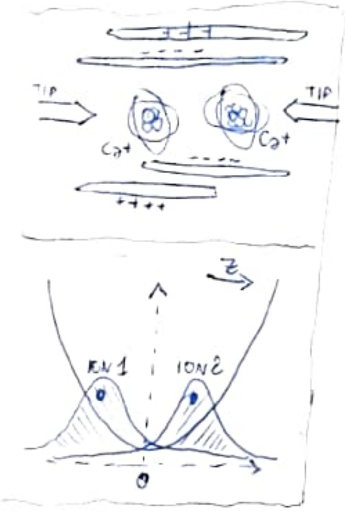
TWO IONS IN A Linear Trap

↳ A PATHWAY TOWARDS ION-QUBIT QUANTUM COMPUTATION
(TRAPPED ION)



X AND Y DIMENSIONS → TIGHTLY CONFINED $x=y=0$ THUS

$$H=\underbrace{\frac{p_1^{z\,2}+p_2^{z\,2}}{2m}}_{\text{KINETIC}}+\underbrace{\frac{1}{2}m\omega_T^2\left(z_1^2+z_2^2\right)}_{\text{TRAP}}+\underbrace{\frac{e^2}{4\pi\varepsilon_0\,|z_1-z_2|}}_{\text{COULOMB REPULSION}}$$

↑ ION MASS $Ca^+\approx m_{\text{proton}}\cdot 40$

↑ NO ANALYTICAL EXACT SOLUTION: I CAN DO SMALL-OSCILLATION EXPANSION (AND CHECK "A POSTERIORI" CONSISTENCY)

$$\hat{z}_j\longrightarrow\hat{z}_j'=\hat{z}_j-\bar{z}_j\,\hat{\mathbb{1}}$$

↑ CLASSICAL EQUILIBRIUM POSITION (JUST A NUMBER, NOT AN OP.)

COMMUTATORS ARE THE SAME

$$[z_i',p_j]=[z_i,p_j]-\bar{z}_i\cancel{[\mathbb{1},p_j]}$$

SAME

LET US FIND $\bar{z}_j$

FIRST OF ALL, 2 LEFT ⟷ 1 RIGHT SO $\bar{z}_1>\bar{z}_2$ THUS $\frac{1}{|\bar{z}_1-\bar{z}_2|}=\frac{1}{\bar{z}_1-\bar{z}_2}$

CLASSICAL EQUILIBRIUM: $\dot{p}=0$ (NO ACCELER.) $\quad 0=-\dot{p}=+\frac{\partial H}{\partial q}\ \rightsquigarrow\ \left.\frac{\partial H}{\partial z_j}\right|_{\bar{z}_j}\overset{!}{=}0$

$$\left.\frac{\partial H}{\partial z_1}\right|_{\bar{z}_1,\bar{z}_2}\quad m\omega_T^2\bar{z}_1-\frac{e^2}{4\pi\varepsilon_0}\frac{1}{(\bar{z}_1-\bar{z}_2)^2}=0$$

$$\left.\frac{\partial H}{\partial z_2}\right|_{\bar{z}_1,\bar{z}_2}\quad m\omega_T^2\bar{z}_2+\frac{e^2}{4\pi\varepsilon_0}\frac{1}{(\bar{z}_1-\bar{z}_2)^2}=0$$

SUMMING THE EQUATIONS $\bar{z}_2=-\bar{z}_1$ SUBSTITUTE

$$m\omega_T^2\bar{z}_1-\frac{e^2}{16\pi\varepsilon_0}\frac{1}{\bar{z}_1^2}=0$$

$$\bar{z}_1=\left(\frac{e^2}{16\pi\varepsilon_0 m\omega_T^2}\right)^{1/3}=-\bar{z}_2$$

CLASS. EQUILIBRIUM AND NOW

$$\hat{z}_j=\bar{z}_j\,\mathbb{1}+\hat{z}_j'\quad\text{SO}$$

$$\hat{z}_1^2=\bar{z}_1^2\,\hat{\mathbb{1}}+2\bar{z}_1\hat{z}_1'+\hat{z}_1'^2\qquad \hat{z}_2^2=\bar{z}_2^2\,\hat{\mathbb{1}}+2\bar{z}_2\hat{z}_2'+\hat{z}_2'^2$$

