Stochastic Rounding, Sharkovskii's Theorem, and the Future of Floating Point

Useful Notation and Terminology

- $f^n = f \circ f \circ \cdots \circ f, f^2 = f \circ f$
- An orbit of $f(x)$ is $|f(x)| = n \text{ s.t. } f^n(x) = x$
- A static point is a point $x \text{ s.t. } f(x) = x$
- The circle is the $[0,1)$ domain wrapped back on itself
  - $S' \equiv [0,1]/\sim, 0 \sim 1$

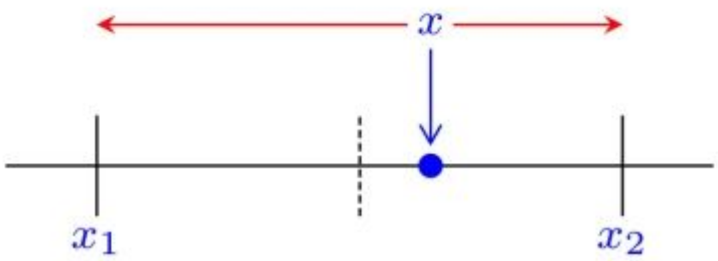
    $\equiv \mathbb{R}/\sim, x \sim w \text{ if } x - w \in \mathbb{Z}$

    $\equiv (\mathbb{R},+)/(\mathbb{Z},+)$

Floating Point Rounding

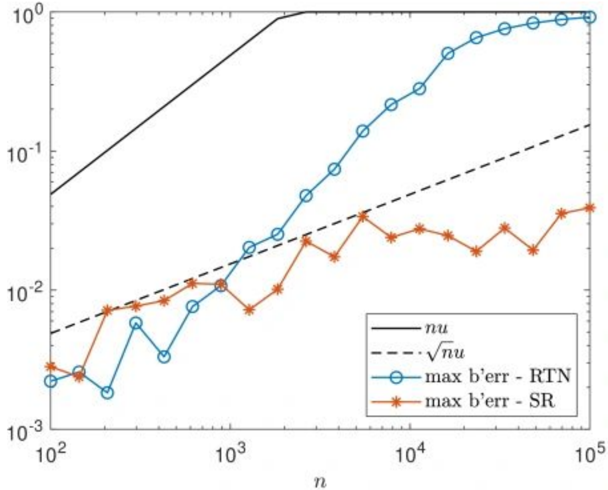
- Rounding is a very important issue in any system that isn't closed.
- Computers are one of those systems, since you can obtain decimals that are not representable in base 2
- The IEEE standard for floating point rounds to the nearest representable float, with ties going to the even in order to make the last bit a 0
- This standard is deterministic and has issues with chaotic systems
- The rounding error can build up over time leading to divergent behavior

Stochastic Rounding



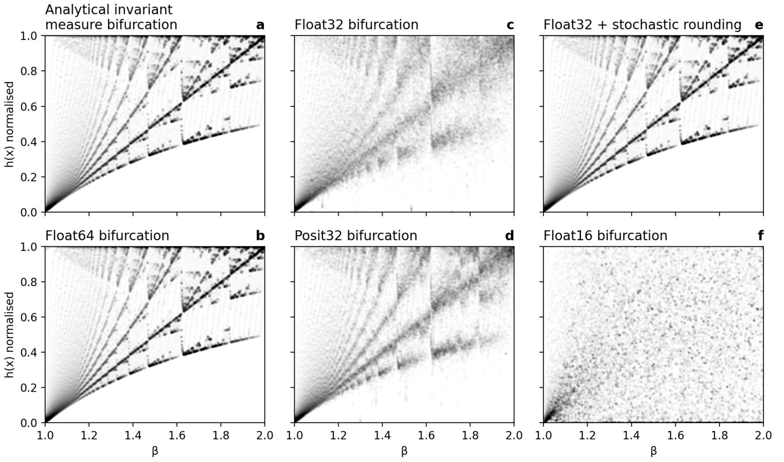
- Stochastic rounding aims to solve this issue by introducing randomness into rounding
- Stochastic rounding performs the operation in a higher precision, then rounds to the lower precision using a weighted distribution
- Rounding up in this example has a probability of $\frac{x - x_1}{x_2 - x_1}$ and has a probability of rounding down equivalent to $\frac{x - x_2}{x_2 - x_1}$
- Note that these probabilities sum to 1
- This leads to an accurate expected value and the goal is to have the errors cancel out to maintain better accuracy with chaotic systems

