Warning: These are my notes. Don't expect this document to be self-contained and correct.

1 ADER-DG for linear acoustics

The linearized acoustic equations are given as (see Finite Volume book by LeVeque)

$$\frac{\partial Q_p}{\partial t} + A_{pq}\frac{\partial Q_q}{\partial x} + B_{pq}\frac{\partial Q_q}{\partial y} = 0, \tag{1}$$

where

$$q = \begin{pmatrix} p \\ u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{pmatrix}. \tag{2}$$

The corresponding weak form is

$$\int_\Omega \phi_k \frac{\partial Q_p}{\partial t}\,\mathrm{d}V + \int_{\partial\Omega} \phi_k \left(n_x A_{pq} + n_y B_{pq}\right) Q_q\,\mathrm{d}S - \int_\Omega \left(\frac{\partial \phi_k}{\partial x} A_{pq} Q_q + \frac{\partial \phi_k}{\partial y} B_{pq} Q_q\right)\mathrm{d}V = 0, \tag{3}$$

where $n = (n_x, n_y)$ is the outward unit surface normal.

We discretise the weak form with finite elements, which are axis aligned rectangles $\mathcal{R}^{(m)}$, and obtain

$$\int_{\mathcal{R}^{(m)}} \phi_k \frac{\partial Q_p}{\partial t}\,\mathrm{d}V + \int_{\partial\mathcal{R}^{(m)}} \phi_k \left(\left(n_x A_{pq} + n_y B_{pq}\right) Q_q\right)^*\,\mathrm{d}S - \int_{\mathcal{R}^{(m)}} \left(\frac{\partial \phi_k}{\partial x} A_{pq} + \frac{\partial \phi_k}{\partial y} B_{pq}\right) Q_q\,\mathrm{d}V = 0, \tag{4}$$

where we a numerical flux (indicated with *).

Suppose we are given a grid of points $P_{i,j} = (ih_x, jh_y)$, where $(i,j) \in [0,X] \times [0,Y]$ and $h_x, h_y > 0$. Then a rectangle $R^{(m)}$ with $m = (i,j)$, where $i < X$ and $j < Y$, is given by the four points $\{P_{i,j}, P_{i+1,j}, P_{i,j+1}, P_{i+1,j+1}\}$.

We approximate Q with a modal basis, i.e.

$$Q_p^h(x,y,t) = \hat{Q}_{lp}(t)\,\phi_l\left(\xi^{(m)}(x,y), \eta^{(m)}(x,y)\right), \tag{5}$$

where

$$\xi^{(m)}(x,y) = \frac{x - (P_m)_1}{h_x}, \quad \eta^{(m)}(x,y) = \frac{y - (P_m)_2}{h_y}. \tag{6}$$

Then we obtain, using the substitution rule,

$$\begin{aligned}
|J|\frac{\partial \hat{Q}_{lp}}{\partial t}(t) &\int_0^1\int_0^1 \phi_k(\xi,\eta)\phi_l(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&+ h_x \int_0^1 \phi_k(\xi,1)\left(B_{pq}Q_q\right)^*\,\mathrm{d}\xi - h_x \int_0^1 \phi_k(\xi,0)\left(B_{pq}Q_q\right)^*\,\mathrm{d}\xi \\
&+ h_y \int_0^1 \phi_k(1,\eta)\left(A_{pq}Q_q\right)^*\,\mathrm{d}\eta - h_y \int_0^1 \phi_k(0,\eta)\left(A_{pq}Q_q\right)^*\,\mathrm{d}\eta \\
&- |J|\hat{Q}_{lp}(t)\int_0^1\int_0^1 \left(\frac{1}{h_x}\frac{\partial \phi_k}{\partial \xi}(\xi,\eta)A_{pq} + \frac{1}{h_y}\frac{\partial \phi_k}{\partial \eta}(\xi,\eta)B_{pq}\right)\phi_l(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi = 0,
\end{aligned} \tag{7}$$

where $|J| = h_x h_y$.

We turn now to the flux term. First, note that we may use rotational invariance:

$$n_x A + n_y B = TAT^{-1}, \tag{8}$$

where

$$T(n_x,n_y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & n_x & -n_y \\ 0 & n_y & n_x \end{pmatrix}, \quad T^{-1}(n_x,n_y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & n_x & n_y \\ 0 & -n_y & n_x \end{pmatrix}, \tag{9}$$

i.e. we only need to solve the Riemann problem in x-direction. In the homogeneous case we have

$$A^{+}=\frac{1}{2}\begin{pmatrix} c & K_0 & 0\\ 1/\rho_0 & c & 0\\ 0 & 0 & 0\end{pmatrix},\quad A^{-}=\frac{1}{2}\begin{pmatrix} -c & K_0 & 0\\ 1/\rho_0 & -c & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{10}$$

where $c=\sqrt{K_0/\rho_0}$.

