Differential Geometry Lecture Notes

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Contents

Preface

Textbook Reference

(1) Do Carmo: Differential Geometry of Curves and Surfaces.

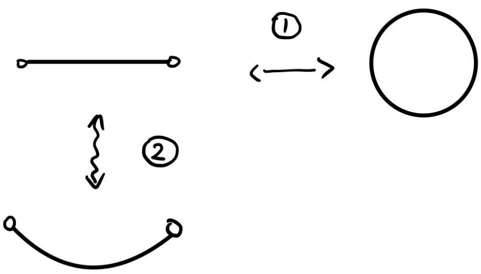
(2) Sebastián Montiel, Antorio Ros: Curves and Surfaces.

(3) *Chinese Title, add later*

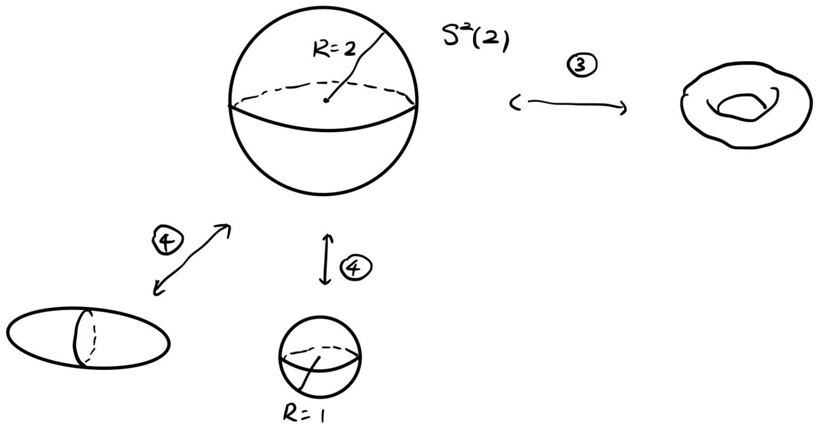
Course Introduction

The Goal of this course is to study the "differential geometry of curves and surfaces".
• **Geometry**: How is a geometric object curved / How to measure the curvedness of a geometric object?

Example. *In the illustration below, (1) differs by "topology". In (2), they are topologically the same, while the lower curve is "more curved" than the upper curve.*



Example. *(3) differs by "topology", but in (4)* $\mathbb{S}^2(1)$ *is more curve than* $\mathbb{S}^2(2)$*, even topologically they are the same.(either homeomorphically or diffeomorhically).*



The "Curved property" also affects geometric quantities, like length, area, volume, angle between the curves, etc.

Local Geometry: How does a "curved " space look like in a neighborhood of a point?

Global Geometry: If we know how a "curved space" is look like at each point, can we observe how such space looks like globally? This is usually related to topological problems.

• **Differential**: In this course, by "smoothness" we mean the geometric objects we'll study are "nice" enough so we can apply "calculus" tools to study them.

Main tool: Calculus! We'll see how powerful calculus is in this course, especially, like the maximal principle, integration by parts(stoke's theorem), Taylor's expansion, implicit function theory, etc.

Queastion: How to tell the "smoothness"?(Need to find good parametrization)Finding a good "gauge"(that is "coordinate") to work with is also an important question in geometry.

•Curves: 1-d geometric object.

Surfaces: 2-d geometric object.

Remark. In this course, we only focus on curves and surfaces in $\mathbb{R}^3$.However, as a training on preparing for later geometry course, I suggest you also try to think about the ambiant space is $\mathbb{S}^3$ or $\mathbb{H}^3$.

•**Intrinsic geometry**: Study the geometric object without considering the ambient space. This begins from the Gauss's elegant theorem and was developed by Riemann.

Example. *Consider the unit sphere* $\mathbb{S}^2$

Extrinsic geometry: view it as $x^2+y^2+z^2=1$ *in* $\mathbb{R}^3$.

Intrinsic geometry: (θ,φ) *or* (φ,θ) *are "essential" coordinates on* $\mathbb{S}^2$.

$$\mathrm{d}s^2 = \mathrm{d}\varphi^2 + (\sin\varphi)^2\,\mathrm{d}\theta^2$$

(Caution: (θ,φ) *is outer normal, while* (φ,θ) *is inner normal.)*

• Useful / Common techniques:

1) Comparison: compare the studied geometric object with "model space". It's very important to study examples in geometry. As a suggestion you are expected to spend time to play with $\mathbb{S}^2$.For example: How is $\mathbb{S}^2$ curved? What's the shortest line in $\mathbb{S}^2$? How many symmetries are there on $\mathbb{S}^2$? Can you add "extra structure" on$\mathbb{S}^2$ to make it a complex object? Is this "extra structure" "rigid"?What/s the "moment map" on $\mathbb{S}^2$? Does there exist a "holomorphic" map from $\mathbb{S}^2$ to a torus, or a surface of arbitrary genus?

