MAP 555 : Signal Processing

Part 1 : Fourier analysis and analog filtering

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November 19, 2021

Full course overview

1. **Fourier analysis and analog filtering**
   1.1 Fourier Transform
   1.2 Convolution and filtering
   1.3 Applications of analog signal processing
2. **Digital signal processing**
   2.1 Sampling and properties of discrete signals
   2.2 z Transform and transfer function
   2.3 Fast Fourier Transform
3. **Random signals**
   3.1 Random signals, stochastic processes
   3.2 Correlation and spectral representation
   3.3 Filtering and linear prediction of stationary random signals
4. **Signal representation and dictionary learning**
   4.1 Non stationary signals and short time FT
   4.2 Common signal representations (Fourier, wavelets)
   4.3 Source separation and dictionary learning
   4.4 Signal processing with machine learning

Course overview

Signal and function in $L_p(\mathrm{R})$ space

L_p space

$L_p(S)$ is the set of functions whose absolute value to the power of p has a finite integral or equivalently that

$$\|x\|_p = \int_S |x(t)|^p dt < \infty \tag{1}$$

- $L_1(\mathbf{R})$ is the set of absolute integrable functions
- $L_2(\mathbf{R})$ is the set of quadratically integrable functions (finite energy)
- $L_\infty(\mathbf{R})$ is the set of bounded functions

Signal and images

In this course we will mostly study

- 1D temporal signal with $x(t) \in \mathbf{R}, \forall t \in \mathbf{R}$ (or complex valued function).
- 2D images with $x(\mathbf{v}) \in \mathbf{R}, \forall \mathbf{v} \in \mathbf{R}^2$

Some properties of signals

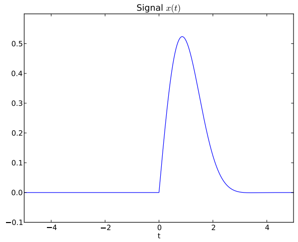
Causality

A signal $x(t)$ is causal if

$$x(t) = 0, \quad \forall x < 0$$

Example:

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \sin(t)\exp\left(-\frac{t^2}{2}\right) & \text{for } t \geq 0 \end{cases}$$

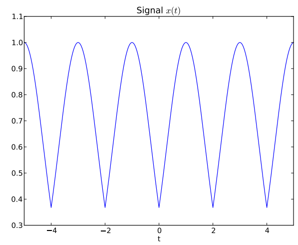


Periodicity

A signal $x(t)$ is periodic of period T_0 is

$$x(t - kT_0) = x(t), \forall t \in \mathbb{R}, \forall k \in \mathbb{N}$$

Example:
$x(t) = \exp\left(-\frac{(t-kT_0-1)^2}{2}\right)$ for $kT_0 < t < (k+1)T_0, \quad \forall k \in \mathbb{N}$

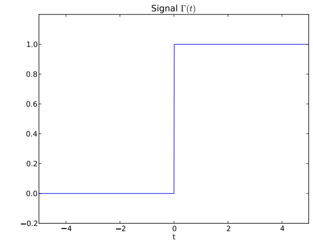


Classical signals (1)

Heaviside function

$$\Gamma(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/2 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases} \tag{2}$$

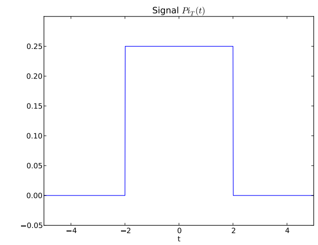
Also known as the step function.



Rectangular function

$$\Pi_T(t) = \begin{cases} 1/T & \text{if } |t| < T/2 \\ 1/2T & \text{if } |t| = T/2 \\ 0 & \text{else} \end{cases} \tag{3}$$

- $\Pi(t) = \frac{1}{T}(\Gamma(t - \frac{T}{2}) - \Gamma(t + \frac{T}{2}))$.
- Finite energy signal (finite support).



Classical signals (2)

Complex exponential

let $e_z(t)$ be the following function $\mathbf{R} \to \mathbf{C}$

$$e_z(t) = \exp(zt) \tag{4}$$

where z is a complex number. When $z = \tau + wi$ the,

$$e_z(t) = (\cos(w * t) + i * \sin(w * t))\exp(\tau t)$$

Special cases:

- $z = \tau$ real, then we recover the classical exponential.

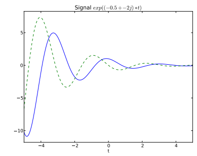
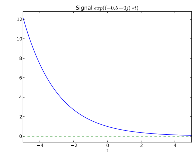
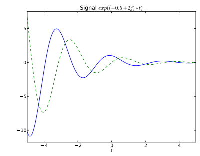
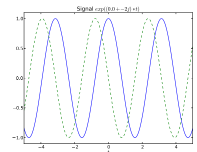
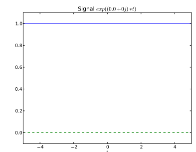
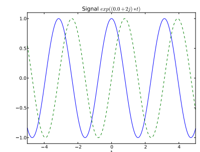
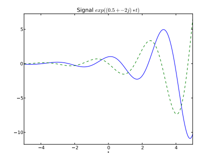
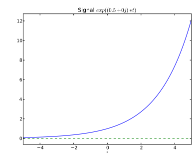
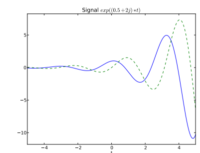
$$e_z(t) = \exp(\tau t)$$

- $z = wi$ imaginary then

$$e_z(t) = \cos(w * t) + i * \sin(w * t)$$

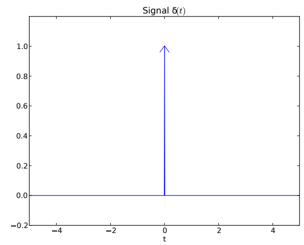
Classical signals (3)

Complex exponantial with $z = \tau + wi$

$\tau \backslash w$	-2	0	2
-0.5			
0			
$.5$			

Dirac delta

Main properties of Dirac delta



- Model point mass at 0.
- Value outside 0 : $\delta(t) = 0, \forall t \neq 0$
- δ is a tempered distribution.
- Very useful tool in signal processing
- Can be seen as the derivative of the Heavyside function $1_{t \geq 0}(t)$
- Integral

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1, \qquad \int_{-\infty}^{+\infty} x(t)\delta(t)dt = x(0) \tag{5}$$

- Dirac and function evaluation for signal $x(t)$ and $t_0 \in \mathbf{R}$:

$$\delta(t - t_0)x(t) = \delta(t - t_0)x(t_0)$$

$$\langle x(t), \delta(t - t_0) \rangle = \int_{-\infty}^{+\infty} x(t)\delta(t - t_0)dt = x(t_0) \tag{6}$$

Dirac delta (2)

Dirac delta definition

- Let ϕ a function supported in $[-1, 1]$ of unit mass: $\int_{-\infty}^{\infty} \phi(u)du = 1$
- $\phi_T(t) = \frac{1}{T}\phi(\frac{t}{T})$ has support on $[-T, T]$ and unit mass.
- We can define the dirac delta δ as

$$\delta(t) = \lim_{T \to 0} \phi_T(t)$$

Dirac delta in practice

- Theoretical object in signal processing (impulse).
- Used to model signal sampling for digital signal processing.
- Used to model point source in Astronomy/image processing, point charge in Physics.
- Has a bounded discrete variant.

Convolution operator

Definition

Let two signals $x(t)$ and $h(t)$. The convolution between the two signals is defined as

$$x(t) \star h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau \tag{7}$$

- Convolution is a bilinear mapping between x and h.
- It models the relation between the input and the output of a Linear Time Invariant system.
- If $f \in L_1(\mathbf{R})$ and $h \in L_p(\mathbf{R}), p \geq 1$ then

$$\|f \star h\|_p \leq \|f\|_1 \|h\|_p$$

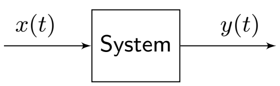
- The dirac delta δ is the neutral element for the convolution operator:

$$x(t) \star \delta(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau = x(t) \tag{8}$$

- It can also be used to model a temporal delay:

$$x(t) \star \delta(t - t_0) = x(t - t_0) \tag{9}$$

Linear Time-Invariant (LTI) systems



Definition

- A system describes a relation between an input $x(t)$ and an output $y(t)$.
- Properties of LTI systems:
  - Linearity $\qquad x_1(t) + ax_2(t) \to y_1(t) + ay_2(t)$
  - Time invariance $\qquad x(t - \tau) \to y(t - \tau)$
- A LTI system can most of the time be expressed as a convolution of the form:

$$y(t) = x(t) \star h(t)$$

  where $h(t)$ is called the impulse response (the response of the system to an input $x(t) = \delta(t)$)

Examples

- Passive electronic systems (resistor/capacitor/inductor) .
- Newtonian mechanics, Fluid mechanics, Fourier Optics.

LTI systems and Ordinary Differential Equation

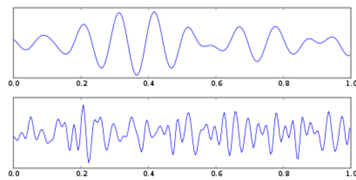
Ordinary Differential Equation (ODE)

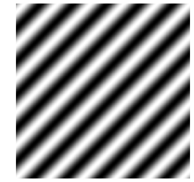
The system is defined by a linear equation of the form:

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + a_n \frac{d^n y(t)}{dt^n} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_m \frac{d^m x(t)}{dt^m} \quad (10)$$

- ODE based system with linear relations are an important class of LTI systems.
- Also called homogeneous linear differential equation.
- n is the number of derivatives for $y(t)$ and m for $x(t)$.
- $\max(m, n)$ is the order of the system.
- The output of the system can be computed from the input by solving Eq. (10).
- Linearity and time invariance are obvious from the equation.

Signal and frequencies





- A signal is $x(t)$ a function of time, an image $x(\mathbf{v})$ a function of space.
- Those functions are what we measure/observe but can be hard to interpret/process automatically.
- Another representation for a signal is in the frequency domain $(1/t)$.
- Better representation for numerous applications.

Applications

- Signal processing (biomedical, electrical).
- Image processing (2D signals), filtering, reconstruction.
- Colors are combination of waves of different frequencies.

Fourier Series (1)

MÉMOIRE

SUR LA

PROPAGATION DE LA CHALEUR

DANS LES CORPS SOLIDES,

PAR M. FOURIER (1).

(2) $\varphi(y) = a \cos\frac{\pi y}{2} + a' \cos 3\frac{\pi y}{2} + a'' \cos 5\frac{\pi y}{2} + \ldots.$

Multipliant de part et d'autre par $\cos(2i+1)\frac{\pi y}{2}$, et intégrant ensuite depuis $y = -1$ jusqu'à $y = +1$, il vient

$$a_i = \int_{-1}^{+1} \varphi(y) \cos(2i+1)\frac{\pi y}{2}\, dy,$$

History

- Trigonometric series used by Euler, d'Alembert, Bernoulli and Gauss.
- Introduced by Joseph Fourier in [Fourier, 1807].
- Fourier claimed that these series could approximate any function.

