Handbook of

Special Relativity

A Representational Approach

Phy-Hub

"Aether you believe in a universal medium that light propagates through or you don't"

—**Anonymous**

Contents

Preface

Your moma didn't raise you right, so now its my turn to whip you into shape. By the end of this book you will also be able to understand yourself a little more, (because there will be a subsection on retarded jerks). You already have something in common with this topic of relativity, cause you are special!... The only thing that can undo Lorentz length contraction is your moma

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This book will start with a visual and non mathematical introduction to special relativity, giving you a good conceptual basis for when it comes to the mathematics. We will then move on to the mathematical derivation of how space and time work in special relativity, followed by velocities, relativistic beaming/aberration, special relativistic fields, hopefully Liénard-Wiechert-Potentials, maybe quarternions in SR if you are lucky, and so on.

*** the math will be shown in as plainly/simple a way as it can

*** I may include some derivations that follows more closely the first chapters sections

*** SR can be more difficult to visualise than quantum mechanics

This book is not an inertial frame of reference unless in free fall (but gravity and that is for a different day). Damage caused to this book by allowing it to enter a short lived inertial frame of reference before making an abrupt exit from it, is not covered by our returns policy (do not throw book at ground!).

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Chapter 1

Conceptual Overview

Newtonian physics serves as an excellent approximation at low speeds, accurately predicting the behavior of objects in our everyday world, but it gives way to different effects as we approach the speed of light. It turns out that an observer moving at high speed relative to another observer would notice lengths contract, time run slower, and events that were simultaneous to the other observer, no longer simultaneous to them. There is however one thing that remains constant, the speed that light travels relative to each observer is the same. To understand how all this works, we have to redefine our understanding of time and space, with special relativity.

Special relativity is vital for particle, nuclear, and astrophysics. It is needed for calculating the power output when it comes to nuclear fusion and fission or measuring the speed of galaxies and stars from the relativistic effects on the frequency of their emitted light. It is also needed for precision timekeeping for the likes of the fast-moving GPS satellites which need it for accurately calculating positions.

To understand how special relativity works we will need to first look at our classical understanding of the world, and using the fact that light somehow always moves at constant speed relative to everything, we will see how we must change our laws of physics for this to be true. Which in turn will lead to many interesting consequences. We will start with an overview of all the main concepts, leaving the mathematical details and deeper insights for later chapters.

1.1 Classical Addition of Velocities

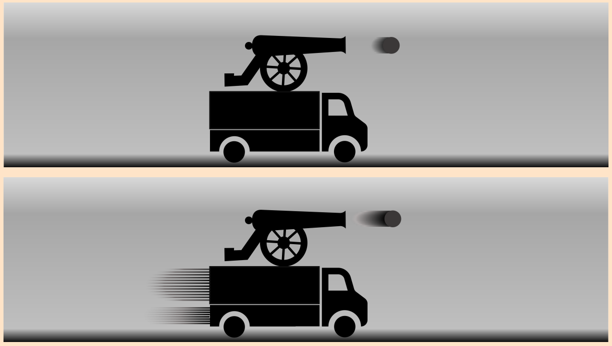


Figure 1.1: Diagram, showing the speed of cannonball from two different perspectives, (top) is from someone's perspective who is on the truck who sees the cannon at rest with the road moving backward, the cannonball is shot and is moving forward. (bottom) is from someone's perspective who is at rest on the road and sees the truck and cannon moving relative to them, when the cannonball is shot it looks like it is moving forward faster for them, as it seems to have the cannonball's velocity plus that of the truck it was shot from.

Let us imagine that a truck is moving forward on a road at a constant speed of 20 m/s (meters per second), on the top of that truck, is a cannon that fires a cannonball in the same direction the truck is moving. The ball travels at a speed of 300 m/s relative to the truck. Classically to find out how fast the ball is moving relative to the road, we add the speed of the truck and the speed of the ball together: $20 + 300 = 320$ m/s. So we find the ball is moving at a speed of 320 m/s relative to the road, and this is what we observe. So in classical physics, velocities directly add, the ball relative to the road would move at the speed of the ball relative to the truck plus the truck's speed relative to the road. But this turns out to just be a very good approximation for objects in our normal day life which we are used to seeing, which are moving much slower than the speed of light (which is roughly 300 000 000 m/s), but we will get to this later.

For now, we will explain more about the classical view, so that you will be able to see more clearly the differences when it comes to special relativity.

1.2 Inertial Reference Frame

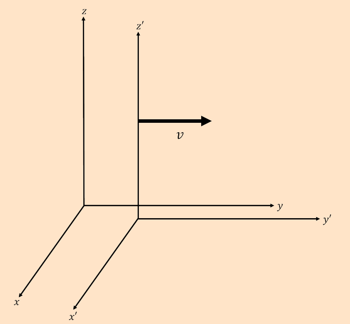


Figure 1.2: Diagram of a reference frame moving relative to another.

A reference frame can be thought of as an abstract coordinate system. The origin of its axis, its orientation, and its scale are specified by a set of points in space. The purpose of it is to provide a standardized means of measuring and describing the coordinates of objects within that frame at any instant of time.

An inertial reference frame, is a reference frame that is not undergoing any acceleration. You can tell if you are being accelerated, as you will feel a force, for example, in an accelerating car you will feel the chair being accelerated into you, with your body slightly lagging behind in the acceleration, and its this basic principle for how accelerometers measure acceleration as shown in the diagram.

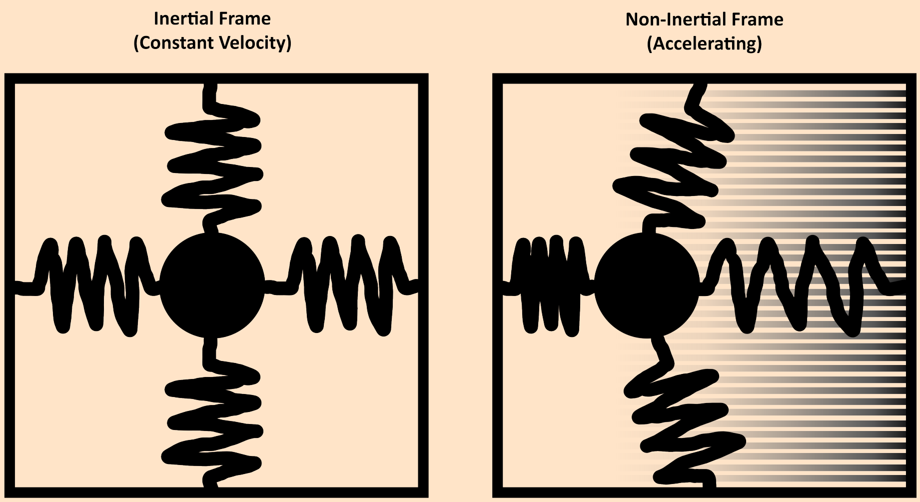


Figure 1.3: Diagram of a ball, attached to the walls of a box by springs, with the ball centered in the box in the inertial frame, i.e. with no acceleration (left), and in a non-inertial frame (right) where the box is now accelerating to the right, The ball lags behind as the box accelerates.

1.3 Classical Reference Frames

To see how to swap between reference frames in special relativity we will first have to introduce what the classical swapping between two reference frames looks like. To help us understand this, we will use a system of a moving platform, like we find on a treadmill. The two frames of reference will be that of the treadmill's platform and the room in which it is in.

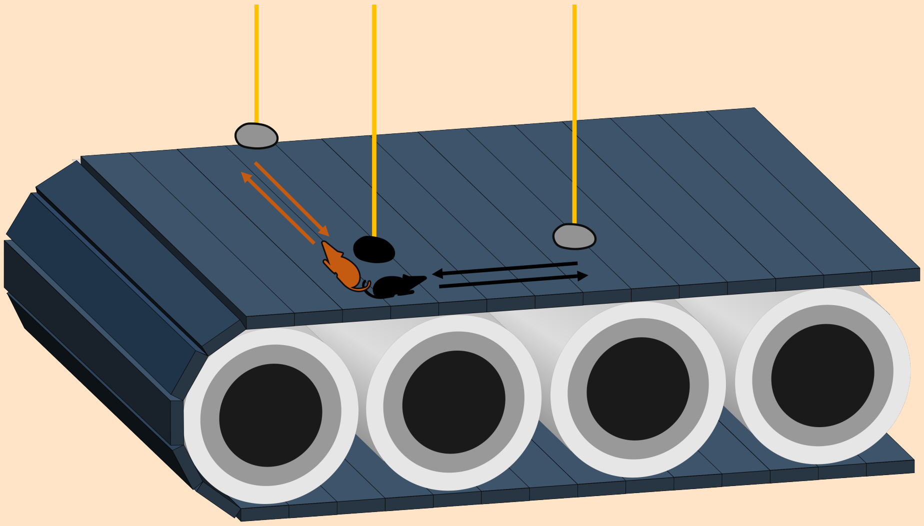


Figure 1.4: Diagram of 3d view of treadmill platform, with two rats and three hanging rocks.

To illustrate swapping between frames classically, We will look at a setup of two rats on a treadmill as shown in Figure 1.4, with three rocks hanging above the platform, at rest relative to the room. Both rats start under the same rock, one runs to a rock positioned in the forward direction of the treadmill, and the other runs to an equally distanced rock to the side, they then return to the starting rock. Here the platform can be seen as the medium in which the rats move. Both rats travel at the same constant speed relative to the platform, with the platform at a lower speed than the rats to allow them to get to the rocks.

If the platform is at rest, they will return to the starting rock at the same time. But if the treadmill is turned on and the platform is now moving, the rats will now have to also work against the movement of the platform to get to the rocks, this will lead to different distances the rats have to travel, and as a result, the rats get back to the starting rock at different times. The figures 1.5 and 1.5 show what the direction of movement of the rats and rocks look like in each reference frame.

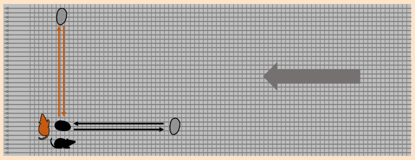


Figure 1.5: Rats on moving platform, from room's reference frame.

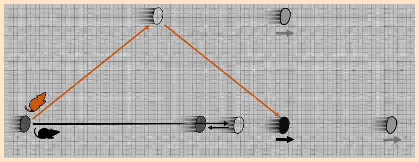


Figure 1.5: Rats on moving platform, from platform's reference frame.

In the room's reference frame we have the platform moving backward, the rat moving straight ahead will have impedance to its movement but after it turns around it now has a boost from the platform to get back to the starting rock, were as the sideways moving rat will have to go sideways and also balances out the backward pull of the platform to keep a perpendicular movement towards the hanging rocks. This gives different lengths of paths depending on the reference frame, as shown in figures 1.5 and 1.5. It also gives different times for the rats to return to the starting rock. This is what we would expect and see with this classical example of frame swapping, but next, we will look at whether we can have the same sort of frame swapping when it comes to light traveling in two different frames of reference, and the notion of a universal rest frame referred to as the aether.

1.4 The Aether

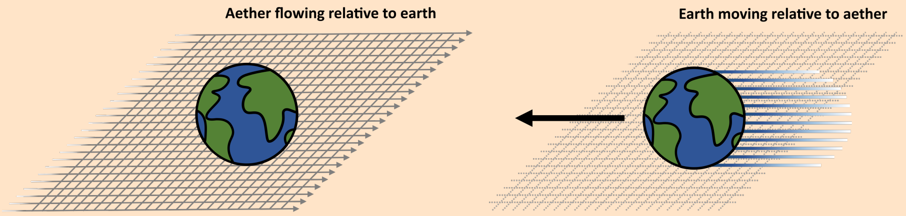


Figure 1.6: A diagram showing the aether's and earth's movement relative to each other.

In the 1800s, the theory was that light was a wave and therefore would need a medium for it to travel through that filled the vacuum of space, called "the luminiferous aether", and that light would travel at a constant speed relative to this aether, like how the rats in the previous section moved at a constant speed relative to the medium of the treadmill's platform.

An experimental setup by Michelson and Morley, shown in Fig. 1.7, was devised to measure earth's movement through the aether [1], by measuring how it affected the speed of light in different directions when observed in earth's reference frame.

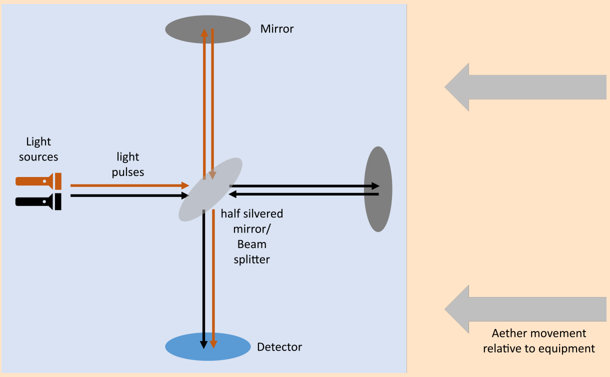


Figure 1.7: A diagram showing the Michelson-Morley experiment, (we can take the part of the paths between the the beam splitter and two mirrors to be analogous to the case of the paths in the previous rat and treadmill diagrams)

It did this by splitting a single light beam into two perpendicular paths, that are then reflected back to be recombined and sent towards a light detector. By rotating the whole interferometer setup, the two light paths could be aligned either parallel or perpendicular to the Earth's motion through the presumed aether. They reasoned that if the speed of light was constant with respect to the proposed aether, that just as in the rat experiment from the previous section, the split light beams would recombine at different times. From the previous section the room is analogous to the room the experiment was carried out on earth, the treadmill's platform is analogous to the aether and the rats analogous to the light. However, when Michelson and Morley performed the experiment, they found no difference in travel time to the detector for both paths, indicating that there was no difference in the speed of light in any direction in earth's reference frame. hence no dependence of lights speed on the supposed aether. This null result seriously discredited the aether theories and ultimately led to the proposal by Einstein in 1905 that the speed of light (in a vacuum) is a universal constant and independent on motion of observer or source. And to allow for us to have this universal speed of light (in a vacuum), it will require us to change our ideas of how time and positions are perceived by different observers.

1.5 Speed of Light

The experiments showed light does not have a medium that it travels at a constant velocity relative too, but instead travels at a constant velocity in a vacuum in all reference frames, independent of how fast the source of that light is moving in the frame, e.g. a moving truck's head lights. Light only moves slower in objects such as glass due to being impeded by the interactions between it and the material it is travelling through.

Light itself moves extremely fast compared to any other everyday speeds we are used to (roughly 300 000 000 m/s, it can travel the diameter of the world in the blink of an eye). This is why we do not notice any delay in things we see around us in everyday life, though we should keep in mind that there is this delay, e.g. the light we see from the sun was emitted by it eight minutes ago for it to reach us now, and the further an object is located from us, the further back in time we are seeing it because of this delay.

When we look at the same truck setup as the previous sections, but in a vacuum this time, where the cannon firing a cannonball is swapped for the headlights emitting light, it gives the same speed of light when measured relative to the truck or the road, but how can this be true?

For this to be true, we need a new way of thinking about velocity addition, since the velocities of objects have to be added in a way that is consistent with the requirement that the speed of light is constant, but also give approximate classical addition at speeds of objects at much less than the speed of light, like we observe with the cannon and truck. Since the speed of the light depends only

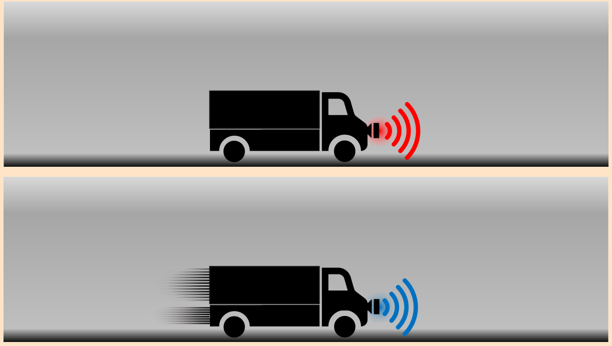


Figure 1.8: A diagram, showing light emitted from a truck in two different reference frames, with the emitted light having the same speed in each frame, though with different frequencies/energies due to the Doppler effect, which will be explained later.

on the units of time and positions, the only way to correct for this is to have the measured positions and times of objects transform differently when swapping between reference frames than the classical way. This is what we will talk about next. For the curious, it was the experiment by Ole Rømer that showed that showed that light travelled at a finite speed rather than being instantaneously emitted and received.

1.6 Positions and Time

1.6.1 Time Dilation

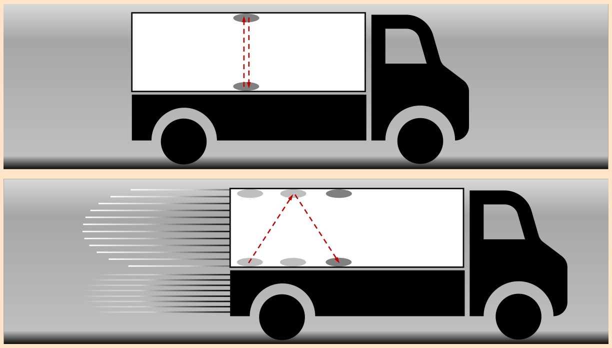


Figure 1.9: Diagram showing the the extra distance light travels in the road's frame with truck moving

Let us imagine a simple clock, as shown in Figure 1.9, made of a light pulse moving back and forth between two mirrors on a moving truck, with a mirror on the truck's floor and on the roof directly above it, we will keep time by taking a tick of this clock to be when the light travels from one mirror to the other mirror and back again. For an observer in the truck, they will see the light go straight up and down between the mirrors, but an observer stationary relative to the road will see not just the light travelling up and down but also with the direction of the truck as well, hence it will travel a longer distance. But since the distance travelled by the light in the moving frame is longer and the speed of light is constant, we are only left with the possibility that the perceived travel time of the light has to be longer in this reference frame, i.e. the clock ticks slower to the observer on the road watching the moving truck. How much it is moving slower, can be solved using ratio of the lengths of paths in each frame as they are shown in the diagram. This difference in the flow of time will later lead to more odd consequences. The slowing of the ticking is also the same for any type of clock, and if you were to play a movie on the truck, it would take a longer amount of time for it to play through from the roads reference frame, it is the actual time itself being slowed down.

