Weak form of Laplace equation:

$$\Delta u(x,y) = 0. \tag{1}$$

$$\forall w, \quad 0 = \int_\Omega w(\Delta u) d\Omega = \int_\Omega w[\underline{\nabla}\cdot(\underline{\nabla}u)]d\Omega = \int_{\partial\Omega} w\underline{\nabla}u\cdot\underline{n}dS - \int_\Omega \underline{\nabla}w\cdot\underline{\nabla}u d\Omega$$

$$\forall w, \quad \int_\Omega \underline{\nabla}w\cdot\underline{\nabla}u d\Omega = \int_{\partial\Omega} w\underline{\nabla}u\cdot\underline{n}dS$$

We approximate the solution by

$$u(x,y) = \sum_{i=0}^{m} N_i(x,y)u_i$$
$$w(x,y) = \sum_{i=0}^{m} N_i(x,y)w_i$$

$$\underline{\nabla}u(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x}u_1 + \cdots + \frac{\partial N_m}{\partial x}u_m \\ \frac{\partial N_1}{\partial y}u_1 + \cdots + \frac{\partial N_m}{\partial y}u_m \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \dots & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_1}{\partial y} & \dots & \frac{\partial N_m}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$\underline{\nabla}w(x,y) = \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x}w_1 + \cdots + \frac{\partial N_m}{\partial x}w_m \\ \frac{\partial N_1}{\partial y}w_1 + \cdots + \frac{\partial N_m}{\partial y}w_m \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \dots & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_1}{\partial y} & \dots & \frac{\partial N_m}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

$$\underline{\nabla}w\cdot\underline{\nabla}u = \begin{bmatrix} w_1 & \dots & w_m \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} \\ \vdots & \vdots \\ \frac{\partial N_m}{\partial x} & \frac{\partial N_m}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial x} & \dots & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_1}{\partial y} & \dots & \frac{\partial N_m}{\partial y} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = [w]B^TB[u]$$

$$\boxed{\int_\Omega [w]B^TB[u]d\Omega = \int_{\partial\Omega} w\underline{\nabla}u\cdot\underline{n}dS}.$$

For a local element:

$$1 = (x_1, y_1)$$
$$2 = (x_2, y_2)$$
$$3 = (x_3, y_3)$$
$$4 = (x_4, y_4)$$

$$J^{(1)} = \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} (\eta-1)(x_1-x_2)+(1+\eta)(x_3-x_4) & (\eta-1)(y_1-y_2)+(1+\eta)(y_3-y_4) \\ (\xi-1)(x_1-x_4)+(1+\xi)(x_3-x_2) & (\xi-1)(y_1-y_4)+(1+\xi)(y_3-y_2) \end{bmatrix}$$

$$\det(J^{(1)}) = \frac{1}{16}\left[(\eta-1)(x_1-x_2)+(1+\eta)(x_3-x_4)\right]\left[(\xi-1)(y_1-y_4)+(1+\xi)(y_3-y_2)\right] - \frac{1}{16}\left[(\eta-1)(y_1-y_2)+(1+\eta)(y_3-y_4)\right]\left[(\xi-1)(x_1-x_4)+(1+\xi)(x_3-x_2)\right]$$

$$(J^{(1)})^{-1} = \frac{1}{\det(J^{(1)})}\begin{bmatrix} (\xi-1)(y_1-y_4)+(1+\xi)(y_3-y_2) & -(\xi-1)(x_1-x_4)-(1+\xi)(x_3-x_2) \\ -(\eta-1)(y_1-y_2)-(1+\eta)(y_3-y_4) & (\eta-1)(x_1-x_2)+(1+\eta)(x_3-x_4) \end{bmatrix}$$

$$B^{(1)} = \frac{1}{4}(J^{(1)})^{-1}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}$$

$$K_{11}^{(e)} = \left|J^{(1)}\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\right|(B^{(1)})^T\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)B^{(1)}\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$
$$K_{12}^{(e)} = \left|J^{(1)}\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right|(B^{(1)})^T\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)B^{(1)}\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$
$$K_{21}^{(e)} = \left|J^{(1)}\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\right|(B^{(1)})^T\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)B^{(1)}\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$
$$K_{22}^{(e)} = \left|J^{(1)}\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right|(B^{(1)})^T\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)B^{(1)}\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

For a rectangular element,

$$\begin{bmatrix} \frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{x}{y}-2\frac{y}{x}\right) & -\frac{1}{6}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{y}{x}-2\frac{x}{y}\right) \\ \frac{1}{6}\left(\frac{x}{y}-2\frac{y}{x}\right) & \frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{y}{x}-2\frac{x}{y}\right) & -\frac{1}{6}\left(\frac{x}{y}+\frac{y}{x}\right) \\ -\frac{1}{6}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{y}{x}-2\frac{x}{y}\right) & \frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{x}{y}-2\frac{y}{x}\right) \\ \frac{1}{6}\left(\frac{y}{x}-2\frac{x}{y}\right) & -\frac{1}{6}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{x}{y}-2\frac{y}{x}\right) & \frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}\right) \end{bmatrix}$$

