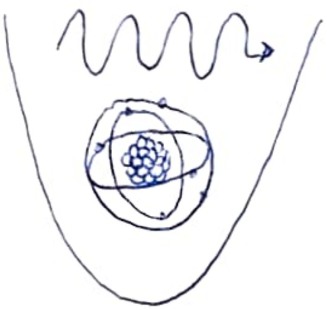
OPTICALLY-COUPLING ELECTRONIC LEVELS WITH CENTER-OF-MASS MOTION (IN A TRAP) A.K.A.

THE Lamb-Dicke REGIME



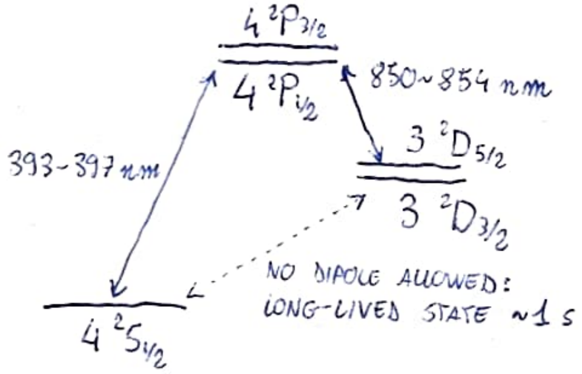
$\boxed{3}$ INGREDIENTS

↳ 1. Hydrogen-Like Atom [EXAMPLE: Ca^+ ION]

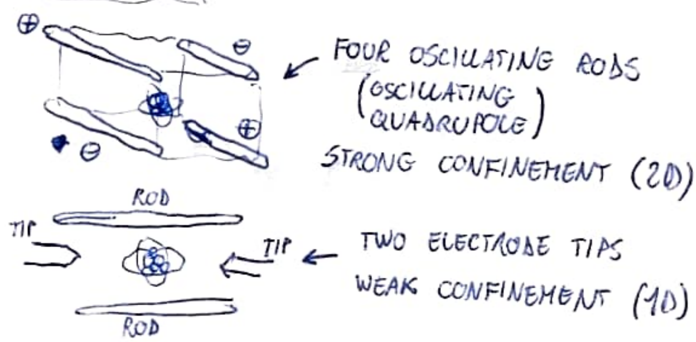
↳ 2. HARMONIC TRAP FOR THE CENTER-OF-MASS MOTION
⇝ BASICALLY 1D [EXAMPLE: PAUL TRAP FOR IONS]

↳ 3. LASER W/ FREQUENCY CLOSE TO OPTICAL ATOMIC TRANSITIONS

THE Ca^+ EXAMPLE



PAUL TRAP



NO TIME TO EXPLAIN ☹

3. INGREDIENTS ⟹ 3 LENGTHSCALES !

1. TYPICAL ATOMIC RADIUS = TYPICAL RELATIVE COORDINATE OF OUTER ELECTRON ≈ "BOHR RADIUS" OF THE OUTER ELECTRON

HYDRO BOHR RAD.

$$\tilde{a}_0 \approx \left(\frac{n^2}{Z_{eff}}\right) a_0 = \frac{n^2}{Z_{eff}} \frac{4\pi\epsilon_0 \hbar}{m e^2} \lesssim 1\,nm$$

m: REDUCED $\approx m_e$

$Ca^+ \rightarrow \frac{(4)^2}{(Z_{eff}=2)} \cdot 53\,pm \approx \cancel{}\ 0{,}42\,nm$

$n=4$ (4s)

3. LASER WAVELENGTH $\lambda_{LASER} = \frac{2\pi c}{\omega_L}$ ↙ FOR OPTICAL TRANSITIONS $\lambda \approx 100\,nm$ (UV) ~ $1\,\mu m$ (IR)

Ca^+ TRANSITIONS 400 nm; 850 nm

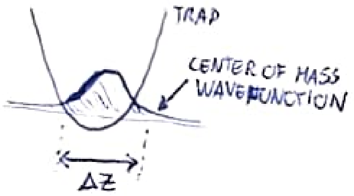
FIRST LENGHTSCALE SEPARATION

$$\tilde{a}_0 \ll \lambda$$

⟵[ATOMS ARE SMALLER THAN THE WAVELENGTHS OF LASERS THAT EXCITE THEM (MARCO USES THIS)

(20) UNCERTAINTY IN THE POSITION OF THE ATOM $\Delta z_{CM} \Leftarrow$ HOW MUCH IS THIS ?

CLEARLY, IT DEPENDS ON THE TRAPPING POTENTIAL



TRAPPING POTENTIAL $V(z_{C.M.}) = \frac{1}{2} m_A \omega^2_{TRAP} z^2$ (HARMONIC APPROX.)

