Open Systems

A QUANTUM STATE CAN HOLD LESS INFORMATION THAN A DETERMINISTIC STATE. IN THAT CASE THE STATE IS NOT PURE AN WE DESCRIBE IT AS A PROBABILISTIC DISTRIBUTION (MIXTURE) OF PURE STATES

$$\overset{\text{PURE}}{|\psi\rangle} \longrightarrow \overset{\text{MIXED}}{(p_\alpha, \underset{\substack{\uparrow \\ \text{NOT NECESSARILY ORTHOGONAL}}}{|\psi_\alpha\rangle})} \quad \text{WITH} \quad \sum_\alpha p_\alpha = 1$$

OBSERVABLES

$$\langle O \rangle = \sum_\alpha p_\alpha \langle \psi_\alpha | O | \psi_\alpha \rangle = \sum_\alpha p_\alpha \, \mathrm{Tr}[O \, \overbrace{|\psi_\alpha\rangle\langle\psi_\alpha|}^{\text{RANK-1 PROJECTOR}}]$$

$$= \sum \mathrm{Tr}[O (p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|)] = \mathrm{Tr}[O (\sum p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|)]$$

$$= \mathrm{Tr}[O \rho] \qquad \boxed{\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|}$$

THE DENSITY MATRIX CONTAINS ALL THE INFORMATION

PROPERTIES

① ρ IS POSITIVE (SEMIDEFINITE) $\quad \langle \phi | \rho | \phi \rangle \geqslant 0$

- ↳ IT FOLLOWS THAT ρ IS HERMITIAN & ALL $\geqslant 0$ EIGENVALUES
- ↳ ITS EIGENBASIS SHOWS THE MIXTURE IN DISTINGUISHABLE STATES

② ρ CAN BE NORMALIZED $\quad \mathrm{Tr}[\rho] = \langle \mathbb{1} \rangle = 1$

③ STATES IN THE MIXTURE DO NOT INTERFERE

$$\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$$

PROBABILITY OF MEASURING STATE $|\phi\rangle$

$$P_1 = \mathrm{Tr}[|\phi\rangle\langle\phi| \rho_1] = \langle \phi | \rho_1 | \phi \rangle$$

$$P_2 = \langle \phi | \rho_2 | \phi \rangle$$

$$P_{50/50} = \langle \phi | \rho | \phi \rangle = \langle \phi | \left(\frac{\rho_1}{2} + \frac{\rho_2}{2} \right) | \phi \rangle = \frac{P_1}{2} + \frac{P_2}{2}$$

CLASSICAL STATISTICS

QUANTIFIERS OF DETERMINISM

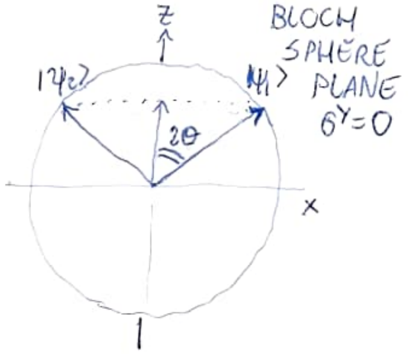
PURITY $P = \mathrm{Tr}[\rho^2]$ $\qquad \frac{1}{\dim_S} \leq \mathrm{Tr}[\rho^2] \leq 1$ $\qquad \frac{\text{IFF} = 1}{\text{PURE STATE}}$

ENTROPY $S = -\mathrm{Tr}[\rho \log_{\uparrow} \rho]$ $\qquad 0 \leq S \leq \log(\dim_S)$ $\qquad \frac{\text{IFF} = 0}{\text{PURE STATE}}$

THE BASE OF THE LOG DEFINES THE UNIT OF ENTROPY

EXAMPLE (QUBIT) $\quad |\psi_1\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad |\psi_2\rangle = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$

$\langle \sigma^z \rangle$	$\cos 2\theta$	$\cos 2\theta$
$\langle \sigma^x \rangle$	$\sin 2\theta$	$-\sin 2\theta$
$\langle \sigma^y \rangle$	0	0



50/50 MIXTURE $\quad \begin{cases} |\psi_1\rangle & p = \frac{1}{2} \\ |\psi_2\rangle & p = \frac{1}{2} \end{cases}$

$$\rho_0 = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2| = \frac{1}{2}\begin{pmatrix} \cos^2\theta & \cos\sin \\ \cos\sin & s^2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} c^2 & -cs \\ -cs & s^2 \end{pmatrix}$$

$$\rho = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad \begin{matrix} \mathrm{Tr}[\rho] = 1 \\ \rho \geq 0 \end{matrix}$$

ρ - IS ALSO MIXTURE OF $|0\rangle$ AND $|1\rangle$ $\Big\}$ $\begin{cases} |0\rangle & p = \cos^2\theta \\ |1\rangle & p = \sin^2\theta \end{cases}$ $\quad \rho = \cos^2\theta\, |0\rangle\langle 0| + \sin^2\theta\, |1\rangle\langle 1|$