   If we consider an "Energy minimizing map" from $\mathbb{S}^2$ to $\mathbb{S}^2$, what can we say about such map?(It is holomorphic/antiholomorphic.)

   After you have learned Riemann Geometry, you'll see an energy minimizing map from $\mathbb{S}^2$ to a Riemannian manifold must be an angle-preserving map(conformal map).

   What kinds of 2-d geometric space could be $\mathbb{S}^2$?(this is a global geometry problem.)(i.e. what kinds of geometric conditions can characterize $\mathbb{S}^2$?)

2) To study higher dimensional objects,it's also important to understand lower dimensional objects, and it's also important to understand lower dimensional objects contained in the studied objects.

3) Study "functions" (more generally sections, including functions, vector fields, differential forms, etc.) on a given geometric object.

Example. *On a closed surface (* $\mathbb{S}^2$,$\mathbb{T}^2$,Σ_g*)(compact without boundary) there is no non-constant harmonic function.(i.e.* $\Delta u = 0$*)(Analysis will get involved.)*

We usually care about those functions related to geometry, such as distance functions, curvature-related functions, etc.

Example (More trivial than the last one)**.** *Consider* $f''(x)=0$, *what can you say of the solution of it when* x *lies on a line and when* x *lies on a circle?*

Chapter 1

Differential Geometry of Curves

1.1 Linear algebra convention and its geometric explanation

- We use "ROW VECTOR" in this course, i.e

$$v \in \mathbb{R}^n, v = (v_1, v_2, \cdots, v_n)$$

- let $e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 1)$ be the standard basis, then

$$v = \sum_{i=1}^{n} v^i e_i = \begin{bmatrix} v^1 & v^2 & \cdots & v^n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

- $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ (non-degenerate) linear map

$$v \mapsto \varphi(v) = v \cdot A.$$

This corresponds to the right action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$.

$$\Rightarrow \varphi(e_i) = e_j \cdot A = \sum_{i=1}^{n} A_j^{\ i} e_i \text{ (taking the j-th row of } A)$$

$$A_j^{\ i} \begin{cases} \text{upper index: column index} \\ \text{lower index: row index} \end{cases}$$

$$\Rightarrow \varphi \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \varphi(e_1) \\ \varphi(e_2) \\ \vdots \\ \varphi(e_n) \end{bmatrix} = \begin{bmatrix} e_1 \cdot A \\ e_2 \cdot A \\ \vdots \\ e_n \cdot A \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Remark (Important!). In row vector convention, a non-degenerate linear map corresponds to the right action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$. But this induces left action of $GL(n, \mathbb{R})$ on the orthonormal basis (frame) $\{e_1, e_2, \ldots, e_n\}$. This phenomenon provides an important example in differential geometry, which will be explained later in the theory of principle bundle.(i.e. let G be a lie group, $G \curvearrowright M$ being a right action, where M is a differentiable manifold, then this right action induces a left action of G on the frame bundle of M.)

Let $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ be another basis of $\mathbb{R}^n$. Let f be the corresponding linear map, i.e.

$$f\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \vdots \\ \tilde{e}_n \end{bmatrix} = B \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$\Rightarrow \tilde{e}_k = \sum_{j=1}^{n} {B_k}^j e_j$$

We compare the matrix of φ in terms of $\{\tilde{e}_1 \cdots \tilde{e}_n\}$

$$\varphi \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_n \end{bmatrix} = \varphi \left[B \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ e_n \end{bmatrix} \right] = B \cdot \varphi \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \text{(linearity of } \varphi\text{)}$$

$$= BA \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = BAB^{-1} \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ e_n \end{bmatrix}$$

Note in this case.

$$(\varphi \circ f) \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = BA \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

In terms of entries,

$$\begin{aligned} \varphi(\tilde{e}_k) &= \varphi(\sum_{j=1}^{n} {B_k}^j e_j) = \sum_{j=1}^{n} {B_k}^j \varphi(e_j) \qquad \text{(linearity)} \\ &= \sum_{i,j=1}^{n} {B_k}^j {A_j}^i e_i = \sum_{i,j,p=1}^{n} {B_k}^j {A_j}^i {(B^{-1})_i}^p \tilde{e}_p \end{aligned}$$

Remark. This computation tells that the row vector convention yields to the fact that $GL(n, \mathbb{R})$ acting on itself from the right when we consider the action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$. In modern Geometry, it's more common to use column vector as convention. This row vector convention was adopted by S.S. Chern and also Do Cormo's book.

