

# Differential Geometry Lecture Notes



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# Preface

## Textbook Reference

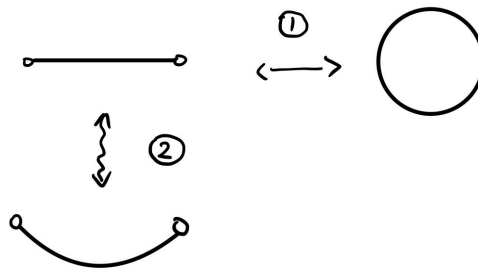
- (1) Do Carmo: Differential Geometry of Curves and Surfaces.
- (2) Sebastián Montiel, Antorio Ros: Curves and Surfaces.
- (3) *Chinese Title, add later*

## Course Introduction

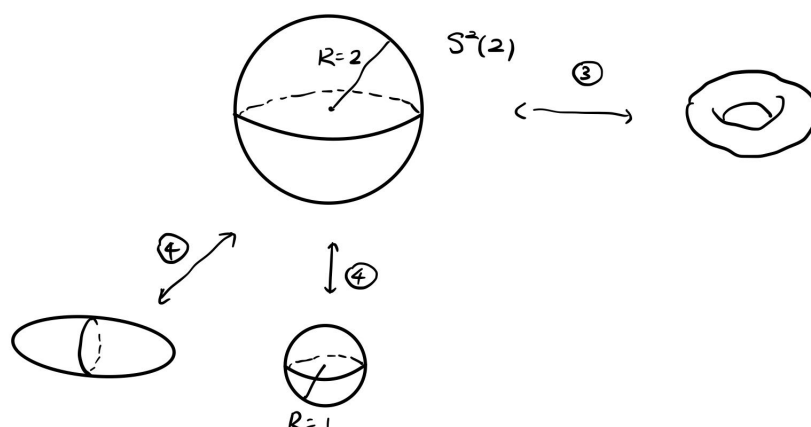
The Goal of this course is to study the “differential geometry of curves and surfaces”.

• **Geometry:** How is a geometric object curved / How to measure the curvedness of a geometric object?

**Example.** In the illustration below, (1) differs by “topology”. In (2), they are topologically the same, while the lower curve is “more curved” than the upper curve.



**Example.** (3) differs by “topology”, but in (4)  $\mathbb{S}^2(1)$  is more curve than  $\mathbb{S}^2(2)$ , even topologically they are the same.(either homeomorphically or diffeomorphically).



The “Curved property” also affects geometric quantities, like length, area, volume, angle between the curves, etc.

**Local Geometry:** How does a “curved” space look like in a neighborhood of a point?

**Global Geometry:** If we know how a “curved space” is look like at each point, can we observe how such space looks like globally? This is usually related to topological problems.

• **Differential:** In this course, by “smoothness” we mean the geometric objects we’ll study are “nice” enough so we can apply “calculus” tools to study them.

**Main tool:** Calculus! We’ll see how powerful calculus is in this course, especially, like the maximal principle, integration by parts(stoke’s theorem), Taylor’s expansion, implicit function theory, etc.



Queastion: How to tell the “smoothness”? (Need to find good parametrization) Finding a good “gauge”(that is “coordinate”) to work with is also an important question in geometry.

• Curves: 1-d geometric object.

Surfaces: 2-d geometric object.



*Remark.* In this course, we only focus on curves and surfaces in  $\mathbb{R}^3$ . However, as a training on preparing for later geometry course, I suggest you also try to think about the ambient space is  $\mathbb{S}^3$  or  $\mathbb{H}^3$ .

• **Intrinsic geometry:** Study the geometric object without considering the ambient space. This begins from the Gauss's elegant theorem and was developed by Riemann.

**Example.** Consider the unit sphere  $\mathbb{S}^2$

*Extrinsic geometry:* view it as  $x^2 + y^2 + z^2 = 1$  in  $\mathbb{R}^3$ .

*Intrinsic geometry:*  $(\theta, \varphi)$  or  $(\varphi, \theta)$  are “essential” coordinates on  $\mathbb{S}^2$ .

$$ds^2 = d\varphi^2 + (\sin \varphi)^2 d\theta^2$$

(Caution:  $(\theta, \varphi)$  is outer normal, while  $(\varphi, \theta)$  is inner normal.)

• Useful / Common techniques:

- 1) Comparison: compare the studied geometric object with “model space”. It's very important to study examples in geometry. As a suggestion you are expected to spend time to play with  $\mathbb{S}^2$ . For example: How is  $\mathbb{S}^2$  curved? What's the shortest line in  $\mathbb{S}^2$ ? How many symmetries are there on  $\mathbb{S}^2$ ? Can you add “extra structure” on  $\mathbb{S}^2$  to make it a complex object? Is this “extra structure” “rigid”? What/s the “moment map” on  $\mathbb{S}^2$ ? Does there exist a “holomorphic” map from  $\mathbb{S}^2$  to a torus, or a surface of arbitrary genus?

If we consider an “Energy minimizing map” from  $\mathbb{S}^2$  to  $\mathbb{S}^2$ , what can we say about such map? (It is holomorphic/antiholomorphic.)

After you have learned Riemann Geometry, you'll see an energy minimizing map from  $\mathbb{S}^2$  to a Riemannian manifold must be an angle-preserving map (conformal map).

