$\boxed{4.}$ PERTURBATION THEORY (TIME INDEPENDENT)

DONE IN A WAY THAT IS USEFUL

↳ DEGENERATE UNPERTURBED EIGENSPACES

HYDROGEN ATOM LEVELS $\xrightarrow{\text{SPECIAL RELATIVITY EFFECTS}}$ FINE STRUCTURE $\xrightarrow{N \leftrightarrow e^- \text{ MAGNETIC COUPLING EFFECTS}}$ HYPERFINE STRUCTURE

UNPERTURBED HAMILTONIAN

$$H_0 |\mathcal{E}^{(0)}, J\rangle = \mathcal{E}^{(0)} |\mathcal{E}^{(0)}, J\rangle$$

$\mathcal{E}^{(0)}$: UNP. ENERGY LEVE

J: DEGENERACY

EXAMPLE

$$\begin{pmatrix} \mathcal{E}_0 & & \\ & \mathcal{E}_0 & \\ & & \mathcal{E}_1 \end{pmatrix} \begin{array}{l} \rightarrow |\mathcal{E}_0, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \rightarrow |\mathcal{E}_0, 2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \rightarrow |\mathcal{E}_1, 1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

PERTURBATION λV

λ → SMALL PARAMETER

V ← THIS CAN BE ANYTHING (BUT $\lesssim 1$)

$$\begin{cases} \tilde{\mathcal{E}}_n = \mathcal{E}_n^{(0)} + \lambda \mathcal{E}_n^{(1)} + \lambda^2 \mathcal{E}_n^{(2)} + \dots \\ |\hat{\mathcal{E}}_n\rangle = |\mathcal{E}_n^{(0)}\rangle + \lambda |\mathcal{E}_n^{(1)}\rangle + \dots \end{cases}$$

ALSO WITH J

$$(H_0 + \lambda V)\left(|\mathcal{E}^{(0)}, J\rangle + \lambda |\mathcal{E}^{(1)}, J'\rangle + \dots\right) = \left(\mathcal{E}^{(0)} + \lambda \mathcal{E}^{(1)} + \dots\right)\left(|\mathcal{E}^{(0)}, J\rangle + \lambda |\mathcal{E}^{(1)}, J\right.$$

$\boxed{\text{ORDER } 1 = \lambda^0}$

$$H_0 |\mathcal{E}^{(0)}, J\rangle = \mathcal{E}^{(0)} |\mathcal{E}^{(0)}, J\rangle$$

WELL, AT LEAST IT IS CONSISTENT

ORDER $\lambda = \lambda^1$

$$V|\varepsilon^{(0)}, J\rangle + H_0|\varepsilon^{(1)}, J'\rangle = \varepsilon^{(1)}|\varepsilon^{(0)}, J\rangle + \varepsilon^{(0)}|\varepsilon^{(1)}, J'\rangle \quad *$$

$\Pi = \Pi^{+} = \Pi^2$

I NOW DEFINE THE PROJECTOR ONTO THE $\varepsilon^{(0)}$ EIGENSPACE OF H_0 $\Big\}$ $\Pi_{\varepsilon_0} = \sum_J |\varepsilon^{(0)}, J\rangle\langle\varepsilon^{(0)}, J|$

$$\Pi_{\varepsilon_0} H_0 = H\Pi_0 = \varepsilon^{(0)}\Pi$$

... AND I MULTIPLY LEFT.

$$\Pi_{\varepsilon_0} V|\varepsilon^{(0)}, J\rangle + \underbrace{\Pi_{\varepsilon_0} H}_{}|\varepsilon^{(1)}, J'\rangle = \varepsilon^{(1)}\Pi_{\varepsilon_0}|\varepsilon^{(0)}, J\rangle + \cancel{\varepsilon^{(0)}\Pi_{\varepsilon}|\varepsilon^{(1)}, J'\rangle}$$

$$\cancel{\varepsilon^{(0)}\Pi_{\varepsilon^{(0)}}|\varepsilon^{(1)}, J'\rangle}$$

$$\Pi_{\varepsilon^{(0)}}|\varepsilon^{(0)}, J\rangle = |\varepsilon^{(0)}, J\rangle$$

$$|\varepsilon^{(0)}, J\rangle = \Pi_{\varepsilon^{(0)}}|\varepsilon^{(0)}, J\rangle$$

$$\underbrace{\left(\Pi_{\varepsilon_0} V \Pi_{\varepsilon_0}\right)}_{\text{HERMITIAN}}|\varepsilon^{(0)}, J\rangle = \varepsilon^{(1)}|\varepsilon^{(0)}, J\rangle \qquad \forall J$$

$$(\Pi V \Pi)^{+} = \Pi^{+}V^{+}\Pi^{+} = \Pi V \Pi$$

Eigenvalue Equation

↳ IT TELLS US HOW THE DEGENERACY IS REMOVED

AND

↳ HOW THE RESOLVED STATES LOOK LIKE

NOTICE → THE RESOLVED STATES ARE NOT A SMALL CORRECTION FROM AN ARBITRARY $|\varepsilon, J\rangle$ (SEE EXAMPLE LATER)

$$\lambda H^{(1)}_{\varepsilon^{(0)}}|\varepsilon^{(0)}, J\rangle = \lambda\varepsilon^{(1)}|\varepsilon^{(0)}, J\rangle$$

where $\lambda H^{(1)}_{\varepsilon_0} = \lambda\left(\Pi_{\varepsilon^{(0)}} V \Pi_{\varepsilon^{(0)}}\right)$

