Models TP, 3/10/25

(3) $$\int_{-\infty}^{+\infty} e^{-\frac{ax^2}{2}+bx}\,dx = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}} \qquad a>0\,,\ b\in\mathbb{C}$$

(4) $$\varphi(k) = \int e^{ikx} p(x)\,dx = \langle e^{ikx}\rangle$$

(5) $$(-i)^n \frac{d^n\varphi}{dk^n}\bigg|_{k=0} = \langle x^n\rangle$$

(6) $$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}\, e^{ikx}\,dx = e^{-\frac{\sigma^2 k^2}{2}}$$

Because of eq. (5)

$$\langle x\rangle = -i\frac{d}{dk} e^{-\frac{\sigma^2k^2}{2}}\bigg|_{k=0} = 0$$

Important Gaussian integrals:

$$\langle x^n\rangle = (-i)^n \frac{d^n}{dk^n} e^{-\frac{\sigma^2k^2}{2}}\bigg|_{k=0} = 0 \qquad \text{if } n \text{ is odd (by symmetry)}$$

however

$$\langle x^4\rangle = \frac{d^4}{dk^4} e^{-\frac{\sigma^2k^2}{2}}\bigg|_{k=0} = \left[\sigma^4\left(3 - 6k^2\sigma^2 + k^4\sigma^4\right)e^{-\frac{\sigma^2k^2}{2}}\right]_{k=0} =$$

$$= 3\sigma^4 = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}}\, x^4\,dx\,.$$

If we want to calculate $\langle x^n\rangle$, we better start from

$$\int_{-\infty}^{+\infty} e^{-\frac{ax^2}{2}}\,dx = \sqrt{\frac{2\pi}{a}}$$

We differentiate both sides wrt a:

once
$$\int_{-\infty}^{+\infty} x^2 e^{-\frac{ax^2}{2}} dx = \frac{\sqrt{2\pi}}{a^{3/2}} \longrightarrow \langle x^2 \rangle$$

twice
$$\int_{-\infty}^{+\infty} x^4 e^{-\frac{ax^2}{2}} dx = \frac{3\sqrt{2\pi}}{a^{5/2}}$$

n times
n is even
$$\boxed{\int_{-\infty}^{+\infty} x^n e^{-\frac{ax^2}{2}} dx = \frac{(n-1)(n-3)\dots 5\cdot 3\cdot 1\ \sqrt{2\pi}}{a^{(n+1)/2}}}$$

From this find the expression of $\langle x^n \rangle$ as a function of σ and n (even).

Multidimensional Gaussian Integrals

Example:
$$\int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2\ e^{-\frac{3}{2}(x_1^2+x_2^2)+x_1x_2} = ?$$

Ex: write down the exponent in the form $-\frac{1}{2}\vec{x}^T A \vec{x}$.

More generally,

$$Z(A) = \int_{\mathbb{R}^n} d\vec{x}^{\,n}\ e^{-\frac{1}{2}\vec{x}^T A \vec{x}}$$

where $\vec{x} = (x_1, \dots, x_n)$ and the matrix A is diagonalizable with strictly positive eigenvalues (positive definite).

Then there exist an orthogonal matrix O ($\Rightarrow O \cdot O^T = O^T \cdot O = \mathbb{1}$) such that we can define $\vec{y} = O\vec{x}$ and $OAO^T = \Lambda$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \qquad \lambda_i > 0 \qquad i = 1, 2, \cdots n$$

It follows that

$$\vec{x}^T A \vec{x} = \vec{x}^T O^T \Lambda O \vec{x} = \vec{y}^T \Lambda \vec{y}$$

$$Z(A) = \int d\vec{x}^n \, e^{-\frac{1}{2}\vec{x}^T A \vec{x}} = \int d\vec{y}^n \left\| \frac{\partial \vec{x}}{\partial \vec{y}} \right\| e^{-\frac{1}{2}\vec{y}^T \Lambda \vec{y}}$$

determinant of the Jacobian : $\left\| \frac{\partial \vec{x}}{\partial \vec{y}} \right\| = \det(O^T) = 1$

