conclusion follows immediately. □

Definition 19.7 (Graph)**.** Let X, Y be normed space and $T : X \to Y$ is a linear operator. The *graph* of T is the set

$$G(T) = \{(x, Tx) : x \in X\}.$$

Then $G(T)$ is a subset of $X \times Y$ on which we define a norm as

$$\left\|(x, y)\right\| = \|x\|_X + \|y\|_Y , \quad \text{for every } (x, y) \in X \times Y.$$

If T is a linear operator, then $G(T)$ is clearly a linear subspace of $X \times Y$. T is said to be a *closed operator* if $G(T)$ is a closed subspace of $X \times Y$.

Proposition 19.3. *Let X, Y be normed spaces and $T : X \to Y$ be a linear operator. Then T is a closed operator if and only if for every sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$,*

$$x_n \to x, \quad Tx_n \to y \quad (n \to \infty) \implies y = Tx.$$

Proof. This is a direct result from the definition of closed operator. □

Theorem 19.14 (Closed Graph Theorem)**.** *Let X, Y be Banach space and $T : X \to Y$ is a linear operator. Then T is bounded if and only if its graph $G(T)$ is closed in $X \times Y$,* i.e., *T is a closed operator.*

Proof. ($\Rightarrow$) Suppose T is bounded and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $(x_n, Tx_n) \to (x, y)$ as $n \to \infty$. Then

$$y = \lim_{n\to\infty} Tx_n = T\left(\lim_{n\to\infty} x_n\right) = Tx,$$

which implies $(x, y) = (x, Tx) \in G(T)$. Hence T is a closed operator.

($\Leftarrow$) Suppose that T is a closed operator. Then $G(T)$ is a linear closed subspace in $X \times Y$. Since X, Y are Banach spaces, so is $X \times Y$. Define

$$\begin{aligned} \tilde{T} : G(T) &\to X, \\ (x, Tx) &\mapsto x. \end{aligned}$$

It is clear that $\tilde{T}$ is bijective. As $\left\|\tilde{T}(x, Tx)\right\| = \|x\| \le \|x\| + \|Tx\| = \left\|(x, Tx)\right\|$, $\tilde{T}$ is bounded. By the Inverse Mapping Theorem, there exists $C > 0$ such that

$$\left\|\tilde{T}^{-1}x\right\| = \left\|(x, Tx)\right\| \le C\|x\|,$$

which is

$$\|Tx\| \le (C-1)\|x\|.$$

This completes the proof. □

19.4 Spectrum of Linear Operators

We consider the linear operator

$$T_\lambda = \lambda I - T.$$

Definition 19.8. Let X be a normed space and T be a linear operator from $D(T) \subseteq X$ to X. For $\lambda \in \mathbb{K}$, there are several possible cases:

(1) $R(T_\lambda)$ has a dense range in X and admits a bounded inverse. We call λ a *regular point* of T and denote by $\rho(T)$ the set of all regular points of T. Then $\rho(T)$ is called the *resolvent set* of T and $\sigma(T) := \mathbb{K}\backslash\rho(T)$ is called the *spectrum* of T. And the inverse $(I - \lambda T)^{-1}$ denoted by $R(\lambda, T)$ is called the *resolvent* at λ of T.

(2) T_λ is not injective, namely, there exists $x \in X, x \neq 0$ such that $(T - \lambda I)x = 0$. In this case, T_λ does not have an inverse. We call λ is an *eigenvalue* of T and x an *eigenvector* of T. The nullspace of T_λ is called the *eigenspace* of T corresponding to the eigenvalue λ of T. The dimension of the eigenspace corresponding to the eigenvalue λ of T is called the *multiplicity* of λ. The set $\sigma_p(T)$ of all eigenvalues of T is called the *point spectrum* of T.

(3) T_λ has an unbounded inverse whose domain $R(\lambda I - T)$ is dense in X. We call the totality of such λ the *continuous spectrum* of T, denoted by $\sigma_C(T)$.

(4) T_λ has an inverse whose domian is not dense in X. We call the totality of such λ the *residual spectrum* of T, denoted by $\sigma_R(T)$.

Theorem 19.15. *Let X be a Banach space and $T \in B(X)$. Then $\sigma(T)$ is bounded and closed. In fact, $\sigma(T) \subseteq \{\lambda \in \mathbb{K} : |\lambda| \le \|T\|\}$.*

Proof. For $|\lambda_0| > \|T\|$, then $\lambda_0 I - T = \lambda_0(I - \frac{1}{\lambda_0}T)$. Since $\left\|\frac{1}{\lambda_0}T\right\| = \frac{\|T\|}{\lambda_0} < 1$, by Theorem 19.11, $I - \lambda_0 T$ has an bounded inverse and

$$(\lambda_0 I - T)^{-1} = \frac{1}{\lambda_0}\sum_{k=0}^{\infty}\left(\frac{T}{\lambda_0}\right)^k.$$

Therefore, $\lambda_0 \in \rho(T)$ and $\sigma(T) \subseteq \{\lambda \in \mathbb{K} : |\lambda| \leq \|T\|\}$. To prove that $\sigma(T)$ is closed, it suffices to show that $\rho(T)$ is open. Letting λ_0 be a resolvent point of T, then by Corollary 19.1, for $\delta < \left\|(\lambda I - T)^{-1}\right\|$, $\lambda I - T - \delta I$ has bounded inverse. Thus $(\lambda_0 - \delta, \lambda_0 + \delta)$ is contained in $\rho(T)$, whence $\rho(T)$ is a open set. ☐

Example 19.6. Consider $T : C[a,b] \to C[a,b], T(f)(t) = tf(t)$.

For $\lambda \in \mathbb{R}$, $\left[(T - \lambda T)f\right](t) = (t-\lambda)f(t)$. Thus if $\lambda \notin [a,b]$, $\left[(T-\lambda T)^{-1}f\right](t) = \frac{1}{t-\lambda}f(t)$. If $\lambda \in [a,b]$, then $\left[(T-\lambda T)f\right](\lambda) = (\lambda - \lambda)f(\lambda) = 0$. Thus $T - \lambda I$ is not surjective and $\sigma(T) = [a,b]$. Suppose that $(I - \lambda T)f = 0$. Then

$$(t-\lambda)f(t) \equiv 0 \implies f(t) = 0 \quad \text{for every } t \neq \lambda \implies f(t) \equiv 0.$$

Hence, $\sigma_p(T) = \varnothing$.

19.5 Hahn-Banach Theorem and its Applications

For convenience, we introduce the notation of restriction of operators. Let X, Y be sets and let $f : X \to Y$. If D is a subset of X, then we refer to f_{χ_D} as a function from D to Y such that $f_{\chi_D}(x) = f(x)$ for all $x \in D$. If we further let X be equipped with a norm $\|\cdot\|$, then $(G, \|\cdot\|_D)$ denotes the normed subspace of $(X, \|\cdot\|)$ with $\|x\|_D = \|x\|$ for all $x \in D$. If $f \in D^*$, then the operator norm of f is denoted by $\|f\|_D$.

Theorem 19.16 (Hahn-Banach Theorem over $\mathbb{R}$)**.** *Let M be linear subspace of a linear space X over $\mathbb{R}$, and let $p : X \to \mathbb{R}$ be a sublinear functional,* i.e.*, for every $x, y \in X$ and $\alpha > 0$, it holds true that*

(1) $p(\alpha x) = \alpha p(x)$.

(2) $(x + y) \leq p(x) + p(y)$.

If f is a linear functional on M such that $f(x) \leq p(x)$ for all $x \in M$, then there exists a linear functional F on X such that

(1) $F_{\chi_M} = f$.

(2) $F(x) \leq p(x)$ for all $x \in X$.

Proof. Let

$$\Omega = \{(Y, F_Y) : Y \text{ is a linear subspace of } X, F_Y \text{ is a linear functional on } Y,$$
$$M \subseteq Y, F_{Y\chi_M} = f, F_Y(x) \leq p(x) \text{ for all } x \in X\}.$$

We introduce an ordering relation $\leq$ on Ω as

$$(Y_1, F_{Y_1}) \leq (Y_2, F_{Y_2}) \iff Y_1 \subseteq Y_2, \quad F_{Y_1\chi_{Y_1}} = F_{Y_2}.$$

We shall use the Zorn's Lemma 1.3 on Ω. To this end, we first show that every totally ordered subset of Ω has an upper bound. Let Ω_0 be a totally ordered subset of Ω, and define

$$Y_0 = \bigcup_{Y \subseteq \Omega_0} Y$$
$$F_{Y_0}(x) = F_{Y_n}(x), \quad x \in Y_n.$$

Then $(Y_0, F_{Y_0}) \subseteq \Omega$ and for every $Y \subset \Omega$, $(Y, F_Y) \leq (Y_0, F_{Y_0})$. It follows from Zorn's Lemma that Ω contains a maximal element (Y_∞, F_{Y_∞}).

We contend that $Y_\infty = X$. For the sake of contradiction we assume that there exists $x_0 \in X$ such that $x_0 \in Y_\infty$. Letting $\tilde{Y} = \operatorname{span} Y \cup \{x_0\}$, then every $x \in tildeY$ can be written uniquely as the form $x = y + \alpha x_0$ with $y \in Y_\infty$ and $\alpha \in \mathbb{R}$. Define a linear functional extending F_{Y_∞} on $\tilde{Y}$ as

$$\tilde{F}(x) = \tilde{F}(y + \alpha x_0) = \tilde{F}(y) + \alpha \tilde{F}(x_0) = F_\infty(y) + \alpha\beta$$

where $\beta = \tilde{F}(x_0)$ should be a constant real number. Besides, we hope that $\tilde{F}$ also obeys

$$\tilde{F}(y + \alpha x_0) = F_{Y_\infty}(y) + \alpha\beta \leq p(y + \alpha x_0) \quad \text{for all } y \in Y_\infty, \alpha \in \mathbb{R}. \tag{19.2}$$

If $\alpha = 0$, then Equation (19.2) is satisfied by definition; if $\alpha > 0$, then it follows that

$$\begin{aligned}
&F_{Y_\infty}(y) + \alpha\beta \leq p(y + \alpha x_0) \quad \text{for all } y \in Y_\infty, \alpha \in \mathbb{R} \\
\iff &\beta \leq -F_{Y_\infty}(\frac{y}{\alpha}) + p(\frac{y}{\alpha} + x_0) \quad \text{for all } y \in Y_\infty, \alpha \in \mathbb{R} \\
\iff &\beta \leq -F_{Y_\infty}(y) + p(y + x_0) \quad \text{for all } y \in Y_\infty;
\end{aligned}$$