In the inhomogeneous case we have

$$A^{+}=\begin{pmatrix} \frac{K_0^{+}c^{-}c^{+}}{K_0^{-}c^{+}+K_0^{+}c^{-}} & \frac{K_0^{-}K_0^{+}c^{+}}{K_0^{-}c^{+}+K_0^{+}c^{-}} & 0\\ \frac{K_0^{+}c^{-}}{\rho_0^{+}\left(K_0^{-}c^{+}+K_0^{+}c^{-}\right)} & \frac{K_0^{-}K_0^{+}}{\rho_0^{+}\left(K_0^{-}c^{+}+K_0^{+}c^{-}\right)} & 0\\ 0 & 0 & 0\end{pmatrix},\quad A^{-}=\begin{pmatrix} -\frac{K_0^{-}c^{-}c^{+}}{K_0^{-}c^{+}+K_0^{+}c^{-}} & \frac{K_0^{-}K_0^{+}c^{-}}{K_0^{-}c^{+}+K_0^{+}c^{-}} & 0\\ \frac{K_0^{-}c^{+}}{\rho_0^{-}\left(K_0^{-}c^{+}+K_0^{+}c^{-}\right)} & -\frac{K_0^{-}K_0^{+}}{\rho_0^{-}\left(K_0^{-}c^{+}+K_0^{+}c^{-}\right)} & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{11}$$

Hence, the flux is given as

$$TA^{-}T^{-1}Q^{-}+TA^{+}T^{-1}Q^{+}. \tag{12}$$

For ease of notation, we define

$$\mathcal{A}^{x,y,\pm}=T(x,y)A^{\pm}T(x,y)^{-1}. \tag{13}$$

Note, that we need the transposed version as we multiply the matrix from the right. Hence, with abuse of notation we define

$$\mathcal{A}^{x,y,\pm}=T^{-T}\left(A^{\pm}\right)^{T}T^{T}=T\left(A^{\pm}\right)^{T}T^{-1}, \tag{14}$$

where we used that $T^{-1}=T^{T}$.

We are now able to obtain the complete semi-discrete scheme:

$$\begin{aligned}
|J|\frac{\partial\hat{Q}_{lp}}{\partial t}(t)&\int_0^1\int_0^1\phi_k(\xi,\eta)\phi_l(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\\
&+h_x\mathcal{A}^{0,1,+}_{pq}\hat{Q}_{lp}(t)\int_0^1\phi_k(\xi,1)\phi_l(\xi,1)\,\mathrm{d}\xi+h_x\mathcal{A}^{0,1,-}_{pq}\hat{Q}^{(i,j+1)}_{lp}(t)\int_0^1\phi_k(\xi,1)\phi_l(\xi,0)\,\mathrm{d}\xi\\
&+h_x\mathcal{A}^{0,-1,+}_{pq}\hat{Q}_{lp}(t)\int_0^1\phi_k(\xi,0)\phi_l(\xi,0)\,\mathrm{d}\xi+h_x\mathcal{A}^{0,-1,-}_{pq}\hat{Q}^{(i,j-1)}_{lp}(t)\int_0^1\phi_k(\xi,0)\phi_l(\xi,1)\,\mathrm{d}\xi\\
&+h_y\mathcal{A}^{1,0,+}_{pq}\hat{Q}_{lp}(t)\int_0^1\phi_k(1,\eta)\phi_l(1,\eta)\,\mathrm{d}\eta+h_y\mathcal{A}^{1,0,-}_{pq}\hat{Q}^{(i+1,j)}_{lp}(t)\int_0^1\phi_k(1,\eta)\phi_l(0,\eta)\,\mathrm{d}\eta\\
&+h_y\mathcal{A}^{-1,0,+}_{pq}\hat{Q}_{lp}(t)\int_0^1\phi_k(0,\eta)\phi_l(0,\eta)\,\mathrm{d}\eta+h_y\mathcal{A}^{-1,0,-}_{pq}\hat{Q}^{(i-1,j)}_{lp}(t)\int_0^1\phi_k(0,\eta)\phi_l(1,\eta)\,\mathrm{d}\eta\\
&-|J|\hat{Q}_{lp}(t)\int_0^1\int_0^1\left(\frac{1}{h_x}\frac{\partial\phi_k}{\partial\xi}(\xi,\eta)A_{pq}+\frac{1}{h_y}\frac{\partial\phi_k}{\partial\eta}(\xi,\eta)B_{pq}\right)\phi_l(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi=0,
\end{aligned} \tag{15}$$

