Models 14-10-25

$$I(\lambda) \simeq g(x_0)\, e^{\lambda f(x_0)} \sqrt{\frac{2\pi}{\lambda |f''(x_0)|}}$$

interior max $\quad I(\lambda) \sim \lambda^{-1/2} e^{\lambda f(x_0)}$

endpoint (not flat) $\quad I(\lambda) \sim \lambda^{-1} e^{\lambda f(x_0)}$

Lp-norm in real analysis

The quantity

$$\|g\|_p := \left(\int_a^b |g(t)|^p dt \right)^{1/p} \qquad p>0$$

is called p-norm when the integral exists (in Lesbegue sense). We want to study the behavior of $\|g\|_p$ as $p \longrightarrow \infty$. We assume that g has a unique maximum in t_0 which is an interior point, and $g \in C^4(a,b)$.

We first study

$$I(p) \equiv \int_a^b |g(t)|^p dt \qquad \|g\|_p = I^{1/p}$$

and

$$I = \int_a^b e^{p \ln |g(t)|} dt$$

When we applied the Laplace method we have always assumed that f is continuously differentiable. However, if g vanishes somewhere in $[a,b]$ then $\ln |g| \longrightarrow -\infty$. However, every neighborhood of points where $g=0$ will yield a negligible contribution to the integral (i.e. to I for $p \gg 1$).

Thus such discontinuities can be neglected. Now we use eq. (22)

$$I(p) = e^{p \ln|g(t_0)|} \sqrt{\frac{2\pi\, g(t_0)}{p\,|g''(t_0)|}} \left(1 + O\left(\frac{1}{p}\right)\right)$$

$$= |g(t_0)|^p \sqrt{\frac{2\pi\, g(t_0)}{p\,|g''(t_0)|}} \left(1 + O\left(\frac{1}{p}\right)\right) \quad \text{as } p \to \infty$$

Obs: $a > 0$ $\quad a^{1/p}\, p^{-\frac{1}{2p}} = e^{\frac{\ln a}{p}}\, e^{-\frac{\ln p}{2p}} =$

$$= \left(1 + \frac{\ln a}{p} + \dots\right)\left(1 - \frac{\ln p}{2p} + \dots\right) \simeq 1 - \frac{\ln p}{2p} + \dots \quad p \to \infty$$

$$I^{1/p} = |g(t_0)| \left(\frac{2\pi\,|g|}{p\,|g''|}\right)^{\frac{1}{2p}} + \text{hot} = |g(t_0)| \left(1 - \frac{\ln p}{2p} + o\left(\frac{1}{p}\right)\right)$$

hence

$$\|g\|_p = \max_{t \in [a,b]} |g(t)| \left\{1 - \frac{\ln p}{2p} + \dots\right\} \quad \text{as } p \to \infty$$

which justifies the usual definition

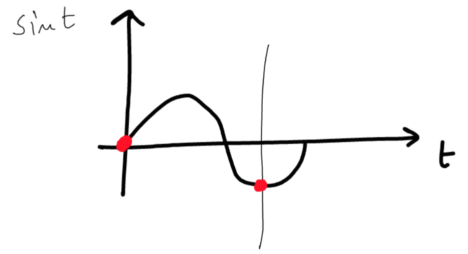
$$\|g\|_\infty = \max_{t \in [a,b]} |g|$$

Example:

Consider the integral

$$\int_0^{\frac{3\pi}{2}} e^{-\lambda \sin t} f(t)\, dt \qquad \text{as } \lambda \to \infty$$

where f is cont. and diff. in $[0, \frac{3\pi}{2}]$.



We want to consider the contributions from the endpoint minima $t=0$, $t=\frac{3\pi}{2}$. For this we write

$$I(\lambda) = \underbrace{\int_0^{\pi/2} e^{-\lambda \sin t} f(t)\, dt}_{I_1} + \underbrace{e^{\lambda} \int_{\pi/2}^{3\pi/2} e^{-\lambda \sin t - \lambda} f(t)\, dt}_{I_2}$$

eq. (24) left endpoint $\qquad I_1 = f(0) \frac{e^{\lambda \cdot 0}}{\lambda \cos(0)} = \frac{f(0)}{\lambda}$ as $\lambda \to \infty$ (leading order)

eq. (23) flat endpoint $\qquad I_2 = f\left(\frac{3\pi}{2}\right) e^{\lambda \cdot 0} \sqrt{\frac{\pi}{2\lambda \sin\left(\frac{3\pi}{2}\right)}} \simeq f\left(\frac{3\pi}{2}\right) \sqrt{\frac{\pi}{2\lambda}}$

Hence the leading contribution comes from I_2 and

$$I(\lambda) \simeq f\left(\frac{3\pi}{2}\right) e^{\lambda} \sqrt{\frac{\pi}{2\lambda}} + \text{h.o.t.} \quad \text{as } \lambda \to 0$$

The min a $t=0$ is subleading and it should be taken into account only at higher order.

<u>Ex</u>: Calculate the leading contribution to

$$\int_0^{\pi} e^{-\lambda \sin t} f(t)\, dt \quad \text{as } \lambda \to \infty.$$

Review of Dynamical Systems

In many applications (physics, biology, chemistry ...) one has to study nonlinear systems of (autonomous) ODEs:

$$\text{①} \quad \begin{cases} \dot{\vec{x}}(t) = \vec{f}(\vec{x}(t)) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

where $\vec{x}(t) = (x_1(t), \dots, x_N(t)) \in U \subseteq \mathbb{R}^N$, U is some open connected set of reals.

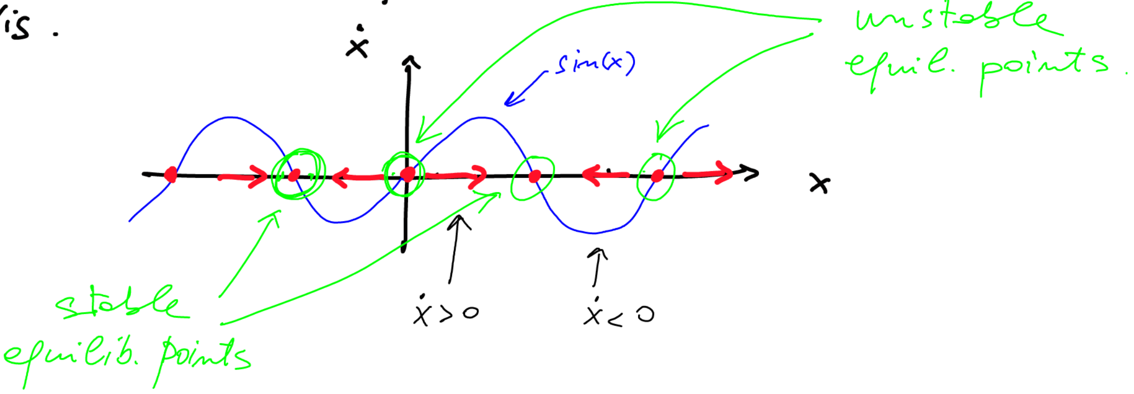
With any $\vec{x}$ we associate a vector $\vec{f}$, the set of these vectors is called vector field. The domain U, where $\vec{f}$ is supposed to be continuous and differentiable, is called the phase space.

The solutions $\vec{x}(t, x_0)$ of the system ① describe smooth curves as t changes: they are called trajectories which are parametric curves in the phase space.

Example: let's consider $(N=1)$

$$\dot{x}(t) = \sin[x(t)]$$

In this case the vector field is 1-dim. and coincides with the x-axis.

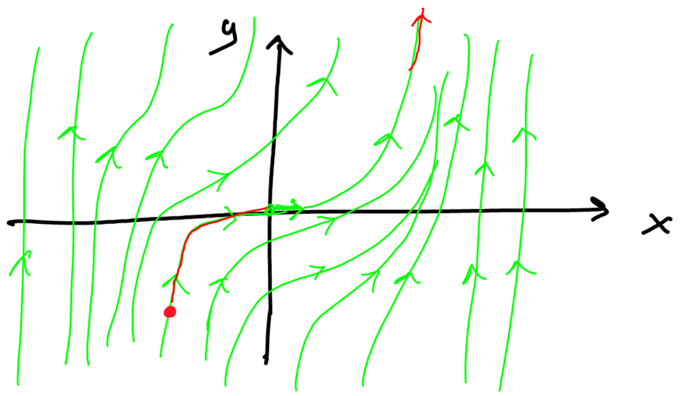


In the $N=2$ case

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = x^2+y^2 \end{cases}$$

The vector field is 2 dim. and we have a map

$$\vec{x} = (x,y) \longrightarrow \vec{f}(\vec{x}) = (\underset{\dot{x}}{\underset{\uparrow}{1}},\ \underset{\dot{y}}{\underset{\uparrow}{x^2+y^2}})$$

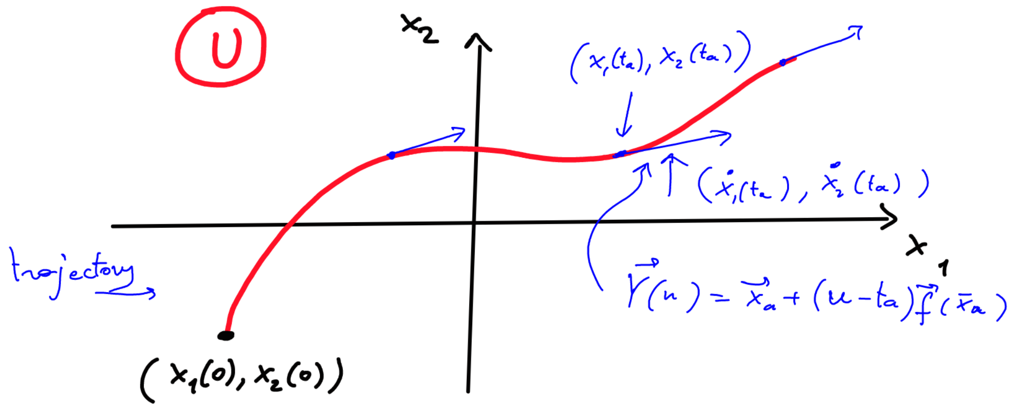


$(0,0) \longrightarrow (1,0)$

In general the vector field $\vec{f}$ is <u>tangent</u> to a trajectory in every point. In fact, let $\vec{x}(t)$ be a trajectory, then the eq. of a tangent line to the trajectory at a point $\vec{x}_a \equiv \vec{x}(t_a)$ is

$$\vec{Y}(u) = \vec{x}_a + (u-t_a)\dot{\vec{x}}(t_a) = \vec{x}_a + (u-t_a)\vec{f}(\vec{x}_a)$$

thus $\vec{f}$ is the directional vector of the straight line $\vec{Y}(u)$.



$N=2$

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \quad (2)$$

By flowing along the vector field, a point traces out the trajectory $\vec{x}(t)$, a curve in the phase space U or a sol. of (2).

A phase portrait is a set of trajectories with the indication of the directions of the vector field.

As we cannot solve the system ① in general, we would like to know at least some properties.

Sometimes it is useful to study nullclines, defined as subspaces (or manifolds) where $\dot{x}_i = 0$ for a give i. If $N=2$ we may get simple curves

$$\dot{x}_1 = 0 \Rightarrow f_1(x_1, x_2) = 0 \Rightarrow x_2^{(1)} = g^{(1)}(x_1)$$

possibly $x_2^{(2)} = g^{(2)}(x_1)$

Two interesting examples

1) Find a solution of $\dot{y} = y^2$ with i.c. $y(0) = 1$.
   Can we then find $y(2)$?

From $\int \frac{dy}{y^2} = t - c$, $y = \frac{1}{c-t}$, $y(0) = 1 \Rightarrow c = 1$

$$y(t) = \frac{1}{1-t} \longrightarrow y(2) = -1$$

As $\dot{y}(t) \geq 0$, y is an increasing function of time but $y(2) = -1$! Actually, the solution exists only in the interval $(0,1)$ as it blows up a $t = 1$.

Solutions may exist only within finite intervals of time or the may not exist for some initial conditions or may not exist at all.

2) Find a solution of $\dot{y} = \sqrt{y}$ for which $y(0) = 0$. Find $y(2)$.

$$\int \frac{dy}{\sqrt{y}} = 2\sqrt{y} = t - c, \quad y = \frac{(t-c)^2}{4} \text{ but } y(0) = 0, \; y(t) = \frac{t^2}{4}$$

hence $y(2) = 1$. But!

However $y(t)=0$ satisfies the eq. and the init. cond.
Even worse:

$$y(t) = \begin{cases} 0 & 0 \le t \le T \\ \frac{(t-T)^2}{4} & t > T \end{cases} \qquad T \text{ is arbitrary } T > 0.$$

is a solution with the <u>same</u> initial condition!
So there are infinitely many solutions, so asking $y(2)$ is meaningless.
Sol. to init. val. problems may not be unique.

Picard's theorem

If $\vec{f}$ is continuous and $\frac{\partial f_i}{\partial x_j}$ are also continuous for all indexes i, j in U, then for any $\vec{x}_0 \in U$ the initial value problem defined in (1) admits a solution on some interval $t \in [-\delta, \delta]$, $\delta > 0$, and this solution is unique.

<u>Obs</u>: in this case trajectories do not intersect.

Fixed points

The fixed points of eq. (1) are the simplest to study as time is not relevant. Indeed, fixed points are the zeros of $\vec{f}$: namely, $\vec{x}^*$ is a fixed point if

$$\vec{f}(\vec{x}^*) = \vec{0} \qquad (3)$$

Warning: even though an ODE has some fixed points, that does not imply that the dynamics (the init. val. solut.) reach them!