Also the closer the truck goes to the speed of light, the bigger the difference in how fast time flows relative to the truck 's time (this can be seen as a longer horizontal stretch in the path taken for the light from the perspective of someone standing still on the road).

1.6.2 Simultaneity

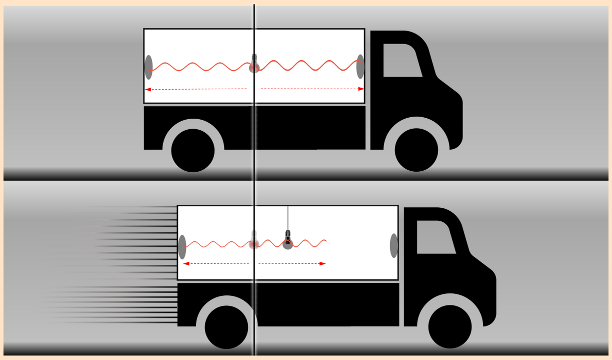


Figure 1.10: A diagram showing how the two events of light reaching the two walls of the truck are simultaneous in one frame but happen at different times in another, due to truck's movement in the second frame and the speed of light remaining the same.

Let us imagine a truck in its rest frame, with a light bulb in the middle and mirrors on the front and back walls, now if the light bulb gives off a pulse of light, the light will travel from the centre of the truck to reach the mirrors simultaneously and back to the light bulb also simultaneously. But the observer on the road watching the truck drive past will see the light bulb simultaneously emit light in both directions and also simultaneously return to the bulb again. However, to do this the light reflects off each wall's mirrors at different times in this frame as shown in the diagram. This is because the speed of light is the same for both directions but the truck is moving, meaning the back of the truck is moving towards where the bulb was when the pulse was emitted making the distances travelled shorter, and due to the front wall moving away from it, the light travels a longer distance to get to the front wall. Due to this and the speed of light being the same in both reference frames, the observer on the road will see the light hit the mirrors at different times. So, times of events (like when the light reaches either mirror) are different for observers in different reference frames, there is no one true order of events, e.g. an observer in a faster moving truck moving in an opposite direction would see the light reach the front wall first. However for all observers the light will return to the central light bulb simultaneously. If two events happen in the same position at the same time (i.e. the light returning to the centre of the light bulb simultaneously) then this happens simultaneously in all frames of reference.

I have left out any mention of length contraction, which will be introduced in the next section, and why it needs to be equal in both directions from light bulb, i.e. for the light to simultaneously return to the light bulb they need to maintain equal distances of the walls from the central light bulb.

1.6.3 Length Contraction

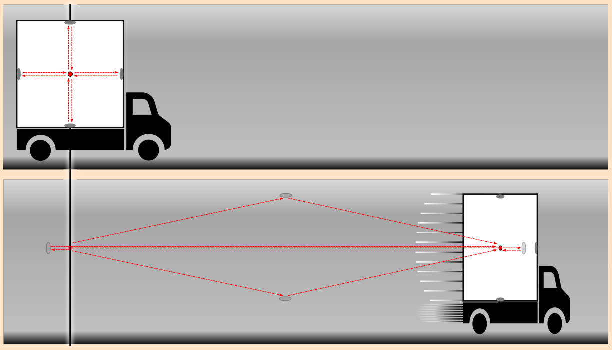


Figure 1.11: A diagram, showing a truck with a square container in it's rest frame (top), emitting light from a central bulb in the four directions, with all light being reflected by the mirrored sides back to the centre. In the second frame with the truck now moving (bottom) we see the truck 's length contracted and what the light paths would be in this frame

If we have a truck in its rest frame with square container with light emitted in all four directions from the centre, so that it will bounce off the mirrored sides of the truck and return to the centre simultaneously. we require that in the moving frame they also all return to the centre simultaneously, as multiple events that happen at a single point simultaneously in one frame, must happen simultaneously in all other frames. This time between light being emitted and absorbed will be the dilated time, that was described in the time dilation section.

To achieve this simultaneity in the return of the light to the bulb, the length of the path of light in each direction has to be the same (as its speed is same in all directions), we can work out the length of the upward path from the time dilation section, and this is the length the path needs to be in the horizontal directions as well, the paths can only have this length if the truck length is contracted when moving, the exact amount of contraction can be worked geometrically. And it turns out that the ratio of the increase in the amount of time that passes before light is reabsorbed is inversely proportional to the amount the truck is contracted in the direction of its movement.

i.e. time passes by a factor slower in the moving frame, and the change in distances in the direction of movement are inversely proportional to the same factor.

The reason that we do not worry about contraction in the distance between the floor and ceiling mirrors, will be explained in a later section.

3 car system way of explaining length contraction (may delete)

Say we have 3 cars that are equally distanced and at rest on the road, the middle car sends out a pulse of light that reaches the front and back car at the same time to the road observer and the observers in the car, when it reaches them both, all the cars accelerate for a predetermined fixed amount of time, so that after this acceleration period the cars are now moving but still equally spaced relative to the observers in the cars and the observer on the road. As for all observers the light reaches the front and back cars at the same time.

now if we do this a second time and the middle car releases another pulse, in the frame of the observers in the cars: again the front and back drivers receive

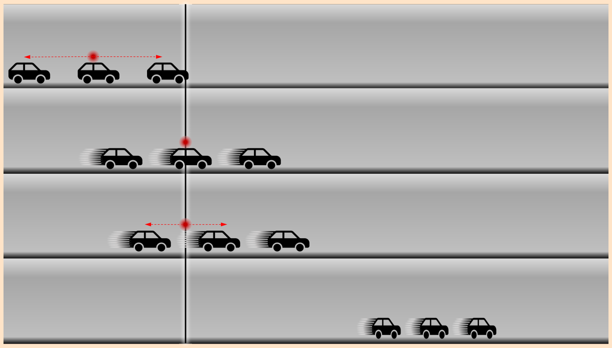


Figure 1.12: Diagram that illustrates an experiment with three cars initially at rest and equally spaced on a road. The middle car emits light that reaches the front and back cars simultaneously, triggering all cars to accelerate for a predetermined amount of time. Now another light is emitted from the middle car after the acceleration has finished, the road observer sees the light reach the back car first, as it is moving towards where the light had been emitted, this causes it to begin accelerating before the front car. This results in the cars being closer together after acceleration, demonstrating Lorentz length contraction, However, to the car observer, the distance between the cars remains unchanged from the initial distances.

the pulses at the same time and accelerate, and again after the acceleration they have the same equal distance between the cars. But for the observer on the road, they see the back car receive the signal first as that car is moving towards the point where the light pulse was emitted and the front car is moving away from it. This would mean the back car would start accelerating first to get to the final constant velocity and get closer to the front car before the front car begins to accelerate, hence the cars end up closer together after all of this. The observers in the car and the road observer, do not agree with what the distances between the cars is. This contraction of length between cars, in the frame in which the cars are moving, is called Lorentz length contraction, which means that objects that are moving faster become shorter, the distances between the objects also become shorter.

1.6.4 No Width Contraction

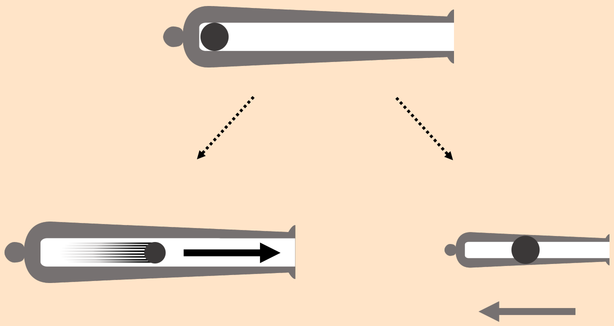


Figure 1.13: A diagram showing why there must not be contraction in perpendicular direction to frames relative motion. The top figure shows a ball and cannon at rest, the bottom figures show the cannon ball being fired in the frame of cannon (left) were the moving ball has a contracted width, and shows the frame of the ball (right) were now the canon is now moving and has a contract width, with the ball at rest being the same size, both frames would contradict each other, if there was a change in width of moving objects, as the walls of the cannon and the balls surface would overlap in one frame and not be touching in the other. So we require that there is no change in size perpendicular to the movement of the object.

1.7 Doppler Effect

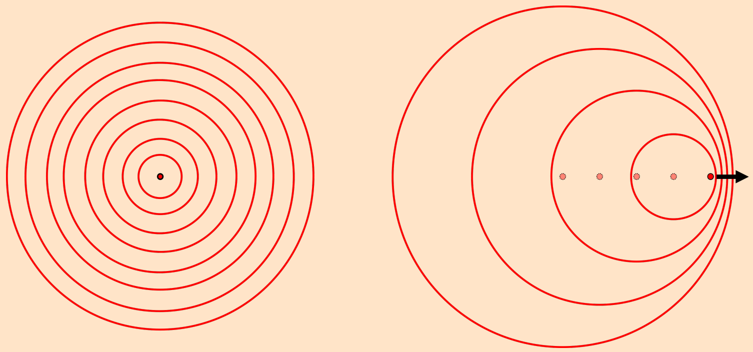


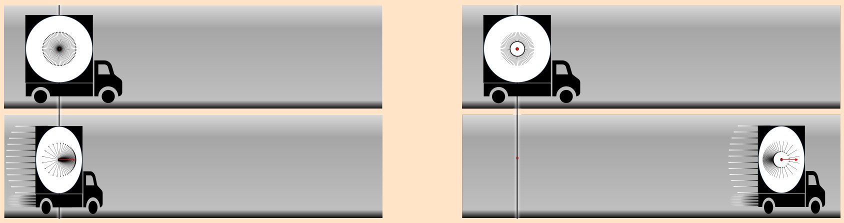
Figure 1.14: A diagram showing (left) a central source at rest emitting several circular pulses of light with equal time between each pulse, (right) the same source in a frame where it is now moving and emitting circular pulses of light, but each subsequential pulse is emitted from a different position as the source is moving, marked by a faded dot

If we have a source at rest, emitting circular pulses of light with equal times between each pulse, we will have concentric circular pulses in this frame, but if we move to a frame where the source is now moving to the right, each circular pulse is now being emitted from a different position as the source is moving. Due to the source moving, each pulse will be emitted closer to the right hand side of the previous pulses, creating a bunching up of the pulses (increase in frequency) in the direction of movement and a spreading out (decrease in frequency) of the pulses in the opposite direction, that is what happens in the classical version of the Doppler effect, that you will notice that an ambulance or police car sounds different driving towards you and when driving away, this is due to the bunching up of the sounds waves in the direction of the moving vehicle. In special relativity, we also have to take the time dilation of the pulses into account, as there will be a longer time between each subsequential circular pulse, Due to perceived time moving more slowly for objects moving relative to the observer, this has a decreasing effect on the frequency in all directions, but directly in the direction of movement of the source this is outweighed by the frequency increase from the previous bunching up effect. Since the energy of the light is proportional to the frequency it is increased in the direction of motion of the source due to the Doppler effects and decreased in the opposite direction.

One thing not mentioned yet in this picture so far is how the light is also effected by what is called the aberration, which is the change in the direction of each part of the emitted light pulse, which leads to light not being circularly symmetric in each of the spherical pulses, which will be explained in the next section.

1.8 Aberration

The picture painted of the Doppler effect in the previous section has yet to show the effect on the angular distribution of the light in each of the spherical pulses. Here we will show that there is also a higher concentration of light in the direction of the source's movement, as shown in Figure 1.15.



(a) Pulse emitted

(b) Pulse absorbed

Figure 1.15: Two diagrams of a truck with a spherical container in two different frames, a light pulse is emitted from the centre of the container (left) and reflected back by the mirrored wall to be absorbed in the centre again (right), the truck in its rest frame (top of both figures) shows a evenly distributed light outward pulse, but in the frame with the moving truck (bottom of both figures) the emitted light is now more concentrated in the direction of the truck , which is required to allow the light to be reflected and returned to the centre of the truck simultaneously.

With the help of the previous length contraction section, let us imagine a truck with a spherical container with mirrored walls, a central bulb emits light in all directions, in the rest frame all light reaches the spherical wall at the same time and returns to the center bulb simultaneously. Then in the moving frame we have the spherical container length contracted and the light moving at the same speed but reflecting off the walls at different times, returning to the centre bulb simultaneous, for this to be true, the directions of the light have to be aberrated in the way shown in the diagram to allow for this simultaneous return to the bulb.

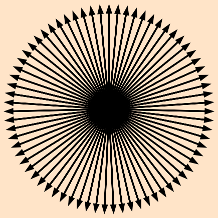
From the diagrams you can see how light is aberrated when emitted from a moving source relative to its rest frame and also when being absorbed by a source, the faster the source is moving, the more the direction of each part of the light pulse is aberrated.

If the speed of the source was to approach the speed of light, all emitted light would approach the direction of movement of the source, if the source was theoretically able to reach the speed of light, all light would be emitted in the direction of the source but also move at the same speed of the source, i.e. the source would move with the emitted light and it would not leave the vicinity of the source. Though the rate at which it emits it would also tend to zero. If a photon theoretically had mass and its influence of the gravitational force moved at the speed of light, then we would not be able to feel any gravitational effects outside the vicinity of it, due to all of its gravitational field being propagated in the direction of its movement and at the same speed as the photon.

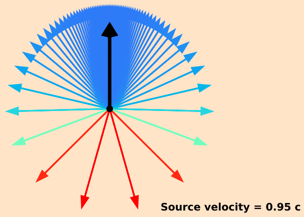
Also in astrophysics it has to be taken into account that the earths view of the universe is distorted by aberration, as shown in diagram ref... *** diagram of earths view of universe is distorted from the aberration (example on wiki) ***

1.9 Relativistic Beaming

If we now take both the Doppler picture and the aberration picture from the previous sections together, so that if we have a source that in its rest frame is emitting a spherical pulse of light with evenly distributed angles. Then in a frame with the source now moving, we will have this spherical pulse's wavelength bunched up in the direction of movement (giving a change in the colour of the light), with the angular spread of the light at a higher concentration in the direction of the movement of the source.



(a) rest frame



(b) Moving frame

Figure 1.16: Outward pulse of light from source in a rest and moving frame

(a) rest frame (b) Moving frame

Figure 1.17: Inward pulse of light from source in a rest and moving frame

1.9.1 Retarded Field

If we were to have multiple pulses with a constant time between each pulse, and remembering that each pulses origin was from a point in the sources past position (the retarded position of the source), we can work out what the full view of all the pulses in the rest frame is in the moving frame, shown in Figure 1.18. From this we can work out the distribution and hence concentration of the light for any moving source relative to that of it at rest.

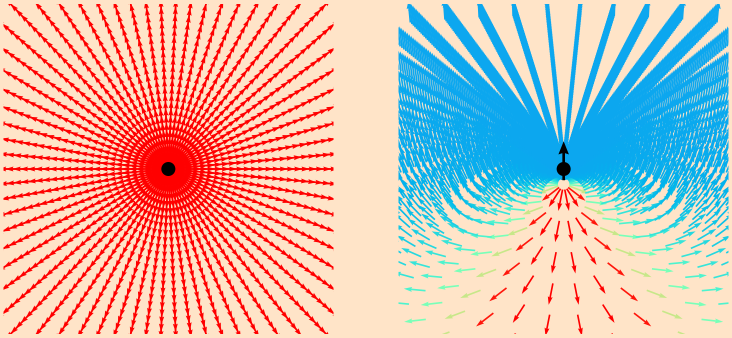


Figure 1.18: A diagram showing multiple spherically pulses of light from a source at rest that have evenly distributed angles (left), and what it becomes in a corresponding frame where the source is moving (right)

1.9.2 Perceived vs Actual Speed

Imagine that a ball one light year (the distance it takes light to travel in a year) away, is fired directly towards you. It will take one year for the first light from the oncoming ball to reach you. If the ball moves at three-quarters of the speed of light, the ball will hit you in four-thirds of a year (a year and four months) after it is fired. The last light from the ball will reach you just as the ball hits you. As you see it, the time between the first and last light from the ball is four months. During those four months, you will see the ball start at its initial position and travel a distance of one light year. So the ball appears to you to have been moving three times faster than light. This is just how it appears to the observer due to the delay in the light signal/the retarded view of the system, leading to latency on how the system is observed.

This shows how important it is to take the delay in the light from objects into account when observing relativistic systems, this view is called the delayed/retarded view, i.e what we see now is objects in their past positions, and the further things are from you the further into the past we are currently seeing them.

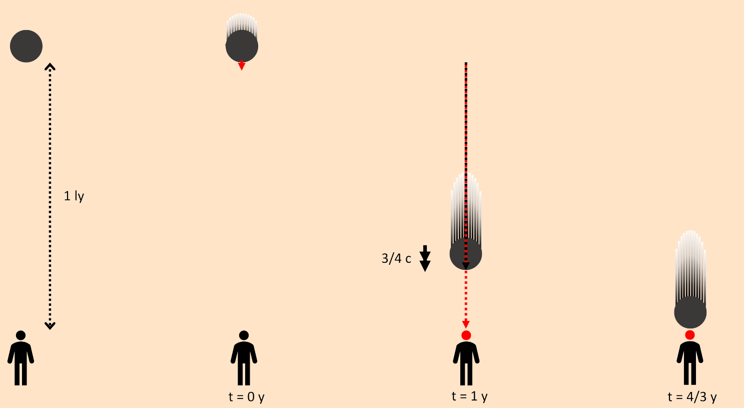


Figure 1.19: A diagram demonstrating the perceived speed of a ball vs its actual speed

1.10 Velocity Transformation

Previously, we explored how velocities at the speed of light change between different frames of reference, that is that their directions are aberrated, However, for objects moving at speeds less than light, we need the transforming speeds to be different as well as aberrated. As we need the transform to tend towards the classical transform as the speeds tend towards zero. To understand these velocity transformations, we will need to first delve into the mathematical details which will be covered in later chapters.

1.11 The More Abstract Properties...

The transformation of more abstract quantities that depend on position and time, such as momentum, energy and mass do not have a simple visual interpretation, and this is the point at which the mathematics is needed for a proper understanding of how they transform. With the maths you will find new conservation of momentum and energy laws.

1.12 Summary

A key early idea was that light travelled through a medium called the luminiferous aether. The Michelson-Morley experiment attempted to measure Earth's motion relative to this aether. Surprisingly, no difference was found in the speed of light regardless of direction. This conflicted with intuitive addition of velocities, discredited the aether theory, and led Einstein to propose light's constant speed in all frames as a fundamental principle.