Stochastic Rounding in Action



- Two n vectors are sampled in the $[0, 1)$ distribution
- Their inner products are then computed using the IEEE floating point standard where $u \approx 4.9 \cdot 10^{-4}$
- Connolly, Higham, and Mary (2020) showed that stochastic rounding error is bound by $\sqrt{nu}$ while round to nearest is only bounded by nu

Bernoulli Map



- The Bernoulli Map is a simple map for the circle that multiplies each number by 2
- The generalized Bernoulli Map is the same mapping, but instead it multiplies the number by an arbitrary factor
- This is a simple chaotic system, and a system that Klöwer, M, PV Coveney, EA Paxton and TN Palmer showed is significantly improved by using stochastic floating point
- We could also improve computation on this by taking advantage of Sharkovskii’s Theorem

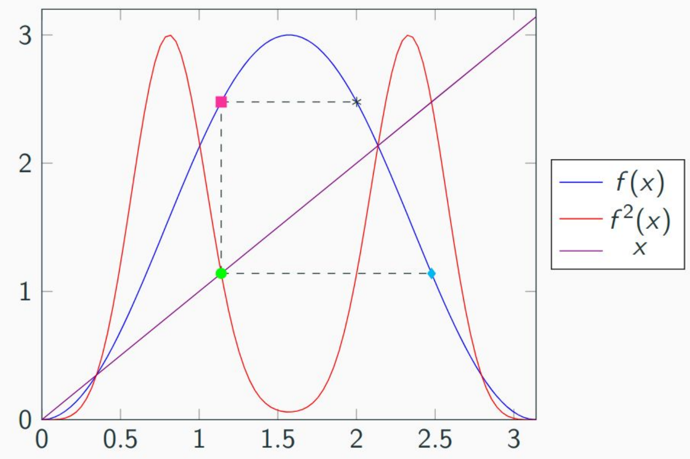
Sharkovskii’s Theorem, Limits of Digital Computers

- Peter V. Coveney mentions in his paper *Sharkovskii’s theorem and the limits of digital computers for the simulation of chaotic dynamical systems* that Sharkovskii’s Theorem proves that floating point numbers cannot accurately simulate these dynamical systems
- Sharkovskii’s Theorem shows that any orbit that is not a power of 2 implies infinitely many orbits, as is shown in the famous paper *Period 3 Implies Chaos*
- However, we can try to take advantage of Sharkovskii’s theorem to build better data structures for chaotic systems

What is Sharkovskii’s Theorem

- Sharkovskii’s Theorem gives us an ordering of numbers.
- The theorem states that for a continuous mapping on a finite domain, we have an ordering of numbers such that if we find an orbit with length n, then we know there exists orbits of every length that comes after n in the ordering.
- This proof is a really cool relatively simple proof that can analyzed graphically using the intermediate value theorem

Sneak Peak of the Proof



- This is the main idea for the entire proof
- This is the simplest case, where we show that an orbit of length 2 implies a stable point
- All that is needed is the intermediate value theorem
- While the proofs for any orbit of length not equal to 2 or 1 implies an orbit of length 2 and that an orbit of odd length m implies and orbit of length m+2 are conceptually a lot harder to grasp, they pretty much use this same idea and visualization

Sharkovskii's Ordering

- The order first goes as $2^n(2m+3) \succ 2^n(2(m+1)+3) \succ \cdots \succ 2^{n+1}(2m+3) \succ \ldots$
- Then after, $m \to \infty, n \to \infty$, we get the ordering $2^n \succ 2^{n-1} \succ \ldots 2 \succ 1$
- So the full ordering is $3 \succ 5 \succ 7 \succ \ldots$

$$6 \succ 10 \succ 14 \succ \ldots$$

$$2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \succ \ldots$$

$$2^4 \succ 2^3 \succ 2^2 \succ 2 \succ 1$$

Sources

- https://www.sciencedirect.com/science/article/pii/S1877750324002424?via%3Dihub
- https://nhigham.com/2020/07/07/what-is-stochastic-rounding/
- https://arxiv.org/html/2410.10517v1
- https://github.com/milankl/BernoulliMap?tab=readme-ov-file
- https://www.nature.com/articles/s41598-023-37004-4