If $\alpha < 0$, likewise it follows that

$$\begin{aligned}
&F_{Y_\infty}(y) + \alpha\beta \leq p(y + \alpha x_0) \quad \text{for all } y \in Y_\infty, \alpha \in \mathbb{R} \\
\iff &F_{Y_\infty}(-\frac{y}{\alpha}) - p(-\frac{y}{\alpha} - x_0) \leq \beta \quad \text{for all } y \in Y_\infty, \alpha \in \mathbb{R} \\
\iff &F_{Y_\infty}(y) - p(y - x_0) \leq \beta \quad \text{for all } y \in Y_\infty.
\end{aligned}$$

Therefore, such β exists if and only if for any $y, y' \in Y_\infty$,

$$F_{Y_\infty}(y') - p(y' - x_0) \leq \beta \leq -F(y) + p(y + x_0)$$

which is equivalent to

$$\sup_{y \in Y_\infty} (F_{Y_\infty}(y) - p(y - x_0)) \leq \beta \leq \inf_{y \in Y_\infty} (-F_{Y_\infty}(y) + p(y + x_0)).$$

We claim that this is indeed the case by checking that for any $y, y' \in Y_\infty$,

$$\begin{aligned}
-F_{Y_\infty}(y) &+ p(y + x_0) - F_{Y_\infty}(y') + p(y' - x_0) \\
&= -F_{Y_\infty}(y + y') + p(y + x_0) + p(y' - x_0) \\
&\geq -F_{Y_\infty}(y + y') + p(y + y') \geq 0.
\end{aligned}$$

This means $(\tilde{Y}, \tilde{F}) \in \Omega$ and $(Y_\infty, F_{Y_\infty}) < (\tilde{Y}, \tilde{F})$, contradicting the fact that (Y_∞, F_{Y_∞}) is an maximal element. Hence $Y_\infty = X$, which completes the proof. $\square$

Theorem 19.17 (Hahn-Banach Theorem)**.** *Let X be a normed space, $G \subseteq X$ be a linear subspace, and let $f \in G^*$. Then there exists $F \in X^*$ such that $F_{\chi G} = f$ and $\|F\| = \|f\|_G$.*

Proof. We first prove the case that $\mathbb{K} = \mathbb{R}$. Define $p : X \to \mathbb{R}, x \mapsto \|f\|_G \|x\|$. Then for any $x \in G$,

$$f(x) \leq \left|f(x)\right| \leq \|f\|_G \|x\| = p(x).$$

By the Hahn-Banach Theorem over $\mathbb{R}$ 19.16, there exists a linear functional F such that $F_{\chi G} = f$ and $F(x) \leq p(x)$ for all $x \in X$. This means $F(x) \leq p(x) = \|f\|_G \|x\|$ and $F(-x) \leq p(x) = \|f\|_G \|x\|$, implying $\|F\| \leq \|f\|_G$ and $F \in X^*$. Since $\|F\| \geq \|f\|_G$, we have $\|f\|_G = \|F\|$. This is exactly what we want.

Next, we proceed with the case that $\mathbb{K} = \mathbb{C}$. Suppose $f(x) = \phi(x) + \mathrm{i}\psi(x)$, where ϕ, ψ are real-valued functions, which can be viewed as linear functionals over $\mathbb{R}$. Noticing that

$$f(\mathrm{i}x) = \phi(\mathrm{i}x) + \mathrm{i}\psi(\mathrm{i}x) = \mathrm{i}\phi(x) - \psi(x) = \mathrm{i}f(x),$$

we get $\phi(x) = \psi(ix)$ for all $x \in X$. Then it follows from the fact that for $x \in G$,

$$\phi(x) \leq \left|f(x)\right| \leq \|f\|_G \|x\| = p(x)$$

that there exists a bounded linear functional Φ such that

(1) $\Phi_{\chi G} = f$.

(2) $\Phi(x) \leq p(x) = \|f\|_G \|x\|$ for all $x \in X$.

by the Hahn-Banach Theorem over $\mathbb{R}$.

Define $F(x) = \Phi(x) - \mathrm{i}\Phi(\mathrm{i}x)$. Then it is easy to show the following three facts:

(1) F is a linear functional over $\mathbb{K}$.

(2) $F_{\chi G} = f$.

(3) $\left|f(x)\right| \leq p(x)$ for all $x \in X$. Indeed, for any $x \in X$, assuming that $F(x) = \mathrm{e}^{\mathrm{i}\theta}\left|F(x)\right|$, then

$$\left|F(x)\right| = e^{-\mathrm{i}\theta}F(x) = F(\mathrm{e}^{-\mathrm{i}\theta}x) = \Phi(\mathrm{e}^{-\mathrm{i}\theta}x) \leq \|f\|_G \left\|\mathrm{e}^{-\mathrm{i}\theta}x\right\| = \|f\|_G \|x\|.$$

Repeating the process with which we proved the case of real linear space finally completes the proof. □

Corollary 19.3. *LEt X be a normed space, and let $x_0 \in X$ with $x_0 \neq 0$. Then there exists an $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.*

Proof. Let $G = \operatorname{span}\{x_0\}$ and define $f_0 : G \to \mathbb{K}$ as $f_0(\alpha x_0) = \alpha\|x_0\|$ for all $x \in X$. Then $f_0 \in G^*$. By the Hahn-Banach Theorem, f_0 extends to some $f \in X^*$ that satisfies our requirement. □

Corollary 19.4. *Let X be a normed space, and let G be a subspace of X. If $x \in X$ satisfies $d = d(x_0, G) = \inf_{x \in G}\|x, x_0\| > 0$, then there exists $f \in X^*$ such that $f(x_0) = 1$, $f(x) = 0$ for any $x \in G$ and $\|f\| = \frac{1}{\alpha}$.*

Proof. Let $G_1 = \operatorname{span} G \cup \{x_0\}$ and define $f_1 : G_1 \to \mathbb{R}$ as $(\alpha x_0 + x) = \alpha$, which clearly belongs to G_1. We compute

$$\|f\|_{G_1} = \sup_{\alpha x_0 + x \neq 0} \frac{\left|f_1(\alpha x_0 + x)\right|}{\|\alpha x_0 + x\|} = \sup_{\substack{\alpha \neq 0 \\ x \in G}} \frac{|\alpha|}{\|\alpha x_0 + x\|}$$
$$= \sup_{\substack{\alpha \neq 0 \\ x \in G}} \frac{1}{\left\|x_0 + \frac{x}{\alpha}\right\|} = \sup_{x \in G} \frac{1}{\|x - x_0\|} = \frac{1}{d}.$$

Then the conclusion follows as a result of the Hahn-Banach Theorem. □

Proposition 19.4. *Let X be a normed space. Then*

$$\|x\| = \sup_{\substack{f \in X^* \\ f \neq 0}} \frac{f(x)}{\|f\|} = \sup_{\substack{f \in X^* \\ \|f\| = 1}} f(x)$$

Proof. Here we only deal with the first equality. We assume without loss of generality that $x \in X$ is such that $x \neq 0$. For any $f \in X^*$ such that $f \neq 0$, $\left|f(x)\right| \leq \|f\|\|x\|$ yields $\frac{\left|f(x)\right|}{\|f\|} \leq \|x\|$. Therefore, $\|x\| \geq \sup_{\substack{f \in X^* \\ f \neq 0}} \frac{f(x)}{\|f\|}$.

On the other hand, by the Hahn-Banach Theorem, there exists an $f_0 \in X^*$ such that $f_0(x) = 1$ and $\|f_0\| = 1$. Hence,

$$\sup_{\substack{f \in X^* \\ f \neq 0}} \frac{f(x)}{\|f\|} \geq \frac{\|x\| \, f_0(x)}{\|f_0\|} = \|x\|,$$

which completes the proof. □

Chapter 20

Hilbert Spaces

Hilbert space is an extension of the Euclidean space. An essential characteristic of Euclidean space is that inner product can be defined on it.

20.1 Inner Product

Definition 20.1 (inner product)**.** [inner product space] An *inner product* $< \cdot, \cdot >$: $X \times X \to \mathbb{K}$ on on a linear space X over $\mathbb{K}$ satisfy the following conditions

(1) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$ (positive definiteness);

(2) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ (conjugate symmetry);

(3) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ (bilinearity),

for any $u, v, w \in X$ and $\alpha, \beta \in \mathbb{K}$.

A linear space equipped with an inner product is called a *pre-Hilbert (inner product space)* space.

Theorem 20.1 (Cauchy-Schwarz inequality)**.** *Suppose X is a pre-Hilbert space. Then for any $u, v \in X$,*

$$\left|\langle u, v\rangle\right| \leq \langle u, u\rangle^{\frac{1}{2}} \langle v, v\rangle^{\frac{1}{2}}. \tag{20.1}$$

Or with notation of norm,

$$\left|\langle u, v\rangle\right| \leq \|u\| \|v\|. \tag{20.2}$$

Proof. Assume that $v \neq 0$; put $\alpha = \frac{\langle u,v\rangle}{\langle v,v\rangle}$ and then we get

$$\begin{aligned} 0 &\leq \langle u - \alpha v, u - \alpha v\rangle \\ &= \langle u, u\rangle - \frac{\left|\langle u, v\rangle\right|^2}{\langle v, v\rangle} - \frac{\left|\langle u, v\rangle\right|^2}{\overline{\langle v, v\rangle}} - \frac{\left|\langle u, v\rangle\right|^2}{\langle v, v\rangle^2}\langle v, v\rangle \\ &= \langle u, u\rangle - \frac{\left|\langle u, v\rangle\right|^2}{\langle v, v\rangle}, \end{aligned} \tag{20.3}$$

which is exactly what we want. $\square$

Theorem 20.2. *Each pre-Hilbert space X over $\mathbb{K}$ is a normed space with respect to the norm*

$$\|v\| := \langle u, u\rangle^{\frac{1}{2}}. \tag{20.4}$$

Proof. Suppose $u, v \in X$ and $\alpha \in \mathbb{K}$. Then by positive definiteness of inner product, we get

$$\|\alpha u\| \geq 0 \text{ and } \|u\| = 0 \text{ if and only if } u = 0. \tag{20.5}$$

By conjugate symmetry of inner product,

$$\|\alpha u\| = \langle \alpha u, \alpha u\rangle^{\frac{1}{2}} = \Big(|\alpha|^2 \langle u, u\rangle\Big)^{\frac{1}{2}} = |\alpha| \|u\|. \tag{20.6}$$

And finally, by Cauchy-Schwarz inequality,

$$\Re(\langle u, v\rangle) \leq \big|\langle u, v\rangle\big| \leq \langle u, u\rangle^{\frac{1}{2}} \langle v, v\rangle^{\frac{1}{2}} = \|u\| \|v\|. \tag{20.7}$$

So

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + \|v\|^2 + 2\Re(\langle u, v\rangle) \\ &\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\ &= (\|u\| + \|v\|)^2. \end{aligned} \tag{20.8}$$

□

Hence, every pre-Hilbert space is a normed space together with the norm induced by its inner product. Henceforth, by the norm of a pre-Hilbert space we implicitly mean the induced norm.