Special relativity emerged from the insight that light's speed in a vacuum is constant for all observers, regardless of the lights source's speed relative to each observer. This required rethinking the concepts of time intervals and distances between points to accommodate light's fixed speed and led to requiring that a clock moving relative to an observer ticks slower from the observer's perspective while also being contracted in the direction of its motion. This was shown in figure 1.9 and figure 1.11.

It also led to the simultaneity of events not being absolute, e.g. light emitted from the middle of a moving truck reaches the front and back walls at the same time to someone at rest in the truck. But someone standing still on the road sees the light hit both walls at different times, as shown in figure 1.10. There is no universal "now" at a distance - observers relate events differently.

Classical physics provides an excellent approximation of reality at low speeds. Relativistic effects only become readily noticeable at speeds approaching light's. We don't normally experience them because motions in everyday life are very slow relative to light's which is roughly three hundred million metres per second. The faster an object moves relative to an observer, the more it's lengths contract and it's time dilates, to the observer.

When it comes to velocities, they combine differently than in classical physics. Though we must have that the transforms reduce to the intuitive addition of velocities of classical physics at low speeds while accommodating light's constant rate in all frames of reference. This leads to initially bizarre outcomes like **apparent** faster-than-light motion emerging from how relativistic optical effects play out across distances, due to the delay in light signals from objects at a distance. Nevertheless, special relativity gives a truer, more accurate picture of reality.

We also seen that lights frequencies shift when emitted from moving sources due to space-time effects on wave propagation, the Doppler effect, and that light concentrates in direction of source's motion, the aberrational effect. Both together intensely focus light and radiation in the direction of rapidly moving objects, relativistic beaming.

These effects will all be explained more fully with maths in the coming chapters, along with other effects that require the maths to fully understand, as well as new conservation laws of energy, mass and momentum. We will derive everything step by step starting with the transformation of positions and times between reference frames.

*** explain what photon sees, i.e. when its emitted it sees a universe thats infinitly thin in direction of movement and time between emition and absorption is then zero

1.13 Definitions

[**Mathematical Framework**]

Reference frame: An abstract coordinate system taken to be at rest consisting of three spatial and one time axis. The origin of its axis, its orientation, and its scale are specified by a set of points in space. The purpose of it is to provide a standardised means of defining the position of objects within that frame at any instant of time.

Inertial reference frame: A reference frame that is not undergoing any acceleration. An observer's reference frame is inertial if they do not feel a force being acted on then.

Proper/Rest frame: The Proper/Rest frame of an object, is the reference frame of the object itself.

Primed frame: A reference frame that is moving relative to the current frame.

frame velocity: The velocity of a reference frame relative to the current reference frame.

Frame transform: The changing of coordinates and other quantities in a reference frame into their corresponding values in another reference frame.

Galilean/Classical frame transform: Transformation of coordinates and physical quantities according to Newtonian physics.

Relativistic frame transform: The changing of the spatial coordinates, time coordinate, and other physical quantities into their corresponding values in a different inertial reference frame using equations from special relativity.

Observer: An entity/thing that has an associated a frame of reference/system of coordinates in which to measure the position, orientation, and other properties of objects with respect to.

Event: An occurrence that can be that can be specified by a unique combination of the three spatial and one time coordinates. These four coordinates, together, form a point in spacetime.

Simultaneity: Within a reference frame, two or more events are simultaneous if they occur at the same time in it. Classically if any two events are simultaneous in one reference frame, they are also simultaneous in any other reference frame. In special relativity, this is not true, and the order of events can depend on the frame of reference.

Retarded P position: The previous position of a source of a field that an observer sees it at at, due to the time delay between when a source emits a field and when that field is detected by the observer

[The Mechanics]

Aether: A proposed medium that filled the vacuum of space that light propagated through.

Vacuum: Empty space with no matter and negligible amount of effects from fields.

Lorentz length contraction: The shortening of an object along the direction of its motion relative to an observer (relative to the object at rest).

Time dilation: The slowing of the passing of time for objects in motion relative an observer, compared to the passing of time for the observer.

Doppler effect: The change of a wave's frequency due to the relative motion of its source to an observer. This is due to the bunching up of the wave in the direction of movement of the source and the stretching of the wave in the opposite direction, as well as the dilation of time between two parts of the emitted wave.

Aberration of light: The changing of the direction of the propagation of light when moving into another reference frame.

Relativistic beaming: The accumulation of the Doppler effect and aberration of light, and how it effects the amount of light and its frequency emitted at difference angles from a moving source. This effect can be extended to any wave or field being emitted at the speed of light from a moving source.

Retarded field: A moving source's field, where at each point has been propagated at the speed of light from the retarded position of the source...

Field:

[**Defs for later**]

Invariant: ...

Chapter 2

Positions, Time and Velocity

Now that we have learned about the basic concepts of special relativity, we can let the mathematics begin, starting with classical relativity as a starting point, it will give us an understanding of how to transform between reference frames, we can then compare special relativity to this to help with our intuition.

2.1 Classical Relativity

For classical (Galilean) relativity, if we have two observers moving relative to each other at constant velocity and want to find how each observer would describe the positions and motion of a particle relative to them, we need to know the positions and times in one observer's frame and how to transform them into the other observers frame. i.e. if we have how one observer would describe the coordinates of a particle over time relative to them and we want to now know the coordinates of the particle with respect to the other observer, we need to know how the transformation of coordinates works.

Before special relativity, it was assumed that distances between two positions was the same for all observers, i.e. the first observer would see a stick and measure it to be the same length as another observer moving relative to them would measure it to be. long story short, lengths are the same for all observers in Galilean transform. Another assumption is that time is constant for everyone, i.e. clocks move at same rate for each observer and if a particle was to emit light at a particular time. This light would be emitted at the same time for the other observers too.

So lets see how the mathematics works for the classical coordinate system transform between two observers. If we have the coordinate system in one inertial reference frame $< S >$ to describe an event E, which is given by the set of spatial and time coordinates (x, y, z) and t, and we want to find the coordinates of the event in another "primed" inertial reference frame/coordinate system that is moving at a relative speed v to the first frame in the z-axis, that was initially overlapping with the first frame at time $t = 0$, with the events coordinates now described by x', y', z' at time t'. The direction of second frame is in the upwards z-direction instead of being to the right in the case of the truck diagrams from the previous chapter, this to help us when we use spherical polar coordinates later and to help see the symmetries.

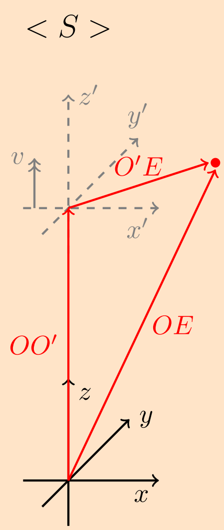


Figure 2.1: Diagram, of a reference frame $< S >$ with its associated (x, y, z) coordinate system at rest in this frame, it also shows a primed coordinate axis (x', y', z') at time $t = t'$ from when the axis where overlapping, associated with another inertial reference frames moving at velocity v in the z-axis, it shows an event with position vector $\overrightarrow{OE}$ in this frame and the vector $\overrightarrow{OO'}$ to this primed axis origin and the vector $\overrightarrow{O'E}$ which is vector from this primed origin to the event in the shown frame.

For frame $< S >$ we have it's origin O and an event E located at

$$\overrightarrow{OE} = (x, y, z) \tag{2.1}$$

at time t from axis overlap, and a prime frame $< S' >$'s origin O' moving at relative velocity v, so that in the original frame we have after a time

$$\overrightarrow{OO'} = (0, 0, vt) \tag{2.2}$$

in $< S' >$ we have the event's coordinates as

$$\overrightarrow{O'E} = (x', y', z') = \overrightarrow{OE} - \overrightarrow{OO'} = (x, y, z - vt) \tag{2.3}$$

for frame $< S >$ we have

$$\overrightarrow{OE} = (x, y, z) = \overrightarrow{OO'} + \overrightarrow{O'E} = (x', y', z' + vt') \tag{2.4}$$

so we have $x = x'$, $y = y'$, $t = t'$ (time is assumed to be the same across reference frames in classical relativity) and for the z-components

$$\begin{aligned} z' &= z - vt \\ z &= z' + vt' \end{aligned} \tag{2.5}$$

This shows the symmetry between the frames, as if we where originally in the $< S' >$ frame we would have the $< S >$ frame moving at -v relative to it, so we would expect that the inverse transformation would replace one frames coordinates with the other and have the the opposite frame velocity instead. So the inverse of going from primed frame to original is just a matter of swapping all primed coordinates with original coordinates and changing the sign on the frame velocity, this can be seen from starting at time $t = 0$ we have the frames origins overlapping and that there is a symmetry from here, we have the freedom to choose either frame to be the original, as no frame is particularly special and depending which one you pick the relative velocity is either v or -v.

The full description of the position and time transform for an event is given as

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= z - vt \\ \text{at time} & \\ t' &= t \end{aligned} \tag{2.6}$$

2.2 Postulates of Special Relativity

1. The laws of physics are the same in all inertial frames of reference (i.e. if an observer carries out an experiment in an inertial reference frame, there would be no difference if the experiment was instead carried out within a different inertial reference frame)

2. The speed of light in a vacuum is the same in all inertial/non-accelerating frames of reference.

They are not the only assumptions, you may notice some small jumps in the logic when it comes to the derivation. But be aware small miss understandings have thrown people down the special relativity rabbit hole when trying to find inconsistencies or prove it wrong, through out this I will try to leave as few logical gaps as possible, and explain as simply and as in depth as possible so that you do not do this yourself.

2.3 Derivations

We will be following a similar thought pattern as in the conceptual introduction for the derivations, an alternative derivation using spherical light pulses is given in Appendix D.7.

2.3.1 Time Dilation

We can have a clock which keeps time using a light pulse being emitted from a bulb towards a mirror and back again, recording a tick on its clock each time it returns to were it was emitted, taking a time $\Delta\tau$ in the rest frame to return. The distance the light travels to the mirror and back in this time is

$$2W = c\Delta\tau, \tag{2.7}$$

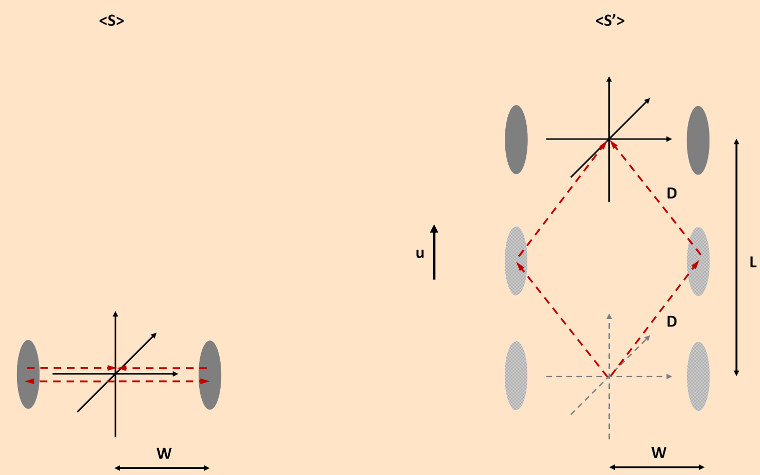


Figure 2.2: A diagram showing a clock keeping time using light, light is emitted from the origin and reflected off a mirror back to the origin, shown here in two different frames (describe what things are in diagram)(need axis labels)

the distance the bulb travels during this in the moving frame is

$$L = u\Delta t' = -v\Delta t', \tag{2.8}$$

where $\Delta t'$ is the time it took light to return in moving frame and u is the speed of the bulb which is in the opposite direction of the primed frame relative to the proper frame, i.e. $u = -v$, the distance light travels in this time is

$$2D = c\Delta t', \tag{2.9}$$

this forms a triangle, so we have the square of the hypotenuse $2D$ equal to the square of the lengths W and L:

$$c^2\Delta t'^2 = v^2\Delta t'^2 + c^2\Delta\tau^2 \tag{2.10}$$

rearranging to find the time it took in the primed frame relative to the time in the proper frame, we find

$$\Delta t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta\tau = \gamma\Delta\tau \tag{2.11}$$

with

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2.12}$$

which means

$$\Delta t' \geq \Delta\tau \tag{2.13}$$

So we take this outcome as the time taken between two events in the same position in the moving frame taking longer than in the rest frame of the clock, due to the longer distance it has to travel at the same speed.

2.3.2 Length Contraction TODO

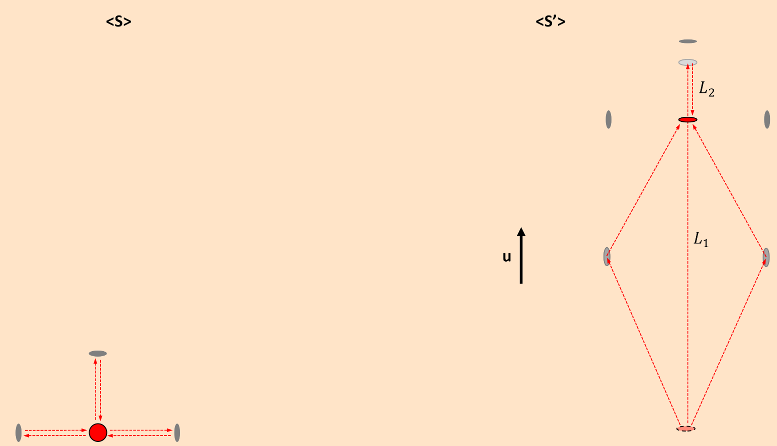


Figure 2.3: A diagram, showing a central bulb emitting light in the three directions, with all light being reflected by the mirrors back to the centre, this is shown at rest on the left and moving in another inertial frame with the system now moving, with its length slightly contracted in the direction of movement.

If we have a system in its rest frame with light emitted in three directions from the centre, one directly infront and the other two to the sides, so that it will bounce off the mirrors return to the centre simultaneously. we require that in the moving frame they also all return to the centre simultaneously, as multiple events that happen at a single point simultaneously in one frame, must happen simultaneously in all other frames. This time between light being emitted and absorbed will be the dilated time from previous section. To achieve this simultaneity in the return of the light to the bulb, the length of the full path of light from there centre to the mirrors and back in each direction has to be the same (as its speed of light is same in all directions and the return time has to be the same), we can work out the length of the sideways paths from the time dilation section, and this is the length the path needs to be in the vertical directions as well, the paths can only have this length if the system length is contracted when moving in the direction of movement. Noting that there is no contraction in the perpendicular direction to the movement due to the paradox explained in chapter 1.6.4.

The pulse sent forward has a distance

$$L_1 = ct'_1 = D' + ut'_1 \tag{2.14}$$

to travel to get to the mirror that is moving away from the point it was emitted, where t'_1 is the time taken to get there and D' is the distance between the mirror in front and the bulb and u being the speed the bulb and mirrors are moving in the primed frame. After this the light then has to travel back to the bulb a distance

$$L_2 = ct'_2 = D' - ut'_2 \tag{2.15}$$

where t'_2 is the time taken to return to the bulb from the mirror, both of these give the two times as

$$t'_1 = \frac{D'}{c-u} \tag{2.16}$$

and

$$t'_2 = \frac{D'}{c+u} \tag{2.17}$$

adding both times together gives the total primed time

$$t' = D'(\frac{1}{c+u} + \frac{1}{c-u}) = \frac{D'}{c}\frac{1}{1-u^2/c^2} = \frac{D'}{c}\frac{1}{1-v^2/c^2} = \gamma^2\frac{D'}{c} \tag{2.18}$$

from the time dilation equation we have

$$ct' = c\gamma t = \gamma D = \gamma^2 D' \tag{2.19}$$

and finally leading to the contracted distance to the front mirror in the primed frame being

$$D' = \frac{1}{\gamma}D \tag{2.20}$$

so we have

$$D' \leq D \tag{2.21}$$

we take from this that distance in the direction of movement of an object are contracted in its direction of movement compared to the rest frame of the object.

2.3.3 Position and Time Transformation

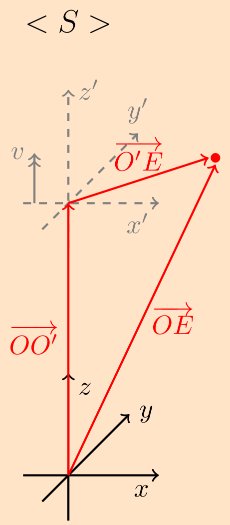


Figure 2.4: A diagram showing an event in a frame with another primed frame's axis shown moving in the z-direction.

If we are in a frame $< S >$ with origin O at position $(0, 0, 0)$ and a primed frame $< S' >$ moving in the z-direction at velocity v such that the position of the primed origin in this frame is

$$\overrightarrow{OO'} = (0, 0, vt) \tag{2.22}$$

at time t from when the origins overlapped, if there is an event (lets say a light or another object reaching a mirror that is moving with the primed frames axis) described in this frames coordinates as

$$E = (x, y, z, t) \tag{2.23}$$

with the spatial coordinates shown in the diagram as

$$\overrightarrow{OE} = (x, y, z) \tag{2.24}$$

in the primed frame we have the event described as

$$E' = (x', y', z', t') \tag{2.25}$$

now in the current frame we have the primed frame's axis moving with the mirror, from previous chapter we should have this primed distance in the z-direction to the event contracted, as the primed z-coordinates are contracted in this frame, as seen from length contraction section; the events z-coordinate is $z = z'/\gamma$, so that distance between the primes origin and the event in this frame (not the primed frame) is

$$\overrightarrow{O'E} = (x, y, z'/\gamma) \tag{2.26}$$

noting that we have the x and y-components unaffected due to the cannon and ball thought experiment from the first chapter. from the diagram we can see

$$\overrightarrow{OE} = \overrightarrow{OO'} + \overrightarrow{O'E} \tag{2.27}$$

subbing in each component into this equation for the z-direction and rearranging, we have

$$z' = \gamma(z - vt) \tag{2.28}$$

If we were instead in frame $< S' >$ we would have $\overrightarrow{OE} = (x, y, z/\gamma)$, $\overrightarrow{OO'} = (0, 0, vt')$, and $\overrightarrow{O'E} = (x, y, z')$, doing the same for these we have

$$z = \gamma(z' + vt') \tag{2.29}$$

Which should be expected as this shows the expected symmetry between the frames, as if we where originally in the $< S' >$ frame we would have the $< S >$ frame moving instead at -v in the z-direction relative to the $< S' >$ frame, so we would expect that the inverse transformation would replace one frames coordinates with the other and have the the negative of the velocity instead.

