Models of TP 17-10-25

First partial exam on 11th November (Tues.)

We started from

$$\begin{cases} \dot{\vec{x}}(t) = \vec{f}(\vec{x}(t)) \\ \vec{x}(0) = \vec{x}_0 \end{cases} \qquad (1)$$

Fixed points

$$\vec{f}(\vec{x}^*) = \vec{0} \qquad (3)$$

Warning: even though an ODE admits some fixed points, that does not imply that the dynamics will reach them!
In particular this is true when a fixed point is unstable.

Local stability of fixed points

When the fixed points are known, we can try to understand whether they are stable or not. Stability means that, if one introduces a small perturbation to the fixed point, then the perturbation decays in time. This stability is local because the perturbation is only close to the fixed point and small. Therefore one cannot claim anything about what happens for away from the fixed point.

Let's consider the system of ODEs:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

The conditions for the fixed points if $f(x^*,y^*) = 0 \; g(x^*,y^*)$.
Let's introduce a small perturbation $u(t) = x(t) - x^*$, $v(t) = y(t) - y^*$ and $|u|$ and $|v|$ are "small": so we can Taylor expand the system:

$$\dot{u} = u \frac{\partial f}{\partial x}\Big|_* + v \frac{\partial f}{\partial y}\Big|_* + O(u^2, v^2, uv)$$

$$\dot{v} = u \frac{\partial g}{\partial x}\Big|_* + v \frac{\partial g}{\partial y}\Big|_* + O(u^2, v^2, uv)$$

calculated at (x^*, y^*)

So at leading order, we get the linearized system:

$$(4) \qquad \boxed{\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}} \qquad A \equiv \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_*$$

A is the jacobian matrix at the fixed point

This can be easily generalized to system (1) with N deg. of f.
$\vec{u} \equiv (u_1, \dots u_N) \dots \quad u_i = x_i - x_i^*$

$$(5) \qquad \dot{\vec{u}} = A \vec{u} \qquad A_{ij} \equiv \frac{\partial f_i}{\partial x_j}\Big|_{\vec{x}^*}$$

Is it safe to neglect the nonlinear terms? Not always.
When the linear system is a local faithful representation of the nonlinear system?

Hyperbolic fixed points : a fixed point of an N-order system is hyperbolic if all eigenvalues of the linearized system are such that $\mathrm{Re}(\lambda_i) \neq 0$ for any $i = 1, \dots N$.

Hartman - Grobman theorem : the local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the corresponding linearized system.

topologically equivalent means that there exists an homeomorphism (a continuous map with a continuous inverse) which maps one phase portrait into the linearized one with the same direction of time.

If the $\mathrm{Re}(\lambda_i) < 0$ for all $i = 1 \dots N$, we say that the fixed point x^* is asymptotically stable.

Let's focus on eq. (4,5) and write $\vec{u}(t) = e^{\lambda t}\, \vec{v}$, then

$$\lambda \vec{v} = A \vec{v}$$

The eigenvalue are given by the solutions of $\det(\lambda \mathbb{1} - A) = 0$

In the case of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we get

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - T\lambda + D$$

where $T = \mathrm{trace}(A) = a+d$ and $D = \det(A) = ad - bc$.

In this case the eigenvalues are given by

$$\boxed{{}_1\lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}}$$

and therefore the full sol. of the lin. equation is

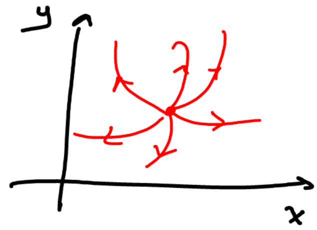
$$\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 \qquad \left(\sum_i e^{\lambda_i t} \vec{v}_i \right)$$

This clearly shows that if $Re(\lambda_i) < 0$ then $\lim_{t \to \infty} \vec{u}(t) = 0$

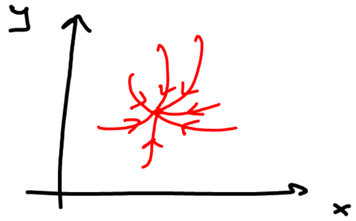
A gallery of phase portraits (2×2 case)

Real eigenvals:

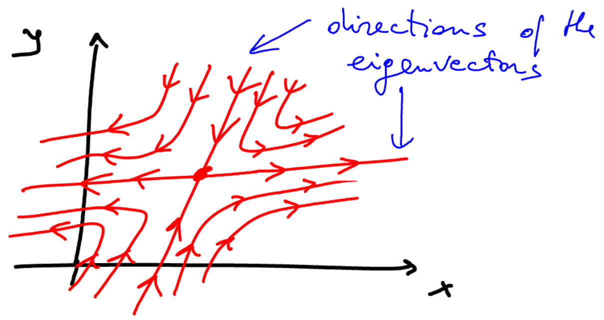
$0 < \lambda_1 < \lambda_2$ unstable node



$\lambda_2 < \lambda_1 < 0$ stable node

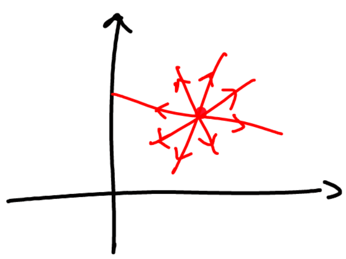


$\lambda_2 < 0 < \lambda_1$ saddle node



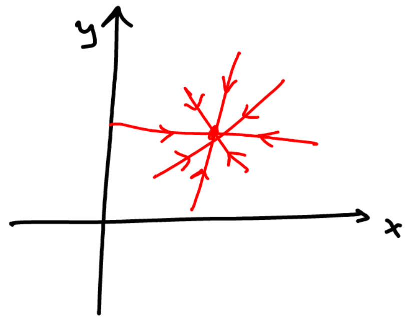
$0 < \lambda_1 = \lambda_2$ unstable star

