Differential Geometry Lecture Notes

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Contents

Preface

Textbook Reference

(1) Do Carmo: Differential Geometry of Curves and Surfaces.

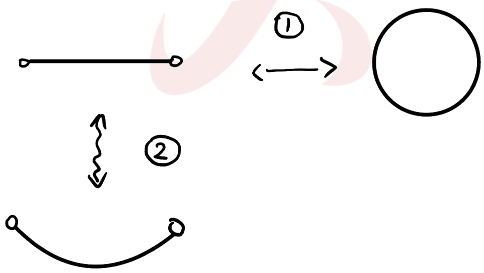
(2) Sebastián Montiel, Antorio Ros: Curves and Surfaces.

(3) *Chinese Title, add later*

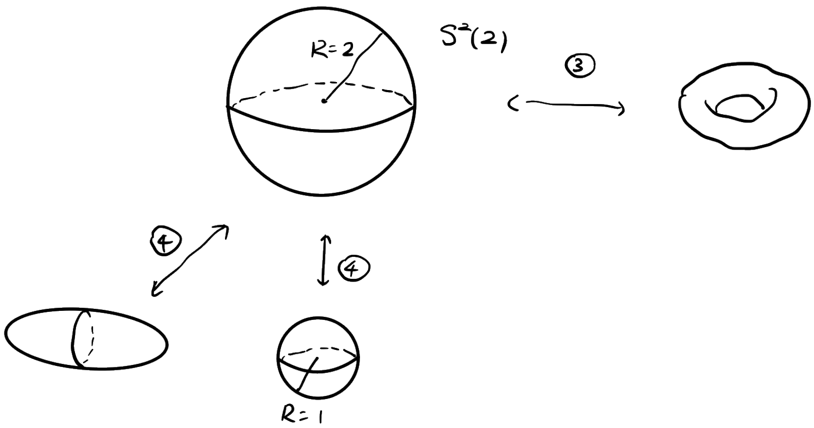
Course Introduction

The Goal of this course is to study the “differential geometry of curves and surfaces”.
• **Geometry**: How is a geometric object curved / How to measure the curvedness of a geometric object?

Example 0.0.1. *In the illustration below, (1) differs by “topology”. In (2), they are topologically the same, while the lower curve is “more curved” than the upper curve.*



Example 0.0.2. *In the following picture, (3) differs by “topology”, but in (4)* $\mathbb{S}^2(1)$ *is more curve than* $\mathbb{S}^2(2)$*, even topologically they are the same.(either homeomorphically or diffeomorhically).*



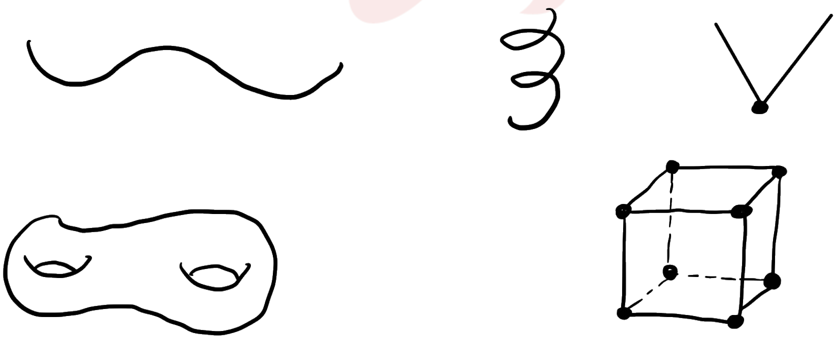
The “Curved property” also affects geometric quantities, like length, area, volume, angle between the curves, etc.

Local Geometry: How does a “curved ” space look like in a neighborhood of a point?

Global Geometry: If we know how a “curved space” is look like at each point, can we observe how such space looks like globally? This is usually related to topological problems.

• **Differential**: In this course, by “smoothness” we mean the geometric objects we’ll study are “nice” enough so we can apply “calculus” tools to study them.

Main tool: Calculus! We’ll see how powerful calculus is in this course, especially, like the maximal principle, integration by parts(stoke’s theorem), Taylor’s expansion, implicit function theory, etc.



Queastion: How to tell the “smoothness”?(Need to find good parametrization)Finding a good “gauge”(that is “coordinate”) to work with is also an important question in geometry.

•Curves: 1-d geometric object.

Surfaces: 2-d geometric object.

Remark. In this course, we only focus on curves and surfaces in $\mathbb{R}^3$.However, as a training on preparing for later geometry course, I suggest you also try to think about the ambiant space is $\mathbb{S}^3$ or $\mathbb{H}^3$.

•**Intrinsic geometry**: Study the geometric object without considering the ambient space. This begins from the Gauss's elegant theorem and was developed by Riemann.

Example 0.0.3. *Consider the unit sphere* $\mathbb{S}^2$
Extrinsic geometry: view it as $x^2 + y^2 + z^2 = 1$ *in* $\mathbb{R}^3$*.*
Intrinsic geometry: (θ, φ) *or* (φ, θ) *are "essential" coordinates on* $\mathbb{S}^2$*.*

$$\mathrm{d}s^2 = \mathrm{d}\varphi^2 + (\sin\varphi)^2 \mathrm{d}\theta^2$$

(Caution: (θ, φ) *is outer normal, while* (φ, θ) *is inner normal.)*

• Useful / Common techniques:

1) Comparison: compare the studied geometric object with "model space". It's very important to study examples in geometry. As a suggestion you are expected to spend time to play with $\mathbb{S}^2$.For example: How is $\mathbb{S}^2$ curved? What's the shortest line in $\mathbb{S}^2$? How many symmetries are there on $\mathbb{S}^2$? Can you add "extra structure" on$\mathbb{S}^2$ to make it a complex object? Is this "extra structure" "rigid"?What/s the "moment map" on $\mathbb{S}^2$? Does there exist a "holomorphic" map from $\mathbb{S}^2$ to a torus, or a surface of arbitrary genus?

   If we consider an "Energy minimizing map" from $\mathbb{S}^2$ to $\mathbb{S}^2$, what can we say about such map?(It is holomorphic/antiholomorphic.)

   After you have learned Riemann Geometry, you'll see an energy minimizing map from $\mathbb{S}^2$ to a Riemannian manifold must be an angle-preserving map(conformal map).

   What kinds of 2-d geometric space could be $\mathbb{S}^2$?(this is a global geometry problem.)(*i.e.* what kinds of geometric conditions can characterize $\mathbb{S}^2$?)

2) To study higher dimensional objects,it's also important to understand lower dimensional objects, and it's also important to understand lower dimensional objects contained in the studied objects.

3) Study "functions" (more generally sections, including functions, vector fields, differential forms, etc.) on a given geometric object.

Example 0.0.4. *On a closed surface (* $\mathbb{S}^2$,$\mathbb{T}^2$,Σ_g*)(compact without boundary) there is no non-constant harmonic function.(i.e.* $\Delta u = 0$*)(Analysis will get involved.)*

We usually care about those functions related to geometry, such as distance functions, curvature-related functions, etc.

Example 0.0.5 (More trivial than the last one)**.** *Consider* $f''(x) = 0$*, what can you say of the solution of it when* x *lies on a line and when* x *lies on a circle?*

Chapter 1

Differential Geometry of Curves

1.1 Linear algebra convention and its geometric explanation

- We use "ROW VECTOR" in this course, *i.e.*

$$v \in \mathbb{R}^n, v = (v_1, v_2, \cdots, v_n)$$

- let $e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 1)$ be the standard basis, then

$$v = \sum_{i=1}^{n} v^i e_i = \begin{bmatrix} v^1 & v^2 & \cdots & v^n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

- $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ (non-degenerate) linear map

$$v \mapsto \varphi(v) = v \cdot A.$$

This corresponds to the right action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$.

$$\Rightarrow \varphi(e_i) = e_j \cdot A = \sum_{i=1}^{n} A_j{}^i e_i \text{ (taking the j-th row of } A\text{)}$$

$$A_j{}^i \begin{cases} \text{upper index: column index} \\ \text{lower index: row index} \end{cases}$$

$$\Rightarrow \varphi \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \varphi(e_1) \\ \varphi(e_2) \\ \vdots \\ \varphi(e_n) \end{bmatrix} = \begin{bmatrix} e_1 \cdot A \\ e_2 \cdot A \\ \vdots \\ e_n \cdot A \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Remark (Important!). In row vector convention, a non-degenerate linear map corresponds to the right action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$. But this induces left action of $GL(n, \mathbb{R})$ on the orthonormal basis (frame) $\{e_1, e_2, \dots, e_n\}$. This phenomenon provides an important example in differential geometry, which will be explained later in the theory of principle bundle.(*i.e.* let G be a lie group, $G \curvearrowleft M$ being a right action, where M is a differentiable manifold, then this right action induces a left action of G on the frame bundle of M.)

Let $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ be another basis of $\mathbb{R}^n$. Let f be the corresponding linear map, *i.e.*

$$f\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \vdots \\ \tilde{e}_n \end{bmatrix} = B \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$\Rightarrow \tilde{e}_k = \sum_{j=1}^n B_k{}^j e_j$$

We compare the matrix of φ in terms of $\{\tilde{e}_1 \cdots \tilde{e}_n\}$

$$\varphi\begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_n \end{bmatrix} = \varphi\left[B\begin{bmatrix} \tilde{e}_1 \\ \vdots \\ e_n \end{bmatrix}\right] = B \cdot \varphi\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \text{(linearity of } \varphi)$$

$$= BA\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = BAB^{-1}\begin{bmatrix} \tilde{e}_1 \\ \vdots \\ e_n \end{bmatrix}$$

Note in this case.

$$(\varphi \circ f)\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = BA\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

In terms of entries,

$$\varphi(\tilde{e}_k) = \varphi(\sum_{j=1}^n B_k{}^j e_j) = \sum_{j=1}^n B_k{}^j \varphi(e_j) \quad \text{(linearity)}$$

$$= \sum_{i,j=1}^n B_k{}^j A_j{}^i e_i = \sum_{i,j,p=1}^n B_k{}^j A_j{}^i (B^{-1})_i{}^p \tilde{e}_p$$

Remark. This computation tells that the row vector convention yields to the fact that $GL(n, \mathbb{R})$ acting on itself from the right when we consider the action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$. In modern Geometry, it's more common to use column vector as convention. This row vector convention was adopted by S.S. Chern and also Do Cormo's book.

1.2 Parametrized Curves

Definition 1.2.1. Let $I = (a, b)$, if $\alpha\colon I \to \mathbb{R}^3$ is a C^∞ map,

$$t \mapsto (x(t), y(t), z(t))$$

then $\alpha(t)$ is a parametrized differentiable curve in $\mathbb{R}^3$. The image of α is called the trace of the curve.

Remark.

1) a, b could be finite number or infinity.

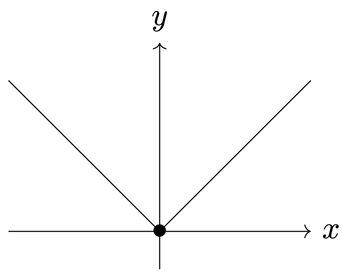
2) Same curve may have different parametrizations.

3) The parametrization automatically gives the direction of the motion on the curve.

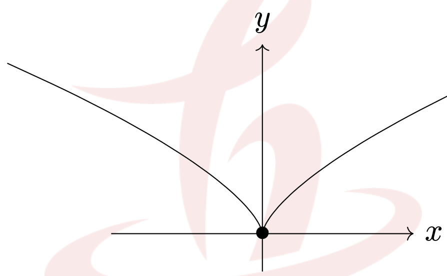
4) "Differentiable" just means $\alpha(t)$ is a C^∞ **map**, it does not say the (trace of) curve can not have singularities.

Example 1.2.2.

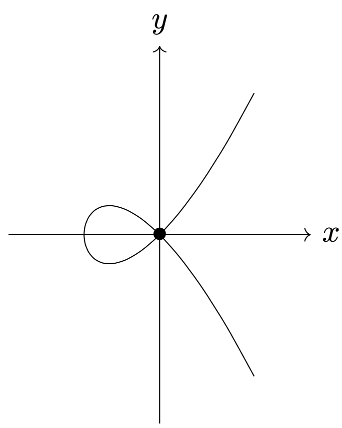
(1) $\alpha(t) = (t, |t|)$ *is not a differentiable curve.*



(2) $\alpha = (t^3, t^2)$ *is a differentiable curve. It can be also given by a equation* $y^3 = x^2$*, which is a cuspidal cubic curve.*



(3) $\alpha(t) = (t^2 - 1, t^3 - t)$*. This parametrization appers in the "blow-up" process of* $y^2 = x^3 + x^2$*. Here "blow-up" is introducing tangents to seperate points.*



Remark. (2) and (3) above may be the first examples you'll see in an algebraic geometry course.

Question: At the origin, what can you obsefve on (2) and (3)?
Answer: (2) $\alpha'(0) = 0$. (3) α is not one to one, but $\alpha'(0) \neq 0$.
Question: Define a differentiable curve in $\mathbb{R}^3$ and $\mathbb{S}^n$.

Remark. Among above differentiable parametrizations, (2) and (3) are differentiable curves. However, if we take $\beta(t) = (t, t^{\frac{2}{3}})$, this also parametrizes (2), but it's not a differentiable curve!

Definition 1.2.3. Let $\alpha(t)\colon I \to \mathbb{R}^3$ be a parametrized differentiable curve, then at $t_0 \in I$.

$$\alpha'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$$

is the velocity of $\alpha(t)$ at t_0.

(1) If $\alpha'(t_0) \neq 0$, we call $\alpha(t_0)$ a regular point.

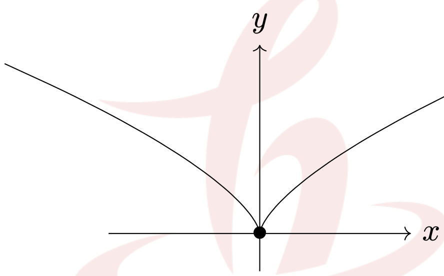
(2) If $\alpha'(t_0) = 0$, we call $\alpha(t_0)$ a singular point.

(3) If for all $t \in I$, $\alpha'(t) \neq 0$, we call $\alpha(t)$ a regular curve.

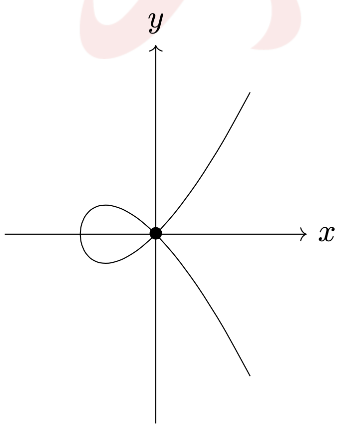
Question: What can you say about C^∞ parametrization for a regular curve?

Regular curve $\Longleftrightarrow$ at each point, there is a unique tangent line.

Example 1.2.4. $\alpha(t) = (t^3, t^2)$ *is not a regular curve. (Since* $\alpha'(0) = 0$*)*

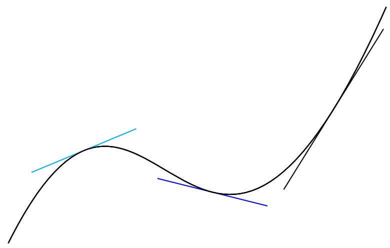


Example 1.2.5. $\alpha(t) = (t^2 - 1, t^3, t)$ *is a regular curve.*



Definition 1.2.6. Let $\alpha(t)$ be a regular curve, then the tangent line at t_0 is

$$l(t) = \alpha(t_0) + \alpha'(t_0)(t - t_0))$$



Definition 1.2.7. Let $\alpha(t)$ be a regular curve, the arc-length of $\alpha(t)$ is

$$s(t) = \int_{t_0}^{t} |\alpha'(t)|\,\mathrm{d}t.$$

Then $s'(t) = |\alpha'(t)|$

Question What's $|\alpha'(t)|$?

$\alpha(t)\colon I \to \mathbb{R}^3$ is a curve in $\mathbb{R}^3$. Here on $\mathbb{R}^3$, as the Euclidean space, we always assume the standard Euclidean inner product on it, *i.e.* $\forall u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{i,j=1}^{3} \delta_{ij}u_iv_j$$

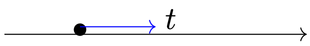
Let $\alpha(t) = (x(t), y(t), z(t)), \alpha'(t) = (x'(t), y'(t), z'(t))$, then $|\alpha'(t)| = \sqrt{\langle\alpha'(t), \alpha'(t)\rangle}$

Exercise. *Review vector Calculations, such as dot product, cross product and their properties, especially geometric meanie of these calculation, such as length, area, volume, angle, orientation, etc.*

Question: Can you define the arclength of a regular curve in $\mathbb{R}^n$? How about on $\mathbb{S}^n$?

• Arclength parameter(an intrinsic parametrization of a curve)

Example 1.2.8. *On a straight line, x=t describes the distance of the point away from the origin.*



On a general curve, we also want "some" parameter, which describes the arclength of point away from the initial point. This can happen iff $|\alpha'(t)| = 1$,*i.e.* a point on the curve moves in a unit speed.

$$\Rightarrow s(t) = \int_0^t \mathrm{d}t = t.$$

Question: For a given regular curve $\alpha(t)\colon I \to \mathbb{R}^3$, how to find such parameter?

Answer: $s(t) = \int_{t_0}^{t} |\alpha'(t)|\,\mathrm{d}t$ is a function in t, and $s'(t) = |\alpha'(t)| \neq 0$(because the curve is regular). By the implicit function theorem, there is a function

$$t = t(s), t'(s) = \frac{1}{|\alpha'(t)|}.$$

This implies that

$$\alpha(t) = \alpha(t(s)) = (x(t(s)), y(t(s)), z(t(s)))$$
$$|\alpha'(s)| = |\alpha'(t)t'(s)| = |\alpha'(t)|\,|t'(s)| = 1$$

Convention In this course, we only consider differentiable regular curves, which are parametrized by the arclength (for convenience).

Remark. In this course, we only consider the curve without self-intersecting points, i.e curves "embedded into" $\mathbb{R}^3$. Here "embedded" means $d\alpha$ is a linear isomorphism and α is homeomorphic to its image.

1.3 Local theory of a regular space curve

Goal. Describe a space curve by using geometric quantities.

Question. How to make a space curve?

Starting with a straight line, we can bend it and twist it in a given way to produce a space curve.

- Bending the line $\longrightarrow$ "curvature".
- Twisting $\longrightarrow$ "torsion".
- Their relations are contained in Frenet formula.
- Conversely, fundamental theorem of the local theory of curves tells, once we're given two function, $\kappa(s), \tau(s)$, we can describe a unique curve in $\mathbb{R}^3$ up to a rigid motion, *s.t.* $\kappa(s)$ is its curvature and $\tau(s)$ is its torsion.

Recall: In Calculus, if $y = f(x)$ represents a curve, then $f''(x)$ tells the convexity of the curve. It measures how fast the velocity changes. It's also related to how straight line is bent.

Let $\alpha\colon I \to \mathbb{R}^2$ be a regular plane curve, parametrized by arc length, *i.e.* $|\alpha'(s)| = 1$. Then $\langle \alpha'(s), \alpha''(s)\rangle = 0$, and hence $\alpha''(s) \perp \alpha'(s)$. For a plane curve, we take normal of the curve to be counterclockwise 90° rotation of the tangent vector.

Let N be the unit normal vector along $\alpha(s)$, we have

$$\langle \alpha''(s), N(s)\rangle = \pm|\alpha''(s)|.$$

Definition 1.3.1. The curvature of a plane curve $\alpha(s)$ is defined as

$$\kappa(s) = \langle \alpha''(s), N(s)\rangle\,.$$

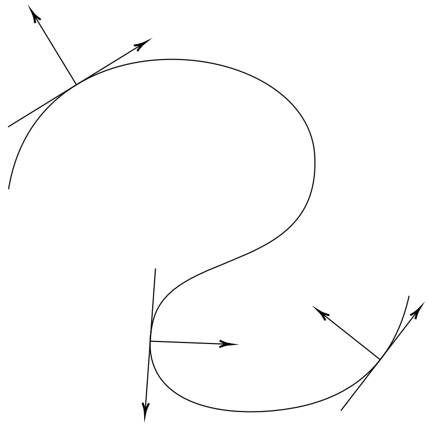


Figure 1.3.1: Example of a plane curve and its tangent and normal

Definition 1.3.2. Further we denote T be the unit tangent vector, then the Frenet equation of $\alpha(s)$ is

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T \end{cases}.$$

Note that

$$\langle T', N\rangle = \kappa \implies \langle T, N'\rangle = -\kappa.$$

- $\kappa > 0 \implies$ the point on the curve moves counterclockwise direction or say "to its left".
- $\kappa < 0 \implies$ the point on the curve moves clockwise direction or say "to its right".

Question. For the curve in fig. 1.3.1, can you tell where $\kappa > 0$ and where $\kappa < 0$ without doing calculation?

Remark. The sign of the curvature of the plane curve is caused by the direction convention of the unit normal vector. This could change according to the orientation of a curve.

Next, we take a look at the geometric meaning of $|\alpha''(s)|$ at some point $\alpha(s_0)$. By definition:

$$|\alpha''(s_0)| = \lim_{h\to 0}\left|\frac{\alpha'(s_0+h) - \alpha'(s_0)}{h}\right|.$$

We have

$$\begin{aligned} |\alpha'(s_0+h) - \alpha'(s_0)| &= \left(|\alpha'(s_0+h)|^2 + |\alpha'(s_0)|^2 - 2\left\langle \alpha'(s_0+h), \alpha'(s_0)\right\rangle\right)^{\frac{1}{2}} \\ &= (2 - 2\cos\theta_h)^{\frac{1}{2}} \\ &= (2 - 2(1 - \frac{1}{2}\theta_h^2) + \tilde{o}(\theta_h)^4)^{\frac{1}{2}} \\ &= (\theta_h^2 + \tilde{o}(\theta_h)^4)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$|\alpha''(s_0)| = \lim_{h\to 0}\left|\frac{\alpha'(s_0+h) - \alpha'(s_0)}{h}\right| = \lim_{h\to 0}\left|\frac{\theta_h}{h}\right| = |\theta'(s_0)|.$$

i.e. $|\alpha''(s)|$ measures the changing rate of angle of tangents.

In fact, for a plane curve, let θ be the angle between $\alpha'(s_0)$ and $\alpha'(s)$, then

$$\langle \alpha'(s), \alpha'(s_0) \rangle = \cos\theta_s \implies \langle \alpha''(s), \alpha'(s_0) \rangle = -\sin\theta_s \cdot \theta'_s.$$

Notice that $\cos\theta_s$ is the projection of $\alpha'(s_0)$ on the tangent $\alpha'(s)$, hence

$$\sin\theta_s = \langle \alpha'(s_0), N(s) \rangle .$$

On the other hand, $\alpha''(s) = T'(s) = \kappa(s)N(s) = \pm|\alpha''(s)|N(s)$, this gives $\theta'_s = \pm|\alpha''(s)|$.

Let $\alpha\colon I \to \mathbb{R}^3$ be a space curve, parametrized by arclength, *i.e.* $|\alpha'(s) = 1|$, we also have $\langle \alpha'(s), \alpha''(s) \rangle = 0$, *i.e.* $\alpha''(s) \perp \alpha'(s)$.

Unlike case of dim 2, it does not make sense to prescribe a normal vector of a curve. However, from above discussion, we see the geometric meaning of $|\alpha''(s)|$ is the measure of how fast the point on the curve leaving the straight line. We came into following definition:

Definition 1.3.3. The *curvature* of a regular space curve $\alpha(s)$ parametrized by arclength is defined as

$$\kappa(s) = |\alpha''(s)|.$$

And the unit normal vector at $\alpha(s)$ is

$$N = \frac{\alpha''(s)}{|\alpha''(s)|}, \quad \text{for } |\alpha''(s)| > 0.$$

Remark.

- If $|\alpha''(s)| \equiv 0$ then α must be a straight line, and all unit normal vectors line on a unit circle $\perp \alpha$.
- If $|\alpha''(s_0)| = 0$, we call s_0 a singular point of order 1. (Note. s_0 *s.t.* $|\alpha(s_0)| = 0$ is called a singular point of order 0) At such points, there is no well-defined normal vector.

Definition 1.3.4. The plane determined by T, N is called the *osculating plane* of $\alpha(s)$. The unit normal vector of the osculating plane

$$B = T \times N$$

is called *binormal vector*.

Remark.

- $\{T, N, B\}$ satisfies the right-hand rule.
- $|B'|$ measures how fast the point leaves the osculating plane.

If we denote θ_h be the angle between $B(s_0+h)$ and $B(s_0)$, similar to former calculation, we have

$$\begin{aligned} |B'(s_0)| &= \lim_{h\to 0} \left| \frac{B(s_0+h) - B(s_0)}{h} \right| \\ &= \lim_{h\to 0} \left| \frac{\sqrt{2 - 2\cos\theta_h}}{h} \right| \\ &= |\theta'_{s_0}|. \end{aligned}$$

As we saw, at each (non-singular) point on a space curve $\alpha(s)$, we can associate an oriented orthonormal frame $\{T, N, B\}$.

Question. How these three vector fields are related to the geometry of the curve?

By definition, we write 0-order info of $\{T, N, B\}$, *i.e.*

$$\begin{cases} T = \alpha' & \\ N = \frac{\alpha''}{|\alpha''|} & \implies T' = \alpha'' = \kappa N \\ B = T \times N & \implies B' = T' \times N + T \times N' = T \times N' \end{cases}.$$

Since $|B| = 1$, we have $B' \perp B$. Also, from above we see $B' \perp T$, hence $B' \parallel N$.

Definition 1.3.5. We define

$$B' = \tau N.$$

Here $\tau(s)$ is called the torsion of $\alpha(s)$.

Next, we also want to find N'. $N = B \times T$ gives

$$\begin{aligned} N' &= B' \times T + B \times T' \\ &= \tau N \times T + B \times (\kappa N) \\ &= -\kappa T - \tau B. \end{aligned}$$

Theorem 1.3.6. *The fundamental equations of a space curve (also called the Frenet equations) is given by*

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{1.3.1}$$

Remark.

(1) $\tau \equiv 0$ but $\kappa \neq 0$ at all points $\implies$ $\alpha(s)$ is a plane curve. (Note this may be not true if we don't assume $\kappa \neq 0$, see ex.10 in Do Carmo's book)
$\kappa \equiv 0 \implies \alpha(s)$ is a straight line.

Note $\tau \equiv 0 \implies B' = 0 \implies B$ is constant vector. Further,

$$(\alpha \cdot B)' = \alpha' \cdot B = T \cdot B = 0,$$

gives $\alpha \cdot B = \text{constant} \implies (\alpha(s) - \alpha(s_0)) \cdot B = 0$. Since $\kappa \neq 0$ at all points, the osculating plane is always well-defined, hence B is always defined, we proved α lie in some plane perpendicular to vector B.

(2) In different textbooks, you may see the definition of τ having a different sign from here.

Friendly warning: When studying the Geometry, (even later in Riemannian Geometry), it happens a lot that different authors use different sign convention for the same definition. It's very important that you should fix your own notation, ant keep it consistently!

Definition 1.3.7. $\{T, N, B\}$ is called Frenet trihedron of $\alpha(s)$, it gives a moving orthonormal basis of $\mathbb{R}^3$ along the curve $\alpha(s)$.

The Frenet equation describes how such moving orthonormal basis moves along $\alpha(s)$.

Remark. Note that in above discussion, we have chosen a special parameter, the arc-length parameter, of $\alpha(s)$. In the study of Geometry, finding a good parametrization can simplify a lot of work and itself an important problem. In more general framework, it's called a "Gauge related" problem.

We have seen that given a regular curve $\alpha(s)$, parametrized by arclength, the Frenet equation is eq. (1.3.1), for some functions $\kappa(s) > 0$ be its curvature and $\tau(s)$ be its torsion. Conversely, we ask

Question. If we're given smooth functions $\kappa(s), \tau(s)$ with $\kappa(s) > 0$,

(1) (Existence) Does there exist a regular curve $\alpha(s)$ *s.t.* $\kappa(s)$ is its curvature and $\tau(s)$ is its torsion?

(2) (Uniqueness) If such curve exists, is it unique in some sense?

The answer is **YES!**

Theorem 1.3.8 (Fundamental theorem of the local theory of curves)**.** *Let $\kappa(s), \tau(s) : I \to \mathbb{R}$ be smooth functions, assume $\kappa(s) > 0$, then*

- *(Existence) There is a regular curve realize κ and τ as its curvature and torsion.*
- *(Uniqueness) If α, β are two such curves parametrized by arclength parameter, then they only differ by a rigid motion of $\mathbb{R}^3$. i.e. $\exists T \in O(3), c \in \mathbb{R}^3$ s.t. $\beta = T\alpha + c$.*

Remark.

(1) Existence follows from a Cauchy problem (initial value problem) of ODE system.

(2) The curve is unique up to a rotation of $\mathbb{R}^3$ and a translation.

Proof. If we denote

$$X(s) = \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad P(s) = \begin{bmatrix} & \kappa & \\ -\kappa & & -\tau \\ & \tau & \end{bmatrix}.$$

Where T, N, B are viewed as **row vectors**. Then the Frenet equation writes as

$$X' = PX.$$

This is a first order linear ODE (of nine unknown functions), then by the existence and uniqueness theorem of ODEs, given any initial value

$$X(0) = \begin{bmatrix} T_0 \\ N_0 \\ B_0 \end{bmatrix}$$

witch form an orthonormal basis, the system has a unique solution that extend to whole domain I.

We need to check the solution actually is orthonormal frame for each s, notice the orthonormal relation can be written as

$$XX^t = I_3.$$

Where I_3 is the identity matrix of dimension three. Take differential on left-hand side of the equation we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(XX^t) &= X'X^t + X(X^t)' = X'X^t + X(X')^t \\ &= PXX^t + X(PX)^t \\ &= PXX^t + XX^tP^t \\ &= PXX^t - XX^tP.\end{aligned}$$

If we denote $Y = XX^t$, we see

$$Y' = PY - YP$$

In coordinates, if we set $T = v_1, N = v_2, B = v_3$, and

$$y_{ij}(s) = \left\langle v_i(s), v_j(s)\right\rangle, \quad P = (a_{ij})_{i,j=1}^3.$$

We have $y_{ij} = y_{ji}, a_{ij} = -a_{ji}$, and then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}y_{ij} &= \left\langle v_i', v_j\right\rangle + \left\langle v_i, v_j'\right\rangle \\ &= \left\langle a_{ik}v_k, v_j\right\rangle + \left\langle v_i, a_{jk}v_k\right\rangle \\ &= a_{ik}y_{kj} + a_{jk}y_{ki} \\ &= a_{ik}y_{kj} - y_{ik}a_{kj}.\end{aligned}$$

This gives again a first order ODE system, with initial value $Y(0) = I_3$, or say $y_{ij}(0) = \delta_{ij}$, but there is an obvious solution $Y \equiv I_3$, so by uniqueness theorem, this is it. This proves $XX^t = I_3$ for any s.

Until now, we have proved the existence of orthonormal moving frame $\{T, N, B\}$. Notice T is just $\alpha'(s)$, so given initial point $\alpha(0) = \alpha_0$, integrate w.r.t s gives a valid solution

$$\alpha(s) = \alpha_0 + \int_0^s T(\xi)\,\mathrm{d}\xi.$$

For the uniqueness, we need to look carefully into the initial condition we chose for the solution α, that is, choice of initial frame $\{T_0, N_0, B_0\}$ and initial point α_0. Given two valid solution curve α, β, with initial condition (X_a, α_0) and (X_b, β_0), we choose an orthogonal matrix $T = X_bX_a^{-1}$, a constant $c = \beta_0 - T\alpha_0$, then we see the curve

$$\tilde{\beta} = T(\alpha - \alpha_0) + \beta_0 = T\alpha + c$$

satisfy exactly the same initial condition as β, so they must agree. This proves the uniqueness up to rigid motion we stated before. □

Remark.

(1) **Exercise:** Check that for solution given above, κ and τ are its curvature and torsion.

(2) The condition $\kappa > 0$ is needed for uniqueness. Can you construct a counterexample when there is one point *s.t.* $\kappa = 0$?

(3) Uniqueness can be proved without knowledge of ODEs, see theorem after this remark.

(4) We can view the ODE problem at a somehow higher point. Consider the space of all orthonormal frames, it is actually a smooth manifold. It's a little non-trivial, but we can identify the space with the three dimensional rotation group $SO(3)$, smoothly embedded into $\mathbb{R}^9$, the space of three dimensional matrices. The equation, can be interpreted to a (time dependent) vector field on $SO(3)$. One can verify the vector field is tangent to the manifold, so it is actually a vector field not only in $\mathbb{R}^9$ but in $SO(3)$ itself. Similar to the existence and uniqueness theorem of ODEs on Euclid spaces, we have a version of such theorem for smooth manifolds. It states that for a smooth manifold M, a (maybe time dependent) smooth vector field X on M, then with any given initial point p, there exists an integral curve on M starting at p, tangent to X everywhere, and it is unique. Using the theorem, we can say that with given initial $\{T_0, N_0, B_0\}$, there exists a unique solution $\{T(s), N(s), B(s)\}$. Note that the solution is automatically lie in $SO(n)$, no need to verify it is orthonormal.

Proof. (Uniqueness, without ODE knowledge).

Let $\alpha(s), \beta(s)$ share the same $\kappa(s), \tau(s)$ as their curvature and torsion, by similarly a rotation and a translation, we assume they have same initial condition, *i.e.* $\alpha(0) = \beta(0)$, and the Frenet frame agree at $s = 0$. We claim now we must have $\alpha(s) = \beta(s), \forall\, s$.

Notice $\alpha(s) = \alpha(0) + \int_{s_0}^{s} \alpha'(s)\,\mathrm{d}s$, so it suffices to show $T_\alpha = T_\beta$, equivalently

$$|T_\alpha - T_\beta|^2 = 0.$$

Take differential we get:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|T_\alpha - T_\beta|^2 &= \left\langle T_\alpha - T_\beta, T_\alpha' - T_\beta' \right\rangle \\ &= -\left\langle T_\alpha, \kappa N_\beta \right\rangle - \left\langle T_\beta, \kappa N_\alpha \right\rangle.\end{aligned}$$

Similar calculation gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|N_\alpha - N_\beta|^2 &= \left\langle N_\alpha, \kappa T_\beta + \tau B_\beta \right\rangle + \left\langle N_\beta, \kappa T_\alpha + \tau B_\alpha \right\rangle \\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|B_\alpha - B_\beta|^2 &= -\left\langle B_\alpha, \tau N_\beta \right\rangle - \left\langle B_\beta, \tau N_\alpha \right\rangle.\end{aligned}$$

Sum the three equation we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(|T_\alpha - T_\beta|^2 + |N_\alpha - N_\beta|^2 + |B_\alpha - B_\beta|^2) = 0.$$

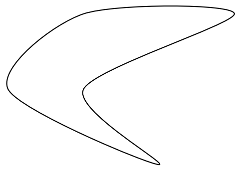
But the sum of square equals 0 at $s = s_0$, so it is identically 0, in particular, $T_\alpha = T_\beta$ for all s. □

1.4 Global theory of plane curves

The global theory is related to "topology" of the geometric objects. For 1-dimensional geometry, *i.e.* curves, it's always oriented. And the simplest distinction in topology is "open" and "closed".

Definition 1.4.1 (Closed curves).

- We say $\alpha\colon I=[a,b]\to\mathbb{R}^3$ (or $\mathbb{R}^2$) is a closed regular curve, if $\alpha(a)=\alpha(b)$ and $\alpha^{(k)}(a)=\alpha^{(k)}(b)$ (in another word, $\alpha\colon\mathbb{S}^1\to\mathbb{R}^3$ is a differentiable curve).
- Furthermore, if α has no self-intersection point other than $\alpha(a)=\alpha(b)$, then we call $\alpha(s)$ to be a simple closed curve.



1.4.1 Isoperimetric inequality

This is one of the oldest and most famous problem in geometry. It's still attracting mathematicians to investigate such problem in various geometric formulations nowadays.

Question. Given a closed plane curve C with. Let D be the region bounded by C. When does the region have the maximal area, if C is among all the curves with fixed length?

Answer. C must be a circle when the maximal area is achieved.

Remark. Even though we'll only handle smooth, simple closed curves in the following discussion, in general we don't have to assume the curve to be simple: ○○ has less area than ○.(caution: their boundaries are intended to have the same length). Thick about how ○○ comes from ○.

Proofs of the Isoperimetric inequality

Proof. 1 (Hurwitz's proof) This relies on the "Wirtinger's inequality".

Let $\alpha(t)$ be a closed, simple smooth curve, where t can be any parameter. The length of it is

$$L=\int_a^b\sqrt{x'(t)^2+y'(t)^2}\mathrm{d}t.$$

Observe that we need to find the lower bound of L^2. Generally, for an integral $L=\int\sqrt{f}\mathrm{d}t$, Hölder inequality (or Cauchy-Schwarz) naturally gives estimate of L. Hence, it's natural to find a "good parameter" to clear. Although the arclength s is a good candidate, it turns out in this case that another good parameter is

$$\theta=\frac{2\pi}{L}s.$$

$s\in[0,L]\Rightarrow\theta\in[0,2\pi]$. (This parameter θ comes from the "Wirtinger's inequality', but of course a rescaling of wirtinger's inequality allows us to use s as usual).

Let's take $\theta=\frac{2\pi}{L}s$, then

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2=\left(\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2=\left(\frac{L}{2\pi}\right)^2.$$

$$\Rightarrow \frac{L^2}{2\pi} = \frac{L^2}{4\pi^2}\cdot 2\pi = \int_0^{2\pi} \left(x'(\theta)^2 + y'(\theta)^2\right) \mathrm{d}\theta.$$

Therefore,

$$\begin{aligned} 2\left(\frac{L^2}{4\pi} - A\right) &= \int_0^{2\pi} \left(x'(\theta)^2 + y'(\theta)^2\right) \mathrm{d}\theta - 2\int_0^{2\pi} x(\theta)y'(\theta)\mathrm{d}\theta \\ &= \int_0^{2\pi} x'(\theta)^2 - x(\theta)^2 + \underbrace{(y'(\theta) - x(\theta))^2}_{\geq 0} \mathrm{d}\theta \\ &\geq \int_0^{2\pi} x'(\theta)^2 - x(\theta)^2 \mathrm{d}\theta. \qquad (\star) \end{aligned}$$

Now, the proof reduces to the following lemma.

Lemma 1.4.2 (Wirtinger's inequality). *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a 2π-periodic smooth function and $\int_0^{2\pi} f(\theta)\mathrm{d}\theta = 0$, Then*

$$\int_0^{2\pi} f(\theta)^2 \mathrm{d}\theta \leq \int_0^{2\pi} f'(\theta)^2 \mathrm{d}\theta,$$

and equality holds iff $f(\theta) = a\cos(\theta) + b\sin(\theta)$.

(Proof of the lemma is left as a homework problem.)

To apply this to $(\star)$, we need to assume $\int_0^{2\pi} x(\theta)\mathrm{d}\theta = 0$. However, we know the center of mass of the curve is $\left(\frac{\int x(\theta)\mathrm{d}\theta}{L}, \frac{\int y(\theta)\mathrm{d}\theta}{L}\right)$, and by choosing the origin of $\mathbb{R}^2$ as the center of mass, we can guarantee $\int_0^{2\pi} x(\theta)\mathrm{d}\theta = 0$, this yields $\star \geq 0$, *i.e.* $L^2 \geq 4\pi A$. Moreover, equality implies

$$x(\theta) = a\cos(\theta) + b\sin(\theta) \text{ and } y'(\theta) = x(\theta) \Rightarrow$$

$$y(\theta) = a\sin(\theta) - b\cos(\theta) + c.$$

So $(x(\theta), y(\theta))$ is a circle. □

Proof. 2 (By Schmidt) See Do Carmo's book (page 33-35). It will be lectured by TA in a recitation. □

Remark.

(1) There are many other proofs of Isoperimetric inequality. In the homework 3, we will use a modern tool-curve shortening flow to give a proof.

(2)

$$L^2 \geq 4\pi A \Rightarrow \frac{L^2}{4\pi} \geq A \Rightarrow \frac{L^2}{4\pi^2 r^2} \geq \frac{A}{\pi r^2} (\text{take } r = 1)$$

$$\Rightarrow \frac{L}{2\pi} \geq \left(\frac{A}{\pi}\right)^{\frac{1}{2}} \; i.e. \; \frac{\text{length of curve}}{\text{length of the unit circle}} \geq \left(\frac{\text{Area bounded by the curve}}{\text{Area of the unit disk}}\right)^{\frac{1}{2}}.$$

• **Generalization**: Let E be a compact domain in $\mathbb{R}^n$ with smooth boundary ∂E, then

$$\frac{\text{Area}(\partial E)}{\text{Area of the unit sphere in } \mathbb{R}^n} \geq \left(\frac{\text{Volume of } E}{\text{Volume of the unit ball}}\right)^{\frac{n-1}{n}}$$

For simplicity, we write

$$\frac{|\partial E|}{\partial B^n} \geq \left(\frac{|E|}{|B^n|}\right)^{\frac{n-1}{n}}.$$

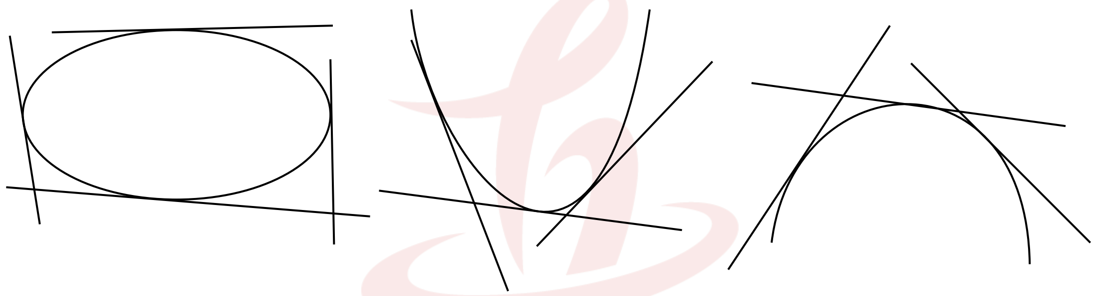
Question. Can you propose some generalizations of isoperimetric inequality? Isoperimetric inequality is one of the motivation to develop geometric measure theory!

1.4.2 Four-vertex theorem

Theorem 1.4.3. *A simple closed convex plane curve has at least four vertices.*

Remark. The four-vertex theorem holds also for simple closed non-convex curves. The proof is harder, however.

Definition 1.4.4 (Convex curves)**.** $\alpha(s)$ is a convex curve, if at each point $\alpha(s_0)$, the whole curve lies on the same side of the tangent line.



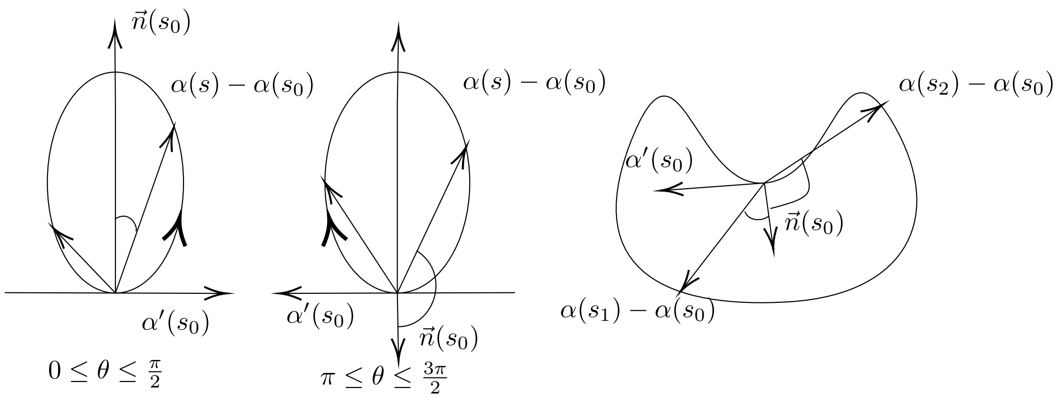
The convex curve has the following useful characterization.

Proposition 1.4.5. *$\alpha(s)$ is a convex curve $\Leftrightarrow$ at each point $\alpha(s_0)$, only one of the following holds:*

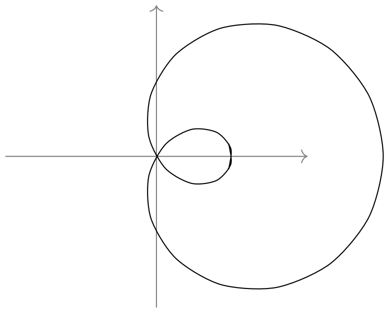
$$\textit{For all } s \in I, \textit{ either } (\alpha(s)-\alpha(s_0))\cdot\vec{n}(s_0) \geq 0 \textit{ or } (\alpha(s)-\alpha(s_0))\cdot\vec{n}(s_0) \leq 0.$$

Geometrically, this means at a convex point, the angle between vector $\alpha(s)-\alpha(s_0)$ and $\vec{n}(s_0)$ should be either $[0, \frac{\pi}{2}]$ or $[\pi, \frac{3\pi}{2}]$.

Example 1.4.6.



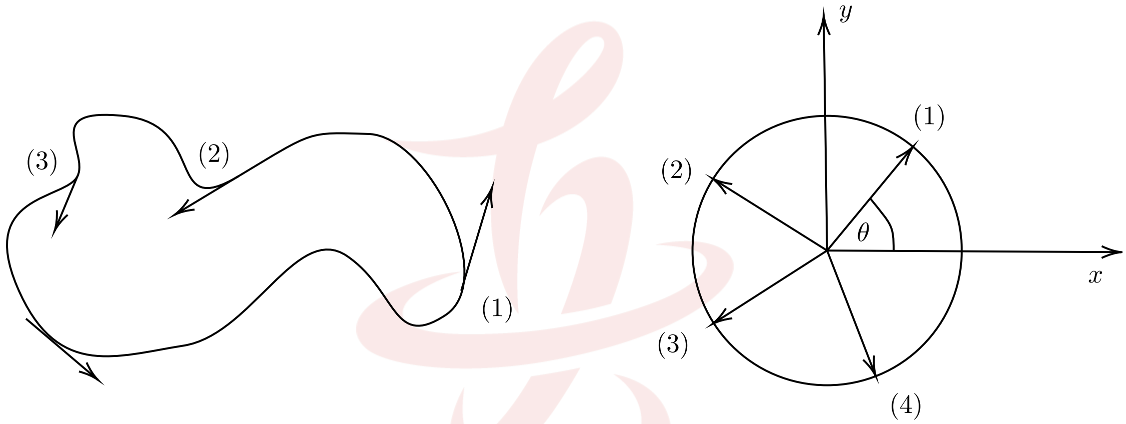
Example 1.4.7. $\alpha(t) = ((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t)\,,\ t \in \mathbb{R}$.



Proposition 1.4.8. *$\alpha(s)$ is a simple closed curve, then*

$$\alpha(s) \text{ is convex} \Leftrightarrow k(s) \geq 0\ \forall s \text{ or } k(s) \leq 0\ \forall s.$$

Previously, we have seen that $k(s)$ measures the rate of change of the angle between tangent vectors. Let's see another similar application. Let $\alpha(s)$ be parametrized by arclength, then $t(s) \equiv \alpha'(s)$ is a unit tangent vector, *i.e.* $|t(s)| = 1$.



Let θ be the angle between $t(s)$ and the x-axis, *i.e.* $t(s) = (\cos\theta, \sin\theta)$

$$\left.\begin{array}{l} t'(s) = (-\sin\theta, \cos\theta)\dfrac{\mathrm{d}\theta}{\mathrm{d}s} = \dfrac{\mathrm{d}\theta}{\mathrm{d}s}\cdot\vec{n} \\ \text{on the other hand, } t'(s) = k\cdot\vec{n} \end{array}\right\} \Rightarrow \boxed{k(s) = \frac{\mathrm{d}\theta}{\mathrm{d}s}}.$$

As an application, if $k(s) \not\equiv 0$, then $s = s(\theta)$ is defined so that $\frac{\mathrm{d}s}{\mathrm{d}\theta} = \frac{1}{k}$, *i.e.* θ can be used as a parameter of $\alpha(s)$. Such θ is called the angle parameter.

! In the study of geometry, the sign of the curvature is a very important thing to keep in mind.

Definition 1.4.9. Let $\alpha\colon I \to \mathbb{R}^3$ be a regular curve. The point at which $k'(t_0) = 0$ is called a vertex of α.(critical point of the curvature $k(t)$)

Proof of proposition 1.4.8.

Claim 1: $\alpha(s)$ is Globally convex $\Rightarrow$ either $k \geq 0$ or $k \leq 0$ locally for all s.