Now substituting the z transforms into eachother we have

$$z = \gamma(\gamma(z - vt) + vt') \tag{2.30}$$

and rearranging gives

$$t' = \gamma\left(\left(\frac{1}{\gamma^2} - 1\right)\frac{z}{v} + t\right) \tag{2.31}$$

with $\gamma^{-2} = 1 - v^2/c^2$ then giving

$$t' = \gamma\left(t - \frac{vz}{c^2}\right) \tag{2.32}$$

leaving full description of the location and time of an event as

$$\boxed{\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \\ \text{at time} & \\ t' &= \gamma\left(t - \frac{vz}{c^2}\right) \\ \text{with} & \\ \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}} \tag{2.33}$$

For the time transform the $\frac{vz}{c^2}$ term is the cause of simultaneous events in one inertial frame no longer being simultaneous in another inertial frame, and the gamma factor γ, is the term that causes the overall change in how fast time flows relative in each frame. In the spatial coordinate transforms the gamma factor is the cause of the change of lengths in the direction of the frame's movement. How the transformation works is shown in figure 2.5.

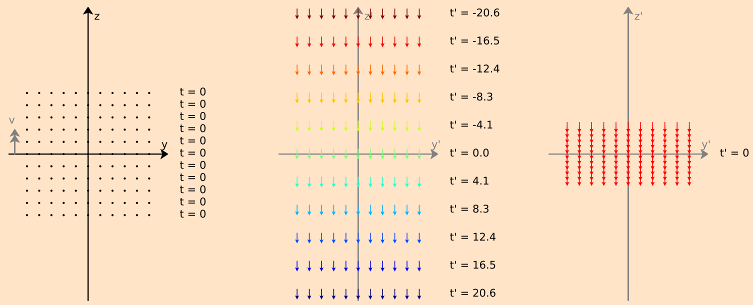


Figure 2.5: A diagram, showing a grid of points at time $t = 0$ that are at rest in their proper frame (left), giving the direct transformation into the moving primed frame (middle) where the points are now moving in the negative z-direction with the colours and labels showing the desynchronisation of the primed times t', the next figure (right) shows what these points would be when they are synchronised with the origin's time $t' = 0$.

2.3.4 Transforms with Low Relative Frame Speed Change

For the coordinate transforms in the previous section we get back the classical coordinates transform for frame velocities much smaller than the speed of light, i.e. with $v \ll c$ we have $\gamma \approx 1$ we get

$$t' \approx t \tag{2.34}$$

$$z' \approx z - vt \tag{2.35}$$

which is the expected result for the classical coordinates transform, and what we are used to seeing things work in everyday life. The influence of the gamma factor is shown in figure 2.6.

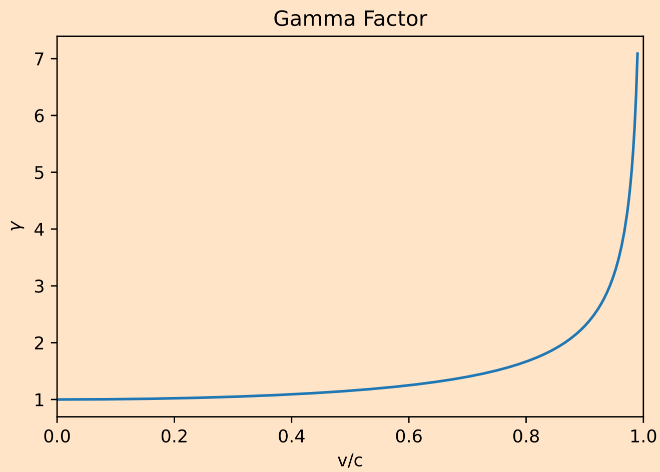


Figure 2.6: A diagram, showing the magnitude of the gamma factor with increasing ratio of frame speed to the speed of light, with γ tending to infinity as $v/c \to 1$.

2.4 Velocity

from taking the infinitesimals of equations from eq. (2.33), we have

$$\begin{aligned} dx' &= dx \\ dy' &= dy \\ dz' &= \gamma(dz - vdt) \\ dt' &= \gamma\left(dt - \frac{vdz}{c^2}\right) \end{aligned} \tag{2.36}$$

giving the velocities as

$$
\begin{aligned}
u'_x &= \frac{dx'}{dt'} = \frac{dx}{\gamma\left(dt - \frac{vdz}{c^2}\right)} \\
u'_y &= \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{vdz}{c^2}\right)} \\
u'_z &= \frac{dz'}{dt'} = \frac{\gamma(dz - vdt)}{\gamma\left(dt - \frac{vdz}{c^2}\right)}
\end{aligned}
\tag{2.37}
$$

now dividing the top and bottom of fraction by dt we can give the equations in their vector form as

$$
\mathbf{U'}_p = \frac{1}{\gamma\left(1 - \frac{v}{c^2}u_z\right)} \begin{pmatrix} u_x \\ u_y \\ \gamma\left(u_z - v\right) \end{pmatrix}
\tag{2.38}
$$

at position $\mathbf{R}'$ and time t'

the generalised velocity transform derived in appendix (todo) is

$$
\mathbf{U'}_p = \frac{1}{\gamma} \frac{\mathbf{U}_p + \left[\frac{\gamma - 1}{\|\mathbf{V}\|^2}(\mathbf{U}_p \cdot \mathbf{V}) - \gamma\right]\mathbf{V}}{1 - \frac{\mathbf{U}_p \cdot \mathbf{V}}{c^2}}
\tag{2.39}
$$

with $\mathbf{V}$ being the frames velocity in any generalised direction, instead of just being in the z-direction

2.5 Variables

E **:** an event given by the set of spatial and time coordinates. *** might be confused with energy later ***

$< S >$ **:** an inertial reference frame.

$< S' >$ **:** a second inertial reference frame moving relative to inertial frame

$< S >$.

v **:** is the velocity of second reference frame relative to the first reference frame

c **:** the speed that light travels in empty space when it is not impeded by matter and fields.

x, y, z, t **:** are the position and time coordinates of event in current inertial reference frame

x', y', z', t' **:** are the corresponding position and time coordinates the event in another inertial reference frame, referred to as the primed frame, and denoted with a $'$.

τ **:** the time of an event in the rest frame of an observer.

$\delta t, \delta t'$ **:** ...

$\delta tau, \delta tau'$ **:** ...

γ **:** a relativistic factor given by $1/\sqrt{1 - v^2/c^2}$ which determines how much of an effect the transform of the coordinates deviates from the classical transform. It is only significant at relative frame speeds close to that of light.

$\vec{U_p}$

$\vec{V}$

$\vec{u}, \vec{v}$

u_x, u_y, u_z

R_p

R

R'

$dxdydzdt$

$dx' dy' dz' dt'$

dU

$\vec{a} ax, ay, az$

$\hat{v}$

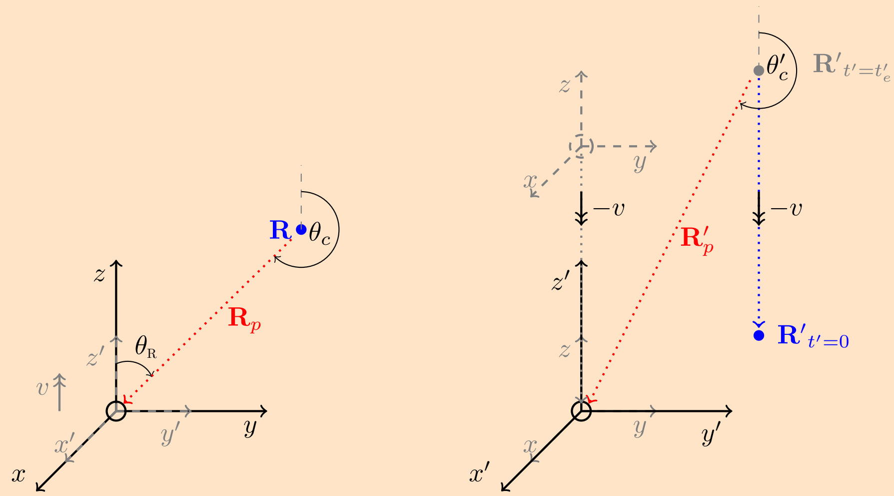
Chapter 3

Observers Delayed World View

Light travels extremely fast (roughly 300,000,000 m/s) compared to other everyday speeds we are used to. In the blink of an eye, light could travel around the world twice. We don't need to normally worry about the delay in the light signal, as the time to propagate the distances in our everyday world is very short, so the signal can be approximated as being instantaneous, therefore ignoring the more complex modeling of the delayed signal. But when it comes to particle physics or astronomy, we might be dealing with speeds close to that of light or also huge distances, e.g. light has taken 152,000 years to travel from the Andromeda galaxy, so we are currently seeing this galaxy where it was and how it looked 152,000 years ago. These past positions that we currently see are called the retarded positions/delayed view positions.

3.1 Relativistic Observers Delayed view

When looking at moving objects we see them where they were in their past positions, due to the delay in the light signal, so let us talk about the transform showing a source particle emitting a photon to an observer, showing the retarded position the observer sees.



(a) Source and observer's rest frame (b) primed frame

Figure 3.1: Diagram of a source particle (blue) emitting light (red), which is absorbed by an observer positioned at the origin at time $t = t' = 0$ with the observer and source at rest in the first inertial frame (left) and then shown moving in a primed frame (right).

system:

If we take the rest frame of an observer and a particle that emits a pulse of light from its position at $\mathbf{R}$ so that it is received by the observer at the origin, at time $t = 0$.

The time the pulse of light was emitted is $T_e = -\frac{|\mathbf{R}|}{c}$ (were $\frac{|\mathbf{R}|}{c}$ is the time it takes light to propagate along $\mathbf{R}$ to the origin), and the velocity of that light that is propagating towards the origin is

$$\mathbf{c} = c \begin{pmatrix} 0 \\ \sin\theta_c \\ \cos\theta_c \end{pmatrix} \tag{3.1}$$

giving the emitted position as

$$\mathbf{R} = cT_e \begin{pmatrix} 0 \\ \sin\theta_c \\ \cos\theta_c \end{pmatrix} \tag{3.2}$$

the position vector and light's velocity are in opposite directions which can be seen due to the emitted time being negative. If you wish to have this in terms of the position vector angle from the z-axis you can use the equation for the angles $\theta_{\mathrm{R}} = \theta_c - \pi$. Now inputting the delayed positions and times (with origin time $t = 0$) in the first frame into the coordinate transform we get the delayed position in the primed frame as

$$\mathbf{R}' = cT_e \begin{pmatrix} 0 \\ \sin\theta_c \\ \gamma\left(\cos\theta_c - \frac{v}{c}\right) \end{pmatrix} \tag{3.3}$$

at time

$$T'_e = \gamma(1 - \frac{v}{c}\cos\theta_c)T_e$$

we can see that this does agree with the light's velocity transform as well, as it is in the direct opposite direction to $\mathbf{R}$ as it should be. With the light being received at the origin of the primed frame axis at time $t' = 0$, this is the delayed position, if you wanted to find the position of where the particle actually is, we would need to propagate this to the current time of zero of the observer, given by

$$\mathbf{R}'_{t'=0} = \mathbf{R}' - \mathbf{v}T'_e = cT_e \begin{pmatrix} 0 \\ \sin\theta_c \\ \frac{1}{\gamma}\cos\theta_c \end{pmatrix} \tag{3.4}$$

3.2 Coordinates delayed view and distribution

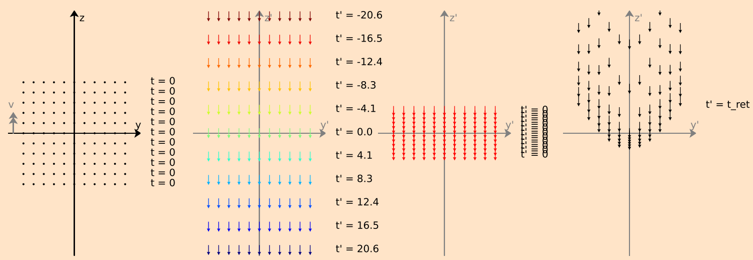


Figure 3.2: A diagram, showing a grid of points at time $t = 0$ that are at rest in their proper frame (left), giving the direct transformation into the moving primed frame (second) where the points are now moving in the negative z-direction with the colors and labels showing the desynchronization of the primed times t', the next figure (third) shows what these points would be when they are synchronized with the origin's time $t' = 0$, the last figure (fourth) shows the same primed frame $< S' >$ but now with the particles positioned where they would be perceived to be by an observer at the origin with the observers time $t' = 0$, these positions are due to the delay in the light from these particles taking time to get to the observer, the retarded positions of the particles.

How the delayed view of coordinates is taken into account after a transformation is shown in Figure 3.2:

subfigure 1: Shows an inital reference frame which is the rest frame of particles laid out in a grid at time $t = 0$ to represent a grid of coordinates, with observer at origin.

subfigure 2: shows primed frame $< S' >$ (primed frame velocity is in the z-direction, $v = 0.9$), with all coordinates transformed using Lorentz transformations to the positions at transformed time time $t' = -\gamma \frac{zv}{c^2}$, and now have a velocity of $-v = -0.9$

subfigure 3: shows the same primed frame $< S' >$ but all particles are where they would be positioned at the observers time $t' = 0$, i.e propagated forward or backwards from their the transformed positions until they are they share the same time at the observer at the origin $t' = 0$

subfigure 4: Shows the same primed frame $< S' >$ but now with the particles positioned where they would be perceived to be by an observer at the origin with the observers time $t' = 0$, these delayed view positions are due to the delay in the light from these particles taking time to get to the observer, these are also known as the retarded positions of the particles.

* could include plot of larger grid of points to show distribution of delayed view of particles better, or give heat map of desity of delayed view coordinates

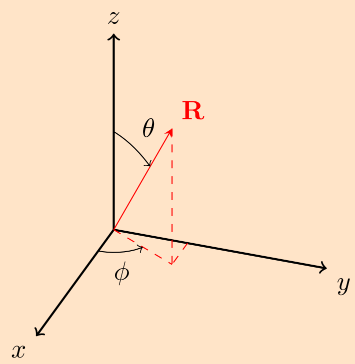
3.2.1 Delayed Coordinate Distribution

wanna get the relative delayed view proper coordinate density in primed frame, there should also be a rotational aspect to it due to the non uniform distribution of delayed proper coordinates

Chapter 4

Spherical light pulses

4.1 Spherical Polar Coordinates



For a given coordinate $\mathbf{R}$ we have it shown how to get spherical polar coordinates (r, θ, ϕ) from the Cartesian coordinates (x, y, z), where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ are the angles from the Z-Axis and X-axis respectively, and r is the distance from the origin, i.e. the magnitude

$$\boxed{\mathbf{R} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}} \tag{4.1}$$

with $r = \sqrt{x^2+y^2+z^2}$, $\phi = \arctan\frac{y}{x}$, and $\theta = \arccos\frac{z}{r} = \arctan\frac{\sqrt{x^2+y^2}}{z}$

with the unit vectors for the spherical polar coordinates

$$\hat{r} = \begin{pmatrix} \sin\theta\,\cos\phi \\ \sin\theta\,\sin\phi \\ \cos\theta \end{pmatrix}, \hat{\theta} = \begin{pmatrix} \cos\theta\,\cos\phi \\ \cos\theta\,\sin\phi \\ -\sin\theta \end{pmatrix}, \hat{\phi} = \begin{pmatrix} -\,\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \tag{4.2}$$

$$\begin{aligned} \frac{\partial \mathbf{R}}{\partial r} &= \hat{r} \\ \frac{\partial \mathbf{R}}{\partial \theta} &= \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{pmatrix} = r\hat{\theta} \\ \frac{\partial \mathbf{R}}{\partial \phi} &= \begin{pmatrix} -r\sin\theta\sin\phi \\ r\sin\theta\cos\phi \\ 0 \end{pmatrix} = r\sin\theta\hat{\phi} \end{aligned} \tag{4.3}$$

4.2 Light Speed Velocity Transform

for the case where the speed is that of light, $|u| = c$, with the direction in the first frame is at an angle α from the z-axis and φ, so that in spherical polar coordinates

$$\mathbf{c} = c\begin{pmatrix} \cos\varphi\sin\alpha \\ \sin\varphi\sin\alpha \\ \cos\alpha \end{pmatrix}, \tag{4.4}$$

we then have from the velocity transform in Eq. (2.38)

$$\mathbf{c}' = \frac{c}{\gamma\left(1 - \frac{v}{c}\cos\alpha\right)}\begin{pmatrix} \sin\alpha\cos\varphi \\ \sin\alpha\sin\varphi \\ \gamma\left(\cos\alpha - \frac{v}{c}\right) \end{pmatrix} \tag{4.5}$$

at position $\mathbf{R}'$ and time t'

which still has the same speed of light (can be shown easily by taking magnitude of vector) but has rotated / aberrated propagation angle.
Speed of light: (easy mistake: speed of light is constant, not the velocity as direction of movement does change)

4.3 Aberration in Special Relativity

4.3.1 Setup of Spherical light Pulse

In the rest frame of a point source particle positioned at the origin $O = (0, 0, 0)$ and at time $t = 0$ we have a spherical pulse of light emanated from the source, which is propagated at velocity $\mathbf{c}$ (from last section) in all direction

for a time $t = T_{prop} \geq 0$, so that the position of any part of the light pulse is given as

$$\mathbf{R} = ct \begin{pmatrix} \cos\varphi \sin\alpha \\ \sin\varphi \sin\alpha \\ \cos\alpha \end{pmatrix}. \tag{4.6}$$

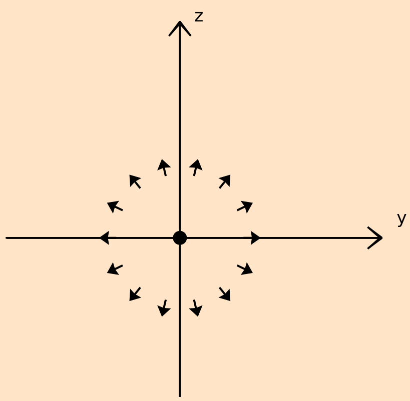


Figure 4.1: Diagram showing, position of light pulse after a certain amount time

4.3.2 Aberration

Now changing to a primed frame, moving at speed v in the z-direction, relative to the sources rest frame