Fourier series (2)

Decomposition as trigonometric series

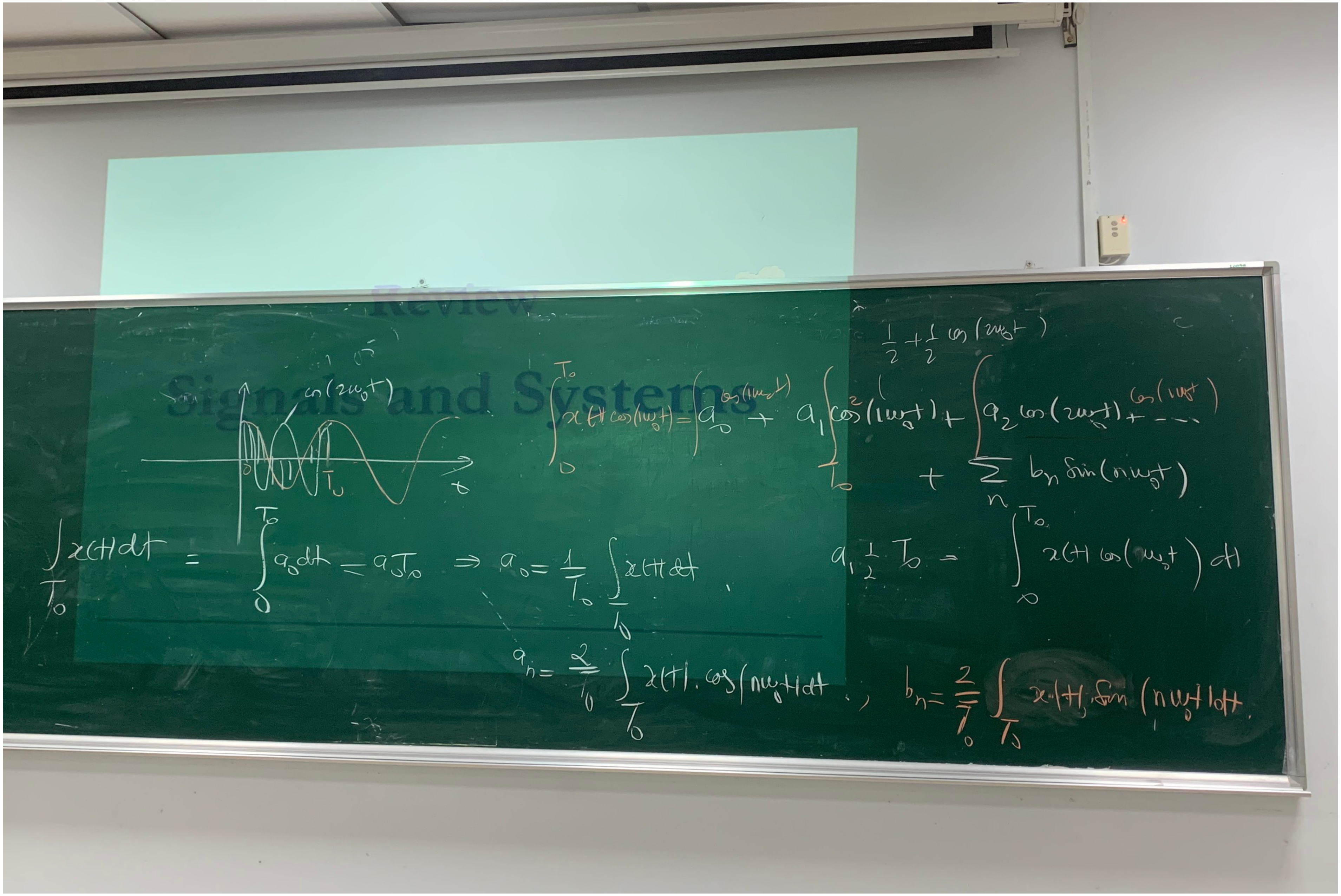
One can express periodic $x(t)$ of period $T_0 = \frac{2\pi}{w_0}$ integrable on the period as

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kw_0 t) + b_k \sin(kw_0 t)]$$

where a_k and b_k are the Fourier coefficients that can be computed as

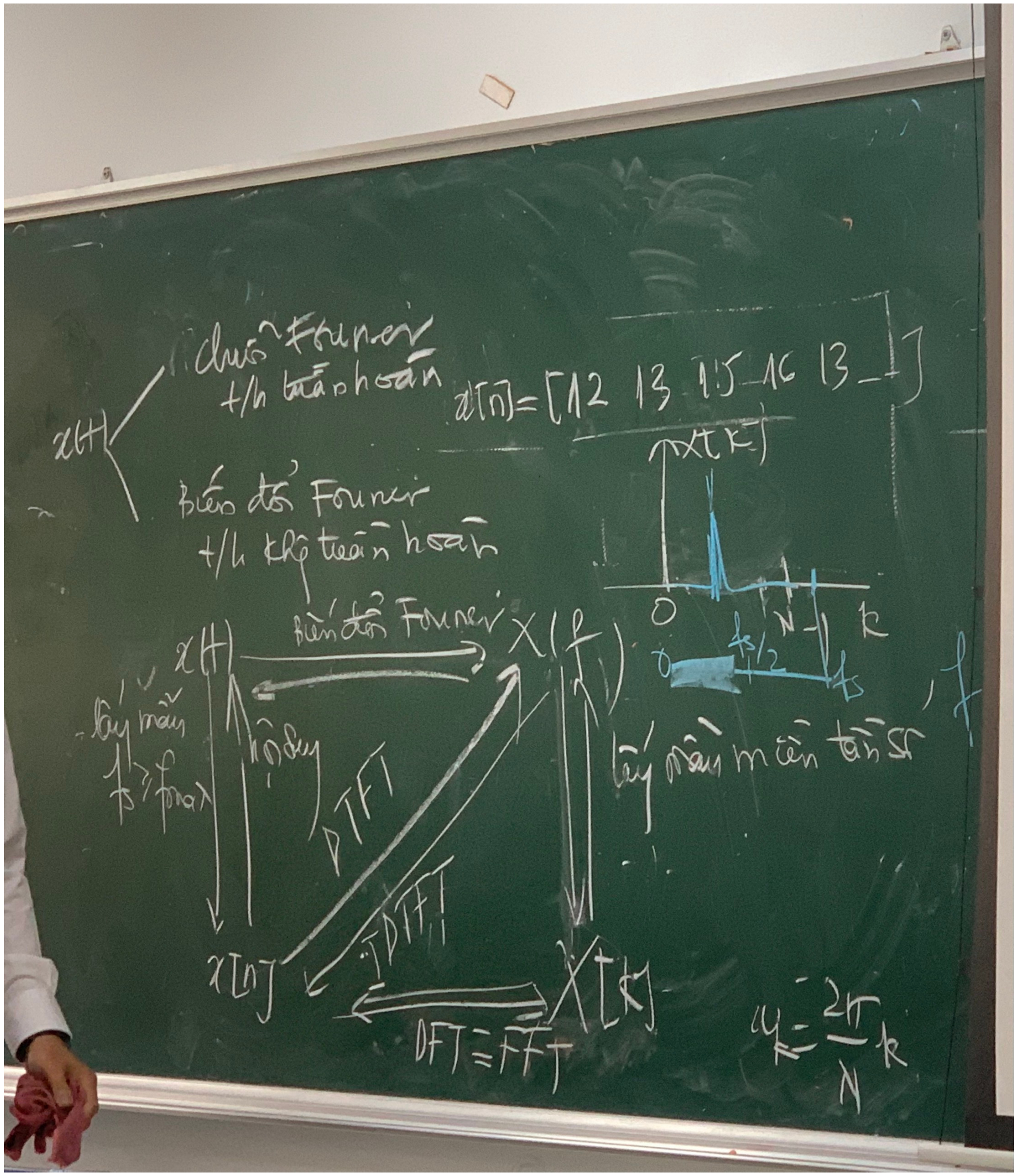
$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(kw_0 t) dt \qquad b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin(kw_0 t) dt$$

- Representation of a periodic signal as an infinite number of coefficients corresponding to harmonic frequencies.
- Can be interpreted as a change of basis from temporal to frequencies.
- Functions can be approximated with a finite number N of terms.
- Gibbs phenomenon appears for discontinuous functions [Hewitt and Hewitt, 1979].

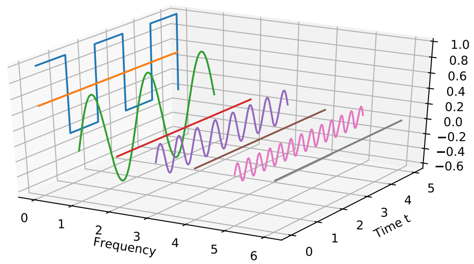

Review
Signals and Systems
$\cos(2\omega_0 t)$
0
T_0
t
$\frac{1}{2}+\frac{1}{2}\cos(2\omega_0 t)$
$\int_0^{T_0} x(t)\cos(\omega_0 t) = \int a_0 \cos(\omega_0 t) + a_1 \int \cos^2(\omega_0 t) + \int a_2 \cos(2\omega_0 t)\cos(\omega_0 t) + \cdots$
$+ \sum_n b_n \sin(n\omega_0 t)$
$\int_{T_0} x(t)dt = \int_0^{T_0} a_0 dt = a_0 T_0 \Rightarrow a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$
$a_1 \frac{1}{2} T_0 = \int_0^{T_0} x(t)\cos(\omega_0 t)dt$
$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cdot\cos(n\omega_0 t)dt$, $b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)dt$

Phân tích các tín hiệu tuần hoàn thành các tín hiệu có thành phần sin cos

Đối với tín hiệu tuần hoàn khác ví dụ vuông or tam giác thì F = 1 /T chỉ là tần số cơ bản thôi.



Example of Fourier series



Example : Square wave

- Square wave with $T_0 = 2$

Complex Fourier series

Complex harmonic decomposition

One can express periodic $x(t)$ of period $T_0 = \frac{2\pi}{w_0}$ integrable on the period as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \qquad \text{avec } w_0 = \frac{2\pi}{T_0}$$

where the coefficients c_k are called the **complex Fourier coefficients** and can be computed with

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-ikw_0t}dt = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-ikw_0t}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-ikw_0t}dt$$

Relations between decompositions

Using the Euler formula we can show that a_k and b_k and the c_k coefficients are related by

$$\frac{a_0}{2} = c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = i(c_k - c_{-k})$$

Note that if $x(t)$ is an even function then the $b_k = 0$, and if $x(t)$ is odd then $a_k = 0$.

Fourier transform

Definition (Fourier Transform)

The Fourier Transform (FT) of a signal $x(t)$ can be expressed as

$$\mathcal{F}[x(t)] = X(f) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi f t} x(t)\mathrm{d}t \tag{11}$$

When it exists the inverse Fourier transform is defined as

$$\mathcal{F}^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi f t} X(f)\mathrm{d}f \tag{12}$$

- Note that the $\hat{}$ operator is also often used for the Fourier transform $\hat{x}$ of x.
- In signal processing and electrical engineering the references often use j instead of i for the imaginary number (i is a measure of current).
- The FT is a change of representation for the function x from the temporal representation to the harmonic (frequency) representation.

Interpretation of the Fourier transform

$$x(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi f t} X(f)\mathrm{d}f$$

Harmonic representation

- The FT represents the signal in the frequency domain.
- $|X(f)|$ is the magnitude of a sinusoidal signal for frequency f.
- $Arg(X(f)$ is the phase of the sinusoidal signal.
- For a real signal $x(t)$, $X(f) = X(-f)^*$ and an informal interpretation would be

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{i2\pi f t}df = \int_{-\infty}^{+\infty} |X(f)|e^{i2\pi(ft + Arg(X(f)))}df \tag{13}$$

$$\approx X(0) + 2\int_{0+}^{+\infty} |X(f)|\cos(2\pi(ft + Arg(X(f)))) \tag{14}$$

- The modulus and argument of the FT allow identification of the frequency content of the signal and its phase.