1.2 Parametrized Curves

Definition 1.2.1. Let $I = (a, b)$, if $\alpha\colon I \to \mathbb{R}^3$ is a C^∞ map,

$$t \mapsto (x(t), y(t), z(t))$$

then $\alpha(t)$ is a parametrized differentiable curve in $\mathbb{R}^3$. The image of α is called the trace of the curve.

Remark.

1) a, b could be finite number or infinity.

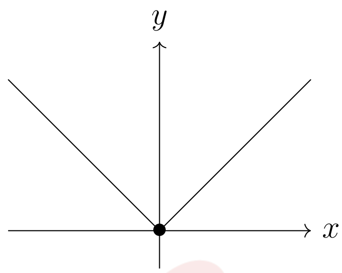
2) Same curve may have different parametrizations.

3) The parametrization automatically gives the direction of the motion on the curve.

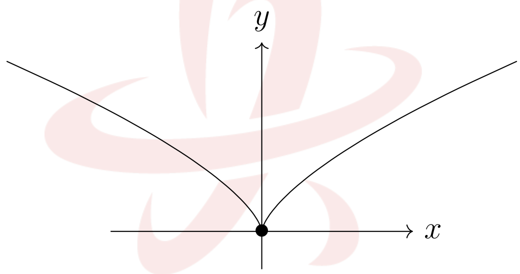
4) "Differentiable" just means $\alpha(t)$ is a C^∞ **map**, it does not say the (trace of) curve can not have singularities.

Example.

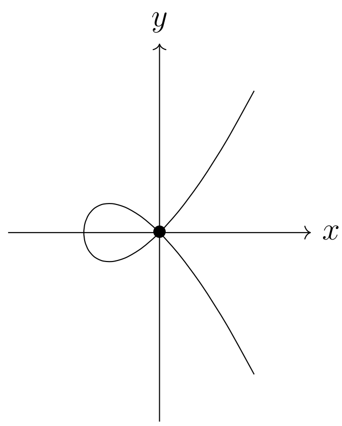
(1) $\alpha(t) = (t, |t|)$ *is not a differentiable curve.*



(2) $\alpha = (t^3, t^2)$ *is a differentiable curve. It can be also given by a equation* $y^3 = x^2$*, which is a cuspidal cubic curve.*



(3) $\alpha(t) = (t^2 - 1, t^3 - t)$*. This parametrization appers in the "blow-up" process of* $y^2 = x^3 + x^2$*. Here "blow-up" is introducing tangents to seperate points.*



Remark. (2) and (3) above may be the first examples you'll see in an algebraic geometry course.

Question: At the origin, what can you obsefve on (2) and (3)?
Answer: (2) $\alpha'(0) = 0$. (3) α is not one to one, but $\alpha'(0) \neq 0$.
Question: Define a differentiable curve in $\mathbb{R}^3$ and $\mathbb{S}^n$.

Remark. Among above differentiable parametrizations, (2) and (3) are differentiable curves. However, if we take $\beta(t) = (t, t^{\frac{2}{3}})$, this also parametrizes (2), but it's not a differentiable curve!

Definition 1.2.2. Let $\alpha(t)\colon I \to \mathbb{R}^3$ be a parametrized differentiable curve, then at $t_0 \in I$.

$$\alpha'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$$

is the velocity of $\alpha(t)$ at t_0.

(1) If $\alpha'(t_0) \neq 0$, we call $\alpha(t_0)$ a regular point.

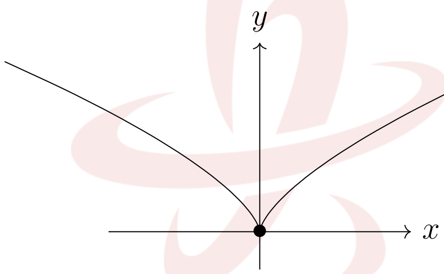
(2) If $\alpha'(t_0) = 0$, we call $\alpha(t_0)$ a singular point.

