Background:

The field of chaotic dynamical systems has an issue with computer simulations. Due to the inaccuracy of the IEEE floating point standard, and the general limitations of using a finitely represented base two number system, these systems can not be accurately simulated on modern computers. Even extremely simple systems such as the Bernoulli Map of the circle aren't simulatable to any degree of accuracy past a certain number of iterations. The Bernoulli Map simply doubles every number, and on the circle this is performed in the reals where two numbers are equivalent if their difference is an integer. We can easily compute that using modulo arithmetic. In the Bernoulli Map, all dyadic rationals are pre orbits of 0, a stable point, while every other rational is chaotic. Due to the floating point rounding rules, after a certain number of iterations the floating point errors accumulate until every number eventually becomes a dyadic rational, leading to the collapse of the system. While according to Sharkovskii's Theorem, we will never be able to design a generalized float that can work with these orbits, we have a few pathways in order to better attack the problem.

Description:

One way to better simulate these systems is to use stochastic rounding. While there doesn't currently exist a stochastic rounding library for Python, a couple exist for Julia, C, and Fortran. This rounding performs the calculations in a higher precision, and rounds into the lower precision using a probability distribution with the two closest points in the lower precision. This means that the expected value of the rounding should still be the original number, and with enough repeated calculations the errors should cancel out rather than accumulate. This can be hard to implement, as you will see in my Jupyter Notebook. Another way is to look at each chaotic system individually and try to analyse the orbits in order to design custom data types to more accurately simulate the systems we want. Sharkovskii's Theorem comes in handy for this, and I was able to make a custom fraction class that simulates the Bernoulli Map on the circle accurately for as many iterations as required and much faster than the default fraction class.

Application:

This can be applied to any chaotic system on a finite domain that has an orbit whose length is not a power of 2. This means that the chaotic system has infinite orbits. This is when our custom Sharkovskii data types can come in handy. Stochastic rounding, on the other hand, is a lot more versatile and can be used on any chaotic system where you have fractions whose denominator is not a power of two and you are worried about accuracy of the decimal part with repeated iterations

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