W.L.O.G., we assume c is oriented counterclockwise, $\vec{n}$ is the inner unit normal vector. We'll show

$$\text{convex} \Rightarrow k \geq 0 \text{ for all s.}$$

Assuming not, then $\exists s_0$ such that $k(s_0) < 0$. By the continuity of k(s), we can assume $k(s_0) = \min k(s)$. Establish a coordinate system at $\alpha(s_0)$ such that $\alpha(s_0)$ is the origin, $\alpha'(s_0)$ corresponds to the x-axis and $\vec{n}(s_0)$ to the y-axis. We'll show that $\exists s_1, s_2$ such that

$$\langle \alpha(s_1), \vec{n}(s_0) \rangle < 0, \quad \langle \alpha(s_2), \vec{n}(s_0) \rangle > 0.$$

Consider the function

$$f(s) = \langle \alpha''(s), \vec{n}(s_0) \rangle,$$

then $f(s_0) = k(s_0) \leq 0$, which implies that there exists a neighborhood $I_\epsilon = (s_0 - \epsilon, s_0 + \epsilon)$, so that $f(s) < 0$ for $s \in I_\epsilon$.

$$\Rightarrow \langle \alpha''(s), \vec{n}(s_0) \rangle < 0 \Rightarrow \langle \alpha'(s), \vec{n}(s_0) \rangle < \langle \alpha(s_0), \vec{n}(s_0) \rangle = 0$$

$$\Rightarrow \langle \alpha(s), \vec{n}(s_0) \rangle < \langle \alpha(s_0), \vec{n}(s_0) \rangle = 0.$$

So there exists an s_1 such that $\langle \alpha(s_1), \vec{n}(s_0) \rangle < 0$.

If for all $s \in I$, $\langle \alpha(s), \vec{n}(s_0) \rangle \leq 0$, then this means that all points lie on the opposite side of $\vec{n}$. Hence, $\vec{n}$ is "outer" normal, a contradiction to our assumption on the direction of $\vec{n}$. So $\exists s_2$ such that $\alpha(s_2) > 0$. But this contradicts the assumption on convexity.

Claim 2: $k \geq 0 \Rightarrow$ global convexity.

If not, there exists an s_0 such that the curve has points on both sides of the tangent line of $\alpha(s_0)$. Consider the height function

$$h(s) = \langle \alpha(s) - \alpha(s_0), \vec{n}(s_0) \rangle,$$

then $\exists\ s_1, s_2$ such that $h(s_1) < 0 = h(s_0) < h(s_2)$. We can assume that $s_0 < s_1 < s_2 < s_0 + l$, where l is the length of $\alpha(s)$. By the continuity of h, we can further assume

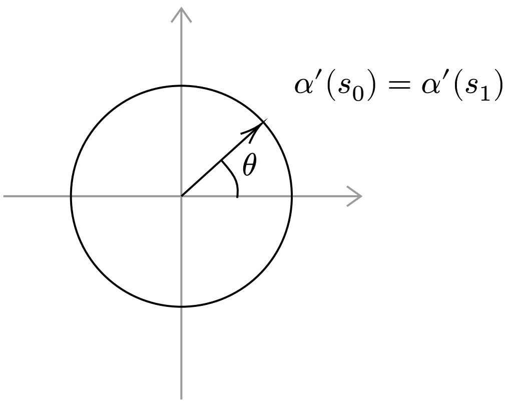
$$\begin{aligned} & h(s_1) = \min h(s), \ \ h(s_2) = \max h(s). \\ \Rightarrow & h'(s_1) = \langle \alpha'(s_1), \vec{n}(s_0) \rangle = 0 \Rightarrow \alpha'(s_1) \perp \vec{n}(s_0) \\ & h'(s_2) = \langle \alpha'(s_2), \vec{n}(s_0) \rangle = 0 \Rightarrow \alpha'(s_2) \perp \vec{n}(s_0), \end{aligned}$$

and we also know $\alpha'(s_0) \perp \vec{n}(s_0)$.

$\therefore$ at least two of $\alpha'(s_0), \alpha'(s_1), \alpha'(s_2)$ have the same direction. Let's assume.

$$\alpha'(s_0) = \alpha'(s_1) \quad (\because \text{they have the same length})$$

Note that they are unit vectors, *i.e.* images are on $\mathbb{S}^1$.



As we have discussed in the lecture, if θ is the angle between $t(s)$ and a fixed direction

$$k=\frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

Hence, we can consider a function:

$$\theta(s)=\int_{s_0}^{s} k(s)\mathrm{d}s.$$

By assumption, $\theta(s)$ is non-decreasing $(k\geq 0)$ and $\theta\left(s_0\right)=0$

$$\theta\left(s_0+L\right)=\int_{s_0}^{s_0+L} k(s)\mathrm{d}s=2\pi.$$

(Fact: for a simple closed curve in $\mathbb{R}^2$, $\int_c k\mathrm{d}s=2\pi$)

Since for each unit vector $\alpha'(s)$, we have a unique $\theta(s)\in[0,2\pi)$

$$\alpha'\left(s_0\right)=\alpha'\left(s_1\right)\Rightarrow\theta\left(s_0\right)=\theta\left(s_1\right)\in[0,2\pi)\quad\left(\because\theta:\left[s_0,s_0+L\right)\xrightarrow{\nearrow}[0.2\pi)\right).$$

But

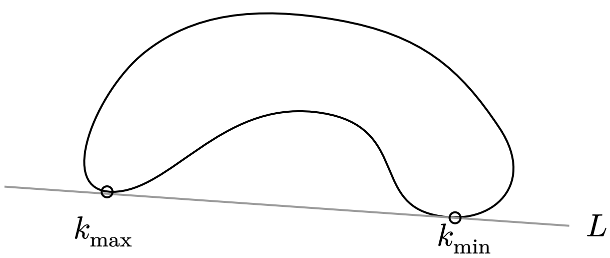
$$\begin{aligned}
& s_0<s_1\Rightarrow\theta(s_0)=\text{constant on }\left[s_0,s_1\right]\\
& \Rightarrow\alpha'(s)=\text{constant on}\left[s_0,s_1\right],\ \alpha'(s)=\alpha'(s_0)\\
\Rightarrow\quad & \int_{s_0}^{s_1}\left\langle\alpha'(s),\vec{n}_0\right\rangle ds=\left\langle\alpha(s_1)-\alpha(s_0),\vec{n}_0\right\rangle=h(s_1).
\end{aligned}$$

This contradicts $h(s_1)<0$ □

Further explanation of the four-vertex theorem(sketch)

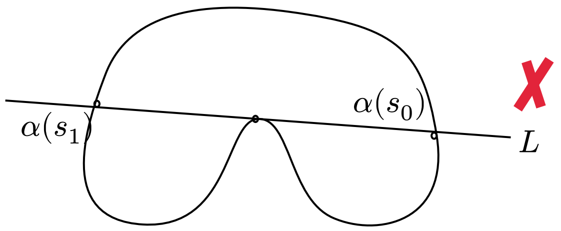
Let L be the line passing through $\alpha(s_0)$ and $\alpha(s_1)$, and $\alpha(s_0)$ is a k_{min} point and $\alpha(s_1)$ is a k_{max} point.

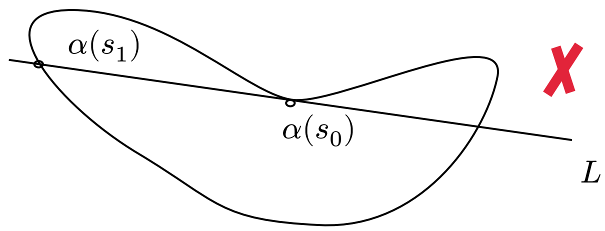
Claim 1: It can't happen that all points lie on the same side of L, *i.e.* the configuration in this illustration is impossible.



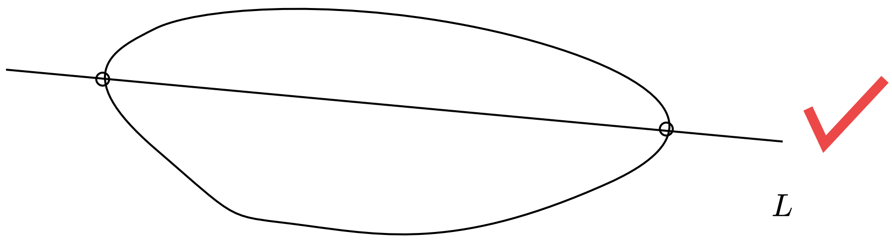
(Reason: simple closed + convexity $\Rightarrow\theta(s)$ is increasing on $[0,2\pi]$, the same argument as the previous page.) This implies that there must be points on both sides of L.

Claim : No other points of C meet L.





(Reason: same as Claim 1.) Hence, Claim 1 + Claim 2⇒

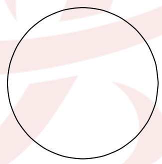


Claim 3: ∃ a third and a fourth vertex. (See the proof)

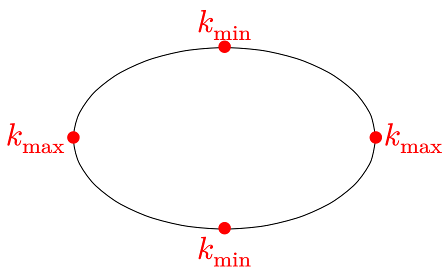
Exercise. *Let $\alpha(s) = (x(s), y(s))$ be a simple closed curve in $\mathbb{R}^2$. Let $\tilde{\alpha}(s)$ be the image of $\alpha(s)$ under stereographic projection. Show that if $\alpha(s_0)$ is a vertex of $\alpha(s)$, then $\tilde{\alpha}(s_0)$ has vanishing torsion.*

Example 1.4.10.

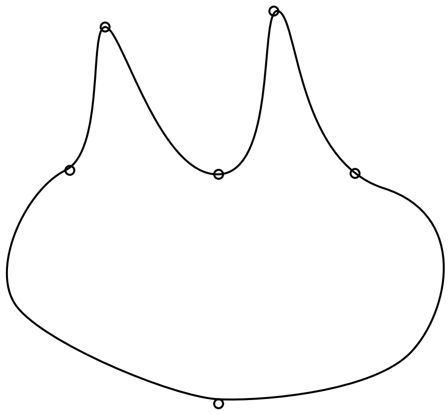
- *The circle with radius r and curvature $k = \frac{1}{r}$ has infinitely many vertices.*



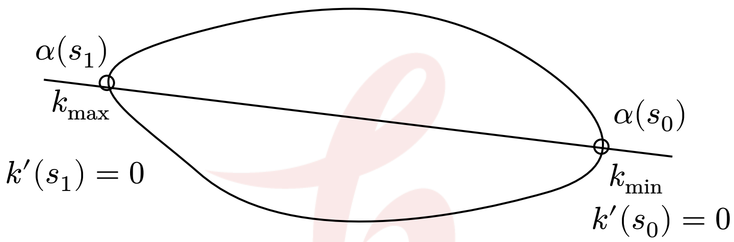
- *An ellipse has four vertices.*



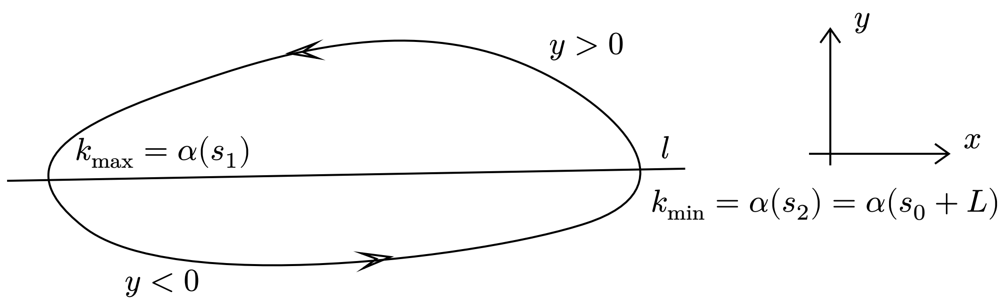
- *Although this is nonconvex, it has more than four vertices.*



Proof of theorem 1.4.3. Let $\alpha(s)$ be parametrized by arclength. First, since the curvature $k(s)$ is a continuous function on I, it must have maximum and minimum, at which $k'(s) = 0$, *i.e.* $\alpha(s)$ has at least 2 vertices. Let $\alpha(s_0)$ be a $k_{\min}$ point, $\alpha(s_1)$ be a $k_{\max}$ point. Consider a line l connecting $\alpha(s_0)$ and $\alpha(s_1)$. For convenience, we assume line l coincides with x-axis.



The first observation is: on l, there is no other point of $\alpha(s)$. Hence, $\alpha(s)$ is divided into two pieces. If not, assume $\alpha(s_2)$ is a third point, and W.L.O.G. assume $k'(s_2) = 0$. The tangent line at $\alpha(s_2)$ must be the same as l. Since the curve α is convex, the whole curve $\alpha(s)$ must lie on the same side of l. This forces the tangent lines of $\alpha(s_0)$ and $\alpha(s_1)$ can only be l. But $\alpha(s_0)$ is a $k_{\min}$ point and $\alpha(s_1)$ is a $k_{\max}$ point, which implies $k(s_0) = k(s_1) = 0$. Therefore, $k \equiv 0$ on α, a contradiction.



Next, we look for the third vertex. If $\alpha(s)$ has only two vertices at $\alpha(s_0)$ and $\alpha(s_1)$, then from s_0 to s_1, $k'(s) > 0$ and from s_1 to $s_0 + L$, $k'(s) < 0$

$$\Rightarrow y \cdot k'(s) \geq 0, \ \forall s$$

$$\Rightarrow 0 < \int_\alpha y \cdot k'(s) \mathrm{d}s = -\int_\alpha y'(s) k \mathrm{d}s.$$

Note that if

$$\alpha(s) = (x(s), y(s)), t(s) = \alpha'(s) \left(x'(s), y'(s)\right).$$

$$t'(s) = (x''(s), y''(s)) = k\vec{n} = k(-y', x') \Rightarrow -k'(s)k = x''$$
$$\therefore \int_\alpha y'k\mathrm{d}s = \int_\alpha x''\mathrm{d}s = 0.$$

A contradiction!. Hence, there must be a third vertex, say $\alpha(s_2)$, at which $k'(s_2) = 0$.

Note that $k'(s)$ changes its sign at vertices, so the number of vertices must be even. Then there are at least 4 vertices. □

Remark. The proof of the four-vertex theorem for non-convex case can be found in Montiel-Ros's book Chapter 9.6. (4-vertex theorem for space curves: simple closed curve on a convex surface has at least four points with vanishing torsion.)

1.4.3 Minkowski problem (1-dim)

Theorem 1.4.11 (1-dim Minkowski problem)**.** *Given a periodic, strictly positive function k, satisfying the following condition:*

$$\int_0^{2\pi} \frac{\cos\theta}{k}\mathrm{d}\theta = \int_0^{2\pi} \frac{\sin\theta}{k}\mathrm{d}\theta = 0.$$

There is an oval in $\mathbb{R}^2$ (i.e. simple closed strictly convex curve) such that the curvature function is k.

Definition 1.4.12. A plane curve $\alpha(t)$ is strictly convex iff $\alpha(t)$ is convex and at each point, the tangent line meets with $\alpha(t)$ at only one point.

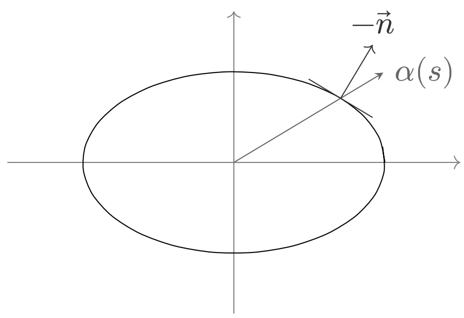
Proposition 1.4.13. *A simple closed curve is strictly convex iff with inward unit normal vector field, the curvature function $k > 0$.*

Minkowski problem: Given a strictly positive, periodic function k, does there exist a simple closed convex curve α with k as the curvature function?

Remark. This is a prescribed curvature problem. There are a lot of similar questions in geometry. Such problems are usually related to solving certain P.D.E.

Let's derive a differential equation for the above problem. Let α be a strictly convex curve, then $k > 0$, and we can use the angle parameter θ, *i.e.*

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = k, \ \frac{\mathrm{d}s}{\mathrm{d}\theta} = \frac{1}{k}$$



Consider a function

$$h(s) = -\langle \alpha(s), \vec{n}(s) \rangle \text{ support function.}$$

(Recall $\int_C h(s)\mathrm{d}s = 2 \cdot \text{Area}$).
Clearly $h(0) = h(2\pi)$ if we use $h(\theta) = -\langle \alpha\left(s(\theta)\right), \vec{n}\left(s(\theta)\right) \rangle$.

$$\begin{aligned} h'(\theta) &= -\langle \alpha'(s)\frac{\mathrm{d}s}{\mathrm{d}\theta}, \vec{n}(s) \rangle - \langle \alpha(\theta), \frac{\mathrm{d}\vec{n}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\theta} \rangle \\ &= -\langle \alpha(\theta), -k \cdot \vec{t} \cdot \frac{1}{k} \rangle = \langle \alpha(\theta), \vec{t}\left(s(\theta)\right) \rangle. \end{aligned}$$

Hence, $h'(0) = h'(2\pi)$.
We also conclude that

$$\begin{aligned} \alpha(\theta) &= \langle \alpha(\theta), \vec{t} \rangle \cdot \vec{t} + \langle \alpha(\theta), \vec{n} \rangle \cdot \vec{n} \\ &= h'(\theta)\vec{t} - h(\theta)\vec{n}, \end{aligned}$$

i.e. the curve is determined by the support function h.
$\left(\alpha(s) = h'(s)\dfrac{\mathrm{d}s}{\mathrm{d}\theta}\vec{t} - h(s)\vec{n} = h'(s)\dfrac{1}{k}\vec{t} - h(s)\vec{n} \right)$

$$\begin{aligned} h''(\theta) &= \langle \alpha'(s)\frac{\mathrm{d}s}{\mathrm{d}\theta}, \vec{t} \rangle + \langle \alpha(\theta), \frac{\mathrm{d}\vec{t}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\theta} \rangle \\ &= \frac{1}{k} + \langle \alpha(\theta), k\vec{n} \cdot \frac{1}{k} \rangle = \frac{1}{k} - h. \end{aligned}$$

i.e. $\boxed{h''(\theta) + h(\theta) = \dfrac{1}{k}}$.

Hence, if $\alpha(s) = \alpha(\theta)$ is a strictly convex closed curve, the support function $h(\theta) = -\langle \alpha, \vec{n} \rangle$ satisfies a second linear o.d.e.

$$h''(\theta) + h = \frac{1}{k}.$$

Observation: If $\exists\ h$ that satisfies the equation above. Note $\theta \in [0, 2\pi)$, then

$$\begin{aligned} \int_0^{2\pi} \cos\theta \frac{1}{k}\mathrm{d}\theta &= \int_0^{2\pi} \cos\theta \cdot \left(h''\theta + h\right)\mathrm{d}\theta \\ &= \int_0^{2\pi} \sin\theta \cdot h'(\theta) + \int_0^{2\pi} \cos\theta \cdot h \\ &= -\int_0^{2\pi} \cos\theta \cdot h(\theta) + \int_0^{2\pi} \cos\theta \cdot h = 0. \end{aligned}$$

Similarly,

$$\int_0^{2\pi} \sin\theta \cdot \frac{1}{k}\mathrm{d}\theta = 0,$$

i.e. if k is the curvature of a strictly convex curve, it must satisfy

$$\int_0^{2\pi} \cos\theta \cdot \frac{1}{k}\mathrm{d}\theta = \int_0^{2\pi} \sin\theta \cdot \frac{1}{k}\mathrm{d}\theta = 0.$$

In fact, from o.d.e, we can directly construct the solution like this:

$$h(\theta) = -\cos\theta \int_0^\theta \frac{\sin\psi}{k}\mathrm{d}\psi + \sin\theta \int_0^\theta \frac{\cos\psi}{k}\mathrm{d}\psi.$$

Recall that $\vec{t}(s) = (\cos\theta, \sin\theta)$, $\vec{n}(s) = (-\sin\theta, \cos\theta)$. Since

$$\begin{aligned}
\alpha(\theta) &= h'(\theta)\vec{t} - h(\theta)\vec{n} \\
&= \left(\cos\theta\sin\theta \int_0^\theta \frac{\sin\psi}{k} + \cos^2\theta \int_0^\theta \frac{\cos\psi}{k}, \sin^2\theta \int_0^\theta \frac{\sin\psi}{k} + \sin\theta\cos\theta \int_0^\theta \frac{\cos\psi}{k}\right) \\
&\quad - \left(\sin\theta\cos\theta \int_0^\theta \frac{\sin\psi}{k} - \sin^2\theta \int_0^\theta \frac{\cos\psi}{k}, -\cos^2\theta \int_0^\theta \frac{\sin\psi}{k} + \sin\theta\cos\theta \int_0^\theta \frac{\cos\psi}{k}\right) \\
&= \left(\int_0^\theta \frac{\cos\psi}{k}, \int_0^\theta \frac{\sin\psi}{k}\right)
\end{aligned}$$

$\alpha(s)$ is closed $\Leftrightarrow$ $h(0) = h(2\pi)$, $h'(0) = h'(2\pi) \Leftrightarrow \int_0^{2\pi} \cos\theta \cdot \frac{1}{k}\mathrm{d}\theta = \int_0^{2\pi} \sin\theta \cdot \frac{1}{k}\mathrm{d}\theta = 0.$

Remark. In general (higher dimensional case) solving a similar P.D.E. equation is highly nontrivial!

Cheng-Yau 1976 CPAM: given a $C^{k,\alpha}$ positive function K on the sphere $\mathbb{S}^n$ $(k \geq 3)$, which satisfies

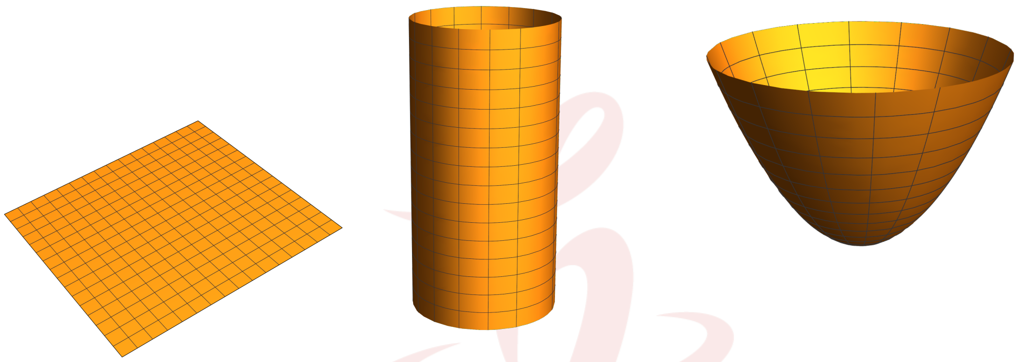
$$\int_{\mathbb{S}^n} \frac{x_i}{K}\mathrm{d}V_{\mathbb{S}^n} = 0,$$

where $x_1, x_2, \cdots, x_{n+1}$ are coordinate functions on $\mathbb{S}^n$. Then there is a strictly convex closed hypersurface $M^n \hookrightarrow \mathbb{R}^{n+1}$ such that the Gaussian curvature is K.

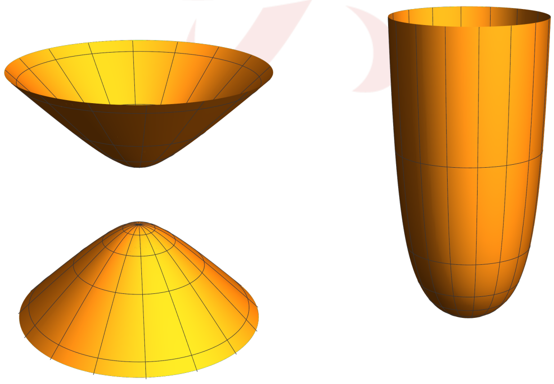
Chapter 2

Differential Geometry of Surfaces

A First Look



(a) Plane (b) Cylinder (c) Paraboloid

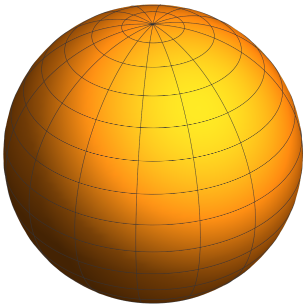


(d) Hyperboloid (e) Cigar soliton

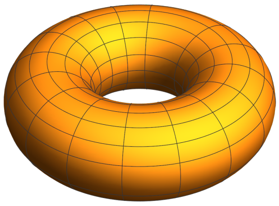
Surface collection 1

Example 2.0.1.

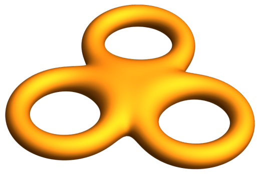
- *Collection 1 are complete non-compact surfaces.*
- *Collection 2 are compact surfaces without boundary (closed).*



(a) $\mathbb{S}^2$

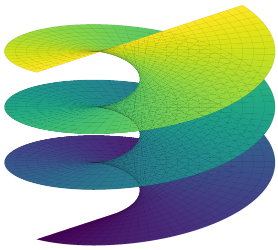


(b) $\mathbb{T}^2$

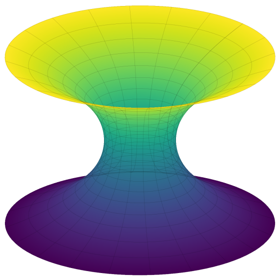


(c) Σ_g for $g = 3$

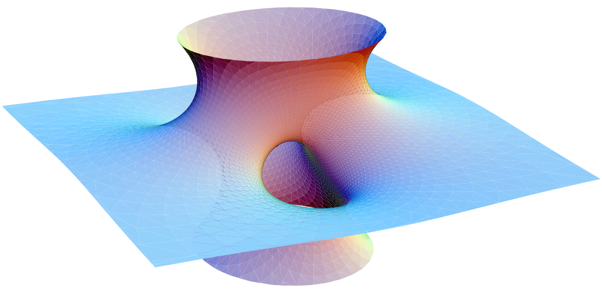
Surface collection 2



(a) Helicoid



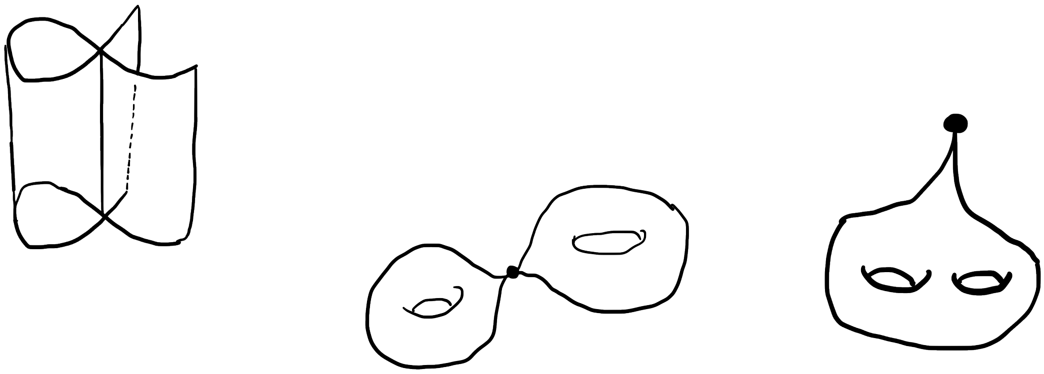
(b) Catenoid



(c) Costa minimal surface

(d) Soap bubble

Surface collection 3



(a) Self-intersected (b) Nodal surfaces (c) Cusp

"Surface" collection 4

- *Collection 3 are so called minimal surfaces, a very important class of surfaces. The term "minimal" intuits smallest area in certain sense.*
- *There are surfaces will NOT be investigated in this course, the ones with self-intersection, node points or cusps, and non-orientable surfaces.*

2.1 Definition of Regular Surface

Definition 2.1.1 (Regular surfaces in $\mathbb{R}^3$).

A subset $S \subset \mathbb{R}^3$ is called a regular surface, if $\forall\, p \in S$, $\exists\, V \subset \mathbb{R}^3$ neighborhood of p, an open set $U \subset \mathbb{R}^2$ and a trivialization map

$$F\colon U \to V \cap S.$$

s.t. F is smooth, homeomorphism onto its image, and regular.

Remark.

(1) F is homeomorphism means both F and F^{-1} are continuous map.

(2) F is "regular" means $\forall\, p \in U$, $\mathrm{d}F_p$ is an injection as linear map $\mathbb{R}^2 \to \mathbb{R}^3$.

Let's see what the term "regular" means:
Assume F is written as $F(u,v) = (x(u,v), y(u,v), z(u,v))$, at $p \in U$,

$$\mathrm{d}F_p : T_pU \to T_{F(p)}S$$

is a linear map. On $\mathbb{R}^2$, coordinate vector fields $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ form a basis. On $\mathbb{R}^3$ we also have standard basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$. Then

$$\mathrm{d}F_p \begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}.$$

Hence

$$
\begin{aligned}
\mathrm{d}F_p \text{ is injective } &\iff \ker \mathrm{d}F_p = 0 \\
&\iff \frac{\partial(x,y,z)}{\partial(u,v)} \text{ has rank 2} \\
&\iff \frac{\partial F}{\partial u} \ \&\ \frac{\partial F}{\partial v} \text{ are linearly independent} \\
&\iff \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq 0. \\
&\quad \textit{(Geometrically this defines the normal vector field} \\
&\quad \textit{of the tangent plane)} \\
&\iff \text{One of the following minors is non-zero:} \\
&\qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right|, \quad \left|\frac{\partial(x,z)}{\partial(u,v)}\right|, \quad \left|\frac{\partial(y,z)}{\partial(u,v)}\right|
\end{aligned}
$$

(Geometrically, this means (u,v) can be viewed as coordinate at $p \in S$ via F. In fact, since "$\mathrm{d}F_p$ is injective" is an open condition, (u,v) serves as a local coordinate chart in a neighborhood of p)

We also call F to be a local parametrization of S. Note that such F is usually not globally defined.

From the definition, we see a regular surface in $\mathbb{R}^3$ is characterized by at each point, we can find a "smooth" slice chart in a neighbourhood of the point. The term "slice chart" means coordinate chart with local part of the surface containing in the chart as a slice.

Question. Consider two points p, q on the surface, live close to each other. It might happen that their corresponding coordinate chart overlap. Then in the intersection of two charts, there are two different parametrizations. What relation between these two parametrizations should be?

Set-up: $F_1 : U_1 \to V_1 \cap S,\ (u,v) \mapsto F_1(u,v), \quad F_2 : U_2 \to V_2 \cap S,\ (\alpha,\beta) \mapsto F_2(\alpha,\beta)$

Let $W = V_1 \cap V_2 \cap S$, since F_i is homeomorphism, $F_1^{-1}(W) \subset U,\ F_2^{-1}(W) \subset U_2$.

Claim: (Very important).

$G = F_2^{-1} \circ F_1 : F_1^{-1}(W) \to F_2^{-1}(W)$ is a diffeomorphism, *i.e.* both G and G^{-1} are smooth functions.

The importance of this claim leads us to give an intrinsic definition of a regular surface S. *i.e.* a regular surface is obtained by padding up open sets in $\mathbb{R}^2$, in a smooth way. Later in differential geometry course, we'll define a smooth manifold by such intrinsic definition. The diffeomorphism G above is called the transition map. Different property of G determines different structure. If G is only a homeomorphism, then S is a topological surface. If G is a bi-holomorphism, then S is a complex surface.

The proof of the claim needs the inverse function theorem.

Theorem 2.1.2 (Inverse function thm). *$U \subset \mathbb{R}^n$ open. $F : U \to \mathbb{R}^n$ is a C^1 map, $p \in U$. If $\mathrm{d}F_p : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there is a neighbourhood of p and a neighbourhood of $F(p)$. s.t. $F : V \to W$ is invertible. Moreover F^{-1} is also C^1. If condition is substituted to F smooth, then F^{-1} has same smoothness.*

Remark. From linear algebra, a linear operator on finite dimensional vector space is injective iff it's surjective. Hence, it's sufficient to check $\det\left(\mathrm{d}F_p\right) \neq 0$, *i.e.* $\mathrm{d}F_p$ is non-singular.

!! A priori, we don't know if F_2^{-1} is smooth, since we have not defined what "smooth map" on a surface mean.

Proof of claim. Since F_1 and F_2 are homeomorphism, G, G^{-1} are continuous. S is a regular surface, so at $p \in U_1$, $(\mathrm{d}F_1)_p \colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective. W.L.O.G. we can assume

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0 \text{ at } p.$$

Consider a map $h\colon F_1^{-1}(W) \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3, (u,v,t) \mapsto (x(u,v), y(u,v), z(u,v)+t)$. Then h has Jacobian

$$\det\begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix} \neq 0 \text{ at } p.$$

By inverse function theorem, $\exists$ a neighbourhood $D \subset \mathbb{R}^3$ of (p,t) *s.t.* h is invertible on D, and h^{-1} smooth. Now since $F_1^{-1} \circ F_2 = h^{-1} \circ F_2\big|_{t=0}$, and RHS is smooth, we conclude that $F_1^{-1} \circ F_2$ is smooth. Similarly, G^{-1} is smooth. □

Now we give an intrinsic definition (No need to assume $S \subset \mathbb{R}^3$).

Definition 2.1.3. Topological space S (second countable, Hausdorff) is called a regular surface if S has a covering $\{V_\alpha, f_\alpha\}$ *s.t.*

(1) $f_\alpha \colon V_\alpha \to f_\alpha(V_\alpha) \overset{\text{open}}{\subset} \mathbb{R}^2$ is a homeomorphism.

(2) If $V_\alpha \cap V_\beta \neq \emptyset$, then

$$f_\beta \circ f_\alpha^{-1} \colon f_\alpha(v_\alpha \cap V_\beta) \to f_\beta(V_\alpha \cap V_\beta)$$

is a diffeomorphism, called the transition map.

Remark. In higher dimension, this definition yields "smooth manifold".

2.2 Examples of Regular Surfaces

Example 2.2.1. $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ *is a regular surface with (trivial) global parametrization*

$$F(x,y) = (x,y,0).$$

Example 2.2.2. *Standard 2-sphere* $\mathbb{S}^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$. *This is a very important example, we'll give (local) parametrization for* $\mathbb{S}^2$ *in 3 ways.*

(a) Parametrization induced from $\mathbb{R}^3$.
If the point is on upper hemisphere, let $U = \{x^2+y^2<1\}$,

$$F_1 \colon U \to \mathbb{S}^2,\ (x,y) \mapsto (x,y,\sqrt{1-x^2-y^2}).$$

Check definition:

F_1 is smooth □.

$F_1\colon U \to F_1(U)$ is homeomorphism, since F_1^{-1} is projection onto xy-plane, is also continuous.

F_1 is regular:

$$\frac{\partial F_1}{\partial x} = (1, 0, -\frac{x}{\sqrt{1-x^2-y^2}}), \quad \frac{\partial F_1}{\partial y} = (0, 1, -\frac{y}{\sqrt{1-x^2-y^2}}).$$

Clearly they are linearly independent.

Similarly, if the point is on lower hemisphere, we have

$$F_2\colon U \to \mathbb{S}^2, \ (x, y) \mapsto (y, x, -\sqrt{1-x^2-y^2}).$$

However, $F_1(U) \cup F_2(U)$ can not fully cover $\mathbb{S}^2$, points on the equator are left. To cover them, we add 4 more charts:

$$\begin{aligned} F_3(y,z) &= (\sqrt{1-y^2-z^2}, y, z) && \text{(front hemisphere)} \\ F_4(y,z) &= (-\sqrt{1-y^2-z^2}, z, y) && \text{(back)} \\ F_5(z,x) &= (x, \sqrt{1-x^2-z^2}, z) && \text{(right)} \\ F_6(z,x) &= (z, -\sqrt{1-x^2-z^2}, x) && \text{(left)} \end{aligned}$$

We can check F_2-F_6 also satisfy the definition. Hence, we have given each point a smooth chart, and $\mathbb{S}^2$ is regular.

Exercise. *Check transition maps between F_1-F_6 are smooth.*

(b) Geographical parametrization.

Let $U = \{(\theta, \varphi) \in \mathbb{R}^2 : 0 < \theta < 2\pi, 0 < \varphi < \pi\}$,

$$\begin{aligned} &F_1\colon U \to \mathbb{S}^2, \ (\theta,\varphi) \mapsto (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) && (\text{missing half of } \{y=0\} \cap \mathbb{S}^2) \\ &F_2\colon U \to \mathbb{S}^2, \ (\theta,\varphi) \mapsto (\sin\theta\sin\varphi, \cos\varphi, \cos\theta\sin\varphi) && (\text{missing half of } \{x=0\} \cap \mathbb{S}^2) \\ &F_3\colon U \to \mathbb{S}^2, \ (\theta,\varphi) \mapsto (\cos\varphi, \cos\theta\sin\varphi, \sin\theta\sin\varphi) && (\text{missing half of } \{z=0\} \cap \mathbb{S}^2). \end{aligned}$$

Clearly $\mathbb{S}^2 \subset F_1(U) \cup F_2(U) \cup F_3(U)$, each F_i is smooth and regular. To see they are homeomorphism, we can compute e.g.

$$F_1^{-1}(x, y, z) = (\arccos\frac{x}{\sqrt{x^2+y^2}}, \arccos z).$$

(c) Stereographic parametrization.

Consider the ray connecting North Pole $(0, 0, 1)$ and point (x, y, z) on $\mathbb{S}^2$. Then there is a unique point (u, v) on xy-plane on the ray, the projection is given by

$$p_N\colon \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2, \quad (x,y,z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z}) =: (u, v)$$

which is rational map, with inverse

$$p_N^{-1}(u, v) = (\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, 1 - \frac{2}{1+u^2+v^2})$$

also rational and normal.

Similarly consider the projection from $S = (0, 0, -1)$, $p_S(x, y, z) = (\frac{y}{1+z}, \frac{x}{1+z})$, these two charts can cover $\mathbb{S}^2$.

Exercise. *Check the transition function*

$$p_S \circ p_N^{-1} = (\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}).$$

Remark. In the stereographic projection parametrization, if we identify $\mathbb{R}^2$ with $\mathbb{C}^1$, and introduce complex coordinate

$$w_1 = \frac{x}{1-z} + \frac{y}{1-z}i, \quad w_2 = \frac{x}{1+z} - \frac{y}{1+z}i.$$

Then we can check

$$w_1 \cdot w_2 = 1 \quad \text{and} \quad p_S \circ p_N^{-1}(w_1) = w_2.$$

In this way, we have given a "complex structure" on $\mathbb{S}^2$, the resulting surface is an 1 dim complex manifold, called $\mathbb{CP}^1$. You'll learn more abort $\mathbb{CP}^n$ in complex / algebraic geometry.

!! $\mathbb{S}^2$ is a very important example in geometry. One should keep this example in mind and try to understand and explore it in studding geometry. $\mathbb{S}^2$ is a (strictly) convex surface, has maximal number of $\mathbb{S}^1$ action (This tells the moment polytope is a line segment). It has 3-rotation axis (3 Killing vector fields). It looks the same everywhere (homogeneous). However, since every circle on $\mathbb{S}^2$ can shrink to a point, without obstruction, *i.e.* $\pi_1(\mathbb{S}^2) = 0$, the fundamental group does not play a role here.

Example 2.2.3. *Graph of a function, i.e.* $(x, y, z = f(x, y))$ *is a regular surface. More generally, if a surface* S *can be expressed (locally or globally) as graph of smooth functions, then* S *is a regular surface.*

Parametrization is given by

$$F\colon U \to S, \quad (x, y) \to (x, y, f(x, y)).$$

We have

(a) F is smooth □

(b) F is homeomorphism, $F^{-1}(x, y, z) = (x, y)$.

(c) F is regular $\det \mathrm{Jac}(\mathrm{d}F) = 1$.

Example 2.2.4. *More generally, the level set of a smooth function* $f(x, y, z)$ *at a* <u>*regular value*</u> *is a regular surface.*

Definition 2.2.5 (Critical point (value) & regular point (value))**.** Let $F\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ be smooth map.

- If $\mathrm{d}F_p : \mathbb{R}^m \to \mathbb{R}^n$ is surjective, *i.e.*rank $\mathrm{d}F_p = n$, then p is called a regular point. $c \in \mathbb{R}^n$ is called a regular value if $\forall\, p \in F^{-1}(c)$ is regular point. If $F^{-1}(c)$ is empty we also call c a regular point.
- If $\mathrm{d}F_P$ is not surjective, then p is called a critical point, and $F(p)$ is called a critical value.

Remark. If c is a regular value, then $F^{-1}(c)$ is a smooth embedded submanifold of $\mathbb{R}^n$, with codimension n.

From linear algebra, $\mathrm{d}F_p$ is surjective $\iff$ $\dim\ker \mathrm{d}F_p = m-n$. Hence, when we talk abort "regular" point / value, it only makes sense when $m \geq n$.

Now, let's consider the case we're interested in. Let

$$\begin{aligned} f\colon \mathbb{R}^3 &\longrightarrow \mathbb{R} \quad \text{is smooth function} \\ (x,y,z) &\longmapsto f(x,y,z). \end{aligned}$$

We compute $\mathrm{d}f_p$, $\quad p \in U \subset \mathbb{R}^3 = \operatorname{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$, $\quad \mathbb{R}^1 = \{\frac{\partial}{\partial w}\}$:

$$\left[\mathrm{d}f_p\left(\frac{\partial}{\partial x}\right)\right] h = \frac{\partial}{\partial x}(h\circ f)(p) = \frac{\partial h}{\partial w}\frac{\partial f}{\partial x}, \quad \forall\, h \text{ smooth function in } w.$$

Hence,

$$\mathrm{d}f_p\left(\frac{\partial}{\partial x}\right) = \frac{\partial f}{\partial w}\frac{\partial}{\partial w}.$$

i.e.

$$\mathrm{d}f_p \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \frac{\partial}{\partial w}.$$

Hence, p is critical value $\iff$ $\operatorname{rank} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}_p < 1 \iff \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$ at p.

This is exactly how we define the "critical point" of $f(x,y,z)$ formerly. p is a regular point $\iff$ at least one of partial derivatives is nonzero at p.

Proposition 2.2.6 (Level surface is regular)**.** *Let $f\colon U \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. c is regular value of f, then*

$$f^{-1}(c) = \{(x,y,z) \in U : f(x,y,z) = c\}$$

is a regular surface (in U).

For example, $f(x,y,z) = x^2+y^2+z^2$, then $\forall\, c > 0$, level surface $f^{-1}(c)$ is regular. This is exactly the sphere of radius $\sqrt{c}$.

Proof. $\forall\, p \in U$ *s.t.* $f(p) = c$, write $p = (x,y,z)$. Since c is regular value, at least one of $f_x(p), f_y(p), f(z)$ is non-vanishing. W.L.O.G. assume $f_z(p) \neq 0$. Then by implicit function theorem, $\exists$ a neighbourhood of (x,y) *s.t.* $z = h(x,y)$ and $f(x,y,h(x,y)) = c$ locally. Hence, near p we have a local parametrization given by graph $(x,y,h(x,y))$, then $f^{-1}(c)$ is regular by example 2.2.3. □

Theorem 2.2.7 (Implicit function)**.** *Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open sets, $F\colon U\times V \to \mathbb{R}^n$ be smooth (Only need C^1) map*

$$(x^1,\dots,x^m,y^1,\dots,y^n) \mapsto (f^1,\dots,f^n).$$

If $f(x_0,y_0) = 0$ and $\frac{\partial(f^1,\dots,f^n)}{\partial(y^1,\dots,y^n)}$ has full rank (i.e. rank n) at (x_0,y_0), then $\exists$ a neighbourhood $x_0 \in U_0 \subset U$, $y_0 \in V_0 \subset V$ and $h\colon U_0 \to V_0$ a smooth map, s.t. locally $(y^1,\dots,y^n) = h(x^1,\dots,x^m)$ and $F(x^1,\dots,x^m,g(x^1,\dots,x^m)) = 0$.

Remark. The implicit function theorem is one of the most frequently used theorems in geometry and PDE theory. As an exercise, please review the version you learned in previous class. In geometry, roughly speaking, the implicit function theorem can be understood as: the tangent plane determines the "small" neighbourhood of a point on a regular surface.

In example 2.2.3, we see the graph of a smooth function $f(x, y)$ is a regular surface. The converse is also true locally.

Proposition 2.2.8. *S is a regular surface in $\mathbb{R}^3$, $p \in S$, then there is a neighbourhood U of p s.t. U is a graph of a smooth function (Either $z = f_1(x, y)$ or $y = f_2(z, x)$ or $x = f_3(y, z)$).*

Proof. S is regular $\implies$ near p we have local parametrization

$$\begin{aligned} F\colon U \subset \mathbb{R}^2 &\longrightarrow S \subset \mathbb{R}^3 \\ (u, v) &\longmapsto (x(u, v), y(u, v), z(u, v)). \end{aligned}$$

Since $\mathrm{d}F_p$ is injective $\implies$ at least one of

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right|, \quad \left|\frac{\partial(y, z)}{\partial(u, v)}\right|, \text{ or } \left|\frac{\partial(z, x)}{\partial(u, v)}\right|$$

is non-vanishing. If we assume $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0$, then by the inverse (implicit) function theorem, u, v are functions of x, y *i.e.*

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases} \longleftrightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases} .$$

Hence, above local parametrization F can be rewritten as

$$\tilde{F}(x, y) = (x, y, z(u(x, y), v(x, y))) = (x, y, z(x, y)).$$

□

On a regular surface S, one prefers to find a "nice" coordinate (local parametrization) to carry out computation to analyze geometric properties. If there is a possible local parametrization, the following proposition says it suffices to check (1) and (3) in the definition, and only part of (2), then this parametrization is indeed a parametrization on S.

Proposition 2.2.9. *S is a regular surface, $F\colon U \to \mathbb{R}^3$ is a (local) parametrization, near some point p, if F is smooth and regular, also F is bijective, then F^{-1} is also continuous.*

Proof. Essentially by the implicit function theorem, refer to Do Carmo's book. □

Example 2.2.10. *We have checked $\mathbb{S}^2$ is a regular surface by using the local parameter induced from $\mathbb{R}^3$. Hence, once we write the spherical coordinate, then clearly it gives a local parametrization by above proposition.*

Example 2.2.11 (Torus). *Rotating $(y-\sqrt{2})^2+z^2=1$ around z-axis, then the resulting surface is a torus*

$$(\sqrt{x^2+y^2}-\sqrt{2})^2+z^2=1.$$

This is a level surface of $f(x,y,z)=(\sqrt{x^2+y^2}-\sqrt{2})^2+z^2$ and 1 is a regular value. Hence, it's a regular surface. It's convenient to use the following parametrization

$$\begin{cases} x=(\sqrt{2}+\cos\theta)\sin\varphi \\ y=(\sqrt{2}+\cos\theta)\cos\varphi \\ z=\sin\theta \end{cases}.$$

Exercise (Homework). *Show that the two-sheeted cone, with its vertex at origin*

$$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2-z^2=0\}$$

is not a regular surface.

Exercise. *$P=\{(x,y,z):x=y\}$, let $X\colon U\to\mathbb{R}^3$ given by*

$$X(u,v)=(u+v,u-v,uv)$$

where $U=\{(u,v)\in\mathbb{R}^2 : u>v\}$. Clearly $X(U)\subset P$. Is X a parametrization of P?

2.3 Differential Functions on a regular surface

Question (1)**.** How to define a function on $\mathbb{S}^2$?

In calculus, we have seen this is just $f(x,y,z)|_{\mathbb{S}^2}$, but what does this mean? If $p\in$ upper half semi-sphere $\Rightarrow f\left(x,y,\sqrt{1-x^2-y^2}\right)\Rightarrow f$ is just a function on

$$U=(x,y)\in\mathbb{R}^3\colon x^2+y^2<1.$$

More precisely, if we let $F\colon U\to\mathbb{S}^2$ be the local parametrization,

$$(x,y)\mapsto(x,y,\sqrt{1-x^2-y^2}),$$

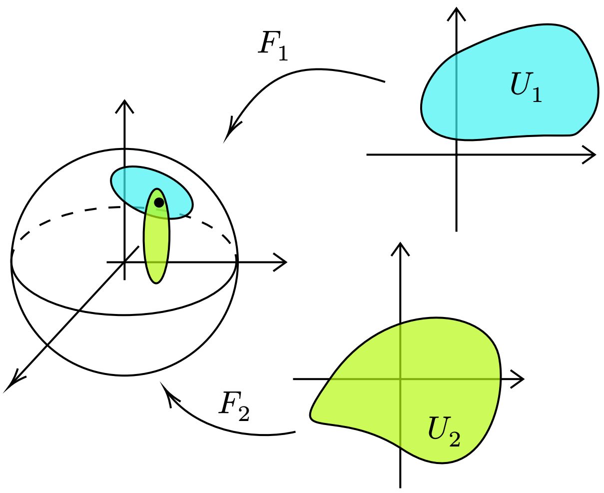
then $\tilde{f}(x,y)\colon U\to\mathbb{R}, \tilde{f}(x,y)=f\circ F$.

