Models of T.P. 30/9/25

Gaussian Integrals

Consider the Gaussian distribution (PDF):

$$p(x) = c e^{-\frac{a x^2}{2}} \qquad a > 0$$

p must be normalized on $(-\infty, +\infty)$:

$$\int_{-\infty}^{+\infty} p(x)\, dx = 1 \quad \Rightarrow \quad c \int_{-\infty}^{+\infty} e^{-\frac{a x^2}{2}} dx = 1$$

The simplest Gaussian integral:

(1) $$\boxed{\int_{-\infty}^{+\infty} e^{-\frac{a x^2}{2}} dx = \sqrt{\frac{2\pi}{a}}}$$

A more general Gaussian integral is

(2) $$\int_{-\infty}^{+\infty} e^{-\frac{a x^2}{2} + b x} dx = ?$$

To calculate this integral we use a change of vars.
The max of the exponent has changed. Let's find it:

$$\frac{d}{dx}\left(-\frac{a x^2}{2} + b x\right) = -a x + b = 0 \quad \Rightarrow \quad x = \frac{b}{a} \quad (b \text{ real})$$

We introduce $y = x - \frac{b}{a}$

$$-\frac{a x^2}{2} + b x = -\frac{a}{2}\left(y + \frac{b}{a}\right)^2 + b\left(y + \frac{b}{a}\right) = \overset{\text{verify}}{\downarrow} \cdots = -\frac{a y^2}{2} + \frac{b^2}{2a}$$

(3) $$\int_{-\infty}^{+\infty} e^{-\frac{a x^2}{2} + b x} dx = \int_{-\infty}^{+\infty} e^{-\frac{a y^2}{2} + \frac{b^2}{2a}} dy \overset{(1)}{=} \underline{\sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}} \qquad \begin{matrix} a > 0 \\ b \in \mathbb{C} \end{matrix}$$

b can also be a complex number.

Let's calculate the integral in (3) when $b = it$ $t \in \mathbb{R}$.

Let's define

$$\varphi(t) = \int_{-\infty}^{+\infty} dx \, e^{ixt} \sqrt{\frac{a}{2\pi}} \, e^{-\frac{ax^2}{2}}$$

<u>Note</u>:

$\frac{d}{dt} \int f(x,t)\, dx = \int \frac{\partial f(x,t)}{\partial t} dx$ if $f, \partial_t f$ are continuous and

$|f(x,t)| < A(x)$, $|\partial_t f| < B(x)$, $\int A(x)\, dx < \infty$ $\int B(x)\, dx < \infty$

(f is continuous and uniformly bounded).

$$\varphi'(t) = \sqrt{\frac{a}{2\pi}}\, i \int dx \; x \, e^{ixt} e^{-\frac{ax^2}{2}}$$

$$= -\frac{i}{\sqrt{2\pi a}} \int dx \; e^{ixt} \frac{d}{dx} e^{-\frac{ax^2}{2}} \qquad \frac{d}{dx} e^{-ax^2} = -ax e^{-ax^2}$$

$$= -\frac{t}{\sqrt{2\pi a}} \underline{\int dx \; e^{ixt} \; e^{-\frac{ax^2}{2}}} \qquad \text{(by parts)}$$

$$= -\frac{t}{a} \varphi(t)$$

$$\Rightarrow \quad \varphi' = -\frac{t}{a} \varphi$$

This differential eq. is linear and the solution is $\varphi(t) = c e^{-\frac{t^2}{2a}}$ (check this out!). It must be $\varphi(0) = 1 \Rightarrow c = 1$

$$\varphi(t) = e^{-\frac{t^2}{2a}} \underset{b=it}{=} e^{\frac{b^2}{2a}}$$

Characteristic Functions

If we are given a PDF $p(x)$, its char. func. is defined as

$$④ \qquad \varphi(k) = \int e^{ikx}\, p(x)\, dx \equiv \langle e^{ikx} \rangle_p$$

In general $\qquad \langle f \rangle_p \equiv \int f(x)\, p(x)\, dx$

$\varphi(k)$ has a very nice property:

$$\frac{d\varphi}{dk} = \int (ix)\, e^{ikx} p(x)\, dx \longrightarrow -i \left.\frac{d\varphi}{dk}\right|_{k=0} = \int x\, p(x)\, dx = \langle x \rangle$$

and in general:

$$⑤ \qquad (-i)^n \left.\frac{d^n\varphi}{dk^n}\right|_{k=0} = \int x^n\, p(x)\, dx = \langle x^n \rangle$$

n-th moment of P
$n = 1, 2, \ldots$

What is the c.f. of the Gaussian distribution?
Substitute $b \to ik$ in eq. ②, then you get

$$\int_{-\infty}^{+\infty} e^{-\frac{ax^2}{2} + bx}\, dx = \int e^{-\frac{ax^2}{2} + ikx} \underset{\uparrow ③}{=} \sqrt{\frac{2\pi}{a}}\, e^{-\frac{k^2}{2a}}$$

hence the c.f. of the Gaussian is $\left(\sqrt{\frac{1}{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}},\ a = \frac{1}{\sigma^2} \right)$

$$\boxed{\varphi(k) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}\, e^{ikx}\, dx = e^{-\frac{\sigma^2 k^2}{2}}} \qquad ⑥$$

Ex : - show that when the mean of the Gaussian is μ

$$\varphi(k) = e^{ik\mu - \frac{\sigma^2}{2}k^2}$$

- Calculate the c.f. of the uniform distr. $U([a,b])$

and the γ-distribution $p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ $\quad \alpha > 0$, $\beta > 0$.

gamma function