$\langle \sigma^z \rangle = \cos 2\theta \qquad \langle \sigma^x \rangle = \langle \sigma^y \rangle = 0$ ✓ CHECKS OUT

$$\rho = \begin{pmatrix} \frac{\cos 2\theta + 1}{2} & \\ & \frac{1 - \cos 2\theta}{2} \end{pmatrix} = \frac{\mathbb{1}}{2} + \frac{1}{2}\cos 2\theta\, \sigma^z$$

$$\mathrm{Tr}[\rho^2] = \frac{1}{2}\left(1 + \cos^2 2\theta\right) \begin{matrix} \nearrow 1 \begin{cases} 2\theta = 0 \\ 2\theta = \pi \end{cases} \\ \searrow 0 \left\{ 2\theta = \frac{\pi}{2} \right. \end{matrix}$$

EXERCISE

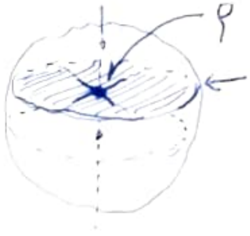
CALCULATE S

$$|\psi(\varphi)\rangle = \begin{pmatrix}\cos\theta \\ e^{i\varphi}\sin\theta\end{pmatrix}$$

COMPLETE UNCERTAINTY OVER φ

$$dp(\varphi) = \frac{1}{2\pi}\,d\varphi$$

$$\rho = \int |\psi(\varphi)\rangle\langle\psi(\varphi)|\,dp(\varphi) = \int_0^{2\pi}\begin{pmatrix}\cos^2\theta & e^{-i\varphi}\sin\theta\cos\theta \\ c.c. & \sin^2\theta\end{pmatrix}\frac{d\varphi}{2\pi} = \begin{pmatrix}\cos^2\theta & 0 \\ 0 & \sin^2\theta\end{pmatrix}$$



THAT IS WHAT HAPPENS WHEN YOU LOSE INFO ON THE RELATIVE PHASE, I.E. WHEN YOU "DEPHASE"

CLOSED SYSTEM DYNAMICS FOR DENSITY MATRICES

$\rho = \sum p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ NO INTERFERENCE ⇒ EVERY MEMBER EVOLVES BY ITSELF
PROBABILITIES ARE STATIC $\dot{p}_\alpha = 0$

$$\dot\rho = \sum p_\alpha |\dot\psi_\alpha\rangle\langle\psi_\alpha| + \sum p_\alpha |\psi_\alpha\rangle\langle\dot\psi_\alpha| = \sum p_\alpha\left(-\frac{i}{\hbar}H|\psi\rangle\langle\psi| + \frac{i}{\hbar}|\psi\rangle\langle\psi|H\right)$$

$$= \sum p_\alpha \frac{i}{\hbar}\left[|\psi_\alpha\rangle\langle\psi_\alpha|, H\right] = \left[\sum p_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|, H\right]\frac{i}{\hbar} = +\frac{i}{\hbar}[\rho, H]$$

$$\boxed{\dot\rho = \frac{i}{\hbar}[\rho, H]}$$

SCHRÖDINGER EQ. FOR DENSITY MATRICES (SCHR. PICTURE)

(SIMILAR TO HEISENBERG PICTURE BUT WITH A MINUS)

$[\rho, H] = 0$ → ρ IS STATIONARY

GIBBS/BOLTZMANN ENSEMBLE

$$\rho = \frac{\exp\left(-\frac{H}{k_B T}\right)}{\mathrm{Tr}\left[\exp\left(-\frac{H}{k_B T}\right)\right]}$$

↑ QUANTUM SYSTEM IN CONTACT WITH FIXED T RESERVOIR EQUILIBRATES HERE

① STATIONARY ⟷ EQUILIBRIUM

② MAXIMISES S AT FIXED INTERNAL ENERGY $\mathrm{Tr}[H\rho]$

-OR-

③ MINIMIZES THE FREE ENERGY

$F = \mathrm{Tr}[H\rho] - T\,S(\rho)$ AT FIXED T TEMPERATURE

➔ BUT HOW TO GET THERE ?