Definition 20.2 (Hilbert space)**.** A *Hilbert space* is a complete pre-Hilbert space.

There is no much difficulty in verifying the following facts.

Example 20.1. $\mathbb{R}^n$ is a Hilbert space together with the inner product defined as

$$\langle x, y\rangle = \sum_{k=1}^{n} x_i y_i$$

for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

Example 20.2. l^2 is a Hilbert space together with the inner product

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_i \overline{y_i}$$

for $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in l^2$.

Example 20.3. $L^2[a, b]$ is a Hilbert space together with the inner product

$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}\, \mathrm{d}t, \quad \text{for all } x, y \in L^2[a, b].$$

Example 20.4. Denote by H the linear space of all complex valued continuous functions on $[a, b]$, and define

$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}\, \mathrm{d}t, \quad \text{for all } x, y \in H.$$

Then H is a pre-Hilbert space but not a Hilbert space.

Theorem 20.3. *The inner product of a pre-Hilbert space H is a continuous function on $H \times H$,* i.e., *for any $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset H$,*

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y \implies \lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

Proof. This follows from the fact

$$\begin{aligned}\left|\langle x_n, y_n\rangle - \langle x, y\rangle\right| &= \left|\langle x_n, y_n\rangle - \langle x, y_n\rangle + \langle x, y_n\rangle - \langle x, y\rangle\right| \\ &\le \left|\langle x_n, y_n\rangle - \langle x, y_n\rangle\right| + \left|\langle x, y_n\rangle - \langle x, y\rangle\right| \\ &\le \|x_n - x\|\|y\| + \|x\|\|y_n - y\| \to 0\end{aligned}$$

as $n \to \infty$. □

Theorem 20.4 (Parallelogram Law)**.** *Let H be a Hilbert space. If $x, y \in H$, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof. It follows from

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + \Re\langle x, y\rangle + \|y\|^2$$

that

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + \Re\langle x, y\rangle + \|y\|^2, \\ \|x-y\|^2 &= \|x\|^2 - \Re\langle x, y\rangle + \|y\|^2.\end{aligned} \tag{20.9}$$

Then we add and complete the proof. □

In particular, if $\langle x, y\rangle = 0$, then $\|x+y\| = \|x-y\|$ since $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\Re\langle x, y\rangle$. It follows that $\|x+y\| = \|x\|^2 + \|y\|^2$, which is the famous Pythagorean Theorem in general case.

Theorem 20.5 (Polarization Identity)**.** *Let H be a pre-Hilbert space over $\mathbb{K}$.*

(1) If $\mathbb{K} = \mathbb{R}$ and $x, y \in H$, then

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2). \tag{20.10}$$

(2) If $\mathbb{K} = \mathbb{C}$ and $x, y \in H$, then

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\|x - \mathrm{i}y\| - \mathrm{i}\|x + \mathrm{i}y\|). \tag{20.11}$$

Proof. They are straightforward results of the Equation (20.9). □

In fact, a normed space that obeys parallelogram law is essentially pre-Hilbert space.

Theorem 20.6. *Let X be a normed space. If the norm $\|\cdot\|$ satisfies parallelogram* *law, then there exists an inner product* $\langle\cdot,\cdot\rangle$ *by which* $\|\cdot\|$ *is induced,* i.e.,

$$\|x\| = \sqrt{\langle x, x\rangle} \quad \text{for all } x \in X.$$

Proof. If the assumption of the theorem is satisfied, then it can be shown that inner product of form 20.10 ($\mathbb{K} = \mathbb{R}$) or of form 20.11 ($\mathbb{K} = \mathbb{C}$) obeys the requirements of inner product. □

20.2 Orthogonality

Definition 20.3 (Orthogonal)**.** Let H be a pre-Hilbert space. We say $x, y \in H$ are orthogonal, denoted by $x \perp y$, if $\langle x, y\rangle = 0$. Let M, N be subsets of H, then by $x \perp M$ we mean x is orthogonal to every element in H, and by $M \perp N$ we mean $x \perp y$ for any $x \in M$ and $y \in N$.

Letting M be a subset in a pre-Hilbert space H, then we call

$$M^{\perp} = \{x \in H : x \perp M\}$$

the orthogonal complement of M.

Theorem 20.7. *Let H be a pre-Hilbert space.*

(1) If $x, y, z \in H$ are such that $x = y + z$ and $y \perp x$, then

$$\|x\|^2 = \|y\|^2 + \|z\|^2 .$$

(2) If L is a dense subset in H and $x \in H$ is such that $x \perp L$, then $x = 0$.

(3) If M is a subset in H, then $M^{\perp}$ is a closed subspace of H.

Proof. Trivial. □

Theorem 20.8 (Projection Theorem)**.** *Let M be a complete convex subset in a pre-Hilbert space. Then for each $x \in H$, there exists $x_0 \in M$ such that*

$$\|x - x_0\| = d(x, M).$$

Proof. With no harm to generality, assume that $M \subsetneq H$. Let $\alpha = d(x, M)$. Then we can find a sequence $\{x_n\}_{n=1}^{\infty}$ in M such that

$$\|x_n - x_0\| \to 0, \quad n \to \infty.$$

Since M is convex, $\frac{x_n + x_m}{2} \in M$ for all $m, n \in \mathbb{N}^*$. Further, we have $\left\|x_0 - \frac{x_n+x_m}{2}\right\| \geq \alpha$. By the parallelogram law, it follows that

$$\begin{aligned}
\|x_n - x_m\| &= \|x_m - x + x - x_n\| \\
&= 2\|x_n - x\|^2 + 2\|x - x_n\| - 4\left\|x - \frac{x_n + x_m}{2}\right\| \\
&\leq 2\|x_n - x\|^2 + 2\|x - x_n\| - 4\alpha.
\end{aligned}$$

Letting $n, m \to \infty$, we obtain $\|x_n - x_m\| \to \infty$. Hence, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. The completeness of M finishes the proof. □

In particular, since closed subspace of a Hilbert space is a complete convex set, the theorem above is valid. In this case, it is known as the *Hilbert Projection Theorem.* If M is a complete convex subset of a Hilbert space H, then we can define an operator $P_M : H \to M$ as follows:

$$P_M(x) = x_0 \quad \text{such that } x_0 \in M \text{ and } \|x - x_0\| = d(x, M).$$

P_M is called a projection operator and $P_M x$ is the projection of x to subset M.

Theorem 20.9 (Orthogonal Decomposition Theorem)**.** *Let M be a closed subspace of a Hilbert space H. Then for each $x_0 \in H$, there exists a unique decomposition $x_0 = x_1 + x_2$ where $x_1 \in M$ and $x_2 \in M^\perp$.*

Proof. Without loss of generality, assume that $x_0 \notin M$. We apply Projection Theorem 20.8 to M, which yields that there exists a unique $x_1 \in M$ such that

$$\|x_0 - x_1\| = d(x_0, M).$$

Set $\|x_0 - x_1\| = \alpha$. For any $z \in M$ and any $\lambda \in \mathbb{K}$, clearly $x_1 + \lambda z \in M$, and then

$$\alpha^2 \leq \|x_0 - x_1 - \lambda z\|^2 = \|x_0 - x_1\|^2 - 2\Re(\lambda\langle z, x_0 - x_1\rangle) + |\lambda|^2 \|x\|^2 .$$

Letting $\lambda = \frac{\langle x_0 - x_1, z\rangle}{\|z\|^2}$, then the inequality above reduces to $\left|\langle x_0 - x_1, z\rangle\right|^2 \leq 0$. Hence, $\langle x_0 - x_1, z\rangle = 0$, namely, $x_0 - x_1$ is Orthogonal to M. Set $x_2 = x_0 - x_1$, then $x_2 \in M^\perp$.

Now we verify uniqueness. Suppose that $x_0 = x_1' + x_2'$ where $x_1' \in M$ and $x_2' \in M^\perp$, then by subtraction we obtain

$$x_1 - x_1' = x_2' - x_2.$$

Since $x_1 - x_1' \in M^\perp$ and $x_2' - x_2 \in M^\perp$, it must be the case $x_1 = x_1'$ and $x_2' = x_2$. $\square$

In fact, the preceding theorem states that a Hilbert space H can be written as the direct sum of two orthogonal subspaces, *i.e.*, $H = M \oplus M^\perp$.

Definition 20.4 (Orthogonal system, orthonormal set and orthonormal basis)**.** Let H be a pre-Hilbert space, and let $\{x_\alpha\}_{\alpha \in i}\}$ be a subset non zero elements of H where I is an index set. If $\langle x_\alpha, x_\beta\rangle = 0$ for all pairs $(\alpha, \beta) \in I \times I$ with $\alpha \neq \beta$, then $\{x_\alpha\}_{\alpha \in I}$ is said to be an *orthogonal system* of H. If further $\|x_\alpha\| = 1$ for all $\alpha \in I$, then $\{x_\alpha\}_{\alpha \in I}$ is said to be an *orthonormal system* of H.

A complete orthogonal (resp. orthonormal) system of a pre-Hilbert space is called an *orthogonal (orthonormal) basis*, *i.e.*,

$$\overline{\operatorname{span}\left(\{x_\alpha\}_{\alpha \in I}\right)} = H.$$

Example 20.5. Consider the space $L^2[0, 2\pi]$, and let

$$e_n(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{int}.$$

One can show without difficulty that $\{e_n\}_{n=1}^\infty$ is a orthonormal system of $L^2[0, 2\pi]$. Indeed, $\langle x_n, e_n\rangle$ is the classic Fourier coefficients of $x \in L^2[0, 2\pi]$.

More generally, if $\{e_\alpha\}_{\alpha \in I}$ is an orthonormal system of a Hilbert space H, then $\{\langle x, e_\alpha\rangle\}_{\alpha \in I}$ are called the Fourier coefficients of $x \in H$. For the sake of simplicity, we consider the case that I is at most countable henceforth. A problem that concerns us is the convergence of Fourier series $\sum_{n=1}^\infty \langle x, e_n\rangle e_n$. We solve this problem by the following theorems.