What kinds of 2-d geometric space could be  $\mathbb{S}^2$ ? (this is a global geometry problem.) (i.e. what kinds of geometric conditions can characterize  $\mathbb{S}^2$ ?)

- 2) To study higher dimensional objects, it's also important to understand lower dimensional objects, and it's also important to understand lower dimensional objects contained in the studied objects.
- 3) Study “functions” (more generally sections, including functions, vector fields, differential forms, etc.) on a given geometric object.

**Example.** On a closed surface  $(\mathbb{S}^2, \mathbb{T}^2, \Sigma_g)$  (compact without boundary) there is no non-constant harmonic function. (i.e.  $\Delta u = 0$ ) (Analysis will get involved.)

We usually care about those functions related to geometry, such as distance functions, curvature-related functions, etc.

**Example** (More trivial than the last one). Consider  $f''(x) = 0$ , what can you say of the solution of it when  $x$  lies on a line and when  $x$  lies on a circle?



# Chapter 1

## Differential Geometry of Curves

### 1.1 Linear algebra convention and its geometric explanation

- We use “ROW VECTOR” in this course, i.e

$$v \in \mathbb{R}^n, v = (v_1, v_2, \dots, v_n)$$

- let  $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 1)$  be the standard basis, then

$$v = \sum_{i=1}^n v^i e_i = \begin{bmatrix} v^1 & v^2 & \dots & v^n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

- $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  (non-degenerate) linear map

$$v \mapsto \varphi(v) = v \cdot A.$$

This corresponds to the right action of  $GL(n, \mathbb{R})$  on  $\mathbb{R}^n$ .

$$\Rightarrow \varphi(e_i) = e_j \cdot A = \sum_{i=1}^n A_j^i e_i \text{ (taking the } j\text{-th row of } A)$$

$$A_j^i \begin{cases} \text{upper index: column index} \\ \text{lower index: row index} \end{cases}$$

$$\Rightarrow \varphi \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \varphi(e_1) \\ \varphi(e_2) \\ \vdots \\ \varphi(e_n) \end{bmatrix} = \begin{bmatrix} e_1 \cdot A \\ e_2 \cdot A \\ \vdots \\ e_n \cdot A \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

*Remark* (Important!). In row vector convention, a non-degenerate linear map corresponds to the right action of  $GL(n, \mathbb{R})$  on  $\mathbb{R}^n$ . But this induces left action of  $GL(n, \mathbb{R})$  on the orthonormal basis (frame)  $\{e_1, e_2, \dots, e_n\}$ . This phenomenon provides an important example in differential geometry, which will be explained later in the theory of principle bundle.(i.e. let  $G$  be a lie group,  $G \curvearrowright M$  being a right action, where  $M$  is a differentiable manifold, then this right action induces a left action of  $G$  on the frame bundle of  $M$ .)



Let  $\{\tilde{e}_1, \dots, \tilde{e}_n\}$  be another basis of  $\mathbb{R}^n$ . Let  $f$  be the corresponding linear map, i.e.

$$f \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \vdots \\ \tilde{e}_n \end{bmatrix} = B \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$\Rightarrow \tilde{e}_k = \sum_{j=1}^n B_k^j e_j$$

We compare the matrix of  $\varphi$  in terms of  $\{\tilde{e}_1 \cdots \tilde{e}_n\}$

$$\varphi \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_n \end{bmatrix} = \varphi \left[ B \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ e_n \end{bmatrix} \right] = B \cdot \varphi \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \quad (\text{linearity of } \varphi)$$

$$= BA \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = BAB^{-1} \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ e_n \end{bmatrix}$$

Note in this case,

$$(\varphi \circ f) \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = BA \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

In terms of entries,

$$\begin{aligned} \varphi(\tilde{e}_k) &= \varphi\left(\sum_{j=1}^n B_k^j e_j\right) = \sum_{j=1}^n B_k^j \varphi(e_j) \quad (\text{linearity}) \\ &= \sum_{i,j=1}^n B_k^j A_j^i e_i = \sum_{i,j,p=1}^n B_k^j A_j^i (B^{-1})_i^p \tilde{e}_p \end{aligned}$$

*Remark.* This computation tells that the row vector convention yields to the fact that  $GL(n, \mathbb{R})$  acting on itself from the right when we consider the action of  $GL(n, \mathbb{R})$  on  $\mathbb{R}^n$ . In modern Geometry, it's more common to use column vector as convention. This row vector convention was adopted by S.S. Chern and also Do Carmo's book.

## 1.2 Parametrized Curves

**Definition 1.2.1.** Let  $I = (a, b)$ , if  $\alpha: I \rightarrow \mathbb{R}^3$  is a  $C^\infty$  map,

$$t \mapsto (x(t), y(t), z(t))$$

then  $\alpha(t)$  is a parametrized differentiable curve in  $\mathbb{R}^3$ . The image of  $\alpha$  is called the trace of the curve.