MASS OF THE ATOM/ION (m_A) — TYPICAL TRAPS (ω_{TRAP})

$m_{Ca^+} \approx 40\, m_{PROTON}$ $\qquad \frac{\omega}{2\pi} = 0.1 \sim 10$ MHz

ASSUMING THAT THE CENTER-OF-MASS IS IN THE GROUND STATE OF THE HARMONIC OSCILLATOR $\rightarrow$

$$\boxed{H = \frac{P^2}{2m_A} + \frac{m_A \omega_T^2}{2} z^2} = \hbar\omega_T (a^\dagger a + \tfrac{1}{2})$$

$$z = \sqrt{\frac{\hbar}{2m_A\omega_T}} (a + a^\dagger) \qquad P = \sqrt{\frac{\hbar m_A \omega_T}{2}}\, i(a^\dagger - a)$$

OUTCOMES

$$\langle 0|z|0\rangle = \sqrt{\frac{\hbar}{2m_A\omega_T}} (\cancel{\langle a\rangle} + \cancel{\langle a^\dagger\rangle}) = 0$$

$$\Delta z = \sqrt{\langle 0|z^2|0\rangle - \cancel{\langle z\rangle^2}} = \sqrt{\left(\frac{\hbar}{2m_A\omega_T}\right)\left(\cancel{\langle a^2\rangle} + \underbrace{\langle a^\dagger a\rangle}_{=0} + \underbrace{\langle a a^\dagger\rangle}_{=1} + \cancel{\langle a^{\dagger 2}\rangle}\right)}$$

$$\Delta z = \sqrt{\frac{\hbar}{2m_A\omega_T}} = \sqrt{\frac{(2\pi\hbar)}{2m_A\left(\frac{\omega_T}{2\pi}\right)}} = \sqrt{\frac{10^{-34}\ \mathrm{Kg\ m^2 s^{-2}}}{2\cdot 40\cdot(1.66\cdot 10^{-27}\mathrm{Kg})\ [10^{+5} \sim 10^{+7}\ \mathrm{s^{-1}}]}}$$

$$= \sqrt{[10^{-16} \sim 10^{-14}]\,\mathrm{m^2}} = 10^{-8} \sim 10^{-7}\,\mathrm{m} = 10 \sim 100\ \mathrm{nm}$$

SECOND LENGHTSCALE SEPARATION ?

$$\Delta z \overset{?}{\ll} \lambda$$

IS THE ATOM POSITION FIXED AT THE LASER WAVELENGTH ?

LAMB-DICKE PARAMETER

$$\eta = \frac{2\pi\,\Delta z}{\lambda_{LASER}} = \Delta z \cdot K_{LASER}$$

$(\cos\theta)$

WAVEVECTOR ω/c

RATIO BETWEEN ATOM C.M. LENGTH / LASER LENGTH

$$\boxed{\eta \ll 1}$$

LAMB DICKE REGIME

BUT NOT TOO SMALL

$\eta \sim 0{,}1$ IS GOOD

~ 0.05 VERY COMMON $\frac{1}{20}$

TUNABLE VIA TRAPPING FREQUENCY

ALSO NOTICE $\tilde{a}_0 \ll \Delta z$

I CAN NOW USE η AS A SMALL PARAMETER AND CARRY OUT EXPANSIONS.

HAMILTONIANS

$$H^{\text{ATOM}}_{\substack{\text{RELATIVE}\\ \text{COORDINATE}}} = \hbar\omega_{eg}\,|e\rangle\langle e|$$

$$H^{\text{ATOM}}_{\substack{\text{CENTER-OF-MASS}\\ \text{COORDINATE}}} = \hbar\,\omega_{TRAP}\,a^{\dagger}a$$

(FORGET THE $+\frac{1}{2}$ CONSTANT) ← ATOM COM VIBRATIONS "PHONONS"

$$H_{\substack{\text{ATOM-LIGHT}\\ \text{INTERACTION}}} = -\,\hat{\vec{d}}\cdot\hat{\vec{E}}(\hat{\vec{r}}_{CM})$$

← ATOM COM COORDINATE OPERATOR

DIPOLE TRANSITION OPERATOR IN RELATIVE COORDINATE SPACE

ELECTRIC FIELD OP. IN PHOTONIC SPACE

HERE I AM USING $\hat{a}_0 \ll \Delta z_0$

AS USUAL →
$$\hat{\vec{d}} = \vec{d}_{eg}\,|e\rangle\langle g| + \vec{d}_{eg}^{\,*}\,|g\rangle\langle e|$$

$$\hat{\vec{E}} = \sum_{k\lambda}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0}}\;\vec{\epsilon}_{k\lambda}\;2\,\mathrm{Im}\left(u_{k\lambda}(\hat{\vec{r}}_{CM})\;\hat{a}_{k\lambda}\right)$$