(3) If for all $t \in I$, $\alpha'(t) \neq 0$, we call $\alpha(t)$ a regular curve.

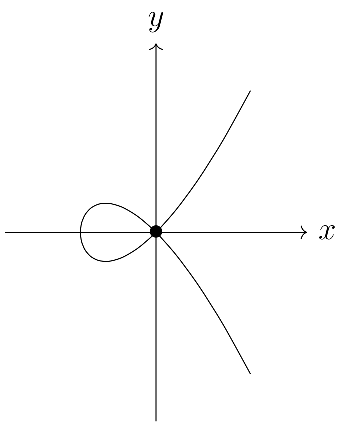
Question: What can you say about C^∞ parametrization for a regular curve?

$$\text{Regular curve} \iff \text{at each point, there is a unique tangent line.}$$

Example. *$\alpha(t) = (t^3, t^2)$ is not a regular curve. (Since $\alpha'(0) = 0$)*

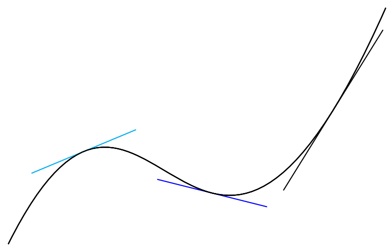


Example. *$\alpha(t) = (t^2 - 1, t^3, t)$ is a regular curve.*



Definition 1.2.3. Let $\alpha(t)$ be a regular curve, then the tangent line at t_0 is

$$l(t) = \alpha(t_0) + \alpha'(t_0)(t - t_0))$$



Definition 1.2.4. Let $\alpha(t)$ be a regular curve, the arc-length of $\alpha(t)$ is

$$s(t) = \int_{t_0}^{t} \left|\alpha'(t)\right| \mathrm{d}t.$$

Then $s'(t) = \left|\alpha'(t)\right|$

Question What's $\left|\alpha'(t)\right|$?

$\alpha(t)\colon I \to \mathbb{R}^3$ is a curve in $\mathbb{R}^3$. Here on $\mathbb{R}^3$, as the Euclidean space, we always assume the standard Euclidean inner product on it, i.e. $\forall u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{i,j=1}^{3} \delta_{ij}u_iv_j$$

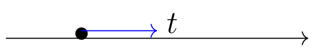
Let $\alpha(t) = (x(t), y(t), z(t)), \alpha'(t) = (x'(t), y'(t), z'(t))$, then $\left|\alpha'(t)\right| = \sqrt{\langle\alpha'(t), \alpha'(t)\rangle}$

Exercise. *Review vector Calculations, such as dot product, cross product and their properties, especially geometric meanie of these calculation, such as length, area, volume, angle, orientation, etc.*

Question: Can you define the arclength of a regular curve in $\mathbb{R}^n$¿ How about on $\mathbb{S}^n$?

- Arclength parameter(an intrinsic parametrization of a curve)

Example. *On a straight line, x=t describes the distance of the point away from the origin.*



On a general curve, we also want "some" parameter, which describes the arclength of point away from the initial point. This can happen iff $|\alpha'(t)| = 1$,i.e. a point on the curve moves in a unit speed.

$$\Rightarrow s(t) = \int_{0}^{t} \mathrm{d}t = t.$$

Question: For a given regular curve $\alpha(t)\colon I \to \mathbb{R}^3$, how to find such parameter?
Answer: $s(t) = \int_{t_0}^{t} \left|\alpha'(t)\right| dt$ is a function in t, and $s'(t) = \left|\alpha'(t)\right| \neq 0$(because the curve is regular). By the implicit function theorem, there is a function

$$t = t(s), t'(s) = \frac{1}{\left|\alpha'(t)\right|}.$$

This implies that

$$\alpha(t) = \alpha(t(s)) = (x(t(s)), y(t(s)), z(t(s)))$$
$$|\alpha'(s)| = |\alpha'(t)t'(s)| = |\alpha'(t)|\,|t'(s)| = 1$$

Convention In this course, we only consider differentiable regular curves, which are parametrized by the arclength (for convenience).

Remark. In this course, we only consider the curve without self-intersecting points, i.e curves "embedded into" $\mathbb{R}^3$. Here "embedded" means $d\alpha$ is a linear isomorphism and α is homeomorphic to its image.

1.3 Local theory of a regular space curve

Goal. Describe a space curve by using geometric quantities.