Similarly, if the point lies on other five charts(see example 2.2.2). There is also a function $\tilde{f}$ essentially defined on an open set of $\mathbb{R}^2$ to associate f.

Question (2)**.** What does it mean by a "smooth" (or differentiable) function on $\mathbb{S}^2$?

$$\begin{aligned} f \text{ is smooth near } p \Leftrightarrow & \tilde{f}\colon U\subset\mathbb{R}^2\to\mathbb{R}\text{is smooth,} \\ & (\because F\colon U\subset\mathbb{R}^2\to\mathbb{S}^2\text{is smooth}) \end{aligned}$$

Question (3)**.** p could lie on two charts, (*i.e.* we can associate two coordinate charts near p, will the "smoothness" of f be affected?)



$$\begin{aligned}\tilde{f}(x,y) &= f\circ F_1\\ &= f\left(x,y,\sqrt{1-x^2-y^2}\right)\end{aligned}$$

$$\begin{aligned}\tilde{f}(x,z) &= f\circ F_2\\ &= f\left(x,\sqrt{1-x^2-z^2},z\right)\end{aligned}$$

Smoothness is not affected by the coordinate change. Moreover, it's easy to apply the chain rule to find relation of differentials of f between different coordinate charts.

Definition 2.3.1 (Differential function)**.** S is a regular surface in $\mathbb{R}^3$. A function f is said to be differentiable at $p\in S$, if for a local parametrization near p

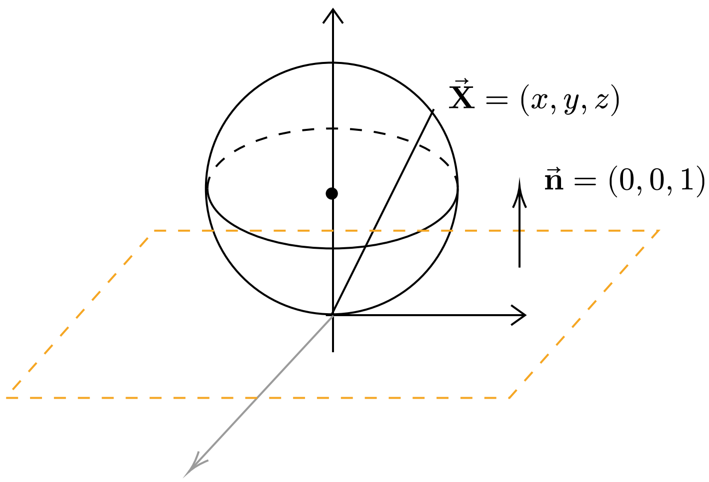
$$x\colon U\subset\mathbb{R}^2\to\mathbb{R}^3,$$

the function $f\circ x\colon\mathbb{R}^2\to\mathbb{R}$ is differentiable at $x^{-1}(p)$

Remark. $f\circ x$ is well-defined, since the differentiability is not affected by a change of parameter, e.g. if $y\colon V\subset\mathbb{R}^2\to$ $mathbbR^3$ is another parametrization near p, $f\circ y = f\circ x\circ(\underbrace{x^{-1}\circ y}_{\text{change of parameter}})$ is differentiable.

Example 2.3.2. *Consider* $x^2+y^2+(z-1)^2$*=1.*

(1) Consider $\vec{\mathbf{X}}=(x,y,z)$ *be the position vector field in* $\mathbb{R}^3$*. Let* $f(x,y,z)=\vec{\mathbf{X}}\cdot\vec{\mathbf{n}}$ *where* $\vec{\mathbf{n}}=(0,0,1)$ *is the unit normal of xy plane.*



$\Rightarrow f(x,y,z)=z$*. At point* $(x,y,z)\in\mathbb{S}^2$, $z>1$, $f(x,y,z)=1+\sqrt{1-x^2-y^2}$*, where* $x^2+y^2<1$*, it's a smooth function.*

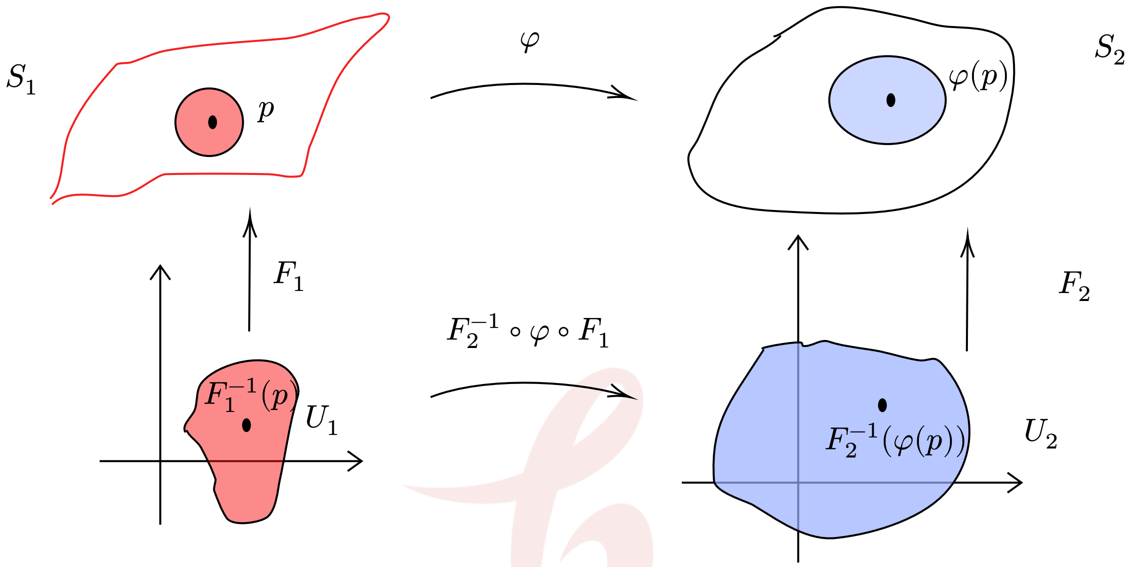
(2) Consider $d^2(x,y,z)=x^2+y^2+z^2=|\vec{\mathbf{X}}|^2$*, d is the distance function on* $\mathbb{R}^2$*.* $d^2|_{\mathbb{S}^2}=2z=2f$ *in (1), hence it's a smooth function.*

Exercise. *How does the function f look like in other coordinate charts?*

Definition 2.3.3 (Differentiable mapping between two surfaces)**.** S_1, S_2 are regular surfaces, $\varphi\colon S_1 \to S_2$ is a mapping that maps $p \in S_1$ to $\varphi(p) \in S_2$. We say φ is differentiable at $p \in S$, if for some parametrization

$$F\colon U_1 \to S_1(\ni p), \quad F_2\colon U_2 \to S_2(\ni \varphi(p))$$

$$F_2^{-1} \circ \varphi \circ F_1\colon U_1 \to U_2 \text{ is differentiable at } F_1^{-1}(p).$$



Locally, if (u, v) serves as coordinate of p, $F_2^{-1} \circ \varphi \circ F_1 = (\varphi_1, \varphi_2)$, then φ is differentiable iff φ^1 and φ^2 are differentiable Functions.

Remark. In the future, we won't explicitly write F_1 and F_2, but simply write $\varphi\colon S_1 \to S_2$ locally as $\varphi(u, v) = \big(\varphi^1(u, v), \varphi^2(u, v)\big)$.

Definition 2.3.4 (Diffeomorphism)**.** S_1 and S_2 are regular surfaces. S_1 is diffeomorphic to S_2 if $\exists f\colon S_1 \to S_2$ such that both f and f^{-1} are differentiable.

Remark. Diffeomorphism is the equivalent relation we often use in differential geometry. Later, we'll introduce a "stronger" equivalent relation after defining the 1st fundamental form.

Question. Can you give an intuitive example of "Diffeomorphism" between two surfaces?

Example 2.3.5. *Identity map/ $\mathbb{R}^2$ with translation action, rotation action/ $\mathbb{S}^2(1)$ by changing radius.*

! All diffeomorphism of a surface is a group. (Lie group) (because the composition (multiplication of the group) of two diffeomorphism is also a diffeomorphism).

Example 2.3.6. $x^2 + y^2 + z^2 = 1$ *is diffeomorphic to* $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1(a, b, c \neq 0)$.

Proof. Consider $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$

$$(x, y, z) \mapsto (ax, by, cz).$$

This is a diffeomorphism of $\mathbb{R}^3$.(what is $\varphi|_{\mathbb{S}^2}$?)

If $p \in \mathbb{S}^2$ lies on the upper semi-sphere, we use parametrization

$$F(x, y) = (x, y, \sqrt{1 - x^2 - y^2})\colon U \subset \mathbb{R}^3 \to \mathbb{R}^3.$$

So near p

$$\varphi|_{\mathbb{S}^2} = (ax, by, cz) = (u, v, w) \Rightarrow \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 + \left(\frac{w}{c}\right)^2 = 1.$$

By checking points in other five charts on $\mathbb{S}^2$, we obtain $\varphi\colon \mathbb{S}^2 \to$ ellipsoid is a diffeomorphism. (φ^{-1} is self-clear in the above expression.) □

2.4 Tangent plane and differential of a map

• Let S be a regular surface, $p_0 \in S$. In Calculus, we defined the tangent plane at p_0: $T_{p_0}S$ as follows: if $F\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a local parametrization near p_0

$$(u, v) \mapsto (x(u, v), y(u, v), z(u, v)).$$

Since F is regular, $\Rightarrow \frac{\partial F}{\partial u} \wedge \frac{\partial F}{\partial v}$ is a normal vector of the tangent plane. Hence, $T_{p_0}S$ has the equation

$$\left(\frac{\partial F}{\partial u} \wedge \frac{\partial F}{\partial v}\right) \cdot (p - p_0) = 0, \quad p \in \mathbb{R}^3.$$

However, this definition has some deficiency:

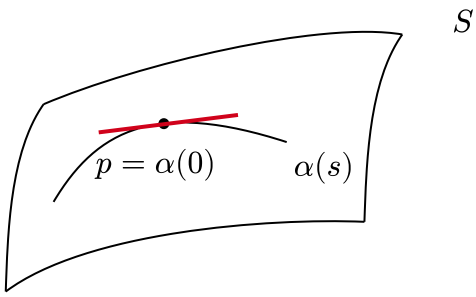
(1) We "have" not checked what happened if we change the parameter.

(In geometry, whenever you define a concept by using some parametrization, you should always check that the concept doesn't depend on the choice of your parametrization, so it's a well-defined definition.)

(2) When we write the equation of the tangent plane, we have used Euclidean inner product in $\mathbb{R}^3$.

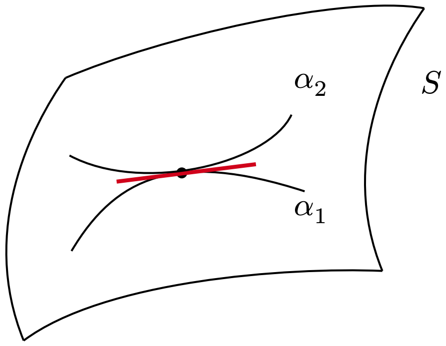
(The tangent space should be an intrinsic object, *i.e.* it depends on the surface itself only.)

Definition 2.4.1 (Tangent vector)**.** V is called a tangent vector at $p \in S$ if $\exists$ a smooth curve $\alpha\colon (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$ such that $\alpha'(0) = V$.



Definition 2.4.2 (Tangent space). T_pS = all tangent vectors at p. (*i.e.* Tangent space is the collection of all tangent vectors of equivalent classes of curves passing through p.)

Here, we say two curves $\alpha_1\colon (-\epsilon,\epsilon)\to S, \alpha_2\colon(-\epsilon,\epsilon)\to S$ are equivalent at p if $\alpha_1(0)=\alpha_2(0)$ and $\alpha_1'(0)=\alpha_2'(0)$.



Remark.

(1) This definition is sufficient in the following study of the course. Later, we will give another definition of tangent space by viewing a tangent vector as an operator acting on some equivalent class of C^∞ functions at a point.(*i.e.* a tangent vector is a "directional derivative", which satisfies "Linearity" and "Leibniz rule") This will tell us how to take derivative of a function on a surface.

(2) In algebraic geometry course, you'll also see another definition of tangent vector (space) of an algebraic variety. You should make a comparison with these definitions, This is more abstract.

Claim: Local expression of tangent vector is independent of the choice of local parametrization.

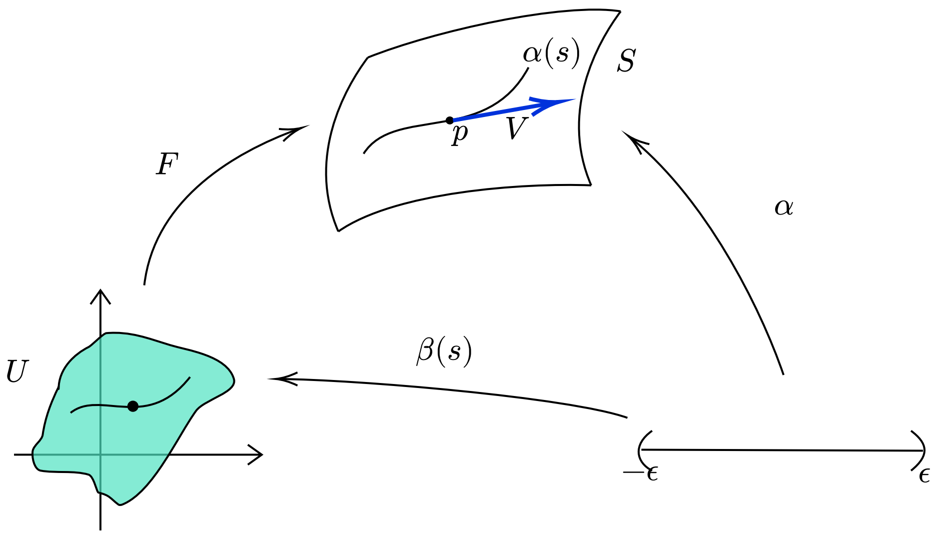
Proof. Let $\alpha(s)\colon(-\epsilon,\epsilon)\to S$ be the regular curve such that $\alpha'(0)=V$ for the given tangent vector V at p. Choose a local parametrization near p,

$$\begin{aligned} F\colon U\subset\mathbb{R}^2 &\to S\\ (u,v)&\mapsto (x(u,v),y(u,v),z(u,v))\,, \end{aligned}$$

then $\alpha(s)=(x\,(u(s),v(s))\,,y\,(u(s),v(s))\,,z\,(u(s),v(s)))$. In fact, if we define

$$\beta(s)=F^{-1}\circ\alpha(s)\colon(-\epsilon,\epsilon)\to U\subset\mathbb{R}^2,$$

then $\alpha(s)=F\circ\beta(s)$.



$$V=\alpha'(0)=\frac{\partial(x,y,z)}{\partial(u,v)}\cdot\left.\frac{\partial(u,v)}{\partial s}\right|_{s=0}. \tag{1}$$

Now, let G be another parametrization

$$G\colon V\to S$$
$$(\alpha,\beta)\mapsto(x(\alpha,\beta),y(\alpha,\beta),z(\alpha,\beta)).$$

Let $\gamma\colon(-\epsilon,\epsilon)\to V$ be $\gamma(s)=G^{-1}\circ\alpha(s)\colon(-\epsilon,\epsilon)\to V$, then $\alpha(s)=G\circ\gamma(s)$, and

$$\alpha'(0)=\frac{\partial(x,y,z)}{\partial(\alpha,\beta)}\cdot\left.\frac{\partial(\alpha,\beta)}{\partial s}\right|_{s=0}. \tag{2}$$

(1) and (2) are essentially the same by the chain rule, *i.e.*

$$\frac{\partial(x,y,z)}{\partial(u,v)}\cdot\left.\frac{\partial(u,v)}{\partial s}\right|_{s=0}=\frac{\partial(x,y,z)}{\partial(\alpha,\beta)}\cdot\underbrace{\frac{\partial(\alpha,\beta)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(\alpha,\beta)}}_{\text{identity matrix}}\cdot\left.\frac{\partial(\alpha,\beta)}{\partial s}\right|_{s=0}$$

□

Proposition 2.4.3. *Let $F\colon U\subset\mathbb{R}^2\to S$ be a local parametrization near p with $F(q)=p$, then $dF_q\left(\mathbb{R}^2\right)=T_pS$.*

Proof.

(1) $dF_q\left(\mathbb{R}^2\right)\subset T_pS$

$\forall V\in\mathbb{R}^2$, let $\beta(t)=q+tV\in\mathbb{R}^2$, then $\beta'(0)=v$. Let $\alpha(t)=F(\beta(t))$ is a C^∞-curve passing through p, and

$$T_pS\ni\alpha'(0)=dF_q(\beta'(0))=dF_q(V)$$

(2) $T_pS\subset dF_q\left(\mathbb{R}^2\right)$

$V\in T_pS.\ \Rightarrow\exists\alpha(t)\colon(-\epsilon,\epsilon)\to S$ such that $\alpha(0)=p$ and $\alpha'(0)=V$. Let $\beta(t)=F^{-1}(\alpha(t))$, then $\alpha(t)=F(\beta(t))$. We remark $\beta(t)$ is a C^∞ -curve in $\mathbb{R}^2$.(Because we have checked the transition function is C^∞ and F^{-1} comes from the composition of transition function) Thus,

$$V=\alpha'(0)=dF_q(\beta'(0))\in\operatorname{im}dF_q$$

□

Remark.

(1) T_pS is the full image of linear space under a linear map, thus T_pS is a linear space of dimension 2.

(2) dF_q is injective $\Rightarrow F_u(q), F_v(q)$ are linearly independent, and they are tangent vectors of coordinate curve $F(u, v_0)$ and $F(u_0, v)$ at $(u_0, v_0) = q$, which implies that $T_pS = \operatorname{span}\{F_u(q), F_v(q)\}$.

(3) $TS = \bigsqcup_{p\in S} T_pS = \left\{(p, V) | p \in S, V \in T_pS\right\}$. This set can be given a smooth coordinate covering, and is called the tangent bundle of S, which is a 4-d smooth manifold. The projection

$$\begin{aligned} \pi : TS &\to S \\ (p, V) &\mapsto p \end{aligned}$$

is C^∞

Tangent vector, tangent space, tangent bundle and tangent vector field

(1) On $\mathbb{R}^n$.

- A vector is $V = (v^1, v^2, \dots, v^n)$.
- A standard basis is $\{e_1, e_2, \dots, e_n\}$, where $e_i = (0, \dots, \overset{i^{th}}{1}, \dots, 0)$.
  $\Rightarrow V = \sum_{i=1}^n v^i e_i$.
- f is smooth in $\mathbb{R}^n$, the directional derivative along a vector v at p is

$$(D_V f)(p) = \langle \nabla f, V \rangle_p = \sum_{i=1}^n v^i \left.\frac{\partial f}{\partial x^i}\right|_p = \left.\left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\right) f\right|_p .$$

  Hence, we can identify $e_i = \frac{\partial}{\partial x^i}$ for $i = 1, 2, \dots, n$ and shall call $\frac{\partial}{\partial x^i}$ the coordinate vector (field).

$$\Rightarrow V = \sum_{i=1}^n v^i e_i = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}.$$

- Note in this way, we actually view a vector V as a first order linear operator acting on smooth functions, *i.e.* for $p \in \mathbb{R}^n$

$$V_p : C^\infty(\mathbb{R}^n) \to \mathbb{R},$$

$$f \mapsto V(f)(p) = (D_V f)(p) = \left(v^i \frac{\partial f}{\partial x^i}\right)(p).$$

  Furthermore, this defines

$$V : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

$$f \mapsto V(f).$$

The operator V satisfies

$$\begin{cases} (V+kW)f = V(f)+kW(f) & \text{(Linearity)} \\ V(fg) = V(f)g + fV(g) & \text{(Leibniz rule)} \end{cases}$$

(2) S is a regular surface in $\mathbb{R}^3$, $p \in S$

- $C^\infty(S) =$ all smooth functions on S.
- We have defined $V \in T_pS$, if $\exists$ a smooth curve $\alpha\colon (-\epsilon,\epsilon) \to S$ such that $\alpha(t) = p$, $\alpha'(0) = V$.
- To compute V, we choose a local parametrization

$$\begin{aligned} F\colon U(\subset \mathbb{R}^2) &\longrightarrow S \subset \mathbb{R}^3 \\ (y^1, y^2) &\longmapsto (x^1, x^2, x^3). \end{aligned}$$

$\Rightarrow \alpha(s) = F\left(y^1(s), y^2(s)\right)$ (This is to say we consider $\alpha\colon (-\epsilon,\epsilon) \to U \xrightarrow{F} S$).

$$\Rightarrow V = \alpha'(0) = \left(\sum_{i=1}^{2} \frac{\partial x^1}{\partial y^i}\frac{\mathrm{d}y^i}{\mathrm{d}s}, \sum_{i=1}^{2} \frac{\partial x^2}{\partial y^i}\frac{\mathrm{d}y^i}{\mathrm{d}s}, \sum_{i=1}^{2} \frac{\partial x^3}{\partial y^i}\frac{\mathrm{d}y^i}{\mathrm{d}s} \right)\Bigg|_{p(s=0)}.$$

Notice V is a vector in $\mathbb{R}^3$, by using notation in item 1

$$V_p = \sum_{\alpha=1}^{3} \left(\sum_{i=1}^{2} \frac{\partial x^\alpha}{\partial y^i}\frac{\mathrm{d}y^i}{\mathrm{d}s} \right) \frac{\partial}{\partial x^\alpha}\bigg|_p.$$

Clearly, the linearity and Leibniz rule hold for the

$$V_p\colon C^\infty(S) \to \mathbb{R}$$

$$f \mapsto V_p(f) = \left(\left(\sum_{\alpha=1}^{3}\sum_{i=1}^{2} \frac{\partial x^\alpha}{\partial y^i}\frac{\mathrm{d}y^i}{\mathrm{d}s}\frac{\partial}{\partial x^\alpha} \right) f \right)(p). \tag{a}$$

Note that during the lecture, we have shown

$$T_pS = dF_{F^{-1}(p)}\left(\mathbb{R}^2\right) = \operatorname{span}\left\{F_{y^1}, F_{y^2}\right\}.$$

Moreover,

$$F_{y^i} = \sum_{\alpha=1}^{3} \frac{\partial x^\alpha}{\partial y^i}\frac{\partial}{\partial x^\alpha},$$

by using the notation in item 1, and we can write

$$F_{y^i} = \frac{\partial F}{\partial y^i} = dF_{F^{-1}(p)}\left(\frac{\partial}{\partial y^i}\right),\ i = 1, 2.$$

$$\Rightarrow T_pS = \operatorname{span}\left\{ dF_{F^{-1}(p)}\left(\frac{\partial}{\partial y^1}\right), dF_{F^{-1}(p)}\left(\frac{\partial}{\partial y^2}\right) \right\},$$

since $dF_{F^{-1}(p)}\colon \mathbb{R}^2 \to T_pS$ is a linear isomorphism. By abusing notation here (actually not)

$$T_pS = \operatorname{span}\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right\},$$

and

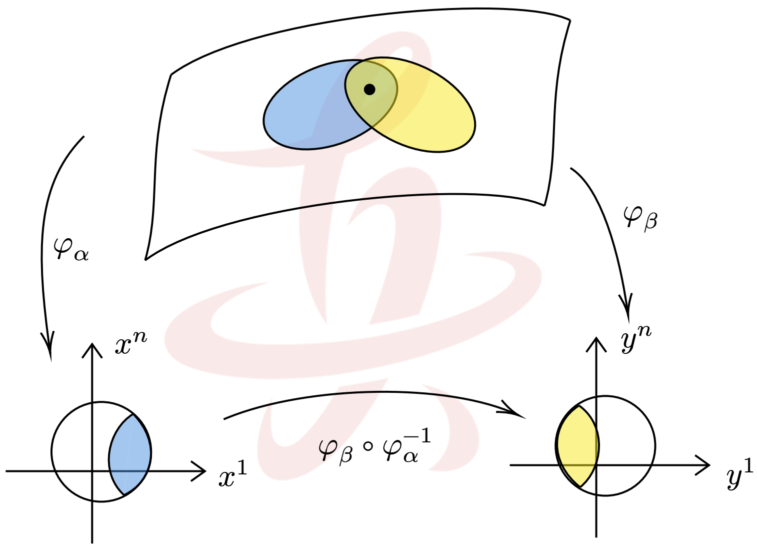
$$V_p(f) = \left(\sum_{i=1}^{2} \frac{\mathrm{d}y^i}{\mathrm{d}s}\frac{\partial}{\partial y^i}\right)(f)(p).$$

Hence,

$$T_pS = \{\text{all tangent vectors } v \text{ at } p\}.$$

Remark. If M is a smooth manifold of dimension n. By definition, M is a topological manifold (*i.e.* Hausdorff + second countable + locally Euclidean) that admits a smooth structure, which means that $\exists$ a covering $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ of M such that

- $\varphi_\alpha\colon U_\alpha \to \varphi(U_\alpha) \subset \mathbb{R}^n$ is a homeomorphism.
- $\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha\left(U_\alpha \cap U_\beta\right) \to \varphi_\beta\left(U_\alpha \cap U_\beta\right)$ is C^∞, $\forall \alpha, \beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$.



Note that $\varphi_\alpha\colon U_\alpha \to \varphi_\alpha(U_\alpha)$ actually endow a coordinate on the chart U_α, *i.e.* $p \in U_\alpha$, $\varphi_\alpha(p) = \left(x^1(p), x^2(p), \dots, x^n(p)\right)$. Hence, $p \in M$, choose U_α as its coordinate chart

$$\Rightarrow T_pM = \operatorname{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}.$$

Note that if at p, there are two coordinate charts

$$\{U, (x^1, \dots, x^n)\} \text{ and } \{V, (y^1, \dots, y^n)\},$$

and a vector $V \in T_pS$ is expressed as

$$V = \sum_{i=1}^{n} V^i \frac{\partial}{\partial x^i} = \sum_{\alpha=1}^{n} \widetilde{V}^\alpha \frac{\partial}{\partial y^\alpha},$$

then

$$\widetilde{V}^\alpha = V^i \frac{\partial y^\alpha}{\partial x^i}$$

(3) Tangent (vector) bundle

In (2), we have seen that the tangent space T_pS of a regular surface does not depend on how we "put" S inside $\mathbb{R}^3$, and that $T_pS = \operatorname{span}\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right\}$ for $\left(y^1, y^2\right)$ a local coordinate in a neighborhood of p. This discussion holds for any smooth manifold. In the following, let's just give the general discussion.

- Let M be a smooth manifold, $\forall p \in M$, T_pM is the tangent space at p. Define

$$TM = \bigsqcup_{p\in M} T_pM = \left\{(p, V) | p \in M, V \in T_pM\right\},$$

then there is a projection map

$$\pi\colon TM \to M$$

$$(p, V) \mapsto p.$$

We shall show that the total space of $TM \to M$ is a $2n$-dim smooth manifold such that $\pi\colon TM \to M$ is a smooth map. To achieve this goal, we need to define a smooth structure on the total space TM.

- M is a smooth manifold, by definition, $\exists$ a countable covering $\{(U\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ such that
  - $\varphi_\alpha\colon U_\alpha \to \varphi\left(U_\alpha\right) \subset \mathbb{R}^n$ is a homeomorphism,

$$\varphi_\alpha(p) = \left(x_\alpha^1(p), x_\alpha^2(p), \ldots, x_\alpha^n(p)\right).$$

  - $\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha\left(U_\alpha \cap U_\beta\right) \to \varphi_\beta\left(U_\alpha \cap U_\beta\right)$ is smooth, $\forall \alpha, \beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$.

- For each U_α, $\pi^{-1}\left(U_\alpha\right)$ contains all pairs $(p, V), p \in U_\alpha, V \in T_pM$. We define local trivialization (local coordinate chart near (p, V)) by letting

$$\widetilde{\varphi}_\alpha\colon \pi^{-1}\left(U_\alpha\right) \to U_\alpha \times \mathbb{R}^n$$

$$(p, V) \mapsto \left(\varphi_\alpha(p), V^1, V^2, \ldots, V^n\right).$$

Note $\widetilde{\varphi}_\alpha(p) = \left(x_\alpha^1(p), x_\alpha^2(p), \ldots, x_\alpha^n(p)\right)$, $V = \sum_{i=1}^n V^i(p)\frac{\partial}{\partial x^i}$, then

$$\widetilde{\varphi}_\alpha^{-1}\left(x_\alpha^1, \ldots, x_\alpha^n; V^1, \ldots, V^n\right) = \left(p, \sum V^i \frac{\partial}{\partial x^i}\right),$$

i.e. $\widetilde{\varphi}_\alpha$ is bijective and $\widetilde{\varphi}_\alpha\left(\pi^{-1}\left(U_\alpha\right)\right)$ is an open set. Then $\left\{\widetilde{\varphi}_\alpha^{-1}\left(U_\alpha \times \mathbb{R}^n\right)\right\}_{\alpha\in\Lambda}$ is a countable covering of TM. Moreover, this defines a topology on TM, such that

$$\widetilde{\varphi}_\alpha\colon \pi^{-1}\left(U_\alpha\right) \to U_\alpha \times R^n$$

is a homeomorphism. Next, if $\pi^{-1}\left(U_\alpha\right) \cap \pi^{-1}\left(U_\beta\right) \neq \emptyset \left(\Leftrightarrow U_\alpha \cap U_\beta \neq \emptyset\right)$, let $p \in U_\alpha \cap U_\beta$ and

$$\widetilde{\varphi}_\alpha(p, V) = \left(x_\alpha^1(p), \cdots, x_\alpha^n(p); V_\alpha^1, \ldots, V_\alpha^n\right)$$

$$\widetilde{\varphi}_\beta(p,V)=\left(x_\beta^1(p),\cdots,x_\beta^n(p);V_\beta^1,\ldots,V_\beta^n\right)$$

$$\begin{aligned}\Rightarrow \widetilde{\varphi}_\beta\circ\widetilde{\varphi}_\alpha^{-1}\left(x_\alpha^1(p),\cdots,x_\alpha^n(p);V_\alpha^1,\ldots,V_\alpha^n\right)\\ =\left(x_\beta^1(p),\cdots,x_\beta^n(p);V_\beta^1,\ldots,V_\beta^n\right),\end{aligned}$$

where

$$V_p=\sum_{i=1}^n V_\alpha^i\frac{\partial}{\partial x_\alpha^i}=\sum_{j=1}^n V_\beta^j\frac{\partial}{\partial x_\beta^j}$$

$$V_\beta^j=\sum_{i=1}^n V_\alpha^i\frac{\partial y_\beta^j}{\partial x_\alpha^i},$$

i.e.

$$\begin{aligned}&\widetilde{\varphi}_\beta\circ\widetilde{\varphi}_\alpha^{-1}\left(x_\alpha^1(p),\cdots,x_\alpha^n(p);V_\alpha^1,\ldots,V_\alpha^n\right)\\ &=\left(\varphi_\beta\circ\varphi_\alpha^{-1}\left(x_\alpha^1,\ldots,x_\alpha^n\right);\sum_{i=1}^n V_\alpha^i\frac{\partial y_\beta^1}{\partial x_\alpha^i},\cdots,\sum_{i=1}^n V_\alpha^i\frac{\partial y_\beta^n}{\partial x_\alpha^i}\right).\end{aligned}$$

Hence, $\widetilde{\varphi}_\beta\circ\widetilde{\varphi}_\alpha^{-1}$ is smooth since $\varphi_\beta\circ\varphi_\alpha^{-1}$ is smooth, and the transition function on the vector part is just linear.

- The Hausdorff property of TM is also clear from the Hausdorff property of M.
- With the smooth structure defined above, Clearly $\pi\colon TM\to M$ is a smooth map since π is just a projection (when expressed in coordinate).

Tangent vector field

Definition 2.4.4. A smooth vector field V is a C^∞-map

$$V\colon M\to TM$$

such that $\pi\circ V=\mathrm{id}_M$. In a coordinate chart $\left(U_\alpha,\left(x_\alpha^1,\ldots x_\alpha^n\right)\right)$,

$$V=\sum_{i=1}^n V_\alpha^i(x)\frac{\partial}{\partial x_\alpha^i}$$

Remark.

(1) A smooth vector field is also called a smooth section of the tangent (vector) bundle.

(2) Since the tangent vector is linear and satisfies the Leibniz rule, so is the vector field. Hence,

$$V\colon C^\infty(M)\to C^\infty(M)$$
$$f\mapsto V(f)$$

is characterized by $V(fg)=V(f)g+fV(g)$.

Exercise.

(1) $\mathbb{S}^1$*: unit circle.* $T\mathbb{S}^1 \simeq \mathbb{S}^1 \times \mathbb{R}$ *is a diffeomorphism.*

Intuitive observation of this diffeomorphism: we can choose local charts on $\mathbb{S}^1$ *by*

$$\theta \in (0, 2\pi) = U, \;\; \varphi \in (\frac{\pi}{2}, \frac{3\pi}{2}) = V$$

i.e. $\varphi = \theta + \frac{\pi}{2}$*, which is simply a rotation.*

$$T\mathbb{S}^1\big|_U = \left\{\left(\theta, a(\theta)\frac{\partial}{\partial\theta}\right)\right\}$$

$$T\mathbb{S}^1\big|_V = \left\{\left(\varphi, a(\varphi)\frac{\partial}{\partial\varphi}\right)\right\}.$$

In this case, the transition function is just linear by rotation. Hence, $T\mathbb{S}^1\big|_U$ *and* $T\mathbb{S}^1\big|_V$ *can be glued by shifting* $\frac{\pi}{2}$*.*

(2) $\mathbb{S}^2$ *unit sphere. However,* $T\mathbb{S}^2 \ncong \mathbb{S}^2 \times \mathbb{R}^2$*. To see this, we take stereographical parametrization. (Since there are only two coordinate charts)*

$$p_N \colon \mathbb{S}^2 - \{N\} \to \mathbb{R}^2$$
$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

$$p_S \colon \mathbb{S}^2 - \{S\} \to \mathbb{R}^2$$
$$(x, y, z) \mapsto \left(\frac{y}{1+z}, \frac{x}{1+z}\right).$$

The transition function is given by

$$p_S \circ p_N^{-1}(x, y) = \left(\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = (u, v)$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x} = -2uv\frac{\partial}{\partial u} + \left(v^2 - u^2\right)\frac{\partial}{\partial v} \\ \frac{\partial}{\partial y} = \left(u^2 - v^2\right)\frac{\partial}{\partial u} - 2uv\frac{\partial}{\partial v} \end{cases}.$$

Note $(x, y) = (0, 0)$ *is just the North Pole and* $(u, v) = (0, 0)$ *is just the South Pole. At these two points, the coordinate vector fields are zero vectors, e.g.* $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ *are zero vectors at the South Pole. This means that we can not extend the tangent plane in the trivialization* $p_N^{-1}\left(\mathbb{R}^2\right)$ *passing through the South Pole, hence we can not have a global trivialization of* $T\mathbb{S}^2$*.* $T\mathbb{S}^2 \ncong \mathbb{S}^2 \times \mathbb{R}^2$

Next, we would like to mention a more abstract way to think about the tangent space.

- M is a smooth manifold, $p \in M$
- $C_p^\infty(M)$={all C^∞ functions on M vanishing at p}.
  $C_p^\infty(M)^2$={all finite sums $\sum f_i g_i$ with $f_i, g_i \in C_p^\infty(M)$}

$$\Rightarrow C^\infty(M) = \{\text{constant functions}\} \oplus C_p^\infty(M) = \mathbb{R} \oplus C_p^\infty(M).$$

- Let V be a tangent vector at p, $V\colon C^\infty(M) \to \mathbb{R}$ and $V|_{\mathbb{R}} = 0$. Moreover, it's easy to see by the Leibniz rule that $V|_{C_p^\infty(M)^2} = 0$

$$\Rightarrow V\colon C_p^\infty(M)/C_p^\infty(M)^2 \to \mathbb{R}$$

is a linear map, *i.e.* $V \in \left(C_p^\infty(M)/C_p^\infty(M)^2\right)^*$

$\Rightarrow T_pM \simeq \left(C_p^\infty(M)/C_p^\infty(M)^2\right)^*$ is an isomorphism. Hence, we can define tangent space T_pM as $\left(C_p^\infty(M)/C_p^\infty(M)^2\right)^*$

This definition tells that a tangent vector is a linear map on $C^\infty(M)$ vanishing on constants and only depends on the first order Taylor's expansion of a function f at p.

In algebraic geometry language, the vanishing ideals $C_p^\infty(M)$ are the maximal ideals in the algebra of smooth functions, with $C_p^\infty(M)^2$ the second power. More generally, for any maximal ideal I in a commutative algebra A, one may define the tangent space as $\left(I/I^2\right)^*$.

Remark. This last part is not required in this course.

Definition 2.4.5 (Differential of a map)**.** Let S_1, S_2 be regular surfaces, $f\colon S_1 \to S_2$ is a C^∞ map. Its differential at p

$$df_p\colon T_pS_1 \to T_{f(p)}S_2$$

is defined as follows:

$\forall v \in T_pS_1$, take an $\alpha(s)\colon (-\epsilon, \epsilon) \to S_1$ such that $\alpha'(0) = v, \alpha(0) = p$. Then $df_p(v)$ is defined as

$$df_p(v) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f\left(\alpha(s)\right) \in T_{f(p)}S_2. \tag{1}$$

Remark. Although in the definition above, we do not involve coordinate, in practice we usually need to do computation with coordinate.

- $df_p\colon T_pS_1 \to T_{f(p)}S_2$ is a linear map.

Proof. On S_1 near p, we choose local parametrization $F\colon U \to S_1$. On S_2 near $f(p)$, we choose local parametrization $G\colon V \to S_2$. We have seen

$$dF_p\left(\mathbb{R}^2\right) = T_pS_1$$

$$dG_{f(p)}\left(\mathbb{R}^2\right) = T_{f(p)}S_2,$$

and dF_p, $dG_{f(p)}$ are both invertible linear maps. Hence, we have the following diagram.

$$\begin{tikzcd}
S_1 \arrow[r, "f"] & S_2 & & T_pS_1 \arrow[r, "df_p"] & T_{f(p)}S_2 \\
\mathbb{R}^2 \supset U \arrow[u, "F"] \arrow[r, "d(G^{-1}\circ f\circ F)"'] & V \subset \mathbb{R}^2 \arrow[u, "G"] \arrow[r, phantom, ""] & {} \arrow[r, "\text{linearize at } p"] & \mathbb{R}^2 \arrow[u, "dF_p"] \arrow[r, "d(G^{-1}\circ f\circ F)_p"'] & \mathbb{R}^2 \arrow[u, "dG_{f(p)}"]
\end{tikzcd}$$

By the chain rule,

$$d\left(G^{-1}\circ f\circ F\right)_p = dG^{-1}_{f(p)}\circ df_p\circ dF_{F^{-1}(p)}.$$

From calculus, we know $d\left(G^{-1}\circ f\circ F\right)_p : \mathbb{R}^2\to\mathbb{R}^2$ is a linear map and dF_p, $dG^{-1}_{f(p)}$ are linear maps. Hence, we conclude $df_p\colon T_pS_1\to T_{f(p)}S_2$ is also linear. $\square$

Remark.

(a) In eq. (1), if taking $S_2=\mathbb{R}$, then we have defined the "differential of a smooth function f"

$$df_p(v)=\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left(f\left(\alpha(s)\right)\right),$$

where $df_p(v)$ should be understood as the "directional derivation" of f at p. Now df_p is a covector, belonging to T_p^*S, the dual space of T_pS ={all linear maps $\omega\colon T_pS\to\mathbb{R}$}. T_p^*S is also a linear space. If $T_pS=\mathrm{span}\{e_1,e_2\}$, then $T_p^*S=\mathrm{span}\{e_1^*,e_2^*\}$ where

$$e_i^*\left(e_j\right)=\begin{cases}1 & i=j\\ 0 & i\neq j.\end{cases}$$

Note: globally not all covectors can be written as df Due to the obstruction from topology, *i.e.* $H^1(M,\mathbb{R})$.

(b) Later, you may also see (with other definition of tangent vector) $df_p(V)=V(f)(p)$, where$V(f)$ means taking derivative of f along V.

(c) $d(f+g)_p(V)=df_p(V)+dg_p(V)$
$d(f\cdot g)_p(V)=df_p(V)g(p)+f(p)dg_p(V)$ (Leibniz rule)

Exercise.

(1) $\mathrm{Id}\colon M\to M$ *is the identity map.* $\Rightarrow d(\mathrm{Id})_p(V)=V$.

(2) Compute the differential of a rotational map on $\mathbb{S}^2$. *We use local parametrization on* $\mathbb{S}^2$*:*

$$\left(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi\right)^t=p.$$

Consider the rotation of $\mathbb{S}^2$ *around the* z*-axis in counterclockwise direction about degree* α

$$\begin{aligned}\Rightarrow R(\alpha)(p)&=\left(\sin\varphi\cos(\theta+\alpha),\sin\varphi\sin(\theta+\alpha),\cos\varphi\right)^t\\ &=\left(\sin\varphi\left(\cos\theta\cos\alpha-\sin\theta\sin\alpha\right),\sin\varphi\left(\sin\theta\cos\alpha+\cos\theta\sin\alpha\right),\cos\varphi\right)^t\\ &=\underbrace{\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}}_{A}\begin{pmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{pmatrix}.\end{aligned}$$

Note $R(\alpha)\colon\mathbb{S}^2\to\mathbb{S}^2$ *is a linear action (map). To compute* dR_p*, we need to know for* $v\in T_p\mathbb{S}^2$*, what is* $dR_p(v)$*?*

Let $\gamma(s)$ be the curve on S such that $\gamma'(0) = v$, then

$$\begin{aligned} dR_p(v) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} R_p\left(\gamma(s)\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left(A \cdot \gamma(s)\right) = A \cdot \gamma'(0) = A \cdot v, \end{aligned}$$

this means $dR_p(v)$ is the rotation of the vector v about degree alpha.(see)

(3) *Let $f\colon S_1 \to S_2$ be a diffeomorphism of two regular surfaces. Then at $p \in S_1$, $df_p\colon T_pS_1 \to T_{f(p)}S_2$ is a linear isomorphism. The converse is only true locally.*

Definition 2.4.6 (Local diffeomorphism). S_1, S_2 are regular surfaces, $p \in S_1$, the map $\varphi\colon S_1 \to S_2$ is called a local diffeomorphism at p, if $\exists$ a neighborhood U of p such that $\varphi\colon U \to \varphi(U)$ is a diffeomorphism.

Proposition 2.4.7 (Inverse function (mapping) theorem). *$\varphi\colon S_1 \to S_2$ is a differentiable map, $p \in S_1$. If $\varphi\colon T_pS_1 \to T_{\varphi(p)}S_2$ is an isomorphism, then $\exists$ a neighborhood U of p such that $\varphi\colon U \to \varphi(U)$ is a local diffeomorphism at p.*

Exercise. *(Following item 2)*

(1) *Find the rotational map of $\mathbb{S}^2$ around other 2 axes.*

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \quad C = \begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix}$$

(2) *What can you observe from matrices A, B and C?*

(Their determinant is 1 and form a basis of $SO(3)$ ={all rotational maps of $\mathbb{S}^2$}. Observe that the rotations of $\mathbb{S}^2$ should preserve the lengths and angles over $\mathbb{S}^2$. This is just based on "observation" later, and after we define the lengths and angles in $\mathbb{S}^2$ we'll check this.)

(3) *Let $f\colon S_1 \to S_2$ be a diffeomorphism of two regular surfaces, then at $p \in S_1$, $\mathrm{d}f_p : T_pS_1 \to T_{f(p)}S_2$ is a linear isomorphism. The converse is only true locally.*

2.5 Orientation of a regular surface

Question. Consider $\mathbb{S}^2$ with spherical parametrization, are following two parametrizations the same?

- $(\varphi, \theta) \longmapsto (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$
- $(\theta, \varphi) \longmapsto (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

The answer is NO! They give different normal directions which decide "inside" and "outside" of $\mathbb{S}^2$.

Recall: $\mathbb{R}^2$ is orientable. It has two orientations. Given a basis $\{e_1, e_2\}$, let

$$J_+ = \Bigg\{E_+ = \{e_1^+, e_2^+\} : E_+ \text{ has same orientation as } E.$$

$$\text{i.e. } \begin{bmatrix} e_1^+ \\ e_2^+ \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A_+} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \text{ and } \det A_+ > 0\Bigg\};$$

$$J_- = \Bigg\{E_- = \{e_1^-, e_2^-\} : E_- \text{ has same orientation as } E.$$

$$\text{i.e. } \begin{bmatrix} e_1^- \\ e_2^- \end{bmatrix} = A_- \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \text{ and } \det A_- > 0\Bigg\}.$$

Then either J_+ or J_- gives an orientation of $\mathbb{R}^2$.

Question.

(1) Let S be a regular surface, how can we define orientation?

(2) Are all surfaces orientable?

Assume we have two parametrizations around p:

$$\begin{aligned} F\colon U &\to S \qquad & T_pS &\cong \mathbb{R}^2 = \operatorname{span}\{F_u, F_v\} \\ G\colon V &\to S \qquad & T_pS &\cong \mathbb{R}^2 = \operatorname{span}\{G_\alpha, G_\beta\}. \end{aligned}$$

Note near p, $F(u,v) = G(\alpha, \beta) \implies$

$$\begin{bmatrix} G_\alpha \\ G_\beta \end{bmatrix} = \begin{bmatrix} u_\alpha & v_\alpha \\ u_\beta & v_\beta \end{bmatrix} \begin{bmatrix} F_u \\ F_v \end{bmatrix}.$$

i.e. Two parametrizations F and G give the same orientation iff

$$\left|\frac{\partial(u,v)}{\partial(\alpha,\beta)}\right| > 0.$$

Definition 2.5.1 (Orientation of a regular surface)**.** We say a regular surface S is orientable if there exists a coordinate chart covering of S, *s.t.* if a point p belongs to two charts, then the induced basis on T_pS by the two parametrizations have the same orientation in above sense.

Remark.

(1) "Orientation" is a global intrinsic property. "Orientability" is essentially reflected by the topology of the surface. In algebraic topology course, you'll see that the orientability is determined by the 1st Stiefel-Whitney class in $H^1(M, \mathbb{Z}/2\mathbb{Z})$ for a vector bundle on topological manifold M. With additional "smooth" structure on M, we have more ways to check the orientability, for example, $\exists$ a nowhere vanishing top differential form (or say, a volume form).

(2) One of important applications of "orientation" in this course (and later in Reimann Geometry) is allowing us to define "integration" on a regular surface.

Exercise. *Give a definition of orientation of $\mathbb{R}^n$ as a vector space.*

Example 2.5.2. $\mathbb{S}^2$ *is orientable.*

Consider the stereographic parametrization

$$\begin{aligned}\pi_N\colon \mathbb{S}^2 \setminus \{N\} &\longrightarrow \mathbb{R}^2\\ (x,y,z) &\longmapsto (u,v) = (\frac{x}{1-z}, \frac{y}{1-z}).\end{aligned}$$

$$\pi_N^{-1}(u,v) = (\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, 1 - \frac{2}{1+u^2+v^2}).$$

$$\begin{aligned}\pi_S\colon \mathbb{S}^2 \setminus \{S\} &\longrightarrow \mathbb{R}^2\\ (x,y,z) &\longmapsto (\alpha,\beta) = (\frac{y}{1+z}, \frac{x}{1+z}).\end{aligned}$$

$$\pi_N^{-1}(\alpha,\beta) = (\frac{2\beta}{1+\alpha^2+\beta^2}, \frac{2\alpha}{1+\alpha^2+\beta^2}, \frac{2}{1+\alpha^2+\beta^2} - 1).$$

Clearly $\mathbb{S}^2 = \pi_N^{-1}(U) \cup \pi_S^{-1}(V)$. To see the orientation, we compute the Jacobian of change form one chart to another. $(u,v) \to (\alpha,\beta)$. Now

$$\begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases}, \quad \begin{cases} \alpha = \frac{y}{1+z} \\ \beta = \frac{x}{1+z} \end{cases}.$$

Hence

$$\begin{cases} u\alpha = v\beta \\ u\beta + v\alpha = 1 \end{cases} \implies \begin{cases} u = \frac{\beta}{\alpha^2+\beta^2} \\ v = \frac{\alpha}{\alpha^2+\beta^2} \end{cases}.$$

Then

$$\frac{\partial(u,v)}{\partial(\alpha,\beta)} = \begin{pmatrix} -\frac{2\alpha\beta}{(\alpha^2+\beta^2)^2} & \frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)^2} \\ \frac{\beta^2-\alpha^2}{(\alpha^2+\beta^2)^2} & -\frac{2\alpha\beta}{(\alpha^2+\beta^2)^2} \end{pmatrix} \implies \det > 0.$$

Example 2.5.3. *The graph $(x,y,f(x,y))$ is always orientable.*

- Normal vector field and orientation.