Theorem 20.10 (Orthogonal projection)**.** *Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system of a Hilbert space H, and let $x_0 \in H$. Then*

$$\left\| x_0 - \sum_{k=1}^{N} \alpha_k e_k \right\|$$

obtains its minimun if and only if $\alpha_k = \langle x_0, e_k \rangle$ for all $k = 1, 2, \ldots, N$.

Proof. Observe that $x_0 - \sum_{k=1}^{N} \perp e_k$ for all $k = 1, 2, \ldots, N$. This theorem follows directly from the computation

$$\begin{aligned}
\left\| x - \sum_{k=1}^{N} \alpha_k e_k \right\| &= \left\| x - \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k + \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k - \sum_{k=1}^{N} \alpha_k e_k \right\|^2 \\
&= \left\| x_0 - \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k \right\|^2 + \left\| \sum_{k=1}^{N} (\langle x_0, e_k \rangle - \alpha_k) e_k \right\|^2 \\
&= \left\| x_0 - \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k \right\|^2 + \sum_{k=1}^{N} \left| \langle x_0, e_k \rangle - \alpha_k \right|^2,
\end{aligned}$$

which achieves its minimum if and only if $\alpha_k = \langle x_0, e_k \rangle$ for all $k = 1, 2, \ldots, N>$ □

Set $M = \operatorname{span}\left(\{e_n\}_{n=1}^{N}\right)$, the preceding theorem demonstrates that

$$P_M x = \sum_{k=1}^{N} \langle x, e_k \rangle e_k.$$

Corollary 20.1 (Bessel's Inequality)**.** *Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system of a pre-Hilbert space H, and let $x_0 \in H$. Then*

$$\sum_{k=1}^{\infty} \left| \langle x_0, e_k \rangle \right|^2 \leq \|x_0\|^2 .$$

Proof. This follows from the deduction

$$\left\| x_0 - \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k \right\|^2 = \|x_0\|^2 - \sum_{k=1}^{N} \left| \langle x_0, e_k \rangle \right|^2 \geq 0$$

since $x_0 - \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k \perp e_k$ for all $k = 1, 2, \ldots, N$. □

Theorem 20.11 (Parseval's Identity)**.** *If and only if $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of a pre-Hilbert space H, the* Parseval's Identity *holds true for all $x_0 \in H$:*

$$\sum_{k=1}^{\infty} \left| \langle x_0, e_n \rangle \right|^2 = \|x_0\|^2 . \tag{20.12}$$

Proof. ($\Longrightarrow$) Assume that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of H. Then for any $\epsilon > 0$, there exist an $N_0 \in \mathbb{N}^*$ and coefficients α_k for $k = 1, 2, \ldots, N_0$ such that

$$\left\| x_0 - \sum_{k=1}^{N_0} \alpha_k e_k \right\| \leq \epsilon.$$

By Theorem 20.10, it follows that

$$\left\| x_0 - \sum_{k=1}^{N_0} \langle x_0, e_k \rangle e_k \right\| \leq \left\| x_0 - \sum_{k=1}^{N_0} \alpha_k e_k \right\| \leq \epsilon.$$

Thus for $N > N_0$, we have

$$\left\| x_0 - \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k \right\|^2 = \|x_0\|^2 - \sum_{k=N+1}^{\infty} \left| \langle x_0, e_k \rangle \right|^2 \leq \left\| x_0 - \sum_{k=1}^{N_0} \langle x_0, e_k \rangle e_k^2 \right\| \leq \epsilon^2.$$

Therefore, $\sum_{k=1}^{\infty} \langle x_0, e_k \rangle e_k = x_0$ and the Parseval's Identity follows.

($\Longleftarrow$) Conversely, assume that Parseval's Identity (20.12) holds true for all $x_0 \in H$. Then

$$\left\| x_0 - \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k \right\|^2 = \|x_0\|^2 - \left\| \sum_{k=1}^{N} \langle x_0, e_k \rangle e_k \right\| = \|x_0\|^2 - \sum_{k=1}^{N} \left| \langle x_0, e_k \rangle \right|^2 \to 0$$

as $N \to \infty$, which implies that $\{e_n\}_{n=1}^{\infty}$ is complete. $\square$

Theorem 20.12. *Let H be a Hilbert space, and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system of H. Then $\{x_n\}_{n=1}^{\infty}$ is complete if and only if $x \perp e_n$ for all $n \in \mathbb{N}^*$ implies $x = 0$.*

Proof. Assume that $\{e_n\}_{n=1}^{\infty}$ is complete and $x_0 \in H$ is such that $x_0 \perp e_n$ for all $n \in \mathbb{N}^*$. Then by the Parseval;s Identity,

$$\|x_0\| = \sum_{n=1}^{\infty} \left| \langle x_0, e_n \rangle \right|^2 = 0,$$

implying that $x_0 = 0$.

Conversely, assume that $x \perp e_n$ for all $n \in \mathbb{N}^*$ implies $x = 0$. And suppose for the sake of contradiction that $\{e_n\}_{n=1}^{\infty}$ is not complete. That is, setting $M = \operatorname{span}\left(\{e_n\}_{n=1}^{\infty}\right)$, $\overline{M}$ is not dense in H. Hence, we can find an $x_0 \in H \backslash \overline{M}$. By the theorem of orthogonal decomposition 20.9 (This requires completeness of H), there exists $x_1 \in \overline{M}$ and $x_2 \in \overline{M}^{\perp}$ such that $x_0 = x_1 + x_2$, $x_2 \neq 0$. However, since $x_2 \in \overline{M}^{\perp}$, x_2 is orthogonal to e_n for all $n \in \mathbb{N}^*$, which is a contradiction to the hypothesis. $\square$

Theorem 20.13 (Riesz-Fischer)**.** *Let H be a Hilbert space, and let $x = \{e_n\}_{n=1}^{\infty}$ be an orthonormal system of H. If $\{\xi_n\}_{n=1}^{\infty} \in l^2$, then there exists $u_0 \in H$ such that for all $n \in \mathbb{N}^*$,*

$$\xi_n = \langle u_0, e_n \rangle,$$

and
$$\|u_0\|^2 = \|x\|^2$$

Proof. Set for $n \in \mathbb{N}^*$,
$$u_n = \sum_{k=1}^{n} \xi_k e_k.$$
Then $\langle u_n, e_k\rangle = \xi_k$ if $k = 1, 2, \ldots, n$ and $\langle u_n, e_k\rangle = 0$ if $k = n+1, n+2, \ldots$. For $m \in \mathbb{N}^*$,
$$\|u_{n+m} - u_n\| = \left\| \sum_{k=n+1}^{n+m} \xi_k e_k \right\| = \sum_{k=n+1}^{n+m} |\xi_k|^2 .$$
Since $x_0 = \{\xi_n\}_{n=1}^{\infty} \in l^2$, we know $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence, which has a limit $u_0 \in H$ by completeness of H.

Letting $n \to \infty$ in the following
$$\langle u_0, e_k\rangle = \langle u_0 - u_n, e_k\rangle + \langle u_n, e_k\rangle = \langle u_0 - u_n, e_k\rangle + \xi_k \quad (n > k)$$
yields that $\langle u_0, e_k\rangle = \xi_k$ for all $k \in \mathbb{N}^*$. Furthermore,
$$\begin{aligned} \|u_0 - u_n\|^2 =& \left\| x_0 - \sum_{k=1}^{n} \xi_k e_k \right\|^2 \\ =& \|u_0\|^2 - \left\| \sum_{k=1}^{n} \xi_k e_k \right\|^2 = \|u_0\|^2 - \sum_{k=1}^{n} |\xi_k|^2 . \end{aligned}$$
Letting $n \to \infty$ yields that $\|u_0\|^2 = \sum_{k=1}^{\infty} |\xi_k|^2 = \|x\|^2$. □

Theorem 20.14 (The Gram-Schimit Orthogonalization Process)**.** *Let H be a Hilbert space, and let $\{x_n\}_{n=1}^{\infty}$ be a linearly independent subset of H. Then there exists an orthonormal set $\{e_1, e_2, \ldots, e_n\}$ such that for every $n \in \mathbb{N}^*$,*
$$\mathrm{span}\{x_1, x_2, \ldots, x_n\} = \mathrm{span}\{e_1, e_2, \ldots, e_n\}.$$

Proof. It is easy to verify that the following process satisfies the requirement:
$$\begin{aligned} h_n &= x_n - \sum_{k=1}^{n-1} \langle x_k, e_k\rangle, \\ e_n &= \frac{h_n}{\|h_n\|}. \end{aligned}$$
□

Corollary 20.2. *There exists an orthonormal basis for any sepearable Hilbert space.*

Proof. Since H is sepearable, there is a countable dense subset of H. Then by the Gram-Schimit Orthogonalization Process, a orthonormal basis $\{e_1, e_2, \ldots\}$ can be constructed, of which the span is dense in H. □

We point out that any infinite dimensional sepearable Hilbert space is essentially space l^2. Before that, we give the definition of isomorphism on HIlbert spaces.

Definition 20.5 (isomorphism, isomorphic on Hilbert spaces)**.** If H_1 and H_2 are Hilbert spaces, an *isomorphism* between H_1 and H_2 is a linear surjection $U : H_1 \to H_2$ such that

$$\langle Ux, Uy\rangle = \langle x, y\rangle$$

for all $x, y \in H_1$. In this case, H_1 and H_2 are said to be *isomorphic*.

The concept of isomorphism is universal in Mathematics, which not only builds an bijection between two sets but also preserves the operation between them.

Theorem 20.15. *Let H be a sepearable HIlbert space of infinite dimension. Then there exists an isomorphism between H and l^2.*

Proof. Since H is sepearable and infinite dimensional, there exists an orthonormal basis $\{e_n : n \in \mathbb{N}^*\}$ of H. Define $\phi(x) = \{\langle x, e_n\rangle\}_{n=1}^{\infty}$ for all $x \in H$. Then by the Bessel's Inequality, $\phi(x) \in l^2$ for all $x \in H$. And the linearity of ϕ comes from that of inner product. By the Riesz-Fischer's Theorem 20.13, ϕ is also onto. Finally, we compute that

$$\begin{aligned}\langle x, y\rangle &= \langle\sum_{k=1}^{\infty}\langle x, e_k\rangle e_k, \sum_{k=1}^{\infty}\langle y, e_k\rangle e_k\rangle \\ &= \sum_{k=1}^{\infty}\langle x, e_k\rangle\overline{\langle y, e_k\rangle} = \langle\phi(x), \phi(y)\rangle.\end{aligned}$$

Therefore, ϕ is an isomorphism between H and l^2. □

20.3 The Riesz Representation Theorem and Conjugate Spaces of Hilbert Spaces

Theorem 20.16 (F.Riesz Representation Theorem)**.** *Let H be a Hilbert space. If $f \in H^*$, then there exists a unique $y_f \in H$ such that $\|y_f\| = \|f\|$ and*

$$f(x) = \langle x, y_f\rangle$$

for every $x \in H$.