Examples of Fourier Transform (1)

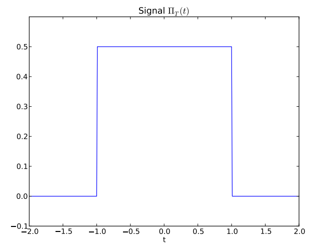
Rectangular function

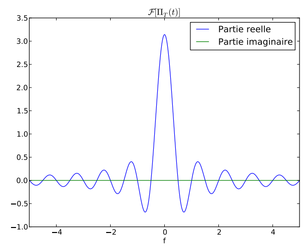
$$\Pi_T(t) = \begin{cases} 1/T & \text{if } |t| < T/2 \\ 1/2T & \text{if } |t| = T/2 \\ 0 & \text{else} \end{cases} \qquad (15)$$

The Fourier transform is

$$\begin{aligned} \mathcal{F}[\Pi_T(t)] =& \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t \\ =& \left[\frac{-\mathrm{e}^{-\mathrm{i}2\pi f t}}{\mathrm{i}2\pi f T}\right]_{-T/2}^{T/2} \\ =& \frac{\mathrm{e}^{\mathrm{i}\pi f T} - \mathrm{e}^{-\mathrm{i}\pi f T}}{\mathrm{i}2\pi f T} \\ =& \frac{\sin(\pi f T)}{\pi f T} = \mathrm{sinc}(\pi f T) \end{aligned}$$

with $\mathrm{sinc}(t) = \frac{\sin(t)}{t}$ and $\mathrm{sinc}(0) = 1$





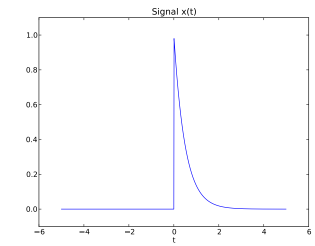
Examples of Fourier Transform (2)

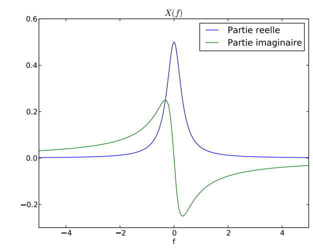
Decreasing exponential

$$x(t) = e^{-at}\Gamma(t), \quad \Gamma(t) = \begin{cases} 1 & \text{for } t > 0 \\ 1/2 & \text{for } t = 0 \\ 0 & \text{else} \end{cases}$$

with $a > 0$
The Fourier transform is

$$\begin{aligned} \mathcal{F}[e^{-at}\Gamma(t)] &= \int_0^\infty e^{-at}\mathrm{e}^{-\mathrm{i}2\pi f t}dt \\ &= \int_0^\infty e^{-(a+\mathrm{i}2\pi f)t}dt \\ &= \left[\frac{\mathrm{e}^{-(a+\mathrm{i}2\pi f)t}}{-(a+\mathrm{i}2\pi f)}\right]_0^\infty \\ &= \frac{1}{a+i2\pi f} \end{aligned}$$





Properties of the Fourier Transform

Linearity

Let $x_1(t)$ and $x_2(t)$ two signals of TF $X_1(f)$ and $X_2(f)$ respectively.
For $a \in \mathbf{R}$ and $b \in \mathbf{R}$, we have :

$$\mathcal{F}[ax_1(t) + bx_2(t)] = aX_1(f) + bX_2(f)$$

Proof. Comes from the linearity of the integration.

Time shift

Let $x(t)$ be a signal of FT $X(f)$.
For $t_0 \in \mathbf{R}$, let $x(t - t_0)$ a time shift of $x(t)$ then we have:

$$\mathcal{F}[x(t - t_0)] = e^{-i2\pi t_0 f}X(f)$$

Proof. Change of variable in the integral.

Properties of the Fourier Transform (2)

Frequency shift

Let $x(t)$ be a signal of FT $X(f)$ then we have

$$\mathcal{F}\left[e^{i2\pi f_0 t}x(t)\right] = X(f - f_0)$$

Multiplication by a complex exponential of frequency f_0, translates the TF by f_0.
Proof. Regroup exponentials in the integral.

Time scaling

Let $x(t)$ be a signal of FT $X(f)$ and a a scaling $a \neq 0$ then we have

$$\mathcal{F}[x(at)] = \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

Proof. Change of variable for separate cases $a > 0$ and $a < 0$.

Properties of the Fourier Transform (3)

Derivation

Let $x(t)$ be a signal of FT $X(f)$ then we have

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = i2\pi f X(f)$$

Integration

Let $x(t)$ be a signal of FT $X(f)$ such that $\int_{-\infty}^{\infty} x(t)dt = 0$ then we have

$$\mathcal{F}\left[\int_{-\infty}^{t} x(u)du\right] = \frac{1}{i2\pi f}X(f)$$

If $\int_{-\infty}^{\infty}(x(t) - c)dt = 0$ where c is often called the constant term, we have

$$\mathcal{F}\left[\int_{-\infty}^{t} x(u)du\right] = \frac{1}{i2\pi f}X(f) + c\delta(f)$$

where $\delta(f)$ is the Dirac delta.

Those two properties can be used to solve Ordinary Differential Equations (ODE).

Properties of the Fourier Transform (4)

Even and odd signals

$x(t)$	$X(f)$
Even real	Even real
Odd real	Odd imaginary
Even imaginary	Even imaginary
Odd imaginary	Odd real

For a real signal $x(t)$: $X(f) = X(-f)^*$

Conjugate signal

Let $x(t)$ be a signal of FT $X(f)$ and $x^*(t)$ its complex conjugate, then we have

$$\mathcal{F}[x^*(t)] = X^*(-f)$$

Duality of the Fourier Transform

Let $x(t)$ be a signal of FT $X(f)$. When the inverse Fourier transform exists we can write

$$x(-t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi f(-t)}df = \int_{-\infty}^{+\infty} X(f)e^{-j2\pi ft}df = \mathcal{F}[X(f)]$$

- The last term is the TF of function $X(f)$.
- This means that if $\mathcal{F}[x(t)] = X(f)$ then

$$\mathcal{F}[X(t)] = x(-f)$$

- Applying twice the TF operator to $x(t)$ returns $x(-t)$: $\mathcal{F}[\mathcal{F}[x(t)]] = x(-t)$

Example

For the rectangular function $\Pi_T(t)$:

$$\Pi_T(t) \to \text{sinc}(\pi f T)$$
$$\text{sinc}(\pi T t) \to \Pi_T(-f) = \Pi_T(f)$$

Fourier Transform in $L_p(\mathrm{R})$

- For $1 \leq p \leq 2$ the FT maps from $L_p(\mathbf{R})$ to $L_q(\mathbf{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$.
- Consequence of the Riesz–Thorin theorem.
- The TF of an absolute integrable function is bounded (Example : rectangle).

Parseval-Plancherel identity in L_2

The TF of an L_2 function is L_2. Note that L_2 is a Hilbert space of inner product:

$$< x, y >= \int_{-\infty}^{\infty} x(t)y^*(t)dt$$

For two functions $x, y \in L_2(\mathbf{R})^2$ of respective TF $X, Y \in L_2(\mathbf{R})^2$ the Parseval-Plancherel identity states that

$$< x, y >= \int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df \tag{16}$$

$$< x, x >= \int_{-\infty}^{\infty} |x(t)|^2dt = \int_{-\infty}^{\infty} |X(f)|^2df \tag{17}$$

which means that the energy of a signal is preserved by FT.

More details in [Hunter, 2019, Chap. 5.A] and [Mallat et al., 2015, Chap. 1]

Convolution and Fourier Transform

Convolution and Fourier Transform

Let two signals $x(t)$ and $h(t)$ of respective Fourier transform $X(f)$ and $H(f)$ then

$$\mathcal{F}[x(t) \star h(t)] = X(f)H(f) \tag{18}$$

- The TF of a convolution is a pointwise multiplication in frequency.
- The complex exponential function is the eigenvector for the convolution operator.
- Easy interpretation of the effect of a linear filtering.

Proof

$$\begin{aligned}\mathcal{F}[x(t) \star h(t)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2i\pi ft}x(u)h(t-u)dudt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2i\pi f(u+v)}x(u)h(v)dudv \\ &= \left\{\int_{-\infty}^{\infty} e^{-2i\pi fu}x(u)du\right\}\left\{\int_{-\infty}^{\infty} e^{-2i\pi fv}h(v)dv\right\} = X(f)H(f)\end{aligned}$$

with the change of variable $v = t - u$.

Dirac delta and Fourier Transform

Fourier transform and Dirac delta

- Fourier Transform of $\delta(t)$ and $\delta(t - t_0)$:

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t)e^{-i2\pi ft}dt = e^0 = 1$$

$$\mathcal{F}[\delta(t - t_0)] = e^{-i2\pi ft_0}$$

- By duality of FT we have:

$$\mathcal{F}[1] = \delta(t)$$
$$\mathcal{F}[e^{i2\pi f_0 t}] = \delta(f - f_0)$$

- Convolution

$$\mathcal{F}[x(t) \star \delta(t)] = 1X(f) = X(f)$$
$$\mathcal{F}[x(t)\delta(t)] = X(f) * 1 = \int_{-\infty}^{\infty} X(f)df = x(0)$$

The dirac comb

- The dirac comb is expressed as

$$\text{Ш}_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{19}$$

  where Ш is the Cyrilic Sha symbol.
- The Fourier Transform of the dirac comb is

$$\mathcal{F}[\text{Ш}_T(t)] = \sum_{k=-\infty}^{\infty} e^{2i\pi kTf} = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) = \frac{1}{T}\text{Ш}_{\frac{1}{T}}(f) \tag{20}$$

  where the second equality comes from the Poisson summation formula.
- The dirac comb is used to perform a regular temporal sampling.
- Multiplying a signal by the dirac comb corresponds to a convolution by a dirac comb in the Frequency domain (and vice versa).

Fourier transform of periodic signals

Cosine

$$x(t) = \cos(2\pi f_0 t) \qquad \text{with } f_0 > 0$$

- Bounded signal with unbounded energy.
- Intuitively this signal contains only one frequency (f_0)
- Its TF can be computed using to the dirac distribution.

FT of trigonometric functions

$$\mathcal{F}\left[\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right] = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$
$$\mathcal{F}\left[\sin(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2i}\right] = \frac{1}{2i}\delta(f - f_0) - \frac{1}{2i}\delta(f + f_0)$$

The FT of sine and cosine is equal to 0 everywhere except on the frequency f_0 of the functions.

Fourier transform of periodic signals (2)

Fourier transform of periodic signal

Let $x(t)$ be a periodic signal of period T_0, it can be expressed as the following complex Fourier series:

$$x(t) = \sum_k c_k e^{i2\pi \frac{k}{T_0} t}$$

Its Fourier transform can be expressed as

$$X(f) = \mathcal{F}[x(t)] = \sum_k c_k \delta\left(f - \frac{k}{T_0}\right)$$

- The FT of a periodic signal of period is null except on frequencies $\frac{k}{T_0}$, $k \in \mathbb{N}$.
- $\frac{1}{T_0}$ is the fundamental frequency, $\frac{k}{T_0}$ with $|k| \geq 2$ are called the harmonics.
- The TF of a periodic function is a weighted sum of diracs.

Fourier Transform in R^d

The Fourier Transform can be naturally extended to functions in $\mathbf{R}^d$.