EXCERCISE

CONTRACT $*$ WITH $\langle\varepsilon^{(0)}_n, J|$ AND LEARN SOMETHING ABOUT $|\varepsilon^{(1)}, J\rangle$

4A | HIGHER ORDERS OF DEG-REMOVING HAMILTONIANS

↳ USUALLY THE LOWEST NONZERO ORDER COUNTS

NON-DEG

$$H^{(1)}_{\mathcal{E}_0} = \Pi_{\mathcal{E}_0} V \Pi_{\mathcal{E}_0}$$

$$\mathcal{E}^{(1)} = \langle \mathcal{E}_0 | V | \mathcal{E}_0 \rangle$$

$$\mathcal{E}^{(2)} = \sum_{\mathcal{E}_n \neq \mathcal{E}_0} \frac{|\langle \mathcal{E}_n | V | \mathcal{E}_0 \rangle|^2}{\mathcal{E}^{(0)}_0 - \mathcal{E}^{(0)}_n}$$

$$H^{(2)}_{\mathcal{E}_0} = \Pi_{\mathcal{E}_0} V R_{\mathcal{E}_0} V \Pi_{\mathcal{E}_0}$$

$$= \langle \mathcal{E}_0 | V \underbrace{\left(\sum_{\mathcal{E}_n \neq \mathcal{E}_0} \frac{|\mathcal{E}_n \rangle\langle \mathcal{E}_n|}{\mathcal{E}^{(0)}_0 - \mathcal{E}^{(0)}_n} \right)}_{\text{MOORE-PENROSE PSEUDOINVERSE}} V | \mathcal{E}_0 \rangle$$

MOORE-PENROSE PSEUDOINVERSE (INVERT ONLY THE IMAGE)

$$R_{\mathcal{E}_0} = \left(\mathcal{E}_0 \mathbb{1} - H^{(0)} \right)^{\text{"}-1\text{"}}$$

$$H^{(0)} = \begin{pmatrix} \mathcal{E}_0 & & \\ & \mathcal{E}_1 & \\ & & \mathcal{E}_2 \end{pmatrix}$$

$$\downarrow$$

$$R_{\mathcal{E}_0} = \begin{pmatrix} 0 & & \\ & \frac{1}{\mathcal{E}_0 - \mathcal{E}_1} & \\ & & \frac{1}{\mathcal{E}_0 - \mathcal{E}_2} \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 2 \end{pmatrix} \quad A^{\text{"}-1\text{"}} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1/2 \end{pmatrix}$$

$$\mathcal{E}^{(2)} = \langle \mathcal{E}_0 | V \left(\mathcal{E}^{(0)}_0 \mathbb{1} - H_0 \right)^{-1} V | \mathcal{E}$$

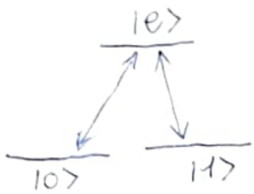
$$H^{(3)}_{\mathcal{E}_0} = \Pi_{\mathcal{E}_0} V R_{\mathcal{E}_0} V R_{\mathcal{E}_0} V \Pi_{\mathcal{E}_0} - \Pi_{\mathcal{E}_0} V \Pi_{\mathcal{E}_0} V R^2_{\mathcal{E}_0} V \Pi_{\mathcal{E}_0}$$

$$H^{(4)}_{\mathcal{E}_0} = \Pi_{\mathcal{E}_0} V R_{\mathcal{E}_0} V R_{\mathcal{E}_0} V R_{\mathcal{E}_0} V \Pi_{\mathcal{E}_0} - \Pi_{\mathcal{E}_0} V R^2_{\mathcal{E}_0} V \Pi_{\mathcal{E}_0} V R_{\mathcal{E}_0} V \Pi_{\mathcal{E}_0}$$
$$- \Pi V \Pi V R V R^2 V \Pi - \Pi V \Pi V R^2 V R V \Pi$$
$$+ \Pi V \Pi V \Pi V R^3 V \Pi$$

☺ NON SENSE

NOT JUST AN EXERCISE

THE LAMBDA SYSTEM (USEFUL FOR RAMAN COUPLING)