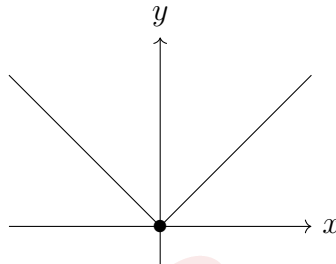
*Remark.*



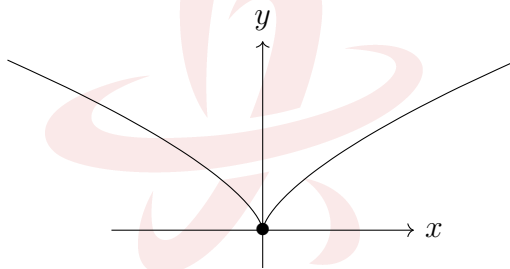
- 1)  $a, b$  could be finite number or infinity.
- 2) Same curve may have different parametrizations.
- 3) The parametrization automatically gives the direction of the motion on the curve.
- 4) “Differentiable” just means  $\alpha(t)$  is a  $C^\infty$  **map**, it does not say the (trace of) curve can not have singularities.

**Example.**

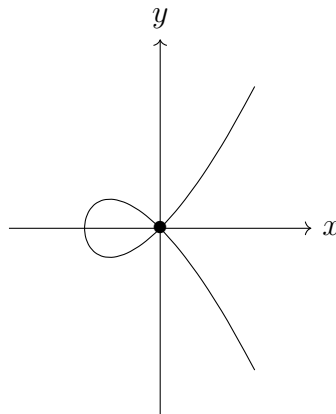
- (1)  $\alpha(t) = (t, |t|)$  is not a differentiable curve.



- (2)  $\alpha = (t^3, t^2)$  is a differentiable curve. It can be also given by a equation  $y^3 = x^2$ , which is a cuspidal cubic curve.



- (3)  $\alpha(t) = (t^2 - 1, t^3 - t)$ . This parametrization appears in the “blow-up” process of  $y^2 = x^3 + x^2$ . Here “blow-up” is introducing tangents to separate points.



*Remark.* (2) and (3) above may be the first examples you’ll see in an algebraic geometry course.





**Question:** At the origin, what can you observe on (2) and (3)?

**Answer:** (2)  $\alpha'(0) = 0$ . (3)  $\alpha$  is not one to one, but  $\alpha'(0) \neq 0$ .

**Question:** Define a differentiable curve in  $\mathbb{R}^3$  and  $\mathbb{S}^n$ .

*Remark.* Among above differentiable parametrizations, (2) and (3) are differentiable curves. However, if we take  $\beta(t) = (t, t^{\frac{2}{3}})$ , this also parametrizes (2), but it's not a differentiable curve!

**Definition 1.2.2.** Let  $\alpha(t): I \rightarrow \mathbb{R}^3$  be a parametrized differentiable curve, then at  $t_0 \in I$ .

$$\alpha'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$$

is the velocity of  $\alpha(t)$  at  $t_0$ .

(1) If  $\alpha'(t_0) \neq 0$ , we call  $\alpha(t_0)$  a regular point.

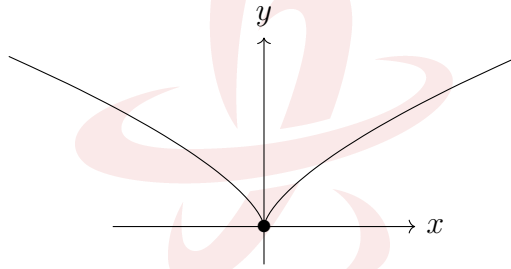
(2) If  $\alpha'(t_0) = 0$ , we call  $\alpha(t_0)$  a singular point.

(3) If for all  $t \in I$ ,  $\alpha'(t) \neq 0$ , we call  $\alpha(t)$  a regular curve.

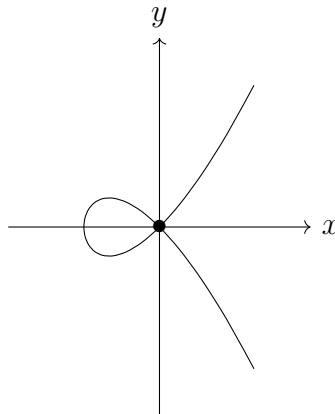
**Question:** What can you say about  $C^\infty$  parametrization for a regular curve?

Regular curve  $\iff$  at each point, there is a unique tangent line.

**Example.**  $\alpha(t) = (t^3, t^2)$  is not a regular curve. (Since  $\alpha'(0) = 0$ )

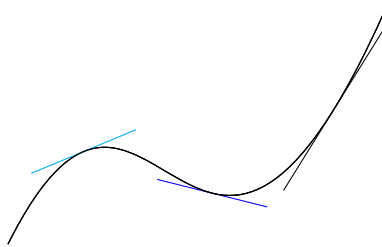


**Example.**  $\alpha(t) = (t^2 - 1, t^3, t)$  is a regular curve.



**Definition 1.2.3.** Let  $\alpha(t)$  be a regular curve, then the tangent line at  $t_0$  is

$$l(t) = \alpha(t_0) + \alpha'(t_0)(t - t_0)$$



**Definition 1.2.4.** Let  $\alpha(t)$  be a regular curve, the arc-length of  $\alpha(t)$  is

$$s(t) = \int_{t_0}^t |\alpha'(t)| dt.$$

Then  $s'(t) = |\alpha'(t)|$

**Question** What's  $|\alpha'(t)|$ ?

$\alpha(t): I \rightarrow \mathbb{R}^3$  is a curve in  $\mathbb{R}^3$ . Here on  $\mathbb{R}^3$ , as the Euclidean space, we always assume the standard Euclidean inner product on it, i.e.  $\forall u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$

$$\langle u, v \rangle = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{i,j=1}^3 \delta_{ij}u_iv_j$$

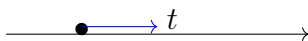
Let  $\alpha(t) = (x(t), y(t), z(t))$ ,  $\alpha'(t) = (x'(t), y'(t), z'(t))$ , then  $|\alpha'(t)| = \sqrt{\langle \alpha'(t), \alpha'(t) \rangle}$

**Exercise.** Review vector Calculations, such as dot product, cross product and their properties, especially geometric meaning of these calculation, such as length, area, volume, angle, orientation, etc.

