0. Being Sloppy with the constants

SOMETIMES
(BUT NOT ALWAYS)
WE MAY "DROP" $\hbar$ OR c OR K_B

OR WE SAY

$$\underset{\substack{\uparrow\\ \text{PLANCK CONSTANT}}}{\hbar = 1}\ ,\quad \underset{\substack{\uparrow\\ \text{SPEED OF LIGHT}}}{c = 1}\ ,\quad \underset{\substack{\uparrow\\ \text{BOLTZMANN CONSTANT}}}{K_B = 1}$$

THAT IS BECAUSE THEY ARE NOT ACTUAL CONSTANTS:
THEY SET A CONVERSION RATE OF A UNIT TO ANOTHER

$$c \simeq 3 \cdot 10^8 \ \frac{m}{s}$$

UNIT OF LENGTH (m)
UNIT OF TIME (s)

ALLOWS US TO EXPRESS LENGTHS IN SECONDS

$$1 \text{ LIGHT SECOND} = 3 \cdot 10^8 \ m$$
$$47 \times \text{EARTH RADII}$$
$$0.78 \times \text{EARTH-MOON DISTANCE}$$

$$1 \text{ meter} = \overbrace{3.3 \cdot 10^{-9}}^{1/c} \text{ seconds}$$

$$\hbar = (1.055) \cdot 10^{-34} \ Kg \cdot m^2 \cdot s^{-1}$$

ALLOWS US TO EXPRESS MASSES IN SECONDS

$$\frac{\hbar}{c^2} = 1.17 \cdot 10^{-51} \ \frac{Kg}{rad/s}$$

$$1 \text{ "MASS" SECOND}^{-1} = 1.17 \cdot 10^{-51} \ Kg$$

YOU NEVER HEAR ABOUT THIS UNIT CUZ IT'S UNPRACTICAL

$$\underbrace{\hbar\omega}_{\substack{\omega:\ rad/s}} \overset{\text{FREQUENCY ENERGY}}{} = \overset{\text{REST MASS}}{mc^2}$$

ELECTRON MASS $\quad 9{,}11 \cdot 10^{-31}$ Kg

$= 7{,}79 \cdot 10^{20}$ rad/s

(2π Hz)

$= 1.24 \cdot 10^{20}$ Hz

BIG !?

COMPARE TO VISIBLE LIGHT $\} \quad 4 \sim 7.9 \cdot 10^{14}$ Hz

$$K_B = 1.38 \cdot 10^{-23} \ \frac{m^2\,Kg}{s^2\,K}$$

EXCERCISE

- WHAT'S THE DIRECT CONVERTION RATE KELVIN – (SECOND)$^{-1}$?
- WHAT'S THE WATER MELTING TEMPERATURE IN (SECONDS)$^{-1}$

CAREFUL » NOT ALL CONSTANTS ARE CONVERSIONS, SOME CARRY ACTUAL INFORMATION

CONVERSION CONSTANTS	TRUE CONSTANT(S)
c	$\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}$
$\hbar$	JUST A NUMBER
K_B	

$$\overset{e^-}{\ominus} \xleftrightarrow{\;d\;} \overset{e^-}{\ominus}$$

THERE IS A SPECIFIC RATIO BETWEEN

→ ELECTROSTATIC ENERGY

→ REST MASS ENERGY

THE RATIO IS A FUNCTION OF α

DIFFERENT $\hbar, c, K_B$ } SAME UNIVERSE DIFFERENT UNITS

DIFFERENT α } ANOTHER UNIVERSE, DIFFERENT FROM OURS

Review: QUANTUM MECH

1 The quantum State

→ IMPOSSIBLE TO DEFINE DETERMINISTICALLY EVERY MEASURABLE PROPERTY OF A SYSTEM

TAKE $\dot{x}$ POSITION, p MOMENTUM

IF ONE IS COMPLETELY DETERMINED $x = x_0$ — DISTRIBUTION $\delta(x - x_0)$

THE OTHER ONE IS COMPLETELY UNDETERMINED — DISTRIBUTION "constant"

QUANTUM STATE → COLLECTION OF ALL THE INFORMATION WE POSSESS ABOUT A QUANTUM SYSTEM

Ψ, ρ

PHYSICAL QUANTITIES $\left\{ \begin{array}{l} \text{EXPECTATION VALUES } \langle O \rangle \\ \text{PROBABILITIES} \dots \end{array} \right\}$ ARE FUNCTIONALS $\langle O \rangle = f_O(\Psi)$