Question. How to make a space curve?

Starting with a straight line, we can bend it and twist it in a given way to produce a space curve.

- Bending the line $\longrightarrow$ "curvature".
- Twisting $\longrightarrow$ "torsion".
- Their relations are contained in Frenet formula.
- Conversely, fundamental theorem of the local theory of curves tells, once we're given two function, $\kappa(s), \tau(s)$, we can describe a unique curve in $\mathbb{R}^3$ up to a rigid motion, *s.t.* $\kappa(s)$ is its curvature and $\tau(s)$ is its torsion.

Recall: In Calculus, if $y = f(x)$ represents a curve, then $f''(x)$ tells the convexity of the curve. It measures how fast the velocity changes. It's also related to how straight line is bent.

Let $\alpha\colon I \to \mathbb{R}^2$ be a regular plane curve, parametrized by arc length, *i.e.* $|\alpha'(s)| = 1$. Then $\langle \alpha'(s), \alpha''(s) \rangle = 0$, and hence $\alpha''(s) \perp \alpha'(s)$. For a plane curve, we take normal of the curve to be counterclockwise $90°$ rotation of the tangent vector.

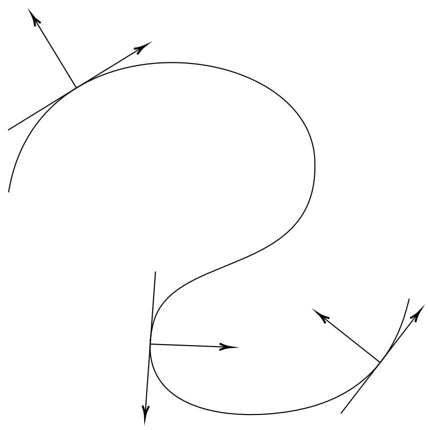


Figure 1.3.1: Example of a plane curve and its tangent and normal

Let N be the unit normal vector along $\alpha(s)$, we have

$$\langle \alpha''(s), N(s) \rangle = \pm|\alpha''(s)|.$$

Definition 1.3.1. The curvature of a plane curve $\alpha(s)$ is defined as

$$\kappa(s) = \langle \alpha''(s), N(s) \rangle .$$

Definition 1.3.2. Further we denote T be the unit tangent vector, then the Frenet equation of $\alpha(s)$ is

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T \end{cases} .$$

Note that

$$\langle T', N \rangle = \kappa \implies \langle T, N' \rangle = -\kappa.$$

- $\kappa > 0 \implies$ the point on the curve moves counterclockwise direction or say "to its left".
- $\kappa < 0 \implies$ the point on the curve moves clockwise direction or say "to its right".

Question. For the curve in fig. 1.3.1, can you tell where $\kappa > 0$ and where $\kappa < 0$ without doing calculation?

Remark. The sign of the curvature of the plane curve is caused by the direction convention of the unit normal vector. This could change according to the orientation of a curve.

Next, we take a look at the geometric meaning of $|\alpha''(s)|$ at some point $\alpha(s_0)$. By definition:

$$|\alpha''(s_0)| = \lim_{h\to 0} \left| \frac{\alpha'(s_0+h) - \alpha'(s_0)}{h} \right| .$$

We have

$$\begin{aligned} |\alpha'(s_0+h) - \alpha'(s_0)| &= \Big(|\alpha'(s_0+h)|^2 + |\alpha'(s_0)|^2 - 2\left\langle \alpha'(s_0+h), \alpha'(s_0) \right\rangle \Big)^{\frac{1}{2}} \\ &= (2 - 2\cos\theta_h)^{\frac{1}{2}} \\ &= (2 - 2(1 - \frac{1}{2}\theta_h^2) + \tilde{o}(\theta_h)^4)^{\frac{1}{2}} \\ &= (\theta_h^2 + \tilde{o}(\theta_h)^4)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$|\alpha''(s_0)| = \lim_{h\to 0} \left| \frac{\alpha'(s_0+h) - \alpha'(s_0)}{h} \right| = \lim_{h\to 0} \left| \frac{\theta_h}{h} \right| = |\theta'(s_0)|.$$

i.e. $|\alpha''(s)|$ measures the changing rate of angle of tangents.