**Question:** Can you define the arclength of a regular curve in  $\mathbb{R}^n$ ? How about on  $\mathbb{S}^n$ ?

• Arclength parameter (an intrinsic parametrization of a curve)

**Example.** On a straight line,  $x=t$  describes the distance of the point away from the origin.



On a general curve, we also want “some” parameter, which describes the arclength of point away from the initial point. This can happen iff  $|\alpha'(t)| = 1$ , i.e. a point on the curve moves in a unit speed.

$$\Rightarrow s(t) = \int_0^t dt = t.$$

**Question:** For a given regular curve  $\alpha(t): I \rightarrow \mathbb{R}^3$ , how to find such parameter?

**Answer:**  $s(t) = \int_{t_0}^t |\alpha'(t)| dt$  is a function in  $t$ , and  $s'(t) = |\alpha'(t)| \neq 0$  (because the curve is regular). By the implicit function theorem, there is a function

$$t = t(s), t'(s) = \frac{1}{|\alpha'(t)|}.$$



This implies that

$$\begin{aligned}\alpha(t) &= \alpha(t(s)) = (x(t(s)), y(t(s)), z(t(s))) \\ |\alpha'(s)| &= |\alpha'(t)t'(s)| = |\alpha'(t)| |t'(s)| = 1\end{aligned}$$

**Convention** In this course, we only consider differentiable regular curves, which are parametrized by the arclength (for convenience).

*Remark.* In this course, we only consider the curve without self-intersecting points, i.e. curves “embedded into”  $\mathbb{R}^3$ . Here “embedded” means  $d\alpha$  is a linear isomorphism and  $\alpha$  is homeomorphic to its image.

## 1.3 Local theory of a regular space curve

**Goal.** Describe a space curve by using geometric quantities.

**Question.** How to make a space curve?

Starting with a straight line, we can bend it and twist it in a given way to produce a space curve.

- Bending the line  $\rightarrow$  “curvature”.
- Twisting  $\rightarrow$  “torsion”.
- Their relations are contained in Frenet formula.
- Conversely, fundamental theorem of the local theory of curves tells, once we’re given two function,  $\kappa(s), \tau(s)$ , we can describe a unique curve in  $\mathbb{R}^3$  up to a rigid motion, s.t.  $\kappa(s)$  is its curvature and  $\tau(s)$  is its torsion.

**Recall:** In Calculus, if  $y = f(x)$  represents a curve, then  $f''(x)$  tells the convexity of the curve. It measures how fast the velocity changes. It’s also related to how straight line is bent.

Let  $\alpha: I \rightarrow \mathbb{R}^2$  be a regular plane curve, parametrized by arc length, i.e.  $|\alpha'(s)| = 1$ . Then  $\langle \alpha'(s), \alpha''(s) \rangle = 0$ , and hence  $\alpha''(s) \perp \alpha'(s)$ . For a plane curve, we take normal of the curve to be counterclockwise  $90^\circ$  rotation of the tangent vector.

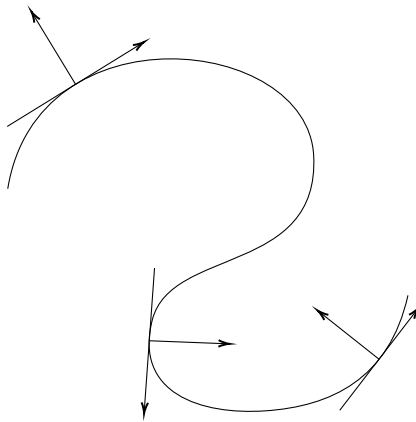


Figure 1.3.1: Example of a plane curve and its tangent and normal

Let  $N$  be the unit normal vector along  $\alpha(s)$ , we have

$$\langle \alpha''(s), N(s) \rangle = \pm |\alpha''(s)|.$$



**Definition 1.3.1.** The curvature of a plane curve  $\alpha(s)$  is defined as

$$\kappa(s) = \langle \alpha''(s), N(s) \rangle.$$

**Definition 1.3.2.** Further we denote  $T$  be the unit tangent vector, then the Frenet equation of  $\alpha(s)$  is

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T \end{cases}.$$

Note that

$$\langle T', N \rangle = \kappa \implies \langle T, N' \rangle = -\kappa.$$

- $\kappa > 0 \implies$  the point on the curve moves counterclockwise direction or say “to its left”.
- $\kappa < 0 \implies$  the point on the curve moves clockwise direction or say “to its right”.