Fourier Tansform in R^d

Let $x(\mathbf{v}) : \mathbf{R}^d \rightarrow \mathbb{C}$, the Fourier Transform of x can be expressed as

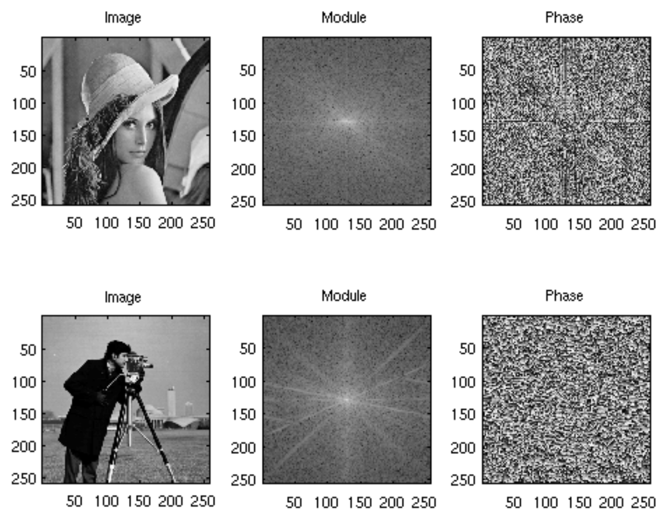
$$\mathcal{F}[x(\mathbf{v})] = X(\mathbf{u}) = \int_{\mathbf{R}^d} x(\mathbf{v}) e^{-2i\pi <\mathbf{v},\mathbf{u}>} \mathrm{d}\mathbf{v} \tag{21}$$

When it exists the Inverse FT is defined as

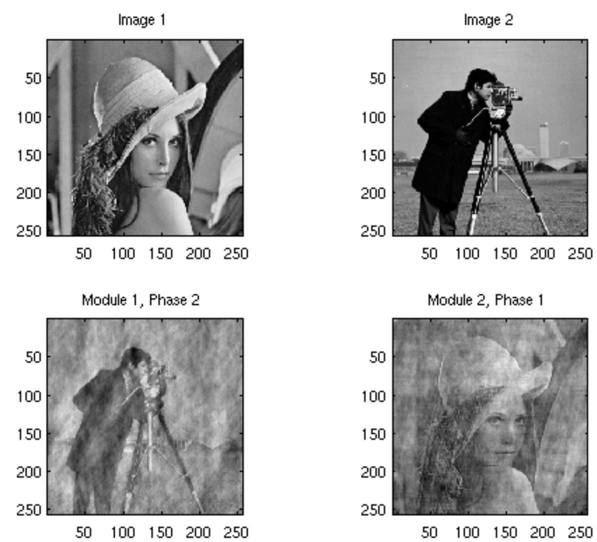
$$\mathcal{F}^{-1}[X(\mathbf{u})] = x(\mathbf{v}) = \int_{\mathbf{R}^d} X(\mathbf{u}) e^{2i\pi <\mathbf{v},\mathbf{u}>} \mathrm{d}\mathbf{u} \tag{22}$$

- $\mathbf{u} \in \mathbf{R}^d$ is a directional frequency.
- All the properties of the 1D FT are preserved (duality, convolution, ...)
- With $d = 2$, frequency representation of black and white images.
- With large d, approximation for efficient kernel approximation in machine learning [Rahimi and Recht, 2008].

Examples of Fourier Tranform in 2D



Examples of Fourier Tranform in 2D (Modulus and Phase)



Fourier transform and angular frequency

- The FT in this course is a function of frequency f (in Hz).
- Another common way to represent frequency is the angular frequency w (in rad/s) such that
$$w = 2\pi f, \qquad f = \frac{w}{2\pi}$$
- When using angular frequency the FT is non-unitary meaning that :
$$\tilde{\mathcal{F}}[x(t)] = \tilde{X}(w) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}wt} x(t) \mathrm{d}t$$
$$\tilde{\mathcal{F}}^{-1}[\tilde{X}(f)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}wt} \tilde{X}(w) \mathrm{d}w$$
- There exists a unitary angular frequency FT that scales both FT and IFT by $\frac{1}{\sqrt{2\pi}}$.
- In the following we will sometime use the FT as a function of the angular frequency:
$$\tilde{X}(w) = X\left(\frac{w}{2\pi}\right)$$

How to compute a Fourier Transform ?

Usual steps

1. Use known FT pairs if possible.
2. Express the function as a composition of operations with known properties:
    - Linearity, time shift
    - Convolution
    - Duality
3. Use the properties of FT on the composition.
4. Check properties (FT of even/odd function) to detect easy mistakes.

As a rule : try to avoid computing the integral but sometime you have to do it.

Frequency response of LTI systems

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Impulse response and frequency response

- Most LTI systems can be expressed as a convolution of the form:
$$y(t) = x(t) \star h(t)$$
where $h(t)$ is called the impulse response (the response of the system to an input $x(t) = \delta(t)$)
- The Fourier transform of the LTI system relation between x and y is
$$Y(f) = H(f)X(f) \tag{23}$$
- The frequency response $H(f)$ (also called transfer function) of the LTI system is the Fourier transform of $h(t)$:
$$H(f) = \frac{Y(f)}{X(f)} \tag{24}$$

Frequency response and static gain

Response to a mono-frequency signal

- For a system of impulse response $h(t)$ with an input $x(t) = e^{2j\pi f_0 t}$
$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau) e^{2j\pi f_0 h(t-\tau)} d\tau \\ &= e^{2j\pi f_0 t} \int_{-\infty}^{+\infty} h(\tau) e^{-2j\pi f_0 h \tau} d\tau \\ &= e^{2j\pi f_0 t} H(f_0) = x(t) H(f_0) \end{aligned}$$
- An input signal with unique frequency f_0 is multiplied by $H(f_0)$.
- Its amplitude is multiplied by $|H(f_0)|$ and a phase $Arg(H(f_0))$ is added.
- The complex exponential is an eigenvector of the convolution operator.

Static gain

The complex static gain is the constant K such that
$$K = H(0) = \int_{-\infty}^{+\infty} h(t) dt$$

LTI systems and Ordinary Differential Equation

Ordinary Differential Equation (ODE)

The system is defined by an equation of the form:

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + a_n \frac{d^n y(t)}{dt^n} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_m \frac{d^m x(t)}{dt^m} \quad (25)$$

Frequency response of an ODE

- We recall the properties of the FT for th n-th derivative of a function:

$$\mathcal{F}\left[\frac{d^{(n)}x(t)}{dt^n}\right] = (2i\pi f)^n X(f) = (iw)^n X(w)$$

- The Frequency response of the ODE can be expressed as

$$H(w) = \frac{Y(w)}{X(w)} = \frac{b_0 + b_1 jw + \cdots + b_m (jw)^m}{a_0 + a_1 jw + \cdots + a_n (jw)^n} \quad (26)$$

Representation of the frequency response

Frequency interpretation of the frequency response

- The frequency response of a system gives information on the transformations due to the system.
- Quantities that can be plotted :

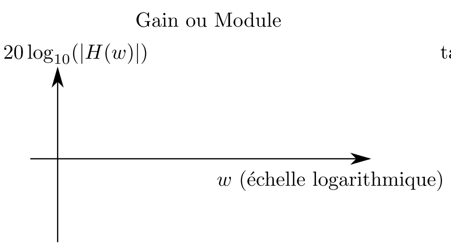
$$\begin{aligned} \tilde{H}(w) &= Re(\tilde{H}(w)) + jIm(\tilde{H}(w)) \\ &= |\tilde{H}(w)| e^{jArg(\tilde{H}(w))} \end{aligned}$$

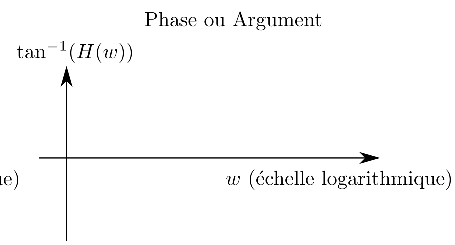
  - $|\tilde{H}(w)|$ modulus of the frequency response.
  - $Arg(\tilde{H}(w)) = \angle\tilde{H}(w) = \tan^{-1}\left(\frac{Im(\tilde{H}(w)}{Re(\tilde{H}(w)}\right)$ phase in radian.

Graphical representation of systems

- Bode plot (Modulus+Argument).
- Nichols/Black plot (Modulus VS Argument).
- Nyquist plot (Real VS Imaginary)

Bode plot





Definition

The Bode plot of a system is composed of two plots that are function of w:

- Magnitude (or gain) in decibels (dB)

$$\tilde{G}(w) = 20\log_{10}(|\tilde{H}(w)|)$$

- Phase in degrees or radians

$$\tilde{\Phi}(w) = Arg(\tilde{H}(w)|) = \angle|\tilde{H}(w)|$$

The scale of the radial frequency w is logarithmic, which means that for a rational frequency response H one will be mostly piecewise linear.

Properties of the Bode plot

The logarithm and the argument allows for simple diagrams for combination of systems

Multiplication

If two LTIs $\tilde{H}_1(w)$ and $\tilde{H}_2(w)$ are in series the the equivalent system is $\tilde{H}(w) = \tilde{H}_1(w)\tilde{H}_2(w)$

- $\tilde{G}(w) = \tilde{G}_1(w) + \tilde{G}_2(w)$
- $\tilde{\Phi}(w) = \tilde{\Phi}_1(w) + \tilde{\Phi}_2(w)$

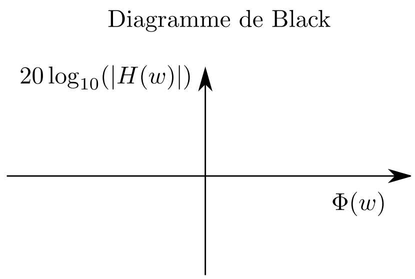
Division

If and LTI can be expressed as $\tilde{H}(w) = \dfrac{\tilde{H}_1(w)}{\tilde{H}_2(w)}$ then

- $\tilde{G}(w) = \tilde{G}_1(w) - \tilde{G}_2(w)$
- $\tilde{\Phi}(w) = \tilde{\Phi}_1(w) - \tilde{\Phi}_2(w)$

This is particularly useful for rational frequency responses such as ODE.

Diagramme de Black

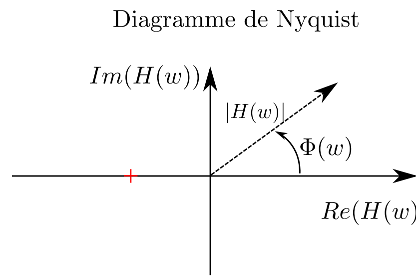


Définition

The Nichols plot (Diagramme de Black in France) is a parametric plot of $\tilde{H}(w)$ with $20\log_{10}|\tilde{H}(w)|$ on y-axis and phase $\tilde{\Phi}(w)$ on x-axis.

- Show the Modulus/Phase trajectory as a function of w.
- Can be plotted following the Bode plot w.

Nyquist plot



Definition

The Nyquist plot is a parametric plot of $\tilde{H}(w)$ with $Real(\tilde{H}(w))$ on x-axis and $Imag(\tilde{\Phi}(w))$ on y-axis.

- Show the trajectory of $\tilde{H}$ in the complex plane.
- Used in system control to study the stability of systems.

Frequency response of electronic systems

Principle

Ohm's law can be extended to capacitors and inductors using what is called complex electrical impedance.

The linear system $i(t) \rightarrow u(t)$ is expressed as

$$\tilde{U}(w) = \tilde{H}(w)\tilde{I}(w) = \tilde{Z}(w)\tilde{I}(w)$$

For electronic systems j is used instead if i as the imaginary number.

Resistor

- $u(t) = Ri(t)$
- $\tilde{U}(w) = R\tilde{I}(w)$
- $Z_R = R$

Capacitor

- $u(t) = \frac{1}{C}\int_{-\infty}^{t} i(u)du$
- $\tilde{U}(w) = \frac{1}{jCw}\tilde{I}(w)$
- $Z_C = \frac{1}{jCw}$

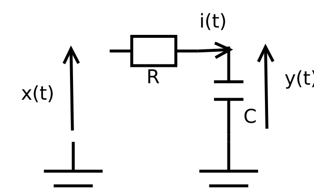
Inductor

- $u(t) = L\frac{di(t)}{dt}$
- $\tilde{U}(w) = jLw\tilde{I}(w)$
- $Z_L = jLw$

The frequency response of passive electronic systems can be computed with simple computation of complex numbers.

First order system (1)

- System



$$x(t) = Ri(t) + y(t)$$
$$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(v)dv$$
$$x(t) = RCy'(t) + y(t)$$

- Frequency response

$$\tilde{H}(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + RC2j\pi f}$$

- Using complex impedance

$$\tilde{Y}(w) = Z_c\tilde{I}(w) \quad \text{et} \quad X(w) = (Z_R + Z_C)\tilde{I}(w)$$
$$\tilde{H}(w) = \frac{\tilde{Y}(w)}{\tilde{X}(w)} = \frac{Z_c}{Z_C + Z_R} = \frac{1}{1 + \frac{Z_R}{Z_C}} = \frac{1}{1 + RCjw}$$

First order system (2)

Normalized system

We reformulate the frequency response as as:

$$H(w) = \frac{1}{1 + j\frac{w}{w_0}} \tag{27}$$

where $w_0 = \frac{1}{\tau} = \frac{1}{RC}$.