To be precomputed:

$$\begin{aligned}
M_{kl}&=\int_0^1\int_0^1\phi_k(\xi,\eta)\phi_l(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\\
F^{x,-,s}_{kl}&=\int_0^1\phi_k(\xi,s)\phi_l(\xi,s)\,\mathrm{d}\xi\\
F^{x,+,s}_{kl}&=\int_0^1\phi_k(\xi,s)\phi_l(\xi,1-s)\,\mathrm{d}\xi\\
F^{y,-,s}_{kl}&=\int_0^1\phi_k(s,\eta)\phi_l(s,\eta)\,\mathrm{d}\eta\\
F^{y,+,s}_{kl}&=\int_0^1\phi_k(s,\eta)\phi_l(1-s,\eta)\,\mathrm{d}\eta\\
K^{\xi}_{kl}&=\int_0^1\int_0^1\frac{\partial\phi_k}{\partial\xi}(\xi,\eta)\phi_l(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\\
K^{\eta}_{kl}&=\int_0^1\int_0^1\frac{\partial\phi_k}{\partial\eta}(\xi,\eta)\phi_l(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi
\end{aligned}$$

1.1 L^2 projection

It might become necessary to project a function on the basis functions, i.e. we require the integral

$$|J| \int_0^1 \int_0^1 \phi_k(\xi,\eta) f\left(x(\xi,\eta), y(\xi,\eta)\right) \,\mathrm{d}\eta \,\mathrm{d}\xi \tag{16}$$

which can be approximated with a quadrature rule (χ, ω) (on $[-1,1]$).

$$\begin{aligned}
&|J| \int_0^1 \int_0^1 \phi_k(\xi,\eta) f\left(x(\xi,\eta), y(\xi,\eta)\right) \,\mathrm{d}\eta \,\mathrm{d}\xi = \\
&\qquad \frac{|J|}{2} \int_0^1 \sum_{j=0}^{N} \omega_j \phi_k\left(\xi, \frac{\chi_j+1}{2}\right) f\left(x\left(\xi, \frac{\chi_j+1}{2}\right), y\left(\xi, \frac{\chi_j+1}{2}\right)\right) \mathrm{d}\xi = \\
&\qquad \frac{|J|}{4} \sum_{i=0}^{N} \sum_{j=0}^{N} \omega_i \omega_j \phi_k\left(\frac{\chi_i+1}{2}, \frac{\chi_j+1}{2}\right) f\left(x\left(\frac{\chi_i+1}{2}, \frac{\chi_j+1}{2}\right), y\left(\frac{\chi_i+1}{2}, \frac{\chi_j+1}{2}\right)\right) \mathrm{d}\xi
\end{aligned} \tag{17}$$

1.2 Convergence test

We assume homogeneous material parameters and we assume that our solution is a plane wave of the form

$$Q_p(x,y,t) = Q_p^0 \sin\left(\omega t - k_x x - k_y y\right). \tag{18}$$

Inserting the solution into **??** yields

$$\left(\omega I_{pq} - k_x A_{pq} - k_y B_{pq}\right) Q_q^0 \cos\left(\omega t - k_x x - k_y y\right) = 0, \tag{19}$$

where I is an identity matrix. So either the cosine term is zero or

$$\left(\omega I_{pq} - k_x A_{pq} - k_y B_{pq}\right) Q_q^0 = 0 \tag{20}$$

must hold. This is equivalent to the following eigenvalue problem:

$$\left(k_x A_{pq} + k_y B_{pq}\right) Q_q^0 = \omega Q_q^0 \tag{21}$$

So ω must be an eigenvalue and Q_q^0 must be an eigenvector of the matrix

$$k_x A + k_y B = \|k\| T\left(\frac{k_x}{\|k\|}, \frac{k_y}{\|k\|}\right) A T\left(\frac{k_x}{\|k\|}, \frac{k_y}{\|k\|}\right)^{-1}, \tag{22}$$

where we used rotational invariance. Let (λ_i, r_i) be eigenvalue and corresponding eigenvector of A. Then

$$Q^0 = T\left(\frac{k_x}{\|k\|}, \frac{k_y}{\|k\|}\right) r_i, \text{ and } \omega = \|k\| \lambda_i \tag{23}$$

solves the homogeneous problem with initial condition $Q_p(x,y,0) = Q_p^0 \sin(-k_x x - k_y y)$.

For the convergence test we choose $r_i = (K_0, c, 0)^T, \omega = c, k_x = 2\pi, k_y = 2\pi$. Then

$$T\left(\sqrt{2}/2, \sqrt{2}/2\right) r_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} K_0 \\ c \\ 0 \end{pmatrix} = \begin{pmatrix} K_0 \\ \sqrt{2}c/2 \\ \sqrt{2}c/2 \end{pmatrix}, \text{ and } \omega = 2\sqrt{2}\pi c. \tag{24}$$

Due to periodic boundary conditions, after 1/2 seconds the solution is equal to the initial condition.

1.3 Source terms

Assume we have a source term of the form

$$S_p(t)\delta(x_s, y_s) \tag{25}$$

then we need to add

$$\frac{1}{|J|} M_{kq}^{-1} \phi_q\left(\xi(x_s,y_s), \eta(x_s,y_s)\right) \int_{t_n}^{t_n+\Delta t} S_p(t) \,\mathrm{d}t \tag{26}$$

to the cell that contains the source term.