EXACT AND

$$|\hat{z}_1-\hat{z}_2|^{-1}\overset{\substack{\text{SMALL OSC.}\\ z_1>z_2}}{\approx}(\hat{z}_1-\hat{z}_2)^{-1}\overset{\text{EXPANSION}}{=}\frac{\hat{\mathbb{1}}}{(\bar{z}_1-\bar{z}_2)}-\frac{(\hat{z}_1'-\hat{z}_2')}{(\bar{z}_1-\bar{z}_2)^2}+\frac{(\hat{z}_1'-\hat{z}_2')^2}{(\bar{z}_1-\bar{z}_2)^3}+\mathcal{O}(\hat{z}_j'^3)$$

SHOULD BE SMALL

$$H_{\text{small oscillations}} = \frac{p_1^{z2} + p_2^{z2}}{2m} + \frac{m\omega_T^2}{2}\left(\cancel{\bar z_1^2 \mathbb{1}} + \cancel{\bar z_2^2 \mathbb{1}} + 2\bar z_1 \hat z_1' + 2\bar z_2 \hat z_2' + \hat z_1'^2 + \hat z_2'^2\right) +$$

CONST.

$$\frac{e^2}{4\pi\varepsilon_0}\left(+ \cancel{\frac{\mathbb{1}}{\bar z_1 - \bar z_2}} - \frac{\hat z_1' - \hat z_2'}{(\bar z_1 - \bar z_2)^2} + \frac{(\hat z_1' - \hat z_2')^2}{(\bar z_1 - \bar z_2)^3}\right) + \mathcal{O}(\hat z_j'^3)$$

SHIFT AWAY THE CONSTANTS

ISOLATING FIRST-ORDER TERMS $\hat z_j'$

$$\mathrm{I}^{\circ} \gg \hat z_1' \underbrace{\left(m\omega_T^2 \bar z_1 - \frac{e^2}{4\pi\varepsilon_0 (\bar z_1 - \bar z_2)^2}\right)}_{=0} + \hat z_2' \underbrace{\left(m\omega_T^2 \bar z_2 + \frac{e^2}{4\pi\varepsilon_0 (\bar z_1 - \bar z_2)^2}\right)}_{=0}$$

BY CONSTRUCTION

ISOLATING SECOND-ORDER TERMS

$$\mathrm{II}^{\circ} \gg \frac{m\omega_T^2}{2}\left(\hat z_1'^2 + \hat z_2'^2\right) + (\hat z_1' - \hat z_2')^2 \frac{e^2}{4\pi\varepsilon_0}(\bar z_1 - \bar z_2)^{-3}$$

$$\frac{e^2}{4\pi\varepsilon_0}(\bar z_1 - \bar z_2)^{-3} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{(2\bar z_1)^3} = \frac{e^2}{32\pi\varepsilon_0}\left(\frac{16\pi\varepsilon_0 m\omega_T}{e^2}\right) = \frac{1}{2} m\omega_T \quad !$$

$$H_{\text{small oscillations}} = \frac{1}{2m}\left(p_1^{z2} + p_2^{z2}\right) + \frac{m\omega_T^2}{2}\left(z_1'^2 + z_2'^2 + (z_1' - z_2')^2\right) + \mathcal{O}(z_j'^3)$$

COUPLED HARMONIC OSCILLATORS (YEAH, NO BIG SURPRISE)

	CENTER-OF-MASS	& RELATIVE
COORDINATE	$Z = \frac{z_1' + z_2'}{2}$	$z = z_1' - z_2'$
MOMENTUM	$P = p_1^z + p_2^z$	$p = \frac{p_1^z - p_2^z}{2}$

$m_A = m_B$ IDENTICAL MASSES

WHICH MEANS

$$p_1^z = \frac{P}{2} + p \qquad p_2^z = \frac{P}{2} - p \qquad z_1' = Z + \frac{z}{2} \qquad z_2' = Z - \frac{z}{2}$$