Since in this course, we consider the regular surface S in $\mathbb{R}^3$, at each $p \in S$, there is a unique tangent plane T_pS, which is a 2 dim linear subspace of $\mathbb{R}^3$. Hence we can talk abort the normal vector field on S. Now we want a "well-defined" normal by taking "unit normal vector $\vec{n}$" at p.

Let's check $\vec{n}$ under change of parameter. Let $F\colon U \to S$ be a parametrization around p. Then

$$T_pS = \operatorname{span}\{F_u, F_v\} \implies \vec{n}_F = \frac{F_u \times F_v}{|F_u \times F_v|}.$$

(Note the cross product here already assigns a direction of $\vec{n}$).

If $G\colon V \to S$ is another parametrization around p. Then

$$T_pS = \operatorname{span}\{G_\alpha, G_\beta\}, \quad \vec{n}_G = \frac{G_\alpha \times G_\beta}{|G_\alpha \times G_\beta|}.$$

Let $\Phi = G^{-1} \circ F \colon U \cap V \to U \cap V$ be the transition map. Then

$$\begin{cases} F_u = G_\alpha \alpha_u + G_\beta \beta_u \\ F_v = G_\alpha \alpha_v + G_\beta \beta_v \end{cases} \quad i.e. \quad \begin{pmatrix} F_u \\ F_v \end{pmatrix} = \begin{pmatrix} \alpha_u & \beta_u \\ \alpha_v & \beta_v \end{pmatrix} \begin{pmatrix} G_\alpha \\ G_\beta \end{pmatrix}.$$

$\implies F_u \times F_v = \det\left(\frac{\partial(\alpha,\beta)}{\partial(u,v)}\right) G_\alpha \times G_\beta$. Hence

$$\vec{n}_F = \operatorname{sign}\det\left(\frac{\partial(\alpha,\beta)}{\partial(u,v)}\right) \cdot \vec{n}_G.$$

Definition 2.5.4 (Unit normal vector field)**.** $U \subset S$ open subset, a unit normal vector field on U is a smooth map

$$\begin{aligned} \vec{n} \colon U &\longrightarrow \mathbb{R}^3 \\ p &\longmapsto \vec{n}_p \text{ (unit normal at } p). \end{aligned}$$

Proposition 2.5.5. *A regular surface is orientable $\iff$ there is a unit normal vector field $\vec{n} \colon S \to \mathbb{R}^3$ defined everywhere on S.*

Remark. On an orientable regular surface, the "unit" normal vector field has image on $\mathbb{S}^2$, *i.e.* there is a smooth map

$$\begin{aligned} \vec{n} \colon S &\longrightarrow \mathbb{S}^2 = \{x^2+y^2+z^2=1\} \\ p &\longmapsto \vec{n}_p. \end{aligned}$$

This $\vec{n}$ is called the *Gauss map* of the surface. Later we'll see that the differential of $\vec{n}$ determines how the surface is curved in $\mathbb{R}^3$.

Proof. "$\implies$": If S is orientable, then $\exists$ a coordinate chart covering $\{F_\alpha(U_\alpha)\}_{\alpha\in\Lambda}$ of S, $(F_\alpha \colon U_\alpha \to S, S = \bigcup_{\alpha\in\Lambda} F_\alpha(U_\alpha))$. *s.t.* on $U_\alpha \cap U_\beta \neq \emptyset$,

$$\det\left(\frac{\partial(u_\alpha, u_\beta)}{\partial(u_\beta, v_\beta)}\right) > 0.$$

Hence at $p \in U_\alpha \cap U_\beta$,

$$\vec{n}_p = \frac{F_{u_\alpha} \times F_{v_\alpha}}{|F_{u_\alpha} \times F_{v_\alpha}|} = \frac{F_{u_\beta} \times F_{v_\beta}}{|F_{u_\beta} \times F_{v_\beta}|}.$$

(This guarantees that we can extend the normal vector in on chart to another)

Hence we can well define $\vec{n} \colon S \to \mathbb{R}^3$. The smoothness of $\vec{n}$ can be checked in local coordinates.

"$\impliedby$": Let $\vec{n} \colon S \to \mathbb{R}^3$ be the unit normal vector field defined on S. Let $\{F_\alpha \colon U_\alpha \to S\}$ be a coordinate chart covering of S. Then on each $F_\alpha(U_\alpha)$,

$$\vec{n}_\alpha = \frac{F_{u_\alpha} \times F_{v_\alpha}}{|F_{u_\alpha} \times F_{v_\alpha}|}.$$

On $U_\alpha \cap U_\beta$ at some point p, we have

$$\frac{F_{u_\alpha} \times F_{v_\alpha}}{|F_{u_\alpha} \times F_{v_\alpha}|} = \operatorname{sign}\det\left(\frac{\partial(u_\beta, v_\beta)}{\partial(u_\alpha, v_\alpha)}\right) \cdot \frac{F_{u_\beta} \times F_{v_\beta}}{|F_{u_\beta} \times F_{v_\beta}|}.$$

Since $\vec{n}_\alpha$ and $\vec{n}_\beta$ agree at p, we see

$$\det\left(\frac{\partial(u_\beta, v_\beta)}{\partial(u_\alpha, v_\alpha)}\right) > 0.$$

$\square$

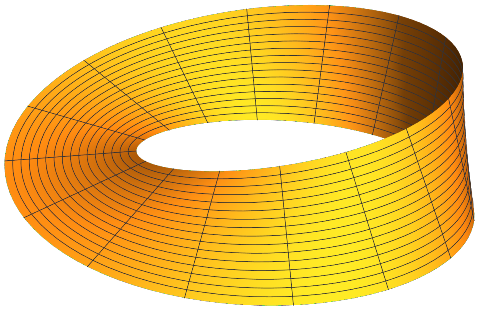


Figure 2.5.1: Möbius strip

Example 2.5.6 (Non-orientated surface)**.** *Möbius strip M (fig. 2.5.1)*

Consider the line segment AB with equation $-\frac{1}{2} \leq z \leq \frac{1}{2}, y = 1, x = 0$, rotating along a circle $x^2 + y^2 = 1$. When the center c moves about angle u away from y-axis, the line $A'B'$ is in the plane determined by z-axis and c' and the angle between $A'B'$ and z-axis is $\frac{u}{2}$.

Let $U = \{(u, v) : 0 < u < 2\pi, -\frac{1}{2} < v < \frac{1}{2}\}$, then

$$F(u, v) = ((1 - v\sin\frac{u}{2})\sin u, (1 - v\sin\frac{u}{2})\cos u, v\cos\frac{u}{2}).$$

This almost covers M, except at $u = 0$. To cover this, we consider rotation AB when the center lies on x-axis. We have parametrization

$$G(\alpha, \beta) = ((1 - \beta\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right))\cos\alpha, (1 - \beta\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right))\sin\alpha, \beta\cos\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)).$$

$$V = \{(\alpha, \beta) : 0 < \alpha < 2\pi, -\frac{1}{2} < \beta < \frac{1}{2}\}.$$

Now the missing part is on $u = \frac{\pi}{2}$.

Consider $F(U) \cap G(V)$, it contains two pieces

$$W_1 = \{F(u, v) : 0 < u < \frac{\pi}{2}\}, \quad W_2 = \{F(u, v) : \frac{\pi}{2} < u < 2\pi\}.$$

On W_1,

$$\begin{cases} \alpha = u + \frac{3}{2}\pi \\ \beta = -v \end{cases} \implies \left|\frac{\partial(\alpha, \beta)}{\partial(u, v)}\right| = -1.$$

On W_2,

$$\begin{cases} \alpha = u - \frac{\pi}{2} \\ \beta = v \end{cases} \implies \left|\frac{\partial(\alpha,\beta)}{\partial(u,v)}\right| = 1.$$

No matter how we change parametrization in W_1 or W_2, we can not guarantee that $\left|\frac{\partial(\alpha,\beta)}{\partial(u,v)}\right|$ is positive everywhere. (*i.e.* $\vec{n}$ has to change sign). Hence M is non-orientable.

Remark. In the following of this course, we only care about orientable surfaces. *i.e.* surfaces having well-defined unit normal vector field everywhere.

2.6 The 1$^{\text{st}}$ fundamental form on S

The 1$^{\text{st}}$ fundamental form on S inherits from the Euclidean inner product of $\mathbb{R}^3$. It allows us to measure geometric quantities like

- Distance between pairs;
- Angle between curves;
- Area of a region;
- etc.

It is an intrinsic, globally defined concept on S. In the language of Riemannian Geometry, the 1st fundamental form is also called the "pull-back" Riemannian metric on S. It's related to the "isometric immersion (embedding)" in Riemannian Geometry course.

! The 1st fundamental form determines the "intrinsic geometry".

We first give the definition of the 1st fundamental form, then explain where it comes from.

Let $S \subset \mathbb{R}^3$ be a regular surface. $\forall\, p \in S$, we take local parametrization

$$\begin{aligned} \varphi\colon U \subset \mathbb{R}^2 &\longrightarrow S \\ (u,v) &\longmapsto \varphi(u,v) = (x,y,z). \end{aligned}$$

and $p = \varphi(u_0, v_0)$, then

$$\varphi_u(u_0,v_0) = (x_u, y_u, z_u)\Big|_{(u_0,v_0)} \in \mathbb{R}^3$$

$$\varphi_v(v_0,v_0) = (x_v, y_v, z_v)\Big|_{(u_0,v_0)} \in \mathbb{R}^3.$$

And $T_pS \cong \mathbb{R}^2 = \operatorname{span}\{\varphi_u(u_0,v_0), \varphi_v(u_0,v_0)\}$.

Define

$$E = \langle \varphi_u, \varphi_u \rangle_p \quad F = \langle \varphi_u, \varphi_v \rangle_p \quad G = \langle \varphi_v, \varphi_v \rangle.$$

The 1st fundamental form at $p \in S$ is defined as

$$I_p = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2.$$

To understand this, let's start with inner product in $\mathbb{R}^3$.

$$\mathbb{R}^3 = \mathrm{span}\{e_1, e_2, e_3\} = \mathrm{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\} = \mathrm{span}\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\}.$$

Here, since for each vector v with length 1, it uniquely determines a directional derivative ∇_v and vice versa. Hence we can identify $e_1 = (1,0,0)$ with $\frac{\partial}{\partial x^1}$, and $e_2 = \frac{\partial}{\partial x^2}$, $e_3 = \frac{\partial}{\partial x^3}$.

The inner product on $\mathbb{R}^3$ is

$$\begin{aligned} g_{\mathbb{R}^3} : \mathbb{R}^3 \times \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto \langle v, w \rangle, \quad v, w \text{ vectors in } \mathbb{R}^3. \end{aligned}$$

Satisfy:

- $g(v, w) = g(w, q)$
- $g(av_1 + bv_2, w) = ag(v_1, w) + bg(v_2, w)$
- $g(v, v) \geq 0$ and $g(v, v) = 0 \iff v = 0$.

If $v = \sum_{i=1}^3 v^i \frac{\partial}{\partial x^i}, w = \sum_{j=1}^3 w^j \frac{\partial}{\partial x^j}$, then

$$g_{\mathbb{R}^3}(v, w) = \sum_{i,j=1}^3 \delta_{ij} v^i w^j, \quad \text{where } \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

$(\mathbb{R}^3)^*$, dual space of $\mathbb{R}^3$ is defined as follows:
For a vector $v \in \mathbb{R}^3$, we define an element $\alpha_v \in (\mathbb{R}^3)^*$ by

$$\alpha_v = g_{\mathbb{R}^3}(v, \cdot) : \mathbb{R}^3 \to R, \quad w \mapsto \alpha_v(w) = g_{\mathbb{R}^3}(v, w).$$

The dual basis of $\{e_1, e_2, e_3\} = \{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\}$ is

$$\{e_1^*, e_2^*, e_3^*\} = \{\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\}.$$

s.t. $e_i^*(e_j) = g(e_i, e_j) = \delta_{ij}$. Then $(\mathbb{R}^3)^* = \mathrm{span}\{e_1^*, e_2^*, e_3^*\} = \mathrm{span}\{\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\}$. Hence

$$\alpha_v = \sum_{i=1}^3 v^i \, \mathrm{d}x^i, \quad v^1 = \left\langle v, \frac{\partial}{\partial x^i} \right\rangle, \quad g_{\mathbb{R}^3}(v, \cdot) = \alpha_v \in (\mathbb{R}^3)^*.$$

By relaxing v in $g_{\mathbb{R}^3}(v, \cdot)$, we can view $g_{\mathbb{R}^3}$ as a symmetric element in $(\mathbb{R}^3)^* \otimes (\mathbb{R}^3)^*$. From linear algebra,

$$\mathrm{Sym}((\mathbb{R}^3)^* \otimes (\mathbb{R}^3)^*) = \mathrm{span}\{\mathrm{d}x^i \, \mathrm{d}x^j = \mathrm{d}x^j \, \mathrm{d}x^i\}_{i,j=1,2,3}.$$

Hence we can write

$$g_{\mathbb{R}^3} = \delta_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2.$$

Now let $S \subset \mathbb{R}^3$ with local parametrization given at the beginning of this section, *i.e.* $x^i = x^i(u, v)$. We have

$$\mathrm{d}x^1 = x_u^1 \, \mathrm{d}u + x_v^1 \, \mathrm{d}v, \quad \mathrm{d}x^2 = x_u^2 \, \mathrm{d}u + x_v^2 \, \mathrm{d}v, \quad \mathrm{d}x^3 = x_u^3 \, \mathrm{d}u + x_v^3 \, \mathrm{d}v.$$

Then we consider restrict $g_{\mathbb{R}^3}$ on TS,

$$\begin{aligned}
g_{\mathbb{R}^3}|_{TS} &= (x_u^1\,\mathrm{d}u + x_v^1\,\mathrm{d}v)^2 + (x_u^2\,\mathrm{d}u + x_v^2\,\mathrm{d}v)^2 + (x_u^3\,\mathrm{d}u + x_v^3\,\mathrm{d}v)^2 \\
&= \left((x_u^1)^2 + (x_u^2)^2 + (x_u^3)^2\right)\mathrm{d}u^2 + 2(x_u^1 x_v^1 + x_u^2 + x_v^2 + x_u^3 x_v^3)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \left((x_v^1)^2 + (x_v^2)^2 + (x_v^3)^2\right)\mathrm{d}v^2 \\
&= \langle \varphi_u, \varphi_u\rangle\,\mathrm{d}u^2 + 2\,\langle \varphi_u, \varphi_v\rangle\,\mathrm{d}u\,\mathrm{d}v + \langle \varphi_v, \varphi_v\rangle\,\mathrm{d}v^2 \\
&= E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 .
\end{aligned}$$

Therefore, we see the 1st fundamental form is just the restriction of Euclidean inner product of $\mathbb{R}^3$ on TS.

<u>Recall:</u> $\varphi\colon U \subset \mathbb{R}^2 \to S, p \in S$, then

$$\mathrm{d}\varphi_p : T_{(u_0,v_0)}U \to T_pS, \quad T_{(u_0,v_0)}U \cong \mathbb{R}^2 = \mathrm{span}\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}.$$

Then

$$\mathrm{d}\varphi_p\,(\frac{\partial}{\partial u}) = \varphi_u, \quad \mathrm{d}\varphi_p\,(\frac{\partial}{\partial v}) = \varphi_v.$$

We get

$$\begin{aligned}
I_p(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) &= E = g_{\mathbb{R}^3}(\varphi_u, \varphi_u) = g_{\mathbb{R}^3}(\mathrm{d}\varphi_p\,(\frac{\partial}{\partial u}), \mathrm{d}\varphi_p\,(\frac{\partial}{\partial u})) \\
I_p(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) &= E = g_{\mathbb{R}^3}(\varphi_u, \varphi_v) = g_{\mathbb{R}^3}(\mathrm{d}\varphi_p\,(\frac{\partial}{\partial u}), \mathrm{d}\varphi_p\,(\frac{\partial}{\partial v})) \\
I_p(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) &= E = g_{\mathbb{R}^3}(\varphi_v, \varphi_v) = g_{\mathbb{R}^3}(\mathrm{d}\varphi_p\,(\frac{\partial}{\partial v}), \mathrm{d}\varphi_p\,(\frac{\partial}{\partial v})).
\end{aligned}$$

Therefore, the 1st fundamental form is a positive definite bilinear form defined on $T_pS \cong T_{(u_0,v_0)}U$.

Remark. In Riemannian Geometry, the generalization of above defines a "pull-back" metric $\varphi^* g$ by $\varphi\colon M \to N$.

Exercise. *Consider a vector $v \in T_pS$, compute $I_p(v, v)$.*

Remark. Since $I_p = (\mathrm{d}u\,, \mathrm{d}v)\begin{pmatrix} E & F \\ F & E \end{pmatrix}\begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}$, we might also refer $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ as the 1st fundamental form.

Theorem 2.6.1. *I is independent of local parametrization.*

Proof. Let $\varphi(u, v)$ and $\psi(\alpha, \beta)$ be two local parametrizations near p.

$$\begin{cases} u = u(\alpha, \beta) \\ v = v(\alpha, \beta) \end{cases} \quad \text{has inverse} \quad \begin{cases} \alpha = \alpha(u, v) \\ \beta = \beta(u, v) \end{cases} .$$

Then

$$\begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} = \begin{pmatrix} u_\alpha & u_\beta \\ v_\alpha & v_\beta \end{pmatrix}\begin{pmatrix} \mathrm{d}\alpha \\ \mathrm{d}\beta \end{pmatrix} = \frac{\partial(u, v)}{\partial(\alpha, \beta)}\begin{pmatrix} \mathrm{d}\alpha \\ \mathrm{d}\beta \end{pmatrix}.$$

$$\begin{pmatrix} \langle\varphi_u,\varphi_v\rangle & \langle\varphi_u,\varphi_v\rangle \\ \langle\varphi_v,\varphi_u\rangle & \langle\varphi_v,\varphi_v\rangle \end{pmatrix} = \begin{pmatrix} \alpha_u & \beta_u \\ \alpha_v & \beta_v \end{pmatrix} \begin{pmatrix} \langle\psi_\alpha,\psi_\beta\rangle & \langle\psi_\alpha,\psi_\beta\rangle \\ \langle\psi_\beta,\psi_\alpha\rangle & \langle\psi_\beta,\psi_\beta\rangle \end{pmatrix} \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}.$$

This gives

$$\begin{aligned} &(\mathrm{d}u\,,\mathrm{d}v) \begin{pmatrix} \langle\varphi_u,\varphi_v\rangle & \langle\varphi_u,\varphi_v\rangle \\ \langle\varphi_v,\varphi_u\rangle & \langle\varphi_v,\varphi_v\rangle \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} \\ =&(\mathrm{d}\alpha\,,\mathrm{d}\beta) \underbrace{\begin{pmatrix} u_\alpha & u_\beta \\ v_\alpha & v_\beta \end{pmatrix} \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}}_{\mathrm{Id}} \begin{pmatrix} \langle\psi_\alpha,\psi_\beta\rangle & \langle\psi_\alpha,\psi_\beta\rangle \\ \langle\psi_\beta,\psi_\alpha\rangle & \langle\psi_\beta,\psi_\beta\rangle \end{pmatrix} \underbrace{\begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix} \begin{pmatrix} u_\alpha & u_\beta \\ v_\alpha & v_\beta \end{pmatrix}}_{\mathrm{Id}} \\ =&(\mathrm{d}\alpha\,,\mathrm{d}\beta) \begin{pmatrix} \langle\psi_\alpha,\psi_\beta\rangle & \langle\psi_\alpha,\psi_\beta\rangle \\ \langle\psi_\beta,\psi_\alpha\rangle & \langle\psi_\beta,\psi_\beta\rangle \end{pmatrix} \begin{pmatrix} \mathrm{d}\alpha \\ \mathrm{d}\beta \end{pmatrix}. \end{aligned}$$

□

Remark. By this theorem, on a regular surface, to compute the 1st fundamental form, it suffices to work it out in a coordinate chart. As you will see, in this course and later in Riemannian Geometry course, there will be a lot of "local computation".

Example 2.6.2 (Plane). *$P \subset \mathbb{R}^3$ a plane through $p_0 = (x_0, y_0, z_0)$, consisting two orthonormal vectors w_1 and w_2, viewed as vectors in $\mathbb{R}^3$. Then*

$$\varphi(u,v) = p_0 + uw_1 + vw_2.$$

$\Longrightarrow$

$$I = \mathrm{d}u^2 + \mathrm{d}v^2.$$

Let v be a vector in P, $v = a\varphi_u + b\varphi = aw_1 + bw_2$. Then

$$I(v,v) = (a,b) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^2 + b^2 = |v|^2.$$

Example 2.6.3 (Cylinder). $\varphi(u,v) = (\cos u, \sin u, v)$, $\{0 < u < 2\pi, v \in \mathbb{R}\}$.
Direct calculate shows

$$I = \mathrm{d}u^2 + \mathrm{d}v^2.$$

Question. It looks both the plane and cylinder have the same 1st fundamental form. This means a vector at a point $p \in P$ will have the same length when we "move" the point p and vector onto the cylinder (Think about the cylinder as gluing two edges of a paper). On the other hand, cylinder is clearly not the plane. What can you conclude?

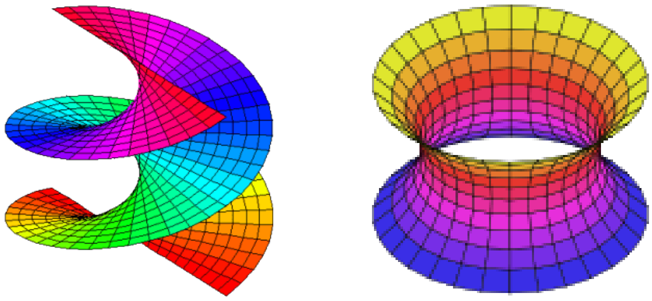
(The difference lies in "topology". There is non-shrinking circles on the cylinder. The plane and cylinder locally look the "same" (with the same measurement). But globally they are different).

Example 2.6.4. $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$, *compute the 1st fundamental form in three parametrizations introduced before. (Left as exercise)*

Example 2.6.5. *Helicoid and catenoid (from homework)*

In your homework you have found local parametrizations of helicoid and catenoid

$$\begin{aligned} H(u,v) &= (v\cos u, v\sin u, au) \\ C(\varphi,\theta) &= (a\cosh\theta\cos\varphi, a\cosh\theta\sin\varphi, a\theta)\,. \end{aligned}$$



(a) Helicoid (b) Catenoid

(1) Compute the 1st fundamental form of them.

$$I_H = \left(v^2 + a^2\right) \mathrm{d}u^2 + \mathrm{d}v^2$$

$$I_C = \left(a^2 \cosh^2 \theta\right) \mathrm{d}\varphi^2 + \left(a^2 \cosh^2 \theta\right) \mathrm{d}\theta^2$$

(2) Show that there is a parametrization on the helicoid, $\tilde{H}\left(\tilde{u}, \tilde{v}\right)$, such that

$$I_{\tilde{H}} = \left(a^2 \cosh^2 \tilde{v}\right) \mathrm{d}\tilde{u}^2 + \left(a^2 \cosh^2 \tilde{v}^2\right) \mathrm{d}\tilde{v}^2.$$

($u = \tilde{u}, v = a \sinh \tilde{v}$)

Remark. In both example 1 and example 3, we have seen that near a point, the two surfaces considered have the same 1st fundamental form (after a change of parametrization). Such property is called “local isometry”. We’ll make this definition more clear later.

•Application of the 1st fundamental form

(1) Arclength of a curve on S.

Note that for a vector $v \in \mathbb{R}^2$, $I(v, v) = |v|^2$. Let $\alpha(t)\colon (0, t) \to S$ be a curve in S and $\varphi\colon U \to S$, $(u, v) \mapsto \varphi(u, v)$ be a local parametrization satisfying $\alpha(t) \in \varphi(U)$.

$$\implies \alpha(t) = \varphi\left(u(t), v(t)\right).$$

$$\implies \alpha'(t) = \varphi_u u'(t) + \varphi_v v'(t).$$

$$\implies \left|\alpha'(t)\right|^2 = I\left(\alpha'(t), \alpha'(t)\right) = E u_t^2 + 2F u_t v_t + G v_t^2.$$

The arclength of $\alpha(t)$ is defined by

$$s(t) = \int_0^t \left|\alpha'(t)\right| \mathrm{d}t = \int_0^t \sqrt{E u_t^2 + 2F u_t v_t + G v_t^2} \mathrm{d}t.$$

$$\implies \mathrm{d}s = \sqrt{E u_t^2 + 2F u_t v_t + G v_t^2} \mathrm{d}t.$$

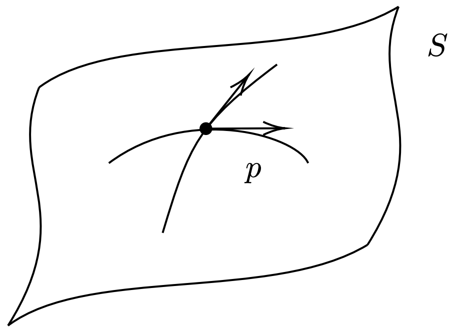
$$\implies \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = E u_t^2 + 2F u_t v_t + G v_t^2.$$

$$\implies \mathrm{d}s^2 = E \mathrm{d}u^2 + 2F \mathrm{d}u \mathrm{d}v + G \mathrm{d}v^2.$$

This explains the “geometric meaning” of I, *i.e.* it measures the infinitesimal arclength.

(2) Angle between two curves intersecting at t_0.

Let $\alpha\colon I \to S$, $\beta\colon I \to S$ be two curves on S, $\alpha(t_0) = \beta(\bar{t}_0) = p \in S$.



Define

$$\cos\theta = \frac{\langle \alpha'(t_0), \beta'(\bar{t}_0)\rangle_{\mathbb{R}^3}}{|\alpha'(t_0)|\,|\beta'(\bar{t}_0)|}.$$

Question. Given a parametrization $\varphi(u,v)$, we have two coordinate curve $\varphi(u,v_0)$, $\varphi(u_0,v)$. What's the angle between them?

$$u\text{-curve: } \alpha(t) = \varphi\left(u(t), c\right) \implies \alpha'(t) = \varphi_u u'.$$

$$v\text{-curve: } \beta(t) = \varphi\left(c, v(t)\right) \implies \beta'(t) = \varphi_v v'.$$

$$\implies \cos\theta = \frac{\langle \varphi_u u', \varphi_v v'\rangle}{|\varphi_u u'|\,|\varphi_v v'|} = \pm\frac{\langle \varphi_u, \varphi_v\rangle_{\mathbb{R}^3}}{|\varphi_u|\,|\varphi_v|} = \pm\frac{F}{\sqrt{EG}}.$$

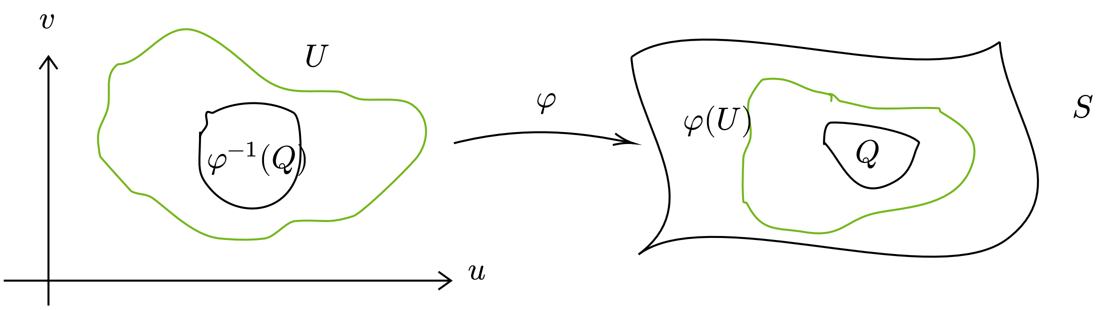
In particular, we conclude that

$$F = 0 \Leftrightarrow \text{Two coordinate curves are orthogonal},$$

and such parametrization is called an orthogonal parametrization. (e.g. For $\mathbb{S}^2$, (θ,φ) and the stereographic projection are two such parametrizations)

(3) (Surface) Area.

Let S be a regular surface. Choose a parametrization $\varphi\colon U \to S$. Let $Q \subset S$ be a bounded domain. Assume $Q \subset \varphi(U)$, so $\varphi^{-1}(Q) \subset U$ is a bounded set in $\mathbb{R}^2$.



Let's assume the boundary of Q is a differentiable curve with singularities lying in a measure zero set.

Definition 2.6.6 (Area of Q)**.**

$$Area(Q) = \iint_{\varphi(Q)} |\varphi_u \wedge \varphi_v| \, \mathrm{d}u\mathrm{d}v \quad \text{(double integral in } \mathbb{R}^2\text{)}.$$

Here, we give the definition in terms of a "parametrization". However, the area of Q is a number only depending on Q itself. We should check that our definition does not depend on the parametrization.

Claim: $Area(Q)$ defined above is independent of the choice of parametrizations.

Proof. (Left as exercise)

Let $\psi(\alpha, \beta) \colon V \to S$ be another parametrization. Let $H = \psi^{-1} \circ \varphi$ be the change of parametrization, $H(u, v) = (\alpha, \beta) \implies \varphi = \psi \circ H$. We compute $\iint_{\varphi(Q)} |\varphi_u \wedge \varphi_v| \, \mathrm{d}u\mathrm{d}v$.

By chain rule

$$\begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} = \frac{\partial(\alpha, \beta)}{\partial(u, v)} \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix}.$$

$$\begin{aligned} \implies \varphi_u \wedge \varphi_v &= (\alpha_u \beta_v)\, \psi_\alpha \wedge \psi_\beta + (\alpha_v \beta_u)\, \psi_\beta \wedge \psi_\alpha \\ &= (\alpha_u \beta_v - \alpha_v \beta_u)\, \psi_\alpha \wedge \psi_\beta \\ &= \det\left(\frac{\partial(\alpha, \beta)}{\partial(u, v)}\right) \psi_\alpha \wedge \psi_\beta. \end{aligned}$$

$$\implies |\varphi_u \wedge \varphi_v| = \left|\det\left(\frac{\partial(\alpha, \beta)}{\partial(u, v)}\right)\right| |\psi_\alpha \wedge \psi_\beta|.$$

On the other hand,

$$\begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} = \frac{\partial(u, v)}{\partial(\alpha, \beta)} \begin{pmatrix} \mathrm{d}\alpha \\ \mathrm{d}\beta \end{pmatrix}.$$

Thus, the change of infinitesimal area element is

$$\mathrm{d}u\mathrm{d}v = |\mathrm{d}u \wedge \mathrm{d}v| = \left|\det\left(\frac{\partial(u, v)}{\partial(\alpha, \beta)}\right)\right| |\mathrm{d}\alpha \wedge \mathrm{d}\beta| = \left|\det\left(\frac{\partial(u, v)}{\partial(\alpha, \beta)}\right)\right| \mathrm{d}\alpha\mathrm{d}\beta.$$

$$\implies |\varphi_u \wedge \varphi_v| \, \mathrm{d}u\mathrm{d}v = |\psi_\alpha \wedge \psi_\beta| \, \mathrm{d}\alpha\mathrm{d}\beta.$$

$$\implies \iint |\varphi_u \wedge \varphi_v| \, \mathrm{d}u\mathrm{d}v = \iint |\psi_\alpha \wedge \psi_\beta| \, \mathrm{d}\alpha\mathrm{d}\beta.$$

□

Remark. By the 1st fundamental form

$$I = E(\mathrm{d}u)^2 + 2F\mathrm{d}u\mathrm{d}v + G(\mathrm{d}v)^2$$

$$\implies Area = \iint |\varphi_u \wedge \varphi_v| \, \mathrm{d}u\mathrm{d}v = \iint \sqrt{EG - F^2} \mathrm{d}u\mathrm{d}v.$$

2.7 Gauss maps and the 2^{nd} fundamental form

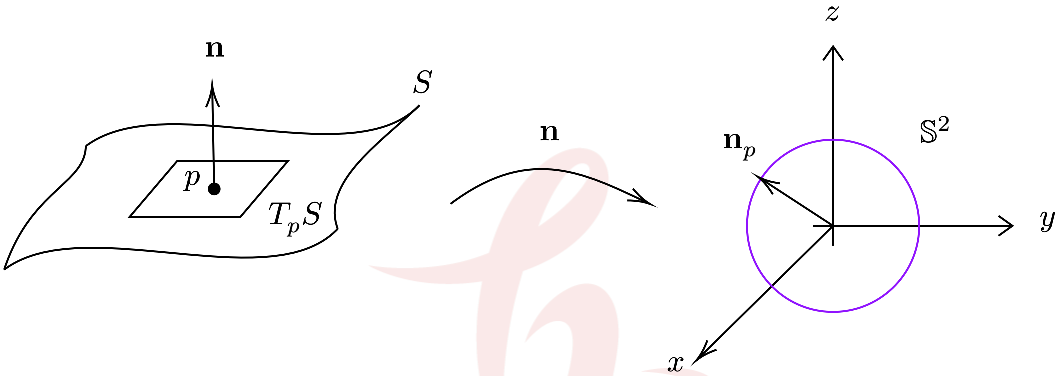
- Gauss maps.

Question. How is a regular surface curved in $\mathbb{R}^3$?

Recall: $S \subset \mathbb{R}^3$ is an oriented regular surface. $\implies$ We can choose a unit vector field

$$\begin{aligned} \mathbf{n}\colon S &\to \mathbb{S}^2(1) \\ p &\mapsto \mathbf{n}_p, \end{aligned}$$

is well-defined everywhere on S. Moreover, $\mathbf{n}$ is a differentiable map with its image lying on the unit sphere, and we called the map to be the Gauss map. $\mathbf{n}$ determines an orientation of S.



Let $\varphi\colon U \to S$ be a local parametrization near $p \in S$, then

$$\mathbf{n} = \frac{\varphi_u \wedge \varphi_v}{|\varphi_u \wedge \varphi_v|}.$$

Let's compute the differential of the Gauss map at p.

$$\mathrm{d}\mathbf{n}_p\colon T_pS \to T_{\mathbf{n}_p}\mathbb{S}^2.$$

$\forall v \in T_pS$, let $\alpha(s)$ be the curve on S such that $\alpha(0) = p$, $\alpha'(0) = v$.

$$\implies \mathrm{d}\mathbf{n}_p(v) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathbf{n}\left(\alpha(s)\right) \text{(changing rate of the Gauss map at } p \text{ along direction } v).$$

Note $\langle \mathbf{n}\left(\alpha(s)\right), \mathbf{n}\left(\alpha(s)\right)\rangle = 1$, taking derivation at $s = 0$:

$$\left\langle \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathbf{n}\left(\alpha(s)\right), \mathbf{n}_p \right\rangle = 0.$$

$$\implies \mathrm{d}\mathbf{n}_p(v) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathbf{n}\left(\alpha(s)\right) \in T_pS.$$

Definition 2.7.1 (The 2^{nd} fundamental form).

- $\forall v \in T_pS$, the 2nd fundamental form

$$II_p(v,v) = -\left\langle \mathbf{dn}_p(v), v \right\rangle_{\mathbb{R}^3}\text{ [1]}.$$

- More generally $\forall v, w \in T_pS$,

$$II_p \colon T_pS \times T_pS \to T_p\mathbb{R}$$
$$(v,w) \mapsto II_p(v,w) = -\left\langle \mathbf{dn}_p(v), w \right\rangle_{\mathbb{R}^3}.$$

$-\mathbf{dn}_p$ is also called the shape operator.

Before we explore II_p, let's compute the Gauss map's differential.
• Let $\varphi(u,v)$ be a local parametrization. Any curve on S has parametrization

$$\alpha(t) = \varphi(u(t), v(t)) = (x(u(t),v(t)), y(u(t),v(t)), z(u(t),v(t))).$$
$$\implies \alpha'(0) = \varphi_u u'(0) + \varphi_v v'(0).$$

$$\begin{aligned}
\mathbf{dn}_p(\alpha'(0)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{n}(\alpha(t)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{n}(x(u(t),v(t)), y(u(t),v(t)), z(u(t),v(t))) \\
&= \left(\mathbf{n}_x x_u + \mathbf{n}_y y_u + \mathbf{n}_z z_u\right) u'(0) + \left(\mathbf{n}_x x_v + \mathbf{n}_y y_v + \mathbf{n}_z z_v\right) v'(0) \\
&= \mathbf{n}_u u'(0) + \mathbf{n}_v v'(0).
\end{aligned}$$

On the other hand, by the linearity of $\mathbf{dn}_p$,

$$\mathbf{dn}_p(\alpha'(0)) = u'(0)\mathbf{dn}_p(\varphi_u) + v'(0)\mathbf{dn}_p(\varphi_v).$$

$$\implies \begin{cases} \mathbf{dn}_p(\varphi_u) = \mathbf{n}_u, \\ \mathbf{dn}_p(\varphi_v) = \mathbf{n}_v. \end{cases}$$

$$\begin{aligned}
\left\langle \mathbf{dn}_p(\varphi_u), \varphi_u \right\rangle &= \langle \mathbf{n}_u, \varphi_u \rangle = \frac{\partial}{\partial u} \underbrace{\langle \mathbf{n}, \varphi_u \rangle}_{=0} - \langle \mathbf{n}, \varphi_{uu} \rangle \\
\left\langle \mathbf{dn}_p(\varphi_u), \varphi_v \right\rangle &= \langle \mathbf{n}_u, \varphi_v \rangle \\
&= \frac{\partial}{\partial u} \underbrace{\langle \mathbf{n}, \varphi_v \rangle}_{=0} - \langle \mathbf{n}, \varphi_{vu} \rangle \\
&= -\langle \mathbf{n}, \varphi_{vu} \rangle \\
\left\langle \mathbf{dn}_p(\varphi_v), \varphi_u \right\rangle &= \langle \mathbf{n}_v, \varphi_u \rangle \\
&= -\langle \mathbf{n}, \varphi_{uv} \rangle \text{ (Note that } \varphi_{uv} = \varphi_{vu} \text{ since } \varphi \text{ is smooth)} \\
\left\langle \mathbf{dn}_p(\varphi_v), \varphi_v \right\rangle &= \langle \mathbf{n}_v, \varphi_v \rangle = -\langle \mathbf{n}, \varphi_{vv} \rangle.
\end{aligned}$$

Hence, we conclude that:

[1] Thus, $\mathbf{dn}_p \colon T_pS \to T_pS$ is a linear map, which is the directional derivation of $\mathbf{n}$ along a tangent direction of S.

Theorem 2.7.2. *$II_p(v,w) = II_p(w,v)$, i.e. II_p is symmetric in v, w, and II_p is a bilinear form.*

Remark.

(1) From the computation above, we see that $\mathrm{d}\mathbf{n}_p$ is self-adjoint, *i.e.* $\left\langle \mathrm{d}\mathbf{n}_p(v), w\right\rangle = \left\langle v, \mathrm{d}\mathbf{n}_p(w)\right\rangle$.

(2) The 2nd fundamental form can be also defined as

$$II_p(v,v) = \left\langle \mathbf{n}_p, \alpha''(0)\right\rangle,$$

where α is a curve with $\alpha'(0) = v$.

Exercise. *Check that this definition coincides with the previous one.*

Proof. Along the curve $\alpha(t) \in S$, $\langle \mathbf{n}(\alpha(t)), \alpha'(t)\rangle = 0$. Therefore,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{n}(\alpha(t)), \alpha'(t)\rangle \\ &= \left\langle \mathrm{d}\mathbf{n}_{\alpha(t)}(\alpha'(t), \alpha'(t))\right\rangle + \langle \mathbf{n}(\alpha(t)), \alpha''(t)\rangle. \end{aligned}$$

□

Just as the 1st fundamental form, we write II as

$$II_p = e\mathrm{d}u^2 + 2f\mathrm{d}u\mathrm{d}v + g\mathrm{d}v^2,$$

where

$$\begin{aligned} e &= -\left\langle \mathrm{d}\mathbf{n}_p(\varphi_u), \varphi_u\right\rangle = -\langle \mathbf{n}_u, \varphi_u\rangle = \langle \mathbf{n}, \varphi_{uu}\rangle \\ f &= -\left\langle \mathrm{d}\mathbf{n}_p(\varphi_u), \varphi_v\right\rangle = -\langle \mathbf{n}_u, \varphi_v\rangle = \langle \mathbf{n}, \varphi_{uv}\rangle \\ (&= -\left\langle \mathrm{d}\mathbf{n}_p(\varphi_v), \varphi_u\right\rangle = -\langle \mathbf{n}_v, \varphi_u\rangle = \langle \mathbf{n}, \varphi_{vu}\rangle) \\ g &= -\left\langle \mathrm{d}\mathbf{n}_p(\varphi_v), \varphi_v\right\rangle = -\langle \mathbf{n}_v, \varphi_v\rangle = \langle \mathbf{n}, \varphi_{vv}\rangle \end{aligned}$$

• Weingarten equations (linear representation of $\mathrm{d}\mathbf{n}$ in $\{\varphi_u, \varphi_v\}$)

We have seen $\mathrm{d}\mathbf{n}_p : T_pS \to T_{\mathbf{n}_p}\mathbb{S}^2$ has image actually lying in $T_pS = \operatorname{span}\{\varphi_u, \varphi_v\}$ in terms of a local parametrization $\varphi(u,v)$.

$$\Longrightarrow \begin{cases} \mathrm{d}\mathbf{n}_p(\varphi_u) = a_{11}\varphi_u + a_{12}\varphi_v \\ \mathrm{d}\mathbf{n}_p(\varphi_v) = a_{21}\varphi_u + a_{22}\varphi_v \end{cases} \quad i.e. \quad \begin{cases} \mathbf{n}_u = a_{11}\varphi_u + a_{12}\varphi_v \\ \mathbf{n}_v = a_{21}\varphi_u + a_{22}\varphi_v \end{cases}$$

We would like to find out the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Recall that

$$I_p = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2,$$

where

$$E = \langle \varphi_u, \varphi_u\rangle, F = \langle \varphi_u, \varphi_v\rangle, G = \langle \varphi_v, \varphi_v\rangle.$$

Now we consider the matrix

$$\begin{pmatrix} \mathbf{n}_u \\ \mathbf{n}_v \end{pmatrix} \begin{pmatrix} \varphi_u & \varphi_v \end{pmatrix} = \begin{pmatrix} \langle \mathbf{n}_u, \varphi_u \rangle & \langle \mathbf{n}_u, \varphi_v \rangle \\ \langle \mathbf{n}_v, \varphi_u \rangle & \langle \mathbf{n}_v, \varphi_v \rangle \end{pmatrix}.$$

On the one hand, the R.H.S. is

$$\begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \text{R.H.S.} &= \begin{pmatrix} a_{11}E + a_{12}F & a_{11}F + a_{12}G \\ a_{21}E + a_{22}F & a_{21}F + a_{22}G \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}. \end{aligned}$$

$$\Longrightarrow -\underbrace{\begin{pmatrix} e & f \\ f & g \end{pmatrix}}_{II} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{I}.$$

$$\begin{aligned} \Longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \\ &= -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \frac{1}{\det} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= -\frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= \frac{1}{EG - F^2} \begin{pmatrix} fF - eG & eF - fE \\ gF - fG & fF - gE \end{pmatrix}. \end{aligned} \tag{$*$}$$

The Weingarten equation is $\mathrm{d}\mathbf{n}_p \sim \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, *i.e.*

$$\begin{pmatrix} \mathbf{n}_u \\ \mathbf{n}_v \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix},$$

with $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ defined in $(*)$.

! For a regular surface $S \subset \mathbb{R}^3$, if we know the local parametrization $\varphi(u, v)$ near a point p, then

$$I_p = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2,$$

and

$$II_p = e\mathrm{d}u^2 + 2f\mathrm{d}u\mathrm{d}v + g\mathrm{d}v^2,$$

are fully understood with

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \varphi_u, \varphi_u \rangle & \langle \varphi_u, \varphi_v \rangle \\ \langle \varphi_v, \varphi_u \rangle & \langle \varphi_v, \varphi_v \rangle \end{pmatrix},$$

and

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle \varphi_{uu}, \mathbf{n} \rangle & \langle \varphi_{uv}, \mathbf{n} \rangle \\ \langle \varphi_{vu}, \mathbf{n} \rangle & \langle \varphi_{vv}, \mathbf{n} \rangle \end{pmatrix}.$$

Example 2.7.3.

(1) Plane: $Ax + By + Cz + D = 0$

$$\mathbf{n} = \frac{(A, B, C)}{\sqrt{A^2 + B^2 + C^2}}$$

is a constant map. Thus, $\mathrm{d}\mathbf{n} = 0$ *and* $II(v, v) = -\langle \mathrm{d}\mathbf{n}(v), v \rangle = 0$.

(2) $\mathbb{S}^2(1) = \{x^2 + y^2 + z^2 = 1\}$. *At point* (x, y, z), *the unit normal vector field is* $\mathbf{n}_\pm = \pm(x, y, z)$, *where* $\mathbf{n}_+$ *means the outer normal vector, and* $\mathbf{n}_-$ *the inner normal vector.*

Let's consider $\mathbf{n}_- = -(x, y, z)$, *let* $\alpha(t) = (x(t), y(t), z(t))$ *be a curve on* $\mathbb{S}^2$, *then*

$$\mathrm{d}\mathbf{n}_- (\alpha'(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{n}_- (\alpha(t)) = -\frac{\mathrm{d}}{\mathrm{d}t}(x(t), y(t), z(t)) = -\alpha'(t).$$

$$\Longrightarrow II(\alpha'(t), \alpha'(t)) = -\langle \mathrm{d}\mathbf{n}_- (\alpha'(t)) \rangle = \langle \alpha'(t), \alpha'(t) \rangle = |\alpha'(t)|^2 \geq 0.$$

If we take $\mathbf{n}_+$, *then*

$$II(\alpha'(t), \alpha(t)) = -|\alpha'(t)|^2 \leq 0.$$

<u>Hence</u>, sign of the 2^{nd} *fundamental form depends on the choice of orientation (i.e. the unit normal).*

(3) Helicoid (in which every point looks like a saddle point).

$$H(u, v) = (v\cos u, v\sin u, au)$$

$$H_u = (-v\sin u, v\cos u, a), \quad H_v = (\cos u, \sin u, 0).$$

$$H_{uu} = (-v\cos u, -v\sin u, 0), H_{uv} = (-\sin u, \cos u, 0), H_{vv} = (0, 0, 0).$$

$$\mathbf{n} = \left(-\frac{a\sin u}{\sqrt{a^2 + v^2}}, \frac{a\cos u}{\sqrt{a^2 + v^2}}, -\frac{v}{\sqrt{a^2 + v^2}}\right).$$

$$II = \frac{2a}{\sqrt{a^2 + v^2}}\mathrm{d}u\mathrm{d}v,$$

i.e.

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} 0 & \frac{2a}{\sqrt{a^2+v^2}} \\ \frac{2a}{\sqrt{a^2+v^2}} & 0 \end{pmatrix}, \quad \lambda = \pm\frac{a}{\sqrt{a^2 + v^2}} \text{ indefinite.}$$

(4) Cylinder. $x^2+y^2=1$

$$c(\theta,v)=(\cos\theta,\sin\theta,v).$$

$$c_\theta=(-\sin\theta,\cos\theta,0),\quad c_v=(0,0,1).$$

$$\mathbf{n}_+=(\cos\theta,\sin\theta,0)(\textit{the outer normal}),$$

$$\mathbf{n}_-=(-\cos\theta,-\sin\theta,0)(\textit{the inner normal}).$$

$$c_{\theta\theta}=(-\cos\theta,-\sin\theta,0),\quad c_{\theta v}=(0,0,0),\quad c_{vv}=(0,0,0).$$

$$\implies II_{\mathbf{n}_-}=d\theta^2\sim\begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

Remark. The cylinder and the plane have the same 1^{st} fundamental form, but different 2^{nd} fundamental form.