Diagramme de Bode

Modulus

1. $\tilde{H}(w) = \frac{1}{1+j\frac{w}{w_0}}$
2. $|\tilde{H}(w)| = \frac{1}{\sqrt{1+\frac{w^2}{w_0^2}}}$
3. $\tilde{G}(w) = 20\log_{10}(|H(w)|) = -10\log_{10}(1+\frac{w^2}{w_0^2})$
4. $\lim_{w\to 0}\tilde{G}(w) = 0$
5. $\lim_{w\to\infty}\tilde{G}(w) = -10\log_{10}(\frac{w^2}{w_0^2}) = -20\log_{10}(w) + 20\log_{10}(w_0)$
6. When $w = w_0$, $\tilde{G}(w) = -10\log_{10}(2) = -3dB$

First order system (3)

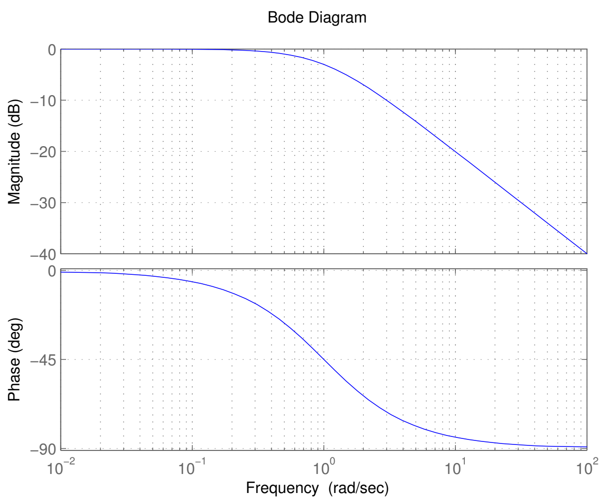
Bode plot

Argument

1. $\tilde{H}(w) = \frac{1}{1+j\frac{w}{w_0}}$
2. $\tilde{\Phi}(w) = \arg(H(w)) = -arg(1+jw) = -tan^{-1}(w)$
3. $\lim_{w\to 0}\tilde{\Phi}(w) = 0$
4. $\lim_{w\to\infty}\tilde{\Phi}(w) = -\pi/2$
5. When $w = w_0$, $\tilde{\Phi}(w) = -tan^{-1}(1) = -\pi/4$ $(-45°)$
   when $w = 10w_0$, $\tilde{\Phi}(w) = -84°$
   when $w = .1w_0$ $\tilde{\Phi}(w) = -6°$

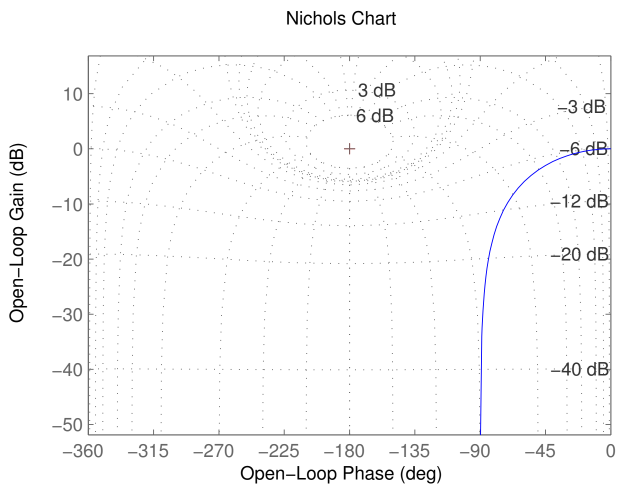
First order system (4)

Bode plot



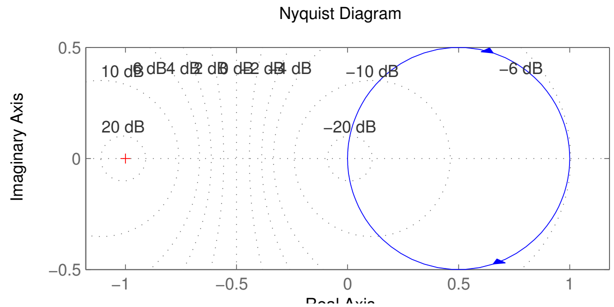
First order system (5)

Nichols plot

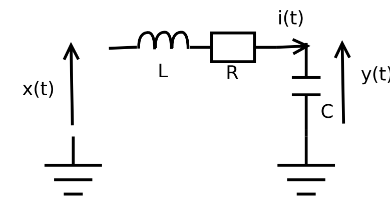


First order system (6)

Nyquist plot



Second order system (1)



- Complex Impedance

$$\tilde{Y}(w) = Z_c \tilde{I}(w)$$
$$\tilde{X}(w) = (Z_L + Z_R + Z_C)\tilde{I}(w)$$

- Frequency response

$$\tilde{H}(w) = \frac{\tilde{Y}(w)}{\tilde{X}(w)} = \frac{Z_C}{Z_L + Z_R + Z_C} = \frac{\frac{1}{jCw}}{\frac{1}{jCw} + R + jLw}$$

- Normalized frequency response

$$\tilde{H}(w) = \frac{1}{1 + RCjw + LC(jw)^2} = \frac{k}{1 + 2z\frac{jw}{w_n} + (\frac{jw}{w_n})^2}$$

  - k Static gain : $k = 1$
  - z damping ratio of the system : $z = \frac{R}{2}\sqrt{\frac{C}{L}}$
  - w_n natural frequency of the system : $w_n = \frac{1}{\sqrt{LC}}$

Second order system (2)

Linear differential equation

The second order differential equation corresponding to the system is

$$\frac{d^2y(t)}{dt^2} + 2zw_n\frac{dy(t)}{dt} + w_n^2 y(t) = kw_n^2 x(t) \qquad (28)$$

Factorization

The second order system can be factorized as

$$\tilde{H}(w) = \frac{kw_n^2}{(jw - c_1)(jw - c_2)} \qquad (29)$$

with

$$c_1 = \qquad (30)$$
$$c_2 = -zw_n - w_n\sqrt{z^2 - 1} \qquad (31)$$

c_1 and c_2 are called the poles of the transfer function.

Second order system (3)

Response of the system for $z > 1$

- c_1 and c_2 are real coefficients.
- The FT can be expressed as

$$\tilde{H}(w) = \frac{M}{jw - c_1} - \frac{M}{jw - c_2} \qquad (32)$$

with $M = \frac{w_n}{2\sqrt{z^2-1}}$,

- The impulse response of the system is

$$h(t) = M(e^{c_1 t} - e^{c_2 t})\Gamma(t)$$

- The step response of the system is

$$e(t) = \left(1 + M\left(\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right)\right)\Gamma(t)$$

Second order system (4)

Response of the system for $z = 1$

The FT becomes:

$$\tilde{H}(w) = \frac{kw_n^2}{(jw + w_n)^2} \tag{33}$$

that is the square of one first order system.
The impulse response fo the system can be expressed as

$$h(t) = w_n^2 t e^{-w_n t}\Gamma(t)$$

The step response can be expressed as

$$e(t) = (1 - e^{-w_n t} - w_n t e^{-w_n t})\Gamma(t)$$

Second order system (5)

Response of the system for $z < 1$

- In this case the damping is weak and oscillations appear.
- This comes from the fact that when $z < 1$ coefficients c_1 and c_2 are complex. The impulse response is

$$h(t) = M(e^{c_1 t} - e^{c_2 t})\Gamma(t)$$

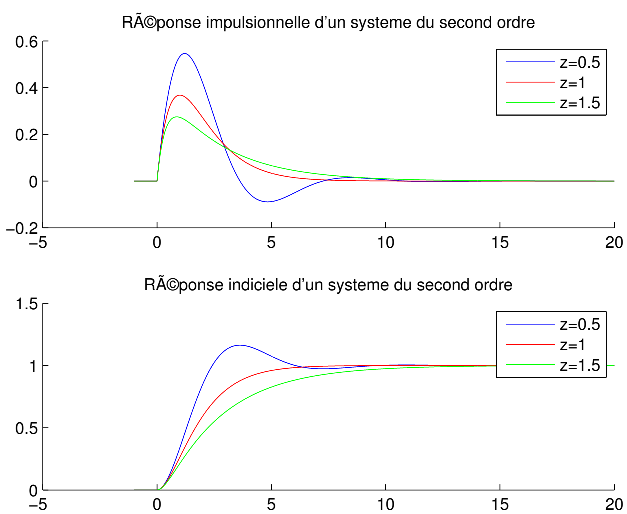
- The step response is

$$h(t) = \frac{w_n e^{-z w_n t}}{\sqrt{1-z^2}} \sin\left(w_n t\sqrt{1-z^2}\right)\Gamma(t)$$

that is a sine with an exponentially decreasing magnitude.

Second order system (6)

Impulse and step responses



Second order system (7)

Bode plot

We can plot the Bode plot using the normalized frequency response:

$$H(w) = \frac{k}{\left(\frac{jw}{w_n}\right)^2 + 2z\left(\frac{jw}{w_n}\right) + 1} \tag{34}$$

Modulus

1. $\tilde{H}(w) = \frac{k}{\left(\frac{jw}{w_n}\right)^2 + 2z\left(\frac{jw}{w_n}\right)+1}$.
2. $|\tilde{H}(w)| = \frac{k}{\sqrt{\left(1-\left(\frac{w}{w_n}\right)^2\right)^2 + 4z^2\left(\frac{w}{w_n}\right)^2}}$.
3. $\tilde{G}(w) = 20\log_{10}(|\tilde{H}(w)|) =$
   $-10\log_{10}\left(\left(1-\left(\frac{w}{w_n}\right)^2\right)^2 + 4z^2\left(\frac{w}{w_n}\right)^2\right) + 20log(k)$
4. $\lim_{w\to 0}\tilde{G}(w) = 20log(k)$
5. $\lim_{w\to\infty}\tilde{G}(w) = -10\log_{10}(\frac{w^4}{w_n^4}) = -40\log_{10}(w) + 40\log_{10}(w_n)$
6. En $w = w_0$, $\tilde{G}(w) = -20\log_{10}(2z) + 20\log(k)$.

Second order system (8)

Properties of the modulus

- The modulus of the frequency response for $z < \sqrt(2)/2$ has a maximum at the following frequency

$$w_{max} = w_n\sqrt{1-2z^2}$$

- The value of the modulus at this frequency is

$$|\tilde{H}(w_{max})| = \frac{k}{2z\sqrt{1-z^2}}$$

- The cutoff frequency at -3dB is equal to

$$w_{-3} = w_n\sqrt{1+2z^2+\sqrt{2-4z^2+4z^4}}$$

Second order system (9)

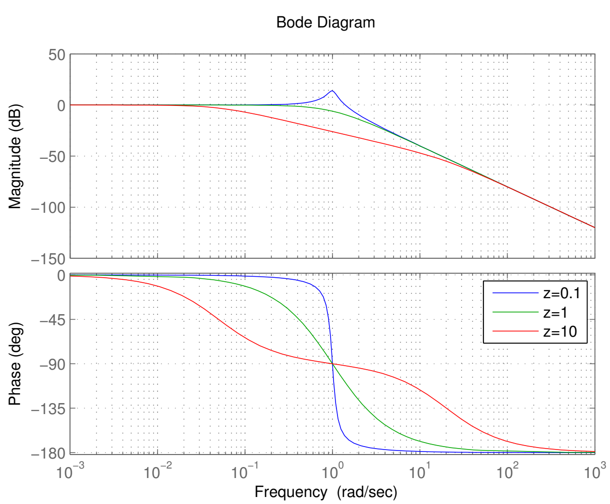
Bode plot

Argument

1. $\tilde{H}(w) = \frac{k}{\left(\frac{jw}{w_n}\right)^2+2z\left(\frac{jw}{w_n}\right)+1}$.
2. $\tilde{\Phi}(w) = \arg(H(w)) = -arg(\left(\frac{jw}{w_n}\right)^2 + 2z\left(\frac{jw}{w_n}\right)+1) = -tan^{-1}\left(\frac{2z\frac{w}{w_n}}{1-\frac{w^2}{w_n^2}}\right)$.
3. $\lim_{w\to 0}\tilde{\Phi}(w) = 0$
4. $\lim_{w\to\infty}\tilde{\Phi}(w) = -\pi(-180^\circ)$
5. En $w = w_0$, $\tilde{\Phi}(w) = -tan^{-1}(1) = -90^\circ$,