Proof. Without loss of generality, we assume that $f \neq 0$. By projection theorem, we can find a $y_0 \in \mathscr{N}(f)^{\perp}$, since $\mathscr{N}(f)$ is a closed subspace of H. Setting

$$y_f = \frac{\overline{f(y_0)}}{\|y_0\|^2}y_0,$$

we verify that y_f satisfies the requirements.

We consider three cases:

(1) If $x \in \mathscr{N}(f)$, then obviously $f(x) = 0 = \langle x, y_f\rangle$.

(2) If $x = \alpha y_0$, then

$$\langle x, y_f \rangle = \alpha \langle y_0, y_f \rangle = \langle y_0, \overline{\frac{f(y_0)}{\|y_0\|^2}} y_0 \rangle = \alpha f(y_0) = f(x).$$

(3) Otherwise, since for every $x \in H$,

$$f(x - \frac{f(x)}{f(y_f)} y_f) = f(x) - \frac{f(x)}{f(y_f)} f(y_f) = 0,$$

it follows that $x - \frac{f(x)}{f(y_f)} y_f \in \mathscr{N}(f)$. Thus

$$\begin{aligned} f(x) &= f\left(x - \frac{f(x)}{f(y_f)} y_f\right) + f\left(\frac{f(x)}{f(y_f)} y_f\right) \\ &= \langle x - \frac{f(x)}{f(y_f)} y_f, y_f \rangle + \langle \frac{f(x)}{f(y_f)} y_f, y_f \rangle \\ &= \langle x, y_f \rangle. \end{aligned}$$

Consequently, ϕ preserves inner product.

On the one hand,

$$\|f\| = \sup_{\|x\| \le 1} |f(x)| = \sup_{\|x\| \le 1} |\langle x, y_f \rangle| \le \|y_f\|.$$

On the other hand,

$$\|f\| = \sup_{\|x\| \le 1} |f(x)| \ge \left| f(\frac{y_f}{\|y_f\|}) \right| = \langle \frac{y_f}{\|y_f\|}, y_f \rangle = \|y_f\|.$$

Therefore, $\|f\| = \|y_f\|$.

Finally, if there is a $y_f' \in H$ such that $f(x) = \langle x, y_f' \rangle$ for all $x \in H$, then

$$\langle x, y_f - y_f' \rangle = 0$$

for all $x \in H$, which implies that $y_f = y_f'$. □

The Riesz Representation Theorem tells us the following very fundamental geometry fact. That is, if f is a non-zero bounded linear operator on a Hilbert space, then the null space of f is a hyper-plane and its orthogonal complement has dimension one.

With the Riesz Representation Theorem, one can construct a bijection between a Hilbert space H and its conjugate space H^* as

$$\Phi : H \to H^*, \quad x \mapsto f_x$$

where f_x is the linear functional defined as $f_x(y) = \langle y, x \rangle$ for all $y \in H$. The mapping Φ is called the *duality mapping* of H. In the sense of Φ, it is reasonable to identify H with H^*, thanks to the following proposition. We say Hilbert spaces are self-conjugate.

Proposition 20.1. *The duality mapping Φ of H is continuous and norm-preserving.*

If H is a real Hilbert space, then Φ is linear. If H is a complex Hilbert space, then Φ is antilinear.

Proof. Omitted. ☐

Now we consider the bounded linear operators of Hilbert spaces. Let H be a Hilbert space, and let $A \in \mathscr{B}(H)$. Define $f_{A,y}(x) = \langle Ax, y \rangle$, then clearly $f_{A,y} \in \mathscr{B}(H)$. By the Riesz Representation Theorem, there exists a unique $z \in H$ such that for all $x \in H$,

$$f_{A,y}(x) = \langle x, z \rangle.$$

Then it is valid to define

$$B : H \to H, \quad y \mapsto z. \tag{20.13}$$

Proposition 20.2. *The operator B defined in (20.13) is a bounded linear operator on H.*

Proof. Supposing $y_1, y_2 \in H$ and $\alpha, \beta \in \mathbb{K}$, then

$$\begin{aligned}\langle Ax, \alpha y_1 + \beta y_2 \rangle &= \overline{\alpha} \langle Ax, y_1 \rangle + \overline{\beta} \langle Ax, y_2 \rangle = \overline{\alpha} \langle x, By_1 \rangle + \overline{\beta} \langle x, By_2 \rangle \\ &= \langle x, \alpha By_1 \rangle + \langle x, \beta By_2 \rangle = \langle x, \alpha By_1 + \beta By_2 \rangle,\end{aligned}$$

which implies the linearity.

And the boundedness follows from the computation

$$\|By\|^2 = \langle By, By \rangle = \langle ABy, y \rangle \leq \|ABy\| \|y\| \leq \|AB\| \|y\|^2 .$$

☐

Definition 20.6 (Adjoint operator)**.** Let H be a Hilbert space, and let $A \in \mathscr{B}(H)$. We call the operator $A^* \in \mathscr{B}(H)$ defined by $\langle Ax, y \rangle = \langle x, A^* y \rangle$ the *adjoint operator* of A.

Proposition 20.3. *Let H be a Hilbert space, and let $A \in \mathscr{B}(H)$. Then $(\alpha A)^* = \overline{\alpha} A^*$.*

Proof. This follows from $\langle \alpha Ax, y \rangle = \alpha \langle Ax, y \rangle = \alpha \langle x, A^* y \rangle = \langle x, \overline{\alpha} A^* y \rangle$. ☐

Remark 13. There's slight difference between the definition of adjoint operator in Hilbert space and that in Banach space. In Banach space, it holds true that $(\alpha A)^* = \alpha A^*$.

Definition 20.7. Suppose that H is a Hilbert space and $A \in \mathscr{B}(H)$. We say A is a *self-adjoint operator* if $A^* = A$.

Proposition 20.4. *Let A be a self-adjoint operator on a Hilbert space H. Then the following statements hold true.*

(1) $\langle Ax, x \rangle \in \mathbb{R}$ *for all* $x \in H$.
(2) $\|A\| = \sup_{\|x\|=1} \langle Ax, x \rangle$.
(3) The eigenvalues of A are real.
(4) If x_1, x_2 are eigenvectors of A with respect to λ_1, λ_2 respectively, then $x_1 \perp x_2$.

Proof. (1) $\langle Ax, x \rangle = \langle x, Ax \rangle = \overline{\langle Ax, x \rangle}$.

(2) On the one hand, for $x \in H$ with $\|x\| = 1$,

$$\langle Ax, x \rangle \leq \|A\| \|x\|^2 \leq \|A\| .$$

Thus $\sup_{\|x\|=1}\langle Ax, x\rangle \leq \|A\|$. On the other hand,

$$\begin{aligned}\langle A(x+y), x+y\rangle &= \langle Ax, x\rangle + \langle Ax, y\rangle + \langle Ay, x\rangle + \langle Ay, y\rangle \\ &= \langle Ax, x\rangle + \langle Ax, y\rangle + \langle y, Ax\rangle + \langle Ay, y\rangle\end{aligned} \tag{20.14}$$

gives that

$$\langle A(x+y), x+y\rangle = \langle Ax, x\rangle + \Re\langle Ax, y\rangle + \langle Ay, y\rangle \tag{20.15}$$

$$\langle A(x-y), x-y\rangle = \langle Ax, x\rangle - \Re\langle Ax, y\rangle + \langle Ay, y\rangle \tag{20.16}$$

Put $M = \sup_{\|x\|=1}\langle Ax, x\rangle$. Substracting one of these two equations from the other yields

$$\begin{aligned}4\Re\langle Ax, y\rangle &= \langle A(x+y), x+y\rangle + \langle A(x-y), x-y\rangle \\ &\leq M(\|x+y\|^2 + \|x-y\|^2) = 2M(\|x\|^2 + \|y\|^2).\end{aligned}$$

Let $x \in B(0,1)$ and $y = \frac{Ax}{\|Ax\|}$. It follows that

$$4\Re\langle Ax, \frac{Ax}{\|Ax\|}\rangle = 4\|Ax\| \leq 4M,$$

which implies that $\|A\| \leq M = \sup_{\|x\|=1}\langle Ax, x\rangle$.

(3) $Ax = \lambda x, x \neq 0 \implies \langle \lambda x, x\rangle = \langle Ax, x\rangle = \langle x, Ax\rangle = \langle x, \lambda x\rangle$. (4) $\langle x_1, x_2\rangle = \frac{1}{\lambda_1}\langle Ax_1, x_2\rangle = \frac{1}{\lambda_1}\langle x_1, Ax_2\rangle = \frac{\lambda_2}{\lambda_1}\langle x_1, x_2\rangle$. $\square$

Chapter 21

Topological Linear Spaces

Norms and inner products are common ways to introduce topology to a linear space. But there are other ways to endow topology. The main aim of this chapter is to generalize the results we have studied in Banach spaces and Hilbert spaces to more general topological linear space.

21.1 Basic Properties of Topological Linear Spaces

Definition 21.1 (Topological linear space)**.** Let X be a linear space over number field $\mathbb{K}$, and let $\mathcal{T}$ be a topology on X. The topological space $(X, \mathcal{T})$ is said to be a topological linear space if the following are satisfied:

(1) (T_1 axiom) Every set of single element is closed.

(2) The linear space operations (addition and scalar multiplication) is continuous with respect to $\mathcal{T}$.

Under these conditions, $\mathcal{T}$ is called a *linear topology* on X.

More specifically, by continuity of addition we mean the mapping

$$(x, y) \mapsto x + y$$

is a continuous function from $X \times X$ to X is continuous: If $x_1, x_2 \in X$ and V is a neighborhood of $x_1 + x_2$, then there should exist neighborhoods V_1, V_2 of x_1, x_2 respectively such that $V_1 + V_2 \subseteq V$. Likewise, by continuity of scalar multiplication we mean for every $x \in X$ nad $\alpha \in \mathbb{K}$, if V is a neighborhood of αx, then there should exist $\delta > 0$ and neighborhood U_1 of x such that $\beta U \subseteq V$ whenever $|\beta - \alpha| < \delta$.