In fact, for a plane curve, let θ be the angle between $\alpha'(s_0)$ and $\alpha'(s)$, then

$$\langle \alpha'(s), \alpha'(s_0) \rangle = \cos\theta_s \implies \langle \alpha''(s), \alpha'(s_0) \rangle = -\sin\theta_s \cdot \theta_s'.$$

Notice that $\cos\theta_s$ is the projection of $\alpha'(s_0)$ on the tangent $\alpha'(s)$, hence

$$\sin\theta_s = \langle \alpha'(s_0), N(s) \rangle .$$

On the other hand, $\alpha''(s) = T'(s) = \kappa(s)N(s) = \pm|\alpha''(s)|N(s)$, this gives $\theta_s' = \pm|\alpha''(s)|$.

Let $\alpha\colon I \to \mathbb{R}^3$ be a space curve, parametrized by arclength, *i.e.* $|\alpha'(s) = 1|$, we also have $\langle \alpha'(s), \alpha''(s) \rangle = 0$, *i.e.* $\alpha''(s) \perp \alpha'(s)$.

Unlike case of dim 2, it does not make sense to prescribe a normal vector of a curve. However, from above discussion, we see the geometric meaning of $|\alpha''(s)|$ is the measure of how fast the point on the curve leaving the straight line. We came into following definition:

Definition 1.3.3. The *curvature* of a regular space curve $\alpha(s)$ parametrized by arclength is defined as

$$\kappa(s) = |\alpha''(s)|.$$

And the unit normal vector at $\alpha(s)$ is

$$N = \frac{\alpha''(s)}{|\alpha''(s)|}, \quad \text{for } |\alpha''(s)| > 0.$$

Remark.

- If $|\alpha''(s)| \equiv 0$ then α must be a straight line, and all unit normal vectors line on a unit circle $\perp \alpha$.
- If $|\alpha''(s_0)| = 0$, we call s_0 a singular point of order 1. (Note. s_0 *s.t.* $|\alpha(s_0)| = 0$ is called a singular point of order 0) At such points, there is no well-defined normal vector.

Definition 1.3.4. The plane determined by T, N is called the *osculating plane* of $\alpha(s)$. The unit normal vector of the osculating plane

$$B = T \times N$$

is called *binormal vector*.

Remark.

- $\{T, N, B\}$ satisfies the right hand rule.
- $|B'|$ measures how fast the point leaves the osculating plane.

If we denote θ_h be angle between $B(s_0 + h)$ and $B(s_0)$, similar to former calculation, we have

$$\begin{aligned} |B'(s_0)| &= \lim_{h \to 0} \left| \frac{B(s_0 + h) - B(s_0)}{h} \right| \\ &= \lim_{h \to 0} \left| \frac{\sqrt{2 - 2\cos\theta_h}}{h} \right| \\ &= |\theta'_{s_0}|. \end{aligned}$$

As we saw, at each (non-singular) point on a space curve $\alpha(s)$, we can associate an oriented orthonormal frame $\{T, N, B\}$.

Question. How these three vector fields are related to the geometry of the curve?

By definition, we write out 0-order info of $\{T, N, B\}$, *i.e.*

$$\begin{cases} T = \alpha' \\ N = \frac{\alpha''}{|\alpha''|} & \Longrightarrow \ T' = \alpha'' = \kappa N \\ B = T \times N & \Longrightarrow \ B' = T' \times N + T \times N' = T \times N' \end{cases}.$$

Since $|B| = 1$, we have $B' \perp B$. Also from above we see $B' \perp T$, hence $B' \parallel N$.

Definition 1.3.5. We define

$$B' = \tau N.$$

Here $\tau(s)$ is called the torsion of $\alpha(s)$.

Next, we also want to find N'. $N = B \times T$ gives

$$\begin{aligned} N' &= B' \times T + B \times T' \\ &= \tau N \times T + B \times (\kappa N) \\ &= -\kappa T - \tau B. \end{aligned}$$

Theorem 1.3.6. *The fundamental equations of a space curve (also called the Frenet equations) is given by*

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{1.3.1}$$

Remark.

(1) $\tau \equiv 0$ but $\kappa \neq 0$ at all points $\implies$ $\alpha(s)$ is a plane curve. (Note this may be not true if we don't assume $\kappa \neq 0$, see ex.10 in Do Carmo's book)
$\kappa \equiv 0 \implies \alpha(s)$ is a straight line.