2.8 Geometric meaning of the 2^{nd} fundamental form and curvatures

Recall:

- Gauss map
$$\vec{N}\colon S\to\mathbb{S}^2(1),$$
$$p\mapsto\vec{N}_p.$$
- $d\vec{N}_p\colon T_pS\to T_{\vec{N}(p)}\mathbb{S}^2(1)\simeq T_pS.$
- $\forall v\in T_pS,\ II(v,v)=-\left\langle d\vec{N}_p(v),v\right\rangle=\left\langle\vec{N}_p,\alpha''(0)\right\rangle$, where α is a curve with $\alpha(0)=p$, $\alpha'(0)=v$.

(Note that at p along direction v, there are indefinitely many, curves passing through p with tangent vector v, each can be obtained by using a plane containing p,v to intersect with S). However, $II(v,v)$ only depends on $\alpha'(0)=v$.

Goal: Understanding how 2^{nd} fundamental form reflects the surface is curved locally near a point.

2.8.1 Normal curvature

Let $\alpha(s)$ be a regular curve parametrized by the arclength In S.

$$\implies \alpha'(s)=t(s),\quad |\alpha'(s)|=1$$

$$\alpha''(s)=t'(s)=k(s)\mathbf{n}(s),\quad k(s):\text{curvature of }\alpha(s),\quad \mathbf{n}:\text{unit normal vector of }\alpha(s)$$

$$\begin{aligned}\left\langle\alpha''(s),\vec{N}(\alpha(s))\right\rangle&=\frac{\mathrm{d}}{\mathrm{d}s}\langle\underbrace{\alpha'(s)}_{\in T_{\alpha(s)}S}\ \vec{N}(\alpha(s))\rangle-\left\langle\alpha'(s),\frac{\mathrm{d}}{\mathrm{d}s}\vec{N}(\alpha(s))\right\rangle\\&=-\left\langle\alpha'(s),d\vec{N}_{\alpha(s)}(\alpha'(s))\right\rangle\\&=II(\alpha'(s),\alpha'(s))\end{aligned}$$

On the other hand,

$$\left\langle \alpha''(s), \vec{N}(\alpha(s)) \right\rangle = \left\langle k(s)\mathbf{n}(s), \vec{N}(s) \right\rangle = k(s)\cos\theta.$$

This is the projection of "curvature" of $\alpha(s)$ on the normal vector of the surface. We call the value

$$k_{\mathbf{n}} = k(s)\cos\theta$$

to be the "normal curvature of curve $\alpha(s)$".

Hence, the 2nd fundamental form

$$I\!I\left(\alpha'(s), \alpha'(s)\right) = \text{normal curvature of } \alpha(s) = k_{\mathbf{n}}.$$

But $I\!I\left(\alpha'(s), \alpha'(s)\right)$ only depends on $\alpha'(s)$ but not a particular curve. Hence, at $p \in S$, all curves passing through p with the same unit tangent vector v have the same normal curvature. A canonical choice of such curve is called the "normal section" which is a curve obtained by intersecting the plane $\operatorname{span}\{\underbrace{v}_{unit}, N\}$ with S

$$\implies k_{\mathbf{n}} = \pm\text{curvature of normal section along direction } v.$$

$$\implies I\!I(v,v) = \pm\text{curvature of normal section along direction } v.$$

Exercise. *Compute $k_{\mathbf{n}}$ in a local parametrization.*

Let $\alpha(s) = \varphi\left(u(s), v(s)\right)$, $\alpha'(s) = \varphi_u u'(s) + \varphi_v v'(s)$, where s is the arclength parameter. Then,

$$\alpha''(s) = \varphi_{uu}\left(u'(s)\right)^2 + 2\varphi_{uv}u'(s)v'(s) + \varphi_{vv}\left(v'(s)\right)^2 + \underbrace{\varphi_u u''(s) + \varphi_v v''(s)}_{tangential}.$$

$$\begin{aligned}
&\left\langle \alpha''(s), \vec{N}\left(\alpha(s)\right) \right\rangle \\
=&\left\langle \varphi_{uu}, \vec{N} \right\rangle \left(u'(s)\right)^2 + 2\left\langle \varphi_{uv}, \vec{N} \right\rangle u'(s)v'(s) + \left\langle \varphi_{vv}, \vec{N} \right\rangle \left(v'(s)\right)^2 \\
=&e\left(u'(s)\right)^2 + 2fu'(s)v'(s) + g\left(v'(s)\right)^2.
\end{aligned}$$

Therefore,

$$k_{\mathbf{n}}\left(\alpha'(s)\right) = e\left(u'(s)\right)^2 + 2fu'(s)v'(s) + g\left(v'(s)\right)^2.$$

It's also interesting to obtain $k_{\mathbf{n}}$ in an arbitrary parametrization. Let $\alpha(\tau)$ be some parameter,

$$\implies S = (\tau)\int_0^\tau |\alpha'(\tau)|\,\mathrm{d}\tau, \quad \tau'(s) = \frac{1}{|\alpha'(\tau)|}.$$

$$\begin{aligned}
\alpha'(s) &= \alpha'(\tau)\tau'(s), \\
\alpha''(s) &= \alpha''(\tau)\left(\tau'(s)\right)^2 + \alpha'(\tau)\tau''(s).
\end{aligned}$$

$$\begin{aligned}
\left\langle \alpha''(s), \vec{N}\left(\alpha(\tau)\right) \right\rangle &= \left\langle \alpha''(\tau), \vec{N}\left(\alpha(\tau)\right) \right\rangle \left(\tau'(s)\right)^2 \\
&= \frac{I\!I\left(\alpha'(\tau), \alpha'(\tau)\right)}{I\left(\alpha'(\tau), \alpha'(\tau)\right)}.
\end{aligned}$$

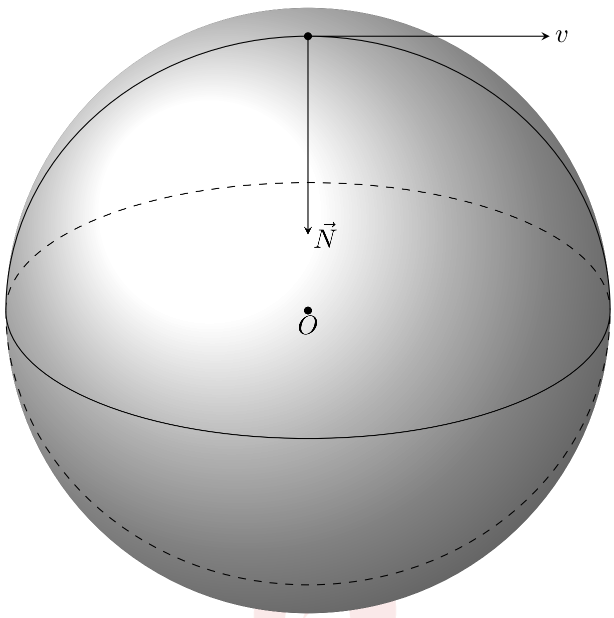
$$\therefore k_{\mathbf{n}}\left(\alpha'(\tau)\right) = \frac{I\!I\left(\alpha'(\tau), \alpha'(\tau)\right)}{I\left(\alpha'(\tau), \alpha'(\tau)\right)}.$$

! The normal curvature along direction v is $I\!I(v,v)$ tells us that how the surface is curved along direction v at p.

Example 2.8.1. *Sphere:* $x^2+y^2+z^2=1$.

Normal sections are the great circles of radius 1.

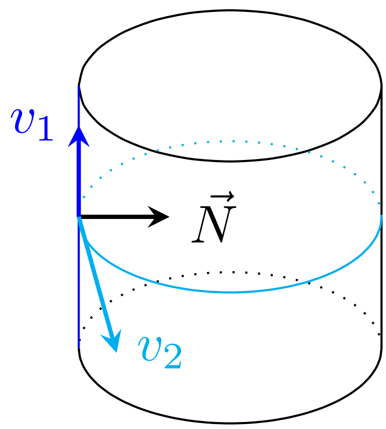
$$\therefore I\!I(v,v)=1, \quad \forall |v|=1.$$



Example 2.8.2. *Cylinder:* $x^2+y^2=1$.

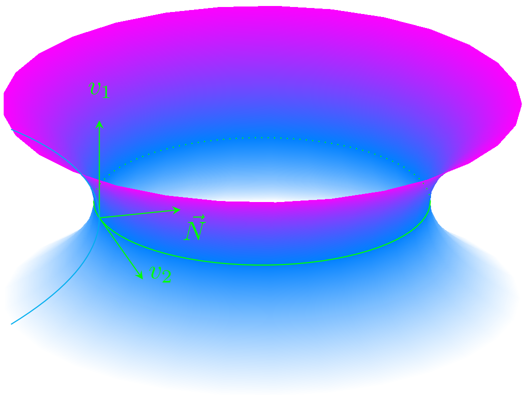
$$\vec{N}: \text{ inner normal.}$$

Let v_1 be the unit normal along vertical lines. $\implies$ normal section is just the line along v_1 $\implies$ $k_1=0$. Let v_2 be the unit normal parallel to xy plane. $\implies$ normal section is a horizontal circle. $\implies$ $k_2=1$ (that is the curvature of a circle). As we move the horizontal circle to the vertical line (*i.e.* $v_2\to v_1$), the normal curvature is decreasing.



Example 2.8.3. *Catenoid:* $\varphi(u,v)=\left(c\cosh\frac{v}{c}\cos u, c\cosh\frac{v}{c}\sin u, v\right)$

The normal section obtained by $S\cap\operatorname{span}\{v_1,\vec{N}\}$ is a catenary, with $k_1<0$. The normal section obtained by $S\cap\operatorname{span}\{v_2,\vec{N}\}$ is a circle, with $k_2>0$.



As you may notice, in example 2.8.1, all normal curvatures are the same at all points and along any direction. In example 2.8.2 and example 2.8.3, at any point there are extremal directions at which the normal curvature achieves the maximum and the minimum. Let's discuss more on these two special normal curvatures.

2.8.2 Principle curvature and principle direction

Since $\mathrm{d}N_p : T_pS \to T_pS$ is linear and symmetric (self-adjoint), for any orthonormal basis e_1, e_2,

$$\mathrm{d}N_p \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{symmetric} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$\implies \exists$ an orthonormal basis $\tilde{e_1}, \tilde{e_2}$ of T_pS such that

$$\mathrm{d}N_p \begin{pmatrix} \tilde{e_1} \\ \tilde{e_2} \end{pmatrix} = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix} \begin{pmatrix} \tilde{e_1} \\ \tilde{e_2} \end{pmatrix} \quad (k_1 \geq k_2)$$

Definition 2.8.4 (Principle curvature and principle direction)**.** k_1 and k_2 above are called the principle curvature and $\tilde{e_1}, \tilde{e_2}$ are called the principle directions at p.

Remark. k_1 and k_2 are the maximum and minimum values of the 2^{nd} fundamental form $I\!I$ restricted on the unit vectors of T_pS.

Proof.
$$I\!I\left(\tilde{e_1}, \tilde{e_1}\right) = -\left\langle \mathrm{d}N_p\left(\tilde{e_1}\right), \tilde{e_1}\right\rangle = \langle k_1\tilde{e_1}, \tilde{e_1}\rangle = k_1.$$

Similarly, $I\!I\left(\tilde{e_2}, \tilde{e_2}\right) = k_2$.

For any unit vector $v \in T_pS$, $v = v_1\tilde{e_1} + v_2\tilde{e_2}$, with $|v| = 1$.

$$\begin{aligned} I\!I(v, v) &= -\left\langle \mathrm{d}N_p(v), v\right\rangle \\ &= -\left\langle v_1\mathrm{d}N_p\left(\tilde{e_1}\right) + v_2\mathrm{d}N_p\left(\tilde{e_2}\right), v_1\tilde{e_1} + v_2\tilde{e_2}\right\rangle \\ &= k_1v_1^2 + k_2v_2^2 = \begin{cases} k_1 + (k_1 - k_2)v_2^2 \leq k_1, \\ (k_1 - k_2)v_1^2 + k_2 \geq k_2. \end{cases} \end{aligned}$$

Hence, k_1 and k_2 are maximum and minimum values of normal curvature. Moreover, $\forall$ unit vector $v \in T_pS$ and e_1, e_2 the principle direction, we can write

$$v = \cos\theta e_1 + \sin\theta e_2,$$

$$k_{\mathbf{n}}(v) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

The last is called "Euler formula". □

Definition 2.8.5 (Gaussian curvature & mean curvature)**.** Follow the definition of principal curvature, we define

- Gaussian curvature: $K = k_1 k_2$.
- Mean curvature: $H = \frac{k_1+k_2}{2}$.

Remark. (1) These two curvatures are very import ant in understanding the surface.

(2) Above definition is in terms of orthonormal basis $\{e_1, e_2\}$ on T_pS, at each $p \in S$.

$$\mathrm{d}N_p \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad K \sim \det, \ H \sim \text{trace}.$$

Exercise. *Find the expression of K and H in arbitrary parametrization. The answer is*

$$K = \frac{\det II}{\det I} = \frac{eg - f^2}{EG - F^2}, \qquad H = \frac{1}{2}\operatorname{tr}_I II = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

Once we know the Gaussian curvature K and mean curvature H, by the fact that $K = \det A, H = -\frac{1}{2}\operatorname{tr} A$, we can write the characteristic polynomial of A:

$$\det(\lambda I - A) = \lambda^2 - \operatorname{tr} A\lambda + \det A = \lambda^2 + 2H\lambda + K.$$

Since the principal curvature $k = -\lambda$, we have

$$k^2 - 2Hk + K = 0.$$

And principal curvatures are:

$$k = H \pm \sqrt{H^2 - K}.$$

Remark.

(1) The Gaussian curvature is an "intrinsic" curvature, it's only determined by the surface itself. The Gaussian elegant theorem tell us that the Gaussian curvature is only determined by 1st fundamental form. This already sheds light on the Riemannian Geometry. (We'll see the theorem later). The most beautiful result in surface theory is the Gauss-Bonnet's theorem: If S is an oriented compact surface without boundary, then

$$\int_S K = 2\pi\chi(S) = 2\pi(2 - 2g).$$

(2) The mean curvature is an "extrinsic curvature". It depends on the ambient space. (Here, our ambient space is $\mathbb{R}^3$). One of important problems in Differential Geometry is studying the surfaces with vanishing mean curvature. Such surface is called "minimal surface". (We will explain "minimal" later). This problem heavily depends on PDE theory (of 2nd order elliptic type).

Here, let's have a further understanding of principal curvature & principal direction from analysis point of view. We have seen:

(1) Normal curvature at $p \in S$:

$$\begin{aligned} k_n : \mathbb{S}^1(T_pS) &\longrightarrow \mathbb{R} \\ v &\longmapsto k_n(v). \end{aligned}$$

(2) Principal direction is at which k_n attains maximum / minimum.

(3) $\forall\, v$ not necessary a unit vector, then

$$k_n(v) = \frac{II_p(v,v)}{I_p(v,v)}.$$

Let $v = v_1\varphi_1 + v^2\varphi_2$, where $\varphi(u^1,u^2)$ is a local parametrization. Then

$$k_n(v) = \frac{(v^1)e^2 + 2v^1v^2f + (v^2)^2g}{(v^1)^2E + 2v^1v^2F + (v^2)^2G}.$$

WLOG assume $v^1 \neq 0$, $\lambda = \frac{v^2}{v^1}$, then

$$k_n(\lambda) = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2}.$$

(4) Since principal curvatures are critical values of k_n, $k_n'(\lambda) = 0$,

$$\iff (2f + 2g\lambda)(E + 2F\lambda + G\lambda^2) - (2F + 2G\lambda)(e + 2f\lambda + g\lambda^2) = 0. \tag{2.8.1}$$

i.e.

$$\frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{f + g\lambda}{F + G\lambda}.$$

Hence

$$k_n = \frac{f + g\lambda}{F + G\lambda}.$$

On the other hand, eq. (2.8.1) also implies

$$\begin{aligned} &(f + g\lambda)(E + F\lambda) + \lambda(f + g\lambda)(F + G\lambda) \\ =&(F + G\lambda)(e + f\lambda) + \lambda(F + G\lambda)(f + g\lambda). \end{aligned} \tag{2.8.2}$$

i.e.

$$\frac{f + g\lambda}{F + G\lambda} = \frac{e + f\lambda}{E + F\lambda}.$$

Hence

$$k_n = \frac{f + g\lambda}{F + G\lambda} = \frac{e + f\lambda}{E + F\lambda} \iff \text{principal curvature.}$$

Then
$$\begin{cases} f+g\lambda = k_n(F+G\lambda) \\ e+f\lambda = k_n(E+F\lambda) \end{cases} \implies \begin{cases} f-Fk_n = (Gk_n-g)\lambda \\ e-Ek_n = (Fk_n-f)\lambda \end{cases}.$$
This linear system has solution λ $\iff$
$$\det \begin{bmatrix} e-k_nE & f-k_nF \\ f-k_nF & g-k_nG \end{bmatrix} = 0.$$
This gives an equation to solve k_n. To see the principal direction, *i.e.* solving λ,
$$\begin{aligned} \text{eq. (2.8.2)} &\implies (gF-fG)\lambda^2 + (gE-eG)\lambda + (fE-eF) = 0 \\ &\iff \det \begin{bmatrix} \lambda^2 & -\lambda & 1 \\ E & F & G \\ e & f & g \end{bmatrix} = 0. \end{aligned}$$

Next, we would like to introduce several special points on a surface by using curvatures introduced previously.

Definition 2.8.6 (Classification of points on surface)**.**
A point p on a regular surface S is called a

(1) Elliptic point, if $K_p > 0$. (All normal section have the same normal vectors)

(2) Hyperbolic point, if $K_p < 0$. (There exist two normal sections with opposite normal vectors)

(3) Parabolic point if $K_p = 0$ but $\mathrm{d}N_p \neq 0$. (One of principal directions is "flat")

(4) Planar point, if $\mathrm{d}N_p = 0$, *i.e.* $k_1 = k_2 = 0$.

(5) Umbilical point, if $k_1 = k_2$.

Remark.

(1) Apparently, umbilical points can only occur at elliptic point or planar point.

(2) At umbilical points, $I\!I = kI$.

(3) On a minimal surface, $H = 0 \implies k_1 = -k_2$, so $K \leq 0$, and there is no elliptic point.

Definition 2.8.7. S is called totally umbilical, if all points are umbilical. For example, both plane and sphere are totally umbilical.

Theorem 2.8.8. *If $S \hookrightarrow \mathbb{R}^3$ is totally umbilical and connected. Then S is either contained in a plane or a sphere.*

Proof. Umbilical $\implies$ at each point, $k_1 = k_2 = k$, then $I\!I_p = kI_p$.
Let $\varphi(u,v)$ be a local parametrization near p. By Weingarten equation,
$$\begin{bmatrix} \mathbf{n}_u \\ \mathbf{n}_v \end{bmatrix} = A \begin{bmatrix} \varphi_U \\ \varphi_v \end{bmatrix} = \begin{bmatrix} -k & 0 \\ 0 & -k \end{bmatrix} \begin{bmatrix} \varphi_u \\ \varphi_v \end{bmatrix}, \quad A = I\!I \cdot I^{-1}.$$

Hence

$$\begin{cases} \mathbf{n}_u = -k\varphi_u \\ \mathbf{n}_v = -k\varphi_v \end{cases} \implies \begin{cases} \mathbf{n}_{uv} = -k_v\varphi_u - k\varphi_{uv} \\ \mathbf{n}_{vu} = -k_u\varphi_v - k\varphi_{vu} \end{cases}.$$

So we must have

$$k_v\varphi_u = k_u\varphi_v.$$

φ_u, φ_v linearly independent $\implies k_u = k_v = 0$. So k is constant on the chart.

Case 1: $k = 0$, then $\mathbf{n}_u = \mathbf{n}_v = 0$. Hence $\mathbf{n}$ is a constant vector field. Then

$$\langle \varphi(u, v), \mathbf{n} \rangle$$

is constant since it has zero differentials. This shows S lie in a plane locally.

Case 2: $k \neq 0$, wlog assume $k > 0$, otherwise we can reverse the orientation. Then

$$\begin{cases} \varphi_u = -\frac{1}{k}\mathbf{n}_u \\ \varphi_v = -\frac{1}{k}\mathbf{n}_v \end{cases}.$$

So $\varphi + \frac{1}{k}\mathbf{n}$ is constant. (Note φ is the position vector). Let $c = \varphi + \frac{1}{k}\mathbf{n}$. Then

$$|\varphi - c| = \frac{1}{k}.$$

This shows S lie in a sphere locally.

So far we proved the result only in a local coordinate chart. Since the surface is connected and smooth, one can easily use a topology (covering) arguement to extend above result globally. $\square$

Remark. So far, we can characterize $\mathbb{S}^2(R)$ in $\mathbb{R}^3$ in the following two ways geometrically:

(1) $\mathbb{S}^2(R) =$ the set of points having the same distance R to a fixed point c. *i.e.*

$$|x - c| = R.$$

(2) $\mathbb{S}^2(R) =$ all points are umbilical. *i.e.* $II = kI, k \neq 0$.

Both of them are very useful in proving some surface is sphere.

Remark. $H^2 - K = \frac{1}{4}(k_1 - k_2)^2 \geq 0$. It vanishes at p when $k_1 = k_2$, *i.e.* p umbilical. Together with Gauss-Bonnet thm, we have

$$\int_S H^2 \geq \int_S K = 2\pi(2 - 2g)$$

on a closed oriented surface, with equality holds iff S is sphere. In fact, we can get a stronger integral inequality.

Theorem 2.8.9 (Yau contest 2014). *For a closed (oriented) surface $S \subset \mathbb{R}^3$,*

$$\int_S H^2 \geq 4\pi.$$

Equality holds $\iff$ S is a sphere.

Proof. Recall the Gauss map $\mathbf{n}\colon S \to \mathbb{S}^2(1)$,

$$\mathrm{d}\mathbf{n}\begin{bmatrix}\varphi_u\\ \varphi_v\end{bmatrix} = A\begin{bmatrix}\varphi_u\\ \varphi_v\end{bmatrix}.$$

$$A = \mathit{II}\cdot I^{-1} \implies K = \det A.$$

For S closed surface, $\mathbf{n}$ is surjective to $\mathbb{S}^2(1)$. Hence

$$\int_S H^2 \geq \int_S |K| \geq \int_{\mathbb{S}^2} 1 = 4\pi.$$

(To see why $\mathbf{n}$ is surjective, notice for any direction vector v_0, there is a point $p \in S$ *s.t.* $\langle p, v_0\rangle$ maximal. For this point one can prove the normal vector coincident with v_0). □

Definition 2.8.10. $W(S) = \int_S H^2\,\mathrm{d}\sigma$ is called the Willmore energy of surface S.

Now we know $W(S) \geq 4\pi$, it's natural to ask what's the exact lower bounded of $W(S)$ for given topology.

Conjecture 2.8.11 (Willmore). *Given any smooth "immersed" torus T in $\mathbb{R}^3$,*

$$W(T) \geq 2\pi.$$

This conjecture is settled in the case of "embedded" by Marques & Neves (2014 Annals). They also showed the equality holds iff the torus is obtained by the stereographic projection of the Clifford torus which is an "embedded" torus in $\mathbb{R}^4$.

$$\frac{1}{\sqrt{2}}\mathbb{S}^1 \times \frac{1}{\sqrt{2}}\mathbb{S}^1 = \{\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq 2\pi\}.$$

Conjecture 2.8.12 (Lawson, proved by Brendle). *Clifford torus is only minimally embedded torus in $\mathbb{S}^3$. (i.e. as a minimal surface in metric of $\mathbb{S}^3$).*

Finally, we introduce some other geometric concepts in term of the 2nd fundamental form.

Definition 2.8.13 (Curvature lines). The curvature lines are integral curves of principal directions: $\gamma(s) \subset S$ *s.t.* $\gamma'(s)$ is principal direction. *i.e.*

$$\mathbf{n}'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{n}(\gamma(s)) = -\lambda(s)\gamma'(s).$$

Remark. At an umbilical point, any normal section provides a principal direction, there are infinity many of them. However, if a point is not umbilical, *i.e.* $k_1 \neq k_2$, then we can find a local parametrization near p, *s.t.* coordinate curves are curvature lines which are orthogonal to each other.

Exercise. *Let $\varphi(u,v)$ be a local parametrization, show that coordinate curves are curvature lines $\iff F = f = 0$.*

Let's derive an equation of curvature lines.

If $\alpha(t) = \varphi(u(t), v(t))$ is a curvature line, then

$$\mathbf{dn}\,(\alpha'(t)) = -\lambda(t)\alpha'(t), \quad \text{while } \alpha'(t) = \varphi_u u'(t) + \varphi_v v'(t).$$

Then

$$\mathbf{n}_u u' + \mathbf{n}_v v' = -(\lambda\varphi_u u' + \lambda\varphi_v v').$$

Product with φ_u and φ_v, we get

$$\begin{cases} eu' + fv' = \lambda E u' + \lambda F v' \\ fu' + gv' = \lambda F u' + \lambda G v' \end{cases}.$$

Hence

$$\begin{cases} \dfrac{fF - eG}{EG - F^2} u' + \dfrac{gF - fG}{EG - F^2} v' = \lambda u' \\ \dfrac{eF - fE}{EG - F^2} u' + \dfrac{fF - gE}{EG - F^2} v' = \lambda v' \end{cases} \qquad \left(\Longleftrightarrow \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} = \lambda \mathrm{Id} \right).$$

Eliminating λ, then

$$(fE - eF)(u')^2 + (gE - eG)u'v' + (gF - fG)(v')^2 = 0.$$

So $\alpha(t)$ is curvature line $\Longleftrightarrow$

$$\det \begin{bmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{bmatrix} = 0.$$

Example 2.8.14 (Catenoid)**.** *Parametrization:* $(a\cosh\theta\cos\varphi, a\cosh\theta\sin\varphi, a\theta)$.

$$I = (a^2\cosh^2\theta)\,\mathrm{d}\varphi^2 + (a^2\cosh^2\theta)\,\mathrm{d}\theta^2$$
$$II = -\,\mathrm{d}\varphi^2 + \mathrm{d}\theta^2\,.$$

$\varphi(u, v_0)$ *is curvature line* $\Longleftrightarrow$ $Ef - eF = 0$,
$\varphi(u_0, v)$ *is curvature line* $\Longleftrightarrow$ $Fg - fG = 0$.
u *curve* $\perp$ v *curve* $\Longrightarrow$ $F = 0$. *Also* $EG > F^2$, *so* $E, G \neq 0$, *then* $f = 0$.

Conversely if $F = f = 0$, *then for* u *curve, i.e.* $v' = 0$, *clearly we have*

$$\det \begin{bmatrix} 0 & 0 & (u')^2 \\ E & 0 & G \\ e & 0 & g \end{bmatrix} = 0.$$

Similarly for v *curve. i.e.* u, v *curves are curvature lines.*

Definition 2.8.15 (Conjugate directions)**.** Direction vectors $v, w \in T_pS$ are called conjugate if

$$II(v, w) = 0 \quad (\Longleftrightarrow \langle \mathbf{dn}_p\,, w\rangle = 0).$$

Example 2.8.16. *(1) Tow principal directions at a non-umbilical point are always conjugate, since for a real symmetric matrix, eigenvector assigned to different eigenvalues are perpendicular.*

(2) $p \in S$ is a non-planar umbilical point, then any two orthogonal directions are conjugate.

(3) $p \in S$ is a planar umbilical point, since $\mathrm{d}\mathbf{n} = 0$, each direction is conjugate to other directions.

Exercise. *Find a necessary and sufficient condition for two unit vectors are conjugate to each other.*

Definition 2.8.17 (Asymptotic direction and asymptotic curve)**.**
If $v \in T_pS$ *s.t.* $II(v,v) = 0$, then v is called asymptotic direction. The integral curve of asymptotic direction is called asymptotic curve.

Remark. (1) Asymptotic direction is conjugate to itself.

(2) $II(v,v) = 0 \implies k_1 \cos^2 \alpha + k_2 \sin^2 \alpha = 0$. So either $\tan^2 \alpha = -\frac{k_1}{k_2}$ or $\cot^2 \alpha = -\frac{k_2}{k_1}$. $\implies$ two values of α.

Exercise. *Let p be a hyperbolic point, $\varphi(u,v)$ is a local parametrization, then $\varphi(u,v_0)$ and $\varphi(u_0,v)$ are asymptotic curves $\iff$ $e = g = 0$.*

Definition 2.8.18 (Dupin indicatrix)**.**

$$D_p = \{v \in T_pS : II(v,v) = \pm 1\} \subset T_pS.$$

Remark. Write $v = v_1e_1 + v_2e_2 \in T_pS$, $\mathrm{d}\mathbf{n}_p(e_i) = -k_ie_i$, then

$$II(v,v) = \pm 1 \iff k_1v_1^2 + k_2v_2^2 = \pm 1.$$

(1) $K_p > 0$ ellipse in T_pS.

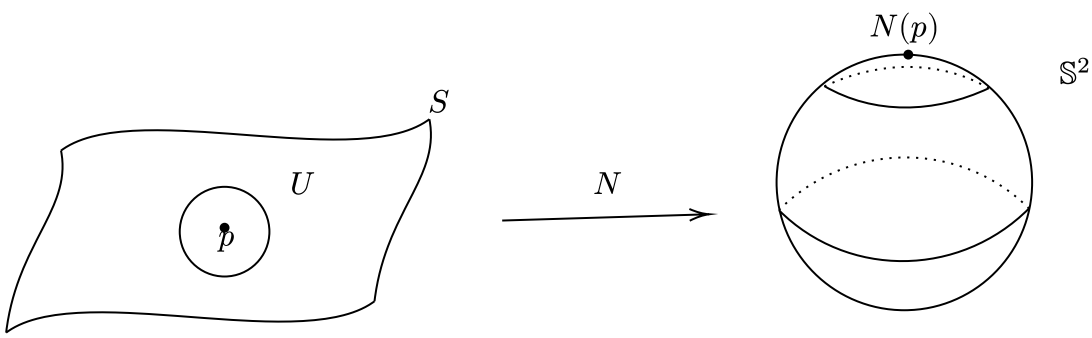
(2) $K_p < 0 \implies$ hyperbolas in T_pS.

(3) $K_p = 0 \implies$ crossing lines.

2.8.3 Geometric interpretation of Gauss curvature

Assume $K_p \neq 0$.

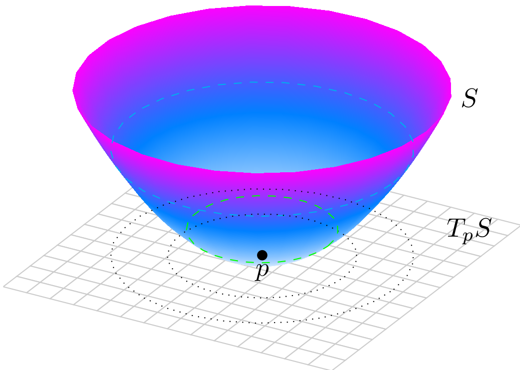
(a) $p \in S$, U is a neighborhood of p. $N\colon S \to \mathbb{S}^2$ is the Gauss map.



Let A= Area of U, δA= Area of $N(U)$, then

$$|K(p)| = \lim_{A\to 0}\frac{\bar{A}}{A}. \tag{1}$$

(b) $p \in T_pS$=tangent plane $\simeq \mathbb{R}^2$. Consider a circle of radius r in T_pS and a "circle" of radius r on S.



Then

$$K(p) = \lim_{r\to 0} 3\frac{2\pi r - c(r)}{\pi r^3}, \tag{2}$$

where $c(r)$= circumference of the circle of radius r on S.

(c) Consider a disk around p of radius r in T_pS. Also consider a "disk" around p of radius r on S.

$$K(p) = \lim_{r\to 0} 12\frac{\pi r^2 - A(r)}{\pi r^4}, \tag{3}$$

where $A(r)$= Area of disk on S.

Remark. From expression of eq. (2) and eq. (3), one can prove them by considering the Taylor's expansion of $c(r)$ and $A(r)$. The proof of these two facts will be postponed after we step into intrinsic geometry.(Need Taylor's expansion of metric tensor)

Proof. Let $F(u, v)$ be the local parametrization on U.

$$\Rightarrow Area(U) = \iint_{F^{-1}(U)} |F_u \wedge F_v| \,\mathrm{d}u\mathrm{d}v,$$

then $T_{N(p)}N(U) = \mathrm{span}\{N_u, N_v\}$,

$$\bar{A} = \iint_{F^{-1}(U)} |N_u \wedge N_v| \,\mathrm{d}u\mathrm{d}v.$$

<u>Recall</u>: $dN\begin{pmatrix}F_u\\F_v\end{pmatrix} = \begin{pmatrix}N_u\\N_v\end{pmatrix} = A\begin{pmatrix}F_u\\F_v\end{pmatrix}$, where A is the Weingarten matrix.

$$\Rightarrow |N_u \wedge N_v| = |\det A|\,|F_u \wedge F_v| = |K|\,|F_u \wedge F_v|.$$

$$\frac{\bar{A}}{A} = \frac{\iint |K| \left|F_u \wedge F_v\right| \mathrm{d}u\mathrm{d}v}{\iint \left|F_u \wedge F_v\right| \mathrm{d}u\mathrm{d}v}$$
$$= \frac{|K(q)| \left|F_u \wedge F_v\right| (q) Area(F^{-1}(U))}{\left|F_u \wedge F_v\right| (\bar{q}) Area(F^{-1}(U))} \quad \text{(Middle value theorem)}.$$

As $A \to 0$, $q \to p$, $\bar{q} \to p$,

$$\Rightarrow |K(p)| = \lim_{A \to 0} \frac{\bar{A}}{A}.$$

□

2.9 More Examples

Exercise. *Graph* $z = h(x, y) = S$.

(Recall: For a regular surface, by the implicit (inverse) function theorem, it can be written as a graph locally.)

In the HW 8, you may have computed

$$I = (1 + h_x^2)\, \mathrm{d}x^2 + 2h_x h_y \mathrm{d}x\mathrm{d}y + (1 + h_y^2)\mathrm{d}y^2$$

$$II = \frac{h_{xx}}{\sqrt{1 + |\nabla h|^2_{\mathbb{R}^2}}}\mathrm{d}x^2 + 2\frac{h_x h_y}{\sqrt{1 + |\nabla h|^2}}\mathrm{d}x\mathrm{d}y + \frac{h_{yy}}{\sqrt{1 + |\nabla h|^2}}\mathrm{d}y^2.$$

$$N = \frac{\left(-h_x, -h_y, 1\right)}{\sqrt{1 + |\nabla h|^2}}.$$

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{\left(1 + |\nabla h|^2\right)^2} = \frac{\det \nabla^2 h}{\left(1 + |\nabla h|^2\right)^2}.$$

$$2H = \frac{(1 + h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1 + h_y^2)h_{xx}}{\left(1 + |\nabla h|^2\right)^{\frac{3}{2}}}.$$ [2]

Note that the last term equals

$$\mathrm{div}_{\mathbb{R}^2}\left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right) = \sum_{i,j=1}^{2} L^{ij}\left(\nabla f\right) \nabla_i \nabla_j h,$$

where

$$L^{ij}\left(\nabla f\right) = \frac{1}{\sqrt{1 + |\nabla h|^2}}\left(\delta_{ij} - \frac{\nabla_i h \nabla_j h}{1 + |\nabla h|^2}\right).$$

Let $p \in S$, and assume p is the origin in $\mathbb{R}^3$, and N agrees with the unit normal of z-axis

$$h_x = h_y = 0 \text{ at p}.$$

[2] A basic observation one should make is how K and H depend on the 2^{nd} derivative of h.

$$e = h_{xx}(0,0), f = h_{xy}(0,0), g = h_{yy}(0,0).$$

Hessian of h at p is

$$\begin{pmatrix} h_{xx}(0,0) & h_{xy}(0,0) \\ h_{yx}(0,0) & h_{yy}(0,0) \end{pmatrix}.$$

Hence, at p

$$II = \text{Hess } h.$$

Moreover, at p

$$\partial H = \delta h.$$

As an application, we give a geometric interpretation of the Dupin indicatrix.

Claim: If $p \in S$ is not a planar point, consider the intersection of a plane parallel to T_pS with S, and the plane is close enough to T_pS, then the obtained curve is approximated by the Dupin indicatrix.

Proof. Since we only care about local behavior at p, W.L.O.G assume near p, the surface is parametrized by the graph $z = h(x,y)$, such that p is the origin and z-axis is the normal direction and x-axis, y-axis are principal directions. Let the intersection curve be $h(x,y) = \epsilon$, for sufficiently small ϵ. Consider the Taylor's expansion of $h(x,y)$ at p

$$\begin{aligned} h(x,y) =& h(0,0) + h_x(0,0)x + h_y(0,0)y \\ &+ \frac{1}{2}\left(h_{xx}(0,0)x^2 + \underbrace{2h_{xy}(0,0)}_{=0(\text{principal direction})} xy + h_y y(0,0)y^2 \right) \\ &+ \text{Remainder} \\ =& \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} h_{xx}(0,0) & 0 \\ 0 & h_{yy}(0,0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \text{Remainder}, \end{aligned}$$

□

where

$$\lim_{(x,y)\to(0,0)} \frac{\text{Remainder}}{x^2+y^2} = 0.$$

Hence, the intersection curve is given by

$$\frac{1}{2}\left(h_{xx}(0,0)x^2 + h_{yy}(0,0)y^2\right) + R = \epsilon.$$

Note: in the 2$^{\text{nd}}$ fundamental formII $f = 0$, and in the 1$^{\text{st}}$ fundamental form I $F = 0$, then at p one has

$$\begin{cases} k_1 = \frac{e}{E} = h_{xx}(0,0) \\ k_2 = \frac{g}{G} = h_{yy}(0,0) \end{cases}$$

using $k_i = H \pm \sqrt{H^2 - K}$. Then, $k_1 x^2 + k_2 y^2 = 2\epsilon$ can be viewed as the 1$^{\text{st}}$ order approximation of the intersection curve. Renormalized it, then

$$k_1 \bar{x}^2 + k_2 \bar{y}^2 = 1,$$

is just the Dupin indicatrix.

Exercise. *Surface of revolution.*

Let $\alpha(s) = \varphi(s), 0, \psi(s)$ be a regular curve, with $\varphi(s) > 0$ and $s \in (a, b)$ is the arclength parametrization. $\Rightarrow (\varphi'(s))^2 + (\psi'(s))^2 = 1 \Rightarrow \varphi'\varphi'' + \psi'\psi'' = 0$. Rotate it about z-axis

$$\Rightarrow \alpha(\theta, s) = (\varphi(s)\cos\theta, \varphi(s)\sin\theta, \psi(s)),\ 0 < \theta < 2\pi.$$

$$\begin{aligned} I &= \mathrm{d}s^2 + \varphi(s)^2\mathrm{d}\theta^2 \\ II &= (\psi'\varphi'' - \psi''\varphi')\,\mathrm{d}s^2 - \varphi\psi'\mathrm{d}\theta^2. \end{aligned}$$

$\because F = f = 0$, $\therefore$ s-curve and θ-curve are curvature lines.

$$K = \frac{e}{E}\frac{g}{G} = -\frac{\varphi''}{\varphi},\ H = \frac{1}{2}\left(\frac{e}{E} + \frac{g}{G}\right).$$

Since principal curvatures satisfy $\lambda^2 - 2\lambda H + K = 0$.

$$\Rightarrow \lambda_1 = \frac{e}{E} = -\frac{\varphi''}{\varphi},\ \lambda_2 = \frac{g}{G} = \psi'\varphi'' - \psi''\varphi'.$$

The mean curvature is

$$H = \frac{1}{2}\frac{-\psi' + \varphi(\psi'\varphi'' - \psi''\varphi')}{\varphi}.$$

Question (Homework)**.** (1) Assume $K = 1$, $\lim_{s\to 0^+}\alpha(s) = (0, 0, 1)$, $\alpha(\frac{\pi}{2}) = (1, 0, 0)$, $s \in (0, \pi)$. Describe the surface and determine the 1st fundamental form ($\mathrm{d}s^2 + \sin^2 s\mathrm{d}\theta^2$).

(2) Assume $K = 0$, $\lim_{s\to 0^+}\alpha(s) = (0, 0, 1)$, $\alpha(1) = (\frac{\sqrt{2}}{2}, 0, \sqrt{2}2)$, $s \in (0, \infty)$, answer the same question as in (1) ($\mathrm{d}s^2 + \frac{1}{2}s^2\mathrm{d}\theta^2$).

(3) Assume $H = 0$, $\alpha(0) = (1, 0, 0)$, $\alpha(1) = (\sqrt{2}, 0, \operatorname{arcsin}(1))$, $s > 0$, answer the question in (1). ($\alpha(s) = \left(\sqrt{1+s^2}, 0, \arcsin s\right)$, $I = \mathrm{d}s^2 + (1+s^2)^2\mathrm{d}\theta^2 \sim$ Catenoid).

Exercise (Ruled surface)**.** *A line passing through a point $\alpha_0 \in \mathbb{R}^3$ with direction $v \in \mathbb{R}^3$ can be written as*

$$\ell(t) = \alpha_0 + tv.$$

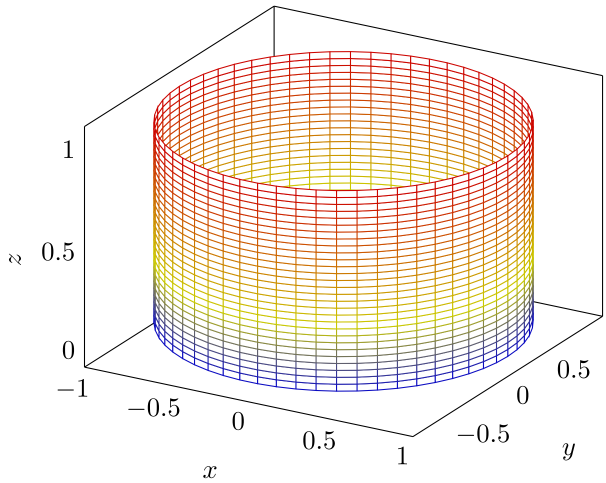
Now we let α_0, v are two vector-valued functions, i.e. $\alpha(s), v(s)$, then the collection of lines form a surface

$$R(s, t) = \alpha(s) + tv(s).$$

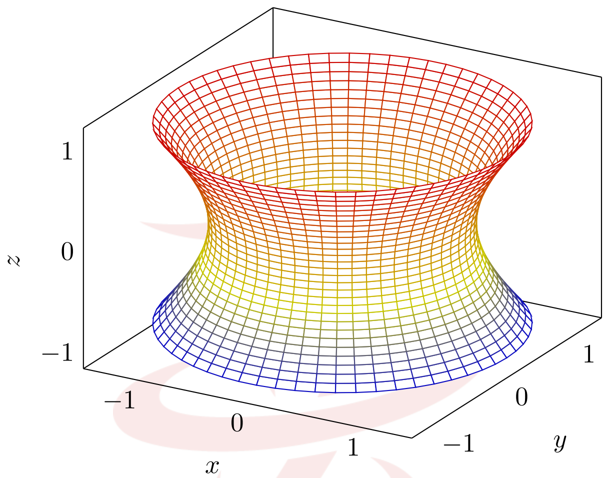
Surfaces with such parametrization is called the ruled surfaces, where $\alpha(s)$ is called the base curve of S and $v(s)$ is called the director curve.

Remark. Ruled surface may contain singular points.

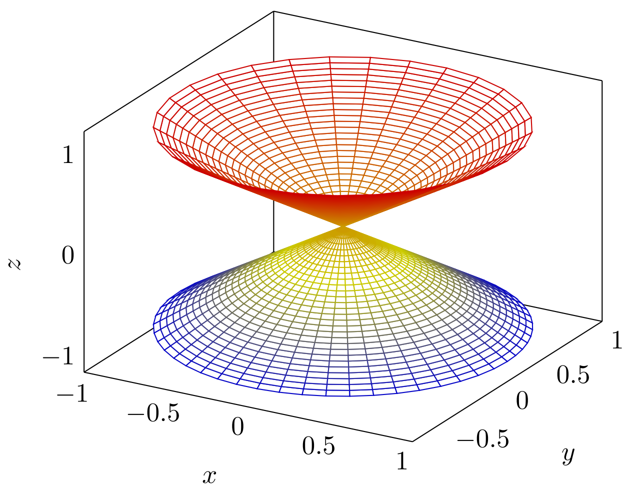
Example 2.9.1. *$R(s, t) = \alpha(s) + te_3$. $\alpha(s)$ is the circle on $x - y$ plane, $e_3 = (0, 0, 1)$. ($x^2 + y^2 = 1$)*



Example 2.9.2. $R(s,t) = \alpha(s) + t\,(\alpha'(s) + e_3)$, *where* $\alpha(s)$ *and* e_3 *are the same as before.*



Example 2.9.3. $R(s,t) = 0 + t\,(u(s) + e_3)$, *where* $u(s) = (\cos\theta, \sin\theta, 0)$



In the following discussion, we assume the ruled surface S

$$R(s,t) = \alpha(s) + tv(s),\ |v(s)| = 1,\ v'(s) \neq 0 \forall s.$$

Direct computation shows:

$$
\begin{aligned}
I =& \left|\alpha_s + tv_s\right|^2 \mathrm{d}s^2 + 2\left\langle \alpha_s, v\right\rangle \mathrm{d}s\mathrm{d}t + \mathrm{d}t^2 \\
N =& \frac{\alpha_s \wedge v + tv_s \wedge v}{\left|\alpha_s \wedge v + tv_s \wedge v\right|} \\
II =& \mathrm{e}\mathrm{d}s^2 + 2f\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Hence,

$$
K = -\frac{f^2}{\det I} \leq 0.
$$

Note that

$$
\begin{aligned}
f &= \left\langle R_{st}, N\right\rangle \\
&= v_s \cdot \frac{\alpha_s \wedge v + tv_s \wedge v}{\left|\alpha_s \wedge v + tv_s \wedge v\right|} \\
&= \frac{(v_s, \alpha_s, v)}{\left|R_s \wedge R_t\right|}.
\end{aligned}
$$

Definition 2.9.4. S is called developable on "flat" surface if $(v_s, v, \alpha_s) \equiv 0$. In this case $K = 0$.

Remark. Physically, a developable surface can be flattened onto a plane without "stretching" or "compressing", but allowing unfolding or cutting along a line. Later, we'll see on page 414, section 5-8 in Do Carmo: If on a developable surface S, all lines can be extended on both sides, the surface can only be a plane or a cylinder.

Remark. On a non-cylindrical ruled surface S, *i.e.*

$$
R(s,t) = \alpha(s) + tv(s), v'(s) \neq 0 \forall s,
$$

one can choose a spherical base curve $\beta(s)$ such that

$$
R(s,t) = \beta(s) + tv(s),
$$

where $\langle \beta'(s) + v'(s)\rangle = 0$, such curve is called the line of striction. Points on the line of striction is called the central points of the ruled surface.

Example 2.9.5. *In example 2.9.2, $\alpha(s)$ is the line of striction.*

Exercise. *Find the line of restriction $(\beta(s) - \alpha(s) - \frac{\langle \alpha', v'\rangle}{|v'|^2} v)$. Then $R(s,u) = \beta(s) + uv(s)$.*

Hence, if we let $\lambda = \frac{(\beta', v, v')}{|v'|^2}$, then $K = -\frac{\lambda^2}{(\lambda^2 + u^2)^2}$.

Exercise (Homework)**.** *Prove that there are no closed smooth minimal surface in $\mathbb{R}^2$.*

Proof. On a closed surface, there must exist an elliptic point, at which $k > 0$, this contradicts to minimal surface. □

Theorem 2.9.6. *(1) If a surface of revolution M is minimal, then M is contained in either a plane or a catenoid.*

(2) If a ruled surface M is minimal, then M is contained in either a plane or a helicoid.

Proof. See homework. □

2.10 Isometries between surfaces

From this lecture on, we'll study the intrinsic geometry of a regular surface. More precisely, we'll see how the 1^{st} fundamental form determines the geometry.

- S_1, S_2 are regular Surfaces, $f\colon S_1 \to S_2$.

(1) f is isomorphism (1-1 and onto) (of sets)$\Leftrightarrow$ S_1 and S_2 are identified as sets.

(2) f is homeomorphism (f is an isomorphism + f and f^{-1} continuous)$Leftrightarrow$ S_1 and S_2 are identified as topological spaces.

(3) f is diffeomorphism (f is a homeomorphism+ f, f^{-1} are smooth) $\Leftrightarrow$ S_1 and S_2 are identified with considering smooth structures on them.

(4) f is isometry

$\Leftrightarrow$ f is a diffeomorphism and f preserves the arclength, area, angles,...(everything defined by the 1^{st} fundamental form).