The Lorentz transformations of the spacial and time coordinates gives the primed positions of the light pulse to be

$$\mathbf{R}' = ct \begin{pmatrix} \cos\varphi\sin\alpha \\ \sin\varphi\sin\alpha \\ \gamma\left(\cos\alpha - \dfrac{v}{c}\right) \end{pmatrix}, \tag{4.7}$$

at primed time

$$t' = \gamma\left(t - \frac{v}{c^2}ct\cos\alpha\right) = \gamma\left(1 - \frac{v}{c}\cos\alpha\right)t \tag{4.8}$$

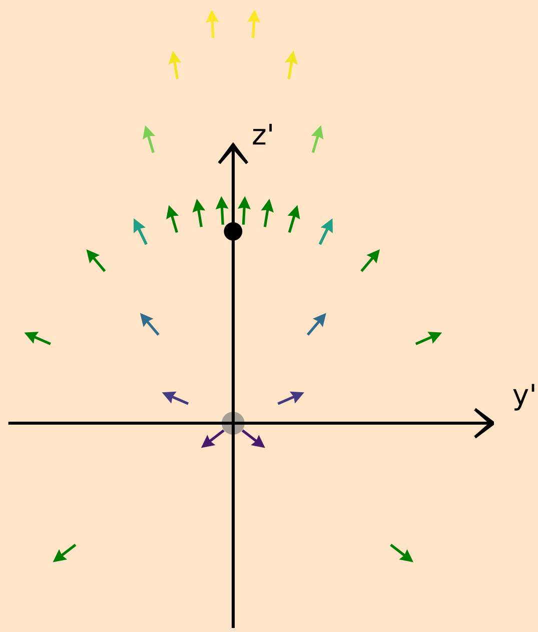


Figure 4.2: *** since have better images, these are same pulse, the elipsoid has its pulse transformed to it at different times, and the circular pulse is when you account for this and propagate each part of the pulse, to back or forward in time so that each part has the same time

*** Here you can see that this primed propagation velocity is in the same direction as the primed displacement vector, the primed frames origin is located at the location the pulse was emitted from

*** $v = -u_p$

say about particles approaching speed of light emitting light only in the direction its moving at the speed its moving

*** some diagrams with inward coming light *** and what math is changed from before

The propagation directions of the light as seen in figure (REF), are transformed such that the angles between them and the Z-Axis, have the aberrational

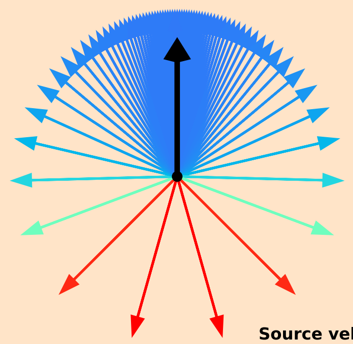


Figure 4.3: primed velocities of a evenly distributed pulse of light in rest frame of the source

relationship

$$\boxed{\cos\theta' = \frac{\mathbf{U}'_z}{\|\mathbf{U}'\|} \equiv \frac{\mathbf{R}'_z}{\|\mathbf{R}'\|} = \frac{\cos\theta + \frac{u_p}{c}}{1 + \frac{u_p}{c}\cos\theta}.} \tag{4.9}$$

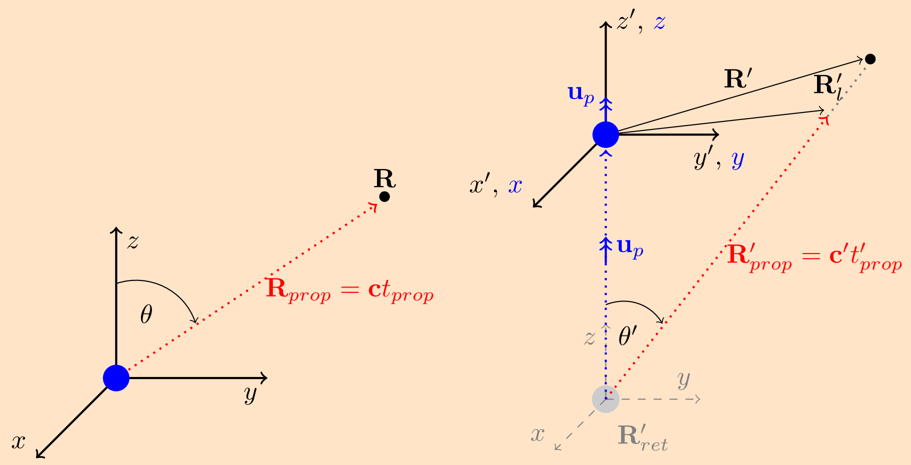
This is the relativistic aberration formula [2], it shows how the field's propagation direction transforms

4.4 Transform of Light Pulse from Origin

As visualised in figure (4.4). If we have a light source particle p, in its rest frame that had emitted a pulse of light in one such that it propagates along $\mathbf{R}_{prop} = \mathbf{c}t_{prop}$ at an angle θ from the z-axis and is currently at position $\mathbf{R}$ at time $t = 0$, where $\mathbf{c}$ is the velocity of light and t_{prop} is the time it took to propagate to $\mathbf{R}$. This system is then transformed as follows; the coordinate $\mathbf{R}$ is transformed to

$$\mathbf{R}' = \begin{pmatrix} x \\ y \\ \gamma z \end{pmatrix}, \tag{4.10}$$

the light pulse's velocity using eq. (REF) in Cartesian coordinates and



(a) proper frame (b) primed frame with moving source

Figure 4.4: *** should have particle at R in proper frame moving in primed frame and at a different position *** The diagram shows a particle p in blue at rest in the proper frame, Fig (a), with a light pulse that has propagated along $\mathbf{R}_{prop}$ shown in red, at an angle θ from the z-axis, to coordinate $\mathbf{R}$ shown in black, with the current time for whole system being $t = 0$. Fig (b), shows the system transformed to a primed frame moving at velocity $\mathbf{v} = (0, 0, v)$ in the z-direction relative to the proper frame, such that the particle and its attached proper axis in this frame are moving at velocity $\mathbf{u}_p = -\mathbf{v}$ and is currently at the origin with time $t' = 0$. when the light was emitted from the particle, they were at the retarded position $\mathbf{R}'_{ret}$ shown in grey, this light was then propagated at an angle θ' from the z-axis, along $\mathbf{R}'_{prop}$ to reach $\mathbf{R}'_l$ at $t' = 0$. it has not yet reached the corresponding Lorentz transformed coordinate $\mathbf{R}'$

particles speed in the primed frame $u_p = -v$

$$\mathbf{c}' = \frac{c}{\text{Å}} \begin{pmatrix} \frac{x}{||\mathbf{R}||} \\ \frac{y}{||\mathbf{R}||} \\ \gamma\left(\frac{z}{||\mathbf{R}||} + \frac{u_p}{c}\right) \end{pmatrix} \tag{4.11}$$

The time that the particle emits the light in the proper frame is $t_{ret} = -\frac{||\mathbf{R}_{prop}||}{||\mathbf{c}||}$ which corresponds to the time

$$t'_{ret} = \gamma t_{ret} = -\gamma \frac{||\mathbf{R}_{prop}||}{||\mathbf{c}||} \tag{4.12}$$

then in the primed frame, the retarded position at which the particle emmits the light pulse is

$$\mathbf{R'}_{ret} = -\gamma \frac{||\mathbf{R}_{prop}||}{||\mathbf{c}||} \mathbf{u}_p \tag{4.13}$$

to find the position the light pulse is currently at, at time $t' = 0$, we have to rewind its time from the position $\mathbf{R'}$ that its at when using the Lorentz transform at time $t' = -\gamma \frac{vz}{c^2} = \gamma \frac{u_p z}{c^2}$ taking away the displacement it propagated in this time to get the current position of light to be $((t'_{prop} = -t'_{ret}))$

$$\begin{aligned} \mathbf{R'}_l &= \mathbf{R'}_{ret} + \mathbf{c'} t'_{prop} = \left(\mathbf{u}_p - \mathbf{c'}\right) t'_{ret} \\ &= \gamma \frac{||\mathbf{R}||}{\text{Å}} \begin{pmatrix} \frac{x}{||\mathbf{R}||} \\ \frac{y}{||\mathbf{R}||} \\ \gamma\left(\frac{z}{||\mathbf{R}||} + \frac{u_p}{c}\right) - \text{Å}\frac{u_p}{c} \end{pmatrix} = \dots \\ &= \frac{1}{\text{Å}} \begin{pmatrix} \gamma x \\ \gamma y \\ z \end{pmatrix} \end{aligned} \tag{4.14}$$

we have the ratio of the propagation distance as

$$\frac{||\mathbf{R}'_{prop}||}{||\mathbf{R}_{prop}||} = \frac{t'_{prop}}{t_{prop}} = \gamma \tag{4.15}$$

4.5 Doppler Effect (to come back to)

(a) Light travelling relative to moving particle

(b) Light travelling relative to particle when at rest

Figure 4.5: light being absorbed by moving and rest particles

For a particle p at rest emitting wave at light speed in any direction, we have the length the wave has travelled in time $d\tau$ as being $cd\tau$. If we have a particle

p moving at speed u'_p in the z-direction and wave being emitted in all directions from it, we have the wave emitted at an angle α from the z-axis, travelling distance cdt in a time dt, with the particle moving a distance $u'_p \cos\alpha dt$ in the direction of this emitted part of the wave, this leads to a bunching up of the wave in the direction of the movement of the particle ($cdt - u'_p \cos\alpha dt = (1 - \frac{u'_p}{c}\cos\alpha)cdt$, the ratio of space the wave takes up in that direction compared to the rest frame is

$$Ratio = \frac{\gamma\left(1 - \dfrac{u'_p}{c}\cos\theta\right)cdt}{cd\tau} \tag{4.16}$$

since as wavelength is inversely proportional to the frequency, and this bunching up at the front will proportionally decrease the wavelength

$$\Phi_r = \frac{cd\tau}{\gamma\left(1 - \dfrac{u'_p}{c}\cos\theta\right)cdt} \tag{4.17}$$

with rate of change in time flow for both frames having the relation $dt = \gamma d\tau$ we get the ratio of this

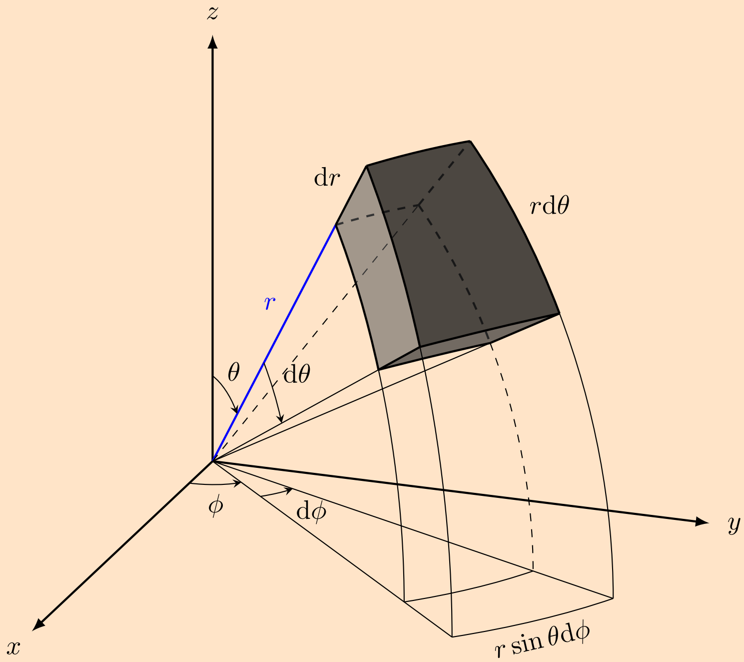
$$\Phi_r = \frac{1}{\gamma\left(1 - \dfrac{u'_p}{c}\cos\theta\right)} \tag{4.18}$$

the vector cosine relation is $\cos\alpha = \frac{\mathbf{u}'_p \cdot \mathbf{c}}{u'_p c}$, giving

$$\Phi_r = \frac{1}{\gamma\left(1 - \dfrac{\mathbf{u}'_p \cdot \mathbf{c}}{c^2}\right)} \tag{4.19}$$

4.6 Relativistic beaming or Flux from Point Source

4.6.1 Surface element



The surface element spanning from θ to $\theta + d\theta$ and ϕ to $\phi + d\phi$ on a spherical surface at (constant) radius r is then

$$\mathrm{d}S = \left\| \frac{\partial \mathbf{R}}{\partial \theta} \times \frac{\partial \mathbf{R}}{\partial \phi} \right\| \mathrm{d}\theta \, \mathrm{d}\phi = \left| r\hat{\theta} \times r\sin\theta\hat{\phi} \right| = r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{4.20}$$

The differential solid angle is

$$\mathrm{d}\Omega = \frac{\mathrm{d}S}{r^2} = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{4.21}$$

A solid angle Ω is a measure of the amount of the field of view from some particular point that a given object covers. The fraction of the field of view covered from a point is given by $\Omega/4\pi$

4.6.2 Field Flux

A proper frame of a particle's differential solid angle element (definition given in previous chapter (maybe move that subsection to here))

$$d\Omega = \sin\theta d\theta d\phi, \tag{4.22}$$

encompasses a certain amount of the wavefront, this is the same amount of the wavefront that is encompassed by the coinciding aberrated differential solid angle

$$d\Omega' = \sin\theta' d\theta' d\phi'. \tag{4.23}$$

We can calculate this element by differentiating both sides of equation (4.9) with respect to θ [3], which gives

$$\sin\theta' d\theta' = \frac{1 - \dfrac{u_p^2}{c^2}}{\left(1 + \dfrac{u_p}{c}\cos\theta\right)^2} \sin\theta d\theta = \frac{1}{\gamma^2 \left(1 + \dfrac{u_p}{c}\cos\theta\right)^2} \sin\theta d\theta \tag{4.24}$$

as $v = -u_p$ now using this and $d\phi' = d\phi$ (as the angle ϕ is always perpendicular to the motion of the particle and hence unaffected by transformation) we have the solid angle in the primed frame given as

$$\boxed{d\Omega' = \frac{1}{\gamma^2 \left(1 + \dfrac{u_p}{c}\cos\theta\right)^2} \sin\theta d\theta d\phi.} \tag{4.25}$$

The relative primed wavefront strength at a given angle is taken as being proportional to the amount of the wavefront per solid angle in the primed frame relative to that in the proper frame, referred to here as the aberrational wavefront strength weighting, given as

$$\Phi_\Omega = \frac{d\Omega}{d\Omega'} = \text{Å}^2. \tag{4.26}$$

(define what I mean by strength weighting)

4.7 Relativistic Beaming of Multiple Wave Fronts of Light

Both of the previous effects can be seen more clearly when looking at how multiple pulses of spherically evenly distributed light transform

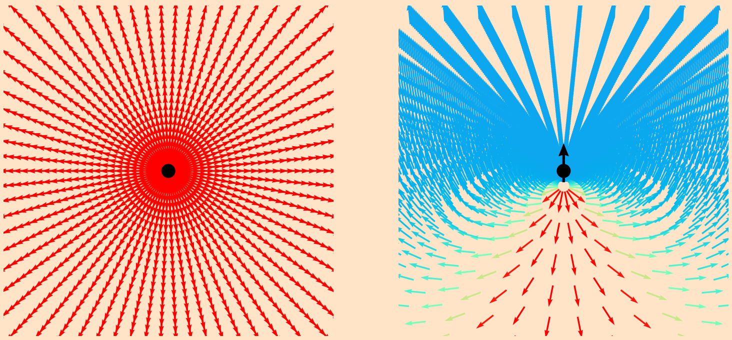


Figure 4.6: A 2D diagram showing multiple spherically symmetrical pulses of light from a source at rest (left), and in a corresponding primed inertial frame where the source is moving (right), the colours show the Doppler effect on lights light

4.8 Variables

θ, ϕ, R

r

$\hat{r}$

$\hat{\theta}$

$\hat{\phi}$

dS

$d\Omega$

Ω

$\varphi, \alpha,$

u_p

R_{prop}, t_{prop}

t_{ret}

R_{ret}

Å

Φ_r

Φ_Ω

θ_c

θ_R

$R'_{t'=0}$

T_{prop}

Chapter 5

Acceleration TODO

To find how the acceleration of a particle transforms for an observer in two different frames we take the differential of the velocity transform eq. (2.38) and use the differentiation rules for two generic functions f and g: $d(gf) = fdg + gdf$, and $d[f(g(x))] = dg(x) * df(g(x))$

$$\begin{aligned} d\mathbf{U}' &= \frac{1}{\gamma\left(1 - \dfrac{v}{c^2}u_z\right)}\begin{pmatrix} du_x \\ du_y \\ \gamma du_z \end{pmatrix} + \frac{\gamma\dfrac{v}{c^2}du_z}{\gamma^2\left(1 - \dfrac{v}{c^2}u_z\right)^2}\begin{pmatrix} u_x \\ u_y \\ \gamma\left(u_z - v\right) \end{pmatrix} \\ &= ... \\ &= \frac{1}{\gamma\left(1 - \dfrac{v}{c^2}u_z\right)^2}\begin{pmatrix} du_x + \frac{v}{c^2}(u_x du_z - u_z du_x) \\ du_y + \frac{v}{c^2}(u_y du_z - u_z du_y) \\ \frac{1}{\gamma}du_z \end{pmatrix} \end{aligned} \tag{5.1}$$

now dividing by the time differential from previous chapter to get the acceleration transform

$$\boxed{\mathbf{a}' = \frac{1}{\gamma^2\left(1 - \frac{v}{c^2}u_z\right)^3}\begin{pmatrix} a_x + \frac{v}{c^2}(u_x a_z - u_z a_x) \\ a_y + \frac{v}{c^2}(u_y a_z - u_z a_y) \\ \frac{1}{\gamma}a_z \end{pmatrix} \quad \text{at position } \mathbf{R}' \text{ and time } t'} \tag{5.2}$$

5.1 Generalised Acceleration Transform

3 acceleration (check this)

$$\mathbf{a}^{'} = \frac{1}{\gamma^3\left(1 - \frac{(\mathbf{u}\cdot\mathbf{v})}{c^2}\right)^3}\left[\gamma\left(1 - \frac{(\mathbf{u}\cdot\mathbf{v})}{c^2}\right)\mathbf{a} + \frac{\gamma(\mathbf{a}\cdot\mathbf{v})}{c^2}\mathbf{u} + (1-\gamma)(\mathbf{a}\cdot\hat{\mathbf{v}})\cdot\hat{\mathbf{v}}\right] \tag{5.3}$$

at position $\mathbf{R}'$ and time t'

Chapter 6

Invariant Quantities TODO

lorentz invariance of different 4 vectors, i.e. 4 position, 4 velocity, and 4 energy momentum vector ... If a physical quantity is lorentz invariant it means it is unchanged by a Lorentz transformation ... having conserved quantities helps us model complex systems and experiments (frame independent quantity)

Most of this is not as straight forward as previous chapters, and more hard to relate to previous coordinates, velocity and acceleration transforms as energy is a more abstract concept, but the main reason we cared about energy in classical physics is because it was a conserved quantity which is extremely important in classical mechanics, as it gives us tools to make calculations.