A SPECIAL CLASS OF STATES

PURE STATES $\xleftrightarrow{\text{A.K.A}}$ VECTOR STATES $\xleftrightarrow{\text{A.K.A}}$ WAVEFUNCTIONS

ARE "THE MOST DETERMINISTIC" STATES: YOU CAN NOT ADD INFORMATION WITHOUT VIOLATING SOMETHING

THEY ARE ▷ $|\psi\rangle$ VECTORS OF A VECTOR SPACE $\mathcal{H}$ $\left\{ \begin{array}{l} |\psi\rangle + |\varphi\rangle \in \mathcal{H} \\ \lambda|\psi\rangle \in \mathcal{H} \end{array} \right.$

- ↳ ON COMPLEX FIELD $\lambda|\psi\rangle, \lambda \in \mathbb{C}$
- ↳ WITH A PRODUCT SCALAR METRIC $\langle \psi | \varphi \rangle$
- ↳ CATCH: $|\psi\rangle$ AND $\lambda|\psi\rangle$ ARE ACTUALLY THE SAME STATE

(BUT $\lambda_1|\psi\rangle + \lambda_2|\varphi\rangle$ AND $\lambda_2|\psi\rangle + \lambda_1|\varphi\rangle$ ARE NOT)

1A

THE HILBERT METRIC

$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$

PHYSICAL MEANING: THE PROBABILITY OF PREPARING $|\psi\rangle$ AND THEN MEASURING $|\varphi\rangle$ IS $\boxed{*}$

$$p = |\langle\varphi|\psi\rangle|^2 = \langle\psi|\varphi\rangle\langle\varphi|\psi\rangle$$

AKA FIDELITY

IT FOLLOWS THAT

ORTHOGONAL STATES $\langle\varphi|\psi\rangle = 0$ ARE FULLY DISTINGUISHABLE

WHY "METRIC"? $\Rightarrow$ BECAUSE IT INTRODUCES A CONCEPT OF "STATE DISTANCE"

EXAMPLE: BURES DISTANCE

$$D_B(\psi,\varphi) = \sqrt{2(1-|\langle\psi|\varphi\rangle|)}$$

$\boxed{1B}$ SUPERPOSITION AND INTERFERENCE

INPUT STATES $|\psi_1\rangle$ OR $|\psi_2\rangle$, MEASURING PROBABILITY OF OUTPUT $|\varphi\rangle$

AMPLITUDES

$\langle\varphi|\psi_1\rangle = c_1$

$\langle\varphi|\psi_2\rangle = c_2$

PROBABILITIES

$|\langle\varphi|\psi_1\rangle|^2 = |c_1|^2 = p_1$

$|\langle\varphi|\psi_2\rangle|^2 = |c_2|^2 = p_2$

FOR SIMPLICITY: ORTHOGONAL $\langle\psi_1|\psi_2\rangle = 0$ $\mapsto$ $|+\rangle = \frac{|\psi_1\rangle + |\psi_2\rangle}{\sqrt{2}}$ IS NORMALIZED

$$p_+ = |\langle\varphi|+\rangle|^2 = \frac{1}{2}|c_1+c_2|^2 \neq \frac{1}{2}(p_1+p_2) = \frac{|c_1|^2+|c_2|^2}{2}$$

REMAINDER

$\mathrm{Re}(c_1^* c_2)$

>0 CONSTRUCTIVE INTERFERENCE

<0 DESTRUCTIVE

$\boxed{*}$ ACTUALLY THE PROBABILITY IS

$$p = \frac{|\langle\psi|\varphi\rangle|^2}{\langle\psi|\psi\rangle\langle\varphi|\varphi\rangle}$$

$\Leftarrow$ PHYSICAL Q. $\Leftrightarrow$ INVARIANT UNDER "GAUGE" TRANSFORM.