Definition 2.10.1 (Isometry). $\varphi\colon S_1 \to S_2$ is a diffeomorphism between regular surfaces. If $\forall p \in S_1$, $v_1, v_2 \in T_pS$

$$\langle v_1, v_2\rangle_p = \left\langle d\varphi_p(v_1), d\varphi_p(v_2)\right\rangle, \tag{1}$$

which is equivalent to saying that φ preserves the 1^{st} fundamental form, then we say φ is an isometry.

Definition 2.10.2 (Local isometry). $\varphi\colon S_1 \to S_2$ is called a local isometry, if $\forall p \in S_1$, $\exists$ a neighborhood U of p such that $\varphi|_U$ is an isometry.

Example 2.10.3. *Catenoid and helicoid are only locally isometric to each other but not global isometric. In fact the helicoid is simply connected, but the catenoid is not (since it comes from revolution).*

Remark. Isometry implies local isometry, however the converse is false.

Example 2.10.4. $\left(\mathbb{R}^2, \mathrm{d}x^2 + \mathrm{d}y^2\right)$ *and the cylinder* $(\cos u, \sin u, v), \mathrm{d}u^2 + \mathrm{d}v^2$ *are only local isometric to each other, but not "globally" isometric.*

(The local isometry map is just the local parametrization of cylinder:

$$\varphi\colon (u, v, 0) \to (\cos u, \sin u, v).$$

Clearly, $\mathbb{R}^2$ *and the cylinder are not diffeomorphic to each other, since the cylinder has a non-shrinkable loop.)*

Exercise. *The condition in eq. (1) is equivalent to*

$$\forall w \in T_pS, \quad |w|_p = \left|d\varphi_p(w)\right|_{\varphi(p)}. \tag{2}$$

Proof. eq. (1) $\Rightarrow$ eq. (2), obviously.

eq. (2) $\Rightarrow$ eq. (1), let $w = v_1 + v_2$, then

$$|w|_p^2 = \left|d\varphi_p(w)\right|^2_{\varphi(p)}, \quad |v_1|_p^2 = \left|d\varphi_p(v_1)\right|^2_{\varphi(p)}, \quad |v_2|_p^2 = \left|d\varphi_p(v_2)\right|^2_{\varphi(p)}$$

$$\Rightarrow \langle v_1, v_2\rangle_p = \left\langle d\varphi_p(v_1), d\varphi_p(v_2)\right\rangle.$$

□

Remark. If $\varphi\colon S_1 \to S_2$ is a local isometry. $\forall p \in S_1$, let U be a coordinate patch of p and $\varphi|_U\colon U \to \varphi(U)$ is a diffeomorphism given by

$$\varphi(u, v) = (\tilde{u}, \tilde{v}).$$

$$I_1 = \begin{pmatrix} \mathrm{d}u & \mathrm{d}v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}$$

$$I_2 = \begin{pmatrix} \mathrm{d}\tilde{u} & \mathrm{d}\tilde{v} \end{pmatrix} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \begin{pmatrix} \mathrm{d}\tilde{u} \\ \mathrm{d}\tilde{v} \end{pmatrix}.$$

Note

$$\begin{pmatrix} \mathrm{d}\tilde{u} \\ \mathrm{d}\tilde{v} \end{pmatrix} = \frac{\partial(\tilde{u}, \tilde{v})}{\partial(u, v)} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} = J_\varphi \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} E & F \\ F & G \end{pmatrix} = J_\varphi^\top \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} J_\varphi.$$

Proposition 2.10.5. *$U \subset \mathbb{R}^2$, assume $\gamma_1(U) \subset S_1$, $\gamma_2(U) \subset S_2$ are two local parametrizations of S_1 and S_2 such that*

$$E_1 = E_2, \; F_1 = F_2, \; G_1 = G_2,$$

then $\varphi = \gamma_2 \circ \gamma_1^{-1}$ is a local isometry.

Proof. γ_1, γ_2 are local parametrizations $\Rightarrow \varphi = \gamma_2 \circ \gamma_1^{-1}$ is a local diffeomorphism. It suffices to show $\forall p \in U$, $w \in T_pU$

$$I_p(w, w) = I_{\varphi(p)}(d\varphi_p(w), d\varphi_p(w)).$$

Let $w = \alpha'(0) \in T_pS$ for a curve $\alpha(t) \in S_1$, $\alpha(t) = \gamma_1(u(t), v(t))$

$$\Rightarrow w = \frac{\partial \gamma_1}{\partial u} u'(0) + \frac{\partial \gamma_1}{\partial v} v'(0)$$

$$I_p(w, w) = E_1 u'(0)^2 + 2F_1 u'(0)v'(0) + G_1 v'(0)^2.$$

Since

$$d\varphi_p(w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi \circ \alpha(t) = \frac{\partial \gamma_2}{\partial u} u'(0) + \frac{\partial \gamma_2}{\partial v} v'(0),$$

$$\therefore I_{\varphi(p)}(d\varphi_p(w), d\varphi_p(w)) = E_2 u'(0)^2 + 2F_2 u'(0)v'(0) + G_2 v'(0)^2.$$

By assumption, $I_p(w, w) = I_{\varphi(p)}(d\varphi_p(w), d\varphi_p(w))$. □

Remark. For two (local) isometric surfaces S_1 and S_2, the 1st fundamental forms might look quite different in the given parametrization. But we can always find a coordinate change on one of then to get the desired isometry between them. (Such coordinate change gives the isometry on the surface to itself)

Exercise. *In your homework 6, you have checked the catenoid and the with*

$$C(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av)\,,\ 0 < u < 2\pi,\quad -\infty < v < +\infty$$

and the helicoid with

$$H(\bar{u},\bar{v}) = (\bar{v}\cos\bar{u}, \bar{v}\sin\bar{u}, a\bar{u})\,,\ 0 < \bar{u} < 2\pi, -\infty < \bar{v} < +\infty$$

are local isometric, and the local isometry is

$$\varphi(u,v) = (\bar{u},\bar{v}),\ \bar{u} = u,\ \bar{v} = a\sinh v.$$

Remark. If $\varphi\colon S_1 \to S_2$ is an isometry, φ preserves distance, length, angle, area et.c.

We often use another weaker equivalent condition, *i.e.* a diffeomorphism only preserving angle.

Definition 2.10.6. A diffeomorphism $\varphi\colon S_1 \to S_2$ is called a conformal map if $\forall p \in S_1$, $v_1, v_2, \in T_pS_1$, we have

$$\left\langle d\varphi_p(v_1), d\varphi_p(v_2)\right\rangle = \lambda^2(p)\left\langle v_1, v_2\right\rangle_p,$$

where λ^2 is a nowhere vanishing smooth function on S_1 and S_1 is said to be conformal to S_2.

Definition 2.10.7 (Locally conformal)**.** $\varphi\colon S_1 \to S_2$ is called a locally conformal map, if $\forall$, $\exists$ a neighborhood U of p such that $\varphi|)U$ is a conformal map.

Proposition 2.10.8. *$\varphi\colon S_1 \to S_2$ is a locally conformal map if and only if φ is an angle preserving smooth map.*

Proof. See homework. □

Remark. (1) Angle-preserving is a geometric description of (locally) conformal condition. In reality, it's more convenient to use definition.

(2) Conformal is also an equivalent relation on regular surfaces.

Example 2.10.9. *In HW 6, you have checked that stereographic map*

$$\pi_N\colon \mathbb{S}^2 - N \to \mathbb{R}^2$$

$$(x,y,z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z}),$$

is a locally conformal map. Recall

$$\begin{aligned} \mathrm{d}s^2_{\mathbb{S}^2} &= \frac{4}{(1+u^2+v^2)^2}\left(\mathrm{d}u^2 + \mathrm{d}v^2\right) \\ &= \frac{4}{(1+u^2+v^2)^2}\mathrm{d}s^2_{\mathbb{R}^2} \end{aligned}$$

Definition 2.10.10 (Locally conformal flat)**.** A regular surface is called locally conformally flat, if $\forall p \in S$ $\exists$ neighborhood U of f such that U is conformally equivalent to an open subset of $\mathbb{R}^2$, *i.e.* on U the first fundamental form of S is

$$I_U = \lambda^2 I_{\mathbb{R}^2}$$

for a non-vanishing smooth function.

A remarkable result in surface theory is:

Theorem 2.10.11. *Any regular surface is locally conformally flat.*

In practice, this is very useful. The theorem tells that $\forall p \in S$, we can choose a local parametrization near p such that

$$\mathrm{d}s^2 = \lambda^2 \mathrm{d}s^2_{\mathbb{R}^2} = \lambda^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2\right).$$

Such parametrization is called Isothermal parametrization. Hence, the study of geometry on S (which is determined by the 1^{st} fundamental form) becomes the study of function λ.

Proof. (Skip) (Reference: S.S. Chern, 1955, proceedings, An elementary proof of the existence of isothermal parameters on a surface). The proof is reduced to solving some complex valued differential equations □

Remark. Later, in Riemannian Geometry course, you'll see there is a "mysterious" curvature (Weyl) tensor to characterize the "locally conformally flat" condition, *i.e.*

$$\text{Weyl tensor} \equiv 0 \Leftrightarrow \text{Locally conformally flat}.$$

However, Weyl tensor is always zero on a manifold of dimension 2 and 3.

Remark. (1) In surface theory, conformal map is closely related to the complex analysis.

(2) In higher dimensional Riemannian Geometry, there is a branch called conformal geometry.

Yamabe Problem:

(1) Let (M, g) be a close Riemannian manifold with $\dim M \geq 3$. Does there exist a Riemannian metric $\tilde{g}$ such that $\tilde{g} = \lambda^2 g$ and the "scalar curvature" of $\tilde{g}$ is a constant c?($c < 0$ and $c = 0$ are solved. $c > 0$ remains open)

(2) In dim $= 2$, this question is equivalent to finding a metric with constant Gaussian curvature on a closed surface under the conformal change.

Chapter 3

Intrinsic geometry

3.1 Einstein convention

Recall $v \in \mathbb{R}^n$, $v = (v^1, \dots, v^n) \implies v = \sum_{i=1}^n v^i e_i$.

(1) The summation is taken over $i = 1, 2, \dots, n$;

(2) $v = \sum_{i=1}^n v^i e_i = \sum_{j=1}^n v^j e_j$, expression of v is independent of choice of summation indices i or j.

(3) One "i" appears as upper index, another "i" appears as lower index.

Einstein convention: Whenever there is pair of same letter as upper and a lower index, then the expression is summation of the index letter from 1 to n, n is usually the dimension of vector space / manifold / etc.

Example 3.1.1.

- *$W = V \cdot A$, $A = (A_i^j)_{n\times n}$ matrix, $v, w \in \mathbb{R}^n$, then*
$$W^j = v^i A_i^j = \sum_i v^i A_i^j.$$

-
$$g^{ij}\omega_{jk} = \sum_{j=1}^n g^{ij}\omega_{jk}..$$
Where g^{ij} is the (i,j)-entry of inverse matrix g^{-1} of g. i.e.
$$g^{ij}g_{jk} = \delta_k^i, \quad g^{ij}g_{ij} = \sum_{i=1}^n\sum_{j=1}^n g^{ij}g_{ij} = \sum_{i=1}^n \delta_i^i = n.$$
We will use Einstein convention from now on.

3.2 Therema Egregium (Gauss)

Goal: The Gaussian curvature K depends only on the 1st fundamental form.

Previously, we have seen once we know local parametrization $\varphi(x^1, x^2)$ of a surface S, then I, II can be computed and the Gaussian curvature is

$$K = \frac{\det II}{\det I} = \frac{eg - f^2}{EG - F^2}.$$

Now, we want to follow the same procedure as we study the Frenet formula of a curve and try to understand the motion of equation of frame φ_1, φ_2, N, where $\varphi_i = \frac{\partial \varphi}{\partial x^i}, N = \frac{\varphi_1 \times \varphi_2}{|\varphi_1 \times \varphi_2|}$.

Fix a local parametrization

$$\varphi \colon U \longrightarrow S \subset \mathbb{R}^3,\ (x^1, x^2) \longmapsto \varphi(x^1, x^2).$$

Then

$$\begin{aligned} I &= g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j\,, \quad g_{ij} = \left\langle \varphi_i, \varphi_j \right\rangle \\ II &= h_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j\,, \quad h_{ij}. \end{aligned}$$

We shall study the differential equation of $\{\varphi_i, N\}$ up to 2nd order.

$$\begin{aligned} \varphi_{11} &= \Gamma^1_{11}\varphi_1 + \Gamma^2_{11}\varphi_2 + h_{11}N \\ \varphi_{12} &= \Gamma^1_{12}\varphi_1 + \Gamma^2_{12}\varphi_2 + h_{12}N \\ \varphi_{21} &= \Gamma^1_{21}\varphi_1 + \Gamma^2_{21}\varphi_2 + h_{21}N \\ \varphi_{22} &= \Gamma^1_{22}\varphi_1 + \Gamma^2_{22}\varphi_2 + h_{22}N \end{aligned}$$

.

Weingarten equation, $A = (a_i^j)$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = A \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}.$$

Then

$$a_i^j = -h_{ik}g^{kj}.$$

Lets write above equations as

$$\begin{cases} \varphi_{ij} = \Gamma^k_{ij}\varphi_k + h_{ij}N \\ N_i = a_i^j = \varphi_j \end{cases} \implies \text{The only unknown are } \Gamma^k_{ij}. \tag{3.2.1}$$

Moreover,

$$\varphi_{ij} = \varphi_{ji} \implies \Gamma^k_{ij} = \Gamma^k_{ji}.$$

$$\left\langle \varphi_{ij}, \varphi_p \right\rangle = \Gamma^k_{ij} \left\langle \varphi_k, \varphi_p \right\rangle = \Gamma^k_{ij} g_{kl}.$$

So

$$\begin{aligned} \frac{\partial g_{ip}}{\partial x^j} - \left\langle \varphi_i, \varphi_{pj} \right\rangle &= \left\langle \varphi_{ij}, \varphi_p \right\rangle = \Gamma^k_{ij} g_{kp} \\ \frac{\partial g_{jp}}{\partial x^i} - \left\langle \varphi_j, \varphi_{pi} \right\rangle &= \left\langle \varphi_{ji}, \varphi_p \right\rangle = \Gamma^k_{ji} g_{kp} \\ \frac{\partial g_{ij}}{\partial x^p} &= \left\langle \varphi_{ip}, \varphi_j \right\rangle + \left\langle \varphi_i, \varphi_{jp} \right\rangle \end{aligned}$$

Hence

$$2\Gamma^k_{ij} g_{kp} = \frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x_p}.$$

$$\Longrightarrow\ 2\Gamma_{ij}^k g_{kp} g^{pq} = (\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x_p}) g^{pq}.$$

We get

$$\Gamma_{ij}^k = \frac{1}{2}(\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} g^{pk} - \frac{\partial g_{ij}}{\partial x_p}).$$

Remark. We multiply g^{pk} form right because of the choice of row vector. In modern convention of the column vector,

$$\Gamma_{ij}^k = \frac{1}{2} g^{kp}(\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x_p}).$$

Definition 3.2.1. Γ_{ij}^k is called the Christoffel symbols. They're uniquely determined by the 1st fundamental form.

Next we derive the Gauss equation, we'll see the Gaussian curvature can be expressed only in 1st fundamental form.

So far, we have obtained

$$\varphi_{ij} = \Gamma_{ij}^k \varphi_k + h_{ij} N \quad \& \quad N_p = a_p^q \varphi_q.$$

Then we have

$$\begin{aligned}
\varphi_{ijp} &= \partial_p \Gamma_{ij}^k \varphi_k + \Gamma_{ij}^k \varphi_{kp} + \partial_p h_{ij} N + h_{ij} N_p \\
&= \partial_p \Gamma_{ij}^k \varphi_k + \Gamma_{ij}^k (\Gamma_{kp}^q + h_{kp} N) + \partial_p h_{ij} N + h_{ij} a_p^q \varphi_q, \\
\varphi_{ipj} &= \partial_j \Gamma_{ip}^k \varphi_k + \Gamma_{ip}^k (\Gamma_{kj}^q + h_{kj} N) + \partial_j h_{ip} N + h_{ip} a_j^q \varphi_q.
\end{aligned}$$

The derivative is taken in $\mathbb{R}^3$, so the two expression should be the same, we got

$$\begin{aligned}
\varphi_{ijp} - \varphi_{ipj} =& (\partial_p \Gamma_{ij}^k - \partial_j \Gamma_{ip}^k)\varphi_k + (\Gamma_{ij}^k \Gamma_{kp}^q - \Gamma_{ip}^k \Gamma_{kj}^q)\varphi_q + (h_{ij} a_p^q - h_{ip} a_j^q)\varphi_q \quad \text{(tangential)} \\
&+ (\Gamma_{ij}^k h_{kp} - \Gamma_{ip}^k h_{kj} + \partial_p h_{ij} - \partial_j h_{ip}) N \quad \text{(normal)} \\
=& 0
\end{aligned}$$

Remark. If the ambient space is not $\mathbb{R}^n$, LHS should give curvature of the ambient space.

Note that φ_i and N are perpendicular, we can split tangential part and normal part of the equation, and use $a_p^k = -h_{pq} g^{qk}$ we get:

Normal part:

$$\partial_p h_{ij} - \Gamma_{pi}^k h_{kj} = \partial_j h_{ip} - \Gamma_{ji}^k h_{kp}. \tag{3.2.2}$$

Tangential part:

$$(h_{ij} h_{pq} - h_{ip} h_{jq}) g^{qk} = \partial_p \Gamma_{ij}^k - \partial_j \Gamma_{ip}^k + \Gamma_{ij}^l \Gamma_{lp}^k - \Gamma_{ip}^l \Gamma_{lj}^k.$$

multiply by g^{kr} we get

$$h_{ij} h_{pq} - h_{ip} h_{jq} = g_{qk} \left(\partial_p \Gamma_{ij}^k - \partial_j \Gamma_{ip}^k + \Gamma_{ij}^l \Gamma_{lp}^k - \Gamma_{ip}^l \Gamma_{lj}^k \right). \tag{3.2.3}$$

eq. (3.2.2) is called the Codazzi equation and eq. (3.2.3) is called the Gauss equation. Take $i = j, p = q$ in Gauss equation, we see LHS becomes

$$h_{ii} h_{pp} - h_{ip} h_{pi} = \det \begin{bmatrix} h_{ii} & h_{ip} \\ h_{pi} & h_{pp} \end{bmatrix}.$$

For 2 dim case, if $i \neq p$, this gives exactly $\det II$. Note RHS is purely determined by I, hence $K = \frac{\det II}{\det I}$ only depends on I, it is an intrinsic geometric quantity.

! Gaussian curvature is the local geometric invariant of surfaces.

We look back Codazzi equation, we can add a term as

$$\partial_p h_{ij} - \Gamma^k_{pi} h_{kj} - \boxed{\Gamma^k_{pj} h_{ki}} = \partial_j h_{ip} - \Gamma^k_{ji} h_{kp} - \boxed{\Gamma^k_{jp} h_{ki}}.$$

In terms of covariant derivative ∇, this writes

$$\nabla_p h_{ij} = \nabla_j h_{ip}.$$

i.e. All 3 index of $\nabla_p h_{ij}$ are symmetric.

Remark. We don't need to memorize the Gauss-Codazzi equations precisely. It suffices to work it out step by step once we know local parametrization. And there is a much more simple form of the equations after we introduced notations in Riemannian geometry.

We also call the Gauss-Codazzi equations are the integrability to solve the equation of motion eq. (3.2.1).

Theorem 3.2.2 (Fundamental theorem of surface theory (local))**.** *Let $U \subset \mathbb{R}^2$ be open, connected set. Given two quadratic form $I = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$, $II = h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$, s.t. I is positively definite. Moreover, the Gauss-Codazzi equations are satisfied. Then there is a surface S in $\mathbb{R}^3$ s.t. I, II are the 1st and 2nd fundamental form of S with U a coordinate chart.*

The surface S is unique up to rigid motion.

Proof. Skip. (Can be found in Do Carmo's book) □

3.3 An invitation of Riemannian Geometry

We have introduced the concept of smooth manifold M, *i.e.* M is a topological manifold together with a smooth structure, given by a collection of coordinate covering $M = \bigcup_\alpha U_\alpha$ *s.t.*

(1) $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$, is homeomorphism.

(2) $\forall\, \alpha, \beta$, $U_\alpha \cap U_\beta \neq \emptyset \implies$ the transition function $\varphi_\beta \circ \varphi_\beta^{-1}$ on $U_\alpha \cap U_\beta$ is smooth.

Each $(U_\alpha, \varphi_\alpha)$ is called a coordinate patch.

Now a basic question is how to take derivative on M? This question is natural to be asked since we want to apply the technique in calculus to study M.

First, we need to know how to differentiate $f \in C^\infty(M)$, this has been done by introducing "tangent vector" & "tangent vector field".

Recall a smooth vector field X on M is a smooth map sending $p \in M$ to a vector $X_p \in T_pM$. Let $\Gamma(TM)$ be set of all smooth vector fields on M. X_p is understood geometrically as $\forall\, f \in C^\infty(M)$,

$$(Xf)(p) = X_p(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\alpha(t)) = \lim_{t\to 0} \frac{f(\alpha(t)) - f(p)}{t}.$$

Where $\alpha(t)$ is any curve with initial value p and initial velocity X_p. We temporarily take this definition and try to generalize it to Lie derivative. Soon we shall take the viewpoint that X_p as first order derivative and generalize it to covariant derivative.

3.3.1 Lie derivative

Now we want to take derivatives of vector fields. First let's consider $M = \mathbb{R}^n$. Let V, W be two smooth vector fields, $p \in \mathbb{R}^n$, and

$$\mathrm{D}_{V_p} W(p) = \lim_{t \to 0} \frac{W(p + tV_p) - W(p)}{t}. \tag{3.3.1}$$

This just means the rate of "change of value of W" w.r.t. the "change of points in the domain".

There're two things to be paid more attention:

(1) $p + tV_p \in \mathbb{R}^n$ is defined via linear structure of $\mathbb{R}^n$.

(2) $W(p + tV_p) - W(p)$, again, the subtraction makes sense by the linear structure of $\mathbb{R}^n$. In which, we can identify

$$T_p\mathbb{R}^n \cong T_{p+tV_p}\mathbb{R}^n \cong \mathbb{R}^n.$$

Next we consider general M as a manifold, $V, W \in \Gamma(TM)$. To generalize eq. (3.3.1) to manifolds, we have to reconsider (1) and (2).

(1) Given p and initial velocity V_p, we can choose a smooth curve $\alpha(t)$ as a replacement of "linear perturbation in $\mathbb{R}^n$"

(2) Then consider the vector field W (restricted on $\alpha(t)$). *i.e.* $W(\alpha(t)) \in T_{\alpha(t)}M$. However $T_{\alpha(t)} \neq T_pM$, not the same linear space. To do the subtraction, we need to "move" $W(\alpha(t))$ back to T_pM. This can be done by viewing

$$\alpha(t) \colon p \mapsto \alpha(t)$$

as a smooth map from M to M (at least locally). Then we can define a map

$$\alpha(-t) \colon \alpha_(t) \mapsto p. \quad \text{(again from } M \text{ to } M)$$

We have

$$\begin{aligned} \mathrm{d}\alpha(-t)_{\alpha(t)} : T_{\alpha(t)}M &\longrightarrow T_pM \\ W(\alpha(t)) &\longmapsto (\mathrm{d}\alpha(-t))_{\alpha(t)} (W(\alpha(t))) \in T_pM. \end{aligned}$$

Then $(\mathrm{d}\alpha(-t))_{\alpha(t)} (W(\alpha(t)))$ and W_p both live in T_pM.

Definition 3.3.1 (Lie derivative).

$$\mathcal{L}_V W(p) := \lim_{t \to 0} \frac{\left(\mathrm{d}\alpha(-t)_{\alpha(t)}\right)(W(\alpha(t))) - W(p)}{t}$$

is called the Lie derivative of W along V, of course, we can write above as

$$\mathcal{L}_V W(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{d}\alpha(-t)_{\alpha(t)}\right)(W(\alpha(t))).$$

Note that $\mathcal{L}_V W$ is still a smooth vector field (Exercise). The computational formula is given by

Proposition 3.3.2.

$$\mathcal{L}_V W = [V, W] = VW - WV.$$

In some books, people just take this as the definition of Lie derivative.

$[V, W]$ is called the Lie bracket, defined by $\forall f \in C^\infty(M)$,

$$[V, W](f) = V(W(f)) - W(V(f)).$$

One should check this do give a derivation and defines a smooth vector field.

Note if we write $\mathcal{L}_V f = V(f)$, then above equation writes

$$(\mathcal{L}_V W)(f) = \mathcal{L}_V(W(F)) - W(\mathcal{L}_V f).$$

i.e.

$$\mathcal{L}_V(W(f)) = (\mathcal{L}_V W)(f) + W(\mathcal{L}_V f).$$

This is Leibniz rule for Lie derivative.

Locally, let $V = V^i \frac{\partial}{\partial x^i}, W = W^j \frac{\partial}{\partial x^j}$, then

$$\begin{aligned}
[V, W] &= [V^i \frac{\partial}{\partial x^i}, W^j \frac{\partial}{\partial x^j}] = V^i \frac{\partial}{\partial x^i}\left(W^j \frac{\partial}{\partial x^j}\right) - W^j \frac{\partial}{\partial x^j}\left(V^i \frac{\partial}{\partial x^i}\right) \\
&= V^i \frac{\partial W^j}{\partial x^i} \frac{\partial}{\partial x^j} + V^i W^j \frac{\partial}{\partial x^i x^j} - W^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial x^i} - W^j V^i \frac{\partial}{\partial x^i x^j} \\
&= V^i \frac{\partial W^j}{\partial x^i} \frac{\partial}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial x^i} \\
&= (V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i}) \frac{\partial}{\partial x^j}.
\end{aligned}$$

Let's pause to introducing Lie derivative until we need more properties of it.

3.3.2 Affine connection & Covariant derivative

Now we take another viewpoint of tangent vector fields. *i.e.* A smooth vector field V on M is understood as a 1st order differential operator acting on $C^\infty(M)$. *i.e.* $V \in \Gamma(TM)$, $f \in C^\infty(M)$, $V(f)$ is a smooth function *s.t.*

$$V(f)(p) = V_p(f).$$

where $V_p \colon C^\infty(M) \to \mathbb{R}$ *s.t.*

(1) $V_p(f + \lambda g) = V_p(f + \lambda V_p(g))$,

(2) $V_p(fg) = V_p(f)g(p) + f(p)V_p(g)$.

Lets also write $V(f) = \nabla_V f$, then $\forall V, W \in \Gamma(TM)$, $h \in C^\infty(M)$, we have

$$\nabla \colon \Gamma(TM) \times C^\infty(M) \to C^\infty(M), \quad (V, f) \mapsto \nabla_V f,$$

satisfies:

(1) $\nabla_{V+hW} f = \nabla_V f + h\nabla_W f$,

(2) $\nabla_V (f + \lambda h) = \nabla_V f + \lambda \nabla_V h$,

(3) $\nabla_V (fh) = (\nabla_V)h + f\nabla_V h$.

In a similar fashion, we define the covariant derivative $\nabla_V W$, by requiring (1)-(3) above, more precisely, $\forall\, V, W, Z \in \Gamma(TM), f \in C^\infty(M), \lambda \in \mathbb{R}$,

(1) $\nabla_{V+fZ} W = \nabla_V w + f\nabla_Z W$. ($C^\infty$-linear in subscript vector filed)

(2) $\nabla_V (W + \lambda Z) = \nabla_V W + \lambda \nabla_V Z$. ($\mathbb{R}$-linear in vector fields)

(3) $\nabla_V (fW) = (\nabla_V f)W + f\nabla_V W$. (Leibniz rule)

Definition 3.3.3.

$$\nabla \colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$(X, Y) \mapsto \nabla_X Y$$

is called the affine connection on M.

<u>Fact</u>: There are many such affine connections!

In practice, we need local expressions of $\nabla_V W$. Let $(x^1, \dots, x^n)$ be a local coordinate on $U \subset M$, $V = V^i \frac{\partial}{\partial x^i}, W = W^i \frac{\partial}{\partial x^i}$, then

$$\begin{aligned}
\nabla_V W &= \nabla_{V^i \frac{\partial}{\partial x^i}} \left(W^j \frac{\partial}{\partial x^j} \right) \\
&= V^i \nabla_{\frac{\partial}{\partial x^i}} \left(W^j \frac{\partial}{\partial x^j} \right) \\
&= V^i \left(\left(\nabla_{\frac{\partial}{\partial x^i}} W^j \right) \frac{\partial}{\partial x^j} + W^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \\
&= V^i \left(\frac{\partial W^j}{\partial x^i} \frac{\partial}{\partial x^j} + W^j \boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}} \right).
\end{aligned}$$

Definition 3.3.4. On U, we define the Christoffel symbol by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

(Here, one should compare this definition with the Christoffel symbol introduced in the study of equation of motion in previous lectures).

$$\Rightarrow \nabla_V W = V^i \left(\frac{\partial W^j}{\partial x^i} + W^k \Gamma_{ki}^j \right) \frac{\partial}{\partial x^j}.$$

Conventionally, we define

$$\nabla_i W^j = \frac{\partial W^j}{\partial x^i} + \Gamma_{ik}^j W^k$$

- $\nabla_i W^j$: Taking covariant derivative of j-th component of W along the i-th coordinate direction.

- $\frac{\partial W^j}{\partial x^i}$: Euclidean derivative.
- $\Gamma^j_{ik} W^k$: Correction term.

This notion is frequently used in geometry references.

Again, because there are tons of selection of affine connections, this yields many choices of Γ^k_{ij}! After we introduce the Riemannian metric, we shall see there is a unique affine connection compatible with the Riemannian metric.

Definition 3.3.5 (Riemannian metric)**.** Let M be a smooth manifold. A Riemannian metric on M is a smooth map

$$g\colon \Gamma(TM) \times \Gamma(TM) \to C^\infty(M),$$

such that $\forall X, Y, Z \in \Gamma(TM), f \in C^\infty(M)$ there is the following

(1) $g(X, Y) = g(Y, X)$.(symmetry)

(2) $g(X, X) \geq 0$, and equality achieves iff $X = 0$. (positive definite)

(3) $g(fX + Y, Z) = f \cdot g(X, Z) + g(Y, Z)$.($C^\infty$ linearity)

(M, g) is called a Riemannian manifold.

Remark. At each $p \in M$, g defines an inner product g_p on T_pM. If we choose a coordinate chart near p with local coordinate $(x^1, \dots, x^n)$, the local expression of g is written as

$$g = g_{ij} dx^i dx^j,$$

where $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$.

Example 3.3.6. *The 1st fundamental form on M is a Riemannian metric.*

Using the Riemannian metric g, we can define the length, area, angle, et.c.

Definition 3.3.7. (1) $X \in \Gamma(TM)$, $|X| = \sqrt{g(X, X)}$.

(2) The volume density on (M, g) is defined as

$$\mathrm{d}V = \sqrt{\det\left(g_{ij}\right)} dx^1 \wedge \dots \wedge dx^n.$$

(3) The volume of a bounded region B is

$$V(B) = \int_B 1 \mathrm{d}V$$

With the Riemannian metric introduced, we can uniquely determine an affine connection compatible with the Riemannian metric.

Theorem 3.3.8. *Let (M, g) be a Riemannian manifold, then there is a unique affine connection $\nabla\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ satisfying following conditions: $\forall X, Y, Z$*

(1) $\nabla_Z (g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.(Compatible with metric g)[1]

(2) $\nabla_X Y - \nabla_Y X = [X, Y]$. *(Torsion free)*

The connection ∇ is called Levi-Civita connection.

Locally, take $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}, Z = \frac{\partial}{\partial x^k}$

$$(1) \Rightarrow \frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{li}$$

$$(2) \Rightarrow \Gamma^k_{ij} = \Gamma^k_{ji}.$$

Exercise. *Show that*

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

This is just the same expression as we see on.

So far, we have defined 1st order derivatives (Lie derivative, covariant derivative) on functions and vector fields. Next we shall consider 2nd order derivatives of a function and vector fields. In particular, non-commutative nature of 2nd order derivative on vector fields will be captured by the "curvature".

Let's assume ∇ to be the Levi-Civita connection from now on.

- Recall we have defined tangent bundle. $TM = \bigcup_{p \in M} T_p M$ is the collection of all tangent vector spaces. As a vector space, $T_p M$ is isomorphic to $\mathbb{R}^n$. Let $T_p^* M =$\{all covectors at p\}. We also let $T^*M = \bigcup_{p \in M} T_p^* M = \bigcup_p \left\{ (p, \alpha) | \alpha \in T_p^* M \right\}$. Similar to the TM, one can endow a smooth structure on T^*M such that it's a smooth manifold of dimension $2 \dim M$. We shall also call an element of $\Gamma(T^*M)$ a differential 1-form, where $\Gamma(T^*M)$ is the collection of all global covectors.

- We have introduced

$$\nabla \colon \Gamma(TM) \times C^\infty(M) \to C^\infty(M)$$

$$(X, f) \mapsto \nabla_X f.$$

This map induces

$$\nabla f \colon \Gamma(TM) \to C^\infty(M).$$

At each point $p \in M$ this map is

$$X \mapsto \nabla_X f = \nabla f(X).$$

$\nabla f(p) \colon T_p M \to \mathbb{R} \Rightarrow \nabla f(p) \in T_p^* M \Rightarrow \nabla f \in \Gamma(T^*M)$.

Remark. If $(x^1, \dots, x^n)$ is local coordinate at p. $T_p M = \mathrm{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$, then $T_p^* M = \mathrm{span}\{dx^1, \dots, dx^n\}$, $dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j$

$$\Rightarrow \nabla f(p) = \frac{\partial f}{\partial x^i} dx^i = \nabla_i f \cdot dx^i. \quad (\nabla_i f = \frac{\partial f}{\partial x^i})$$

Definition 3.3.9. The gradient vector field of f, written as $\mathrm{grad} f = \mathrm{grad}_g f$ is defined as $\forall X \in \Gamma(TM)$

$$g(\mathrm{grad} f, X) = \nabla_X f = \nabla f(X).$$

[1](1) is also equivalent to $\nabla g = 0$. (We shall talk about this later)

Locally,

$$\mathrm{grad} f = g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i} = \nabla^i f \frac{\partial}{\partial x^i}.$$

Definition 3.3.10 (Hessian of f)**.**

$$\nabla\nabla f \colon \Gamma(TM)\times\Gamma(TM) \to C^\infty(M)$$

$$(X,Y)\mapsto \nabla\nabla f(X,Y).$$

Where $\nabla\nabla f(X,Y) = (\nabla_X \nabla f)(Y)$[2] $= \nabla_X(\nabla f(Y)) - \nabla f(\nabla_X Y) = X(Yf) - ()\nabla_X Y)f$.

Locally:

$$\begin{aligned}\nabla\nabla f\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) &= \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) - \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} f\\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}.\end{aligned}$$

Remark. (1) We conventionally write $\nabla\nabla f(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}) = \nabla_i\nabla_j f$, *i.e.*

$$\nabla_i\nabla_j f = \frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}.$$

- $\nabla_i\nabla_j f$: tensor component.
- $\frac{\partial^2 f}{\partial x^i\partial x^j}$: Euclidean Hessian.
- $\Gamma^k_{ij}\frac{\partial f}{\partial x^k}$ Correction.

(2) $\nabla_i\nabla_j f = \nabla_j\nabla_i f$.(Symmetric in i and j)

(3) In tensor notation we write this as

$$\nabla\nabla f = \left(\nabla_i\nabla_j f\right) dx^i dx^j.$$

Definition 3.3.11 (Laplacian of f)**.** $\Delta f = \mathrm{trace}\left(\nabla\left(\mathrm{grad} f\right)\right)$.

Locally, $\mathrm{grad} f = \nabla^i f\frac{\partial}{\partial x^i} \Rightarrow \nabla\left(\mathrm{grad} f\right) = \nabla_j\nabla^i f dx^j \otimes \frac{\partial}{\partial x^i}$.

$$\begin{aligned}\Delta f &= \nabla_i\nabla^i f = g^{ij}\nabla_i\nabla_j f\\ &= g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right).\end{aligned}$$

Exercise. *Show that*

$$\Delta f = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\cdot g^{ij}\frac{\partial f}{\partial x^j}\right).$$

[2] Note that here we haven't defined the covariant derivative of a 1-form

Note that the expression of R.H.S follows from "integration by parts", *i.e.* $\forall h \in C_c^\infty(M)$,

$$\int_M h\Delta f\sqrt{g}\mathrm{d}x = -\int_M \langle \nabla h, \nabla f\rangle \sqrt{g}\mathrm{d}x.$$

Next, we consider taking 2^{nd} order derivative of a vector field Z along two different vector fields X and Y. Let's do a general local computation.(The following computation should be very familiar with you)

Let $X = X^i\frac{\partial}{\partial x^i}, Y = Y^j\frac{\partial}{\partial x^j}, Z = Z^k\frac{\partial}{\partial x^k}$

$$\begin{aligned}\Rightarrow \nabla_X\nabla_Y Z &= \nabla_X(\nabla_Y Z)\\ &= \nabla_X\left(\left(Y^j\nabla_j Z^k\right)\frac{\partial}{\partial x^k}\right)\\ &= \left(X^i\nabla_i Y^j\nabla_j Z^k + X^iY^j\nabla_j\nabla_i Z^k\right)\frac{\partial}{\partial x^k}\\ \nabla_Y\nabla_X Z &= \left(Y^j\nabla_j X^i\nabla_i Z^k + X^iY^j\nabla_j\nabla_i Z^k\right)\frac{\partial}{\partial x^k}\end{aligned}$$

$$\begin{aligned}&\Rightarrow \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z\\ &= X^iY^j\left(\nabla_i\nabla_j Z^k - \nabla_j\nabla_i Z^k\right)\frac{\partial}{\partial x^k} + \left(\nabla_{\nabla_X Y}Z^k - \nabla_{\nabla_Y X}Z^k\right)\frac{\partial}{\partial x^k}\\ &= X^iY^j\left(\nabla_i\nabla_j Z^k - \nabla_j\nabla_i Z^k\right)\frac{\partial}{\partial x^k} + \left(\nabla_{[X,Y]}Z^k\right)\frac{\partial}{\partial x^k}.\end{aligned}$$

$\therefore \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = X^iY^j\left(\nabla_i\nabla_j Z^k - \nabla_j\nabla_i Z^k\right)\frac{\partial}{\partial x^k}$.

Definition 3.3.12. The (1,3) Riemannian curvature tensor is defined by a map

$$R\colon \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$$
$$(X,Y,Z)\mapsto R(X,Y)Z$$
$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

Locally,

$$R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R_{ij\ k}^{\ \ l}\frac{\partial}{\partial x^l}$$

Exercise. *(1) Check R is C^∞ in X, Y, Z.*

(2) Find a local expression for $R_{ij\ k}^{\ \ l}$ in terms of Christoffel symbols.

Computation: take $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$,

$$\begin{aligned}R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} &= \nabla_i\nabla_j\frac{\partial}{\partial x^k} - \nabla_j\nabla_i\frac{\partial}{\partial x^k}\\ &= \nabla_i\left(\Gamma_{jk}^l\frac{\partial}{\partial x^l}\right) - \nabla_j\left(\Gamma_{ik}^l\frac{\partial}{\partial x^l}\right)\\ &= \left(\partial_i\Gamma_{jk}^l\right)\frac{\partial}{\partial x^l} + \Gamma_{jk}^l\Gamma_{il}^p\frac{\partial}{\partial x^p} - \partial_j\Gamma_{ik}^l\frac{\partial}{\partial x^l} - \Gamma_{ik}^l\Gamma_{jl}^p\frac{\partial}{\partial x^p}\\ &= \left(\partial_i\Gamma_{jk}^l + \Gamma_{jk}^p\Gamma_{ip}^l - \partial_j\Gamma_{ik}^l - \Gamma_{ik}^p\Gamma_{jp}^l\right)\frac{\partial}{\partial x^l}\\ &= \left(\partial_i\Gamma_{jk}^l - \partial_j\Gamma_{ik}^l + \Gamma_{jk}^p\Gamma_{ip}^l - \Gamma_{ik}^p\Gamma_{jp}^l\right)\frac{\partial}{\partial x^l}\end{aligned}$$

Notation: $R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}=R_{ij\ k}^{\ \ l}\frac{\partial}{\partial x^l}$.

$$R_{ij\ k}^{\ \ l}=\partial_i\Gamma_{jk}^l-\partial_j\Gamma_{ik}^l+\Gamma_{jk}^p\Gamma_{ip}^l-\Gamma_{ik}^p\Gamma_{jp}^l$$

Definition 3.3.13. The (0,4) Riemannian curvature tensor is defined as

$$Rm:\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\to C^\infty(M)$$

$$(X,Y,W,Z)\mapsto Rm(X,Y,W,Z)=g\left(R(X,Y),Z,W\right)$$

Locally, $R_{ijlk}=R(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^k})=g\left(R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)=g\left(R_{ij\ k}^{\ \ p}\frac{\partial}{\partial x^p},\frac{\partial}{\partial x^l}\right)=R_{ij\ k}^{\ \ p}g_{pl}$

$$\Rightarrow R_{ijlk}=R_{ij\ k}^{\ \ p}g_{pl}\ \text{(pull-down the 3}^{\text{rd}}\text{ index)},$$

$$\Rightarrow R_{ij\ k}^{\ \ p}=R_{ijlk}g^{lp}\ \text{(lift-up the 3}^{\text{rd}}\text{ index)}.$$

Remark. For a pair of vector fields X,Y, $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ is an antisymmetric (in X and Y) operator from $\Gamma(TM)\to\Gamma(TM)/$

$$\Rightarrow R(X,Y)\in End\left(\Gamma(TM)\right)=\Gamma\left(T^*M\otimes TM\right).$$

Definition 3.3.14. At $p\in M$, let π be a 2-plane generated by $v,w\in T_pW$, the sectional curvature of π is

$$K_p(\pi)=\frac{Rm(v,w,v,w)}{\|v\wedge w\|^2},$$

where $\|v\wedge w\|^2=g(v,v)g(w,w)-g(v,w)^2$. (Area of π)

Important properties of (0,4) Riemannian curvature tensor:

(1) $Rm(X,Y,W,Z)=-Rm(Y,X,W,Z)$.

(2) $Rm(X,Y,W,Z)+Rm(Y,Z,W,X)+Rm(Z,X,W,Y)=0$ This is called the 1st Bianchi identity.

(3) $Rm(X,Y,W,Z)=Rm(W,Z,X,Y)$.

Locally,

(a) $-R_{jikl}\overset{(1)}{=}R_{ijkl}\overset{(3)}{=}R_{klij}\overset{(1)}{=}-R_{lkij}\overset{(3)}{=}-R_{ijlk}$.

(b) $R_{ijlk}+R_{jkli}+R_{kilj}=0$.

Remark. There are also 2nd Bianchi identity:

$$\nabla_U R(X,Y,V,W)+\nabla_V R(X,Y,W,U)+\nabla_W R(X,Y,U,V)=0,$$

and contracted 2nd Bianchi identity:

$$\nabla_X\underbrace{Ric(X,Y)}_{\text{Ricci curvature}}=\frac{1}{2}\nabla_Y\underbrace{[1pt][7pt]S}_{\text{Scaler curvature}}$$

We are not going to use these two important Bianchi identities in this course.

Theorem 3.3.15. *In 2-d, the sectional curvature is just the Gaussian curvature.*

Proof. (1) We have seen there are 4 Gaussian equations

$$Kg_{11} = \partial_2\Gamma^2_{11} - \partial_1\Gamma^2_{12} + \Gamma^2_{2p}\Gamma^p_{11} - \Gamma^2_{1p}\Gamma^p_{12} = R_{21}{}^2{}_1$$

$$Kg_{12} = \partial_1\Gamma^1_{12} - \partial_2\Gamma^1_{11} + \Gamma^1_{1p}\Gamma^p_{12} - \Gamma^1_{2p}\Gamma^p_{11} = R_{12}{}^1{}_1$$

$$Kg_{21} = \partial_2\Gamma^2_{12} - \partial_1\Gamma^2_{22} + \Gamma^2_{2p}\Gamma^p_{12} - \Gamma^2_{1p}\Gamma^p_{22} = R_{21}{}^2{}_2$$

$$Kg_{22} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1_{1p}\Gamma^p_{22} - \Gamma^1_{2p}\Gamma^p_{12} = R_{12}{}^1{}_2$$

(2)

$$R_{21}{}^2{}_1 = R_{21p1}g^{p2} = \underbrace{R_{2111}}_{0} g^{12} + R_{2121}g^{22} = R_{1212}g^{22}$$

$$R_{12}{}^1{}_1 = R_{12p1}g^{p1} = \underbrace{R_{1211}}_{0} g^{11} + R_{1221}g^{21} = -R_{1212}g^{21}$$

$$R_{21}{}^2{}_2 = R_{21p2}g^{p2} = R_{2112}g^{12} + \underbrace{R_{2122}}_{0} g^{22} = -R_{1212}g^{12}$$

$$R_{12}{}^1{}_2 = R_{12p2}g^{p1} = R_{1212}g^{11} + \underbrace{R_{1222}}_{0} g^{21} = -R_{1212}g^{11}$$

(1) and (2) $\Rightarrow$

$$K\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22}\end{pmatrix} = R_{1212}\begin{pmatrix} g^{22} & -g^{12} \\ -g^{12} & g^{11}\end{pmatrix}. \qquad (\star)$$

Note $g^{11} = \frac{1}{\det g}g_{22}$, $g^{12} = -\frac{1}{\det g}g_{12}$, $g^{22} = \frac{1}{\det g}g_{11}$

$$K\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22}\end{pmatrix} = R_{1212}\cdot\frac{1}{\det g}\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22}\end{pmatrix}.$$

i.e.

$$\boxed{K = \frac{R_{1212}}{\det g}.}$$

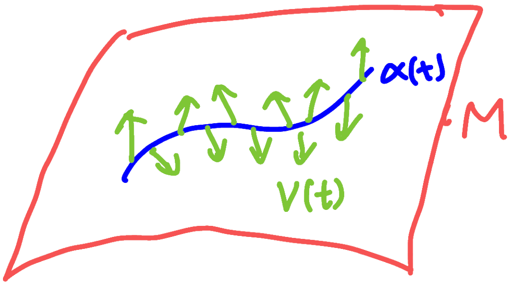
□

Note: $R_{1212} = R\left(\frac{\partial}{\partial 1}, \frac{\partial}{\partial 2}, \frac{\partial}{\partial 1}, \frac{\partial}{\partial 2}\right)$, $\det = g_{11}g_{22} - g_{12}{}^2 =$ Area of plane generated by $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$.

3.4 Parallel transportation and Geodesics

3.4.1 Parallel transport

Let (M, g) be an n-dimensional Riemannian manifold, $v \in \Gamma(TM)$ is a smooth vector field. $\alpha(t) \subset M$ is a regular curve on M. Let $v(t) = v|_{\alpha(t)}$ be the restriction of V on the curve.



We can take "derivatives" of $V(t)$ along $\alpha(t)$, denoted as $\frac{DV(t)}{dt}$. To understand this, we shall compute this locally. Let $(x^1, \ldots, x^n)$ be a local coordinate, write $\alpha(t) = ((x^1(t), \ldots, x^n))$, $V(t) = V^i(t)\frac{\partial}{\partial x^i}$, $\dot{\alpha}(t) = \dot{x}^i(t)\frac{\partial}{\partial x^i}$.

$$
\begin{aligned}
\frac{DV(t)}{dt} &= \frac{D}{dt}\left(V^i(t)\frac{\partial}{\partial x^i}\right) = \dot{V}^i(t)\frac{\partial}{\partial x^i} + V^i(t)\nabla_{\dot{\alpha}(t)}\frac{\partial}{\partial x^i} \\
&= \dot{V}^i(t)\frac{\partial}{\partial x^i} + V^i(t)\dot{x}^j(t)\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} = \dot{V}^i(t)\frac{\partial}{\partial x^i} + V^i(t)\dot{x}^j(t)\Gamma^k_{ji}\frac{\partial}{\partial x^k} \\
&= \left(\dot{V}^i(t) + \Gamma^i_{jk}\dot{x}^j(t)V^k(t)\right)\frac{\partial}{\partial x^i}
\end{aligned}
$$

i.e.

$$
\frac{DV(t)}{dt} = \left(\left(\dot{V}^i(t) + \Gamma^i_{jk}\dot{x}^j(t)V^k(t)\right)\right)\frac{\partial}{\partial x^i} \tag{$\star$}
$$

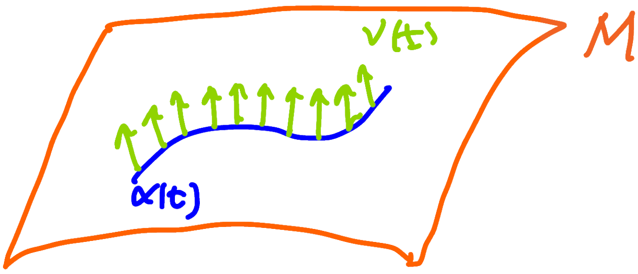
Remark. Let S be a regular surface in $\mathbb{R}^3$, $\alpha(t) \in S$ is a regular curve. V is a smooth vector field on S. $V|_{\alpha(t)}$ is a vector field along $\alpha(t)$. We can also view $V(t)$ as a vector field in $\mathbb{R}^3$. Take the usual derivative of $V(t)$ in $\mathbb{R}^3$, *i.e.* $\frac{\mathrm{d}V(t)}{\mathrm{d}t}$, and let $\frac{DV(t)}{dt}$=tangential part of $\frac{\mathrm{d}V(t)}{\mathrm{d}t}$ on S, *i.e.* taking the projection of $\frac{\mathrm{d}V(t)}{\mathrm{d}t}$ on TS at each point.