**Question.** For the curve in fig. 1.3.1, can you tell where  $\kappa > 0$  and where  $\kappa < 0$  without doing calculation?

*Remark.* The sign of the curvature of the plane curve is caused by the direction convention of the unit normal vector. This could change according to the orientation of a curve.

Next, we take a look at the geometric meaning of  $|\alpha''(s)|$  at some point  $\alpha(s_0)$ . By definition:

$$|\alpha''(s_0)| = \lim_{h \rightarrow 0} \left| \frac{\alpha'(s_0 + h) - \alpha'(s_0)}{h} \right|.$$

We have

$$\begin{aligned} |\alpha'(s_0 + h) - \alpha'(s_0)| &= \left( |\alpha'(s_0 + h)|^2 + |\alpha'(s_0)|^2 - 2 \langle \alpha'(s_0 + h), \alpha'(s_0) \rangle \right)^{\frac{1}{2}} \\ &= (2 - 2 \cos \theta_h)^{\frac{1}{2}} \\ &= (2 - 2(1 - \frac{1}{2} \theta_h^2) + \tilde{o}(\theta_h^4))^{\frac{1}{2}} \\ &= (\theta_h^2 + \tilde{o}(\theta_h^4))^{\frac{1}{2}}. \end{aligned}$$

Hence

$$|\alpha''(s_0)| = \lim_{h \rightarrow 0} \left| \frac{\alpha'(s_0 + h) - \alpha'(s_0)}{h} \right| = \lim_{h \rightarrow 0} \left| \frac{\theta_h}{h} \right| = |\theta'(s_0)|.$$

*i.e.*  $|\alpha''(s)|$  measures the changing rate of angle of tangents.

In fact, for a plane curve, let  $\theta$  be the angle between  $\alpha'(s_0)$  and  $\alpha'(s)$ , then

$$\langle \alpha'(s), \alpha'(s_0) \rangle = \cos \theta_s \implies \langle \alpha''(s), \alpha'(s_0) \rangle = -\sin \theta_s \cdot \theta'_s.$$

Notice that  $\cos \theta_s$  is the projection of  $\alpha'(s_0)$  on the tangent  $\alpha'(s)$ , hence

$$\sin \theta_s = \langle \alpha'(s_0), N(s) \rangle.$$

On the other hand,  $\alpha''(s) = T'(s) = \kappa(s)N(s) = \pm |\alpha''(s)|N(s)$ , this gives  $\theta'_s = \pm |\alpha''(s)|$ .

Let  $\alpha: I \rightarrow \mathbb{R}^3$  be a space curve, parametrized by arclength, *i.e.*  $|\alpha'(s)| = 1$ , we also have  $\langle \alpha'(s), \alpha''(s) \rangle = 0$ , *i.e.*  $\alpha''(s) \perp \alpha'(s)$ .

Unlike case of dim 2, it does not make sense to prescribe a normal vector of a curve. However, from above discussion, we see the geometric meaning of  $|\alpha''(s)|$  is the measure of how fast the point on the curve leaving the straight line. We came into following definition:



**Definition 1.3.3.** The *curvature* of a regular space curve  $\alpha(s)$  parametrized by arclength is defined as

$$\kappa(s) = |\alpha''(s)|.$$

And the unit normal vector at  $\alpha(s)$  is

$$N = \frac{\alpha''(s)}{|\alpha''(s)|}, \quad \text{for } |\alpha''(s)| > 0.$$

*Remark.*

- If  $|\alpha''(s)| \equiv 0$  then  $\alpha$  must be a straight line, and all unit normal vectors line on a unit circle  $\perp \alpha$ .
- If  $|\alpha''(s_0)| = 0$ , we call  $s_0$  a singular point of order 1. (Note.  $s_0$  s.t.  $|\alpha(s_0)| = 0$  is called a singular point of order 0) At such points, there is no well-defined normal vector.

**Definition 1.3.4.** The plane determined by  $T, N$  is called the *osculating plane* of  $\alpha(s)$ . The unit normal vector of the osculating plane

$$B = T \times N$$

is called *binormal vector*.

*Remark.*

- $\{T, N, B\}$  satisfies the right hand rule.
- $|B'|$  measures how fast the point leaves the osculating plane.

If we denote  $\theta_h$  be angle between  $B(s_0 + h)$  and  $B(s_0)$ , similar to former calculation, we have

$$\begin{aligned} |B'(s_0)| &= \lim_{h \rightarrow 0} \left| \frac{B(s_0 + h) - B(s_0)}{h} \right| \\ &= \lim_{h \rightarrow 0} \left| \frac{\sqrt{2 - 2 \cos \theta_h}}{h} \right| \\ &= |\theta'_{s_0}|. \end{aligned}$$

As we saw, at each (non-singular) point on a space curve  $\alpha(s)$ , we can associate an oriented orthonormal frame  $\{T, N, B\}$ .

**Question.** How these three vector fields are related to the geometry of the curve?

By definition, we write out 0-order info of  $\{T, N, B\}$ , i.e.

$$\begin{cases} T = \alpha' \\ N = \frac{\alpha''}{|\alpha''|} \\ B = T \times N \end{cases} \implies \begin{cases} T' = \alpha'' = \kappa N \\ B' = T' \times N + T \times N' = T \times N' \end{cases}.$$

Since  $|B| = 1$ , we have  $B' \perp B$ . Also from above we see  $B' \perp T$ , hence  $B' \parallel N$ .