Whenever the topology is clear according to context, we write a topological linear space $(X, \mathcal{T})$ for convenience. For a topological linear space X, we define the translation operator and scalar product operator as

$$T_y(x) = x + y, \quad M_\alpha(x) = \alpha x.$$

One can easily show that for any $y \in X$ and $\alpha \neq 0$, the operators T_y and M_α is a homeomorphism between X and X.

Since a subset $E \subseteq X$ is open if and only if for every $y \in X$, we have $T_y(E)$ is open, we know that the topology of a topological linear space is represented by its local base at 0. By local base we mean the the class $\mathcal{B}$ of neighborhoods of 0 such that any neighborhood of 0 contains at least one element of $\mathcal{B}$.

21.1.1 Seperation Properties

Lemma 21.1. *Let X be a topological linear space. If W is a neighborhood of 0, then there exists a neighborhood U of 0 which is symmetric (in the sense that $U = -U$) such that $U + U \subseteq W$.*

Proof. Note that $0 = 0+0$, that addition is continuous, and that W is a neighborhood of 0. Therefore, there are neighborhoods V_1, V_2 of 0 such that $V_1 + V_2 \subseteq W$. Setting $U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$, then U is a neighborhood of 0 and has the required properties. □

Theorem 21.1. *Suppose K and C are subsets of a topological linear space X, K is compact, C is closed, and $K \cap C = \varnothing$. Then 0 has a neighborhood V such that*

$$(K + V) \cap (C + V) = \varnothing.$$

Proof. Without loss of generality, assume that $K \neq \varnothing$. Let $x \in K$. Then $x \notin C$. Since C is closed, there exists a neighborhood W_x of 0 such that $x+W_x \cap C = \varnothing$. By Lemma 21.1, 0 has neighborhood V_x which is symmetric such that $V_x + V_x + V_x + V_x \subseteq W$. Hence, $(x + V_x + V_x + V_x + V_x) \cap C = \varnothing$, which is equivalent to $(x + V_x + V_x + V_x) \cap (C + V_x) = \varnothing$. This implies

$$(x + V_x + V_x) \cap (C + V_x) = \varnothing. \tag{21.1}$$

On the other hand, by the comppactness of K, there exists finitely many elements $x_1, x_2, \ldots, x_n$ of K such that $K \subset \bigcup_{k=1}^{n}(x_k + V_{x_k})$. Putting $V = \bigcap_{k=1}^{n} V_{x_k}$, then

$$K + V \subset \bigcup_{k=1}^{n}(x_k + V_{x_k} + V) \subset \bigcup_{k=1}^{n}(x_k + V_{x_k} + V_{x_k}).$$

By the Equation (21.1), $(K+V) \cap (C+V_{x_i}) = \varnothing$, and therefore $(K+V) \cap (C+V) = \varnothing$. □

The theorem above demonstrates that in a topological linear space, disjoint compact set and closed set can be seperated by two disjoint open sets.

Corollary 21.1. *Topological linear spaces are Hausdorff spaces.*

Proof. For any $x, y \in X$, let $K = \{x\}$ and $C = \{y\}$, and apply the preceding theorem. □

Corollary 21.2. *If $\mathcal{B}$ is a local base of a topological linear space X, then every member of $\mathcal{B}$ contains the closure of some member of $\mathcal{B}$.*

Proof. In the proof of Theorem 21.1, since $C + V$ is open, it is true that $\operatorname{cl}(K + V)$ does not intersect $C + V$. This corollary follows if we set $K = \{0\}$. □

21.1.2 Balanced Sets and Bounded Sets

The concept of balanced set is a generalization of the concept of ball in normed space.

Definition 21.2 (Balanced set). Let X be a linear space, and let B be a subset of X. We say B is *balanced* if $\alpha B \subset B$ whenever $|\alpha| < 1$.

Proposition 21.1. *Let X be a topological vector space. The following hold true.*

(1) If $B \subseteq X$ is balanced, then $\operatorname{cl} B$ is balanced.

(2) If $B \subseteq X$ is balanced, and $0 \in \operatorname{int} A$, then $\operatorname{int} B$ is balanced.

(3) Every neighborhood of 0 contains some neighborhood of 0 which is balanced.

(4) Every convex neighborhood of 0 contains some convex neighborhood of 0 which is balanced.

Proof. (1) Omitted.

(2) Suppose that $0 < |\alpha| \leq 1$. Then

$$\alpha \operatorname{int} B = \operatorname{int}(\alpha B) \subseteq \alpha B \subseteq B.$$

Since $\alpha \operatorname{int} B$ is open, it follows that $\alpha \operatorname{int} A \subseteq \operatorname{int} A$. Thanks to the assumption that $0 \in \operatorname{int} A$, $\alpha \operatorname{int} A \subseteq \operatorname{int} A$ holds true for the case $\alpha = 0$. Hence, $\operatorname{int} A$ is also a balanced set.

(3) Let U be a neighborhood of 0. By the continuity of scalar product, there is a $\delta > 0$ and a neighborhood V of 0 such that $\alpha V \subseteq U$ whenever $|\alpha| < \delta$. Put $W = \bigcup_{|\alpha|<\delta} \alpha V$, which is clearly neighborhood of 0 that $W \subseteq U$. Now we verify that W is balanced. Indeed, for any $x \in W$ and β with $|\beta| < 1$, there is an α with $|\alpha| < \delta$ such that $x \in \alpha V$. Hence, $|\beta\alpha| \leq |\alpha| \leq \delta$, and $\beta x \in \beta\alpha V \subset W$.

(4) Let U be a convex neighborhood of 0, and let $A = \bigcap_{|\alpha|=1} \alpha U$. Then $0 \in A$, and A is convex since αU is convex for any $\alpha \in \mathbb{K}$. From (3) in this proposition, there is a balanced neighborhood $W \subseteq U$. For any $\alpha \in \mathbb{K}$ with $|\alpha| = 1$, we have $\alpha^{-1} W \subseteq W$, and thus $W \subseteq \alpha W \subseteq \alpha U$. Therefore, $W \subseteq W$, and $0 \in W \in \operatorname{int} A$. If we can verify that A is a balanced set, then it follows that $\operatorname{int} A$ is balanced by (2) of this proposition. Indeed, let $r \in [0, 1]$, $\beta \in \mathbb{K}$ with $|\beta| = 1$, then

$$r\beta A = \bigcap_{|\alpha|=1} r\beta\alpha U = \bigcap_{|\alpha|=1} r\alpha U \subseteq \bigcap_{|\alpha|=1} \alpha U = A.$$

Finally, it is not hard to show that $\operatorname{int} A$ is convex. Hence, $\operatorname{int} A$ is what we want. □

Definition 21.3 (Locally convex space)**.** Let $\mathcal{B}$ be a local base of a topological linear space X. Then $\mathcal{B}$ is said to be balanced if every member of $\mathcal{B}$ is balanced; $\mathcal{B}$ is said to be convex if every member of $\mathcal{B}$ is convex.

If X has a convex local base, then X is said to be locally convex.

Locally convex topological linear spaces are of considerable interest.

Corollary 21.3. *(1) Every topological linear topological space has a balanced local base.*

(2) Every locally convex topological linear space has a balanced convex local base.

Now we try to generalize the concept of boundedness to topological linear spaces.

Definition 21.4 (Bounded set)**.** Let E be a subset of a topological linear space X. E is said to be bounded if for every neighborhood V of 0, there exists $s > 0$ such that $E \subseteq tV$ whenever $t > s$.

Proposition 21.2. *Let E be a subset of a topological linear space X. Then E is bounded if and only if for every $\{x_n\}_{n=1}^{\infty} \subseteq E$ and $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{K}$,*

$$\alpha_n \to 0 \quad (n \to \infty) \implies \alpha_n x_n \to 0 \quad (n \to \infty). \tag{21.2}$$

Proof. ($\Longrightarrow$) Suppose that E is bounded, V is balanced neighborhood of 0, $\{x_n\}_{n=1}^{\infty} \subseteq E$ and $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{K}$ with $\alpha_n \to \infty \quad (n \to \infty)$. Then there is a $t > 0$ such that $E \subseteq tV$, and there is $N \in \mathbb{N}^*$ such that $|\alpha_n|\, t < 1$ whenever $n > N$. Hence, since V is balanced,

$$\alpha_n x_n = \alpha_n t \times \frac{1}{t} x_n \in V,$$

which implies $\alpha_n x_n \to \infty$ as $n \to \infty$ by definition.

Conversely, assume that condition (21.2) is satisfied, and for the sake of contradiction E is unbounded. Then there exists neighborhood V of 0 such that for every $n \in \mathbb{N}^*$, there is some $x_n \in E$ with $x_n \notin nV$. Then $\frac{1}{n} x_n \notin V$ for all $n \in \mathbb{N}^*$, and thus $\frac{1}{n} x_n$ does not converges 0, which contradicts the condition (21.2). Therefore, E must be bounded. ☐

Theorem 21.2. *Let X be a topological linear space, and let V be a neighborhood of 0. Then the following hold true.*

(1) If $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is a monotonically increasing sequence, then $X = \bigcup_{n=1}^{\infty} \alpha_n V$.

(2) Every compact set in X is bounded.

(3) If $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is a monotonically decreasing sequence converging to 0, then $\{\alpha_n V : n \in \mathbb{N}^*\}$ is a local base of X.

Proof. (1) For any $x \in X$, the mapping $\alpha \mapsto \alpha x$ that maps $\mathbb{K}$ into X is continuous. Then $A_x = \{\alpha \in \mathbb{K} : \alpha x \in V$ is open, since V is open. Noting that $0 \in A_x$, then for sufficient large n, $\frac{1}{\alpha_n} x \in V$, namely, $x \in r_n V$.

(2) Let K be a compact set in X, and let U be a neighborhood of 0. Then there is some balanced neighborhood W of 0 such that $W \subseteq U$. From (1),

$$K \subseteq X = \bigcup_{n=1}^{\infty} nW.$$

By the compactness of K, there exists finitely many $n_1, n_2, \ldots, n_N$ such that

$$K \subseteq \bigcup_{k=1}^{N} n_k W.$$

Assume without loss of generality that $n_1 < n_2 < \cdots < n_N$. Since $0 < \frac{n_k}{n_{k+1}} < 1$ for $k = 1, 2, \ldots, N-1$ and W is balanced,

$$n_1 W \subseteq n_2 W \subseteq \cdots \subseteq n_N W.$$

Hence, $K \subset n_N W \subset n_N U$.