Exercise. *Check: this definition coincides with* $\star$.

(Hint: using the equation of motion of coordinate frame.)

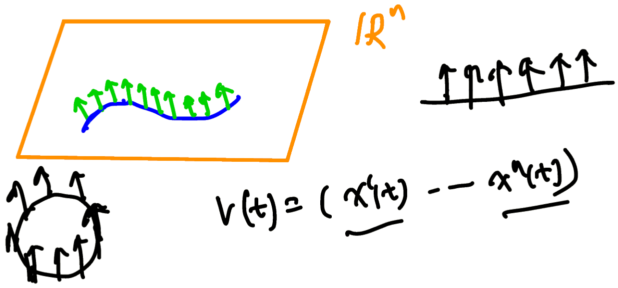
Definition 3.4.1 (parallel vector field)**.** (1) A vector field V along a curve $\alpha(t)\colon I \to M$ is called parallel if

$$
\frac{DV(t)}{dt} = 0.
$$

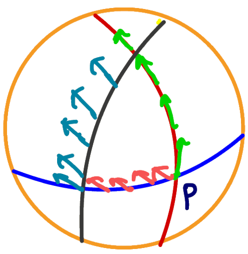


(2) In general, we call a vector field V to be parallel, if $\nabla V = 0$. Equivalently, along any curve $\alpha(t)$ in M, $\frac{DV(t)}{dt} = 0$.

Example 3.4.2. *(1) In* $\mathbb{R}^n$*,* $\frac{DV(t)}{dt} = 0 \Rightarrow V$ *is a constant vector field along the curve.*



(2) $\mathbb{S}^2$*, the tangent vector field of a great circle is a parallel vector field along this circle (Green arrows).*



at p. the initial vector is ↑ and after the parallel transport, coming back at p ending w/ ↑

Let $\alpha(\theta)$ *be the great circle, after a rotation we assume*

$$\alpha(\theta) = (\cos\theta, \sin\theta, 0)$$

to be the equator. The tangent vector field of $\alpha(\theta)$ *is*

$$\alpha'(\theta) = (-\sin\theta, \cos\theta, 0)\,.$$

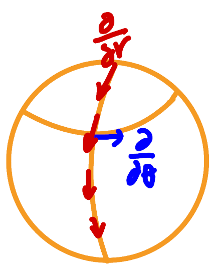
$$\Rightarrow \frac{\mathrm{d}\alpha'(t)}{\mathrm{d}t} = -\alpha(\theta) = \textit{normal field of } \mathbb{S}^2.$$

$\Rightarrow \alpha''(\theta)$ *has no projection part on* $T\mathbb{S}^2$*, i.e.* $\frac{D\alpha'(t)}{dt} = 0$.

Remark. The 1st fundamental form on $\mathbb{S}^2$ is

$$ds^2 = dr^2 + \sin^2 r d\theta^2.$$

At each point p, $T_pS = \mathrm{span}\{\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\}$.



Fixing θ, then $\frac{\partial}{\partial r}$ is the tangent vector field of the great circle, then

$$\frac{D}{dr}\left(\frac{\partial}{\partial r}\right) = \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = \Gamma^r_{rr}\frac{\partial}{\partial r} + \Gamma^\theta_{rr}\frac{\partial}{\partial \theta},$$

where

$$\Gamma^r_{rr} = \frac{1}{2}g^{rk}\left(\frac{\partial g_{rk}}{\partial r} + \frac{\partial g_{rk}}{\partial r} - \frac{\partial g_{rr}}{\partial x^k}\right) = \frac{1}{2}g^{rr}\frac{\partial g_{rr}}{\partial r} = 0.$$

$$\Gamma^\theta_{rr} = \frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{r\theta}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta}\right) = 0$$

$\Rightarrow \frac{D}{dr}\frac{\partial}{\partial r} = 0 \Rightarrow \frac{\partial}{\partial r}$ is parallel along itself.

If we compute

$$\frac{D}{d\theta}\frac{\partial}{\partial \theta} = \nabla_{\frac{\partial}{\partial \theta}}\frac{\partial}{\partial \theta} = \Gamma^r_{\theta\theta}\frac{\partial}{\partial r} + \Gamma^\theta_{\theta\theta}\frac{\partial}{\partial \theta} = -\sin r\cos r\frac{\partial}{\partial r}.$$

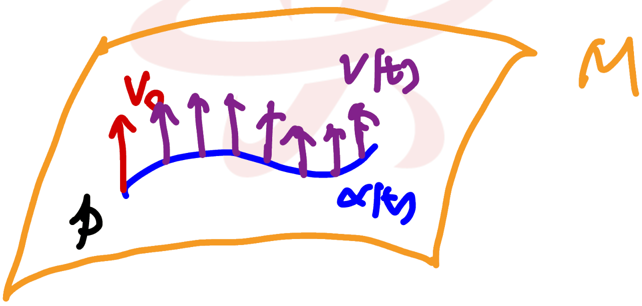
Hence,

$$\frac{D}{d\theta}\frac{\partial}{\partial \theta} = 0 \Longleftrightarrow r = \frac{\pi}{2}.$$

Facts:

$$\frac{DV(t)}{dt} = 0 \Longleftrightarrow \dot{V}^i(t) + \Gamma^i_{jk}(t)\dot{x}^j(t)V^k(t) = 0.$$

This is the 1^{st} order linear O.D.E. system of $V(t)$. Hence, given a curve $\alpha(t)$, let $p \in \alpha(t)$, $v_0 \in T_pM$, then by the Solution to the Cauchy problem, there is a unique parallel vector field $V(t)$ along $\alpha(t)$ such that $V(0) = v_0$.



Consider $p, q \in M$, let $\alpha(t)\colon [0,1] \to M$ such that $\alpha(0) = p, \alpha(1) = q, v_0 \in T_pM$, and $V(t)$ is the parallel vector field along $\alpha(t)$ with $V(0) = v_0$, then $V(1) \in T_qM$ is called the parallel transportation of v_0 at q.

Proposition 3.4.3. *Let V_1, V_2 be two parallel vector fields along a curve $\alpha(t)$, then $\langle V_1, V_2\rangle$ is a constant along $\alpha(t)$.*

Proof. $\frac{\mathrm{d}}{\mathrm{d}t}\langle V_1(t), V_2(t)\rangle = \left\langle \frac{\mathrm{d}V_1(t)}{\mathrm{d}t}, V_2(t)\right\rangle + \left\langle \frac{\mathrm{d}V_2(t)}{\mathrm{d}t}, V_1(t)\right\rangle = 0.$ □

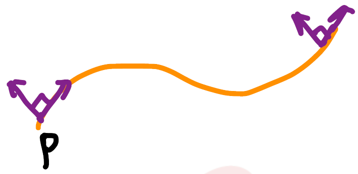
Corollary 3.4.4. *The parallel transport along a curve does not change the length or angle of original vectors.*

In particular, the map

$$P_{\alpha(1)} : T_pM \to T_qM$$

is a linear isometry.

Remark. The advantage of parallel transport is helping us choose an orthonormal frame along a curve $\alpha(t)$. At T_pM, choose orthonormal basis $\{E_1, \dots, E_n\}$ and parallel transport E_i along α, then at each point of the curve $\{E_1(t), \dots, E_n(t)\}$is an orthonormal basis of $T_{\alpha(t)}M$.



For any vector field $V(t)$ along $\alpha(t)$, $V(t) = V^i(t)E_i(t)$,

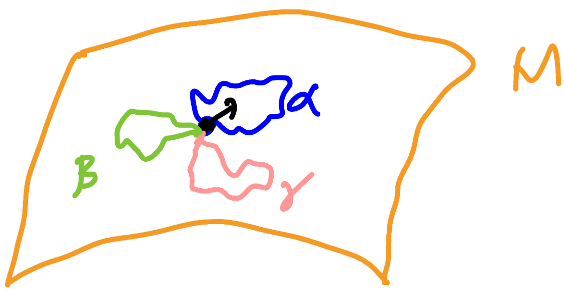
$$\frac{DV(t)}{dt} = \frac{D}{dt}\left(V^i(t)E_i(t)\right) = \dot{V}^i(t)E_i(t) + V^i(t)\underbrace{\frac{DE_i(t)}{dt}}_{=0}.$$

$$\frac{DV(t)}{dt} = \dot{V}^i(t)E_i(t).$$

Remark. $\forall p \in M$, let α be a piecewise smooth closed curve passing through p, then the parallel transport $P_\alpha : T_pM \to T_pM$ yields a linear isometry of T_pM. If β is another piecewise closed curve passing through p, we have another linear isometry P_β.

Definition 3.4.5.

$$\mathrm{Hol}_p(M, g) = \{P_\alpha : T_pM \to T_pM, \alpha\text{: closed curve passing through } p\}.$$



This set has a group structure given by composition of parallel transport. $\mathrm{Hol}_p(M, g)$ is called the Holonomy group of M at p. In fact, it's a Lie group. Note that $\mathrm{Hol}(M, g)$ relies on the Riemannian metric g.

! The classification of Holonomy group is useful to understand the classification of Riemannian manifold (metric). Details can be found in Peter Perterson's Riemannian Geometry book p.388.

Definition 3.4.6 (Geodesics)**.** $\gamma(t)$ is a parametrized curve in M, if

$$\frac{D}{dt}\dot{\gamma}(t) = 0 \left(\Longleftrightarrow \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0 \text{ or simply } \nabla_{\dot{\gamma}}\dot{\gamma}\right).$$

i.e. $\dot{\gamma}(t)$ is a parallel vector field along $\gamma(t)$, then we call $\alpha(t)$ a geodesic.

Remark. By definition, $\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle$ is constant, after reparametrization, $\gamma(t) = \gamma(s)$, $s = ct$ is the arclength parameter, $\therefore$ a curve $\alpha(s)$ with arclength parameter is a geodesic iff $\frac{D}{ds}(\alpha'(s)) = 0$.

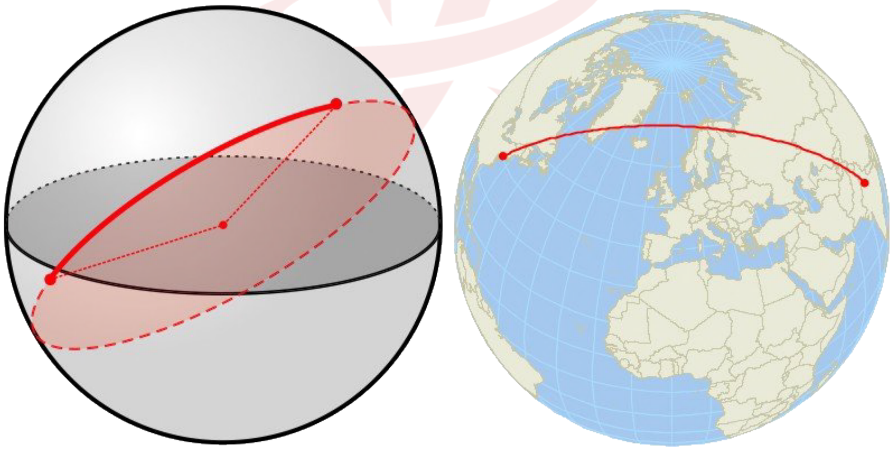
3.4.2 Geodesics on surfaces

Let S be a regular surface in $\mathbb{R}^3$. Let $\alpha(s)$ be a geodesic on S with arclength parameter. Then $\alpha''(s)$ (in $\mathbb{R}^3$) has the following decomposition

$$\begin{aligned}
\alpha''(s) &= \frac{D}{ds}(\alpha'(s)) + \left\langle \alpha''(s), \vec{N}(\alpha(s))\right\rangle \vec{N} \\
&= \left\langle \alpha''(s), \vec{N}(\alpha(s))\right\rangle \vec{N} \\
&= k_{\mathbf{n}}(s)\vec{N}.^{3}
\end{aligned}$$

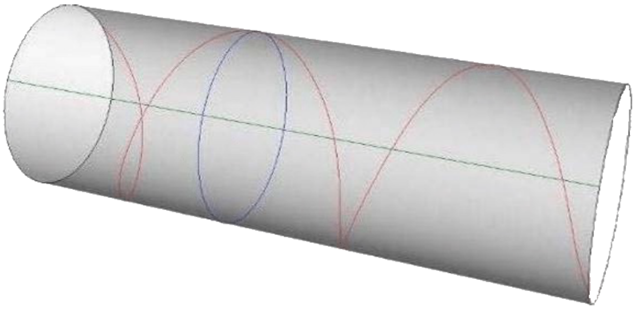
$\therefore \alpha(s)$ is a geodesic $\Longleftrightarrow$ principal normal $\vec{\mathbf{n}}(s)$ is parallel to the normal of surface $\vec{N}$. Note normal section is a geodesic, but geodesic may not be a normal section.

(1) $\mathbb{S}^2$. Great circles are geodesics.



(2) On Cylinder, $x^2 + y^2 = 1$,

[3] Recall that $k_{\mathrm{n}}(s) = II(\alpha'(s), \alpha'(s))$.



The red curve is a geodesic, but not a normal section

consider the local parametrization

$$\gamma(u,v) = (\cos u, \sin u, v)$$

$\Rightarrow$ lines, horizontal circles, and helix curve $(\cos as, \sin as, bs)$ are all geodesics.

3.4.3 Geodesic curvature

Let $\alpha(s)\colon I \to S$ be a regular curve, at each point $p \in \alpha$. We have two o.n. frames:

(1) Frenet's formula $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$;

(2) $\{\vec{t}(s), N(s) \times \vec{t}(s), N(s)\}$, $N(s)$ is unit normal of S at $\alpha(s)$.

Using the second frame, we have

$$\langle \alpha''(s), N(s)\rangle = II(\alpha'(s), \alpha'(s)) = k_n, \text{ normal curvature at } \alpha(s).$$

Definition 3.4.7 (Geodesic curvature)**.**

$$k_g := \left\langle \alpha''(s), N(s) \times \vec{t}(s) \right\rangle$$

is called geodesic curvature.

By the Frenet formula, $\alpha''(s) = \vec{t}'(s) = k\vec{n}(s)$, let θ be the angle between $\vec{n}(s)$ and $N(s)$, then

$$k_n = k\cos\theta = \langle \alpha''(s), N(s)\rangle \qquad \text{normal curvature} \tag{3.4.1}$$

$$k_g = \pm k \sin\theta = \langle \alpha''(s), \rangle \qquad \text{geodesic curvature.} \tag{3.4.2}$$

The sign of expression of k_g depends on the choice of normal of surface and orientation of $\alpha(s)$.

Hence

$$k^2 = k_n^2 + k_g^2.$$

And $\alpha(s)$ is geodesic $\iff \vec{n} \parallel N \iff k_n = k \iff k_g = 0$.

Exercise (Liouvill formula)**.** *Let $\varphi(u,v)$ be an orthogonal parametrization of a regular surface S, $\alpha(s)$ be a regular curve on S with arclength parameter. Let $\theta(s)$ be the angle between u-curve and $\alpha'(s)$. Then*

$$k_g = \frac{\mathrm{d}\theta}{\mathrm{d}s} - \frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta + \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta.$$

3.4.4 Length minimizing curves (variational viewpoint)

(M, g) be Riemannian manifold, let $\gamma(s)\colon [0, l] \to M$ be a curve with arc-length parameter. $\gamma(0) =: p, \gamma(l) =: q$. Let $\Gamma(s,t)$ be a family of smooth variation with endpoints fixed. *i.e.* $\Gamma(s,t)\colon [0,l] \times (-\varepsilon, \varepsilon) \to M$ *s.t.* $\forall\, t \in (-\varepsilon, \varepsilon)$, $\Gamma(s, t_0)$ is a C^∞ curve with initial point p and endpoint q, and, $\Gamma(s, 0) = \gamma(s)$.

Consider the arc-length functional

$$L(\Gamma(s,t)) = \int_0^l \left| \frac{\partial \Gamma(s,t)}{\partial s} \right| \mathrm{d}s\,.$$

Then

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L(\Gamma(s,t)) &= \int_0^l \frac{1}{2} \langle \dot\gamma(s), \dot\gamma(s) \rangle^{-\frac{1}{2}} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \frac{\mathrm{d}\Gamma(s,t)}{\mathrm{d}s}, \frac{\mathrm{d}\Gamma(s,t)}{\mathrm{d}s} \right\rangle \mathrm{d}s \\ &= \int_0^l \left\langle \left.\frac{\mathrm{D}}{\mathrm{d}t}\right|_{t=0} \left(\frac{\partial \Gamma(s,t)}{\partial s} \right), \dot\gamma(s) \right\rangle \mathrm{d}s\,. \end{aligned} \tag{3.4.3}$$

Lemma 3.4.8.

$$\frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial \Gamma(s,t)}{\partial s} \right) = \frac{\mathrm{D}}{\mathrm{d}s}\left(\frac{\partial \Gamma(s,t)}{\partial t} \right).$$

Proof. Let $(x^1, \dots, x^n)$ be local coordinate *s.t.* $\Gamma(s,t)$ is contained in this coordinate chart.

$$\Longrightarrow \Gamma(s,t) = (x^1(s,t), \dots, x^n(s,t)) \Longrightarrow \begin{cases} \frac{\partial \Gamma}{\partial s} = x^i_s \frac{\partial}{\partial x^i} \\ \frac{\partial \Gamma}{\partial t} = x^i_t \frac{\partial}{\partial x^i} \end{cases}.$$

$$\begin{aligned} \Longrightarrow \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial \Gamma}{\partial s} \right) &= \frac{\mathrm{D}}{\mathrm{d}t}\left(x^i_s \frac{\partial}{\partial x^i} \right) \\ &= x^i_{st} \frac{\partial}{\partial x^i} + x^i_s x^j_t \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \\ &= \frac{\mathrm{D}}{\mathrm{d}s}\left(\frac{\partial \Gamma}{\partial t} \right). \end{aligned}$$

□

Then

$$\begin{aligned} \text{eq. (3.4.3)} &= \int_0^l \left\langle \frac{\mathrm{D}}{\mathrm{d}s}\left(\left.\frac{\partial \Gamma}{\partial t}\right|_{t=0} \right), \dot\gamma(s) \right\rangle \mathrm{d}s \\ &= \left.\left\langle \left.\frac{\partial \Gamma}{\partial t}\right|_{t=0}, \dot\gamma(s) \right\rangle\right|_0^l - \int_0^l \left\langle \left.\frac{\partial \Gamma}{\partial t}\right|_{t=0}, \nabla_{\dot\gamma}\dot\gamma \right\rangle \mathrm{d}s\,. \end{aligned}$$

Note the variational family fix endpoints.

$$\Longrightarrow \left.\frac{\partial \Gamma}{\partial t}\right|_{t=0} = \left.\frac{\partial \Gamma}{\partial t}\right|_{t=0} (l) = 0$$

Hence

$$\text{eq. (3.4.3)} = -\int_0^l \left\langle \left.\frac{\partial\Gamma}{\partial t}\right|_{t=0}, \nabla_{\dot\gamma}\dot\gamma \right\rangle \mathrm{d}s$$

Hence Euler-Lagrange equation is $\nabla_{\dot\gamma}\dot\gamma = 0$ (geodesic equation).

We conclude that if $\gamma(s)$ is a curve having shortest length between two points on M, then it must be a geodesic. Conversely, we only have geodesics are local minimizing curves (prove later).

Example 3.4.9. *Every great circle on $\mathbb{S}^2$ is a geodesic.*

Note

$$\nabla_{\dot\gamma}\dot\gamma = 0 \iff \ddot{x}^k(t) + \Gamma^k_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t) = 0.$$

By ODE theory, at each point $p = (x^1(t_0), \dots, x^n(t_0))$ with any prescribed direction $v = \dot{x}^k(t_0)\frac{\partial}{\partial x^k}$, there is a unique geodesic starting at p with initial velocity v. Locally, *i.e.* $\exists\,\gamma(t)\colon (0,\varepsilon) \to M$ geodesic *s.t.* $\gamma(0) = p, \dot\gamma(0) = v$.

Remark. (1) Previously, we have considered the length functional of a family of curves $\Gamma(s,t) = \gamma_t(s)\colon [0,l] \to M$:

$$L(\gamma_t(s)) = \int_0^l |\dot\gamma_t(s)| \,\mathrm{d}s\,.$$

In fact, when we consider the Euler-Lagrange equation, it's more convenient to consider the "energy" functional

$$E(\gamma_t(s)) = \frac{1}{2}\int_0^l |\dot\gamma_t(s)|^2 \,\mathrm{d}s\,. \tag{3.4.4}$$

(Then there won't appear denominator term).

(2) A generalization of such type of functional: let $u\colon (M,g) \to (N,h)$ be C^∞ map, the energy functional is

$$E(u) = \int_M |\mathrm{d}u|^2 \,\mathrm{d}V_g\,.$$

If $(x^1, \dots, x^m)$ is local coordinate of M, $u(x) = (u^1, \dots, u^n)$, then

$$|\mathrm{d}f|^2_{g,h} = g^{ij}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}.$$

As special cases:

(a) If M is an interval $[a,b]$, $u\colon [a,b] \to (N,h)$, $E(u)$ is just eq. (3.4.4).

(b) If $M = \Omega \subset \mathbb{R}^n, N = \mathbb{R}$, *i.e.* u is smooth function $\Omega \to \mathbb{R}$ with compact support,

$$E(u) = \int_{\mathbb{R}^n} |\,\mathrm{d}f\,|^2 \,\mathrm{d}x^1 \cdots \mathrm{d}x^n\,.$$

Then Euler-Lagrange equation is just harmonic function equation.

Definition 3.4.10. The solution of Euler-Lagrange equation of eq. (3.4.4) is called harmonic maps.

3.4.5 Complete manifold

From geodesic equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, $\forall\, p \in M, v \in T_pM$, $\exists$ unique geodesic $\gamma(t)\colon (-\varepsilon, \varepsilon) \to M$ *s.t.* $\gamma(0) = p, \dot{\gamma}(0) = v$. In fact we can write $\gamma(t) = \exp_p(tv)$ locally.

A natural question to ask in how large the defining domain of $\gamma(t)$ could be.

Observation If I_1, I_2 are two interval in $\mathbb{R}, t_0 \in I_1 \cap I_2$, $\gamma_1\colon I_1 \to M, \gamma_2\colon I_2 \to M$ are two geodesics *s.t.* $\gamma_1(t_0) = \gamma_2(t_0)$ and $\dot{\gamma}_1(t_0) = \dot{\gamma}_2(t_0)$. Then γ_1 and γ_2 agree on $I_1 \cap I_2$ by uniqueness of solution of ODE.

Then $\gamma_1 \cup \gamma_2$ is an extension of geodesics γ_1 and γ_2. Furthermore, it is also a geodesic defined on $I_1 \cup I_2$.

Definition 3.4.11 (Maximal geodesic)**.**
$\gamma(t)\colon I \to M$ is called a maximal geodesic if I is the largest defining domain of $\gamma(t)$, *i.e.* it cannot be extended to a geodesic on a larger interval containing I.

Remark. A maximal geodesic can be defined on

(1) $\mathbb{R}$;

(2) Finite open interval (a, b);

(3) Half interval $(-\infty, b)$ or $(a, +\infty)$.

Definition 3.4.12 (Complete manifold)**.**
A Riemannian manifold (M, g) is called (geodesic) complete if all maximal geodesics are define on $\mathbb{R}$.

Remark. This means each geodesic starting from a point p with initial velocity v can be extended from both sides for all time. Furthermore, $\exp_p\colon T_pM \to M$ is defined for all $v \in T_pM$.

Example 3.4.13. $\mathbb{R}^2 \setminus \{0\}$ *is not geodesic complete. Consider geodesic* $\gamma(t) = (t, 0)$, *which can only be defined on* $(-\infty, 0)$.

For the completeness of a manifold, we mention the Hopf-Rinow theorem, the proof of which will be left to Riemannian geometry course.

Note the Riemannian metric g on M defines a distance function

$$d(p, q) = \inf\{L(\gamma) : \gamma \text{ piecewise smooth from } p \text{ to } q\}.$$

Then M became a metric space.

Recall a metric space (M, d) is complete if every Cauchy sequence is convergent.

Theorem 3.4.14 (Hopf-Rinow)**.** *Let* (M, g) *be a manifold without boundary, TFAE:*

(1) (M, g) *is geodesic complete.*

(2) (M, g) *is geodesic complete at some* $p \in M$, *i.e.* $\exp_p$ *is defined on whole* T_pM.

(3) (M, d) *is complete metric space.*

(4) $A \subset M$ *is compact iff* A *is bounded and closed.*

(5) Every closed geodesic ball $\overline{B}(p, r)$ *is compact.*

Corollary 3.4.15. *If (M, g) is complete, then any $p, q \in M$ can be jointed by a minimal geodesic, i.e. the distance $d(p, q)$ is realized by length of a geodesic.*

Corollary 3.4.16. *If (M, g) is closed manifold (compact without boundary), then (M, g) is complete. In particular, any surface without boundary in $\mathbb{R}^3$ which is a closed subset is complete.*

Remark. One of the famous incomplete Riemannian metric lies in the study of geometry of the moduli space (or Teichmüller space) of Riemann surfaces:

$$T_g = \frac{\{\text{Conformal structure of } \Sigma_g\}}{\text{Orientation preserving diffeomorphisms}}.$$

On T_g, each point represents a Riemann surface of genus g with a given conformal structure. There is a metric on such space defined by L^2 pairing of infinitesimal conformal structure at each point. Such metric is called the Weil-Petersson metric an it's not complete.

Key fact: Studying the metric completion of Weil-Petersson metric is related to the compactification of the moduli space! This is one of many correspondence of Differential Geometry $\longleftrightarrow$ Algebraic Geometry.

3.4.6 Exponential map & geodesic spherical coordinate

Let's first see a rescaling argument of geodesic:

We have seen $\forall\, p \in M$, along any prescribed velocity $v \in T_pM$, there is a unique geodesic $\gamma(t)\colon [0, \varepsilon) \to M$ *s.t.* $\gamma(0) = p, \dot{\gamma}(0) = v$. By rescaling the velocity $v \rightsquigarrow \lambda v$, we obtain the geodesic $\beta(\tau)\colon [0, \frac{\varepsilon}{\lambda}) \to M$, which has the same trace with $\gamma(t)$ and $\beta(0) = p, \dot{\beta}(0) = \lambda v$. In particular, we introduce notation

$$\gamma(t, v) = \gamma(t), \quad \beta(\tau, \lambda v) = \beta(\tau), \quad \lambda\tau = t.$$

Then $\beta(\tau, \lambda v) = \beta(\tau) = \gamma(t) = \gamma(t, v) = \gamma(\lambda\tau, v), \forall\, \tau \in [o, \frac{\varepsilon}{\lambda})$.

This rescaling argument allows us to consider geodesics defined on $[0, 1]$.

Definition 3.4.17 (Exponential map)**.** Let $p \in M, v \in T_pM, \gamma\colon [0, 1] \to M$ is a geodesic *s.t.* $\gamma(0) = p, \gamma'(0) = v$. The exponential map is defined as

$$\begin{aligned} \exp_p\colon T_pM &\longrightarrow M \\ v &\longmapsto \exp_p(v) = \gamma(1). \end{aligned}$$

Note the length of γ between p and $\gamma(1)$ is $|v|$, and $\gamma(t) = \exp_p(tv)$ for $t \in [0, 1]$, $\exp_p(0) = p$. Hence geodesic $\gamma(t)$ is the image of ray from 0 in T_pM under $\exp_p$.

We state the following fact without proof:

Proposition 3.4.18. $\exists\, \varepsilon > 0$ *s.t.* $\exp_p\colon B(0, \varepsilon) \to M$ *is a diffeomorphism where* $B(0, \varepsilon)$ *is ball of radius* ε *centered at origin in* T_pM*.*

Remark. The radius ε could be very large or even ∞, but also could be a finite number.

Example 3.4.19. $\exp_p\colon T_pM \to M$ *where p is north pole in $\mathbb{S}^2$.* $\exp_p(v) = q$ *the south pole if $|v| = \pi$, then $\exp_p$ fails to be a diffeomorphism on $B(0, r), r > \pi$.*

Definition 3.4.20. $W \subset M$ is called a normal neighbourhood of p, if for some $U \subset T_pM$ containing 0, $\exp_p\colon U \to W$ is diffeomorphism.

For normal neighbourhood, we can endow coordinate at p by considering the image of Cartesian coordinate and polar coordinate at origin of $T_pM \cong \mathbb{R}^n$.

Normal coordinate

Let $e_1, \dots, e_n$ be orthonormal basis of $T_pM \cong \mathbb{R}^n$, for any vector $v \in U \subset T_pM$, write $v = x^i e_i$. Then we say $q = \exp_p(v)$ has coordinate $(x^1, \dots, x^n)$. In particular, $\exp_p(tv)$ has coordinate $(tx^1, \dots, tx^n)$ and p has coordinate 0.

Since $\gamma(t) = \exp_p(tv)$ satisfies geodesic equation, it writes:

$$\gamma^k(t) = tx^k,\ \Gamma^k_{ij}(\gamma(t))x^i x^j = 0 \quad (\ddot{\gamma}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{x}^i(t)\dot{x}^j(t) = 0).$$

At $p = \gamma(0)$, $\Gamma^k_{ij}(\gamma(0))x^i x^j = 0$. Since $v = x^i e_i$ is chosen arbitrarily, we conclude that $\Gamma^k_{ij} = 0$ at p. Moreover, $g_{ij}(p) = g(e_i, e_j) = \delta_{ij}$.

We summarize that the normal coordinate $(x^1, \dots, x^n)$ in a neighbourhood of p has the following properties:

(1) $p = (0, \dots, 0)$;

(2) $g_{ij}(p) = \delta_{ij}$;

(3) $\Gamma^k_{ij}(p) = 0$;

(4) $R_{ij}{}^k{}_l(p) = (\frac{\partial}{\partial x^i}\Gamma^k_{jl} - \frac{\partial}{\partial x^j}\Gamma^k_{il})(p)$;

(5) $\triangle_g f(p) = g^{ij}(\partial_i\partial_j f - \Gamma^k_{ij}\partial_k f)(p) = \triangle f(p)$.

Geodesic spherical coordinate

Take $(\rho, \theta^1, \dots, \theta^{n-1})$ as polar coordinate in $T_pM \cong \mathbb{R}^n$. Let U be normal neighbourhood of $p \in M$. Then
Geodesic sphere = image of $\{r = r_0\}$ under $\exp_p$;
Radial geodesic = image of $\{\theta = \theta_0\}$ under $\exp_p$.

$(\rho, \theta^1, \dots, \theta^{n-1})$ is called the geodesic spherical coordinate and the vector fields are $\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \theta^1}, \cdots, \frac{\partial}{\partial \theta^{n-1}}$.

Note $\theta = \theta_0$ is a "straight ray" and ρ is just the arc-length parameter of radial geodesic. Hence in $(\rho, \theta^1, \theta^{n-1})$ coordinate the coefficient of the Riemannian metric.

$$g_{11} = g(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho}) = 1 \iff \frac{\partial}{\partial \rho} \text{ is a radial unit vector filed.}$$

Next we study $g_{\rho\theta^i}$, we'll show it's 0. Let's argue for surface since $\theta = \theta_0$ is geodesic,

$$\Longrightarrow \nabla_{\frac{\partial}{\partial \rho}} \frac{\partial}{\partial \rho} = 0 \implies \Gamma^1_{11}\frac{\partial}{\partial \rho} + \Gamma^2_{11}\frac{\partial}{\partial \theta} = 0.$$
$$\Longrightarrow \Gamma^1_{11} = 0, \Gamma^2_{11} = 0$$
$$0 = \Gamma^1_{11} = g^{12}\partial_1 g_{12} = -\frac{1}{\det g} g_{12}\partial_1 g_{12}.$$

Since $g^{22} = \frac{1}{\det g} g^{11} \neq 0$, $\partial_1 g_{12} = 0$, *i.e.* g_{12} is independent of ρ.

Note that $(x_1, x_2) = (\rho\cos\theta, \rho\sin\theta)$,

$$g_{12} = g(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \theta}) = g(\cos\theta\frac{\partial}{\partial x^1} + \sin\theta\frac{\partial}{\partial x^2}, -\rho\sin\theta\frac{\partial}{\partial x^1} + \rho\cos\theta\frac{\partial}{\partial x^2}).$$

Hence $g_{12}(\rho, \theta) = \lim_{\rho\to 0} g_{12}(\rho, \theta) = 0$.

Exercise. *Proof* $g_{\rho\theta^i} = 0$ *for general case.*

We summarize: In geodesic polar coordinate $(\rho, \theta^1, \dots, \theta^{n-1})$,

(1) $\frac{\partial}{\partial\rho}$ is a unit vector field generating a radical geodesic;

(2) $\frac{\partial}{\partial\rho} \perp \frac{\partial}{\partial\theta}$, *i.e.* radial geodesic is orthogonal to geodesic spheres.

These two results are called the *Gauss Lemma.*

Hence in geodesic polar coordinate, the metric is

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \tilde{g}_{ij}(\rho,\theta)\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j .$$

Exercise. *If S is a surface, (ρ,θ) is geodesic polar coordinate, show that*

$$\lim_{\rho\to 0}\sqrt{G} = 0, \quad \lim_{\rho\to 0}(\sqrt{G})_\rho = 1.$$

Remark. In surface case, the Gaussian curvature has expression

$$K = -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}} = -\frac{\partial^2}{\partial\rho^2}\left(\log\sqrt{G}\right) - \left(\frac{\partial}{\partial\rho}\log\sqrt{G}\right)^2.$$

So far, we have seen two useful local expression of Gaussian curvature:

(1) $\mathrm{d}s^2 = \mathrm{d}r^2 + G(r,\theta)\,\mathrm{d}\theta^2 \implies K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$;

(2) $\mathrm{d}s^2 = \lambda(u,v)(\mathrm{d}u^2 + \mathrm{d}v^2) \implies K = -\frac{1}{2\lambda}\triangle\log\lambda.$

Recall:

- $\mathbb{R}^2$: $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \implies K = 0.$
- $\mathbb{S}^2$: $\mathrm{d}s^2 = \mathrm{d}r^2 + \sin^2 r\,\mathrm{d}\theta^2 \implies K = 1.$
- $\mathbb{H}^2$: $\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\theta^2 \implies K = -1.$

Conversely, if we prescribe the Gaussian curvature $K = 0, \pm 1$, then solving ODE

$$\begin{cases} (\sqrt{G})_{rr} + K\sqrt{G} = 0, \\ \lim_{r\to 0}\sqrt{G} = 0, \ \lim_{r\to 0}(\sqrt{G})_r = 1 \end{cases}.$$

Hence, $\mathrm{d}s^2$ can be written in standard form above. This implies if two regular surfaces have the same constant Gaussian curvature, they must be locally isometric. This is *Minding's theorem.*

3.4.7 Applications of Geodesic Polar Coordinate

Recall previously, by comparing the area A of a small neighbourhood of $p \in S$ and the Area $\overline{A}$ of its Gaussian image, then

$$|K(p)| = \lim_{A \to 0} \frac{\overline{A}}{A}.$$

We also mentioned two more results:

I. Let $C(r)$ be the image of circle of radius r centered at 0 in T_pS under exponential map, *i.e.* the geodesic circle of radius r, then

$$K(p) = \lim_{r \to 0} 3 \cdot \frac{2\pi r - L(C(r))}{\pi r^3}.$$

II. Consider similarly the geodesic disk $D(r)$, then

$$K(p) = \lim_{r \to 0} 12 \cdot \frac{\pi r^2 - A(D(r))}{\pi r^4}.$$

The proof of I and II both rely on the Taylor's expansion of metric tensor.

Proof. I.

$$L(C(r)) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{2\pi - \varepsilon} \sqrt{G}\, \mathrm{d}\theta$$

Since we care abort $r \to 0$, we use Taylor's expansion of $\sqrt{G}$.

Recall that $\sqrt{G}\big|_{r=0} = 0$, $(\sqrt{G})_r\big|_{r=0} = 1$ and $K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$. Hence

$$(\sqrt{G})_{rr} + K\sqrt{G} = 0,$$

and $(\sqrt{G})_{rr}\big|_{r=0} = 0$. Take one more derivative we have

$$(\sqrt{G})_{rrr} + K_r\sqrt{G} + K(\sqrt{G})_r = 0.$$

Let $r = 0$, we see

$$K(p) = -(\sqrt{G})_{rrr}\Big|_{r=0}.$$

Hence the expansion is

$$\sqrt{G}(r,\theta) = r - \frac{r^3}{3!}K(p) + o(r^3). \tag{3.4.5}$$

Then $L(C(r)) = 2\pi r - \frac{r^3}{3}\pi K(p) + o(r^3)$, this gives I.

II.Follows by integrating eq. (3.4.5) from $r = 0$ to R. We get

$$A(R) = \int_0^R L(C(r))\, \mathrm{d}r = \pi R^2 - \frac{R^4}{12}\pi K(p) + o(R^4).$$

This gives II. $\square$

3.4.8 Geodesics are Locally Length Minimizing

$\forall\, p \in S$, take a normal neighbourhood U of p. The metric in geodesic polar coordinate is

$$ds^2 = dr^2 + G(r,\theta)\, d\theta^2\,.$$

By shrinking U, we further require $U = \{\exp_p(v) : v \in B(0,\varepsilon) \subset T_pS\}$. For any $q = (r_0, \theta_0) \in U$, we have $r_0 < \varepsilon$, consider all piecewise smooth curve from p to q on S. Let $\gamma(t)\colon [0, t_0] \to S$ be such a curve. First observe that if $\gamma(t)$ is entirely contained in U, we can let $\gamma(t) = (r(t), \theta(t))$, then

$$\gamma'(t) = r'(t)\frac{\partial}{\partial r} + \theta'(t)\frac{\partial}{\partial \theta}.$$

This gives

$$\begin{aligned} L(\gamma(t)) &= \int_0^{t_0} \left(|r'(t)|^2 + |\theta'(t)|^2 G(r,\theta)\right)^{\frac{1}{2}} dt \\ &\geq \int_0^{t_0} |r'(t)|\, dt \geq \int_0^{t_0} r'(t)\, dd = r(t_0) = r_0. \end{aligned}$$

Equality holds iff $\theta'(t) \equiv 0$. *i.e.* γ is the radical ray, which is the geodesic from p to q, given by

$$\gamma(t) = \exp_p(r(t)\frac{\partial}{\partial r}).$$

If the curve is not entirely in U, let t_1 be the first time γ meets the boundary. Then for any $t_2 < t_1$ we know the length of $\gamma\big|_{[0,t_2]}$ must be greater than the geodesic one, *i.e.*

$$L(\gamma|_{[0,t_2]}) = \int_0^{t_2} |\gamma'(t)|\, dt \geq r(t_2).$$

Let $t_2 \to t_1$ we see $L(\gamma) \geq r(t_1) = \varepsilon > r_0$.

We conclude that in U, any piecewise smooth curve has greater length than (radical) geodesic.

Remark. (1) By the Cauchy problem's solution to geodesic equation

$$\begin{cases} \ddot{x}^k(t) + \Gamma_{ij}^k(t)\dot{x}^i(t)\dot{x}^j(t) = 0 \\ \gamma(0) = p, \;\; \dot{\gamma}(0) = v_0 \end{cases}.$$

There is a unique geodesic starting at p, with initial velocity v_0. Hence in this arguement, by shrinking U if necessary, we can see in U, there is unique minimizing geodesic from p to q.

(2) Try to think about how large the normal neighbourhood could be.

This will be answered after we study the second variation of arc-length and this problem is related to the global geometry of the surface.

(3) It's important to study the behavior of a family of geodesics from a point. The local picture should be kept in mind is:

"The smaller curvature is, the faster geodesics will separate"

3.5 Quick introduction to tensor and tensor field

(I) Let V be a vector space, $\dim V = n$. V^* be the covector space={all linear functionals $\alpha\colon V \to \mathbb{R}$}. The pairing is given by

$$V^* \times V \to \mathbb{R}$$

$$(\alpha, v) \mapsto \alpha(v)$$

- A covariant l-tensor on V is a multilinear map

$$\omega\colon V \times \dots \times V \to \mathbb{R}. \tag{1}$$

A contravariant k-tensor is a multilinear map

$$X\colon V^* \times \dots \times V^* \to \mathbb{R}. \tag{2}$$

A tensor of (k,l) type (k-contravariant, l-covariant) is a multilinear map

$$T\colon V^* \times \dots \times V^* \times V \times \dots \times V \to \mathbb{R}. \tag{3}$$

Let

$$\bigotimes^l V^* = \{\text{all covariant } l\text{-tensors}\} \Rightarrow \omega \text{ in } (1) \ \in \bigotimes^l V^*$$
$$\bigotimes^k V = \{\text{all contravariant } k\text{-tensors}\} \Rightarrow X \text{ in } (2) \ \in \bigotimes^k V$$
$$\mathcal{T}^{(k,l)}(V) = \{\text{all } (k,l)\text{-type tensors}\} \Rightarrow T \in \mathcal{T}^{(k,l)}(V)$$

- Next we talk about the tensor product. Let $T \in \mathcal{T}^{(k,l)}(V)$, $S \in \mathcal{T}^{(p,q)}(V)$, then $T \otimes S \in \mathcal{T}^{(k+p,l+q)}(V)$ is defined as

$$\begin{aligned}&T \otimes S\left(\omega^1, \dots, \omega^{k+p}; X_1, \dots, X_{l+q}\right)\\ =&T\left(\omega^1, \dots, \omega^k; X_1, \dots, X_l\right) S\left(\omega^{k+1}, \dots, \omega^{k+p}; X_{l+1}, \dots, X_{l+q}\right).\end{aligned}$$

Let $V = \operatorname{span}\{E_1, \dots, E_n\}$, $V^* = \operatorname{span}\{\alpha^1, \dots, \alpha^n\}$ with $\alpha^i(E_j) = \delta^i_j$, or equivalently $E_j(\alpha^i) = \delta^i_j$.
Then

$$(1) \Rightarrow \omega = \omega_{i_1,i_2,\dots,i_l}\alpha^{i_1} \otimes \dots \otimes \alpha^{i_l}.$$

Moreover,

$$\begin{aligned}\omega\left(E_{j_1}, \dots, E_{j_l}\right) &= \omega_{i_1,i_2,\dots,i_l}\alpha^{i_1} \otimes \dots \otimes \alpha^{i_l}\left(E_{j_1} \otimes \dots \otimes E_{j_l}\right)\\ &= \omega_{i_1,i_2,\dots,i_l}\alpha^{i_1}(E_{j_1}) \cdots \alpha^{i_l}(E_{j_l})\\ &= \omega_{i_1,i_2,\dots,i_l}\delta^{i_1}_{j_1} \cdots \delta^{i_l}_{j_l} = \omega_{j_1 \dots j_l}.\end{aligned}$$

$$(2) \Rightarrow X = X^{i_1 \dots i_k} E_{i_1} \otimes \dots E_{i_k}.$$

$$(3) \Rightarrow T = T^{i_1 \dots i_k}{}_{j_1 \dots j_l} E_{i_1} \otimes \dots E_{i_k} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_l}\text{[4]}$$

(II) On manifold M, we have introduced

[4] Tensor is ordered!

- $TM = \coprod_{p\in M} T_pM$ the tangent bundle.
- $T^*M = \coprod_{p\in M} T_p^*M$ cotangent bundle.

Now, we define (k,l)-tensor bundle

$$\mathcal{T}^{(k,l)}M = \coprod_{p\in M} \mathcal{T}_p^{(k,l)}M,$$

where $\mathcal{T}_p^{(k,l)}M$ is the space of all (k,l)-tensors on T_pM. A tensor field T is a smooth map

$$T\colon M \to \mathcal{T}^{(k,l)}M$$

$$p \mapsto T_p \in \mathcal{T}_p^{(k,l)}M.$$

Let $(x^1,\dots,x^n)$ be a local coordinate, then

$$T = T^{i_1\dots i_k}_{j_1\dots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}.$$

! $T\left(\alpha^1,\dots,\alpha^k; X_1,\dots,X_l\right)$ is C^∞-linear in each variable α^i and X_j.

3.6 Induced Riemannian metric

- Let (M,g) be a Riemannian manifold. Let $p\in M$, and around p we have a local coordinate $(x^1,\dots,x^n)$
- $T_pM = \operatorname{span}\left\{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right\}$, $T_p^*M = \operatorname{span}\left\{dx^1,\dots,dx^n\right\}$. Moreover,

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j.$$

- The Riemannian metric at p is

$$g_p\colon T_pM \times T_pM \to \mathbb{R}$$

$$(V,W) \mapsto g_p(V,W)$$

For a non-zero vector $V\in T_pM$, this induces a linear map

$$\varphi_V\colon T_pM \to \mathbb{R}$$

$$W \mapsto \varphi_V(W)$$

i.e. $\varphi_V \in T_p^*M$={all linear functionals on T_pM}. In particular, let $V = \frac{\partial}{\partial x^i}$, $\forall W = \frac{\partial}{\partial x^j}$

$$\varphi_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}.$$

Hence, as an element in T_p^*M, we can write

$$\varphi_{\frac{\partial}{\partial x^i}} = g_{ij}dx^j.$$

- Discussions above says that we have a linear isomorphism

$$\begin{aligned}\varphi\colon T_pM &\to T_p^*M\\ V &\mapsto \varphi_V\end{aligned}$$

- Let $\psi\colon T_p^*M \to T_pM$ be the inverse of φ, then

$$T_pM \xrightarrow{\varphi} T_p^*M \xrightarrow{\psi} T_pM$$

$$\frac{\partial}{\partial x^i} \mapsto g_{ij}dx^j \mapsto \frac{\partial}{\partial x^i}$$

$i.e.$ $\psi\left(g_{ij}dx^j\right) = g_{ij}\psi\left(dx^j\right) = \frac{\partial}{\partial x^i} \Rightarrow \psi\left(dx^j\right) = \underbrace{g^{ij}}_{\text{inverse of }(g_{ij})}\frac{\partial}{\partial x^i}$.

Moreover, for any covector $\alpha = \alpha_j dx^j$

$$\psi\left(\alpha_j dx^j\right) = \alpha_j g^{ij}\frac{\partial}{\partial x^i}$$

Definition 3.6.1. (1)

$$\begin{aligned}\varphi\colon T_pM &\to T_p^*M\\ v &\mapsto \varphi(v) = \left(v^i g_{ij}\right)dx^j\end{aligned}$$

is called index-lowered down, usually denoted by $\varphi(v) = v_\flat$.[5]

(2)

$$\begin{aligned}\psi\colon T_p^*M &\to T_pM\\ \alpha &\mapsto \psi(\alpha) = \left(\alpha_i g^{ij}\right)\frac{\partial}{\partial x^j}\end{aligned}$$

is called index-lifted up, denoted by $\psi(\alpha) = \alpha^\sharp$.[6]

- The linear isomorphism $\sharp$ enables us to extend the Riemannian metric on T^*M, $i.e.$ $\forall p \in M$

$$\begin{aligned}\tilde{g}\colon T_p^*M \times T_p^*M &\to \mathbb{R}\\ (\alpha,\beta) &\mapsto \tilde{g}(\alpha,\beta)\end{aligned}$$

 such that

$$\tilde{g}(\alpha,\beta) := g\left(\alpha^\sharp, \beta^\sharp\right).$$

 In local coordinate, $\alpha = \alpha_i dx^i, \beta = \beta_j dx^j$

$$\begin{aligned}\tilde{g}(\alpha,\beta) = g\left(\alpha_i g^{ik}\frac{\partial}{\partial x^k}, \beta_j g^{jl}\frac{\partial}{\partial x^l}\right) &= \alpha_i g^{ik}\beta_j g^{jl} g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\\ &= \alpha_i g^{ik}\beta_j \underbrace{g^{jl}g_{kl}}_{\delta_k^j} = \alpha_i\beta_j g^{ij}.\end{aligned}$$

[5] $\flat$ for “flat” in music.
[6] $\sharp$ for “sharp” in music

i.e. $\tilde{g}(\alpha,\beta)=g^{ij}\alpha_i\beta_j$. In particular, $\tilde{g}\left(dx^i,dx^j\right)=g^{ij}$.
In general, we can extend the Riemannian metric to any tensor of the same type, *i.e.* $\forall S,T$ are two p,q tensors, locally

$$S=S^{i_1\cdots i_p}_{j_1\cdots j_q}\frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_q}.$$

$$T=T^{i_1\cdots i_p}_{j_1\cdots j_q}\frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_q}.$$

Then $\tilde{g}(S,T):=S^{i_1\cdots i_p}_{j_1\cdots j_q}T^{k_1\ldots k_p}_{l_1\ldots l_q}g_{i_1k_1}g_{i_2k_2}\cdots g_{i_pk_p}g^{j_1l_1}\cdots g^{j_ql_q}$.