**Definition 1.3.5.** We define

$$B' = \tau N.$$

Here  $\tau(s)$  is called the torsion of  $\alpha(s)$ .

Next, we also want to find  $N'$ .  $N = B \times T$  gives

$$\begin{aligned} N' &= B' \times T + B \times T' \\ &= \tau N \times T + B \times (\kappa N) \\ &= -\kappa T - \tau B. \end{aligned}$$

**Theorem 1.3.6.** *The fundamental equations of a space curve (also called the Frenet equations) is given by*

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \quad (1.3.1)$$

*Remark.*

- (1)  $\tau \equiv 0$  but  $\kappa \neq 0$  at all points  $\implies \alpha(s)$  is a plane curve. (Note this may be not true if we don't assume  $\kappa \neq 0$ , see ex.10 in Do Carmo's book)  
 $\kappa \equiv 0 \implies \alpha(s)$  is a straight line.

Note  $\tau \equiv 0 \implies B' = 0 \implies B$  is constant vector. Further,

$$(\alpha \cdot B)' = \alpha' \cdot B = T \cdot B = 0,$$

gives  $\alpha \cdot B = \text{constant} \implies (\alpha(s) - \alpha(s_0)) \cdot B = 0$ . Since  $\kappa \neq 0$  at all points, the osculating plane is always well-defined, hence  $B$  is always defined, we proved  $\alpha$  lie in some plane perpendicular to vector  $B$ .

- (2) In different textbooks, you may see the definition of  $\tau$  having a different sign from here.

*Friendly warning: When studying the Geometry, (even later in Riemannian Geometry), it happens a lot that different authors use different sign convention for the same definition. It's very important that you should fix your own notation, and keep it consistently!*

**Definition 1.3.7.**  $\{T, N, B\}$  is called Frenet trihedron of  $\alpha(s)$ , it gives an moving orthonormal basis of  $\mathbb{R}^3$  along the curve  $\alpha(s)$ .

The Frenet equation describes how such moving orthonormal basis moves along  $\alpha(s)$ .

*Remark.* Note that in above discussion, we have chosen a special parameter, the arclength parameter, of  $\alpha(s)$ . In the study of Geometry, finding a good parametrization can simplify a lot of work and itself an important problem. In more general framework, it's called a "Gauge related" problem.

We have seen that given a regular curve  $\alpha(s)$ , parametrized by arclength, the Frenet equation is eq. (1.3.1), for some functions  $\kappa(s) > 0$  be its curvature and  $\tau(s)$  be its torsion. Conversely, we ask

**Question.** If we're given smooth functions  $\kappa(s), \tau(s)$  with  $\kappa(s) > 0$ ,



- (1) (Existence) Does there exist a regular curve  $\alpha(s)$  s.t.  $\kappa(s)$  is its curvature and  $\tau(s)$  is its torsion?
- (2) (Uniqueness) If such curve exists, is it unique in some sense?

The answer is **YES!**

**Theorem 1.3.8** (Fundamental theorem of the local theory of curves). *Let  $\kappa(s), \tau(s): I \rightarrow \mathbb{R}$  be smooth functions, assume  $\kappa(s) > 0$ , then*

- (Existence) *There is a regular curve realize  $\kappa$  and  $\tau$  as its curvature and torsion.*
- (Uniqueness) *If  $\alpha, \beta$  are two such curves parametrized by arclength parameter, then they only differ by a rigid motion of  $\mathbb{R}^3$ . i.e.  $\exists T \in O(3), c \in \mathbb{R}^3$  s.t.  $\beta = T\alpha + c$ .*

*Remark.*

- (1) Existence follows from a Cauchy problem (initial value problem) of ODE system.
- (2) The curve is unique up to a rotation of  $\mathbb{R}^3$  and a translation.

*Proof.* If we denote

$$X(s) = \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad P(s) = \begin{bmatrix} & \kappa & \\ -\kappa & & -\tau \\ & \tau & \end{bmatrix}.$$

Where  $T, N, B$  are viewed as **row vectors**. Then the Frenet equation writes as

$$X' = PX.$$

This is a first order linear ODE (of nine unknown functions), then by the existence and uniqueness theorem of ODEs, given any initial value

$$X(0) = \begin{bmatrix} T_0 \\ N_0 \\ B_0 \end{bmatrix}$$

with form an orthonormal basis, the system has a unique solution that extend to whole domain  $I$ .