(show as an exerc.)

$$\vec{y}^T \Lambda \vec{y} = \Sigma_{ij} \, y_i \Lambda_{ij} y_j = \Sigma_{ij} \, y_i \lambda_i \delta_{ij} y_j = \Sigma_i \lambda_i y_i^2$$

From this we get

$$= \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\Sigma_i \lambda_i y_i^2} = \prod_i^n \int_{-\infty}^{+\infty} dy_i \; e^{-\frac{1}{2}\lambda_i y_i^2} \overset{①}{=} \prod_i \sqrt{\frac{2\pi}{\lambda_i}} = \frac{(2\pi)^{n/2}}{\sqrt{\lambda_1 \cdots \lambda_n}}$$

$$\det(A) = \det(O^T \Lambda O) = \det(\Lambda)(\det O)^2 = \det \Lambda = \lambda_1 \cdots \lambda_n$$

Hence

$$⑧ \qquad \boxed{Z(A) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}} = \int_{\mathbb{R}^n} d^n\vec{x} \; e^{-\frac{1}{2}\vec{x}^T A \vec{x}}$$

By using ⑧ show that

$$\int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \; e^{-\frac{3}{2}(x_1^2 + x_2^2) + x_1 x_2} = \frac{\pi}{\sqrt{2}}$$

Exercize: Let $p(x,y) = \frac{\sqrt{\det A}}{2\pi} e^{-\frac{1}{2}(a_{11}x^2 + 2a_{12}xy + a_{22}y^2)}$

where $a_{11}, a_{22} > 0$. Show that $q(x) = \int p(x,y)\,dy$ is still a Gaussian distribution. Find the corresponding variance of x.

This is also true for n-dimension Gaussian variables.

We want now to calculate

$$⑨ \qquad Z(A,\vec{b}) = \int d^n x \; e^{-\frac{1}{2}\vec{x}^T A \vec{x} + \vec{x}^T\cdot\vec{b}}$$

we use the same strategy we used before:

$$\vec{\nabla}_x\left(-\frac{1}{2}\vec{x}^T A\vec{x} + \vec{x}^T\cdot\vec{b}\right) = -A\vec{x} + \vec{b} = 0 \implies \vec{x} = A^{-1}\vec{b}$$

$$(\det A \neq 0)$$

We introduce

$$\vec{y} = \vec{x} - A^{-1}\vec{b}$$

$$-\frac{1}{2}\vec{x}^T A\vec{x} + \vec{x}^T\cdot\vec{b} = -\frac{1}{2}\vec{y}^T A\vec{y} + \frac{\vec{b}^T A^{-1}\vec{b}}{2} \qquad \text{(do the calculation).}$$

Hence

$$Z(A,\vec{b}) = \int_{\mathbb{R}^n} d\vec{y}\; e^{-\frac{1}{2}\vec{y}^T A\vec{y} + \underline{\underline{\frac{\vec{b}^T A^{-1}\vec{b}}{2}}}} = e^{\frac{\vec{b}^T A^{-1}\vec{b}}{2}} \underbrace{\underline{\underline{Z(A,0)}}}_{\text{known from Eq. ⑧}}$$

$$\boxed{Z(A,\vec{b}) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}\, e^{\frac{\vec{b}^T A^{-1}\vec{b}}{2}}} \qquad ⑩$$

Eq. ⑩ allows to find the ch.f. of the multivariate Gaussian distrib.

$$p(\vec{x}) = \frac{1}{Z(A,0)} e^{-\frac{1}{2}\vec{x}^T A\vec{x}} \xrightarrow[\vec{b} = i\vec{k} \text{ (in ⑩)}]{} \boxed{\int_{\mathbb{R}^n} d\vec{x}\; p(\vec{x})\, e^{i\vec{k}\cdot\vec{x}} = e^{-\frac{\vec{k}^T A^{-1}\vec{k}}{2}} = \varphi(\vec{k})} \qquad ⑪$$

What is the meaning of A^{-1} ?