... we will try and find a quantity with the units of energy that will have a conservation principle associated ... the derivations to get the famous $E = mc^2$ use approximations, and assumptions and redefined definitions of energy and momentum, and the energy-momentum relation used quantum mechanics to derive, which would require another handbook to go into ... so instead we use the most important thing we need for energy to be and that is to have a associated conservation rule, which means finding an quantity that is unchanged between frames and has a quantity with the units of energy ... to do this we will start from the "4 vector" "4 position" which we already know the associated conservation rule from ... The lorentz invariant quantities are sometimes called proper time, proper position, proper acceleration.

4-vector will introduce confusion

6.1 recap the conserved quantity from space and time coordinates

We defined the space-time interval between two events:

$$\Delta S^2 = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{6.1}$$

(can be defined as negative of this) (show this is invariant for any primed coordinates) all inertial frames of reference agree on this interval (can show it always goes to proper coordinates in this form when subbed in) the proper time between two events $\Delta\tau$ is also agreed upon in all inertial frames and will also therefore agree on derivatives with respect to τ (where it is the time between two events in the same location?), because they will be invariant we have the position of an event in space-time given as a 4-position vector (what is a 4-vector)

$$\mathbf{S} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{6.2}$$

taking a derivative with respect to τ gives us the 4-velocity 4-vector with $\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma_u \frac{d}{dt}$

$$\mathbf{U} = \frac{d\mathbf{S}}{d\tau} = \gamma_u \frac{d}{dt} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \gamma_u \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} \tag{6.3}$$

know to show that the magnitude of this is invariant by showing all primed velocities give

$$||\mathbf{U}|| = c \tag{6.4}$$

The 4 momentum is defined as

$$\mathbf{P} = m_0\mathbf{U} = \begin{pmatrix} \gamma_u m_0 c \\ \gamma_u m_0 u_x \\ \gamma_u m_0 u_y \\ \gamma_u m_0 u_z \end{pmatrix} \tag{6.5}$$

we have the Taylor series expansion of γ_u factor is

$$\gamma_u = 1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + ... \tag{6.6}$$

when subbed into 4-momentum, we have the time component and multiply by c, we have

$$c\mathbf{P}_t = m_0c^2\gamma_u = m_0c^2(1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + ...) \tag{6.7}$$

$$c\mathbf{P}_t = m_0c^2 + \frac{1}{2}m_0u^2 + \frac{3}{8}\frac{u^4}{c^2} + ...) \tag{6.8}$$

all terms here are in units of energy, the second term is the classical kinetic energy, and the higher-order terms may be seen as higher order kinetic energy terms that are very small for $u << c$, the first term does not rely on the velocity of a mass, but only the mass itself, and this can be taken as a new type of energy all masses have even when at rest, i.e. the energy at rest for a mass is

$$E_{rest} = m_0c^2 \tag{6.9}$$

we will write this new energy into the 4-momentum

$$\mathbf{P} = \begin{pmatrix} \frac{E}{c} \\ \gamma_u m_0 u_x \\ \gamma_u m_0 u_y \\ \gamma_u m_0 u_z \end{pmatrix} \tag{6.10}$$

the magnitude of the 4-momentum can easily be found to be

$$||\mathbf{P}|| = m_0 c \tag{6.11}$$

another conserved quantity that does not depend on the frame and using this and the 4-momentum vector

$$m_0^2 c^2 = -\frac{E^2}{c^2} + (\gamma_u p_x)^2 + (\gamma_u p_y)^2 + (\gamma_u p_z)^2 = -\frac{E^2}{c^2} + (\gamma_u p)^2 \tag{6.12}$$

giving

$$E^2 = m_0^2 c^4 + \gamma^2 p^2 c^2 \tag{6.13}$$

for stationary particles $u = 0$

$$E = m_0 c^2 \tag{6.14}$$

this does not work for a massless particles like photons, we have to use a different method of getting this, we know from before that $\mathbf{P}^2 = m_0^2 c^2 = 0$ leading to the components giving

$$\frac{E^2}{c^2} = |p_{light}|^2 \tag{6.15}$$

and finally

$$E = c p_{light} \tag{6.16}$$

.... not needed: definition of momentum for light borrowed from quantum mechanics

$$p = \frac{hf}{c} \tag{6.17}$$

where h is Planks constant and f is the frequency of the light so that the ....

the reason we use this abstract 4-momentum, is because it is conserved and the Newtonian momentum is not conserved when it comes to special relativity

6.1.1 4-Acceleration

... this is more complex and one difference in this 4-vector here is that the spacial part of the 4-acceleration is not parallel to the 3-acceleration in a primed frame

$$\mathbf{A} = \gamma_u^4 \begin{pmatrix} -c\frac{\mathbf{a}\cdot\mathbf{u}}{c^2} \\ \frac{1}{\gamma_u^2}a_x - u_x\frac{\mathbf{a}\cdot\mathbf{u}}{c^2} \\ \frac{1}{\gamma_u^2}a_y - u_y\frac{\mathbf{a}\cdot\mathbf{u}}{c^2} \\ \frac{1}{\gamma_u^2}a_z - u_z\frac{\mathbf{a}\cdot\mathbf{u}}{c^2} \end{pmatrix} \tag{6.18}$$

this is not really the same thing as what we think of when we think of acceleration, as it does not give the rate of change of velocity in the normal sense in 3d space

$$A = \frac{dV}{d\tau} = \frac{dt}{d\tau}\cdot\frac{dV}{dt} \quad \leftrightarrow \quad \frac{dt}{d\tau}\cdot\frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u u \\ 0 \\ 0 \end{pmatrix} = \gamma_u \begin{pmatrix} \frac{u}{c}\ \gamma_u^3\ a \\ \frac{u^2}{c^2}\ \gamma_u^3\ a + \gamma_u a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{u}{c}\ \gamma_u^4\ a \\ \gamma_u^2\left(\frac{u^2}{c^2}\ \gamma_u^2 + 1\right)a \\ 0 \\ 0 \end{pmatrix} = \gamma_u^4\ a \begin{pmatrix} \frac{u}{c} \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{6.19}$$

6.1.2 4-force

$$\mathbf{F} = \frac{d\mathbf{p}}{d\tau} = m_0\mathbf{A} \tag{6.20}$$

6.2 Energy-Mass Equivalence

*** E=mc2 is derived using an approximation for v¡¡c

Imagine we have a particle at rest emitting light in all directions so that afterwards it is still at rest, but in another frame we have we have this particle moving and the emitted light aberrated, but we still need the particle to remain at a constant velocity after emission. (maybe include a rotation in the impulse/effect of light in each direction on particle of has)

- if we have a ball in its rest frame that emits a flash of light evenly in all directions, we will have the balls energy decrease by the amount of energy of the emitted light, and it will remain at rest after due to the light being emitted evenly in all directions - its kinetic energy is zero in this frame

6.3 Energy-Momentum (Equivalence)

*** Dirac derived this and used quantum mechanics to do so, which is out of this handbooks domain, still consequences of this equation to be explored, such as the "zitterbewegung"

Momentum in special relativity is defined slightly differently, to ensures that the law of conservation of momentum holds true in all inertial frames, as required by the first postulate of relativity (it is still approximately unchanged at low velocities). momentum of a particle with rest mass m_0 is given the

$$\mathbf{p}' = m_0 \frac{dR'}{d\tau} = \gamma_u' m_0 \frac{dR'}{dt'} = \gamma m_0 u' \tag{6.21}$$

where u' is the velocity of the particle in this primed frame and $\gamma_{u'} = 1/\sqrt{1 - u'^2/c^2}$. you will some times the mass of an object referred to as $m = \gamma_{u'} m_0$ which is the apparent mass of a moving particle.

(why not use 3d time differential equation here?)

6.4 Force Transform TODO

*** one of the most confusing topics in special relativity, from the way it is defined, how its derived, and how it relates to other things

*** (a note that the 3-force in special relativity is specifically defined as the time derivative of momentum $\frac{d\mathbf{p}}{dt}$ and not not mass times the acceleration $m\mathbf{a}$, this is due to newtons 2nd law holding only for the first definition in special relativity)

*** This means that acceleration turns out not to be in the same direction as the force in special relativity

force derivation is confusing as most sources give the time derivative of u and v as acceleration, (are they using each force has opposite and equal force, but that would not take into account the retardedness)

$$\mathbf{a} = \frac{1}{m_0\gamma(\mathbf{v})}\left(\mathbf{F} - \frac{(\mathbf{v}\cdot\mathbf{F})\mathbf{v}}{c^2}\right) \tag{6.22}$$

$$\mathbf{F} = \gamma^3 m_0\,\mathbf{a}_\parallel + \gamma m_0\,\mathbf{a}_\perp \tag{6.23}$$

6.4.1 not sure what this is

The force parallel to the velocity of the primed frame is unchanged, $\mathbf{F}_{\parallel\langle'\rangle} = \mathbf{F}_\parallel$, and the force perpendicular to this is $\mathbf{F}_{\perp\langle'\rangle} = \gamma\mathbf{F}_\perp$, with total force, $\mathbf{F} = \mathbf{F}_\parallel + \mathbf{F}_\perp$ and

$$\mathbf{F}_\parallel = \frac{(\mathbf{F}\cdot\mathbf{v})}{|\mathbf{v}|^2}\mathbf{v} \tag{6.24}$$

and substituting this into formula for total force:

$$\mathbf{F}_\perp = \mathbf{F} - \frac{(\mathbf{F}\cdot\mathbf{v})}{|\mathbf{v}|^2}\mathbf{v} \tag{6.25}$$

since $\mathbf{F}_{\langle'\rangle} = \mathbf{F}_{\parallel\langle'\rangle} + \mathbf{F}_{\perp\langle'\rangle}$ we can use equations above to get:

$$\begin{aligned}\mathbf{F}_{\langle'\rangle} &= \frac{(\mathbf{F}\cdot\mathbf{v})}{|\mathbf{v}|^2}\mathbf{v} + \gamma\left(\mathbf{F} - \frac{(\mathbf{F}\cdot\mathbf{v})}{|\mathbf{v}|^2}\mathbf{v}\right)\\ &= \gamma\mathbf{F} + (1-\gamma)\frac{(\mathbf{F}\cdot\mathbf{v})}{|\mathbf{v}|^2}\mathbf{v}\end{aligned} \tag{6.26}$$

this is the same for electric field as it only differs by a magnitude to the force. this is similar to how we get the reverse transform: (is it the primed velocity used in this?)

$$\mathbf{F} = \frac{1}{\gamma}\mathbf{F}_{\langle'\rangle} + (1-\frac{1}{\gamma})\frac{(\mathbf{F}_{\langle'\rangle}\cdot\mathbf{v}_{\langle'\rangle})}{|\mathbf{v}_{\langle'\rangle}|^2}\mathbf{v}_{\langle'\rangle} \tag{6.27}$$

6.5 Variables

δS

$\vec{S}$

γ_u

m_0

$\vec{P}_t$

E_{rest}

E

$\vec{P}$

p_{light}

p, h, f

$\vec{A}$

$\vec{F}$

$\cdots_{parralel} \cdots_{perp}$

$\vec{F}_{<'>}$

Chapter 7

equation for motion in primed frame for arbitrary proper frame motion

Chapter 8

None Point Like Sources

... and seeing moving objects rotated, i.e. can see side the rear side of cube before the rear side has moved past you

Chapter 9

Thought Experiments

Bells Paradox and others

Chapter 10

Applications

e=mc2 and kinematics of relativistic classical particles and that....

... touch on GR, i.e. elevator thought experiment

Chapter 11

Steps for Frame Swapping

steps to take when going between frames

Appendix A

Overview of Aberration equations

A.1 Aberration Equations

A.2 Vectors

$$\vec{R}'_{ret} = \begin{pmatrix} x \\ y \\ \gamma\left(z + \frac{u_p \cdot \|\vec{R}\|}{c}\right) \end{pmatrix} = \|\vec{R}\| \begin{pmatrix} \frac{x}{\|\vec{R}\|} \\ \frac{y}{\|\vec{R}\|} \\ \gamma\left(\frac{z}{\|\vec{R}\|} + \frac{u_p}{c}\right) \end{pmatrix}. \tag{A.1}$$

$$\hat{\vec{U}}' = \hat{\vec{R}}'_{ret} = \frac{1}{\text{Å}} \begin{pmatrix} \frac{x}{\|\vec{R}\|} \\ \frac{y}{\|\vec{R}\|} \\ \gamma\left(\frac{z}{\|\vec{R}\|} + \frac{u_p}{c}\right) \end{pmatrix}, \tag{A.2}$$

$$\|\vec{R}'_{ret}\|^2 = \text{Å}^2 \|\vec{R}\|^2 \tag{A.3}$$

A.3 Trig

$$\text{Å} = \gamma\left(1 + \frac{u_p}{c}\cos\theta\right) = \gamma\left(1 - \beta\cos\theta\right) = \frac{1}{\gamma\left(1 + \beta\cos\theta'\right)} \tag{A.4}$$

$$\cos\theta' = \frac{\cos\theta + \dfrac{u_p}{c}}{1 + \dfrac{u_p}{c}\cos\theta} = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \tag{A.5}$$

$$\cos\theta = \frac{\cos\theta' - \dfrac{u_p}{c}}{1 - \dfrac{u_p}{c}\cos\theta'} = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \tag{A.6}$$

$$\sin\theta = \frac{\sin\theta'}{\gamma\left(1 + \beta\cos\theta'\right)} = \text{Å}\sin\theta' \tag{A.7}$$

A.4 Differential

$$d\theta' = \frac{1}{\text{Å}}d\theta \tag{A.8}$$

Appendix B

Generalised Lorentz Vector Transformations

B.1 Lorentz boost

The general transformation of the coordinates from the initial((proper?) frame, $\vec{X}_0 = (ct_0, x_0, y_0, z_0)$, to coordinates, $\vec{X}_{\langle ' \rangle}$ in frame moving at $\vec{v} = (v_x, v_y, v_z)$ is given by; $\vec{X}_{\langle ' \rangle} = \vec{B}(\vec{v})\vec{X}_0$, where $\vec{B}(\vec{v})$ is:

$$\vec{B}(\vec{v}) = \begin{pmatrix} \gamma & -\frac{\gamma v_x}{c^2} & -\frac{\gamma v_y}{c^2} & -\frac{\gamma v_z}{c^2} \\ -\frac{\gamma v_x}{c^2} & 1 + (\gamma - 1)v_x^2 & (\gamma - 1)v_y v_x & (\gamma - 1)v_x v_z \\ -\frac{\gamma v_y}{c^2} & (\gamma - 1)v_x v_y & 1 + (\gamma - 1)v_y^2 & (\gamma - 1)v_y v_z \\ -\frac{\gamma v_z}{c^2} & (\gamma - 1)v_x v_z & (\gamma - 1)v_y v_z & 1 + (\gamma - 1)v_z^2 \end{pmatrix} \tag{B.1}$$

Lorentz boost must only be used to transform from proper frame to primed frame??????? what is the inverse in this case?