We need to check the solution actually is orthonormal frame for each  $s$ , notice the orthonormal relation can be written as

$$XX^t = I_3.$$

Where  $I_3$  is the identity matrix of dimension three. Take differential on left hand side of the equation we get

$$\begin{aligned} \frac{d}{ds}(XX^t) &= X'X^t + X(X^t)' = X'X^t + X(X')^t \\ &= PXX^t + X(PX)^t \\ &= PXX^t + XX^tP^t \\ &= PXX^t - XX^tP. \end{aligned}$$



If we denote  $Y = XX^t$ , we see

$$Y' = PY - YP$$

In coordinates, if we set  $T = v_1, N = v_2, B = v_3$ , and

$$y_{ij}(s) = \langle v_i(s), v_j(s) \rangle, \quad P = (a_{ij})_{i,j=1}^3.$$

We have  $y_{ij} = y_{ji}, a_{ij} = -a_{ji}$ , and then

$$\begin{aligned} \frac{d}{ds} y_{ij} &= \langle v'_i, v_j \rangle + \langle v_i, v'_j \rangle \\ &= \langle a_{ik} v_k, v_j \rangle + \langle v_i, a_{jk} v_k \rangle \\ &= a_{ik} y_{kj} + a_{jk} y_{ki} \\ &= a_{ik} y_{kj} - y_{ik} a_{kj}. \end{aligned}$$

This gives again a first order ODE system, with initial value  $Y(0) = I_3$ , or say  $y_{ij}(0) = \delta_{ij}$ , but there is an obvious solution  $Y \equiv I_3$ , so by uniqueness theorem, this is it. This proves  $XX^t = I_3$  for any  $s$ .

Until now, we have proved the existence of orthonormal moving frame  $\{T, N, B\}$ . Notice  $T$  is just  $\alpha'(s)$ , so given initial point  $\alpha(0) = \alpha_0$ , integrate w.r.t  $s$  gives a valid solution

$$\alpha(s) = \alpha_0 + \int_0^s T(\xi) d\xi.$$

For the uniqueness, we need to look carefully into the initial condition we chose for the solution  $\alpha$ , that is, choice of initial frame  $\{T_0, N_0, B_0\}$  and initial point  $\alpha_0$ . Given two valid solution curve  $\alpha, \beta$ , with initial condition  $(X_a, \alpha_0)$  and  $(X_b, \beta_0)$ , we choose an orthogonal matrix  $T = X_b X_a^{-1}$ , a constant  $c = \beta_0 - T\alpha_0$ , then we see the curve

$$\tilde{\beta} = T(\alpha - \alpha_0) + \beta_0 = T\alpha + c$$

satisfy exactly the same initial condition as  $\beta$ , so they must agree. This proves the uniqueness up to rigid motion we stated before.  $\square$

*Remark.*

- (1) **Exercise:** Check that for solution given above,  $\kappa$  and  $\tau$  are its curvature and torsion.
- (2) The condition  $\kappa > 0$  is needed for uniqueness. Can you construct a counterexample when there is one point  $s.t.$   $\kappa = 0$ .
- (3) Uniqueness can be proved without knowledge of ODEs, see theorem after this remark.
- (4) We can view the ODE problem at a somehow higher point. Consider the space of all orthonormal frames, it is actually a smooth manifold. It's a little non-trivial but we can identify the space with three dimensional rotation group  $SO(3)$ , smoothly embedded into  $\mathbb{R}^9$ , the space of three dimensional matrices. The equation, can be interpreted to a (time dependent) vector field on  $SO(3)$ . One can verify the vector field is tangent to the manifold, so it is actually a vector field not only in  $\mathbb{R}^9$  but in  $SO(3)$  itself. Similar to the existence and uniqueness theorem of ODEs on Euclid spaces, we have a version of such theorem for smooth manifolds. It states that for a smooth manifold  $M$ , a (maybe time dependent) smooth vector field  $X$  on  $M$ , then





with any given initial point  $p$ , there exists an integral curve on  $M$  starting at  $p$ , tangent to  $X$  everywhere, and it is unique. Using the theorem, we can say that with given initial  $\{T_0, N_0, B_0\}$ , there exists a unique solution  $\{T(s), N(s), B(s)\}$ . Note that the solution is automatically lie in  $SO(n)$ , no need to verify it is orthonormal.

*Proof. (Uniqueness, without ODE knowledge).*

Let  $\alpha(s), \beta(s)$  share the same  $\kappa(s), \tau(s)$  as their curvature and torsion, by similarly a rotation and a translation, we assume they have same initial condition, *i.e.*  $\alpha(0) = \beta(0)$ , and the Frenet frame agree at  $s = 0$ . We claim now we must have  $\alpha(s) = \beta(s), \forall s$ .

Notice  $\alpha(s) = \alpha(0) + \int_{s_0}^s \alpha'(s) ds$ , so it suffices to show  $T_\alpha = T_\beta$ , equivalently

$$|T_\alpha - T_\beta|^2 = 0.$$

Take differential we get:

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} |T_\alpha - T_\beta|^2 &= \langle T_\alpha - T_\beta, T'_\alpha - T'_\beta \rangle \\ &= -\langle T_\alpha, \kappa N_\beta \rangle - \langle T_\beta, \kappa N_\alpha \rangle. \end{aligned}$$

Similar calculation gives

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} |N_\alpha - N_\beta|^2 &= \langle N_\alpha, \kappa T_\beta + \tau B_\beta \rangle + \langle N_\beta, \kappa T_\alpha + \tau B_\alpha \rangle \\ \frac{1}{2} \frac{d}{ds} |B_\alpha - B_\beta|^2 &= -\langle B_\alpha, \tau N_\beta \rangle - \langle B_\beta, \tau N_\alpha \rangle. \end{aligned}$$

Sum the three equation we have

$$\frac{1}{2} \frac{d}{ds} (|T_\alpha - T_\beta|^2 + |N_\alpha - N_\beta|^2 + |B_\alpha - B_\beta|^2) = 0.$$

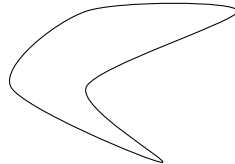
But the sum of square equals 0 at  $s = s_0$ , so it is identically 0, in particular,  $T_\alpha = T_\beta$  for all  $s$ .  $\square$

## 1.4 Global theory of plane curves

The global theory is related to “topology” of the geometric objects. For 1-dimensional geometry, *i.e.* curves, it’s always oriented. And the simplest distinction in topology is “open” and “closed”.