REDUCED DENSITY OPERATOR/MATRIX

BIPARTITE SYSTEM

A | B

$$|\psi_{AB}\rangle = \sum_{ab}{}' c_{ab}\, |\psi_a\rangle_A \underset{\uparrow}{\otimes} |\phi_b\rangle_B$$

TENSOR PRODUCT STRUCTURE

AN OBSERVABE ACTING ONLY ON A

$$\mathcal{O}_{AB} = \mathcal{O}_A \otimes \mathbb{1}_B$$

$$\begin{aligned}
\langle \mathcal{O} \rangle &= \mathrm{Tr}\left[(\mathcal{O}_A \otimes \mathbb{1}_B)\, \rho_{AB} \right] = \\
&= \sum_{ab}{}' \langle a|_A \langle b|_B\, (\mathcal{O}_A \otimes \mathbb{1}_B)\, \rho_{AB}\, |a\rangle_A |b\rangle_B \\
&= \sum_{ab}{}' \langle a|\, \mathcal{O}_A \left(\langle b|_B\, \rho_{AB}\, |b\rangle_B \right) |a\rangle_A = \\
&= \sum_{a}{}' \langle a|\, \mathcal{O}_A\, \rho_A\, |a\rangle = \mathrm{Tr}\left[\mathcal{O}_A\, \rho_A \right]
\end{aligned}$$

WHERE

$$\overset{\substack{\text{REDUCED} \\ \text{D.M.} \\ \downarrow}}{\rho_A} = \sum_{b}{}' \langle b|_B\, \rho_{AB}\, |b\rangle_B = \mathrm{Tr}_B\left[\rho_{AB} \right]$$

PARTIAL TRACE (OVER B)

EXAMPLES

ENTANGLED (QUANTUM CORRELATED)

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle \right) \dashrightarrow \rho_{AB} = \frac{1}{2}\begin{pmatrix} 1 & & & 1 \\ & 0 & & \\ & & 0 & \\ 1 & & & 1 \end{pmatrix} \qquad \rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbb{1}}{2}$$

ENTANGLED STATE

COMPLETELY MIXED

DIFFERENT

CLASSICALLY CORRELATED

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11| \right) = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \qquad \rho_A = \frac{\mathbb{1}}{2}$$

SAME

FLIP A COIN AND WRITE THE RESULT TWICE

$\Uparrow$

NOT ENTANGED

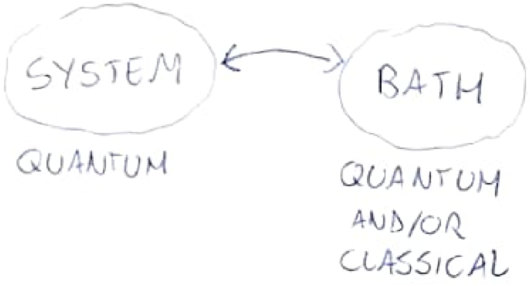
UNCORRELATED

$$\rho = \frac{\mathbb{1}}{2}_A \otimes \frac{\mathbb{1}}{2}_B = \frac{1}{4}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \rho_A = \frac{\mathbb{1}}{2}$$

STILL THE SAME

THE PARTIAL TRACE
HIDES/DELETES/AVERAGES OVER
CORRELATIONS (BETWEEN A AND B)
QUANTUM OR CLASSICAL

The Master Equation

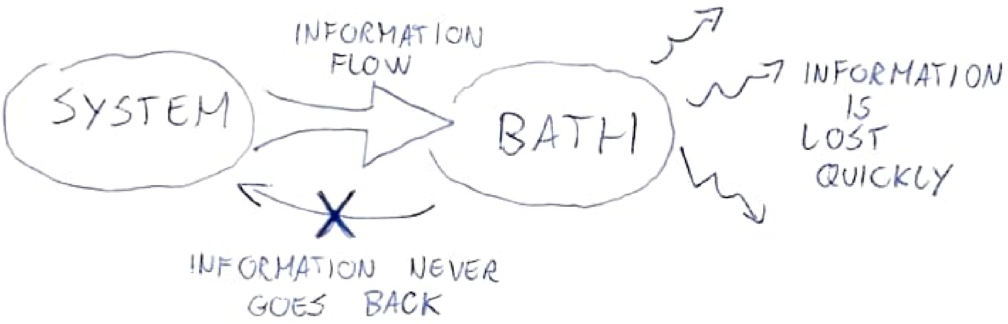


IN GENERAL, TO DESCRIBE THE DYNAMICS, YOU NEED TO TRACK THE (QUANTUM) EVOLUTION OF BOTH SYSTEM + BATH TOGETHER

↳ WHEN CAN WE DESCRIBE THE SYSTEM ALONE AND DESCRIBE ITS EVOLUTION ?

BORN - MARKOV DYNAMICS

! WHEN... THE BATH LOSES IMMEDIATELY MEMORY OF THE SYSTEM (FOR SURE IT WORKS)

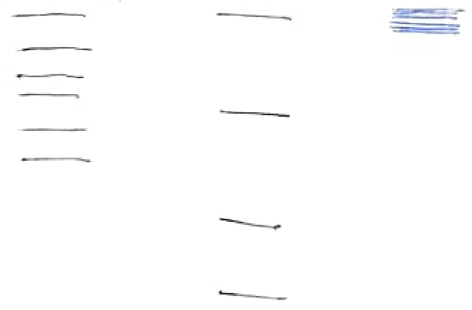


PHYSICAL REQUIREMENT ▷ THE BATH MUST BE FAST 1 AND LARGE 2

1 FAST

$$H_{TOT} = \underbrace{H_{SYS}} + \underbrace{H_{BATH}} + \underbrace{H_{INT}}$$