AND THUS

$$\left(p_1^{z2} + p_2^{z2}\right) = \frac{P^2}{2} + 2p^2 \quad \Big\| \quad \left(z_1'^2 + z_2'^2\right) = 2Z^2 + \frac{z^2}{2} \quad \Big\| \quad (z_1' - z_2')^2 = z^2$$

$$H_{\text{small osc.}} = \left[\frac{P^2}{4m} + m\omega_T^2 Z^2\right] + \left[\frac{p^2}{m} + \frac{m\omega_T^2}{2}\left(\frac{z^2}{2} + z^2\right)\right] + \mathcal{O}(z_j^3)$$

CENTER-OF-MASS HARMONIC OSCILLATOR

(RELATIVE) "STRETCH MODE" HARMONIC OSCILLATOR

DECOUPLED PROBLEMS

COM MODE

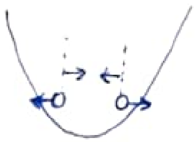
$$Z = \sqrt{\frac{\hbar}{4 m\omega_T}}\left(a_{COM} + a^\dagger_{COM}\right) \; ; \quad P = i\sqrt{\hbar m\omega_T}\left(a^\dagger_{COM} - a_{COM}\right)$$

AND REVERSE $\rightsquigarrow$

$$a_{COM} = \sqrt{\frac{m\omega_T}{\hbar}}\, Z + i\sqrt{\frac{1}{4\hbar m\omega_T}}\, P \qquad [a_s, a^\dagger_s] = 1$$

STRETCH MODE

$$z = \sqrt{\frac{\hbar}{\sqrt{3}\, m\omega_T}}\left(a_s + a^\dagger_s\right) \; ; \quad p = i\sqrt{\frac{\sqrt{3}\,\hbar m\omega_T}{4}}\left(a^\dagger_s - a_s\right)$$



$$\rightsquigarrow \quad a_s = \sqrt{\frac{\sqrt{3}\, m\omega_T}{4\hbar}}\, z + i\sqrt{\frac{1}{\sqrt{3}\,\hbar m\omega_T}}\, p$$

$$H_{\substack{SMALL\\OSC.}} = \hbar\omega_T\Big(a^\dagger_{COM}a_{COM} + \underbrace{\cancel{\tfrac{1}{2}}}_{CONST.}\Big) + \hbar\sqrt{3}\,\omega_T\Big(a^\dagger_s a_s + \underbrace{\cancel{\tfrac{1}{2}}}_{CONST.}\Big) + \cancel{\mathcal{O}}(\ldots)$$

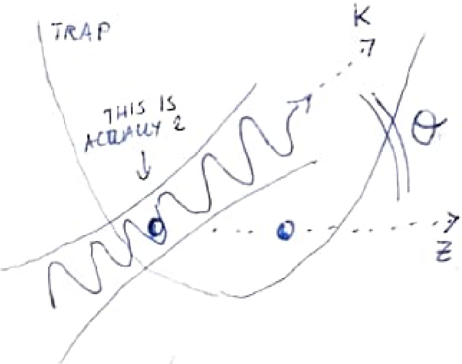
{ THE STRETCH MODE IS $\sqrt{3} \approx 1.73$ TIMES FASTER THAN THE COM MODE, THAT IS, SEVERAL MHz AWAY.

Back to lab coordinates

$$z_{1,2} = \pm \bar{z}_1 + z'_{1,2} = \pm \bar{z}_1 + Z \pm \frac{z}{2}$$

$$z_{1,2} = \pm \bar{z}_1 + \sqrt{\frac{\hbar}{4 m\omega_T}}\left(a_{COM} + a^\dagger_{COM}\right) \pm \sqrt{\frac{\hbar}{4m\sqrt{3}\,\omega_T}}\left(a_s + a^\dagger_s\right)$$