(3) Let U be any neighborhood of 0. By the boundedness of V, there exists $s > 0$ such that $V \subseteq tU$ for $t > s$. Then it is valid to choose $n \in \mathbb{N}^*$ such that $\frac{1}{\delta_n} > s$. In this case, $V \subseteq \frac{1}{\delta_n} U$, namely, $\delta_n V \subseteq U$. ☐

21.1.3 Metriczability

Theorem 21.3. *Let $(X, \mathcal{T})$ be a topological linear space which has a countable base. Then there exists a distance function $d : X \times X \to \mathbb{R}$ such that*

(1) $\mathcal{T}$ is induced by d.

(2) Every open ball (in the sense of d centered at 0 is balanced.

(3) (Translation invariance) $d(x+z, y+z) = d(x, y)$ for any $x, y, z \in X$.

(4) If $(X, \mathcal{T})$ is locally convex, then every open ball in X is convex.

Proof. □

21.1.4 Bounded Linear Operators

Having given the definition of bounded set, it is time to generalize the concept of bounded linear operator to topological linear spaces.

Definition 21.5 (Bounded linear operator in topological linear space)**.** Let X, Y be topological linear space. An linear operator $T : X \to Y$ is *bounded* if T maps every bounded set to a bounded set.

Theorem 21.4. *Let X, Y be topological linear space, and let $T : X \to Y$ be a linear operator. Then consider the following statements:*

(1) T is continuous at 0.

(2) T is continuous.

(3) T is bounded.

(4) If $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to 0, then $\{Tx_n : n \in \mathbb{N}^*\}$ is a bounded set.

(5) If $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to 0, then $Tx_n \to 0$ as $n \to \infty$.

We have the implications:

$$(1) \iff (2) \implies (3) \implies (4).$$

If further X is metrizable, then

$$(4) \implies (5) \implies (1).$$

Proof. (1) $\implies$ (2) Let W be a neighborhood of 0. Then by the continuity of T, there is some neighborhood W of 0 such that $TV \subseteq W$. Hence, for any $x \in X$, if $y - x \in V$, it follows that $Ty - Tx = T(y - x) \subset TV \subset W$, which implies $Ty \subset Tx + W$. Thus, T maps the neighborhood $x + V$ of x into the neighborhood $Tx + W$ of Tx, saying that T is continuous at x.

(2) $\implies$ (1) Trivial.

(2) $\implies$ (3) Suppose that E is a bounded set in X, and $W \subseteq Y$ is a neighborhood of 0. There is a neighborhood $V \subseteq X$ of 9 such that $TV \subseteq W$. Since E is bounded, there exists a positive number $s \in \mathbb{R}$ such that $E \subseteq tV$ whenever $t > s$, which implies that $TE \subseteq T(tV) \subseteq tTV \subseteq tW$. This means TE is a bounded set, and thus T is bounded.

(3) $\implies$ (4) This follows from boundedness of the set $\{x_n : n \in \mathbb{N}^*\}$.

Now assume that T is metrizable in the rest of the proof. (4) $\Longrightarrow$ (5) Suppose that (4) is satisfied, and that $\{x_n\}_{n=1}^{\infty}$ is a sequence in X converging to 0. By Theorem, there is some sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converging to ∞ such that $\alpha_n x_n \to 0$ as $n \to \infty$. Then by assumption, $\{T\alpha_n x_n : n \in \mathbb{N}^*\}$ is a bounded set. The computation

$$Tx_n = \frac{1}{\alpha_n} T(\alpha_n x_n)$$

yields that $Tx_n \to 0$ as $n \to \infty$.

(5) $\Longrightarrow$ (1) Suppose that condition statement (5) is satisfied, but T is discontinuous at 0. That is, there exists a neighborhood $W \subseteq Y$ of 0 such that $T^{-1}(W)$ contains no neighborhood of $0 \in X$. Since X is metrizable, there exists a countable base $\{V_n : n \in \mathbb{N}^*\}$ satisfying $V_{n+1} + V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}^*$. Hence, it is valid to find a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \in V_n \backslash T^{-1}(W)$. Then $x_n \to 0$ while $Tx_n \not\to 0$, contradicting the condition (5). □

Theorem 21.5. *Let f be a non-zero linear functional on a topological linear space X. Then the following following are equivalent.*

(1) Functional f is continuous.

(2) The null space $\mathscr{N}(f)$ is a closed subspace of X.

(3) $\mathscr{N}(f)$ is not dense in X, i.e., $\mathrm{cl}\,(\mathscr{N}(f)) \neq X$.

(4) Functional f is bounded in some neighborhood $V \subseteq X$ of 0.

Proof. (1) $\Longrightarrow$ (2) By the linearity of f, we know $\mathscr{N}(f)$ is subspace of X; And by the continuity of f, we know $\mathscr{N}(f)$ is closed.

(2) $\Longrightarrow$ (3) As $\mathscr{N}(f)$ is closed and f is non-zero, we have $\mathrm{cl}\,(\mathscr{N}(f)) = \mathscr{N}(f) \neq X$.

(2) $\Longrightarrow$ (3) Suppose that $\mathscr{N}(f)$ is not dense in X, then there is some $x_0 \in X$ and neighborhood $V \subseteq X$ of 0 such that $x_0 + V \cap \mathrm{cl}\,(\mathscr{N}(f)) = \varnothing$. With no harm to generality, assume that V is balanced. Now we show that $f(V)$ is bounded by contradiction. Suppose that $f(V)$ is unbounded. Then for any $\alpha \in \mathbb{K}$, there is a $x \in V$ such that $\left|f(x)\right| > \alpha$. Assuming that $f(x) = \left|f(x)\right| \mathrm{e}^{\mathrm{i}\theta}$, we have

$$f\left(\frac{\alpha \mathrm{e}^{-\mathrm{i}\theta} x}{\left|f(x)\right|}\right) = \alpha.$$

Since $\left|\frac{\alpha \mathrm{e}^{-\mathrm{i}\theta}}{\left|f(x)\right|}\right| < 1$, we have $\frac{\alpha \mathrm{e}^{-\mathrm{i}\theta} x}{\left|f(x)\right|} \in V$. Hence, $f(V) = \mathbb{K}$. Therefore, there is some $y_0 \in V$ such that $f(y_0) = -f(x_0)$, or equivalently $x_0 + y_0 \in \mathscr{N}(f)$, in contradiction to our assumption.

(4) $\Longrightarrow$ (1) If statement (4) is holds, then there exists a neighborhood V of 0 and positive real number M such that $\left|f(x)\right| < M$ for any $x \in V$. For any $\epsilon > 0$, put $W = \frac{\epsilon}{M} V$. Then for any $y \in W$, there is a correspongding $x \in V$ such that $y = \frac{\epsilon}{M} x$. Hence, $\left|f(y)\right| = \frac{\epsilon}{M}\left|f(x)\right| < \epsilon$, which implies that f is continuous. □

Chapter 22

Generalized Functions

22.1 Background

Generalized functions (also called distributions) are extensions to the classical notion of functions. Distribution theory reinterprets functions as linear functions acting on a space of *test functions*. The different choices of the spaces of test functions lead to different spaces of distributions. Conventional functions act by integration against a test function, but many other linear functions do not arise in this way, and these are the generalized functions.

In physics, people find it convenient to introduce the so-called dirac δ to describe formally the distributions of physical quantities. The Dirac δ function is defined by

$$\delta_y(x) = \begin{cases} \infty, & x = y \\ 0, & x \neq y, \end{cases} \tag{22.1}$$

along with

$$\int_{-\infty}^{\infty} \delta_y(x)\,\mathrm{d}x = 1. \tag{22.2}$$

However, no classical function satisfies conditions 22.1 and 22.2. We can due with this problem with generalized functions.

22.2 The space $D(\Omega)$

First, let us introduce some concepts and notations.

Definition 22.1. Let a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Denote $\partial_j = \frac{\partial}{\partial_j}$, $j \in \mathbb{Z}_n^+$ and

$$\partial_\alpha = \partial_{\alpha_i}\partial_{\alpha_i}\ldots\partial_{\alpha_n}u. \tag{22.3}$$

For $\alpha = (0, 0, \ldots, 0)$, we set ∂_α the identity mapping.

By $C_0^\infty(\Omega)$, we denote those functions in $C^\infty(\Omega)$ with compact support, *i.e.*, those functions in $C^\infty(\Omega)$ vanishes outside a compact set of Ω.

Example 22.1. A standard example of the space $C_0^\infty(\mathbb{R}^n)$ is

$$j(x) = \begin{cases} C_n \exp(-\frac{1}{1-\|x\|^2}), & \|x\| \leq 1 \\ 0, & \|x\| > 1, \end{cases} \tag{22.4}$$

where C_n is defined as

$$C_n = \left(\int_{\|x\| \le 1} \exp(-\frac{1}{1 - \|x\|^2}) \, \mathrm{d}x \right)^{-1}. \tag{22.5}$$

Definition 22.2. Denote $D(\Omega)$ a space consisting of exactly the functions in $C_0^\infty(\Omega)$ along with the understanding of convergence as $\phi_n \in D(\Omega)$ converges to $\phi \in D(\Omega)$ if the following requirements are satisfied:

(1) there exists a compact subset $K \in \Omega$ such that

$$\phi(x) = 0, \text{for any } x \in \Omega \backslash K \text{ and any } n, \tag{22.6}$$

(2) for any fixed multi-index α, $\partial_\alpha \phi_n$ converges uniformly to $\partial_\alpha \phi$ on K, *i.e.*

$$\max_{x \in K} |\partial_\alpha \phi_n - \partial_\alpha \phi| \to 0 (n \to \infty). \tag{22.7}$$

Chapter 23

Variational Methods

Many problems in functional analysis can be formed in the pattern solving

$$F(u) = 0, \tag{23.1}$$

where u is in a Banach space V.

Definition 23.1. Let X and Y be two normed vector space and $U \in X$ is a open subset of V. A function $f : U \to Y$ is called Fréchet-differentiable at x if there exists a bounded linear operator $A : X \to Y$ such that

$$\lim_{\|h\| \to 0} \frac{\left\|f(x+h) - f(x) - Ah\right\|_X}{\|h\|_V} = 0. \tag{23.2}$$

We write $Df(x) = A$ and call it the Fréchet derivative of f at x. The directional (Gateaux-) derivative of f at x in the direction v is defined as

$$D_v f(x) = \langle v, Df(x) \rangle. \tag{23.3}$$

Part VII

Numerical Optimization

Chapter 24

Convex Sets

24.1 Cones

Definition 24.1. A set C is a cone if for every $x \in C$ and $\theta \geq 0$, $\theta x \in C$ always holds.

