Stress Analysis in Cone

We first solve the small problem of rotating cone. Suppose that the cone has the height H and inner radius r_i and r_e as a function of the cone height z, with the uniform density ρ. This cone rotates with a uniform rotation ω in an airflow at the inlet of a turbojet engine.

We examine the effect of rotation in stress of this cone. By using Cauchy stress, we have

$$\operatorname{div} \underline{\underline{\sigma}} + \rho \underline{f} = 0. \tag{1}$$

We assume that elastic waves is absent, so that the configuration is truly symmetric about the rotation axis z. This assumption maybe oversimplified and need a justification by experiment. But we neglect the question of validity now and attempt to obtain an analytical solution for this problem. The Cauchy stress tensor is expressed:

$$\underline{\underline{\sigma}} := \sigma_{rr} \vec{e}_r \otimes \vec{e}_r + \sigma_{\theta\theta} \vec{e}_\theta \otimes \vec{e}_\theta + \sigma_{zz} \vec{e}_z \otimes \vec{e}_z \tag{2}$$

From this the divergence of this tensor is written as

$$\sigma'_{rr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r},$$

which gives

$$\sigma'_{rr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho\omega^2 r = 0. \tag{3}$$

The compatibility condition gives us

$$\varepsilon_{rr} = \varepsilon_{\theta\theta} + r\varepsilon'_{\theta\theta} \tag{4}$$

If we assume that the materials is isotropic, this is the majority of case of metals, then

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E} - \frac{\nu\sigma_{\theta\theta}}{E} \quad \text{and} \quad \varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu\sigma_{rr}}{E} \tag{5}$$

This gives us

$$r\sigma''_{rr} + 3\sigma'_{rr} + \rho\omega^2 r(3+\nu) = 0. \tag{6}$$

Integrate them gives us

$$\sigma_{rr} = -\frac{\rho\omega^2 r}{8}(3+\nu) + \frac{A}{r^2} + B \quad \text{and} \quad \sigma_{\theta\theta} = -\frac{\rho\omega^2 r}{8}(1+3\nu) - \frac{A}{r^2} + B \tag{7}$$

With the condition $\sigma_{rr}(r_i) = p_0$ and $\sigma_{rr}(r_e) = p_e$, which give us:

$$\sigma_{rr}(r) = p_i \cos\theta + (p_e - p_i)\cos\theta \frac{r_i^2 - r^2}{r_i^2 - r_e^2}\frac{r_e^2}{r^2} + \frac{\rho\omega^2}{8}\frac{3+\nu}{r^2}(r_e^2 - r^2)(r^2 - r_i^2) \tag{8}$$

$$\sigma_{\theta\theta}(r) = p_i \cos\theta + (p_e - p_i)\cos\theta \frac{r_i^2 + r^2}{r_i^2 - r_e^2}\frac{r_e^2}{r^2} + \frac{\rho\omega^2}{8}\left[\frac{3+\nu}{r^2}(r_e^2 + r^2)(r^2 + r_i^2) - 4(1+\nu)r^2\right] \tag{9}$$

$$\tag{10}$$