$(A = \lambda_1 \mathbb{1})$



$\lambda_2 = \lambda_1 < 0$ stable star

$(A = \lambda_1 \mathbb{1})$

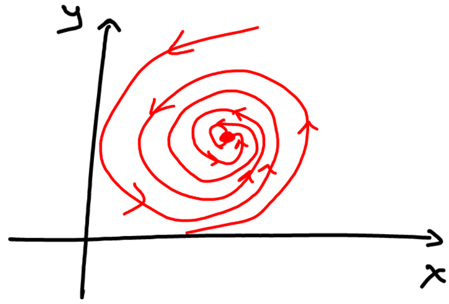


If $A \neq \lambda_1 \mathbb{1}$, then one gets degenerate (un)stable nodes.

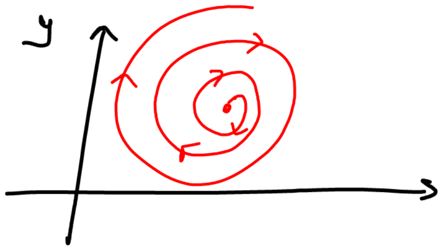
Complex eigenvalues

$$\lambda_{1,2} = \gamma \pm i\omega$$

$\gamma < 0$ stable focus

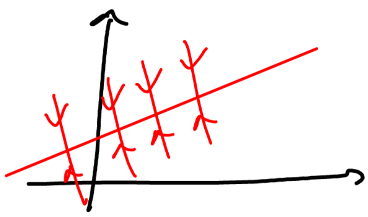


$\gamma > 0$ unstable focus



All the above fixed points are hyperbolic. Here are a few examples of non-hyperbolic fixed points:

$\lambda_1 = 0$ $\lambda_2 > 0$ $(\lambda_2 < 0)$ line of fixed point



$\gamma = 0$ $\lambda_2 = \pm i\omega$

center

In these latter cases the linear stability analysis is not sufficient to determine whether a fixed point is stable or not.

Exercises

1) Find the fixed points of $\begin{cases} \dot{x} = -x + x^3 \\ \dot{y} = -2y \end{cases}$ and classify them.

2) Show that the linearization of the system

$$\begin{cases} \dot{x} = -y + ax(x^2+y^2) \\ \dot{y} = x + ay(x^2+y^2) \end{cases}$$

incorrectly predicts that the origin is a center for all values of a. Indeed, the origin is a stable spiral if $a<0$, and unstable spiral if $a>0$.

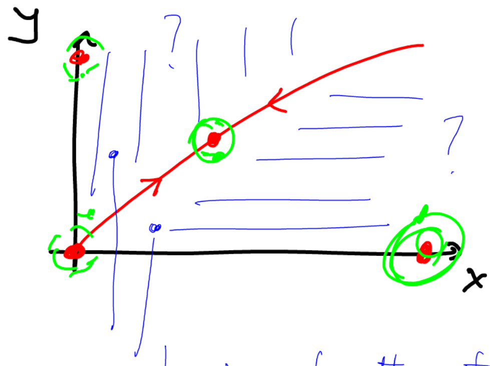
[hint: to analyze the nonlinear system, use polar coord. $x = r\cos\theta$, $y = r\sin\theta$ and find the equations for r and θ. Then interpret the non-lin. system.]

3) The equation of motion of a particle is $\ddot{x} = x - x^3$. Find the fixed points and classify them. Show that the function $E = \frac{\dot{x}^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$ (the energy) is conserved by the dynamics and that the trajectories are closed curves defined by the contours of E. Draw the phase portrait.

4) Study the system of ODEs:

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$
$$x, y \geq 0$$

Find the fixed points and classify them. Show that the phase portrait is

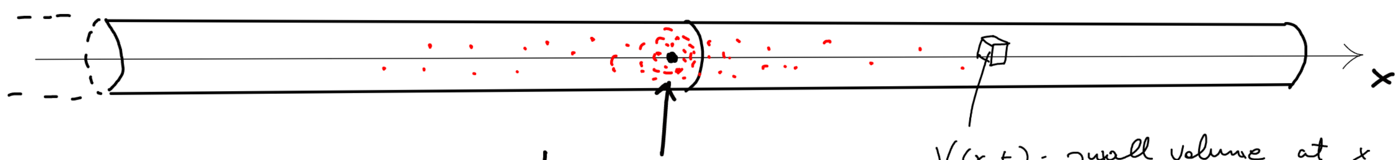


basin of attraction : the set of all initial conditions which end up in one fixed point as $t \to \infty$)

PROPERTIES OF DIFFUSION

A first simple derivation

Let us consider a long and thin tube filled with water.
At time $t=0$ we inject a unit amount of ink at $x=0$



Brownian motion of particles

$$W(x,t) = \frac{\#\text{ of particles in } V(x,t)}{V(x,t)}$$

$W(x,t)$ is the density of "ink" (or Brownian) particles at position $x \in \mathbb{R}$, time $t \geq 0$ as the volume $V \to 0$ and the number of particle $\to \infty$.

$\int_A W(x,t)\,dx$:= prob. to find a particle in the region $A \subseteq \mathbb{R}$.

assuming that $\int_{-\infty}^{+\infty} W(x,t)\,dx = 1$.