$|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad |e\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad |1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$H_0 = \begin{pmatrix}0&&\\&+\Delta&\\&&0\end{pmatrix} \qquad V = \begin{pmatrix}0&\Omega&\\\Omega&0&\Omega\\&\Omega&0\end{pmatrix} = \Omega\begin{pmatrix}&1&\\1&&1\\&1&\end{pmatrix}$$

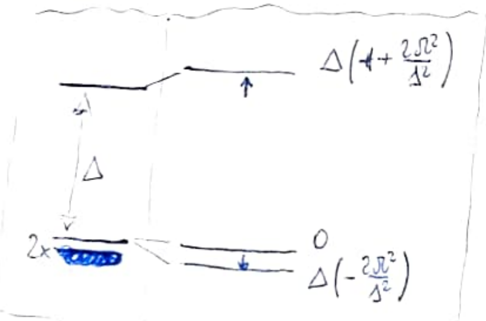
UNPERTURBED HAMILTONIAN

$$\Pi_0 = \begin{pmatrix}1&&\\&0&\\&&1\end{pmatrix} \qquad R_0 = \begin{pmatrix}0&&\\&-\frac{1}{\Delta}&\\&&0\end{pmatrix} \qquad H_0^{(1)} = 0$$

$$\Pi_\Delta = \begin{pmatrix}0&&\\&1&\\&&0\end{pmatrix} \qquad R_\Delta = \begin{pmatrix}+\frac{1}{\Delta}&&\\&0&\\&&+\frac{1}{\Delta}\end{pmatrix} \qquad H_\Delta^{(1)} = 0$$

$$H_0^{(2)} = \Pi_0 V R_0 V \Pi_0 = \frac{\Omega^2}{\Delta}\begin{pmatrix}1&&\\&0&\\&&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&&\\&1&\\&&0\end{pmatrix}\begin{pmatrix}&1&\\1&&1\\&1&\end{pmatrix}\begin{pmatrix}1&&\\&0&\\&&1\end{pmatrix} =$$

$$= -\frac{\Omega^2}{\Delta}\begin{pmatrix}1&0&1\\&&\\1&&1\end{pmatrix}$$



EIGENSTATES $|_0$

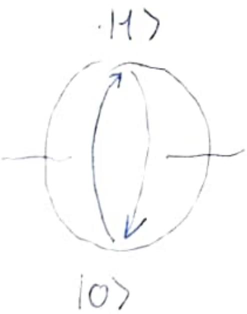
$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \rightsquigarrow \varepsilon^{(2)} = -\frac{2\Omega^2}{\Delta}$

DARK STATE!

$\frac{|0\rangle - |1\rangle}{\sqrt{2}} \rightsquigarrow \varepsilon^{(2)} = \underline{\underline{0}}$ DARK

$$H_2^{(2)} = \ldots\ldots = |e\rangle\langle e|\left(+\frac{2\Omega^2}{\Delta}\right)$$

$$\varepsilon_e^{(0)} + \varepsilon_e^{(2)} = \Delta + \frac{2\Omega^2}{\Delta} = \Delta\left(1+\frac{2\Omega^2}{\Delta^2}\right)$$



FULL RABI FREQUENCY $\left|\left(0-\frac{2\Omega^2}{\Delta}\right)\right| = \frac{2\Omega^2}{\Delta} = \Delta\left(\frac{2\Omega^2}{\Delta^2}\right) \ll \Delta$

SLOW

SLOW MOTION (MACRO)

FAST MOTION (MICRO)

LIKE A SPINNING TOP PRECESSION

$\boxed{\text{SAME}}$ PROBLEM BUT NO PERTURBATION THEORY → EXACT

$\hbar = 1$

$$H_0 = \begin{pmatrix} 0 & & \\ & +\Delta & \\ & & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & \Omega & \\ \Omega & 0 & \Omega \\ & \Omega & 0 \end{pmatrix}$$

$$H_{TOT} = H_0 + V = \Delta \begin{pmatrix} 0 & \eta & \\ \eta & 1 & \eta \\ & \eta & 0 \end{pmatrix} \quad \text{WITH } \eta = \frac{\Omega}{\Delta} \text{ SMALL PARAMETER}$$