Adding a FOCUSED Laser, SAY, ON ION 1



LIGHT-ATOM COUPLING (DIPOLE)

$$H_{\substack{LIGHT\\ATOM\\INT.}} = -\vec{d}\cdot\vec{E}(\vec{r}) = \ldots =$$

ION 1

$$= -\hbar\underset{RABI}{\Omega}\, |e\rangle\langle g|_1 \cos\left(cKt - \cos\theta\, K z_1\right) + h.c.$$

$cK = \omega_L$

$$= -\hbar\Omega\, |e\rangle\langle g|_1 \cos\Big(cKt - \underbrace{\cos\theta\, K\bar{z}_1}_{\phi_0} - \underbrace{\cos\theta\, K\sqrt{\frac{\hbar}{4m\omega_T}}}_{\eta_{COM}}\left(a_{COM} + a^\dagger_{COM}\right) - \underbrace{\cos\theta\, K\sqrt{\frac{\hbar}{4\sqrt{3}\, m\omega_T}}}_{\eta_s}\left(a_s + a^\dagger_s\right)\Big) + h.c.$$

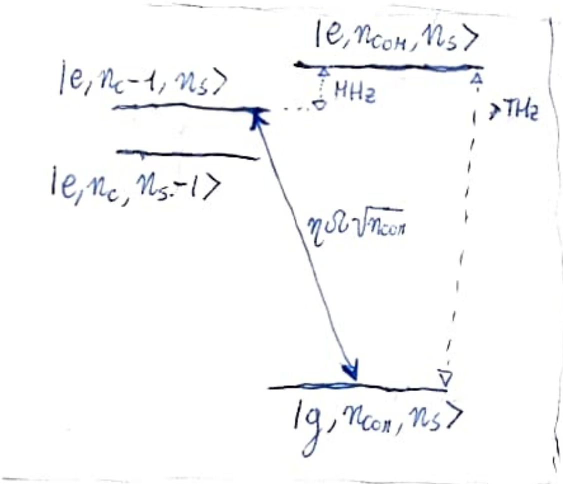
$$H^{(LAB)}_{FULL} = \hbar\omega_{eg}\sum_j |e\rangle\langle e|_j + \hbar\omega_T\, a^\dagger_{CoM} a_{CoM} + \sqrt{3}\,\hbar\omega\, a^\dagger_s a_s + H_{\substack{\text{ATOM}\\ \text{LIGHT}\\ \text{INT.}}}$$

ROTATING FRAME $U(t) = e^{-i\omega_L t \sum_j |e\rangle\langle e|_j + i\phi_0}$ (AND) RWA $\boxed{|\omega_{eg}-\omega_L|, \Omega \ll \omega_{eg}, \omega_L}$

$$H^{(ROTO)}_{RWA} = \hbar\,(\overbrace{\omega_{eg}-\omega_L}^{-\Delta}) \sum_j |e\rangle\langle e|_j + \hbar\omega_T\, a^\dagger_{CoM} a_{CoM} + \sqrt{3}\,\hbar\omega_T\, a^\dagger_s a_s +$$

$$\left\{ -\frac{\hbar\Omega}{2} |e\rangle\langle g|_1\, e^{-i\eta_{CoM}(a_{CoM}+a^\dagger_{CoM})}\, e^{-i\eta_s(a_s+a_s^\dagger)} + h.c. \right.$$

$$\to -\frac{\hbar\Omega}{2} |e\rangle\langle g|_1 \left(1 - i\eta_{CoM}(a_{CoM}+a^\dagger_{CoM}) - i\eta_s(a_s+a_s^\dagger) + \mathcal{O}(\eta^2)\right) + h.c.$$



← ENERGY SELECTION:
I CAN SPECIFICALLY ADDRESS (FOR EXAMPLE) "THE RED SIDEBAND OF THE COM MODE"

NAMELY

$$|\Delta + \omega_T| \approx \text{KHz} \ll \omega_T \approx \text{MHz}$$

BUT THEN

$$|\Delta \pm \sqrt{3}\,\omega_T| \approx |-\omega_T \pm \sqrt{3}\,\omega_T| \approx$$

$$\approx \underbrace{|\sqrt{3} \pm 1|}_{\geq 0.7}\, \omega_T \gg \frac{\omega_T}{2} \approx \text{MHz}$$