**Definition 1.4.1** (Closed curves).

- We say  $\alpha: I = [a, b] \rightarrow \mathbb{R}^3$  (or  $\mathbb{R}^2$ ) is a regular closed curve, if  $\alpha(a) = \alpha(b)$  and  $\alpha^{(k)}(a) = \alpha^{(k)}(b)$  (in another word,  $\alpha: \mathbb{S}^1 \rightarrow \mathbb{R}^3$  is a differentiable curve).
- Furthermore, if  $\alpha$  has no self-intersection point other than  $\alpha(a) = \alpha(b)$ , then we call  $\alpha(s)$  to be a simple closed curve.





### 1.4.1 Isoperimetric inequality

This is one of the oldest and most famous problem in geometry. It's still attracting mathematicians to investigate such problem in various geometric formulations nowadays.

**Question.** Given a closed plane curve  $C$  with. Let  $D$  be the region bounded by  $C$ . When does the region have the maximal area, if  $C$  is among all the curves with fixed length?

**Answer.**  $C$  must be a circle when the maximal area is achieved.

*Remark.* Even though we'll only handle smooth, simple closed curves in the following discussion, in general we don't have to assume the curve to be simple:  $\bigcirc\bigcirc$  has less area than  $\bigcirc$ . (caution: their boundaries are intended to have the same length). Think about how  $\bigcirc\bigcirc$  comes from  $\bigcirc$ .

#### Proofs of the Isoperimetric inequality

*Proof.* 1 (Hurwitz's proof) This relies on the "Wirtinger's inequality".

Let  $\alpha(t)$  be a closed, simple smooth curve, where  $t$  can be any parameter. The length of it is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Observe that we need to find the lower bound of  $L^2$ . Generally, for an integral  $L = \int \sqrt{f} dt$ , Hölder inequality (or Cauchy-Schwarz) naturally gives estimate of  $L$ . Hence, it's natural to find a "good parameter" to clear. Although the arclength  $s$  is a good candidate, it turns out in this case that another good parameter is

$$\theta = \frac{2\pi}{L} s.$$

$s \in [0, L] \Rightarrow \theta \in [0, 2\pi]$ . (This parameter  $\theta$  comes from the "Wirtinger's inequality", but of course a rescaling of wirtinger's inequality allows us to use  $s$  as usual).

Let's take  $\theta = \frac{2\pi}{L} s$ , then

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) \left(\frac{ds}{d\theta}\right)^2 = \left(\frac{L}{2\pi}\right)^2. \\ \Rightarrow \frac{L^2}{2\pi} &= \frac{L^2}{4\pi^2} \cdot 2\pi = \int_0^{2\pi} (x'(\theta)^2 + y'(\theta)^2) d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} 2 \left( \frac{L^2}{4\pi} - A \right) &= \int_0^{2\pi} (x'(\theta)^2 + y'(\theta)^2) d\theta - 2 \int_0^{2\pi} x(\theta)y'(\theta) d\theta \\ &= \int_0^{2\pi} x'(\theta)^2 - x(\theta)^2 + \underbrace{(y'(\theta) - x(\theta))^2}_{\geq 0} d\theta \\ &\geq \int_0^{2\pi} x'(\theta)^2 - x(\theta)^2 d\theta. \end{aligned} \quad (\star)$$

Now, the proof reduces to the following lemma.



**Lemma 1.4.2** (Wirtinger's inequality). *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a  $2\pi$ -periodic smooth function and  $\int_0^{2\pi} f(\theta) d\theta = 0$ , Then*

$$\int_0^{2\pi} f(\theta)^2 d\theta \leq \int_0^{2\pi} f'(\theta)^2 d\theta,$$

*and equality holds iff  $f(\theta) = a \cos(\theta) + b \sin(\theta)$ .*

(Proof of the lemma is left as a homework problem.)

To apply this to (★), we need to assume  $\int_0^{2\pi} x(\theta) d\theta = 0$ . However, we know the center of mass of the curve is  $\left(\frac{\int x(\theta) d\theta}{L}, \frac{\int y(\theta) d\theta}{L}\right)$ , and by choosing the origin of  $\mathbb{R}^2$  as the center of mass, we can guarantee  $\int_0^{2\pi} x(\theta) d\theta = 0$ , this yields ★  $\geq 0$ , i.e.  $L^2 \geq 4\pi A$ . Moreover, equality implies

$$x(\theta) = a \cos(\theta) + b \sin(\theta) \text{ and } y'(\theta) = x(\theta) \Rightarrow$$

$$y(\theta) = a \sin(\theta) - b \cos(\theta) + c$$

So  $(x(\theta), y(\theta))$  is a circle. □

*Proof. 2* (By Schmidt) See Do Carmo's book (page 33-35). It will be lectured by TA in a recitation. □