$|\psi\rangle \to \lambda|\psi\rangle$ WITH $\lambda \in \mathbb{C}$ $(\lambda \neq 0)$

THUS WE WORK WITH NORMALIZED QUANTUM STATES $\langle\psi|\psi\rangle = \langle\varphi|\varphi\rangle = 1$

1C ORTHONORMAL BASES

FOR ALL PRACTICAL PURPOSES

$$\text{DIMENSION}(\mathcal{H}) = \begin{cases} \nearrow \text{FINITE} \\ \searrow \text{COUNTABLE INFINITE} \end{cases}$$

WHY? BECAUSE 1. THE LAB/SAMPLE IS FINITE SIZE
2. WE WORK AT BOUNDED ENERGY BANDWIDTH

DEFINE AN ORTHONORMAL BASIS $|n\rangle$

$\uparrow$ LABEL = ONE OR MORE INTEGERS

$$\langle n | n' \rangle = \delta_{n,n'}$$ KRONECKER DELTA (NOT DIRAC)

$\Uparrow$ HILBERT METRIC GOES IN HERE

$$|\psi\rangle = \sum_n{}' c_n |n\rangle$$ COMPLETENESS

$c_n \downarrow \langle n|\psi\rangle$

1D OPERATORS

THEY ARE ENDOMORPHISMS OF $\mathcal{H}$ (LINEAR AND $\mathcal{H} \to \mathcal{H}$)

NOTATION $\to$ $\hat{A}|\psi\rangle$

$$\langle \psi_1 | \hat{A} | \psi_2 \rangle \overset{\text{def}}{=} (|\psi_1\rangle, \hat{A}|\psi_2\rangle)_{\mathcal{H}}$$

APPLIED TO THE <u>RIGHT</u>

WHERE $\langle \varphi | \psi \rangle = (|\varphi\rangle, |\psi\rangle)_{\mathcal{H}}$

$$= \begin{pmatrix}\text{definition of} \\ \text{HERMITIAN} \\ \text{CONJUGATE}\end{pmatrix} (\hat{A}^\dagger|\psi_1\rangle, |\psi_2\rangle) = \begin{pmatrix}\text{HILBERT} \\ \text{SCALAR} \\ \text{PRODUCT} \\ \text{PROPERTY}\end{pmatrix} (|\psi_2\rangle, \hat{A}^\dagger|\psi_1\rangle)^*$$

$$= (\langle \psi_2 | \hat{A}^\dagger | \psi_1 \rangle)^*$$

2. OBSERVABLES & MEASUREMENT

AN OBSERVABLE IS A HERMITIAN OPERATOR $\mathcal{O} = \mathcal{O}^\dagger$

OR MORE PRECISELY $\langle \psi | \mathcal{O} | \varphi \rangle = (|\psi\rangle, \mathcal{O}|\varphi\rangle) = (\mathcal{O}|\psi\rangle, |\varphi\rangle) = \langle \varphi | \mathcal{O} | \psi \rangle^*$

$\mathcal{O} = \mathcal{O}^\dagger$ SPECTRAL THEOREM $\left\{ \begin{array}{l} \mathcal{O} \text{ CAN BE DIAGONALIZED} \\ \& \\ \text{ITS EIGENBASIS IS ORTHOGONAL} \\ \& \\ \text{EIGENVALUES ARE REAL} \end{array} \right.$ (AT LEAST ONE EIGENBASIS IS ORTHOGONAL)

$$\mathcal{O} = U D U^\dagger$$

$U \to$ UNITARY $UU^\dagger = U^\dagger U = \mathbb{1}$ $\qquad U = \sum_j |j\rangle \overbrace{\langle \varphi_j|}^{\text{EIGENVECTORS}}$

$D \to$ DIAGONAL & REAL $\sum_j \underbrace{\eta_j}_{\text{EIGENVALUES}} |j\rangle\langle j|$ $\quad (\eta_j = \eta_j^*)$

$$\mathcal{O} = \sum_j \eta_j |\varphi_j\rangle\langle\varphi_j|$$

2A THE (HARD) MEASUREMENT PROCESS

PURE STATE $|\psi\rangle$ $\xrightarrow{\text{IS MEASURED UNDER}}$ OBSERVABLE $\mathcal{O}$ $\qquad$ OBSERVABLE SPECTRUM $\mathrm{Spec}\{\mathcal{O}\}$

FOR EVERY POSSIBLE OUTCOME $\lambda \in \mathrm{Spec}\{\mathcal{O}\}$ $\quad \Rightarrow \quad$ OUTCOME PROBABILITY $\displaystyle p_\lambda = \frac{\langle\psi|\Pi_\lambda|\psi\rangle}{\cancel{\langle\psi|\psi\rangle}}$