↑ FAR DETUNED

NO EXCITATION OF STRETCH

$\underbrace{\Delta \approx -\omega_T}_{\text{KHz}}$

↓

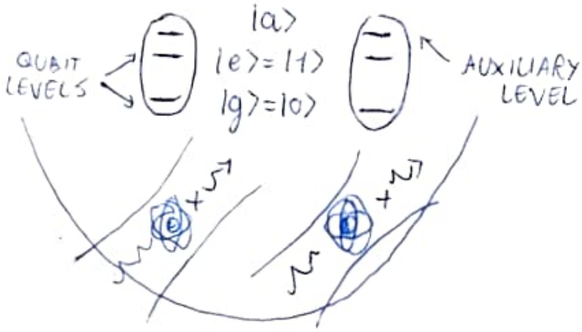
$$H_{final} = \hbar(\omega_T + \varepsilon)|e\rangle\langle e|_1 + \hbar\omega_T\, a^\dagger_{CoM} a_{CoM} +$$

$$+ \frac{\hbar\eta_{CoM}\Omega}{2}\left(i\sigma_1^+ a_{CoM} - i\sigma_1^- a^\dagger_{CoM}\right) + \text{DECOUPLED STUFF.}$$

↑ CENTER-OF-MASS (AXIAL) PHONON JAYNES CUMMINGS ↑

Cirac-Zoller Gate

(THE PIONEER OF TRAPPED ION Q.C.)



Ingredients:

$\underbrace{2\times}$ 3-level Ions (ex. Ca^+)

$\underbrace{2\times}$ Focused Lasers

$\underbrace{1\times}$ HARMONIC ION TRAP
↳ VERY COLD

DIFFICULT REQUIREMENT:

CoM Phonons @ $T=0$!

PRELIMINARY

LASERS WILL BE PERFECTLY RESONANT TO A TRANSITION, WHEN THIS HAPPENS I CAN GO TO AN INTERACTION PICTURE

$$H=\begin{pmatrix}\varepsilon_1 & & & \\ & \boxed{\begin{matrix}\varepsilon_2 & \Omega\\ \Omega & \varepsilon_2\end{matrix}} & \\ & & \varepsilon_3 & \\ & & & \ddots\end{pmatrix}\hbar \rightsquigarrow \underset{\text{FRAME CHANGE}}{\hat{U}}=\begin{pmatrix}e^{i\varepsilon_1 t} & & & \\ & e^{i\varepsilon_2 t} & & \\ & & e^{i\varepsilon_2 t} & \\ & & & \ddots\end{pmatrix} \rightsquigarrow \underbrace{H_{\text{FRAME}}^{\text{NEW}}}_{\text{INTERACTION PICTURE}}=\begin{pmatrix}0 & & & \\ & 0 & \Omega & \\ & \Omega & 0 & \\ & & & 0 \\ & & & & 0\end{pmatrix}$$

BARE HAMILTONIAN GOES AWAY

Laser 1
TUNED TO: $|g\rangle_1 \leftrightarrow |e\rangle_1$ RED SIDEBAND

$$H_{LASER-1}=\frac{\hbar\eta_{COM}\Omega_1}{2}\left(i\,|e\rangle\langle g|_1\, a_{COM}+h.c.\right) \qquad \boxed{\omega_L \simeq \omega_{eg}-\omega_T}$$

Laser 2
TUNED TO: $|g\rangle_2 \leftrightarrow |a\rangle_2$ RED SIDEBAND

$$H_{L2}=\frac{\hbar\eta_{COM}\Omega_2}{2}\left(i\,|a\rangle\langle g|_2\, a_{COM}+h.c.\right) \qquad \boxed{\omega_L \simeq \omega_{ag}-\omega_T}$$