TIMESCALE SEPARATION



$$\tau_B \ll \tau_S \underset{\sim}{\ll} \tau_{INT}$$

REQUIRED FOR BORN-MARKOV APPROX

REQUIRED FOR QUANTUM DYNAMICS

2 LARGE

$$\dim_{BATH} \gg \dim_{SYS}$$

THE BATH MUST HAVE SPACE TO "STORE AND FORGET" INFO ABOUT THE SYSTEM

MICROSCOPIC DERIVATION OF THE MASTER EQUATION

$$H = H_{SYS} + H_{BATH} + H_{INT} \qquad \begin{pmatrix} H_B = \mathbb{1}_S \otimes H_B \\ H_S = H_S \otimes \mathbb{1}_B \end{pmatrix}$$

COMPARE WITH THE FORMAL DERIVATION FROM ADDITIVITY

$$\left(H_{INT} = \sum_{\alpha}' S_\alpha B_\alpha \right)$$

STEP 1 CHANGE REFERENCE FRAME - INTERACTION PICTURE

REFERENCE CHANGE $\quad U(t) = \exp\left(\frac{it}{\hbar}(H_S + H_B)\right)$

HAMILTONIAN OF THE NEW FRAME ▷ $\quad H'(\text{NEW FRAME}) = UHU^\dagger \quad + \quad i\hbar \dot{U} U^\dagger$

OLD HAMILTONIAN TRANSFORMED -(PLUS)- APPARENT FORCES

$[U, H_S \otimes \mathbb{1}] = 0 = [U, \mathbb{1} \otimes H_B]$

EXERCISE → PROVE THAT $i\dot{U}U^\dagger$ IS ALWAYS HERMITIAN FOR GENERIC $U(t)$

$$H'_{\text{NEW FRAME}} = \tilde{H}_{INT} = e^{\frac{it}{\hbar}(H_S+H_B)} H_{INT}\, e^{-\frac{it}{\hbar}(H_S+H_B)}$$

AND NOTHING ELSE

TRANSFORMED INTERACTION

$\rho(t)$ OLD FRAME ⟫ $\rho'(t)$ NEW FRAME $= U(t)\,\rho(t)\,U^\dagger(t)$

THIS WAY $\mathrm{Tr}[\mathcal{O}'\rho'] = \mathrm{Tr}[\mathcal{O}\rho]$

$$\boxed{\dot{\rho}'(t) = +\frac{i}{\hbar}\left[\rho'(t), \tilde{H}_{INT}(t)\right]}$$

↓ SIMPLE INTEGRATION $\quad \int_0^t dt$

STEP 2 INTEGRATE AND PLUG-IN

$$\rho'(t) - \rho'(0) = \frac{i}{\hbar}\int_0^t \left[\rho'(t), \tilde{H}_{INT}(t)\right] dt$$

PLUG BACK INSIDE THE EQUATION AND GET...

$$\rho'(t) = \rho'(0) + \frac{i}{\hbar}\int_0^t \left[\rho'(t), \tilde{H}_{INT}(t)\right] dt$$

$$\dot{\rho}'(t) = \underbrace{\frac{i}{\hbar}\left[\rho'(0), \tilde{H}_{INT}(t)\right]}_{\text{PIECE } \triangle} - \underbrace{\frac{1}{\hbar^2}\int_0^t \left[\left[\rho'(t'), \tilde{H}_{INT}(t')\right], \tilde{H}_{INT}(t)\right] dt'}_{\text{PIECE } \square}$$

BORN APPROXIMATION:

THE STATE OF THE BATH IS UNALTERED

STATIONARY BATH

$$\left(\begin{matrix} [\rho_B, H_B + H_S] \\ \rho_B' = \rho_B \end{matrix} \right)$$

$$\rho(t) = \rho_S(t) \otimes \rho_B(0) \xrightarrow[\text{FRAME}]{\text{INTERACTION}} \rho'(t) = \rho_S'(t) \otimes \rho_B(0)$$

$$\mathrm{Tr}_B[\text{PIECE } \Delta] = \frac{i}{\hbar} \mathrm{Tr}_B \left([\rho_S'(0) \otimes \rho_B(0), \tilde{H}_{INT}(t)] \right) =$$

$$= \frac{i}{\hbar} \sum_\alpha \left([\rho_S'(0), \tilde{S}_\alpha(t)]\, \mathrm{Tr}[\rho_B(0)\, \tilde{B}_\alpha(t)] \right) = \frac{i}{\hbar} \sum \underset{\langle B_\alpha \rangle}{\underset{\shortparallel}{\langle \tilde{B}_\alpha \rangle}} [\rho_S'(0), \tilde{S}_\alpha(t)]$$

