Prof. Dr. Stefan Hofmann Winter term 2016/17

Exercises on General Relativity TVI TMP-TC1

Problem set 5, due November 24 - 28

Exercise 1 – Geodesic equation

Show that the geodesic equation as it was given in the lecture

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \tag{1}$$

is equivalent to the following expression:

$$\dot{x}^\alpha \left(\partial_\alpha \dot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\beta\right) =: \dot{x}^\alpha \nabla_\alpha \dot{x}^\mu = 0 \; , \tag{2}$$

where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$.
Furthermore, consider the metric $g_{\mu\nu} = \eta_{\mu\nu} - 2\Phi\delta_{\mu\nu}$ with the gravitational potential Φ and evaluate the geodesic equation with it in the newtonian limit. Therefore, Φ is small compared to 1, time independent and geodesics are considered in the non-relativistic limit $|\dot{x}^0| \gg |\dot{x}^i|$ with $i \in I(3)$.

Exercise 2 – Area of the 2-sphere

A two-sphere of fixed radius R in three-dimensional Euclidean space is concidered using polar coordinates $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi[$:

$$\begin{aligned} x^1(\theta,\varphi) &= R\sin\theta\cos\varphi \\ x^2(\theta,\varphi) &= R\sin\theta\sin\varphi \\ x^3(\theta,\varphi) &= R\cos\theta \; . \end{aligned}$$

Calculate the matrix

$$M = \begin{pmatrix} \frac{\partial \boldsymbol{x}}{\partial\theta}\cdot\frac{\partial \boldsymbol{x}}{\partial\theta} & \frac{\partial \boldsymbol{x}}{\partial\theta}\cdot\frac{\partial \boldsymbol{x}}{\partial\phi} \\ \frac{\partial \boldsymbol{x}}{\partial\phi}\cdot\frac{\partial \boldsymbol{x}}{\partial\theta} & \frac{\partial \boldsymbol{x}}{\partial\phi}\cdot\frac{\partial \boldsymbol{x}}{\partial\phi} \end{pmatrix} \tag{3}$$

and the area of the two-sphere with

$$A = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \, \sqrt{\det(M)} \; . \tag{4}$$

Exercise 3 – Metric and geodesics of the 2-sphere

Consider the metric of a 2-sphere of radius R:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = R^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{5}$$

with $\mu, \nu \in I(2)$. Why does this describe the metric of a 2-sphere?
The metric encodes all information on the geometry of the manifold. Choosing $x^1 = \theta$ and $x^2 = \phi$, read off the matrix $g_{\mu\nu}$.
The Christoffel symbols are defined as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(2\partial_{(\nu} g_{\mu)\beta} - \partial_\beta g_{\mu\nu}\right) = \frac{1}{2} g^{\alpha\beta} \left(\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}\right) \; . \tag{6}$$

Compute the non-vanishing Christoffel symbols for the two-sphere.
(Hint: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, so only a few components have to be computed explicitly.)
Find the geodesics on a sphere by using these Christoffel symbols.

General information

The lecture takes place on Monday and on Wednesday at 14:00 - 16:00 in A348 (Theresienstraße 37).

There are four tutorials:

Monday at 12:00 - 14:00 in A 249 by Katrin Hammer
Thursday at 16:00 - 18:00 in A 449 by Thomas Steingasser
Friday at 14:00 - 16:00 in A 348 by Andrea Morelato
Friday at 16:00 - 18:00 in A 348 by Elvis Bejko

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_16_17/tvi_tc1_gr/index.html`