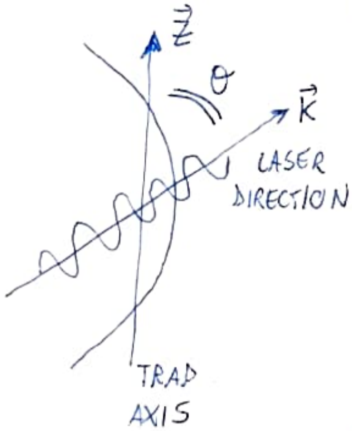
LASER IN

$|0\rangle|0\rangle|0\rangle\ldots$ EMPTY → $\underbrace{|\alpha\rangle}_{\text{COHERENT AT } k\lambda}|0\rangle\ldots$

$$\langle\Psi_{LASER}|\,\hat{\vec{E}}\,|\Psi_{LASER}\rangle(\hat{\vec{r}}) = \ldots = \vec{\xi}\,\cos\left(c|\vec{k}|t - \vec{k}\cdot\hat{\vec{r}}\right)$$

STILL AN OPERATOR IN THE COM COORDINATE

POINT: TRAP IS TIGHT IN XY SO
$$\vec{r} = \begin{pmatrix}0\\0\\z\end{pmatrix}$$
← NO MOTION ALLOWED IN xy



$$\langle\vec{E}\rangle(\hat{\vec{r}}) = \vec{\xi}\,\cos\left(c|\vec{k}|t - \cos\theta\,|\vec{k}|\,z\right)$$

DROP VECTOR $|\vec{k}| \to k$

AS USUAL $\Omega = \dfrac{\vec{d}_{eg}\cdot\vec{\xi}}{\hbar}$ RABI FREQUENCY

$$H^{\text{eff.}}_{\substack{\text{ATOM}\\ \text{LIGHT}}} = -\hbar\Omega\,|e\rangle\langle g|\,\cos\left(ckt - \underbrace{k\hat{z}\cos\theta}_{\text{NEW TERM FROM LAST TIME}}\right) + h.c.$$

I CAN CHANGE THIS SIGN $U = \begin{pmatrix}1 & \\ & -1\end{pmatrix}\begin{matrix}|e\rangle\\|g\rangle\end{matrix}$

$|e\rangle \to -|e\rangle$

THE LASER ACTUALLY COUPLES

RELATIVE ⟷ COM

INTERNAL STATES ⟷ MOTIONAL STATES

$$H^{(LAB)}_{FULL} = \underset{\substack{\uparrow \\ \text{ATOMIC} \\ \text{LEVELS}}}{\hbar\omega_{eg}}|e\rangle\langle e| + \underset{\substack{\uparrow \\ \text{TRAP} \\ \text{LEVELS}}}{\hbar\omega_T}\, a^\dagger a + \underset{\substack{\uparrow \\ \text{RABI}}}{\hbar\Omega}\, |e\rangle\langle g| \cos(ckt - kz\cos\theta) + h.c.$$

Req. $\boxed{\Omega, \Delta \ll \omega_{eg}, \omega_L}$

$\boxed{\text{RWA}}$ ROTATING FRAME WITH LASER FREQUENCY $\omega_L = ck$

$$H^{\left(\substack{\text{ROTATING} \\ \text{FRAME}}\right)}_{FULL} = \hbar\underbrace{(\omega_{eg} - \omega_L)}_{-\Delta}|e\rangle\langle e| + \hbar\omega_T a^\dagger a + \frac{\hbar\Omega}{2}|e\rangle\langle g|\, e^{-i\cos\theta k\hat{z}} + h.c.$$

$$+ \cancel{V(t)}$$ FAST ROTATING TERMS

$\boxed{\text{TIMESCALE SEPARATION} \quad \Delta, \omega_T, \Omega \ll \omega_L}$

BUT

$$\cos\theta\, k\hat{z} = k\cos\theta\sqrt{\frac{\hbar}{2m_A\omega_T}}(a + a^\dagger) = (\Delta z\, k\cos\theta)(a + a^\dagger)$$