Cone is also named *nonnegative homogeneous.* It is easy to prove that a cone is closed.

24.2 Dual cones

24.2.1 Dual cones

Definition 24.2. Let K be a cone. The set is called dual cone of K if

$$K^* = \{y | y^{\mathrm{T}} x \geq 0 \text{ for all } x \in K\}.$$

24.2.2 Examples

Dual of a norm cone

The dual cone of $K = \{(x, t) \in \mathbb{R}^{n+1} | \|x\| \leq t\}$ is

$$K^* = \{(u, v) \in \mathbb{R}^{n+1} | \|u\| \leq v\},$$

where the dual norm is given by $\|u\|_* = \sup\{u^T x | \|x\| \leq 1\}$.

Nonnegative orthant

The cone $\mathbb{R}^n_+$ is self-dual.

Positive semi-definitive matrix

The set of positive semi-definitive matrix $\mathbf{S}^n_+$ is also self-dual.

24.2.3 Property of dual cone

- K^* is a cone and convex.
- K is closed and convex.
- $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$

24.3 Generalized inequalities

24.3.1 Proper cones

Definition 24.3. A cone $K \in \mathbb{R}^n$ is a proper cone if satisfies the following conditions:

- K is convex.
- K is closed.
- K is solid, i.e., it has nonempty interior.
- K is pointed, i.e., it contains no line.

Then we can define generalized inequalities on K, which is a partial ordering on $mathbbR^n$. We write y

Chapter 25

Convex functions

25.1 Convex functions and examples

Definition 25.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathbf{dom} f$ is a convex set and for all $x, y \in \mathbf{dom} f$, and $\theta \geq 0$, the following equation holds:

$$\theta f(x) + (1-\theta) f(y) \leq f(\theta x + (1-\theta) y) \tag{25.1}$$

It can be shown that a function is convex if and only if it is convex when its domain is restrictecd to a line. A convex function is contiuous on the relative interior of its domain.

25.1.1 First-order condition

Theorem 25.1. *Assume f is differentiable, then f is convex if and only if* $\mathbf{dom} f$ *is convex and*

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \tag{25.2}$$

holds for all $x, y \in \mathbf{dom} f$.

25.1.2 Second-order condition

Theorem 25.2. *Supppose that f is twice differentiable and* $\mathbf{dom} f$ *is open, this is, its* Hession Matrix $\nabla^2 f(x)$ *exists at each point in its domain. Then f is convex if and only if $mathbf dom f$ is convex its Hession Matrix is positive semidefinitite.*

For strict convexity, we have that if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom} f$, f is strict convex. However, the converse is not true.

25.1.3 Examples

Here are some convex functions on $\mathbb{R}^2$:

- Norms.
- Max function.

- Quadratic-over-linear function, i.e., $f(x, y) = x^2/y$, with

$$\mathbf{dom} f = \mathbb{R} \times \mathbb{R}_{++}.$$

- Log-sum-exp function, i.e., $f(\mathbf{x}) = \log(e^{x_1} + e^{x_2} + \cdots + e^{x_n})$. In fact, this function can be viewed as a differentiable approximation of the max function, since

$$\max(\mathbf{x}) \leq f(\mathbf{x}) \leq \max(\mathbf{x}) + \log(n) \tag{25.3}$$

  holds for any $\mathbf{x}$.
- Geometric mean, i.e., $f(x) = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ with $\mathbf{dom} f = \mathbb{R}_{++}$ is concave.
- Log-determinant, i.e., $f(\mathbf{X}) = \log\det \mathbf{X}$ on $\mathbf{dom} f = \mathbb{S}^n_{++}$ is concave. *The proof of this part is sophisticated and I don't fully understand so I should turn back later.*

25.1.4 Sublevel sets

Definition 25.2. The α-sublevel set of a function $f : \mathbb{R}^n \leftarrow \mathbb{R}$ is

$$C_\alpha = \{\mathbf{x} \in \mathbf{dom} f \mid f(\mathbf{x}) < \alpha\}.$$

Sublevel sets of a convex function are convex, while the converse is not true, and this can be used to tell a set is convex by expressing it as a sublevel of a convex function.

25.1.5 Epigraph

Definition 25.3. The epigraph of a function $f : \mathbb{R}^n \leftarrow \mathbb{R}$ is

$$\mathbf{epi} = \{(\mathbf{x}, t)) \mid \mathbf{x} \in \mathbf{dom} f, f(\mathbf{x}) \leq t\}.$$

The relation between convex function and convex set is that *a function is convex if and only if its epigraph is a convex set.*

Chapter 26

Gradient Methods

26.1 Conjugate Gradient Method

Consider minimizing the quadratic form

$$f(x) = \frac{1}{2}x^T Gx + b \tag{26.1}$$

where $G \in \mathbb{S}^n_{++}$ and $x \in \setminus^n$. First let's restrict the case to $n = 2$. If G is a diagonal matrix, then intuitively we can find the solution of 26.1 in two steps by exact line searching along with two coordinates, since the contour of 26.1 is a ellipse with its major and minor axes parallel to the coordinates.

In general case, we can diagonalize G by introducing a linear transform, denoted by $D = (d_1, \cdots, d_n)$. Substituting $x = D\tilde{x}$ for 26.1 we get

$$\tilde{f}(\tilde{x}) = f(Dx) = \frac{1}{2}x^T D^T GDx. \tag{26.2}$$

The (i, j)th entry of $D^T GD$ is $d_i^T Gd_j$ and $d_i^T Gd_j = 0$ if $i \neq j$. Here we encounter a very interesting condition, which can be denoted as

$$\langle d_i, d_j \rangle_G = d_i^T Gd_j = 0, \tag{26.3}$$

if we introduce a new inner product

$$\langle x, y \rangle_A = x^T Ay. \tag{26.4}$$

where $A \in \mathbb{S}^n_{++}$. Then condition 26.3, called conjugate, can be viewed as a generalization of orthogonality.

Definition 26.1. Let $G \in \mathbb{S}^n_{++}$ and $x, y \in \setminus^n$ are nonzero vectors. x and y is conjugate with respect to G if

$$\langle x, y \rangle_G = x^T Gy = 0. \tag{26.5}$$

A sequence of vectors $\{d_i\}$, $i = 1, 2, \cdots, m$ is conjugate if any two vectors of them are conjugate.

We have the following simple theorem, of which the proof is straightforward.

Theorem 26.1. *Conjugate vectors are linear independent.*

26.1.1 Connection with linear equations

Chapter 27

Linear Programming

27.1 Standard Form of Linear Progamming

The optimization problem like

$$\begin{aligned} \text{minimize} \quad & f(x) = c^T x \\ \text{subject to} \quad & Ax = b \\ & x \succeq 0. \end{aligned} \tag{27.1}$$

where $x, c \in \setminus^n$, $A \in \setminus^{m\times n}$ and $b \in \setminus^n$, is called the standard form of linear programming. We may always assume that $m < n$. Otherwise, redundant rows of A can be eliminated or one/no solution obeys $Ax = b$, which can be solved by numerical algebra methods, such as LR or QR factorization.

Without loss of generality, we suppose that $c \neq 0$ and $\operatorname{rank}(A) = m$. Obviously, the feasible set of 27.1 $F = \{x \in \setminus^n : Ax = b, x \succeq 0\}$ is a closed convex set.

27.2 Simplex Method

Simplex method is a prevailing approach to solve linear programming problem.

Since A has the attribute that $\operatorname{rank}(A) = m$, we can split A to

$$A = \begin{pmatrix} B, N \end{pmatrix}$$

where $B \in \setminus^{m\times m}$ is non-singular and $N \in \setminus^{m\times(n-m)}$. Correspondingly, x, c are split in the same manner that

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, c = \begin{pmatrix} c_B \\ c_N \end{pmatrix}.$$

Thus $Ax = b$ is equivalent to $Bx_B + Nx_N = b$. Owing to non-singularity of B, we obtain $x_B = B^{-1}(b - Nx_N)$. With the knowledge of linear algebra, we contend that

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}(b - Nx_N) \\ x_N \end{pmatrix} \tag{27.2}$$

is the general solution to equation $Ax = b$, and x_N is the free variable that can be any vector in $\setminus^{n-m}$, x_B is the basic variable. The choice of free variable and basic variable is somewhat arbitrary, since interchanging the position of variables do no harm to correctness of $Ax = b$, as long as B is non-singular. With a fixed non-singular B, setting x_N to zero, then we have

$$x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}, \tag{27.3}$$

which we call a basic solution to 27.1 or a basic feasible solution if further $B^{-1}b \succeq 0$.

Theorem 27.1. *A point $x \in \diagdown^n$ is a basic feasible solution to 27.1 if and only if it is a vertice feasible polytope $F = \{x \in \diagdown^n : Ax = 0, x \succeq 0\}$.*

The basic idea of simplex method is to construct a sequence of vertices $\{x^{(k)}\}$ of F (basic feasible solutions) with descending value of the target function $f(x) = c^T x$. Suppose now we have the kth iterative result $x^{(k)}$, then we are confronted with the problem that *how to decide whether we have obtained an optimal point.*

Assume that

$$x^{(k)} = \begin{pmatrix} x_B^{(k)} \\ x_N^{(k)} \end{pmatrix} = \begin{pmatrix} B^{(k)-1}b \\ 0 \end{pmatrix}$$

where $x_B^{(k)}$ is the basic variable and $x_N^{(k)}$ is the free variable. Otherwise, it is always reasonable to interchange the order of variables and columns of A to adapt it to the form 27.2. Thus $f(x^{(k)}) = c_B^{(k)T} B^{(k)-1} b$. For any $x \in F$, x can be written in the form

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}(b - Nx_N) \\ x_N \end{pmatrix}. \tag{27.4}$$

Then

$$\begin{aligned} f(x) &= c_B^T x_B + c_N^T x_N = c_B^T B^{-1}(b - Nx_N) + c_N^T x_N \\ &= f(x^{(k)}) + (c_N^T - c_B^T B^{-1}) x_N \end{aligned}$$

Part VIII

Probability and Statistics

Chapter 28

Sufficient Statistics

Definition 28.1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution that has pdf or pmf $f(x,)$.

Index