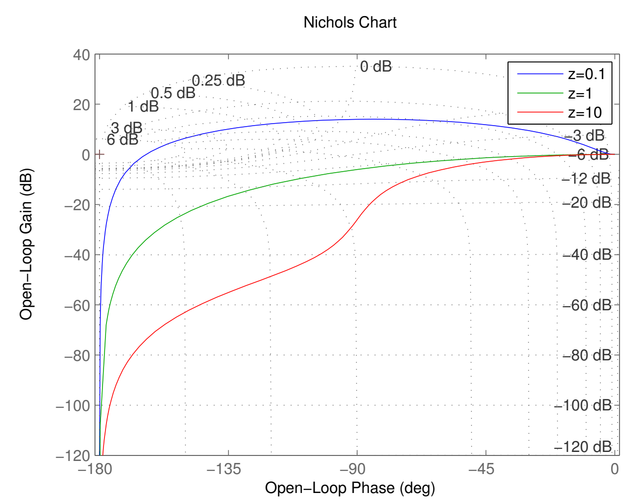
Second order system (10)

Bode plot



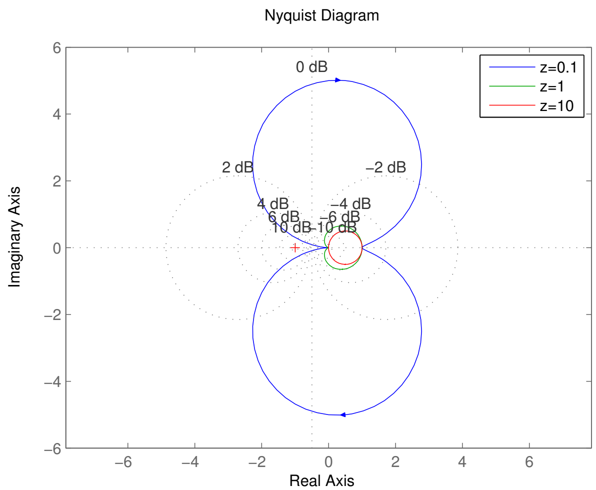
Second order system (11)

Nichols plot

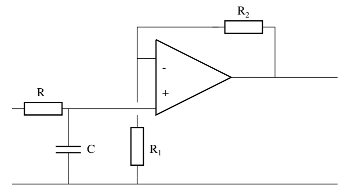


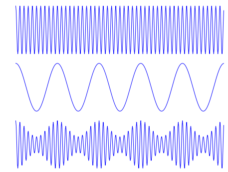
Second order system (12)

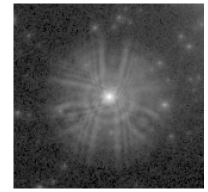
Nyquist plot



Applications of analog signal processing



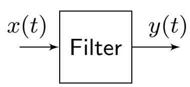




Applications of analog signal processing

- Analog signal filtering.
  - Electronic passive and active filters.
  - Modeling and filtering with physical systems.
- Telecommunications.
  - Amplitude modulation.
  - Multiplexing.
- Fourier opticsL
  - Light propagation in perfect lens/mirror systems.
  - Point spread functions of telescope and cameras.

Analog filtering



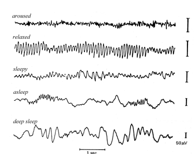
Definition

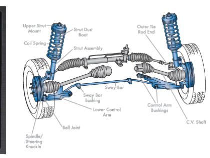
Signal processing system that aim at selecting part of the signal and attenuating another part (noise).
Analog filtering as opposed to digital filtering (next course)

Objectives

- Find a system that transform a signal $x(t)$ to extract pertinent information.
- Attenuate noise in a signal.
- Separate several components of a signal (when different frequency bands).

Applications of analog filtering





- High end audio, amplifiers, (equalizer, echo).
- Car suspension.
- Seismic protection.
- Band-pass before Analog-to-Discrete conversion.
- Fourier optics, telescope modeling.

Signal to Noise Ratio (SNR)

Additive noise

The recording of a signal often contains additive noise:

$$y(t) = x(t) + n(t)$$

$y(t)$ is the recorded signal, $x(t)$ is the signal of interest and $n(t)$ is the noise.

Signal to Noise Ratio

$$SNR = \frac{P_x}{P_n} \quad \text{ou} \quad SNR(dB) = 10\log_{10}(R_{S/B}) \tag{35}$$

- P_x is the power of the signal and P_n is the power of the noise.
- When signals are cosine the SNR is $SNR = \frac{A_x^2}{A_n^2}$ where A_x and A_n are the amplitudes.
- The objective of filtering is often to maximize the SNR.

Filtering and bandwidth

Gain and Attenuation

- In order to characterize a filter one uses its Gain/Phase (Bode plot).

$$\tilde{G}_{DB}(w) = 20\log_{10}(|\tilde{H}(w)|) \quad \text{et} \quad \tilde{\Phi}(w) = Arg(\tilde{H}(w))$$

- Attenuation is also often used $\tilde{A}(w) = -\tilde{G}_{DB}(w)$

Bandwith and passband

The band with of a filter is the set of frequency for which the Gain is over a reference (usually -3dB). Bandwith at $-3dB$:

$$BW = \left\{ w | 20\log\left(\frac{|\tilde{H}(w)|}{\max(|\tilde{H}(w)|)}\right) \geq -3 \right\}$$

Types of filters

- **Low-pass**, $BW = [O, f_c]$ with f_c cutoff frequency
- **High-pass**, $BW = [f_c, \infty]$
- **Band-pass**, $BW = [f_{c_1}, f_{c_2}]$
- **Band-stop**, $BW = [0, f_{c_1}] \cup [f_{c_2}, \infty]$

Filter distortion

Undistorted transmission

A signal is considered undistorted when the output of the system is

$$y(t) = Cx(t - t_0)$$

With

- C a constant gain.
- $t_0 > 0$ is a delay.

A system with no distortion has the following FT and impulse response

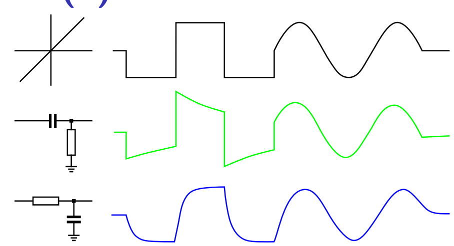
$$\tilde{H}(w) = \frac{\tilde{X}(w)}{\tilde{Y}(w)} = Ce^{-jwt_0} \quad \text{et} \quad h(t) = C\delta(t - t_0)$$

With

- $|\tilde{H}(w)| = C$ or else amplitude distortion.
- $Arg(\tilde{H}(w)) = -wt_0$ or else phase distortion.

Note that the argument of the frequency response varies linearly with the frequency.

Filter distortion (2)



Phase distortion

Let a system of frequency response

$$\tilde{H}(w) = |H(w)|e^{j\phi(w)}$$

We can deduce that for

$$x(t) = \cos(\omega t)$$
$$y(t) = |\tilde{H}(\omega)|\cos(\omega t + \phi(\omega)) = |\tilde{H}(\omega)|\cos(\omega(t + \phi(\omega)/\omega))$$

The delay $\phi(\omega)/\omega$ is also called **propagation time** of **frequency delay**. For it to be independent from frequency it is necessary that

$$\frac{\phi(\omega)}{\omega} = cte = \tau \quad \rightarrow \quad \phi(\omega) = \omega\tau$$

Ideal low pass filter

Definition

- The ideal low-pass filter is often a theoretical object in signal processing.
- Perfect to use when the noise and signal have non-overlapping spectra.
- The frequency response of the ideal filter is

$$H(f) = \begin{cases} 1 & \text{if } |f| < f_c \\ 0 & \text{else} \end{cases}$$

  where f_c is the cutoff frequency.
- The impulse response of the filter is

$$h(t) = 2f_c \frac{\sin(2\pi f_c t)}{2\pi f_c t} = 2f_c \text{sinc}(2\pi f_c t)$$

Realizable filter

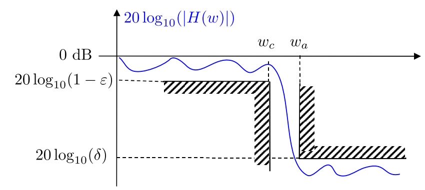
- A realizable temporal filter is **causal** and **stable** (absolute integrable).
- Ideal filter is neither of those and cannot be used for 1D (time) filtering.
- For images (2D) causality is not necessary.

Filter design

Real filter

- Ideal filters are non causal and cannot be implemented in practice .
- We search for an approximation of the ideal filter.
- the approximation has to respect **constraints** (Gabarit in french).

Constraints of a filter



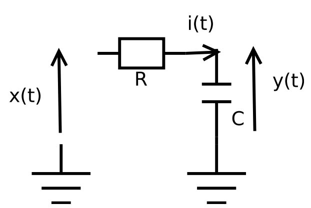
Parameters:

- Bandwidth BP and rejected band
- Oscillations :
  - ε in passing bandwidth
  - δ in attenuated bandwidth

The constraints define the area that are acceptable for a given application.

Simple example of filter design

- Application to Brain computer interface.
- Interesting signal for event related potentials below ≈ 12Hz ($w_s = 2\pi * 12$).
- Electrical noise (EDF) at 50Hz ($w_{edf} = 2\pi * 50$).
- Two low power signals $A_s \approx A_n$.
- Maximum attenuation of signal at -3dB.
- Filtering with first order filter.



- Frequency response

$$\tilde{H}(w) = \frac{1}{1 + j\frac{w}{w_0}}$$

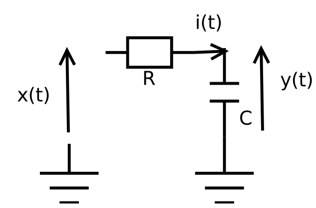
- Gain in Db

$$\tilde{G}(w) = -10\log_{10}\left(1 + \frac{w^2}{w_0^2}\right)$$

- Before filtering:
  SNR$= 20\log_{10}\left(\frac{A_s}{A_n}\right) = 0$
- After filtering :
  SNR$= G(w_s) - G(w_{edf})$
- Choice of w_0?

Simple example of filter design

- SNR$= \tilde{G}(w_s) - \tilde{G}(w_{edf})$
- SNR is a decreasing function of w_0 .
- What is the best value for w_0?

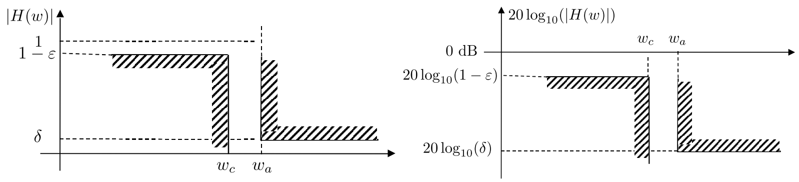


Choix que w_0

- With the maximum attenuation of 3dB constraint. $\rightarrow w_s \leq w_0 \leq \infty$.
- For $w_0 = w_{edf} \rightarrow R_{S/B} = 2.76dB$
- For $w_0 = (w_{edf} + w_s)/2 = 37 * 2 * \pi \rightarrow R_{S/B} = 4.07dB$
- Pour $w_0 = w_s \rightarrow R_{S/B} = 9.63dB$

$\rightarrow w_0 = w_s$ respects the constraint and maximizes the SNR.

Approximating a low pass filter (1)



Constraints for a low-pass filter

- Passband: $1-\varepsilon \leq |\tilde{H}(w)| \leq 1$ pour $w < w_p$
  - w_p: passing frequency.
  - ε: passband margin parameter ($\varepsilon = 1/2 \rightarrow -3dB$).
- Stopband : $|\tilde{H}(w)| \leq \delta$ pour $w > w_a$
  - w_a: attenuation frequency.
  - δ: stopband margin parameter.
- $w_a - w_c$ is the transition band.