Note $\tau \equiv 0 \implies B' = 0 \implies B$ is constant vector. Further,

$$(\alpha \cdot B)' = \alpha' \cdot B = T \cdot B = 0,$$

gives $\alpha \cdot B =$ constant $\implies$ $(\alpha(s) - \alpha(s_0)) \cdot B = 0$. Since $\kappa \neq 0$ at all points, the osculating plane is always well-defined, hence B is always defined, we proved α lie in some plane perpendicular to vector B.

(2) In different textbooks, you may see the definition of τ having a different sign from here.

Friendly warning: When studying the Geometry, (even later in Riemannian Geometry), it happens a lot that different authors use different sign convention for the same definition. It's very important that you should fix your own notation, ant keep it consistently!

Definition 1.3.7. $\{T, N, B\}$ is called Frenet trihedron of $\alpha(s)$, it gives an moving orthonormal basis of $\mathbb{R}^3$ along the curve $\alpha(s)$.

The Frenet equation describes how such moving orthonormal basis moves along $\alpha(s)$.

Remark. Note that in above discussion, we have chosen a special parameter, the arclength parameter, of $\alpha(s)$. In the study of Geometry, finding a good parametrization can simplify a lot of work and itself an important problem. In more general framework, it's called a "Gauge related" problem.

We have seen that given a regular curve $\alpha(s)$, parametrized by arclength, the Frenet equation is eq. (1.3.1), for some functions $\kappa(s) > 0$ be its curvature and $\tau(s)$ be its torsion. Conversely, we ask

Question. If we're given smooth functions $\kappa(s), \tau(s)$ with $\kappa(s) > 0$,

(1) (Existence) Does there exist a regular curve $\alpha(s)$ *s.t.* $\kappa(s)$ is its curvature and $\tau(s)$ is its torsion?

(2) (Uniqueness) If such curve exists, is it unique in some sense?

The answer is **YES!**

Theorem 1.3.8 (Fundamental theorem of the local theory of curves)**.** *Let* $\kappa(s), \tau(s)\colon I \to \mathbb{R}$ *be smooth functions, assume* $\kappa(s) > 0$*, then*

- *(Existence) There is a regular curve realize* κ *and* τ *as its curvature and torsion.*
- *(Uniqueness) If* α, β *are two such curves parametrized by arclength parameter, then they only differ by a rigid motion of* $\mathbb{R}^3$*.* i.e. $\exists T \in O(3), c \in \mathbb{R}^3$ s.t. $\beta = T\alpha + c$*.*

Remark.

(1) Existence follows from a Cauchy problem (initial value problem) of ODE system.

(2) The curve is unique up to a rotation of $\mathbb{R}^3$ and a translation.

Proof. If we denote

$$X(s) = \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad P(s) = \begin{bmatrix} & \kappa & \\ -\kappa & & -\tau \\ & \tau & \end{bmatrix}.$$

Where T, N, B are viewed as **row vectors**. Then the Frenet equation writes as

$$X' = PX.$$

This is a first order linear ODE (of nine unknown functions), then by the existence and uniqueness theorem of ODEs, given any initial value

$$X(0) = \begin{bmatrix} T_0 \\ N_0 \\ B_0 \end{bmatrix}$$

witch form an orthonormal basis, the system has a unique solution that extend to whole domain I.

We need to check the solution actually is orthonormal frame for each s, notice the orthonormal relation can be written as

$$XX^t = I_3.$$

Where I_3 is the identity matrix of dimension three. Take differential on left hand side of the equation we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(XX^t) &= X'X^t + X(X^t)' = X'X^t + X(X')^t \\ &= PXX^t + X(PX)^t \\ &= PXX^t + XX^tP^t \\ &= PXX^t - XX^tP. \end{aligned}$$

If we denote $Y = XX^t$, we see
$$Y' = PY - YP$$
In coordinates, if we set $T = v_1, N = v_2, B = v_3$, and
$$y_{ij}(s) = \left\langle v_i(s), v_j(s) \right\rangle, \quad P = (a_{ij})_{i,j=1}^3.$$
We have $y_{ij} = y_{ji}, a_{ij} = -a_{ji}$, and then
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} y_{ij} &= \left\langle v_i', v_j \right\rangle + \left\langle v_i, v_j' \right\rangle \\ &= \left\langle a_{ik} v_k, v_j \right\rangle + \left\langle v_i, a_{jk} v_k \right\rangle \\ &= a_{ik} y_{kj} + a_{jk} y_{ki} \\ &= a_{ik} y_{kj} - y_{ik} a_{kj}.\end{aligned}$$
This gives again a first order ODE system, with initial value $Y(0) = I_3$, or say $y_{ij}(0) = \delta_{ij}$, but there is an obvious solution $Y \equiv I_3$, so by uniqueness theorem, this is it. This proves $XX^t = I_3$ for any s.