$\Downarrow$

Π_λ PROJECTOR OVER THE EIGENSPACE

$$\boxed{\mathcal{O}\Pi_\lambda|\phi\rangle = \Pi_\lambda|\phi\rangle\lambda}$$

$$[\Pi_\lambda, \mathcal{O}] = 0$$

$$\Pi_\lambda = \Pi_\lambda^2 = \Pi_\lambda^\dagger$$

$\Downarrow$ STATE AFTER MEASURING λ

$$|\psi'_\lambda\rangle = \frac{\Pi_\lambda|\psi\rangle}{\sqrt{\langle\psi|\Pi_\lambda|\psi\rangle}}$$

THIS PROCESS <u>BREAKS</u> TIME REVERSAL

(IT IS FINE BECAUSE IT IS AN EFFECTIVE PICTURE)

<u>EXPECTATION VALUE</u>

$$\langle\mathcal{O}\rangle = \sum_\lambda{}' \lambda p_\lambda = \langle\psi|(\Sigma\lambda\Pi_\lambda)|\psi\rangle = \langle\psi|\mathcal{O}|\psi\rangle$$

<u>VARIANCE</u>

$$\Delta\mathcal{O}^2 = \langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2 = \langle(\mathcal{O} - \langle\mathcal{O}\rangle)^2\rangle$$

2B OPERATORS & OBSERVABLES DO NOT COMMUTE

$$i[A,B]=C$$

NOTICE IF $A=A^\dagger$, $B=B^\dagger \Rightarrow C=C^\dagger$

$C\neq 0$ WE SAY THAT A & B "DO NOT COMMUTE"

$$\Delta A \, \Delta B \geqslant \frac{1}{2}|\langle i[A,B]\rangle|$$

HEISENBERG UNCERTAINTY PRINCIPLE

IN PRACTICE

MEASURE A $\curvearrowright$ A IS DETERMINED $\Longrightarrow$ THEN MEASURE B $\curvearrowright$ B IS DETERMINED $\curvearrowright$ A IS NO MORE DETERMINED

HEISENBERG IS NOT ALWAYS A HARD BOUND

$$\Delta p \, \Delta x \geqslant \frac{\hbar}{2}$$

HARD BOUND

$\Delta p \to 0$ MEANS $\Delta x \to \infty$

BUT CONSIDER

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \sigma^z = \begin{pmatrix}1 & \\ & -1\end{pmatrix}$ $\quad \sigma^x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\quad \sigma^y = \begin{pmatrix} & -i\\ i & \end{pmatrix}$

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\Delta\sigma^z_0 \, \Delta\sigma^x_0 \geqslant \frac{1}{2}|\langle\sigma^y\rangle_0|$$

$$\Delta\sigma^z_0 = \langle 0|(\sigma^z - \langle 0|\sigma^z|0\rangle)^2|0\rangle = \langle 0|(\sigma^z - 1)^2|0\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 4\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$$

$$\Delta\sigma^x_0 = \langle 0|(\sigma^x - \underbrace{\langle 0|\sigma^x|0\rangle}_{0})^2|0\rangle = \langle 0|\sigma^{x^2}|0\rangle = \langle 0|0\rangle = 1$$

$$\langle\sigma^y\rangle_0 = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix} & -i\\ i & \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$$

$$\underbrace{\Delta\sigma^z_0}_{0} \cdot \underbrace{\Delta\sigma^x_0}_{1} \geqslant \frac{1}{2}|\langle\sigma^y\rangle_0|$$

$$0 \cdot 1 \geqslant 0$$

$0 \geqslant 0$

WELL, WHATEVER

NOT REALLY HARD BOUND

INFINITE ACCURACY

FINITE ACCURACY

THIS IS OK

THAT'S ONE REASON WHY Q-INFO MAKES SENSE

2c

PHYSICAL TRANSFORMATIONS OF A CLOSED QUANTUM SYSTEM

IF A SYSTEM IS CLOSED

1. IT STAYS DETERMINISTIC
2. IT PRESERVES TOTAL PROBABILITY

UNDER PHYSICAL TRANSFORMATIONS

$$|\psi\rangle \longrightarrow |\psi'\rangle = T(|\psi\rangle) = |T(\psi)\rangle$$