B.2 Coordinate Transform

Starting with position vector $\vec{r} = (x, y, z)$ and $\vec{r} = \vec{r}_\perp + \vec{r}_\parallel$, where $\vec{r}_\perp$ is component perpendicular to velocity of primed frame, $\vec{v} = (v_x, v_y, v_z)$, in which you wish to transform into, and $\vec{r}_\parallel$ is the component that is parallel to this velocity. We then have from the one dimensional Lorentz transformation:

$$\vec{r}_{\parallel\langle'\rangle} = \gamma(\vec{r}_\parallel - \vec{v}t) \tag{B.2}$$

and

$$\vec{r}_{\perp\langle'\rangle} = \vec{r}_\perp \tag{B.3}$$

where

$$\vec{r}_\parallel = \left(\hat{\vec{v}} \cdot \vec{r}\right)\hat{\vec{v}} = \left(\frac{\vec{v} \cdot \vec{r}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} \tag{B.4}$$

and

$$\vec{r}_{\perp\langle'\rangle} = \vec{r}_\perp = \vec{r} - \vec{r}_\parallel \tag{B.5}$$

now using these and $\vec{r}_{\langle'\rangle} = \vec{r}_{\perp\langle'\rangle} + \vec{r}_{\parallel\langle'\rangle}$ we have:

$$\begin{aligned}
\vec{r}_{\langle'\rangle} &= \gamma(\vec{r}_\parallel - \vec{v}t) + \vec{r} - \vec{r}_\parallel \\
&= \vec{r} + \vec{r}_\parallel(\gamma - 1) - \gamma t\vec{v} \\
&= \vec{r} + \left[\frac{\gamma - 1}{|\vec{v}|^2}(\vec{v} \cdot \vec{r}) - \gamma t\right]\vec{v} \\
&= \begin{pmatrix} x + \left[\frac{\gamma - 1}{|\vec{v}|^2}(\vec{v} \cdot \vec{r}) - \gamma t\right]v_x \\ y + \left[\frac{\gamma - 1}{|\vec{v}|^2}(\vec{v} \cdot \vec{r}) - \gamma t\right]v_y \\ z + \left[\frac{\gamma - 1}{|\vec{v}|^2}(\vec{v} \cdot \vec{r}) - \gamma t\right]v_z \end{pmatrix}
\end{aligned} \tag{B.6}$$

$$d\vec{r}_{\langle'\rangle} = d\vec{r} + \left[\frac{\gamma - 1}{|\vec{v}|^2}(\vec{v} \cdot d\vec{r}) - \gamma dt\right]\vec{v} \tag{B.7}$$

and for time transformation we have: !!!! derive !!!!!!! or state use of $\vec{v}\cdot\vec{r}$

$$\begin{aligned} t_{\langle\prime\rangle} &= \gamma\left(t - \frac{\vec{v}\cdot\vec{r}}{c^2}\right) \\ &= \gamma t - \frac{\gamma}{c^2}(\vec{v}\cdot\vec{r}) \end{aligned} \tag{B.8}$$

$$dt_{\langle\prime\rangle} = \gamma dt - \frac{\gamma}{c^2}(\vec{v}\cdot d\vec{r}) \tag{B.9}$$

B.3 Velocity Transform

$$\begin{aligned} \vec{u}_{\langle\prime\rangle} &= \frac{d\vec{r}_{\langle\prime\rangle}}{dt_{\langle\prime\rangle}} \\ &= \frac{d\vec{r} + \left[\frac{\gamma-1}{|\vec{v}|^2}(\vec{v}\cdot d\vec{r}) - \gamma dt\right]\vec{v}}{\gamma dt - \frac{\gamma}{c^2}(\vec{v}\cdot d\vec{r})} \\ &= \frac{1}{\gamma}\frac{\vec{u} + \left[\frac{\gamma-1}{|\vec{v}|^2}(\vec{u}\cdot\vec{v}) - \gamma\right]\vec{v}}{1 - \frac{\vec{u}\cdot\vec{v}}{c^2}} \\ &= \frac{\frac{\vec{u}}{\gamma} - \vec{v} + \frac{1-\frac{1}{\gamma}}{|\vec{v}|^2}(\vec{u}\cdot\vec{v})\vec{v}}{1 - \frac{\vec{u}\cdot\vec{v}}{c^2}} \end{aligned} \tag{B.10}$$

B.4 Paradoxes and confusions: (possibly for another video)

Before the discovery of Lorentz transformations the force was the same when it was defined by the multiplication of mass times acceleration or rate of change of momentum, but these two things are not equivalent when it comes to special relativity

So the force in special relativity was chosen to be by definition the later explanation (the rate of change of momentum)

This leaves us with the force not being in the same direction as the acceleration of an object in an inertial frame apart from some special cases, (i.e force and acceleration are in the direction velocity of the frame that we are transforming into)

Appendix C

Velocity Transform for propagation at $-c$

If say the field is moving in the opposite direction than we've been calculating so far, then we can just substitute c with $-c$ in the weighting term to get total weight of both M and W to be

$$MW = \frac{1}{\gamma^3}\frac{1}{\left(1 + \frac{v_{proper\langle ' \rangle}}{c}\cos\theta_{\langle ' \rangle}\right)^3} \tag{C.1}$$

which is the same as flipping the effect through the X,Y - Plane

Appendix D

Two Consecutive Velocity Transforms

using the velocity transforms from before we transform a velocity to a primed frame

$$
\begin{aligned}
\vec{U}_{\langle\prime\rangle} &= \frac{c}{\gamma_1}\frac{1}{A_1}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \gamma_1\Big(\cos\theta - \dfrac{v_{1\langle.\rangle}}{c}\Big) \end{pmatrix} \\
&= \frac{1}{\gamma_1}\frac{1}{A_1}\begin{pmatrix} U_1 \\ U_2 \\ \gamma_1\Big(U_3 - v_{1\langle.\rangle}\Big) \end{pmatrix}
\end{aligned} \tag{D.1}
$$

now we wish to transform this to a second primed frame

$$
\begin{aligned}
\vec{U}_{\langle\prime\prime\rangle} &= ... \\
&= \frac{1}{\gamma_2}\frac{1}{1 - \dfrac{v_{2\langle 1\rangle}}{c^2}U_{3\langle\prime\rangle}}\begin{pmatrix} U_{1\langle\prime\rangle} \\ U_{2\langle\prime\rangle} \\ \gamma_2\Big(U_{3\langle\prime\rangle} - v_{2\langle 1\rangle}\Big) \end{pmatrix}
\end{aligned} \tag{D.2}
$$

D.1 Acceleration Transform with consistent time (checked) ((when transformed to another frame and then back, it does not give original acceleration)) (((assumed wrong as velocity due to the time difference due to position not removed)))

differentiating Eq.((2.38)) with respect to the primed time we have the acceleration given as

$$\vec{a'}_p = \frac{d\vec{U'}_p}{dt'} = \frac{1}{\gamma}\frac{d\vec{U'}_p}{dt} = \frac{1}{\gamma}\frac{d}{dt}\left[\frac{1}{\gamma\left(1-\frac{v}{c^2}u_z\right)}\begin{pmatrix} u_x \\ u_y \\ \gamma\left(u_z - v\right)\end{pmatrix}\right] \tag{D.3}$$

... derivation to be continued

$$\vec{a'}_p = \frac{1}{\gamma^2\left(1-\frac{v}{c^2}u_z\right)^2}\begin{pmatrix} \left(1-\frac{v}{c^2}u_z\right)a_x + \frac{v}{c^2}u_x a_z \\ \left(1-\frac{v}{c^2}u_z\right)a_y + \frac{v}{c^2}u_y a_z \\ \frac{1}{\gamma}a_z \end{pmatrix} \tag{D.4}$$

substituting in $\text{Å} = \gamma\left(1-\frac{v}{c^2}u_z\right)$ we then have

$$\vec{a'}_p = \frac{1}{\gamma \text{Å}^2}\begin{pmatrix} \text{Å}a_x + \gamma\frac{v}{c^2}u_x a_z \\ \text{Å}a_y + \gamma\frac{v}{c^2}u_y a_z \\ a_z \end{pmatrix} \tag{D.5}$$

or in terms of primed velocities

$$\vec{a'}_p = \frac{1}{\gamma \text{Å}^2} \begin{pmatrix} \text{Å} a_x + \text{Å}\gamma \dfrac{v}{c^2} u'_x a_z \\ \text{Å} a_y + \text{Å}\gamma \dfrac{v}{c^2} u'_y a_z \\ a_z \end{pmatrix} = \frac{1}{\text{Å}} \begin{pmatrix} \frac{1}{\gamma} a_x + \dfrac{v}{c^2} u'_x a_z \\ \frac{1}{\gamma} a_y + \dfrac{v}{c^2} u'_y a_z \\ a_z + \dfrac{v}{c^2} u'_z a_z \end{pmatrix} \tag{D.6}$$

D.2 Previous acceleration transform

$$\vec{a'}_p = \frac{d\vec{U'}_p}{dt'} = \frac{1}{\text{Å}} \frac{d\vec{U'}_p}{dt} = \frac{1}{\text{Å}^3} \begin{pmatrix} \text{Å} a_x + \gamma \dfrac{v}{c^2} u_x a_z \\ \text{Å} a_y + \gamma \dfrac{v}{c^2} u_y a_z \\ a_z \end{pmatrix} \tag{D.7}$$

or in primed velocity terms

$$\vec{a'}_p = \frac{1}{\text{Å}^2} \begin{pmatrix} a_x + \gamma \dfrac{v}{c^2} u'_x a_z \\ a_y + \gamma \dfrac{v}{c^2} u'_y a_z \\ \gamma a_z + \gamma \dfrac{v}{c^2} u'_z a_z \end{pmatrix} \tag{D.8}$$

D.3 Positional transform with time the same at all coordinates

for a particle p moving at constant velocity $\vec{U}$ we have that its position $\vec{R'}$ above transforms the time component to a different time depending on the proper z coordinate, if we were to find the transform for time that is the same at all coordinates we would need to add the position that will be propagated through in the time $\gamma \frac{vz}{c^2}$ which in the primed frame is moving at velocity $\vec{U'}$ ((correction this is the 3 position and velocity

$$\vec{R'}_{Tconst} = \vec{R'} - \gamma \frac{vz}{c^2} \vec{U'}. \tag{D.9}$$

the generic equation for the position of a particle is

$$\vec{R}_p = \vec{R}_0 + \vec{U}_p t \tag{D.10}$$

where $\vec{R}_0$ is the initial position at time $t = 0$. Then the transform for the particle with consistent primed time is

$$\vec{R}_p' = \begin{pmatrix} x_0 + u_x t \\ y_0 + u_y t \\ \gamma\left((z_0 + u_z t) - vt\right) \end{pmatrix} + \gamma \frac{v}{c^2}\left(z_0 + u_z t\right) \begin{pmatrix} U_x' \\ U_y' \\ U_z' \end{pmatrix}. \tag{D.11}$$

D.4 Lorentz Invariant Quantities

There are certain quantities that remain the same when you move to a different reference frame, these quantities that are independent of the frame that they are measured in are called Lorentz invariant, one that we have already mentioned in the speed of light, and some others are mass of particles (which is sometimes thought of as a changing quantity) and the space time interval $\Delta s = ...$

Do we need this, since next section is on retarded position

The particles position in primed frame from the lorentz transformation is $\vec{P'} = (0, 0, -\gamma v T_{prop}) = (0, 0, -vT^{'}_{prop})$ at time

$$T^{'}_{prop} = \gamma t, \tag{D.12}$$

this is the time from emission of light pulse in the primed frame, and gives the origin of the primed axis as the retarded position of the particle corresponding to the rest frames pulse (and which coincides with the particle being at the origin at time of emission at $t' = 0$). The corresponding pulse from the particle in the primed frame is given by

$$\vec{R'}_{full} = \vec{c'} T^{'}_{prop} = \frac{\gamma ct}{\text{Å}} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \gamma\left(\cos\theta - \beta\right) \end{pmatrix}, \tag{D.13}$$

noting (might be helpful in acceleration transform but also may be nothing) here that from eq.(REF) and eq.(D.12) we have $\|\vec{R}'_{full}\| = \|\vec{R}'\|$ and from eq.(D.13) we have $\|\vec{R}'\| = \text{Å}\|\vec{R}\|$ leading to the ratio

$$\|\vec{R}'_{full}\| = \frac{1}{(1-\beta\cos\theta)}\|\vec{R}'\| \tag{D.14}$$

D.5 ...remove ... Retarded System of Light Pulse From Source at Origin

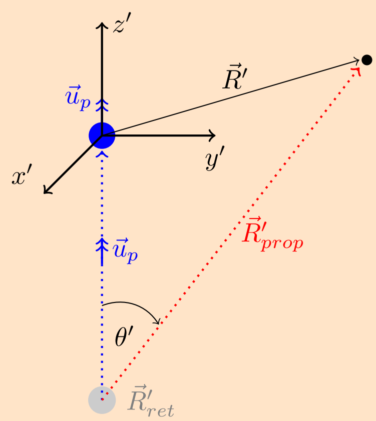


Figure D.1: The diagram shows a particle p in blue, moving at a velocity $\vec{u}_p$, currently positioned at the origin of an axis, it shows light from the source that is currently at a position $\vec{R}'$, this light has propagated from the source when it was at position $\vec{R}'_{ret}$ in grey, this light had propagated from this retarded position, along $\vec{R}'_{prop}$, shown in red, to get to its current position, at an angle of θ' to the Z-axis.

As visualised in figure (D.1). If we have a light source particle p, that is moving at velocity $\vec{u_p} = (0, 0, u_p)$ along the Z-axis and is currently positioned at the origin of an axis at time $t' = 0$ (here we are choosing to show it as a primed coordinate, but we are just preemptively doing this because it will make it easier to introduce the Lorentz transformations later, at the moment we are staying in the same frame and not special relativity is being used). then if we have light that has been propagated from the source to the coordinate $\vec{R}' = (x', y', z')$ at this time, this light would of had to been emitted from P at a previous/retarded point in time T'_{ret}, at a retarded coordinate $\vec{R'}_{ret} = \vec{u}_p T'_{ret} = (0, 0, u_p T'_{ret})$, such that the light has propagated along

$$\vec{R'}_{prop} = -\vec{c}' T'_{ret} = \vec{R'} - \vec{R}'_{ret} = \begin{pmatrix} x' \\ y' \\ z' - u_p T'_{ret} \end{pmatrix}, \tag{D.15}$$

to $\vec{R'}$.

To find this retarded time, we will start with the magnitude of the propagation displacement from previous equation which is equal to the distance the light, propagating at the speed of light, c, travels in the corresponding propagation time $T'_{prop} = -T'_{ret}$. This gives

$$\begin{aligned} \left(cT'_{prop}\right)^2 = (-cT'_{ret})^2 &= \|\vec{R'}_{prop}\|^2 \\ &= (x'^2 + y'^2 + z'^2) + u_p^2 T'^2_{ret} - 2u_p z' T'_{ret}, \end{aligned} \tag{D.16}$$

rearranging this, we get the quadratic

$$T'^2_{ret} + \left(2\gamma^2 \frac{u_p}{c^2} z'\right) T'_{ret} - \frac{\gamma^2}{c^2}(x'^2 + y'^2 + z'^2) = 0, \tag{D.17}$$

where $\gamma = (1 - u_p^2/c^2)^{-1/2}$, we may be preemptively using the constant γ here but we have still not used any special relativity. Now taking the solution for the past time (negative solution) using the quadratic formula, and making use of the identity

$$\gamma^2 = 1 + \gamma^2 \frac{u_p^2}{c^2}, \tag{D.18}$$

we get the result

$$
\begin{aligned}
T'_{ret} &= -\gamma^2 \frac{u_p}{c^2} z' - \sqrt{\left(-\gamma^2 \frac{u_p}{c^2} z'\right)^2 + \frac{\gamma^2}{c^2}(x'^2 + y'^2 + z'^2)} \\
&= -\gamma^2 \frac{u_p}{c^2} z' - \frac{\gamma}{c}\sqrt{x'^2 + y'^2 + \left(1 + \gamma^2 \frac{u_p^2}{c^2}\right) z'^2} \\
&= -\gamma^2 \frac{u_p}{c^2} z' - \frac{\gamma}{c}\sqrt{x'^2 + y'^2 + \gamma^2 z'^2} \\
&= -\gamma^2 \frac{u_p}{c^2} z' - \frac{\gamma}{c}\|\vec{R}\|.
\end{aligned}
\tag{D.19}
$$

Now in the final step, we finally introduced special relativity, with the lorentz transform we requiring $t' = 0$ for all coordinates, meaning that the primed and proper axis overlap at this time, and taking the primed frame as moving at velocity $v = -u_p$ along the z-axis relative to the proper frame of particle, and hence used the Lorentz transform of the spatial coordinates; $\vec{R} = (x, y, z) = (x', y', \gamma z')$, to get $\|\vec{R}\|$. With this we can rewrite the Z-component of equation ((D.15)) as

$$
z' - u_p T'_{ret} \Rightarrow \gamma \left(z + \frac{u_p \cdot \|\vec{R}\|}{c}\right), \tag{D.20}
$$

giving the propagation displacement vector to be

$$
\vec{R}'_{prop} = \begin{pmatrix} x \\ y \\ \gamma\left(z + \dfrac{u_p \cdot \|\vec{R}\|}{c}\right) \end{pmatrix} = \|\vec{R}\| \begin{pmatrix} \frac{x}{\|\vec{R}\|} \\ \frac{y}{\|\vec{R}\|} \\ \gamma\left(\frac{z}{\|\vec{R}\|} + \dfrac{u_p}{c}\right) \end{pmatrix}. \tag{D.21}
$$

Since the light propagates along this in the primed frame, the unit vector of the primed propagation velocity at any general primed coordinate can be worked out to be (show working out in appendix)

$$
\hat{\tilde{U}}' = \hat{\tilde{R}}'_{prop} = \frac{1}{\text{Å}} \begin{pmatrix} \frac{x}{\|\vec{R}\|} \\ \frac{y}{\|\vec{R}\|} \\ \gamma\left(\frac{z}{\|\vec{R}\|} + \dfrac{u_p}{c}\right) \end{pmatrix}, \tag{D.22}
$$

where the factor

$$\text{Å} = \gamma \left(1 + \frac{u_p}{c} \frac{z}{\|\vec{R}\|} \right). \tag{D.23}$$

Now taking the magnitude of the light propagation displacement from equation ((D.21)) and using equation ((D.18)), we have

$$\begin{aligned}
\|\vec{R}'_{prop}\|^2 &= x^2 + y^2 + \gamma^2 \left(z^2 + \frac{u_p^2}{c^2}\|\vec{R}\|^2 + 2\frac{u_p}{c} z \|\vec{R}\| \right) \\
&= \gamma^2 \|\vec{R}\|^2 + \frac{u_p^2}{c^2}\gamma^2 z^2 + 2\frac{u_p}{c}\gamma^2 z \|\vec{R}\| \\
&= \gamma^2 \left(\|\vec{R}\| + \frac{u_p}{c} z \right)^2 \\
&= \gamma^2 \left(1 + \frac{u_p}{c} \frac{z}{\|\vec{R}\|} \right)^2 \|\vec{R}\|^2 \\
&= \text{Å}^2 \|\vec{R}\|^2.
\end{aligned} \tag{D.24}$$

Since the luminal speed of the propagation will be same in both frames we have

$$\begin{aligned}
c = \frac{\|\vec{R}\|}{T_{prop}} &= \frac{\|\vec{R}'_{prop}\|}{T'_{prop}} \\
&= \frac{\text{Å}\|\vec{R}\|}{T'_{prop}} \\
T'_{prop} &= \text{Å} T_{prop},
\end{aligned} \tag{D.25}$$

where T_{prop} is the proper (particles rest frame) time the light takes to propagate to $\vec{R}$ from the particle.