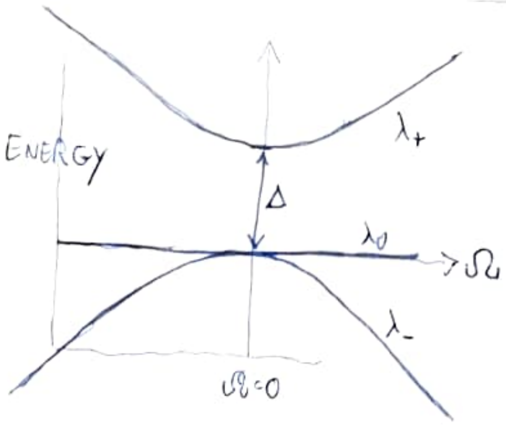
$$\begin{vmatrix} -\lambda & \eta & \\ \eta & 1-\lambda & \eta \\ & \eta & -\lambda \end{vmatrix} = P(\lambda) = \lambda^2(1-\lambda) + 2\lambda\eta^2 = -\lambda(\lambda^2 - \lambda - 2\eta^2)$$

$$P(\lambda) = 0 \rightarrow \lambda = \begin{cases} \frac{1}{2}\left(1 \pm \sqrt{1+8\eta^2}\right) \\ 0 \end{cases}$$

Δ —— $\quad \Delta\left(\frac{1}{2} + \frac{1}{2}\sqrt{1+8\frac{\Omega^2}{\Delta^2}}\right) \approx \Delta\left(\frac{1}{2} + \frac{1}{2}\left(1 + 4\frac{\Omega^2}{\Delta^2}\right)\right) = \Delta\left(1 + \frac{2\Omega^2}{\Delta^2}\right)$ ✓

0 —— $\times 2$ $\quad 0$ ⇐ EXACT

$\Delta\left(\frac{1}{2} - \frac{1}{2}\sqrt{1+8\frac{\Omega^2}{\Delta^2}}\right) \approx \Delta\left(-\frac{2\Omega^2}{\Delta^2}\right)$ ✓



SECULAR MOTION

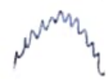
~ LIKE A RABI σ^x

$|0\rangle \leftrightarrow |1\rangle$

FREQUENCY $\left|0 - \frac{2\Omega^2}{\Delta}\right| = \frac{2\Omega^2}{\Delta}$

PERIOD $\frac{\pi\Delta}{\Omega^2}$ (SLOW)

MICROMOTION



see MARCO's LECTURE

WIGGLES OF POPULATION OF $|e\rangle$

FREQUENCY $\sim\Delta$

PERIOD $\frac{2\pi}{\Delta}$ (FAST)

$\boxed{\text{EXERCISE}}$ ⇒ AMPLITUDE OF THE MICROMOTION? MAX $p = |\langle e|\psi_0(t)\rangle|^2$

5. TIME-ORDERED EXPONENTIAL

NOW $H(t)$ TIME DEP

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle \qquad |\psi(t)\rangle = U(t,t_0) |\psi_0\rangle \quad \text{WITH}$$

$$\boxed{\begin{array}{ll} U(t_2,t_1)\,U(t_1,t_0) = U(t_2,t_0) & \\ \text{ADDITIVITY} & \\ U(t,t) = \mathbb{1} & \text{IDENTITY CONNECTION} \end{array}}$$

$$i \frac{d}{dt} U(t,t_0) \cancel{|\psi_0\rangle} = H(t)\, U(t,t_0) \cancel{|\psi_0\rangle}$$

SMALL INCREMENT $\gg$ $\delta t \rightsquigarrow$

$$\underbrace{U(t+\delta t, t_0) = U(t,t_0) + \delta t \frac{dU}{dt}(t,t_0) + \mathcal{O}(\delta t^2)}_{\text{TAYLOR}}$$

$$U(t+\delta t)\,\cancel{U(t,t_0)} = U(t+\delta t, t_0) = \left(\mathbb{1} - i\delta t H(t)\right) \cancel{U(t,t_0)} + \mathcal{O}(\delta t^2)$$

$$U(t+\delta t, t) \approx \exp\left(-iH(t)\,\delta t\right) + \mathcal{O}(\delta t^2)$$

t_0 DIVIDE IN N INTERVALS $t > t_0$

$$\delta t = \frac{t-t_0}{N}$$

$$U(t,t_0) = U(t, t-\delta t)\, U(t-\delta t, t-2\delta t) \cdots U(t_0+\delta t, t_0)$$

THE ORDER IS IMPORTANT

$$= \exp\left(-i\delta t\, H(t-\delta t)\right) \exp\left(-i\delta t\, H(t-2\delta t)\right) \cdots \exp\left(-i\delta t\, H(t_0)\right) + \mathcal{O}(\delta t^2)$$

$$U(t,t_0) = \lim_{\substack{\delta t \to 0 \\ (\text{or } N \to \infty)}} \left(\text{THIS EXPRESSION} \right) = \mathcal{T}\exp\left(-i \int_{t_0}^{t} H(t')\, dt'\right)$$

TIME ORDERED EXPONENTIAL

THE HISTORICAL REASON WHY IT IS WRITTEN LIKE THIS IS DUE TO THE DYSON SERIES

NO FACTORIAL!

ALSO TRUE

$$U(t,t_0) = \mathbb{1} + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, H(t_1) H(t_2) \cdots H(t_n)$$