$$\langle T(\psi)| \mathbb{1} |T(\psi)\rangle = \langle \psi | \mathbb{1} | \psi \rangle = 1$$

$$(T\psi, T\psi)_{\mathcal{H}} = (\psi, \psi)_{\mathcal{H}} \quad \forall \psi$$

THERE ARE ONLY TWO POSSIBILITIES

T ANTI-UNITARY

$$(T\psi, T\varphi) = (\psi, \varphi)^* \quad \forall \psi, \varphi$$

PROBLEM: CANNOT CONTINUOUSLY CONNECT WITH TRIVIAL TRAFO $\mathbb{1}$.
IMPOSSIBLE TO ACHIEVE WITH INCREMENTAL CHANGES
(STILL USEFUL AS A SYMMETRY)

OR

T UNITARY

$$(T\psi, T\varphi) = (\psi, \varphi) \quad \forall \psi, \varphi$$

THIS IS THE COMMON CASE: CONTINUOUSLY CONNECTED TO $\mathbb{1}$

SO TIME EVOLUTION OPERATORS ARE OF THIS CLASS

$$|\psi(t' > t)\rangle = U(t, t') |\psi(t)\rangle$$

$*$ CHANGE OF REFERENCE FRAME CAN BE BOTH UNITARY AND ANTI-UNITARY

UNITARIES FORM A GROUP

$$\underbrace{U V}_{\text{UNITARIES}} = \underbrace{W}_{\text{STILL UNITARY}}$$

MAKES SENSE: STACKING MULTIPLE PHYSICAL OPS IS STILL A PHYSICAL OP.

MATRIX NOTATION ORDER

$\longleftarrow$

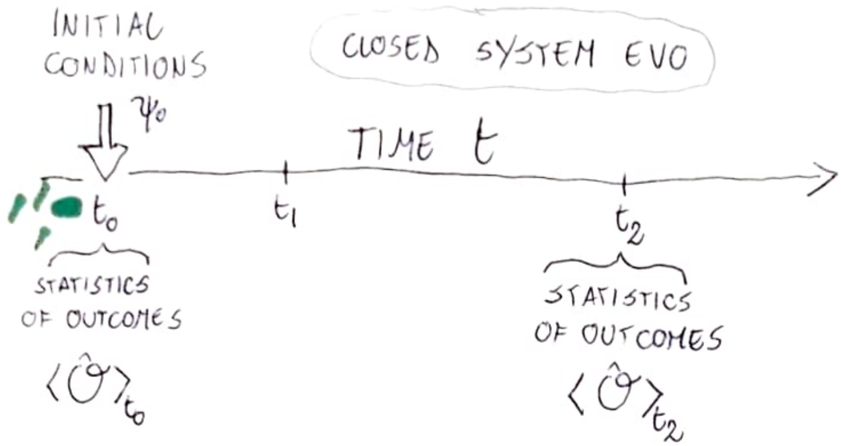
RIGHT TO LEFT

ALWAYS BE MINDFUL!

$$\underbrace{U}_{\text{THEN TRANSFORM WITH } U} \; \underbrace{V}_{\text{FIRST TRANSFORM WITH } V} \; \underbrace{|\psi\rangle}_{\text{INITIAL STATE}}$$

③ Time Evolution

THIS IS WHAT "MECHANICS" IS ABOUT IN QUANTUM MECHANICS



$$\langle \hat{O} \rangle_t = \langle \psi | \hat{O} | \psi \rangle_t$$

EVOLVE $|\psi\rangle$: $\langle \psi(t) | \hat{O} | \psi(t) \rangle$ SCHRÖDINGER PICTURE

EVOLVE O: $\langle \psi_0 | \hat{O}(t) | \psi_0 \rangle$ HEISENBERG PICTURE

THE STATISTICS OF OUTCOMES SHOULD BE THE SAME

3A. Schrödinger Picture – Evolving States

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$$

SCHRÖDINGER'S EQUATION

WHERE $H = H^\dagger$

HOLDS FOR EVERY QUANTUM SYSTEM EVO THAT IS

① PHYSICAL ② CLOSED ③ FIRST-ORDER DIFFERENTIAL IN TIME

(IN A WAY, SCHRÖDINGER IMPLICIT CAN BE RELATIVISTIC IF H IS RELATIVISTIC)

formal Solutions

$\boxed{H \text{ IS } t \text{ INDEPENDENT}}$

A. DIAGONALIZE $H = \sum \varepsilon_j |\varepsilon_j\rangle\langle\varepsilon_j|$ (SO THAT $H|\varepsilon_j\rangle = |\varepsilon_j\rangle \varepsilon_j$)

$$|\varepsilon_j\rangle \xrightarrow{\text{EVOLUTION}} e^{-i\varepsilon_j t/\hbar} |\varepsilon_j\rangle$$