$\boxed{\text{STEP 1}}$ Turn on Laser-1 for a time $T_1=\frac{\pi}{\eta_{COM}\Omega_1}$ (a π-pulse)

$$U_1=\exp\left(\frac{\pi}{\eta_{COM}\Omega_1}(-i)\frac{1}{\hbar}H_1\right)=\exp\left(-i\frac{\pi}{2}\left(i\,|e\rangle\langle g|_1 a_{COM}-i\,|g\rangle\langle e|_1 a^\dagger_{COM}\right)\right)$$

THIS IS BASICALLY A σ^Y ON THE STATES $|g,1\rangle;|e,0\rangle$
AND $\exp\left(-i\frac{\pi}{2}\sigma^Y\right)=-i\sigma^Y$

$$\boxed{e^{i\alpha\sigma}=\cos\alpha\,\mathbb{1}+i\sin\alpha\,\sigma}$$

STARTING STATES

$$|g_1,0\rangle \xrightarrow{U_1} |g_1,0\rangle$$

$$|e_1,0\rangle \longrightarrow \left(|e\rangle\langle g|_1 a-|g\rangle\langle e|_1 a^\dagger\right)|e_1,0\rangle=\overset{\text{MINUS SIGN}}{-}|g_1,1\rangle$$

$\boxed{\text{STEP 2}}$ Turn laser-2 for a time $T_2 = \frac{2\pi}{\eta_{COM}\Omega_2}$ (a 2π pulse)

$$U_2 = \exp\left(-\frac{i}{\hbar}\,\frac{2\pi}{\eta_{COM}\Omega_2}\,H_2\right) = \exp\left(-i\pi\left(i\,|a\rangle\langle g|_2\, a_{COM} + h.c.\right)\right)$$

$$= -\mathbb{1}\Big|_{|g_2,1\rangle,\,|a_2,0\rangle}$$

MINUS THE IDENTITY ON THESE TWO STATES

$\underbrace{\text{POSSIBLE STARTING STATES}}$

$$|g_2,0\rangle \xrightarrow{U_2} |g_2,0\rangle$$
$$|e_2,n\rangle \xrightarrow{U_2} |e_2,n\rangle$$
$$|g_2,1\rangle \longrightarrow -|g_2,1\rangle$$

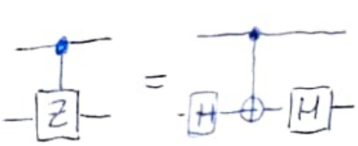
$\boxed{\text{STEP 3}}$ IDENTICAL TO STEP 1

$U_3 = U_1$ AND $U_3|g_1,1\rangle = +|e_1,0\rangle$

Table of CANONICAL states

$\|g_1 g_2 0\rangle$		$\|g_1 g_2 0\rangle$		$\|g_1 g_2 0\rangle$		$\|g_1 g_2 0\rangle$
$\|g_1 e_2 0\rangle$	$\xrightarrow{U_1}$	$\|g_1 e_2 0\rangle$	$\xrightarrow{U_2}$	$\|g_1 g_2 0\rangle$	$\xrightarrow{U_1}$	$\|g_1 e_2 0\rangle$
$\|e_1 g_2 0\rangle$		$-\|g_1 g_2 1\rangle$		$+\|g_1 g_2 1\rangle$		$\|e_1 g_2 0\rangle$
$\|e_1 e_2 0\rangle$		$-\|g_1 e_2 1\rangle$		$-\|g_1 e_2 1\rangle$		$-\|e_1 e_2 0\rangle$

AS A MATRIX $\Big\}$ $G = U_1 U_2 U_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$ ← CONTROL PHASE-FLIP GATE



$\Big\}$ EQUIVALENT TO C-NOT $\longleftrightarrow$ MAXIMALLY ENTANGLING $\longleftrightarrow$ ENABLES UNIVERSAL QUANTUM COMPUT.

$\rightsquigarrow$ EASY TO GENERALIZE TO N IONS [PRL 74, 4091 (1995)]

Big PROBLEM $\rightarrow$ T=0 ZERO TEMPERATURE PHONON IS MICROKELVIN NONSENSE $\Big\}$ SOLUTION: MØLMER-SØRENSEN ('99)