We need to assembly the matrix, where the numbering is 0 - 1 - 2 - 3, counterclockwise for each element. The local stiffness matrices are:

$$K^{(1)} = \begin{bmatrix} K^{(1)}_{00} & K^{(1)}_{01} & K^{(1)}_{02} & K^{(1)}_{03} \\ K^{(1)}_{10} & K^{(1)}_{11} & K^{(1)}_{12} & K^{(1)}_{13} \\ K^{(1)}_{20} & K^{(1)}_{21} & K^{(1)}_{22} & K^{(1)}_{23} \\ K^{(1)}_{30} & K^{(1)}_{31} & K^{(1)}_{32} & K^{(1)}_{33} \end{bmatrix}, \quad K^{(2)} = \begin{bmatrix} K^{(2)}_{00} & K^{(2)}_{01} & K^{(2)}_{02} & K^{(2)}_{03} \\ K^{(2)}_{10} & K^{(2)}_{11} & K^{(2)}_{12} & K^{(2)}_{13} \\ K^{(2)}_{20} & K^{(2)}_{21} & K^{(2)}_{22} & K^{(2)}_{23} \\ K^{(2)}_{30} & K^{(2)}_{31} & K^{(2)}_{32} & K^{(2)}_{33} \end{bmatrix}.$$

The global stiffness matrix K_{global} starts as a zero matrix:

$$K_{\text{global}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Adding $K^{(1)}$: Mapping $\{0,1,2,3\}$ to $\{0,1,2,3\}$: $K_{\text{global}}[i,j] = K^{(1)}[i,j] \quad \forall i,j \in \{0,1,2,3\}$.
After this step:

$$K_{\text{global}} = \begin{bmatrix} K^{(1)}_{00} & K^{(1)}_{01} & K^{(1)}_{02} & K^{(1)}_{03} & 0 & 0 \\ K^{(1)}_{10} & K^{(1)}_{11} & K^{(1)}_{12} & K^{(1)}_{13} & 0 & 0 \\ K^{(1)}_{20} & K^{(1)}_{21} & K^{(1)}_{22} & K^{(1)}_{23} & 0 & 0 \\ K^{(1)}_{30} & K^{(1)}_{31} & K^{(1)}_{32} & K^{(1)}_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then adding $K^{(2)}$: mapping $\{0,1,2,3\}$ to $\{1,4,5,3\}$: $K_{\text{global}}[g_i, g_j] += K^{(2)}[i,j] \quad \forall i,j \in \{0,1,2,3\}$, where $g_i = M^{(2)}[i]$ and $g_j = M^{(2)}[j]$.
After this step:

$$K_{\text{global}} = \begin{bmatrix} K^{(1)}_{00} & K^{(1)}_{01} & K^{(1)}_{02} & K^{(1)}_{03} & 0 & 0 \\ K^{(1)}_{10} & K^{(1)}_{11} + K^{(2)}_{00} & K^{(1)}_{12} & K^{(1)}_{13} + K^{(2)}_{03} & K^{(2)}_{01} & K^{(2)}_{02} \\ K^{(1)}_{20} & K^{(1)}_{21} & K^{(1)}_{22} & K^{(1)}_{23} & 0 & 0 \\ K^{(1)}_{30} & K^{(1)}_{31} + K^{(2)}_{30} & K^{(1)}_{32} & K^{(1)}_{33} + K^{(2)}_{33} & K^{(2)}_{31} & K^{(2)}_{32} \\ 0 & K^{(2)}_{10} & 0 & K^{(2)}_{13} & K^{(2)}_{11} & K^{(2)}_{12} \\ 0 & K^{(2)}_{20} & 0 & K^{(2)}_{23} & K^{(2)}_{21} & K^{(2)}_{22} \end{bmatrix}.$$

Final Global Stiffness Matrix: The final K_{global} combines contributions from all elements:

$$K_{\text{global}} = \sum_{e=1}^{E} \text{Assemble}\left(K^{(e)}\right).$$

The example result:

$$\begin{bmatrix} K^{(1)}_{00} & K^{(1)}_{01} & 0 & K^{(1)}_{03} & K^{(1)}_{02} & 0 & 0 & 0 & 0 \\ K^{(1)}_{10} & K^{(1)}_{11} + K^{(2)}_{00} & K^{(2)}_{01} & K^{(1)}_{13} & K^{(1)}_{12} + K^{(2)}_{03} & K^{(2)}_{02} & 0 & 0 & 0 \\ 0 & K^{(2)}_{10} & K^{(2)}_{11} & 0 & K^{(2)}_{13} & K^{(2)}_{12} & 0 & 0 & 0 \\ K^{(1)}_{30} & K^{(1)}_{31} & 0 & K^{(1)}_{33} + K^{(4)}_{00} & K^{(1)}_{32} + K^{(4)}_{01} & 0 & K^{(4)}_{03} & K^{(4)}_{02} & 0 \\ K^{(1)}_{20} & K^{(1)}_{21} + K^{(2)}_{30} & K^{(2)}_{31} & K^{(1)}_{23} + K^{(4)}_{10} & K^{(1)}_{22} + K^{(2)}_{33} + K^{(3)}_{00} + K^{(4)}_{11} & K^{(2)}_{32} + K^{(3)}_{01} & K^{(4)}_{13} & K^{(3)}_{03} + K^{(4)}_{12} & K^{(3)}_{02} \\ 0 & K^{(2)}_{20} & K^{(2)}_{21} & 0 & K^{(2)}_{23} + K^{(3)}_{10} & K^{(2)}_{22} + K^{(3)}_{11} & 0 & K^{(3)}_{13} & K^{(3)}_{12} \\ 0 & 0 & 0 & K^{(4)}_{30} & K^{(4)}_{31} & 0 & K^{(4)}_{33} & K^{(4)}_{32} & 0 \\ 0 & 0 & 0 & K^{(4)}_{20} & K^{(3)}_{30} + K^{(4)}_{21} & K^{(3)}_{31} & K^{(4)}_{23} & K^{(3)}_{33} + K^{(4)}_{22} & K^{(3)}_{32} \\ 0 & 0 & 0 & 0 & K^{(3)}_{20} & K^{(3)}_{21} & 0 & K^{(3)}_{23} & K^{(3)}_{22} \end{bmatrix}$$

$$\forall w, \quad \int_{\Omega} \underline{\nabla} w \cdot \underline{\nabla} u d\Omega = \int_{\partial\Omega} w \underline{\nabla} u \cdot \underline{n} dS = \int_{\partial\Omega} w(x_\ell, y) g(x,y) dS = \int_{\Gamma_{left}} -w(x_\ell, y) g(x_\ell, y) dy + \int_{\Gamma_{right}} w(x_r, y) g(x_r, y) dy$$

For the right-hand side, we approximate the solution by

$$g(x,y) = \sum_{i=0}^{m} N_i(x,y) g_i$$
$$w(x,y) = \sum_{i=0}^{m} N_i(x,y) w_i$$

$$w(x,y)g(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{m} N_i(x,y) N_j(x,y) w_i g_j.$$

$$\int_{\Gamma_{left}} -w(x_\ell, y) g(x_\ell, y) dy = \int_{y=0}^{y=L_1} -w(x_\ell, y) g(x_\ell, y) dy = -\sum_{i=0}^{m} \sum_{j=0}^{m} w_i g_j \int_{y=0}^{y=L_1} N_i(x_\ell, y) N_j(x_\ell, y) dy.$$

With

$$N_i(\xi, \eta) = \begin{bmatrix} \frac{1}{4}(1-\xi)(1-\eta) & \frac{1}{4}(1+\xi)(1-\eta) & \frac{1}{4}(1-\xi)(1+\eta) & \frac{1}{4}(1+\xi)(1+\eta) \end{bmatrix}$$

$$\begin{aligned} N_i(\xi,\eta) N_j(\xi,\eta) &= \begin{bmatrix} \frac{1}{4}(1-\xi)(1-\eta) \\ \frac{1}{4}(1+\xi)(1-\eta) \\ \frac{1}{4}(1-\xi)(1+\eta) \\ \frac{1}{4}(1+\xi)(1+\eta) \end{bmatrix} \begin{bmatrix} \frac{1}{4}(1-\xi)(1-\eta) & \frac{1}{4}(1+\xi)(1-\eta) & \frac{1}{4}(1-\xi)(1+\eta) & \frac{1}{4}(1+\xi)(1+\eta) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{16}(1-\xi)^2(1-\eta)^2 & \frac{1}{16}(1-\xi^2)(1-\eta)^2 & \frac{1}{16}(1-\xi)^2(1-\eta^2) & \frac{1}{16}(1-\xi^2)(1-\eta^2) \\ \frac{1}{16}(1-\xi^2)(1-\eta)^2 & \frac{1}{16}(1+\xi)^2(1-\eta)^2 & \frac{1}{16}(1-\xi^2)(1-\eta^2) & \frac{1}{16}(1+\xi)^2(1-\eta^2) \\ \frac{1}{16}(1-\xi)^2(1-\eta^2) & \frac{1}{16}(1-\xi^2)(1-\eta^2) & \frac{1}{16}(1-\xi)^2(1+\eta)^2 & \frac{1}{16}(1-\xi^2)(1+\eta)^2 \\ \frac{1}{16}(1-\xi^2)(1-\eta^2) & \frac{1}{16}(1+\xi)^2(1-\eta^2) & \frac{1}{16}(1-\xi^2)(1+\eta)^2 & \frac{1}{16}(1+\xi)^2(1+\eta)^2 \end{bmatrix} \end{aligned}$$