B. EXPAND ANY INITIAL STATE $|\psi_0\rangle$ IN THE EIGENBASIS

$$\langle \varepsilon_j | \psi_0 \rangle = c_j$$

(BANDWIDTH! GAUGE TO $\varepsilon_j - \varepsilon_{j'}$)

$$|\psi_0\rangle = \sum_j c_j |\varepsilon_j\rangle \xrightarrow{\text{EVOLVE}} |\psi(t)\rangle = \sum_j |\varepsilon_j\rangle c_j e^{-i\varepsilon_j t/\hbar}$$

C. FORMAL EXPRESSION WITH THE MATRIX EXPONENTIAL

$$|\psi(t)\rangle = \left(\sum_j |\varepsilon_j\rangle e^{-i\varepsilon_j t/\hbar} \langle \varepsilon_j|\right) |\psi_0\rangle$$

$$= \underbrace{\exp(-iHt/\hbar)}\, |\psi_0\rangle$$

AND THIS IS INDEED UNITARY $U(t)$

AND ADDITIVE $U(t_2)U(t_1) = U(t_1+t_2)$

DON'T FORGET THAT $\exp(A) = \mathbb{1} + A + \frac{A^2}{2} + \frac{A^3}{6} + \ldots = \sum_{j=0}^{\infty} \frac{A^j}{j!}$

BUT ALSO $\exp(\underset{\substack{\uparrow \\ \text{UNITARY}}}{V A V^\dagger}) = V \exp(A) V^\dagger$ SO OFTENTIMES

① DIAGONALIZE

② EXPONENTIATE THE EIGENVALUES

$\boxed{\text{3B.}}$ Heisemberg Picture - Evolving OPS

$$\langle \mathcal{O} \rangle_t = \langle \psi | \mathcal{O} | \psi \rangle_t = \langle \psi_0 | \underbrace{U^\dagger(t)\, \mathcal{O}\, U(t)}_{\tilde{\mathcal{O}}(t)} | \psi_0 \rangle$$

$$\dot{U}(t) = \frac{d}{dt}\left(e^{-iHt/\hbar}\right)$$

$$= \left(-\frac{i}{\hbar} H\right) U(t)$$

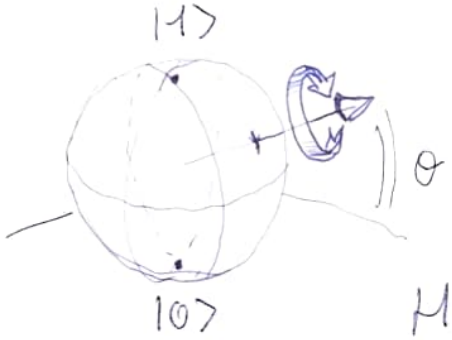
$[H, U] = 0$

$$\frac{d}{dt} \tilde{\mathcal{O}}(t) = \dot{U}^\dagger(t)\, \mathcal{O}\, U(t) + U^\dagger(t)\, \mathcal{O}\, \dot{U}(t)$$

$$= +\frac{i}{\hbar} H U^\dagger \mathcal{O} U - \frac{i}{\hbar} U^\dagger \mathcal{O} U H$$

$$\dot{\tilde{\mathcal{O}}} = \frac{i}{\hbar} [H, \tilde{\mathcal{O}}] \qquad \left(+ U^\dagger \dot{\mathcal{O}} U\right)$$

3c EXAMPLE: DRIVEN 2-LEVEL SYSTEM AND TRANSITIONS



$$|\psi_0\rangle = 0$$

$$\mathcal{H} = \hbar(\Omega\,\sigma^x + \Delta\sigma^z)$$

(RABI FREQ.) (DETUNING) ← WE WILL SEE WHY THIS IS THE CASE

$$H = \hbar\Omega\begin{pmatrix}0&1\\1&0\end{pmatrix} + \hbar\Delta\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \hbar\begin{pmatrix}\Delta & \Omega\\ \Omega & -\Delta\end{pmatrix}$$