The definition of the ch. f. in the multidim. case is :

$$\varphi(\vec{k}) = \int d^s x \; e^{i\vec{k}\cdot\vec{x}} \; p(\vec{x}) \qquad \vec{k} = (k_1, k_2, \dots, k_m)$$

therefore we derive

$$\underbrace{(-i)^s \frac{\partial}{\partial k_i} \frac{\partial}{\partial k_j} \cdots \frac{\partial}{\partial k_\ell}}_{s \text{ many deriv.}} \varphi(\vec{k}) \Big|_{\vec{k}=0} = \int d^s x \; x_i x_j \cdots x_\ell \, p(\vec{x}) = \langle \underbrace{x_i x_j \cdots x_\ell}_{s-\text{variables}} \rangle \qquad (12)$$

s-point correlation function

Let's calculate the 2-point correlation function for a Gaussian distr.

$$\boxed{\langle x_i x_j \rangle = (-i)^2 \frac{\partial}{\partial k_i} \frac{\partial}{\partial k_j} e^{-\frac{\vec{k}^T A^{-1} \vec{k}}{2}} \Big|_{\vec{k}=0} = (A^{-1})_{ij}} \qquad (13)$$

A^{-1} is the 2-point correlation function between a pair of Gauss. r.v.

When A^{-1} is a diagon. matrix, we say that the vars are uncorrel.

In the previous example: $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ $\quad A^{-1} = \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

hence

$$\langle x_1^2 \rangle = \frac{3}{8} = \langle x_2^2 \rangle \qquad \langle x_1 x_2 \rangle = \frac{1}{8} = \langle x_2 x_1 \rangle$$

Notice that, because of symmetry, the s-point correl. funct. for s-variable (s is odd) is zero (the Gaussian remains unchanged when $\vec{x} \to -\vec{x}$).

What happens when we calculate $\langle x_i x_j \cdots x_\ell \rangle$?
Should we do all the derivatives like in eq. (12) ? No!
If the vars are Gaussian then we can use:

$\boxed{\text{Wick's theorem.}}$ Any correlation between an <u>even</u> number of zero-mean Gaussian r.v. can be written down as a sum of products of 2-point correlation functions (A^{-1}).

<u>For instance</u>:

$$\langle x_a x_b x_c x_d \rangle = \underbrace{\langle x_a x_b \rangle}_{(A^{-1})_{ab}} \underbrace{\langle x_c x_d \rangle}_{(A^{-1})_{cd}} + \langle x_a x_c \rangle \langle x_b x_d \rangle + \langle x_a x_d \rangle \langle x_b x_c \rangle$$

In general

(14)
$$\boxed{\langle \underbrace{x_i x_j \cdots x_n x_m}_{s-\text{vars}} \rangle = \sum_{P} (A^{-1})_{i_P j_P} \cdots (A^{-1})_{n_P m_P}}$$

where the sum is over all possible pairings of s indexes, i.e. over all ways of grouping s (even) indexeds $i, j \ldots, n, m$ into pairs (counting pairs even when indexes are equal).

<u>Exercise</u>: show that

$$\langle x_1^2 x_2^2 \rangle = \frac{3}{8} \cdot \frac{3}{8} + \frac{1}{8} \cdot \frac{1}{8} + \frac{1}{8} \cdot \frac{1}{8} = \frac{11}{64}$$

$$\langle x_1^4 \rangle = 3 \left(\frac{3}{8}\right)^2 = \langle x_2^4 \rangle \qquad \langle x_1 x_2^2 \rangle =$$

J. Zinn-Justin, Quantum Field Theory and Critical Phenomena
(Ch. 1).

Important results obtained with characteristic functions

If we are given the joint probability density function $p(x_1, x_2)$ and it happens that $p(x_1, x_2) = p_1(x_1)\, p_2(x_2)$ $(p_1 \neq p_2)$ then the two vars are <u>independent</u>. If, on top of this, $p_1 = p_2$, then we say that the two r.v. are <u>independent and identically distributed</u> (i.i.d.).

If we are given two r.v. that are i.i.d. can we calculate the distribution of their sum?
If $x_1 \sim q(x)$ and $x_2 \sim q(x)$ what is the distribution of $x = x_1 + x_2$?