Example 3.6.2. *(1) If* $S=S_{ij}dx^i\otimes dx^j$, $T=T_{pq}dx^p\otimes dx^q$, *then*

$$\tilde{g}(S,T)=S_{ij}T_{pq}g^{ip}g^{jq}.$$

(2) $\nabla f=df=f_idx^i$, $\mathrm{grad}f=g^{ij}f_j\frac{\partial}{\partial x^i}$.

$$\left.\begin{aligned}\Rightarrow\quad &|\nabla f|^2=g\left(f_idx^i,f_jdx^j\right)=g^{ij}f_if_j.\\ &|\mathrm{grad}f|^2=g^{ij}f_jg^{kl}f_lg_{ik}=g^{ij}f_if_j.\end{aligned}\right\}\Rightarrow|\nabla f|^2=|\mathrm{grad}f|^2.$$

(3) From (1), $\mathrm{Hess}f=\nabla_i\nabla_jfdx^i\otimes dx^j$, [7]

$$\Rightarrow|\mathrm{Hess}f|^2=g^{ik}g^{jl}\nabla_i\nabla_jf\nabla_k\nabla_lf.$$

3.7 Bochner formula in 2-dimensional Riemannian manifold

Theorem 3.7.1. *Let* (S,g) *be a 2-dimensional Riemannian manifold,* $f\in C^3(S)$, *then*

$$\frac{1}{2}\Delta|\nabla f|^2=|\nabla\nabla f|^2+\langle\nabla\left(\Delta f\right),\nabla f\rangle+K|\nabla f|^2.$$

Remark. (1) In your homework, you have shown that on the Euclidean space

$$\frac{1}{2}\Delta|\nabla f|^2=|\nabla\nabla f|^2+\langle\nabla\left(\Delta f\right),\nabla f\rangle.$$

(2) On a general Riemannian manifold, the Bochner formula is

$$\frac{1}{2}\Delta|\nabla f|^2=|\nabla\nabla f|^2+\langle\nabla\left(\Delta f\right),\nabla f\rangle+\mathrm{Ric}(\nabla f,\nabla f).$$

[7] Recall that $\nabla_i\nabla_jf=\frac{\partial^2f}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial f}{\partial x^k}$.

Proof.

$$
\begin{aligned}
\Delta|\nabla f|^2 &= g^{ij}\nabla_i\nabla_j\left(g^{kl}\nabla_k f\nabla_l f\right)\\
&= g^{ij}g^{kl}\left(\nabla_i\nabla_j\nabla_k f\nabla_l f+\nabla_i\nabla_j\nabla_l f\nabla_k f+\nabla_i\nabla_l f\nabla_j\nabla_k f+\nabla_i\nabla_k f\nabla_j\nabla_l f\right)\\
&= 2g^{ij}g^{kl}\nabla_i\nabla_j\nabla_k f\nabla_l f+2g^{ij}g^{kl}\nabla_i\nabla_k f\nabla_j\nabla_l f\\
&= 2|\nabla\nabla f|^2+2g^{ij}g^{kl}\nabla_i\nabla_k\nabla_j f\nabla_l f.
\end{aligned}
$$

To further compute R.H.S., we take geodesic polar coordinate

$$
ds^2=dr^2+G(r,\theta)d\theta^2\Rightarrow g^{-1}=\begin{pmatrix}1&0\\0&\frac{1}{G}\end{pmatrix}.
$$

$$
\begin{aligned}
&g^{ij}g^{kl}\nabla_i\nabla_k\nabla_j f\cdot\nabla_l f=g^{kl}\nabla_i\nabla_k\nabla^i f\nabla_l f\\
=&\nabla_i\nabla_1\nabla^i f\cdot\nabla_1 f+\frac{1}{G}\nabla_i\nabla_2\nabla^i f\nabla_2 f\\
=&\nabla_1\nabla_1\nabla^1 f\cdot\nabla_1 f+\nabla_2\nabla_1\nabla^2 f\cdot\nabla_1 f+\frac{1}{G}\left(\nabla_1\nabla_2\nabla^1 f\nabla_2 f+\nabla_2\nabla_2\nabla^2 f\nabla_2 f\right)
\end{aligned}
$$

(1)

$$
\begin{aligned}
\nabla_2\nabla_1\left(\nabla^2 f\right)\cdot\nabla_1 f&=\nabla_1\nabla_2\left(\nabla^2 f\right)\cdot+R_{21}{}^2{}_i\nabla^i f\nabla_1 f\\
&=\nabla_1\nabla_2\left(\nabla^2 f\right)\cdot\nabla_1 f+R_{212i}g^{22}\nabla^i f\nabla_1 f\\
&=\nabla_1\nabla_2\nabla^2 f\cdot\nabla_1 f+R_{2121}\frac{1}{G}\nabla^1 f\nabla_1 f.
\end{aligned}
$$

(2)

$$
\begin{aligned}
\nabla_1\nabla_2\nabla^1 f\cdot\nabla_2 f&=\nabla_2\nabla_1\nabla^1 f\cdot\nabla_2 f+R_{12}{}^1{}_i\nabla^i f\nabla_2 f\\
&=\nabla_2\nabla_1\nabla^1 f\cdot\nabla_2 f+R_{1212}\nabla^2 f\nabla_2 f.
\end{aligned}
$$

Note that $R_{1212}=R_{2121}$ and $K=\dfrac{R_{1212}}{G}$.

$$
\begin{aligned}
&g^{kl}\nabla_i\nabla_k\nabla^i f\cdot\nabla_l f\\
=&\underbrace{\nabla_1\nabla_1\nabla^1 f\nabla_1 f+\nabla_1\nabla_2\nabla^2 f\nabla_1 f}_{(1)}+\underbrace{K\nabla^1 f\nabla_1 f}_{(2)}\\
&+\underbrace{\frac{1}{G}\left(\nabla_2\nabla_1\nabla^1 f\nabla_2 f+\nabla_2\nabla_2\nabla^2 f\nabla_2 f\right)}_{(1)}+\underbrace{K\nabla^2 f\nabla_2 f}_{(2)}\\
=&\nabla_1\left(\Delta f\right)\nabla_1 f+\frac{1}{G}\left(\nabla_2(\Delta f)\nabla_2 f\right)+K|\nabla f|^2\\
=&\langle\nabla(\Delta f),\nabla f\rangle+K|\nabla f|^2.
\end{aligned}
$$

Hence, $\frac{1}{2}\Delta|\nabla f|^2=|\nabla\nabla f|^2+\langle\nabla\left(\Delta f\right),\nabla f\rangle+K|\nabla f|^2$. □

To obtain interesting geometry results, it(Bochner formula) needs to combine with analysis tools, one of such tools is the following maximum principle.

Theorem 3.7.2 (Hopf-Calabi: Maximum principle)**.**

- (M, g) *is a connected Riemannian manifold.*
- $f\colon M \to \mathbb{R}$*:* C^∞ *subharmonic function, i.e.* $\Delta_g f \geq 0$*.*

$\Rightarrow f$ *attains no maximum unless it's constant.*

The proof is based on the standard maximum principle in the P.D.E. theory. We'll show, if f attains maximum at $p \in M$, then f must be a constant. (Again, we'll consider the dim $M = 2$ case in the following, and in higher dimension arguments are similar.)

Step 1 (Strong version) If $\Delta f > 0$, then f has no local maximum.

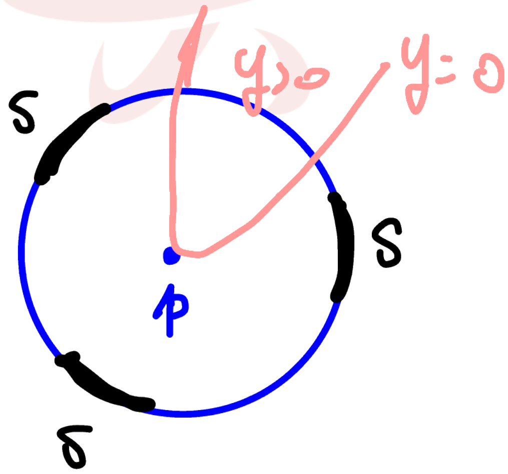
Proof. If f attains local max at $p \in M$, then $\nabla f(p) = 0$ and $\nabla\nabla f \leq 0 \Rightarrow \Delta f(p) \leq 0$. $\square$

Step 2 (Weak version) If $\Delta f \geq 0$, and f attains global maximum at p, then f is constant.

Proof. (By perturbation method) If f attains global maximum at p, then p is also a local maximum of f. Assume f is not a constant in any neighborhood of p, then we may choose a sufficient small coordinate ball(geodesic ball) B of p such that

$$S = \{x \in \partial B | f(x) \geq f(p) - 1\}$$[8]

(You can choose 1 here after replacing f by some λf) is a proper subset of ∂B (*i.e.* there must be some $\bar{q} \in \partial B$, $f(\bar{q}) < f(p)$).



By a suitable choice of coordinate (x, y), we can further assume p lies on $y = 0$ and S lies on $\{y < 0\}$. Consider coordinate $y(q)$ for $\forall q \in B_p(r)$, then y is a smooth function on $\bar{B}$.

Note that

(1) $y(p) = 0$, and $y < 0$ on S.

[8]This is modified slightly by the TA.

(2) $|\nabla y|^2 = |g^{22}| \neq 0$(By positive-definiteness).

(3) $|\Delta y| = |g^{ij}\Gamma^2_{ij}| \leq A$ for some constant $A > 0$.

Let $h = e^{cy} - 1$ for $c \gg 1$, then

(1) $h(p) = 0$.

(2) $\Delta h = ce^{cy}\left(c|\nabla y|^2\right) + \Delta y$. Note that $|\Delta y| \leq A$, $|\nabla y| \neq 0$, we can choose c sufficiently large such that

$$\Delta h > 0 \text{ on } \bar{B}.$$

(3) $h < 0$ on S.

Now consider a perturbation of f

$$f_\epsilon = f + \epsilon h, 0 < \epsilon \ll 1.$$

Then

(1) $f_\epsilon(p) = f(p) + \epsilon h(p) = f(p)$.

(2) $\Delta f_\epsilon = \Delta f + \epsilon \Delta h > 0$ on $\bar{B}$.

(3) f_ϵ must attain maximum inside B. Since $\forall x \in \partial B$, if x lies in $S = \{x | f(x) \geq f(p) - 1\}$, note that p is a local maximum, and $S \subset \{y < 0\}$, then

$$f_\epsilon < f_\epsilon(p) \text{ on } S.$$

If $x \in \partial B - S$, *i.e.* $f(x) < f(p) - 1$, take ϵ sufficiently small such that $|\epsilon h(x)| < 1$ on ∂B, then one has

$$f_\epsilon(x) < f_\epsilon(p) \text{ on } \partial B - S.$$

This contradicts the strong maximum principle.

□

Exercise (Maximum principle in 1-dimensional calculus). *Show that*

(1) $x \in \mathbb{R}$, $f''(x) > 0$, then f has no upper bound.

(2) $x \in \mathbb{R}$, $f''(x) \geq 0$, if $\exists x_0 \in \mathbb{R}$ such that $f(x_0) = A = \max f(x)$, then $f(x) = A$ for all $x \in \mathbb{R}$.

3.8 Induced covariant derivative on tensor fields

Let ∇ be the Levi0Civita connection on (M, g), then for $\forall$ vector field $X \in \Gamma(TM)$, the covariant derivative ∇_X is

$$\nabla_X \colon \Gamma(TM) \to \Gamma(TM)$$

$$Y \mapsto \nabla_X Y$$

and locally

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

Since T^*M is the dual bundle of TM, we can extend ∇_X on T^*M as

$$\nabla_X \colon \Gamma(T^*M) \to \Gamma(T^*M)$$

$$\omega \mapsto \nabla_X \omega$$

such that $\forall Y \in \Gamma(TM)$

$$(\nabla_X \omega)(Y) = \nabla_X (\omega(Y)) - \omega(\nabla_X Y).$$

Locally, $\Gamma(T^*M) = \operatorname{span}\{dx^1, \dots, dx^n\}$

$$\begin{aligned}\left(\nabla_{\frac{\partial}{\partial x^i}} dx^j\right)\left(\frac{\partial}{\partial x^k}\right) &= \nabla_{\frac{\partial}{\partial x^i}} \left(dx^j\left(\frac{\partial}{\partial x^k}\right)\right) - dx^j\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}\right) \\ &= -dx^j\left(\Gamma^p_{ik}\frac{\partial}{\partial x^p}\right) \\ &= -\Gamma^j_{ik}.\end{aligned}$$

$\Rightarrow \boxed{\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\Gamma^j_{ik} dx^k}$. Now, let T be a p,q-tensor

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}.$$

Then

$$\begin{aligned}\nabla_k T &= \nabla_k \left(T^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}\right) \\ &= \partial_k T^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q} \\ &\quad + \sum_{\alpha=1}^{p} T^{i_1 \dots i_\alpha \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \nabla_k\left(\frac{\partial}{\partial x^{i_\alpha}}\right) \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q} \\ &\quad + \sum_{\beta=1}^{q} T^{i_1 \dots i_p}_{j_1 \dots j_\beta \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes \nabla_k\left(dx^{j_\beta}\right) \otimes \dots \otimes dx^{j_q} \\ &= \partial_k T^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q} \\ &\quad + \sum_{\alpha=1}^{p} T^{i_1 \dots \gamma \dots i_p}_{j_1 \dots j_q} \Gamma^{i_\alpha}_{k\gamma} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q} \\ &\quad - \sum_{\beta=1}^{q} T^{i_1 \dots i_p}_{j_1 \dots \theta \dots j_q} \Gamma^{\theta}_{kj_\beta} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}.\end{aligned}$$

$$\begin{aligned}\therefore \nabla_k T = \Bigg(&\partial_k T^{i_1 \dots i_p}_{j_1 \dots j_q} + \sum_{\alpha=1}^{p} T^{i_1 \dots \gamma \dots i_p}_{j_1 \dots j_q} \Gamma^{i_\alpha}_{k\gamma} \\ &- \sum_{\beta=1}^{q} T^{i_1 \dots i_p}_{j_1 \dots \theta \dots j_q} \Gamma^{\theta}_{kj_\beta}\Bigg) \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q},\end{aligned}$$

or

$$\nabla_k T^{i_1\dots i_p}_{j_1\dots j_q} = \partial_k T^{i_1\dots i_p}_{j_1\dots j_q} + \sum_{\alpha=1}^{p} T^{i_1\dots\gamma\dots i_p}_{j_1\dots j_q}\Gamma^{i_\alpha}_{k\gamma} - \sum_{\beta=1}^{q} T^{i_1\dots i_p}_{j_1\dots\theta\dots j_q}\Gamma^{\theta}_{kj_\beta}$$

3.9 The Gauss-Bonnet formula

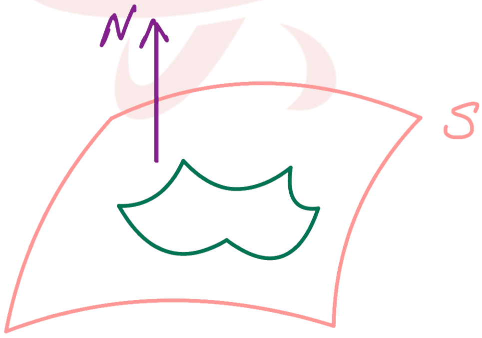
3.9.1 Statements and corollaries

(1) The local version

- S: oriented surface.
- $\gamma\colon U \subset \mathbb{R}^2 \to S$ orthogonal local parametrization. ($i.e.$ $g_{12} = 0$)
- $R \subset \gamma(U)$ is a simply connected region. ($i.e.$ homeomorphic to a disk)
- ∂R: boundary of R, which is a simple closed, piecewise regular positively oriented with the orientation induced from S.
- $\alpha\colon [0, l] \to S$ is the parametrization of ∂R, with arclength parametrization.
- $\alpha(s_0), \dots, \alpha(s_k)$ are the vertices of $\alpha(s)$, $i.e.$ break-up points.
- $\theta_0, \dots, \theta_k$ external angles of α.

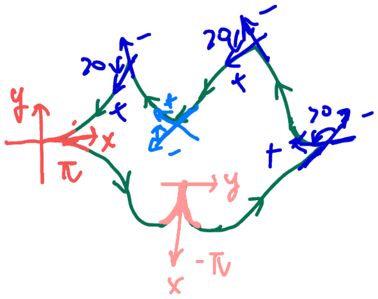
The Gauss-Bonnet formula is

$$\underbrace{\sum_{i=0}^{k}\theta_i}_{0-\mathrm{d(T.C.)}} + \underbrace{\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(s)\mathrm{d}s}_{1-\mathrm{d(T.C.)}} + \underbrace{\iint_R K\mathrm{d}\sigma}_{2-\mathrm{d(T.C.)}} = 2\pi.$$[9]



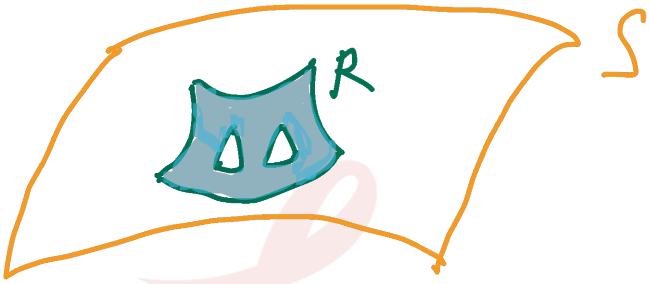
Definition 3.9.1 (External angle)**.** At the vertex $\alpha(s_i)$, if $\alpha(s_i)$ is not a cusp, let θ_i be the angle from $\alpha'(s_i - 0)$ to $\alpha'(s_i + 0)$ along the orientation of the curve, $-\pi < \theta_i < \pi$. If $\alpha(s_i)$ is a cusp, let $\theta_i = \pi$.

[9] T.C.=total curvature.

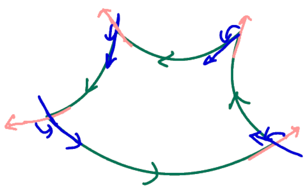


all navy blue angles are positive angle
The cyan angle is negative angle
The pink cusp is $-\pi$
The red cusp is π

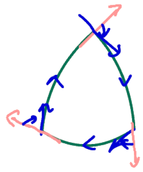
(2) The global version



- S is an oriented surface.
- $R \subset S$ connected regular compact region.
- ∂R are simple closed, piecewise regular curves $\partial R = \bigsqcup_{i=1}^{k} C_i$, where C_i are simple closed piecewise regular curves and $C_i \cap C_j = \emptyset$. ∂R has induced orientation from S.
- $\theta_1, \ldots, \theta_p$ are all external angles of $C_1, \ldots, C_k$.



outer boundary



inner boundary

Theorem 3.9.2 (Gauss-Bonnet theorem)**.**

$$\sum_{i=1}^{p} \theta_i + \sum_{j=1}^{k} \oint_{C_j} k_g(s)\mathrm{d}s + \iint_R K\mathrm{d}\sigma = 2\pi\chi(R).$$

$\chi(R)$ is the Euler characteristic of region R.

To define $\chi(R)$, let's consider a triangulation of R, *i.e.* a family of set of triangles $\cup_{i\in\Lambda} T_i$ in S satisfying.

(1) $R = \cup_{i\in\Lambda} T_i$.

(2) If $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is either a common vertex or a common edge of T_i and T_j.

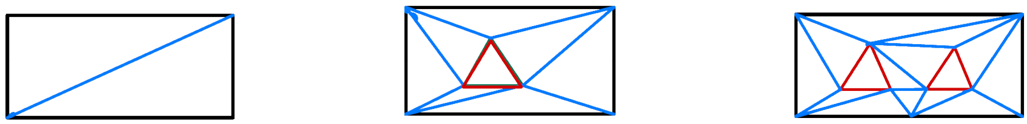
(3) Each common edge has only two angles attaching(attached) to it.

Definition 3.9.3. The Euler characteristic of R is defined as

$$\chi(R) = V - E + F.$$

Here, $V=$ number of vertices in the triangles. $E=$ number of edges in the triangles. $F=$ number of faces in the triangles.

An important topological fact is $\chi(R)$ is independent of the choice of the triangulation.



$\chi = 1$ $\chi = 0$ $\chi = -1$

Corollary 3.9.4. *If S is an oriented compact surface without boundary, then*

$$\underbrace{\iint_S K \mathrm{d}A}_{\text{Integration of local geometry quantity}} = 2\pi\chi(S) = \underbrace{4\pi(1-g)}_{\text{global topological quantity}}.$$ [10]

Remark. This corollary sheds light on how to find a topological invariant by integrating a local geometric quantity (curvature-related function). The beauty of the Gauss-Bonnet is that it links two different subjects in math by just finding the relationship of meaningful subjects.

Example 3.9.5.
- *$g = 0$ sphere $\iint_S K\mathrm{d}A = 4\pi$.*
- *$g = 1$ torus $\iint_S K\mathrm{d}A = 0$.*
- *$g \geq 2$ higher genus torus $\iint_S K\mathrm{d}A < 0$*

Example 3.9.6. *Let's consider a geodesic triangle T on surface S, by local Gauss-Bonnet*

$$\sum_{i=1}^{3} \theta_i + \iint_T K\mathrm{d}\sigma = 2\pi.$$

Let $\psi_i = \pi - \theta_i$ be the interior angles, then

$$\sum_{i=1}^{3} \psi_i = 3\pi - \sum_{i=1}^{3} = \pi + \iint_T K\mathrm{d}\sigma$$

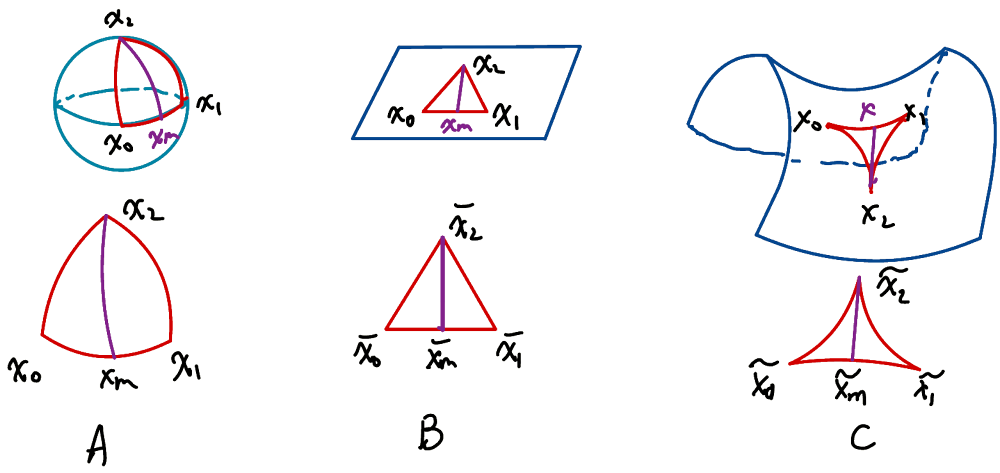
Consequence:

(1) On $\mathbb{S}^2$, $K > 0 \Rightarrow \sum_{i=1}^{3} \psi_i > \pi$.

[10] From algebraic topology, $\chi(S) = 2 - 2g$, where g is the genus.

(2) On a surface with $K = 0 \Rightarrow \sum_{i=1}^{3} \psi_i = \pi$.

(3) On a surface with $K < 0 \Rightarrow \sum_{i=1}^{3} \psi_i < \pi$.



Another important observation is considering geodesic triangles A, B, C*. Let's assume the length of each corresponding edges (geodesics) are the same, i.e.*

$$d_{\mathbb{S}^2}(x_0, x_1) = d_{\mathbb{R}^2}(\bar{x_0}, \bar{x_1}) = d_{\mathbb{H}^2}(\widetilde{x_0}, \widetilde{x_1}).$$

$$d_{\mathbb{S}^2}(x_1, x_2) = d_{\mathbb{R}^2}(\bar{x_1}, \bar{x_2}) = d_{\mathbb{H}^2}(\widetilde{x_1}, \widetilde{x_2}).$$

$$d_{\mathbb{S}^2}(x_0, x_2) = d_{\mathbb{R}^2}(\bar{x_0}, \bar{x_2}) = d_{\mathbb{H}^2}(\widetilde{x_0}, \widetilde{x_2}).$$

Then $d_{\mathbb{S}^2}(x_2, x_m) > d_{\mathbb{R}^2}(\bar{x_2}, \bar{x_m}) > d_{\mathbb{H}^2}(\widetilde{x_2}, \widetilde{x_m})$.

This observation enables us to use surfaces of $K = 1, 0, -1$ *as model spaces to define a general "curved" metric space, these spaces are called* $CAT(k)$ *surface,* $K = 0, \pm 1$*. For example,* $CAT(0)$ *surface means a complete metric space of* $\dim = 2$ *such that any two points can be joint by a geodesic (defined by given metric) and* $CAT(0)$ *(also called NPC space) means* $K \leq 0$*. This condition is characterized by*

$$d^2(x_2, x_m) \leq m d^2(x_1, x_2) + (1-m) d^2(x_0, x_2) - m(1-m) d^2(x_0, x_1) \tag{1}$$

where $d(x_0, x_m) = m \cdot d(x_0, x_1)$.

Question. Can you recognize how the inequality in (1) comes out?

Remark. On a surface (not necessarily compact) with $K = -1$, local Gauss-Bonnet gives

$$\sum_{i=1}^{3} \psi_i = \pi - Area(\Delta).$$

$$\Rightarrow 0 \leq Area(\Delta) \leq \pi.$$

In particular, if $Area(\Delta) \to 0$, then $\sum \psi_i \to \pi$. Then limiting geodesic triangle looks like:

From the Gauss-Bonnet, we also have the following.

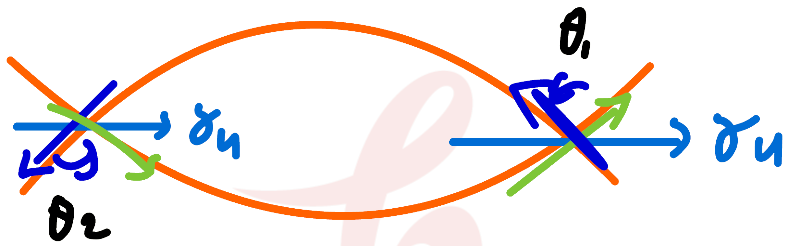
Corollary 3.9.7 (Rotation index). *Let C be a simple closed regular curve in $\mathbb{R}^2$, then $\int_C k\mathrm{d}s = 2\pi$.*

3.9.2 Proof of the (local) Gauss formula

Lemma 3.9.8 (Topological result). *Let $\alpha\colon [0,l] \to S$ be the boundary curve ∂R.*
$\alpha(s_0), \dots, \alpha(s_k)$: vertices of $\alpha(s)$.
$\theta_0, \dots, \theta_k$: external angles of α.
Let $\varphi_i\colon [s_i, s_{i+1}] \to \mathbb{R}$ be the angle from γ_u to $\alpha'(s)$, then

$$\sum_{i=0}^{k} (\varphi_i(s_{i+1}) - \varphi_i(s_i)) + \sum_{i=0}^{k} \theta_i = \pm 2\pi.$$

Where $\pm$ depends on the orientation of α.



Proof of this can be found in Do Carmo's book (sec 5-7, theorem2).

Lemma 3.9.9. *Let $\gamma(u,v)$ be local orthogonal parametrization, $\alpha(s)$: regular curve with arclength parameter. Let $\varphi(s)$ be the angle between γ_u and $\alpha'(s)$, then*

$$k_g = \frac{\mathrm{d}\varphi}{\mathrm{d}s} - \frac{E_v}{2\sqrt{EG}} u'(s) + \frac{G_u}{2\sqrt{EG}} v'(s).$$

Proof. Already obtained in the proof of the Liouville formula. $\square$

Assuming two lemmas above, we'll give a proof of the local Gauss-Bonnet.

Proof. Let $\varphi_i = \varphi_i(s)$ be angle functions from γ_u to $\alpha'(s)$ in $[s_i, s_{i+1}]$. Integrate $k_g(s)$ with expression in lemma 3.9.9

$$\Rightarrow \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)\mathrm{d}s = \underbrace{\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{\mathrm{d}\varphi_i}{\mathrm{d}s}\mathrm{d}s}_{(1)} + \underbrace{\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left(\frac{G_u}{2\sqrt{EG}} v'(s) - \frac{E_v}{2\sqrt{EG}} u'(s) \right) \mathrm{d}s}_{(2)}.$$

(1):

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{\mathrm{d}\varphi_i}{\mathrm{d}s}\mathrm{d}s = \sum_{i=0}^{k} \varphi_i(s_{i+1}) - \varphi_i(s_i) \overset{\textit{lemma } 3.9.8}{=\!=\!=} 2\pi - \sum_{i=0}^{k}.$$

(positive sign is due to $\alpha(s)$ is positively oriented.) (2):

$$\begin{aligned}
&\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}\left(\frac{G_u}{2\sqrt{EG}}v'(s)-\frac{E_v}{2\sqrt{EG}}u'(s)\right)\mathrm{d}s\\
=&\sum_{i=0}^{k}\int_{\alpha_i}\frac{G_u}{2\sqrt{EG}}\mathrm{d}v-\frac{E_v}{2\sqrt{EG}}\mathrm{d}u \xlongequal[\text{formula}]{\text{divergence}} \iint_{\gamma^{-1}(R)}\left(\frac{G_u}{2\sqrt{EG}}\right)_v+\left(\frac{E_v}{2\sqrt{EG}}\right)_u\mathrm{d}u\mathrm{d}v\\
=&-\iint_{\gamma^{-1}(R)}K\sqrt{EG}\mathrm{d}u\mathrm{d}v=-\iint_{\gamma^{-1}(R)}K\mathrm{d}A.
\end{aligned}$$ [11]

□

Proof of global Gauss-Bonnet. Subdivide S by giving a triangulation of S, such that each triangle T_i is contained in a local parametrization with $F=0$, then apply local Gauss-Bonnet formula and $\chi(S)=V-E+F$.(Details are left as exercises.) □

Exercise. *Let $S^2(R)$ be a radius R sphere, verify the Gauss-Bonnet formula directly by computation.*

Exercise. *Let S be a closed convex surface in $\mathbb{R}^3$. Check the Gauss-Bonnet formula.*

Remark. We shall see another nicer proof of the local Gauss-Bonnet for the region bounded by a smooth curve. The proof is due to Simon Donaldson.

3.10 The Hilbert's theorem

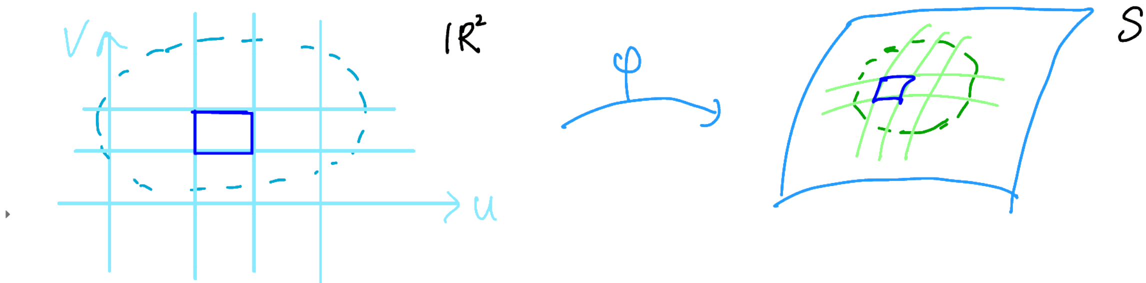
Theorem 3.10.1 (Hilbert's theorem). *Let S be a complete and simply connected regular surface with Gaussian curvature $K=-1$, then there is no isometric embedding of S into $\mathbb{R}^3$.*

Sketch of proof. (1) (S,g) is isometric to the hyperbolic plane in $\mathbb{H}^2$
$\Rightarrow Area(S)=Area(\mathbb{H}^2)=+\infty$.

(2) If $\exists$ an isometric embedding $\varphi\colon S\to\mathbb{R}^3$, then we can construct a global parametrization of S with coordinate curves being asymptotic lines, and the parametrization forms a "chebyshev net" such that any quadrilateral of coordinate curves has area $<2\pi$. This implies
$$Area(S)<2\pi.$$

Then (1) and (2) yields a contradiction. □

Definition 3.10.2. A local parametrization $\phi(u,v)\colon U\subset\mathbb{R}^2\to S$ is called a "chebyshev net" if the length of opposite sides of any quadrilateral formed by coordinate curves are equal.



[11] Use homework 12 Problem 1.

proof of (1). First, from the minding's theorem, we know S and $\mathbb{H}$ are locally isometric to each other. Let $p \in S$, $p' \in \mathbb{H}$, the local isometry yields a linear isometry $l\colon T_pS \to T_{p'}\mathbb{H}$.

To construct a global isometry, let's consider

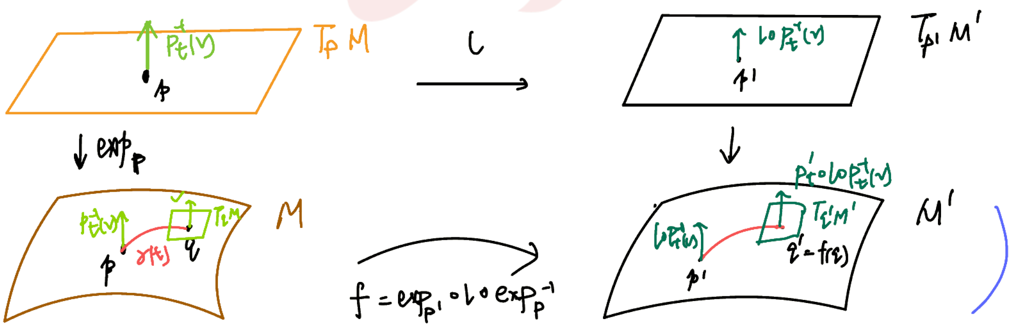
$$\begin{tikzcd} T_pS \arrow[r, "l"] \arrow[d, "\exp_p"] & T_{p'}\mathbb{H} \arrow[d, "\exp_{p'}"] \\ S \arrow[r, dashed] & \mathbb{H} \end{tikzcd}$$

By the Hadamard theorem $\to$ $\exp_p$ and $\exp_{p'}$ are diffeomorphism. [12] Hence $f = \exp_p \circ l \circ \exp_{p'}^{-1}\colon S \to \mathbb{H}$ is a diffeomorphism. We can further conclude that f is a local isometry. This follows from the Cartan's theorem:

- Let (M,g), (M',g') be two Riemannian manifolds, $p \in M$, $p' \in M'$.
- Let $l\colon T_pM \to T_{p'}M'$ be a linear isometry
- U is a normal neighborhood of p on which $\exp_p$ is a diffeomorphism.
- $\forall V \in T_qM$, and let P_t be the parallel transport along $\gamma(t)$, then $P_t^{-1}(V)$ is a vector in T_pM. The linear isometry yields $l \circ P_t^{-1}(V)$ to be a vector in $T_{p'}M'$. Let $q' = \exp_{p'} \circ l \circ \exp_p^{-1} q \in M'$, $\gamma'(t)$ is the geodesic from p' to q' and P_t' is the parallel transport along $\gamma'(t)$. Then using P_t', we get a vector $P_t' \circ l \circ P_t^{-1}(V)$ $inT_{q'}M' \Rightarrow L_t = P_t' \circ l \circ P_t^{-1}\colon T_qM \to T_{q'}M'$ is a linear isometry. If $\forall$ vectors $X, Y, Z, W \in T_qM$ we have

  $$R(X,Y,Z,W) = R'(L_t(X), L_t(Y), L_t(Z), L_t(W))$$

  then $f = \exp_{p'} \circ l \circ \exp_p^{-1}$ is a local isometry.



Hence, we see (S,g) and $\mathbb{H}$ are isometric to each other on $\mathbb{H}^2$. We consider geodesic polar coordinate such that

$$ds^2 = dr^2 + (\sinh r)^2\, d\theta^2.$$

[12]Hadamard's theorem: let (M,g) be a complete Riemannian manifold with $K_M \le 0$, where K_M is the sectional curvature, then $\exp_p\colon T_pM \to M$ is a covering map. Moreover, if M is simply connected, then $\exp_p$ is a diffeomorphism.

Then the radial geodesic $\gamma(t) = \exp_p(t\frac{\partial}{\partial r})$ defines for all time t, $\Rightarrow t \in (0, +\infty)$

$$Area(S) = Area(\mathbb{H}) = \int_0^{2\pi} \int_0^{+\infty} \sinh r dr d\theta = +\infty$$

$\square$

proof of (2). Now we assume S is isometrically embedded into $\mathbb{R}^3$.

Claim 1: $\forall p \in S$, $\exists$ local parametrization (s, t) of S, such that the 1st and 2nd fundamental form are given by

$$I = ds^2 + 2\cos\alpha ds dt + dt^2$$

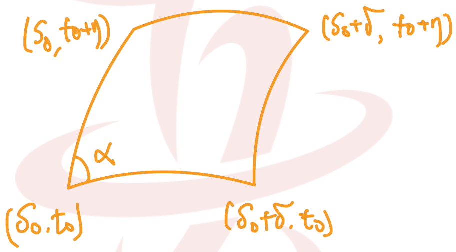
$$II = 2\sin\alpha ds dt$$

where $\alpha = \alpha(s, t)$ satisfies the Sine-Gordan equation

$$\alpha_{st} = \sin\alpha, \quad 0 < \alpha < \pi$$

From this expression of I and II, we see

(1) the length of opposite sides of coordinate quadrilateral are the same $\Rightarrow$ (s, t) forms a chebyshev net, α is the angle between s-curve and t-curve.



(2) coordinate s-curve and t-curve are asymptotic lines

Proof of claim 1. $K = -1$ on $S \Rightarrow$ all pairs are umbilical. Hence, we can first choose local parametrization x^1, x^2 such that

$$I = g_{11}(dx^1)^2 + g_{22}(dx^2)^2$$

$$II = h_{11}(dx^1)^2 + h_{22}(dx^2)^2$$

$\Rightarrow$ principal curvatures $k_1 = \frac{h_{11}}{g_{11}}, k_2 = \frac{h_{22}}{g_{22}}$. The codazzi equation is

$$\frac{\partial_2 k_1}{k_2 - k_1} = \partial_2 \log\sqrt{g_{11}}$$

$$\frac{\partial_1 k_2}{k_1 - k_2} = \partial_1 \log\sqrt{g_{22}}$$

Since $k_1 k_2 = -1$, we set

$$k_1 = \tan\theta, \quad k_2 = -\cot\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$\Rightarrow k_1 - k_2 = \tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta}$. Then the codazzi equation

$$\begin{aligned}\partial_2 \log\sqrt{g_{11}} &= -\sin\theta\cos\theta\partial_2(\tan\theta) = -\sin\theta\cos\theta\frac{\cos\theta\partial_2\sin\theta - \sin\theta\partial_2\cos\theta}{\cos^2\theta}\\ &= -\sin\theta\partial_2\sin\theta + \frac{1-\cos^2\theta}{\cos\theta}\partial_2\cos\theta\\ &= -\partial_2\left(\frac{1}{2}\sin^2\theta + \frac{1}{2}\cos^2\theta\right) + \frac{\partial_2\cos\theta}{\cos\theta}\end{aligned}$$

$\Rightarrow \partial_2\left(\frac{\sqrt{g_{11}}}{\cos\theta}\right) = 0$ Similarly, $\partial_1\left(\frac{\sqrt{g_{22}}}{\sin\theta}\right) = 0$ This enables us to introduce new coordinate

$$y^1 = \int \frac{\sqrt{g_{11}}}{\cos\theta}dx^1 \quad y^2 = \int\frac{\sqrt{g_{22}}}{\sin\theta}dx^2.$$

$$dy^1 = \frac{\sqrt{g_{11}}}{\cos\theta}dx^1 \quad dy^2 = \frac{\sqrt{g_{22}}}{\sin\theta}dx^2$$

$\Rightarrow$ the 1^{st} and 2^{nd} fundamental form in terms of (y^1, y^2) are

$I = \cos^2\theta(dy^1)^2 + \sin^2\theta(dy^2)^2 \quad II = k_1\cos^2\theta(dy^1)^2 + k_2\sin^2\theta(dy^2)^2 = \sin\theta\cos\theta\left((dy^1)^2 - (dy^2)^2\right)$

We consider further coordinate change. Let $y^1 = s+t, y^2 = s-t$ $(dy^1)^2 = ds^2 + dt^2 + 2dsdt(dy^2)^2 = ds^2 + dt^2 - 2dsdt$ $\Rightarrow$In (s,t) coordinate, the 1^{st} and 2^{nd} fundamental form are

$$I = ds^2 + 2\cos 2\theta dsdt + dt^2$$
$$II = 2\sin 2\theta dsdt$$

Let $\alpha = 2\theta$, then

$$I = ds^2 + 2\cos\alpha dsdt + dt^2$$
$$II = 2\sin\alpha dsdt$$

Furthermore, from the Gauss equation, in y^1, y^2 coordinate,

$$I = \cos^2\theta(dy^1)^2 + \sin^2\theta(dy^2)^2$$

$$\begin{aligned}K &= -\frac{1}{\sin\theta\cos\theta}\left(\partial_1\left(\frac{\partial_1\sin\theta}{\cos\theta}\right) + \partial_2\left(\frac{\partial_2\cos\theta}{\sin\theta}\right)\right)\\ &= -\frac{1}{\sin\theta\cos\theta}\left(\frac{\partial^2\theta}{\partial(y^1)^2} - \frac{\partial^2\theta}{\partial(y^2)^2}\right)\end{aligned}$$

$$K = -1 \Leftrightarrow \theta_{11} - \theta_{22} = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta \tag{1}$$

□

Claim 2: Any quadrilateral formed by coordinate courves in the chebyshev net on S has Area less that 2π

Proof. If (s,t) forms a chebyshev net, then

$$I = ds^2 + 2\cos\alpha ds dt + dt^2$$

with $\alpha_{st} = \sin\theta (\because K = -1)$.

Let θ be the quadrilateral formed by coordinate curves, then

$$\begin{aligned}
Area(\theta) &= \int_{s_0}^{s_0+\delta}\int_{t_0}^{t_0+\eta} \sin\alpha ds dt \\
&= \int_{s_0}^{s_0+\delta}\int_{t_0}^{t_0+\eta} \alpha_{st} ds dt \\
&= \alpha(s_0+\delta, t_0+\eta) - \alpha(s_0+\delta, t_0) - \alpha(s_0, t_0+\eta) + \alpha(s_0, t_0) \\
&< 2\pi \quad (\because 0 < \alpha < \pi)
\end{aligned}$$

□

So far, we have shown $\forall p \in S$, we have local parametrization (s,t) near p such that (s,t) forms a chebyshev net and coordinate curves are asymptotic curves.

In fact, we can establish a global parametrization on S, which forms a chebyshev net and coordinate curves are asymptotic curves in the following way.

Let's first define such global parametrization:

$$\varphi \colon \mathbb{R}^2 \to S$$

Take a point $p \in S$, let α_1, α_2 be asymptotic curves through p. Let p be corresponding to the origin $(0,0)$ in $\mathbb{R}^2$. For each $(s,t) \in \mathbb{R}^2$, we consider $\varphi(s,t) \in S$ to be found in the following way. We first find a point p_1 along α_1 with distance s from p. At p_1 we can still choose α_1 as asymptotic line. (by the uniqueness of differential equation $II(\alpha'(s), \alpha'(s))$) Since p_1 is again a hyperbolic point, we have another asymptotic curve α_2 which is obtained by continuous extending α_2 along α_1. Along this α_2', choose point p_2 with distance t from p_1. Let $\varphi(s,t) = p_2 \in S$.

In this way, we define a map

$$\varphi \colon \mathbb{R}^2 \to S$$

$$(s,t) \mapsto \varphi(s,t) \in S$$

There are a few things to be checked.

(1) $\varphi(s,t)$ is well-defined for all $(s,t) \in \mathbb{R}^2$

(Here we need to use the completeness of S. If $\varphi(s,t)$ is not defined for some $\sigma = s_1$,*i.e.* the asymptotes curve α_1 is defined on $[0, s_1)$, let $q = \lim_{s \to s_1} \alpha_1(s)$ by the completeness of S , $q \in S$. Hence, we can establish a local parametrization given by chebyshev net with coordinate curves being asymptotic curves. Hence, $\varphi(s,0)$ is defined similarly for any fixed s. $\varphi(s,t)$ is defined for all t.)

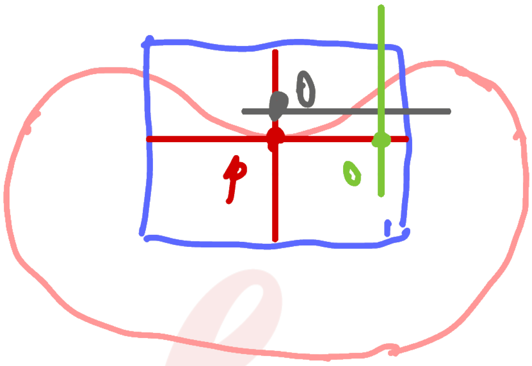
(2) For any fixed t, $\varphi(s,t)$ is an asymptotic curve with s being the arclength.

(First, for any $\varphi(s',t') \in S$, we can find a "rectangular" coordinate chart $s_a < s < s_b, t_a < t < t_b$ such that $\varphi(s,t)$ forms a chebyshev net with coordinate curves being asymptotic curves. By the definition of $\varphi(s,t)$, if for any $t_0 \in (t_a, t_b)$, $\varphi(s,t_0)$ is an asymptotic curve, so for any $t \in (t_a, t_b)$, $\varphi(s,t)$ is an asymptotic curve.

Second, let $\varphi(s_1, t_1) \in S$ be arbitrary point, $\varphi(s_1, t), 0 \le t \le t_1$ is a compact segment, we can cover it by finitely many rectangular neighborhoods and each of them is a chebyshev net. Since $\varphi(s, 0)$ is an asymptotic curve we can iterate the previous step to see $\varphi(s, t)$ is an asymptotic curve.)

(3) $\varphi(s, t) \colon \mathbb{R}^2 \to S$ is a diffeomorphism.

- $\varphi(s, t)$ is a local diffeomorphism.(easy to see)
- φ is injective.
- φ is surjective, we shall see otherwise this would contradics the completeness. If $\varphi(\mathbb{R}^2) = U \neq S$, then U has nonempty boundary ∂U



If $\varphi(\mathbb{R}^2) = U \neq S$, then U has nonempty boundary ∂U

- Let $q \in \partial U$, since $p \in S$, we have local chebyshev net R at p. In R, each coordinate line is an asymptotic curve.
- Take $q \in U \cap R$, then one of the asymptotic curve (coordinate curve) of q (green or grey line) will meet with an asymptotic coordinate curve of p(red lines) at some point $o \in S$. The chebyshev net of o can be chosen to be contained in the intersection of chebyshev nets of p and q
  (1) If $o \in U$, since $\varphi(\mathbb{R}^2)$ is the largest extension of chebyshev net, and $p \notin U$. This implies asymptotic coordinate curves of o can not pass through p (contradiction)
  (2) If $o \notin U$, then asymptotic coordinate line of o can not pass through q, otherwise $q \notin U$ (contradiction).

□

Remark. In the proof of Hilbert's theorem, we have used a fact that the isometric embedding $f \colon S \to \mathbb{R}^3$ is at least C^3(since we need this to compute I,II and codazzi equation). Efimov showed there is no C^2 isometric embedding. There is a counterexample due to Kuiper in 1955, showing that one can find C^1 isometric embedding $f \colon U^+ \to \mathbb{R}^3$, where $U^+ =$ upper half plane with metric $\frac{dx^2 + dy^2}{y}$