... is this correct to say?... Time and length are stretched in the primed frame along $\vec{R}'_{ret}$ relative to the proper frame along $\vec{R}$ by a factor Å, leading to the relative radial density of the light in the primed frame to that of the proper frame, which we will refer to as the radial light strength weighting, given as

$$W_\rho = \frac{1}{\text{Å}}. \tag{D.26}$$

D.6 Acceleration

*** gotta check which theta this is

D.6.1 General Equation

$$\vec{a'} = \frac{1}{\gamma^2}\left[\vec{a_0} + \left(\frac{1}{\gamma} - 1\right)\left(\vec{a_0}\cdot\hat{\vec{v}}\right)\hat{\vec{v}}\right] \tag{D.27}$$

D.6.2 Co-moving Charges

$$\vec{v} = \begin{pmatrix} o \\ v \end{pmatrix} = \begin{pmatrix} o \\ -u_p \end{pmatrix} \tag{D.28}$$

$$\vec{a_0} = \frac{k}{R^2}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \tag{D.29}$$

Substituting into general equation

$$\vec{a'} = \frac{1}{\gamma^3}\frac{k}{R^2}\left[\begin{pmatrix} \gamma\sin\theta \\ \cos\theta \end{pmatrix}\right] \tag{D.30}$$

$$\begin{aligned} \|\vec{a}'\| &= \frac{1}{\gamma^2}\frac{k}{R^2}\sqrt{\sin^2\theta + \gamma^{-2}\cos^2\theta} \\ &= \frac{1}{\gamma^2}\frac{k}{R^2}\sqrt{\sin^2\theta + (1-\beta^2)\cos^2\theta} \\ &= \frac{1}{\gamma^2}\frac{k}{R^2}\sqrt{1-\beta^2\cos^2\theta} \end{aligned} \tag{D.31}$$

using trigonometry and retarded distance transform from above

$$
\begin{aligned}
\|\vec{a}'\| &= \frac{1}{\gamma^2}\frac{k}{R^2}\sqrt{1-\beta^2\cos^2\theta} \\
&= \frac{1}{\gamma^2}\frac{k}{R^2}\sqrt{1-\beta^2\frac{(\cos\theta'+\beta)^2}{(1+\beta\cos\theta')^2}} \\
&= \frac{1}{\gamma^2}\frac{k}{R^2}\sqrt{\frac{(1+\beta\cos\theta')^2-\beta^2(\cos\theta'+\beta)^2}{(1+\beta\cos\theta')^2}} \\
&= \frac{1}{\gamma^2}\frac{k}{R^2}\sqrt{\frac{(1+2\beta\cos\theta')-\beta^2(\beta^2+2\beta\cos\theta')}{(1+\beta\cos\theta')^2}} \\
&= \frac{1}{\gamma^2}\frac{k}{R^2}\sqrt{\frac{(1-\beta^2)2\beta\cos\theta'+(1-\beta^4)}{(1+\beta\cos\theta')^2}}
\end{aligned}
\tag{D.32}
$$

$$
\begin{aligned}
\vec{a'} &= \frac{1}{\gamma^3(1+\beta\cos\theta')}\frac{k}{R^2}\begin{pmatrix}\sin\theta' \\ \cos\theta'+\beta\end{pmatrix} \\
&= \frac{1}{\gamma^3(1+\beta\cos\theta')}\frac{k\left(\gamma\left(1+\beta\cos\theta'\right)\right)^{-2}}{{R'_{ret}}^2}\begin{pmatrix}\sin\theta' \\ \cos\theta'+\beta\end{pmatrix} \\
&= \frac{1}{\gamma^5(1+\beta\cos\theta')^3}\frac{k}{{R'_{ret}}^2}\begin{pmatrix}\sin\theta' \\ \cos\theta'+\beta\end{pmatrix}
\end{aligned}
\tag{D.33}
$$

$$
\begin{aligned}
\|\vec{a'}\| &= \frac{1}{\gamma^3(1+\beta\cos\theta')}\frac{k}{R^2}\sqrt{\sin^2\theta'+(\cos\theta'+\beta)^2} \\
&= \frac{1}{\gamma^3(1+\beta\cos\theta')}\frac{k}{R^2}\sqrt{1+\beta^2+2\cos\theta'\beta}
\end{aligned}
\tag{D.34}
$$

D.7 Derivation from Spherical Light Pulse

*** what about reverse of the spherical wave derivation, so that the spherical pulse of light is emitted towards the origin to reach at time $t = 0 = t'$ (then it would be as if was emitted into the past if we continue the the propagation back through the origin)

*** If you have a point r in an inertial frame then a spherical wave pulse would take $t = |r|/c$ to get there, we will call the light being at coordinate r an event, now to find the corresponding event in the primed frame

*** free to choose the origin of the primed frame, so choosing it to be coinciding at the point light is emitted

*** maybe derive from spherical pulse on lorry, having it return and have its walls be general coordinates, with time dilation

*** derivation in words first: - 2 inertial frames coinciding at $t = t' = 0$ with primed frame moving relativity in z-direction - Describe spherical wave pulse in proper frame and primed frame $|\vec{R}| = ct$ and $|\vec{R'}| = ct'$ -

...

An observer is at the origin of an inertial frame of reference/coordinate system $< S >$. An object moving at a constant velocity v relative to $< S >$ can be described to be stationary in an inertial frame of reference $< S' >$ moving the at the same velocity v shown in Fig. D.2.

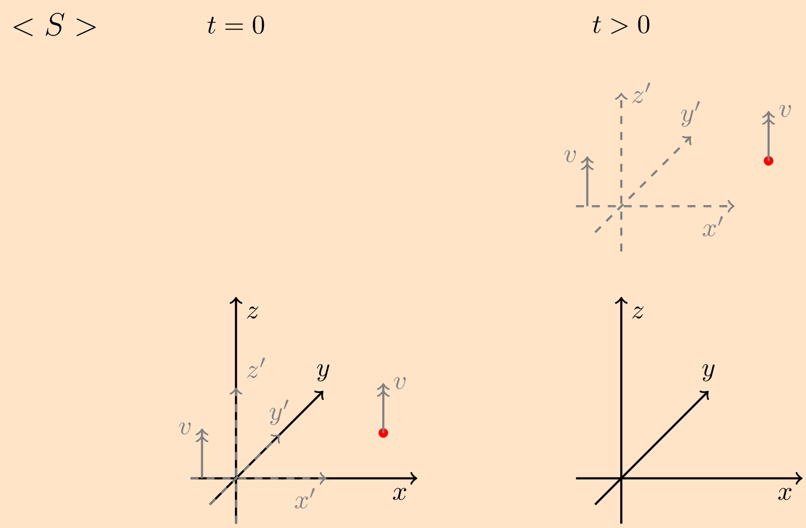


Figure D.2: Proper frame of reference in standard configuration.

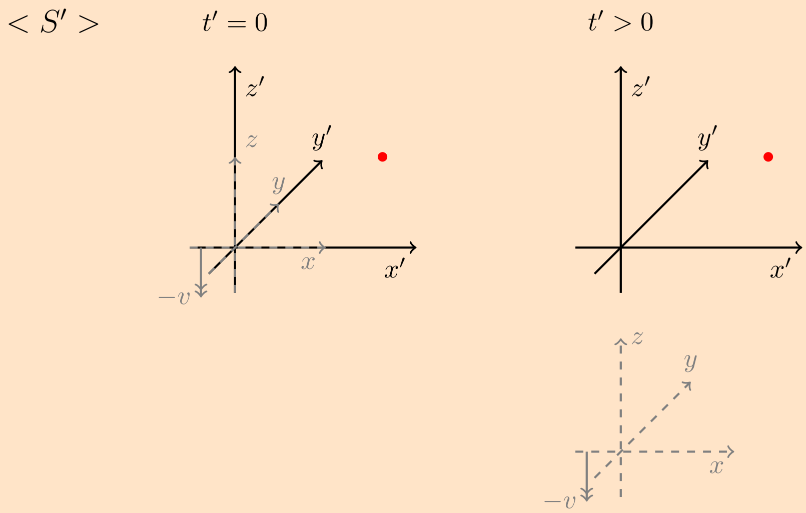


Figure D.3: Primed frame of reference in standard configuration.

- We start with setting the inertial frames up so that the origins of the Cartesian coordinates coincide at $t = 0 = t'$
- It is chosen that the frame of reference $< S' >$ moves in the z-direction (this is chosen as it is easier to move to spherical polar coordinates later and since there is symmetry in the coordinates perpendicular to the movement, and having right-left symmetry is more visually simple)
- *** if there is a spherical light pulse at the time the coordinate axis overlap at $t = t' = 0$ such that the distance that the light has propagated at a time t is ct and ct' respectfully so that the equation describing the spherical pulse in each frame is $ct = \sqrt{x^2 + y^2 + z^2}$ and $ct' = \sqrt{x'^2 + y'^2 + z'^2}$
- We define the s and s' as:

$$\begin{aligned} s^2 &= -c^2t^2 + x^2 + y^2 + z^2 \\ s'^2 &= -c^2t'^2 + x'^2 + y'^2 + z'^2 \end{aligned} \tag{D.35}$$

*** Diagram of wave fronts with both axis in same diagram one moving relative to other showing (maybe having 2 diagrams one with each frame of reference at rest) ***

- At an expanding wave front moving at the speed of light c, in the two frames we have $s = s' = 0$
- We are therefore looking for a transformation s.t. $s = 0 \Leftrightarrow s' = 0$
- for a linear transform it is implied $s'^2 = ks^2$ (where k is a constant) ** why does it have to be linear? (to allow for smooth inverse transformation)
- By symmetry $s^2 = k^2 s'^2 = k^2 s^2$ which leads to $k^2 = 1 \Rightarrow k = 1$ ($k = -1$ is disallowed because as $v \to 0$ we must recover $s = s'$)
- So $s'^2 = s^2$, and by symmetry $x' = x$, $y' = y$ (this is taken from the cannon and ball thought experiment from chapter...)
- This leads to:

$$-c^2t^2 + z^2 = -c^2t'^2 + z'^2 \tag{D.36}$$

- The requirement $z' = 0$ when $z = vt$, the form of a linear equation that gives this for all values of t is

$$z' = \gamma(z - vt) \tag{D.37}$$

- where γ is yet to be determined, we have chosen the linear equation and not one that depends on factors greater than first order, i.e. squared or greater terms of $(z - vt)$, this is because the linear... (to allow for smooth inverse transformation)
- By symmetry if we wanted to instead transform from $< S' >$ to $< S >$, it is the same transform but with the frame change in the opposite direction, so x, y, z, t can just be replaced by x', y', z', t' when the sign in front of v is also changed *** maybe diagram to show this
- Substituting z' into the inverse of eq. (D.37) leads to

$$t' = \gamma\left[t - \frac{z}{v}\left(1 - \frac{1}{\gamma^2}\right)\right] \tag{D.38}$$

- Subbing the previous two equations into eq. (D.36) gives

$$-c^2t^2 + z^2 = -c^2\gamma^2\left[t - \frac{z}{v}\left(1 - \frac{1}{\gamma^2}\right)\right]^2 + \gamma^2(z - vt)^2 \tag{D.39}$$

- This holds true for all z and t
- Equating terms (this is a mathematical technique, were it is known that for an equation like this to be true for all values of t it is required that all terms in front t^2 on each side of the equation must be equal) in front of t^2 leads to

$$\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}} \tag{D.40}$$

- Subbing this into previous equations now completes the Lorentz Transformation:

$$\boxed{\begin{aligned} & x' = x \\ & y' = y \\ & z' = \gamma(z - vt) \\ \text{From eq. (D.38)} \quad & t' = \gamma\left(t - \frac{vz}{c^2}\right) \end{aligned}} \tag{D.41}$$

  *** explain the time transform in terms of the gamma function being the slowing down part and the $\frac{vz}{c^2}$ being the "anti-simultaneity" part

- Galilean coordinates obtained by letting $v \to 0$

we can write this in vector form, multiplying t' by c to give unit of length, this is known as the positional 4 vector and is given as

$$\mathbf{R}' = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma\left(ct - \frac{vz}{c}\right) \\ x \\ y \\ \gamma\left(z - vt\right) \end{pmatrix}. \tag{D.42}$$

an event is given by its spacial coordinates and the time at which it happens.

D.8 Length Contraction

D.9 Time Dilation

Taking two times t_1 and t_2 of a clock at rest at position z in one inertial frame will have transformed times in another reference frame of $t_1' = \gamma(t_1 - \frac{v}{c^2}z_1)$ $t_2' = \gamma(t_2 - \frac{v}{c^2}z_2)$ so that the difference in time is

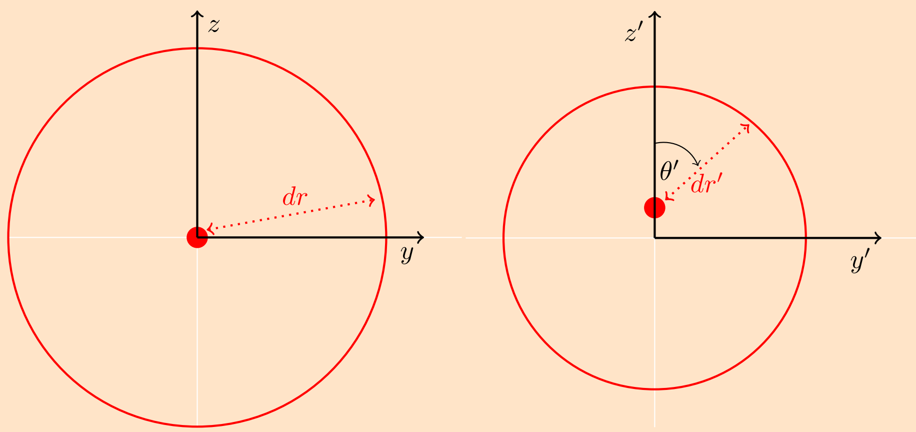
$$\begin{aligned} \Delta t' &= t_2' - t_1' \\ &= \gamma(t_2 - \frac{v}{c^2}z) - \gamma(t_1 - \frac{v}{c^2}z) \\ &= \gamma(t_2 - t_1) \\ &= \gamma\Delta t \end{aligned} \tag{D.43}$$

when we take the change of time at the same location in a rest frame, we always have that $\Delta t' > \Delta t$ i.e the change in time in the rest frame of an observer corresponds to a greater change in time in another moving frame of reference.

D.10 Generalised velocity vector transform

D.10.1 god knows what this is: Doppler Effect

**** this is for receiving particle that is at rest, i.e. $u_p = 0$ ****



(a) Spherical pulse in rest frame (b) Spherical pulse in Primed frame

Figure D.4: since I have everything in opposite directions, need to change equations *** also this is full blown relativity not just retarded field

Figure D.4, shows a particle emitting a spherical wave of light, in the proper frame it begins emitting at time $t = 0$ and after an infinitesimal amount of time the front of the wave has spread out spherically, such that the front is at a distance dr from the origin at all positions,while the wave continues to be emitted from the origin. In the primed frame the wavefront is emitted at $t' = 0$ and reaches a point that is γdr from the origin at all points, with the gamma due to $dr = cdt$ and the the time $dt = \gamma dt'$, but we also now have to take into account that the source particle that is still emitting this wave has moved along the z'-axis to a point $u_p dt' = \gamma u_p dt$ and each part of the wave being emitted from the source is taken as being emitted at an angle θ', such that any point on the wavefront that is at and angle of θ' from the particle, will have the distance dr' between it and the source, and the wave that is currently being emitted from the source particle will go to that position on the wavefront (offal description).

We can get the Doppler effect by taking the ratio of the distances the way will have to travel to get to the current point the wavefront is at, which in the rest frame is $dr = cdt$ and in the primed frame is $|d\vec{dr}'| = u_p/c\gamma dr$

Appendix E

Alternative Derivation

*** diagram ***

The change in position of particle $\Delta\mathbf{R}_p$, with initial position $\mathbf{R}_0$, and a constant velocity

$$\mathbf{U}_p = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \tag{E.1}$$

between time t_1 and t_2 is

$$\Delta\mathbf{R}_p = (\mathbf{R}_0 + \mathbf{U}_p t_2) - (\mathbf{R}_0 + \mathbf{U}_p t_1) = (t_2 - t_1)\mathbf{U}_p = \mathbf{U}_p \Delta t \tag{E.2}$$

(acceleration as well, but these will become negligible when we take an infinitesimal time interval later) The Lorentz transformation for this with $t = 0$ is

$$\Delta\mathbf{R}'_p = \begin{pmatrix} u_x \Delta t \\ u_y \Delta t \\ \gamma\left(u_z \Delta t - v\Delta t\right) \end{pmatrix} \tag{E.3}$$

but this gives primed times to be $\Delta t' = -\gamma \dfrac{v}{c^2} u_z \Delta t$, and we want to find the positions that have the same time, therefore we need to take away the displacement moved in the the primed time $\Delta t' \mathbf{U'}_p$ which is

$$\Delta\mathbf{R}'_p = \begin{pmatrix} u_x \Delta t \\ u_y \Delta t \\ \gamma\left(u_z \Delta t - v \Delta t\right) \end{pmatrix} + \gamma \frac{v}{c^2} u_z \Delta t \mathbf{U'}_p. \tag{E.4}$$

If we divide across by the change in time Δt and take this time change to go to an infinitesimal dt and substituting the time in terms of dilated primed time $dt = \frac{1}{\gamma} dt'$ we then have the primed velocity given as *** the dt' = gamma dt is suspicious ***

$$\frac{d\mathbf{R}'_p}{dt} = \gamma \frac{d\mathbf{R}'_p}{dt'} = \gamma \mathbf{U'}_p = \begin{pmatrix} u_x \\ u_y \\ \gamma\left(u_z - v\right) \end{pmatrix} + \gamma \frac{v}{c^2} u_z \mathbf{U'}_p \tag{E.5}$$

rearranging for primed velocity we have

$$\boxed{\begin{gathered} \mathbf{U'}_p = \frac{1}{\gamma\left(1 - \frac{v}{c^2} u_z\right)} \begin{pmatrix} u_x \\ u_y \\ \gamma\left(u_z - v\right) \end{pmatrix} \\ \text{at position } \mathbf{R}' \text{ and time } t' \end{gathered}} \tag{E.6}$$

Appendix F

Todo for these notes

* Å is not the same Å in velocity and acceleration transform as it is in time transform ((it is the same as its the retarded cosine angel which is same as cosine of angle for primed frame as its at the retarded position

(try to make it as readable as possible for people with poor English, or for the translation)

an introduction sentence... ((relativity cause its the theory of motion for objects relative to other objects, and special because its specifically for the special case where gravity's effects negligible and are not required to be included in calculations and objects are moving at constant velocities)) ((as gravity curves spacetime giving curved coordinates))

decide whether to include minkowski diagram as it is an abstract view rather than representational view

F.1 A note on different notations

*primed notation * my notation for frames

3 postulates/ starting points.

F.2 Astronomy from Earth

... explain how observation from earth works, as you would think the positions of stars at $t' = 0$ would be all over the place as it rotates around the sun

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Phy-Hub