Until now, we have proved the existence of orthonormal moving frame $\{T, N, B\}$. Notice T is just $\alpha'(s)$, so given initial point $\alpha(0) = \alpha_0$, integrate w.r.t s gives a valid solution
$$\alpha(s) = \alpha_0 + \int_0^s T(\xi)\,\mathrm{d}\xi.$$

For the uniqueness, we need to look carefully into the initial condition we chose for the solution α, that is, choice of initial frame $\{T_0, N_0, B_0\}$ and initial point α_0. Given two valid solution curve α, β, with initial condition (X_a, α_0) and (X_b, β_0), we choose an orthogonal matrix $T = X_b X_a^{-1}$, a constant $c = \beta_0 - T\alpha_0$, then we see the curve
$$\tilde{\beta} = T(\alpha - \alpha_0) + \beta_0 = T\alpha + c$$
satisfy exactly the same initial condition as β, so they must agree. This proves the uniqueness up to rigid motion we stated before. □

Remark.

(1) **Exercise:** Check that for solution given above, κ and τ are its curvature and torsion.

(2) The condition $\kappa > 0$ is needed for uniqueness. Can you construct a counterexample when there is one point *s.t.* $\kappa = 0$.

(3) Uniqueness can be proved without knowledge of ODEs, see theorem after this remark.

(4) We can view the ODE problem at a somehow higher point. Consider the space of all orthonormal frames, it is actually a smooth manifold. It's a little non-trivial but we can identity the space with three dimensional rotation group $SO(3)$, smoothly embedded into $\mathbb{R}^9$, the space of three dimensional matrices. The equation, can be interpreted to a (time dependent) vector field on $SO(3)$. One can verify the vector field is tangent to the manifold, so it is actually a vector field not only in $\mathbb{R}^9$ but in $SO(3)$ itself. Similar to the existence and uniqueness theorem of ODEs on Euclid spaces, we have a version of such theorem for smooth manifolds. It states that for a smooth manifold M, a (maybe time dependent) smooth vector field X on M, then

with any given initial point p, there exists an integral curve on M starting at p, tangent to X everywhere, and it is unique. Using the theorem, we can say that with given initial $\{T_0, N_0, B_0\}$, there exists a unique solution $\{T(s), N(s), B(s)\}$. Note that the solution is automatically lie in $SO(n)$, no need to verify it is orthonormal.

Proof. (Uniqueness, without ODE knowledge).

Let $\alpha(s), \beta(s)$ share the same $\kappa(s), \tau(s)$ as their curvature and torsion, by similarly a rotation and a translation, we assume they have same initial condition, *i.e.* $\alpha(0) = \beta(0)$, and the Frenet frame agree at $s = 0$. We claim now we must have $\alpha(s) = \beta(s), \forall\, s$.

Notice $\alpha(s) = \alpha(0) + \int_{s_0}^{s} \alpha'(s)\,\mathrm{d}s$, so it suffices to show $T_\alpha = T_\beta$, equivalently

$$|T_\alpha - T_\beta|^2 = 0.$$

Take differential we get:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|T_\alpha - T_\beta|^2 &= \left\langle T_\alpha - T_\beta, T'_\alpha - T'_\beta \right\rangle \\ &= -\left\langle T_\alpha, \kappa N_\beta \right\rangle - \left\langle T_\beta, \kappa N_\alpha \right\rangle.\end{aligned}$$

Similar calculation gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|N_\alpha - N_\beta|^2 &= \left\langle N_\alpha, \kappa T_\beta + \tau B_\beta \right\rangle + \left\langle N_\beta, \kappa T_\alpha + \tau B_\alpha \right\rangle \\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|B_\alpha - B_\beta|^2 &= -\left\langle B_\alpha, \tau N_\beta \right\rangle - \left\langle B_\beta, \tau N_\alpha \right\rangle.\end{aligned}$$

Sum the three equation we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(|T_\alpha - T_\beta|^2 + |N_\alpha - N_\beta|^2 + |B_\alpha - B_\beta|^2) = 0.$$

But the sum of square equals 0 at $s = s_0$, so it is identically 0, in particular, $T_\alpha = T_\beta$ for all s. □