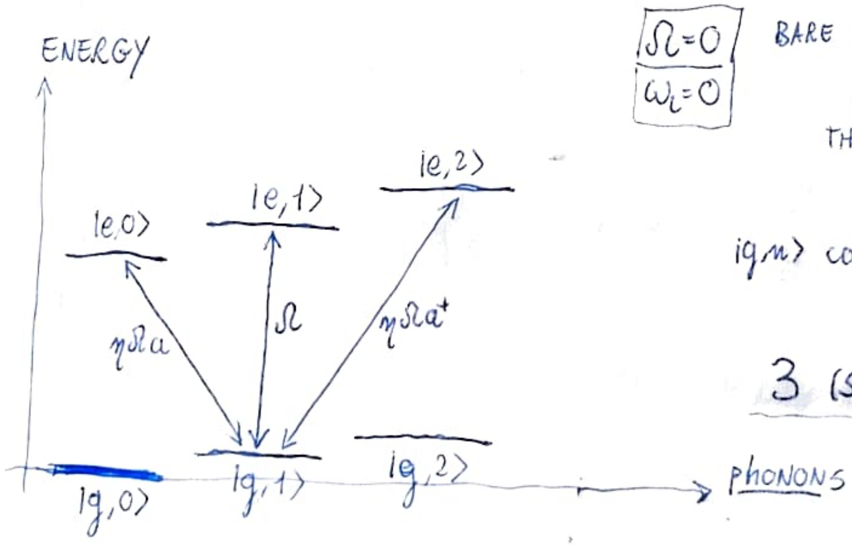
$$= \eta\,(a + a^\dagger)$$ AHA!! AND η SMALL PARAMETER

$$e^{-i\cos\theta k\hat{z}} = e^{-i\eta(a + a^\dagger)} = 1 - i\eta(a^\dagger + a) + O(\eta^2)$$

$$H^{(ROTO)}_{FULL} = \underbrace{\hbar\omega_T a^\dagger a}_{\text{ONLY PHONON}} \underbrace{- \hbar\Delta|e\rangle\langle e| + \frac{\hbar\Omega}{2}\overbrace{(|e\rangle\langle g| + |g\rangle\langle e|)}^{\sigma^x}}_{\text{ONLY INTERNAL}} +$$

$$+ \eta\frac{\hbar\Omega}{2}\underbrace{\left(-i\overset{\sigma^+}{|e\rangle\langle g|} + i\overset{\sigma^-}{|g\rangle\langle e|}\right)}_{\sigma^Y}(a^\dagger + a)$$ $\Leftarrow$ LAMB-DICKE EFFECTIVE COUPLING

$$H^{(ROT)}_{FULL} = \hbar\omega_T a^\dagger a - \hbar\Delta|e\rangle\langle e| + \frac{\hbar\Omega}{2}\sigma^x + \eta\frac{\hbar\Omega}{2}\sigma^Y(a^\dagger + a)$$ NICE



$\boxed{\Omega = 0}$ $\boxed{\omega_L = 0}$ BARE SPECTRUM

$\downarrow$ THEN TURN THE LASER ON

$|g,n\rangle$ COUPLES TO $\begin{cases} |e, n-1\rangle \\ |e, n\rangle \\ |e, n+1\rangle \end{cases}$

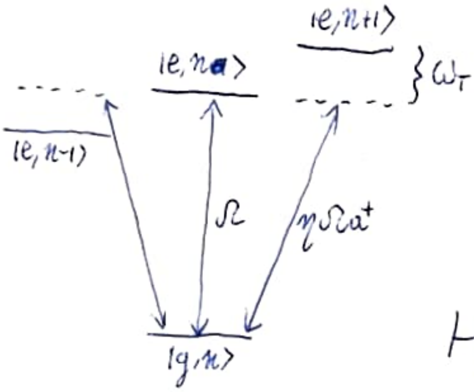
3 (SUB)-REGIMES

SIDEBAND COUPLING

① CARRIER RESONANCE

LASER IN RESONANT WITH THE DIRECT TRANSITION

$\Delta \ll \omega_T$ (AND WEAK $\eta\Omega \ll \omega_T$)



$|g,n\rangle\,;\,|e,n\pm1\rangle$ → MATRIX ELEMENT $\eta\Omega$

→ ENERGY DIFFERENCE (ROT.)