REDEFINE THE MODEL

$$\boxed{\begin{aligned} H_S &\to H_S + \sum S_\alpha \langle B_\alpha \rangle \\ H_{INT} &\to \sum S_\alpha (B_\alpha - \langle B_\alpha \rangle) \end{aligned}}$$

AFTER THIS RE-DEFINITION THE NEW $\langle B_\alpha \rangle = 0$

$$\mathrm{Tr}_B(\text{PIECE } \Delta) = 0.$$

THE SECOND PIECE

$$\dot{\rho}_S'(t) = + \frac{1}{\hbar^2} \int_0^t \mathrm{Tr}_B \left([[\tilde{H}_{INT}(t'), \rho_S'(t') \otimes \rho_B(0)], \tilde{H}_{INT}(t)] \right) dt'$$

NAKAJIMA-ZWANZIG EQUATION

$$H_{INT} = \sum_\alpha S_\alpha \otimes B_\alpha = \sum S_\beta^+ \otimes B_\beta^+$$

$$\dot{\rho}_S'(t) = \frac{1}{\hbar^2} \int_0^t \tilde{S}_\alpha(t')\, \rho_S'(t')\, \tilde{S}_\beta^+(t)\, \underbrace{\mathrm{Tr}[\rho_B(0)\, \tilde{B}_\alpha(t')\, \tilde{B}_\beta^+(t)]}_{\mathrm{Tr}[\rho_B(0)\, \tilde{B}_\alpha(0)\, \tilde{B}_\beta^+(t-t')]}\, dt' + \left(\begin{matrix}\text{OTHER} \\ \text{THREE} \\ \text{COMPONENTS}\end{matrix} \right)$$

$$= \langle \tilde{B}_\alpha(0)\, \tilde{B}_\beta^+(t-t') \rangle = G_{\alpha\beta}(t-t')$$

PUTTING PIECES BACK

$$\dot{\rho}_S'(t) = + \frac{1}{\hbar^2} \sum_{\alpha\beta} \int_0^t \Big\{ G_{\alpha\beta}(t-t') \left(\tilde{S}_\alpha(t')\, \rho_S'(t')\, \tilde{S}_\beta^+(t) - \tilde{S}_\beta^+(t)\, \tilde{S}_\alpha(t')\, \rho_S'(t') \right) +$$

$$+ G_{\beta\alpha}(t'-t) \left(\tilde{S}_\beta^+(t)\, \rho_S'(t')\, \tilde{S}_\alpha(t') - \rho_S'(t')\, \tilde{S}_\alpha(t')\, \tilde{S}_\beta^+(t) \right) \Big\}\, dt$$

MARKOV APPROXIMATION

TIME CORRELATIONS OF THE BATH DECAY SUPER FAST

$$\mathcal{G}_{\alpha\beta}(t) \propto \mathcal{G}_{\alpha\beta}(0)\,\delta(t)$$

« FOR SANITY, LET US DISREGARD DIMENSIONS $\hbar = 1$

$$\dot{\rho}_s'(t) = \sum_{\alpha\beta} \mathcal{G}_{\alpha\beta}\left(\tilde{S}_\alpha \rho_s' \tilde{S}_\beta^+ - \tilde{S}_\beta^+ \tilde{S}_\alpha \rho_s'\right)_t + \mathcal{G}_{\beta\alpha}\left(\tilde{S}_\beta^+ \rho_s' \tilde{S}_\alpha - \rho_s' \tilde{S}_\alpha \tilde{S}_\beta^+\right)$$

BACK TO THE LAB FRAME $\rho_s(t) = e^{-itH_s}\,\dot{\rho}_s\,e^{+itH_s}$

$$\dot{\rho}_s(t) = +i[\rho_s^{(t)}, H_s] + \sum_{\alpha\beta} \underset{\langle B_\alpha B_\beta^\dagger \rangle}{\underset{\shortparallel}{\mathcal{G}_{\alpha\beta}}}\left(S_\alpha \rho_s(t) S_\beta^+ - S_\beta^+ S_\alpha \rho_s(t)\right) + \underbrace{\mathcal{G}_{\beta\alpha}\left(S_\beta^+ \rho_s(t) S_\alpha - \rho_s(t) S_\alpha S_\beta^+\right)}_{= \; \mathcal{G}_{\alpha\beta}\left(S_\alpha \rho S_\beta^+ - \rho S_\beta^+ S_\alpha\right)}$$

USING $\sum_\alpha S_\alpha B_\alpha = \sum_\beta S_\beta^+ B_\beta^+$ EXTEND VECTOR OF OPERATORS

$$\dot{\rho}_s = \frac{i}{\hbar}[\rho_s, H_s] + \sum_{\alpha\beta} \mathcal{G}_{\alpha\beta}\left(2\, S_\alpha \rho S_\beta^+ - \{S_\beta^+ S_\alpha, \rho\}\right)$$

$\mathcal{G}_{\alpha\beta} = \mathrm{Tr}[\rho_B B_\alpha B_\beta^+] \rightarrow \mathcal{G}_{\alpha\beta}$ IS POSITIVE SEMIDEFINITE $\mapsto$ CAN BE DIAGONALIZED $u_{\alpha J}^+ \underset{\geq 0}{\gamma_J} u_{J\beta}$

REDEFINE $L_J = \sum_\alpha u_{J\alpha} S_\alpha \sqrt{2}$

$$\boxed{\dot{\rho}_s = \frac{i}{\hbar}[\rho, H_s] + \sum_J \underset{\substack{\Uparrow \\ \text{DIMENSION OF } t^{-1}}}{\underset{\geq 0}{\gamma_J}}\Big(\overset{\substack{\text{DIMENSIONLESS LINDBLADIANS} \\ \downarrow}}{L_J} \rho L_J^+ - \frac{1}{2}\{L_J^+ L_J, \rho\}\Big)}$$

$\dot{\rho} = \underset{\substack{\uparrow \\ \text{LIOUVILLIAN (LINEAR)}}}{\mathcal{L}}(\rho)$

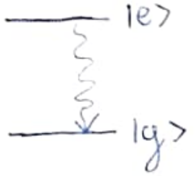
LINDBLAD MASTER EQUATION

NOTICE $\frac{d}{dt}\mathrm{Tr}[\rho] = 0$ ← NO LOSS OF TOTAL PROBABILITY

EXAMPLE 1 [MARKOVIAN DYNAMICS] Decay

2-Level system

$$H_s = \hbar\omega\,|e\rangle\langle e|$$



ONE LINDBLADIAN $L = |g\rangle\langle e|$ WITH RATE γ • { PROBABICISTIC EVENT OF JUMPING TO |g>

$$\rho_0 = \begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix}$$

WHERE $c_0 = 1 - a_0$

$$\underbrace{|b_0|^2 \leq a_0 c_0}_{\text{POSITIVITY CONDITION}}$$

$$\rho(t) = \begin{pmatrix} a(t) & b(t) \\ b^*(t) & \underset{\overset{\shortparallel}{1-a(t)}}{c(t)} \end{pmatrix}$$

$$\dot\rho(t) = -\frac{i}{\hbar} H\rho + \frac{i}{\hbar}\rho H + \gamma L\rho L^\dagger - \frac{1}{2}\gamma L^\dagger L\rho - \frac{\gamma}{2}\rho \underbrace{L^\dagger L}$$

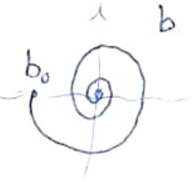
$$H = \hbar\omega\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \dot a & \dot b \\ \dot b & \dot c \end{pmatrix} = i\omega\left(\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ b^* & c \end{pmatrix}\right) + \gamma\left(\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 0 \\ b^* & c \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}\right)$$

DIFFERENTIAL EQUATIONS FOR b AND c → EASY

$$\begin{cases} \dot b = \left(i\omega - \frac{\gamma}{2}\right) b(t) \\ \dot c = -\gamma\, c(t) \end{cases} \Rightarrow \begin{array}{l} b(t) = b_0\, e^{\left(i\omega - \frac{\gamma}{2}\right)t} \\ c(t) = c_0\, e^{-\gamma t} \geq 0 \end{array}$$

⋘ SPIRAL



$a = (1-c)$

$$\rho(t) = \begin{pmatrix} 1 - c_0 e^{-\gamma t} & e^{\left(i\omega - \frac{\gamma}{2}\right)t} b_0 \\ e^{\left(-i\omega - \frac{\gamma}{2}\right)t} b_0^* & c_0 e^{-\gamma t} \end{pmatrix}$$

$$P_e = \langle e|\rho(t)|e\rangle = c_0\, e^{-\gamma t}$$

EXPONENTIAL DECAY

↓ $t \to \infty$ STEADY STATE

$$\rho(t=\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |g\rangle\langle g|$$

UNIQUE!

EXAMPLE 2 Dephasing

$$H = \hbar\omega |e\rangle\langle e|$$

$$L = |e\rangle\langle e| \quad \text{RATE } \gamma \quad \begin{cases}\text{PROBABILISTIC} \\ \text{PHASE} \\ \text{FLIP}\end{cases}$$

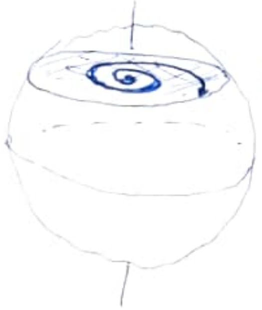
$$L = L^\dagger = L^\dagger L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = H \frac{1}{\hbar\omega}$$

$$\begin{pmatrix} \dot a & \dot b \\ \dot b^* & \dot c \end{pmatrix} = i\omega \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} + \gamma\left(\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 0 \\ b^* & c \end{pmatrix}\right)$$

$$c(t) = c_0 \qquad a(t) = 1 - c_0 \qquad b(t) = e^{(i\omega - \frac{\gamma}{2})t} b_0$$

$\langle \sigma^z \rangle(t) =$ [illegible] $a(t) - c(t) = 1 - 2c_0$ CONSTANT

$$\langle \sigma^x \rangle(t) = \mathrm{Tr}\left[\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}\right] = b + b^* = e^{-\frac{\gamma}{2}t}\,\mathrm{Re}\left(e^{i\omega t} b_0\right)$$



← TRAJECTORY: IN-PLANE SPIRAL

$(b_0 \text{ REAL}) \quad = e^{-\frac{\gamma}{2}t}\cos(\omega t)\, b_0$

STEADY STATE(S)

$$\rho(t=\infty) = \begin{pmatrix} 1-c_0 & 0 \\ 0 & c_0 \end{pmatrix}$$

NON-UNIQUE BECAUSE OF CONSERVED $\langle \sigma^z \rangle$

Dephasing 2.0

NO BATH BUT CLASSICAL NOISE

$$H = |e\rangle\langle e| \hbar \underbrace{\omega}_{\text{RANDOM VARIABLE (STATIC)}}$$

PROBABILITY DENSITY

$$\int \frac{dp}{d\omega} d\omega = \int dp(\omega)$$

$$\frac{dp}{d\omega} = \frac{1}{\pi}\frac{\gamma}{(\omega-\omega_0)^2 + \gamma^2}$$

LORENTZIAN (CAUCHY-LORENTZ DISTR.)

$$\rho_{\substack{\text{FIXED} \\ \omega}}(t) = \begin{pmatrix} 1-c_0 & b_0 e^{+i\omega t} \\ b_0^* e^{-i\omega t} & c_0 \end{pmatrix}$$

$$\rho_{\substack{\text{HAMILTONIAN} \\ \text{ENSEMBLE}}}(t) = \int \rho_\omega(t)\, dp(\omega) = \int \begin{pmatrix} 1-c_0 & b_0 e^{i\omega t} \\ \text{c.c.} & c_0 \end{pmatrix} \frac{dp}{d\omega} d\omega$$

$$= \begin{pmatrix} 1-c_0 & b_0 \int e^{i\omega t} \left(\frac{dp}{d\omega}\right) d\omega \\ \text{c.c.} & c_0 \end{pmatrix}$$

BUT $\displaystyle\int_{-\infty}^{+\infty} \frac{d\omega}{\pi} \frac{\gamma e^{i\omega t}}{(\omega-\omega_0)^2 + \gamma^2} = e^{i\omega_0 t - \gamma|t|}$

$$\rho_{t>0}(t) = \begin{pmatrix} 1-c_0 & e^{(i\omega_0 - \gamma)t} \\ \text{c.c.} & c_0 \end{pmatrix}$$

← IDENTICAL TO PREVIOUS EXERCISE. MARKOVIA DYNAMICS ALSO MODELS NOISE (SOME FORMS)

FORMAL INTEGRATOR OF THE LME

$$\dot{\rho} = \frac{i}{\hbar}[\rho, H] + \sum_j \gamma_j \left(L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\} \right) = \underset{\substack{\uparrow \\ \text{LIOUVILLIAN} \\ \text{SUPEROPERATOR}}}{\mathcal{L}}(\rho)$$

IS 1° ORD & LINEAR

$\mathcal{L}(\rho+\rho') = \mathcal{L}(\rho) + \mathcal{L}(\rho')$

WE KNOW HOW TO FORMALLY INTEGRATE (EQUIVALENT TO SCHRÖDINGER)

TURN ρ FROM MATRIX TO VECTOR $\hat{\rho} \rightarrow |\rho\rangle\rangle$

$\hat{\rho}$ DIMENSION $d_H \times d_H$

$|\rho\rangle\rangle$ DIMENSION $d_H^2 \times 1$

LIKE THIS $|i\rangle\langle j| \longrightarrow |ij\rangle\rangle$

$$\xrightarrow{\text{SO THAT}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

SOMETIMES KNOWN AS CHOI TRANSFORM

PROPERTIES

$$A \rho B \xrightarrow{\text{CHOI}} A \otimes \underbrace{B^t}_{\substack{\text{TRANSPOSE} \\ \text{IN CANONICAL BASIS}}} |\rho\rangle\rangle$$

HILBERT-SCHMIDT PRODUCT ON OPS

$$\mathrm{Tr}[A^\dagger, B] = (A, B) = \langle\langle A | B \rangle\rangle$$

$\mathrm{Tr}[\mathcal{O}\rho] =$ [redacted]

$$= \langle\langle \mathbb{1} | \mathcal{O} \otimes \mathbb{1} | \rho \rangle\rangle = \langle\langle \mathbb{1} | \mathbb{1} \otimes \mathcal{O}^* | \rho \rangle\rangle = \langle\langle \mathcal{O}^{(t)} | \rho \rangle\rangle$$

MATRIX ELEMENT $\rho_{ij} = \langle i | \rho | j \rangle = \langle\langle ij | \rho \rangle\rangle$

THE LIOUVILLIAN AS SUPERMATRIX $\hat{\hat{\mathcal{L}}}$ (IN CHOI TRANSFORM)

$$|\dot{\rho}\rangle\rangle = \underset{\substack{\uparrow \\ \text{IF } \mathcal{L} \text{ IS} \\ \text{TIME INDEPENDENT}}}{\hat{\hat{\mathcal{L}}}} |\rho\rangle\rangle = \left(-\frac{i}{\hbar} H \otimes \mathbb{1} + \frac{i}{\hbar} \mathbb{1} \otimes H^* + \sum_j \gamma_j \left(L_j \otimes L_j^* \right.\right.$$

[redacted]

$$\left.\left. - \frac{1}{2} L_j^\dagger L_j \otimes \mathbb{1} - \frac{1}{2} \mathbb{1} \otimes L_j^t L_j^* \right)\right) |\rho\rangle\rangle$$

$$\boxed{|\rho(t)\rangle\rangle = \exp(\hat{\hat{\mathcal{L}}} t) |\rho_0\rangle\rangle}$$

HOWEVER $\hat{\mathcal{L}}$ IS NOT HERMITIAN.

IFF $\mathcal{L}$ CAN BE DIAGONALIZED $X D X^{-1}$

$$|\rho(t)\rangle\rangle = X e^{Dt} X^{-1} |\rho_0\rangle\rangle$$

Ⓐ EXPLODING SOLUTIONS ARE NOT PHYSICAL, SO $\forall \lambda \in D$, $Re(\lambda) \leq 0$

NEGATIVE REAL PART
FOR THE LIOUVILLIAN SPEC

Ⓑ STEADY STATE $|\dot{\rho}\rangle\rangle = 0 \iff \underbrace{\hat{\hat{\mathcal{L}}}|\rho\rangle\rangle = 0}$

$|\rho\rangle\rangle_{STEADY}$ MUST BE IN THE <u>KERNEL</u> OF $\mathcal{L}$

! NOT ALL EIGENVECTORS ARE DENSITY MATRICES !

IN FACT

$$\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = |\mathbb{1}\rangle\rangle$$

$|\dot{\rho}\rangle\rangle = \mathcal{L}|\rho\rangle\rangle$ PRESERVES THE NORM $Tr[\rho] = Tr[\mathbb{1}^\dagger \rho] = \langle\langle \mathbb{1}|\rho\rangle\rangle$

THIS IMPLIES $\rightarrow \underbrace{\mathcal{L}|v\rangle\rangle = \lambda|v\rangle\rangle}_{\text{EIGENVECTOR}}$ $|v(t)\rangle\rangle = e^{\lambda t}|v_0\rangle\rangle$ if $Re(\lambda) < 0$

$$Tr[v_0] = Tr[v(t)] = \underbrace{e^{\lambda t}}_{<1} Tr[v_0] \Rightarrow Tr[v_0] = 0$$

ALL DECAYING
EIGEN-SUPERVECTORS
ARE TRACELESS

$\hbar = 1$

EXERCISE

PUMP & DECAY

CAN COHERENT PUMPING
FIGHT SPONTANEOUS
EMISSION ?

$$\begin{cases} H = \Omega\sigma^x = \Omega(|g\rangle\langle e| + |e\rangle\langle g|) \\ L = \underset{\sigma^+}{\underset{\parallel}{|g\rangle}}\langle e| \quad \text{WITH RATE } \gamma \end{cases}$$

$$H = \Omega\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad L = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \quad L^\dagger L = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\hat{\hat{\mathcal{L}}} = -i\Omega\overset{H\otimes\mathbb{1}}{\begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}} + i\Omega\overset{\mathbb{1}\otimes H^t}{\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}} + \frac{\gamma}{2}\left(\overset{2L\otimes L^*}{\begin{pmatrix} 0 & 0 & 0 & 2 \\ & 0 & & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}} - \overset{L^\dagger L\otimes\mathbb{1}}{\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}} - \overset{\mathbb{1}\otimes(L^\dagger L)^t}{\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}}\right)$$

$$\hat{\mathcal{L}} = \Omega\left[\begin{pmatrix} 0 & i & -i & \\ i & 0 & & -i \\ -i & & 0 & i \\ & -i & i & 0 \end{pmatrix} + \frac{\gamma}{2\Omega}\begin{pmatrix} 0 & & & 2 \\ & -1 & & \\ & & -1 & \\ & & & -2 \end{pmatrix}\right]$$