Approximating a low-pass filter(2)

- Need for an approximation function that respects the constraints *constrained optimization*.
- Criterion is optimized (for instance maximization of SNR).
- Two approaches are usually used:

Maximally flat frequency response

- Minimal distortion is achieved when the passband is flat.
- Let $|\tilde{H}(w)|$ be the modulus of the frequency response of an order k filter.
- $|\tilde{H}(w)|$ is *maximally flat* in $w=0$ if all the K^{th}derivatives are null

$$\frac{d^K|\tilde{H}(w)|}{dw^K} = 0$$

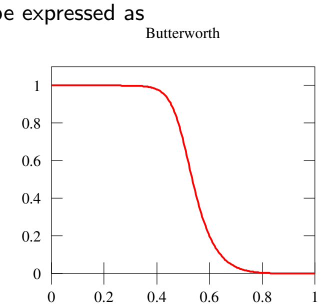
Equiripple filter

- A better rollof (sharper decrease) can be achieved at teh cost of oscillations.
- Oscilations can occur in the passband (leading to distortion) of cutband (limited attenuation).
- An equiripple filter has constant magnitude for its oscillations in the bandpass.

Butterworth filter (1)

- Butterworth filters are *maximally flat* [Butterworth et al., 1930].
- The amplitude of the frequency response can be expressed as

$$|\tilde{H}(w)| = \frac{1}{\sqrt{1+\left(\frac{w}{w_c}\right)^{2n}}} \qquad (36)$$



with

- n: order of the filter.
- w_c: cutoff frequency.

- The passing w_p and attenuation w_a frequencies are:

For $|\tilde{H}(w)| = 1-\varepsilon$

$$w_p = w_c\left(\frac{\varepsilon}{1-\varepsilon}\right)^{1/2n}$$

For $|\tilde{H}(w)| = \delta$

$$w_a = w_c\left(\frac{1-\delta}{\delta}\right)^{1/2n}$$

Butterworth filter (2)

- The Butterworth filter is monotonically decreasing with the frequency.
- The amplitude of the frequency response can be expressed as

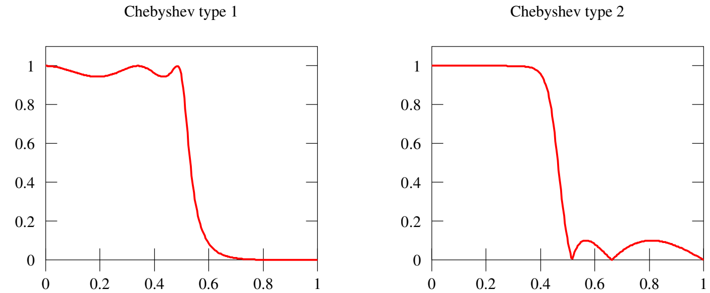
$$|\tilde{H}(w)| = 1 - \frac{1}{2}\left(\frac{w}{w_c}\right)^{2n} + \frac{3}{8}\left(\frac{w}{w_c}\right)^{4n} - \frac{5}{16}\left(\frac{w}{w_c}\right)^{6n} + \dots$$

- The derivative in $w=0$ is then null up to order $k = 2n-1$.
- The frequency response of a (normalized) Butterworth filter can be expressed as $\tilde{H}(w) = \frac{1}{B_n(w)}$ where $B_n(w)$ is a Butterworth polynomial :

$$, \quad B_n(w) = \begin{cases} \prod_{k=1}^{\frac{n}{2}} \left[(jw))^2 - 2jw\cos\left(\frac{2k+n-1}{2n}\pi\right)+1\right] & \text{if } n = \text{ even} \\ (jw+1)\prod_{k=1}^{\frac{n-1}{2}} \left[(jw))^2 - 2jw\cos\left(\frac{2k+n-1}{2n}\pi\right)+1\right] & \text{if } n = \text{ odd} \end{cases}$$

Order	Polynomial
1	$1+jw$
2	$(jw)^2+\sqrt{2}jw+1$
3	$(jw+1)((jw)^2+jw+1)$

Chebyshev filter



- Better rolloff than Butterworth of same order but leads to oscillations in the bandpass (type 1) or in the stopband (type 2).
- *Equiripple* filter.
- Amplitude of the frequency response:

$$|\tilde{H}(w)| = \frac{1}{\sqrt{1+\varepsilon^2 T_n^2\left(\frac{w}{w_c}\right)}}$$

- $T_n(\cdot)$: Chebyshev polynomial of order n.

Filter implementation

Implementation of the filter consist in finding the physical components that recovers the selected frequency response $\tilde{H}(w)$.

Passive filter

- Only passive components (R, C, L).
- No energy source, no amplification (conservation of energy).
- The input and output impedance has an effect on the frequency response (impedance matching).

Active filter

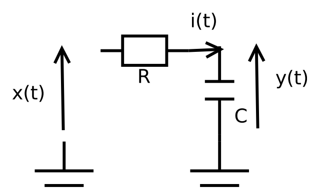
- Use an energy source and Operational Amplifiers (OA).
- OA has near infinite impedance but limited bandwidth (typically 100KHz).
- Saturation can occur (non-linearity).
- Stability can be a problem (due to feedback)

Rarely use inductors in practice (price, resistance, space, mutual inductance) !

Passive filters (1)

Example filter

- Brain-Computer Interface application.
- $w_0 = w_s = 2\pi * 12$
- $w_0 = \frac{1}{RC} \rightarrow RC = \frac{1}{2*\pi*12} \approx 0.01326$
- What to choose for R and C ?
- Price and space constraints.



Filter transformation

- **low-pass $\rightarrow$ high-pass**

$$1/jCw \rightarrow jLw \quad \text{et} \quad jLw \rightarrow 1/jCw$$

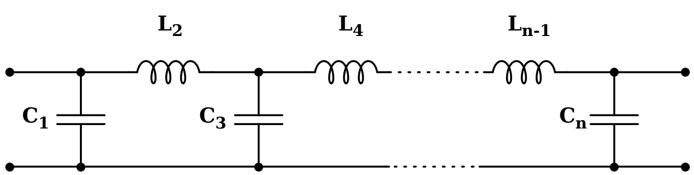
- **low pass $\rightarrow$ band-pass**

$$1/jCw \rightarrow B/C(jw + 1/jw) \quad \text{et} \quad jLw \rightarrow L/B/(jw + 1/jw)$$

Passive filters (2)

Butterworth Filter

- Corresponding frequency response with the Cauer topology.
- For an order n filter with cutoff frequency $w_c = 1$ the following structure:



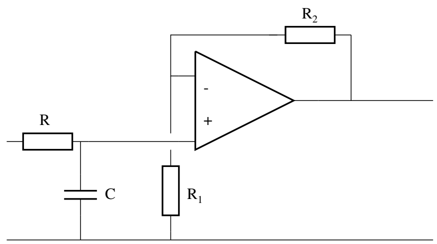
With the values :

  - $C_k = 2\sin(\frac{2k-1}{2n}\pi)$ for k odd.
  - $L_k = 2\sin(\frac{2k-1}{2n}\pi)$ for k even.

- Assuming the input and output have a 1 Ohm resistance.

Active filters (1)

First order active filter (with amplification)



- Frequency response

$$\tilde{H}(w) = \frac{A}{1 + \frac{jw}{w_0}}$$

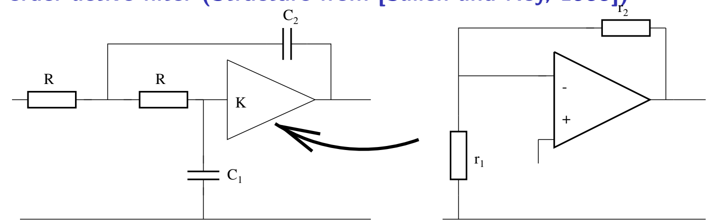
where

$$A = \frac{R_1 + R_2}{R_1} \quad \text{et} \quad w_0 = \frac{1}{RC}$$

- Parameters: R, C, R_1, R_2
- Permute R and C for a high-pass filter.

Active filters (2)

Second order active filter (Structure from [Sallen and Key, 1955])



- Frequency response

$$\tilde{H}(w) = \frac{K}{1 + \frac{2zjw}{w_n} + \frac{(jw)^2}{w_n^2}}$$

where

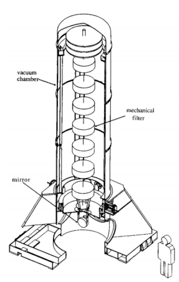
$$w_n = \frac{1}{R\sqrt{C_1 C_2}} \quad \text{et} \quad z = \sqrt{\frac{C_1}{C_2}}\frac{3-K}{2} \quad \text{et} \quad K = \frac{r_1 + r_2}{r_1}$$

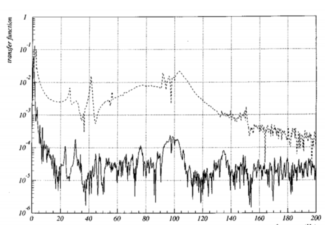
- Parameters: R, C_1, C_2, r_1, r_2.

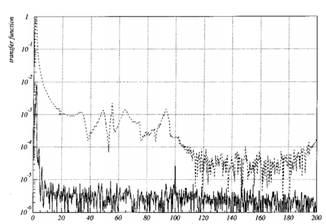
Analog filtering : mechanical filter

Virgo Gravitational waves detector [Acernese et al., 2014]

- Interferometer to detect gravitational waves.
- Attenuate vibrations from the earth [Braccini et al., 1996]
- Objective : attenuations of 10^{-9} for high frequencies.
- Use a mirror in a chamber with mechanical filters.
- Use a series of mechanical filters for the attenuation.
- Active correction for remaining low frequencies.







Modulation

Modulation is an encoding method that allows to transport a band-limited signal. Demodulation is the reverse operation.

Motivations

- Raw signal transmission often not efficient (electromagnetic waves).
- The change in frequencies allow transmitting several band-limited signals in parallel.
- Use only of an authorized bandwidth.

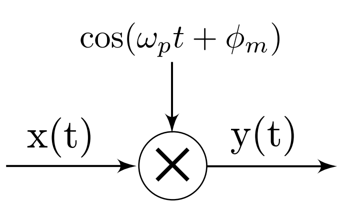
Definitions

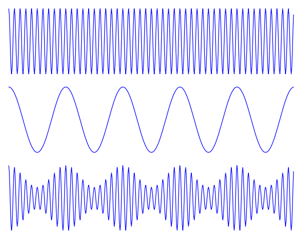
- **Modulating signal** $x(t)$ is a band limited signal we want to transmit ($X(f) = 0 \quad \text{pour} \quad |f| > f_x$).
- **Carrier** is the periodic base signal $p(t)$ used for transportation often :

$$p(t) = \cos(2\pi f_p t)$$

- **Modulated signal** $y(t)$ is a band-limited signal that can be transported in the physical medium (cable, air, optical fiber)

Amplitude Modulation (1)





Definition: Amplitude Modulation (AM)

The carrier is multiplied by the modulating signal $x(t)$

$$y(t) = A_c(1 + k_s x(t)) \cos(2\pi f_p t + \phi_m)$$

- k_s: modulation factor
- f_p: carrier frequency
- ϕ_m: phase (usually added during transmission).

Amplitude Modulation (2)

Modulation index

- Envelope of the modulated signal.

$$a(t) = A_c|1 + k_s x(t)|$$

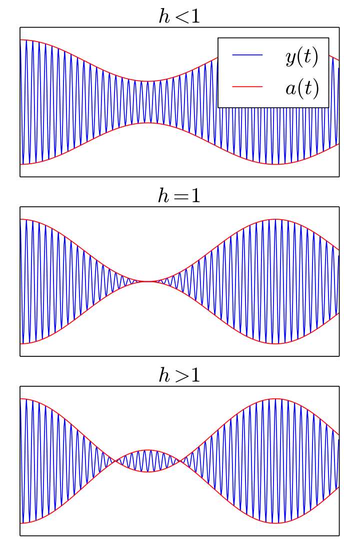
- Maximum amplitude of modulating signal:

$$M_x = \max_t |x(t)|$$

- The index of modulation is defined as

$$h = k_s M_x$$

  - $h < 1$: under-modulation.
  - $h > 1$: over-modulation.