$\underbrace{}_{\text{NO COUPLING}}$

$|\omega_T \pm \Delta| \approx \omega_T \gg \eta\Omega$

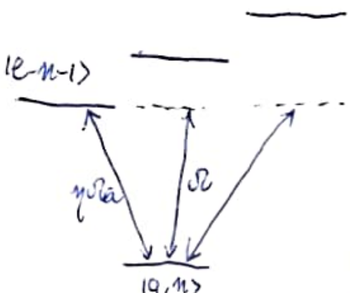
$$H_{CARRIER} = \hbar\omega_T a^+a - \hbar\Delta|e\rangle\langle e| + \frac{\hbar\Omega}{2}\sigma^x$$

② RED SIDEBAND

LASER ω_L RESONANT WITH $\omega_{eg}-\omega_T$, THAT IS

$$|\omega_L - \omega_{eg} + \omega_T| \leqslant \omega_T \qquad \text{(AND WEAK } \Omega \ll \omega_T)$$

$$|\Delta + \omega_T| \leqslant \omega_T$$



THEREFORE $|g,n\rangle \overset{\text{ONLY}}{\longleftrightarrow} |e,n-1\rangle$

$$H_{\substack{RED\\SIDEBAND}} = \hbar\omega_T a^+a - \hbar\Delta|e\rangle\langle e| + \frac{\eta\hbar\Omega}{2}\left(-i|e\rangle\langle g|a + i|g\rangle\langle e|a^+\right)$$

$$= \hbar\omega_T a^+a - \hbar\Delta|e\rangle\langle e| + \frac{\eta\hbar\Omega}{2}\left(-i\sigma^+a + h.c.\right)$$

JAYNES-CUMMINGS MODEL (BETWEEN PHONON AND LEVEL)

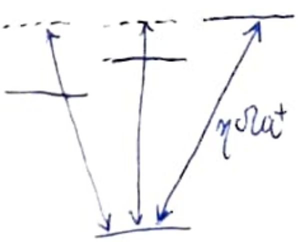
LASER DETUNED TO LOWER ("RED") FREQUENCIES THAN CARRIER

③ BLUE SIDEBAND

ω_L RESONANT WITH $\omega_{eg}+\omega_T$, OR

$$|\omega_L - \omega_{eg} - \omega_T| \leqslant \omega_T \qquad \text{(AND OFC. } \Omega \ll \omega_T)$$

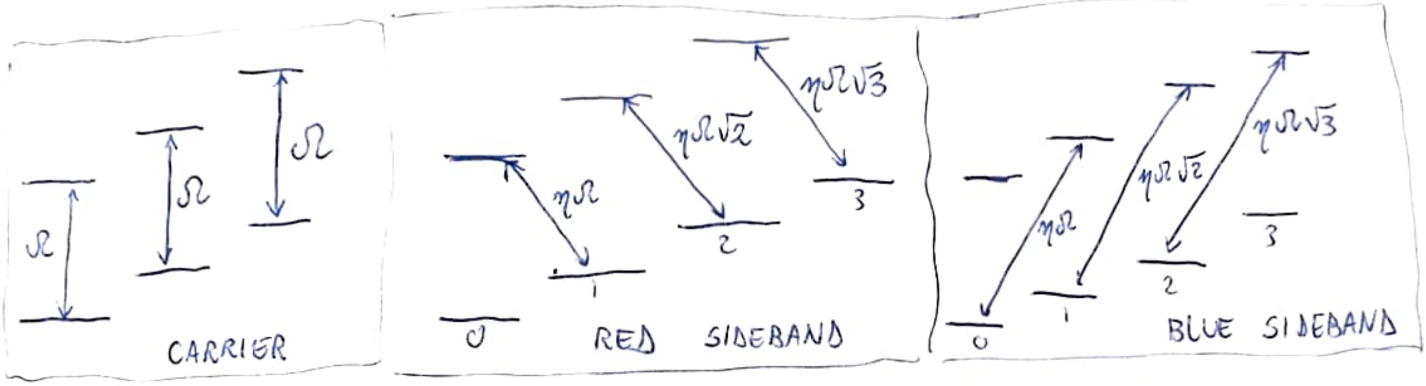
$$|\Delta - \omega_g| \leqslant \omega_T$$



$$H_{\substack{BLUE\\SIDEBAND}} = \hbar\omega_T a^+a - \hbar\Delta|e\rangle\langle e| + \frac{\eta\Omega\hbar}{2}\left(-i\sigma^+a^+ + h.c.\right)$$

"ANTI"-JC MODEL

OBSERVATION ABOUT EFFECTIVE RABI FREQ.



EFF. FREQUENCIES MATCH

EFF. FREQUENCIES MISMATCH
<u>AND</u> INCOMMENSURATE !!

THIS IS AN ISSUE WHEN DESIGNING MULTI-QUBIT GATES ON ION TRAPS: ASYNCHRONY REQURES PERFECT COOLING OR WORKAROUND

$$1\ \text{MHz} = \overbrace{48\ \mu\text{K}}^{\text{THAT'S FRIGGIN COLD}}$$

<u>BTW</u>, THE RED SIDEBAND CAN BE USED TO COOL-DOWN ATOM VIBRATIONS