$$N_i(-1,\eta)N_j(-1,\eta) = \begin{bmatrix} \frac{1}{4}(1-\eta)^2 & 0 & \frac{1}{4}(1-\eta^2) & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4}(1-\eta^2) & 0 & \frac{1}{4}(1+\eta)^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N_i(\xi,\eta)N_j(\xi,\eta) = \begin{bmatrix} \frac{1}{4}(1-\xi)(1-\eta) & \frac{1}{4}(1+\xi)(1-\eta) & \frac{1}{4}(1-\xi)(1+\eta) & \frac{1}{4}(1+\xi)(1+\eta) \end{bmatrix} \begin{bmatrix} \frac{1}{4}(1-\xi)(1-\eta) \\ \frac{1}{4}(1+\xi)(1-\eta) \\ \frac{1}{4}(1-\xi)(1+\eta) \\ \frac{1}{4}(1+\xi)(1+\eta) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{16}(1-\xi)^2(1-\eta)^2 + \frac{1}{16}(1+\xi)^2(1-\eta)^2 + \frac{1}{16}(1-\xi)^2(1+\eta)^2 + \frac{1}{16}(1+\xi)^2(1+\eta)^2 \end{bmatrix}$$

$$N_i(-1,\eta)N_j(-1,\eta) = \begin{bmatrix} \frac{1}{4}(1-\eta)^2 + \frac{1}{4}(1+\eta)^2 \end{bmatrix}$$

Our problem is to calculate the equation:
The first element is a rectangular element, then by set $x = H1$ and $y = L_1$, we have

$$\begin{bmatrix} \frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{x}{y}-2\frac{y}{x}\right) & -\frac{1}{6}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{y}{x}-2\frac{x}{y}\right) \\ \frac{1}{6}\left(\frac{x}{y}-2\frac{y}{x}\right) & \frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{y}{x}-2\frac{x}{y}\right) & -\frac{1}{6}\left(\frac{x}{y}+\frac{y}{x}\right) \\ -\frac{1}{6}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{y}{x}-2\frac{x}{y}\right) & \frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{x}{y}-2\frac{y}{x}\right) \\ \frac{1}{6}\left(\frac{y}{x}-2\frac{x}{y}\right) & -\frac{1}{6}\left(\frac{x}{y}+\frac{y}{x}\right) & \frac{1}{6}\left(\frac{x}{y}-2\frac{y}{x}\right) & \frac{1}{3}\left(\frac{x}{y}+\frac{y}{x}\right) \end{bmatrix}$$

Then, by definition $y/x = \zeta$, we have:

$$\begin{bmatrix} \frac{1}{3}\left(\frac{1}{\zeta}+\zeta\right) & \frac{1}{6}\left(\frac{1}{\zeta}-2\zeta\right) & -\frac{1}{6}\left(\frac{1}{\zeta}+\zeta\right) & \frac{1}{6}\left(\zeta-2\frac{1}{\zeta}\right) \\ \frac{1}{6}\left(\frac{1}{\zeta}-2\zeta\right) & \frac{1}{3}\left(\frac{1}{\zeta}+\zeta\right) & \frac{1}{6}\left(\zeta-2\frac{1}{\zeta}\right) & -\frac{1}{6}\left(\frac{1}{\zeta}+\zeta\right) \\ -\frac{1}{6}\left(\frac{1}{\zeta}+\zeta\right) & \frac{1}{6}\left(\zeta-2\frac{1}{\zeta}\right) & \frac{1}{3}\left(\frac{1}{\zeta}+\zeta\right) & \frac{1}{6}\left(\frac{1}{\zeta}-2\zeta\right) \\ \frac{1}{6}\left(\zeta-2\frac{1}{\zeta}\right) & -\frac{1}{6}\left(\frac{1}{\zeta}+\zeta\right) & \frac{1}{6}\left(\frac{1}{\zeta}-2\zeta\right) & \frac{1}{3}\left(\frac{1}{\zeta}+\zeta\right) \end{bmatrix} = \begin{bmatrix} K^1_{00} & K^1_{01} & K^1_{02} & K^1_{03} \\ K^1_{10} & K^1_{11} & K^1_{12} & K^1_{13} \\ K^1_{20} & K^1_{21} & K^1_{22} & K^1_{23} \\ K^1_{30} & K^1_{31} & K^1_{32} & K^1_{33} \end{bmatrix}$$

For the oblique domain: