Models 10/10/25

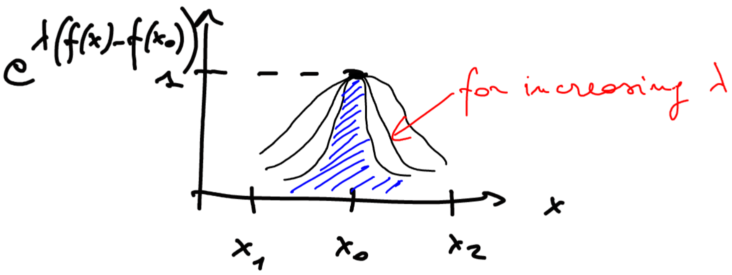
The Laplace method

In several situations one wish to evaluate complicated integrals which have a form

$$I(\lambda) = \int_{x_1}^{x_2} dx \; g(x)\, e^{\lambda f(x)} \qquad \lambda \in \mathbb{R} \tag{21}$$

where g and f are continuous and differentiable functions. Although $I(\lambda)$ cannot be calculated for any arbitrary λ, it happens that it can be well approximated (under appropriate conditions) as $\lambda \to \infty$.

The core idea of Laplace's method is that the major contribution to the integral in (21) as $\lambda \to \infty$ comes from the neighborhood of the point in $[x_1, x_2]$ where $f(x)$ gets its maximum value, which we call x_0.



There are essentially three cases:

1) x_0 is an interior maximum, $x_1 < x_0 < x_2$ and $f'(x_0) = 0$
We assume that $f''(x_0) < 0$ (actually if $f'(x_0) = f''(x_0) = f'''(x_0) = 0$ but $f^{(IV)}(x_0) < 0$ we can apply very similar ideas and the calculations are not more difficult). Also $g(x_0) \neq 0$

and it is finite.

As we expect that the dominant contributions comes from the neighborhood of x_0 and $f(x) = f(x_0) + \frac{(x-x_0)^2}{2} f''(x_0) + O(|x-x_0|^2)$ as $x \to x_0$, we obtain

$$I(\lambda) \simeq \int_{x_1}^{x_2} g(x)\, e^{\lambda\left(f(x_0) + \frac{(x-x_0)^2}{2} f''(x_0)\right)} dx \underset{\text{at lead. order}}{\simeq} g(x_0)\, e^{\lambda f(x_0)} \underline{\underline{\int_{x_1}^{x_2} e^{-\lambda \frac{|f''(x_0)|}{2} (x-x_0)^2} dx}}$$

We change var. $s = (x-x_0)\sqrt{\frac{|f''(x_0)|}{2}\lambda}$ so

$$I(\lambda) = g(x_0)\, e^{\lambda f(x_0)} \sqrt{\frac{2}{\lambda |f''(x_0)|}} \underbrace{\int_{(x_1-x_0)\sqrt{\frac{|f''(x_0)|\lambda}{2}}}^{(x_2-x_0)\sqrt{\frac{|f''(x_0)|\lambda}{2}}} e^{-s^2} ds}_{\text{as } \lambda \to \infty \;\longrightarrow\; \int_{-\infty}^{+\infty} e^{-s^2} ds = \sqrt{\pi}} =$$

(22) $$\boxed{I(\lambda) = g(x_0)\, e^{\lambda f(x_0)} \sqrt{\frac{2\pi}{\lambda |f''(x_0)|}}}$$ as $\lambda \to +\infty$ (leading order)

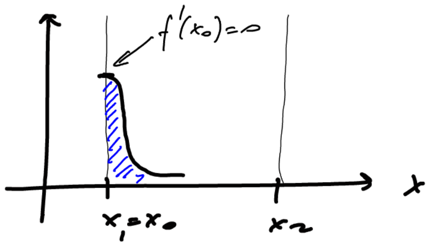
one should prove that this is the leading order (we did not).

Exercise: show that the next to leading order of $I(\lambda)$ in eq. (22) is given by

$$I(\lambda) = e^{\lambda f(x_0)} \sqrt{\frac{2\pi}{\lambda |f''(x_0)|}} \left(g(x_0) + \frac{c}{\lambda} \right) \quad \text{as } \lambda \to \infty$$

where c is a constant that depends on the derivatives of f up to 4^{th} order (at $x = x_0$) and on $g(x_0)$ and $g'(x_0)$.

2) $x_0 = x_1$ or $x_0 = x_2$ (x_0 is a flat endpoint) and $f'(x_0) = 0$.



It is easy to show that the leading order formula is

(23)
$$\boxed{I(\lambda) = g(x_0)\, e^{\lambda f(x_0)} \sqrt{\frac{\pi}{2\lambda |f''(x_0)|}}}$$

Example

The modified Bessel function of second kind is a special function that occurs in many applications. It is given by

$$K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh(\nu t)\, dt \quad , \quad x > 0$$

We wish to estimate $K_\nu(x)$ as $x \to +\infty$ for fixed ν. Since $\cosh'(t) = \sinh(t)$, $t > 0$, the max of $e^{-x \cosh t}$ as a function of t ($x > 0$) occurs at $t = 0$, with zero derivative

So we can apply eq. 23 $\quad (\cosh t \simeq 1 + \frac{t^2}{2} + \dots)$
$f(t) = \cosh t$

$$\boxed{K_\nu(x) = e^{-x}\sqrt{\frac{\pi}{2x}}}$$

as $x \to \infty$
at leading order

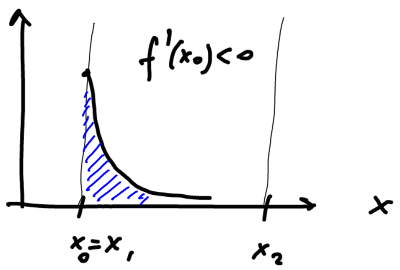
notice that it does not depend on ν!

Exercize: show that

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\left(1 + \frac{c(\nu)}{x} + \dots\right)$$

and $\quad c(\nu) = \frac{4\nu^2 - 1}{8}$.

3) $x_0 = x_1$ or $x_0 = x_2$ (x_0 is an endpoint) but $f'(x_0) \neq 0$
Without loss of generality we take $x_0 = x_1$ and $f'(x_0) < 0$



In this case we get $\quad f(x) = f(x_0) + (x - x_0) f'(x_0) + \dots$
hence

$$I(\lambda) \cong \int_{x_1 = x_0}^{x_2} g(x)\, e^{\lambda[f(x_0) + (x - x_0) f'(x_0)]}\, dx =$$

$$= g(x_0)\, e^{\lambda f(x_0)} \int_{x_0}^{x_2} e^{\lambda (x - x_0) f'(x_0)}\, dx =$$

$$= g(x_0)\, e^{\lambda f(x_0)} \int_0^{x_2-x_0} e^{\lambda f'(x_0) s}\, ds = \frac{g(x_0)\, e^{\lambda f(x_0)}}{\lambda f'(x_0)} \left[e^{\lambda s f'(x_0)} \right]_0^{x_2-x_1} =$$

$\uparrow$ $S = x - x_0$

$$= \frac{g(x_0)\, e^{\lambda f(x_0)}}{\lambda f'(x_0)} \left(e^{\lambda (x_2 - x_1) f'(x_0)} - 1 \right)$$

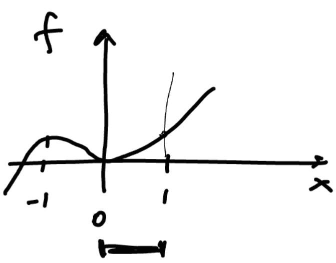
as $\lambda \to \infty$ and $f'(x_0) < 0$ we get

$$\text{(24)} \qquad \boxed{I(\lambda) \cong g(x_1) \frac{e^{\lambda f(x_1)}}{\lambda |f'(x_1)|}} \qquad \text{as } \lambda \to \infty$$

Example:

Obtain the leading order approximation of

$$I(\lambda) = \int_0^1 x^m e^{\lambda(3x^2 + 2x^3)}\, dx \qquad \text{as } \lambda \to +\infty$$



The max is at $x=1$, $f(1) = 5$, $f'(1) = 12$, $g(1) = 1$. We can apply eq. (24)

$$I(\lambda) \simeq \frac{e^{5\lambda}}{12\lambda}$$ at leading order (no dependence on m)

Exercize:

Calculate the leading order approx. when the integral is

$$\int_{-1}^{0} \dots = ? \qquad \int_{-2}^{0} \dots = ?$$

Stirling formula

We want to estimate how fast $N!$ goes to infinity as $N \to \infty$. We will apply the Laplace's method to the gamma function:

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} e^{-x} dx \qquad \lambda > 0 \tag{25}$$

Exercize: Show that $\Gamma(\lambda+1) = \lambda\,\Gamma(\lambda)$ hence $\lambda! = \Gamma(\lambda+1)$ which generalize the factorial to complex numbers.

Let's consider $\Gamma(\lambda+1) = \int_0^\infty x^\lambda e^{-x} dx$. If we write this as $\int_0^\infty e^{-x} e^{\lambda \ln x} dx$ we cannot apply Laplace's method. It is more beneficial to consider the max of the function $f(x) = -x + \lambda \ln x$ and set $g(x) = 1$. The max occurs at $x = \lambda$ which suggests a chang of vars: $x = \lambda t$ (so the max is now fixed w.r.t. t). We get

$$\Gamma(\lambda+1) = \int_0^\infty x^\lambda e^{-x} dx \underset{x=\lambda t}{\uparrow}{=} \lambda^{\lambda+1} \int_0^\infty t^\lambda e^{-\lambda t} dt$$

$t^\lambda e^{-\lambda t} = e^{\lambda(\ln t - t)}$ $\quad h(t) = \ln t - t$, $h'(1) = 0$, $h''(1) = -1$.

In this way we can apply eq. (22) with $g(t) = 1$ $(t_0 = 1)$.

$$\boxed{\Gamma(\lambda+1) = \lambda! \simeq \lambda^{\lambda+1} e^{-\lambda} \sqrt{\frac{2\pi}{\lambda}}} \quad \text{as } \lambda \to \infty. \tag{26}$$

Exercise : Show that at leading order

$$\int_0^\infty e^{-\lambda t} e^{-\frac{1}{t}} \, dt \cong \frac{\sqrt{\pi}\, e^{-2\sqrt{\lambda}}}{\lambda^{3/4}} \qquad \text{as } \lambda \to \infty$$

Example :

Let's consider the class of integrals:

$$I_m(x) = \int_0^\infty t^m e^{-\frac{t^2}{2} - \frac{x}{t}} \, dt \qquad x > 0.$$

We calculate $I_m(x)$ for large x and fixed m.

$\frac{t^2}{2} + \frac{x}{t}$ has a movable min at $\frac{d}{dt}\left(\frac{t^2}{2} + \frac{x}{t}\right) = 0$, $\bar{t} = x^{1/3}$

So we introduce the new variable τ :

$$t = x^{1/3} \tau$$

So

$$I_m(x) = x^{\frac{m+1}{3}} \int_0^\infty \tau^m e^{-x^{2/3}\left(\frac{\tau^2}{2} + \frac{1}{\tau}\right)} \, d\tau$$

the min of the exponent occurs at $\tau = 1$ (interior)

so we can apply eq. (22) :

$$I_m(x) = x^{\frac{m+1}{3}} e^{-\frac{3}{2} x^{2/3}} \sqrt{\frac{2\pi}{x^{2/3} \cdot 3}}$$

$$= x^{m/3} e^{-\frac{3}{2} x^{2/3}} \sqrt{\frac{2\pi}{3}} \qquad \text{as } x \to \infty.$$

leading order, fixed m

Exercises :

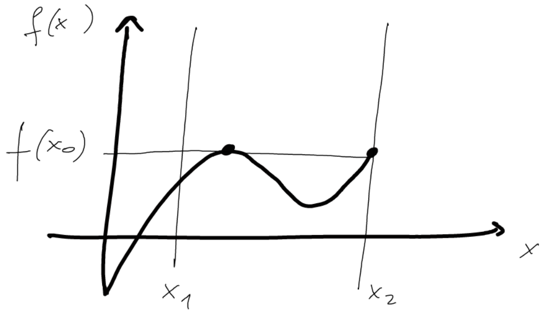
$$\int_{-2}^{0} e^{t} e^{\lambda(3t^2+2t^3)} dt \simeq e^{\lambda-1}\sqrt{\frac{\pi}{3\lambda}}$$

$$\int_{0}^{1} e^{t} e^{\lambda(3t^2+2t^3)} dt \simeq \frac{e^{5\lambda+1}}{12\lambda}$$

$$\int_{0}^{1} \sqrt{1+t}\, e^{\lambda(2t-t^2)} dt \simeq e^{\lambda}\sqrt{\frac{\pi}{2\lambda}}$$

$$\int_{-1}^{2} e^{\lambda(t^3-1)}(1+t^2)dt = \frac{5e^{6\lambda}}{11\lambda}$$

Question : Let's assume that $f(x)$ behaves like in the fig :



Where does the leading order come from ?