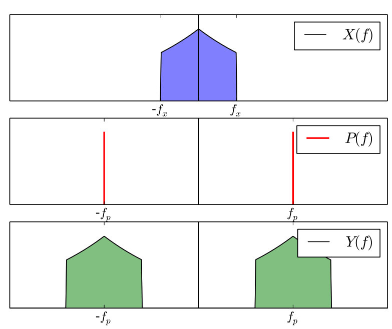
Amplitude Modulation (3)

Interpretation in the Fourier domain

- Multiplication → Convolution.

$$Y(f) = X(f) \star P(f)$$

- The spectrum of the modulating signal is moved around the frequency f_p.
- Simple way to transmit a band limited signal in a given bandwidth.
- Modulated signal spectrum is contained in $f_p \pm f_x$.



Amplitude Modulation (4)

Synchronous demodulation

Done with multiplying the signal with the carrier:

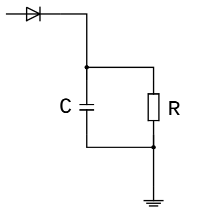
$$\begin{aligned} w(t) &= y(t)\cos(2\pi f_p t + \phi_d) \\ &= A_s(1 + k_s x(t))\cos(2\pi f_p t + \phi_m)\cos(2\pi f_p t + \phi_d) \\ &= \frac{A_s}{2}(1 + k_s x(t))\cos(\phi_m - \phi_d) + \frac{A_s}{2}(1 + k_s x(t))\cos(4\pi f_p t + \phi_m + \phi_d) \end{aligned}$$

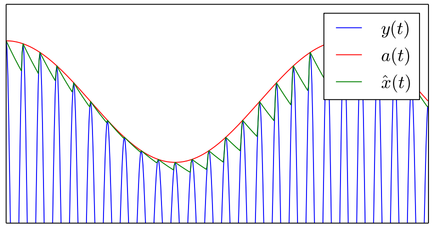
After low pass filtering (and removing of the constant) one can recover

$$\hat{x}(t) = \frac{A_s}{2} k_s x(t) \cos(\phi_m - \phi_d)$$

- $\cos(\phi_m - \phi_d) = 1$ if $\phi_m = \phi_d$.
- Very important to have a good synchronization (requires active components).

Amplitude Modulation (5)





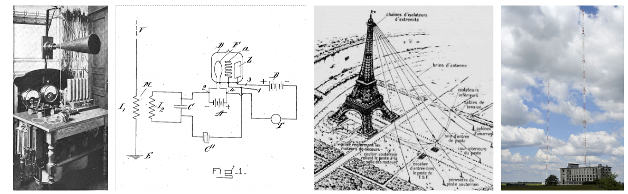
Asynchrone demodulation

- Synchronous demodulation can require complex active components.
- A coarse approximation of the envelope of the signal can be done with a simple diode/RC system.
- Requires under-modulation because if $h < 1$ then

$$a(t) = A_c|1 + k_s x(t)| = A_c + A_c k_s x(t)$$

- Can require a lot of power for transmission.

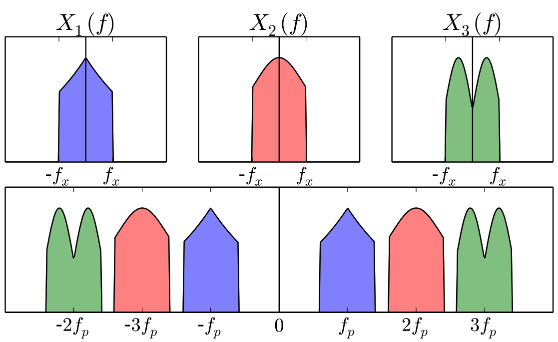
Applications of amplitude modulation (1)



Low frequency radio broadcasting

- First AM transmission R. Fessenden on 23 December 1900 at Cobb Island, Maryland (1.6Km).
- 1907 Lee de Forest invents the triod vacuum tube allowing for a better amplification [De Forest, 1908].
- Weather bulletin emitted from the Eiffel Tower in february 1922.
- **France Inter grandes ondes**
  - Emitted between 1 January 1947 and 31 December 2016.
  - Allouis longwave transmitter (2000KW), now used for TDF time signal.

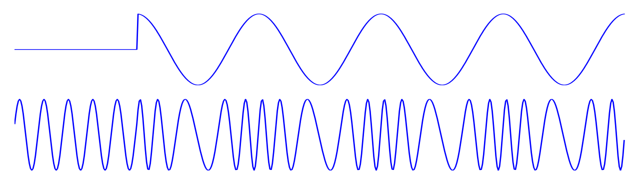
Applications of amplitude modulation (2)



Frequency-division multiplexing

- Multiplexing: transmission of several signals in parallel.
- Use of a different f_p for each signal.
- Every signal is band limited : if $\Delta f_p > 2f_x$ then no loss of information.
- Frequency Hoping: Experimented by G. Marconi, Patent by N. Tesla [Tesla, 1903], proposed for secret communication by [Kiesler and George, 1942].

Frequency modulation (1)



Definition

Frequency modulation (FM) consists in modifying the frequency of the carrier using $x(t)$. The modulated signal has the following form:

$$y(t) = \cos\left(2\pi \int_0^t f(\tau) d\tau\right)$$

- $f(t) = f_p + f_\Delta x(t)$ is the instantaneous frequency of the signal.
  - If $x(t) = 0$ we recover the carrier.
  - When $x(t) \neq 0$ the instantaneous frequency is modified by $x(t)$
- f_Δ is the frequency deviation (equivalent to k_s in AM).

Frequency modulation (2)

Properties of Frequency Modulation

- More robust than AM (noise, atténuation) but propagation distance limited.
- More complex to implement (requires a Voltage Controled Oscillator VCO).
- Intuitively the spectrum of the modulated signal should be $\neq 0$ only in the band $f_p \pm f_\Delta M_x$, BUT
- Continuous variation of the frequencies imply a spectrum on all frequencies.
- The Carson bandwidth rule states that most of the signal power (98%) is in the band

$$b = 2(f_\Delta + f_x)$$

Application of Frequency Modulation

- FM radio broadcasting.
- Frequency modulation synthesis (chiptunes).
- Magnetic tape storage.

Fourier Optics

Principle

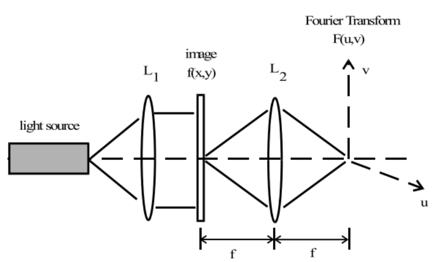
- Introduction to Fourier Optics: [Perrin and Montgomery, 2018, Goodman, 2005]
- Huygens–Fresnel principle for wave propagation.
- When the source is at infinity, one can use the Fraunhofer diffraction (far field).
- Several optical elements corresponds to linear operations and can be defined as LTI and modeled/interpreted through Fourier Transform.
- Difference between coherent VS incoherent sources.

Applications of Fourier transform in optics

- Analog image processing techniques.
- MRI : sampling of an image in the Fourier domain.
- Astronomy : modeling of telescopes, source detection, coronarography.

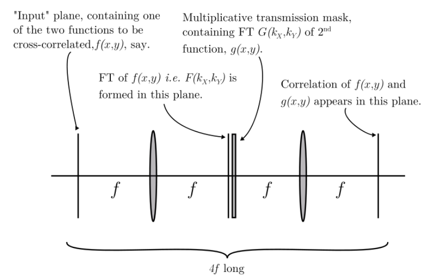
2D Fourier Transform using a lens

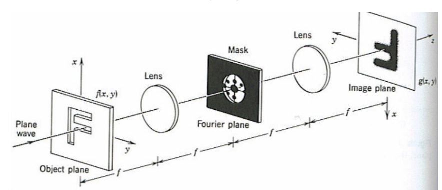
- Place a transmissible object at the focal length of a lens.
- The FT of the object is formed on the focal plane behind the lens.
- FT computed at the speed of light, depends on the precision of the optics.

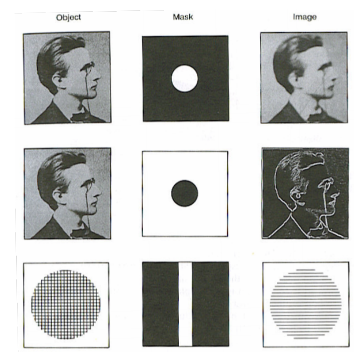


- Let $i(\mathbf{v})$ be the 2D image in the focal plane before the lens and $I(\mathbf{u})$ its FT.
- Here f is the focal length of the lens (in optics ν or v are often used to denote frequency).
- λ is the wavelength of the light source.
- The the image formed in the right focal plane will be $I(\frac{\mathbf{p}}{\lambda f})$ where $\mathbf{p}$ is the position in the focal plane.

The 4F correlator

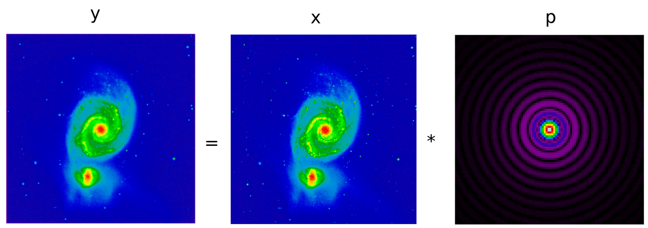






- One can use the FT provided by Optical lenses to perform analog image filtering.
- The Filtering is done with a mask in the Fourier plane of the image.
- Equivalent to a convolution (correlation).
- The output image is mirrored due to the two FT instead of an inverse FT.
- Active research domain in Optical Neural Networks [Zuo et al., 2019]

Telecope and Point Spread Function (PSF)



- The source point in the images are considered incoherent to the observed image (intensity) is the sum of the responses of each source.
- A telescope can be considered as a LTI system (at least close to the axis).
- The relation between the true image and the image observed in the focal plane is always a convolution by what is called the Point Spread Function:

$$y(\mathbf{v}) = x(\mathbf{v}) \star h(\mathbf{v})$$

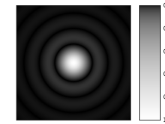
- The PSF h can be obtained as

$$h(\mathbf{v}) = \left|\mathcal{F}^{-1}[A(\mathbf{u})]\right|^2$$

where $A(\mathbf{u})$ is the aperture shape of the telescope.

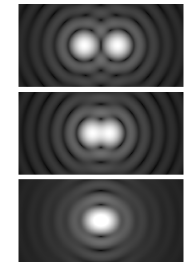
Airy disk and angular resolution

Circular Aperture



- The PSF for a circular aperture often called the Airy disk comes from the FT of the circle $\mathcal{F}_{2D}[circ(r)] = J_1(2_p ir')/r'$.
- The diameter of the circle defines the maximum resolution.

Angular resolution

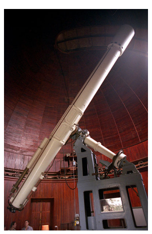


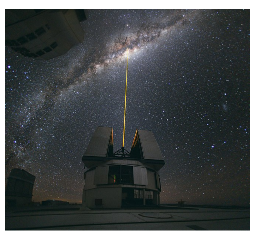
- Minimal angle that allows discriminating two point sources.
- Given by the Rayleigh criterion

$$\theta = 1.22\frac{\lambda}{D}$$

λ is the wavelength and D is the diameter of the telescope.
- It corresponds to the first zero of the Bessel J_1 function.

Fourier Optics in astronomy






Real life telescopes

- New telescopes have several small mirrors : more complex PSF.
- Fourier Optics model only for perfect optics.
- Lenses/mirrors have optical aberrations and a surface roughness introducing scattering.
- Ground telescope have to compensate for atmospheric turbulence (deformable mirrors with adaptive optics).

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