REWRITE $\tilde{\Omega} = \sqrt{\Omega^2 + \Delta^2}$ $\theta = \arctan\left(\frac{\Delta}{\Omega}\right)$

$\hookrightarrow$ $\Omega = \tilde{\Omega}\sin\theta$ $\Delta = \tilde{\Omega}\cos\theta$

$$H = \hbar\tilde{\Omega}\,(\vec{n}\cdot\vec{\hat{\sigma}}) \qquad \vec{n} = \begin{pmatrix}\cos\theta\\0\\\sin\theta\end{pmatrix} \qquad \vec{\sigma} = \begin{pmatrix}\sigma^x\\\sigma^y\\\sigma^z\end{pmatrix}$$

NOTICE THAT $(\vec{n}\cdot\vec{\hat{\sigma}})^2 = \mathbb{1}$

Therefore $$\exp\left(-\frac{iHt}{\hbar}\right) = \sum_J \frac{(-i\tilde{\Omega}t)^J}{J!}(\vec{n}\cdot\vec{\hat{\sigma}})^J$$

$$= \mathbb{1}\sum_J^{\text{even}} \frac{(-i\tilde{\Omega}t)}{J!} + (\vec{n}\cdot\vec{\hat{\sigma}})\sum_J^{\text{odd}}\frac{(-i\tilde{\Omega}t)}{J!}$$

$$= \mathbb{1}\cos(\tilde{\Omega}t) + (\vec{n}\cdot\vec{\hat{\sigma}})(-i\sin(\tilde{\Omega}t))$$

$$\cos(\tilde{\Omega}t)\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\sin(\tilde{\Omega}t)\begin{pmatrix}\sin\theta & \cos\theta\\ \cos\theta & -\sin\theta\end{pmatrix}$$

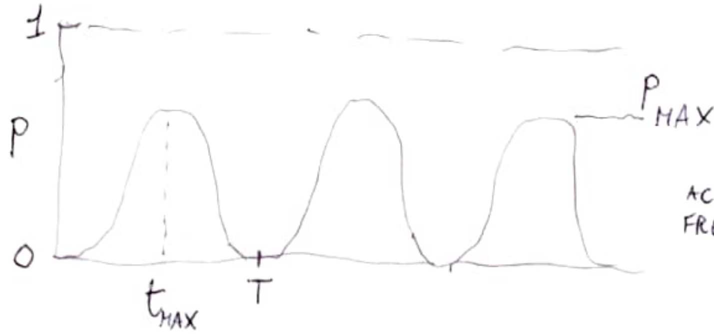
$$|\psi_0\rangle = 0 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|\psi(t)\rangle = \cos(\tilde{\Omega}t)\begin{pmatrix}1\\0\end{pmatrix} - i\sin(\tilde{\Omega}t)\begin{pmatrix}\sin\theta\\\cos\theta\end{pmatrix}$$

probability of measuring $|1\rangle$

$$|\langle 1|\psi(t)\rangle|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\tilde{\Omega}t) - i\sin(\tilde{\Omega}t)\sin\theta \\ -i\sin(\tilde{\Omega}t)\cos\theta \end{pmatrix} \right|^2$$

$$= \sin^2(\tilde{\Omega}t)\cos^2\theta$$



ACTUAL FREQUENCY $(2\tilde{\Omega})$

LEVEL SPACING OF PAULI MATRIX

COUPLING

$$t_{MAX} = (2j+1)\frac{\pi}{2\tilde{\Omega}} = (2j+1)\frac{\pi}{2\sqrt{\Omega^2+\Delta^2}}$$

PERIOD $T = \frac{\pi}{\tilde{\Omega}}$

MAX TRANSITION PROB. $\xrightarrow{}$ $$P_{MAX} = \cos^2\theta = \frac{\Omega^2}{\Omega^2+\Delta^2} = \left(1 - \frac{\Delta^2}{\Omega^2+\Delta^2}\right)$$

$\Downarrow$

QUALITATIVE LESSON:
TO HAVE HIGH TRANSITION YOU NEED $\Delta \ll \Omega$

(IMPORTANT LATER)