

REINFORCEMENT LEARNING

1. THE PROBLEM

S_t	state at time t
A_t	action at time t
R_t	reward at time t
γ	discount rate (where $0 \leq \gamma \leq 1$)
G_t	discounted return at time t ($\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$)
\mathcal{S}	set of all nonterminal states
\mathcal{S}^+	set of all states (including terminal states)
\mathcal{A}	set of all actions
$\mathcal{A}(s)$	set of all actions available in state s
\mathcal{R}	set of all rewards
$p(s', r s, a)$	probability of next state s' and reward r , given current state s and current action a ($\mathbb{P}(S_{t+1} = s', R_{t+1} = r S_t = s, A_t = a)$)

2. THE SOLUTION

π	policy <i>if deterministic:</i> $\pi(s) \in \mathcal{A}(s)$ for all $s \in \mathcal{S}$ <i>if stochastic:</i> $\pi(a s) = \mathbb{P}(A_t = a S_t = s)$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$
v_π	state-value function for policy π ($v_\pi(s) \doteq \mathbb{E}[G_t S_t = s]$ for all $s \in \mathcal{S}$)
q_π	action-value function for policy π ($q_\pi(s, a) \doteq \mathbb{E}[G_t S_t = s, A_t = a]$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$)
v_*	optimal state-value function ($v_*(s) \doteq \max_\pi v_\pi(s)$ for all $s \in \mathcal{S}$)
q_*	optimal action-value function ($q_*(s, a) \doteq \max_\pi q_\pi(s, a)$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$)

3. BELLMAN EQUATIONS

3.1. Bellman Expectation Equations.

$$v_{\pi}(s) = \sum_{a \in \mathcal{A}(s)} \pi(a|s) \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma v_{\pi}(s'))$$

$$q_{\pi}(s, a) = \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma \sum_{a' \in \mathcal{A}(s')} \pi(a'|s') q_{\pi}(s', a'))$$

3.2. Bellman Optimality Equations.

$$v_{*}(s) = \max_{a \in \mathcal{A}(s)} \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma v_{*}(s'))$$

$$q_{*}(s, a) = \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma \max_{a' \in \mathcal{A}(s')} q_{*}(s', a'))$$

3.3. Useful Formulas for Deriving the Bellman Equations.

$$v_{\pi}(s) = \sum_{a \in \mathcal{A}(s)} \pi(a|s) q_{\pi}(s, a)$$

$$v_{*}(s) = \max_{a \in \mathcal{A}(s)} q_{*}(s, a)$$

$$q_{\pi}(s, a) = \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma v_{\pi}(s'))$$

$$q_{*}(s, a) = \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma v_{*}(s'))$$

$$q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t | S_t = s, A_t = a] \quad (1)$$

$$= \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} \mathbb{P}(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a) \mathbb{E}_\pi[G_t | S_t = s, A_t = a, S_{t+1} = s', R_{t+1} = r] \quad (2)$$

$$= \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r | s, a) \mathbb{E}_\pi[G_t | S_t = s, A_t = a, S_{t+1} = s', R_{t+1} = r] \quad (3)$$

$$= \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r | s, a) \mathbb{E}_\pi[G_t | S_{t+1} = s', R_{t+1} = r] \quad (4)$$

$$= \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r | s, a) \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} | S_{t+1} = s', R_{t+1} = r] \quad (5)$$

$$= \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r | s, a) (r + \gamma \mathbb{E}_\pi[G_{t+1} | S_{t+1} = s']) \quad (6)$$

$$= \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r | s, a) (r + \gamma v_\pi(s')) \quad (7)$$

The reasoning for the above is as follows:

- (1) by definition ($q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$)
- (2) Law of Total Expectation
- (3) by definition ($p(s', r | s, a) \doteq \mathbb{P}(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a)$)
- (4) $\mathbb{E}_\pi[G_t | S_t = s, A_t = a, S_{t+1} = s', R_{t+1} = r] = \mathbb{E}_\pi[G_t | S_{t+1} = s', R_{t+1} = r]$
- (5) $G_t = R_{t+1} + \gamma G_{t+1}$
- (6) Linearity of Expectation
- (7) $v_\pi(s') = \mathbb{E}_\pi[G_{t+1} | S_{t+1} = s']$

4. DYNAMIC PROGRAMMING

Algorithm 1: Policy Evaluation

Input: MDP, policy π , small positive number θ

Output: $V \approx v_\pi$

Initialize V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

repeat

$\Delta \leftarrow 0$

for $s \in \mathcal{S}$ **do**

$v \leftarrow V(s)$

$V(s) \leftarrow \sum_{a \in \mathcal{A}(s)} \pi(a|s) \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma V(s'))$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

end

until $\Delta < \theta$;

return V

Algorithm 2: Estimation of Action Values

Input: MDP, state-value function V

Output: action-value function Q

for $s \in \mathcal{S}$ **do**

for $a \in \mathcal{A}(s)$ **do**

$Q(s, a) \leftarrow \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma V(s'))$

end

end

return Q

Algorithm 3: Policy Improvement

Input: MDP, value function V
Output: policy π'
for $s \in \mathcal{S}$ **do**
 for $a \in \mathcal{A}(s)$ **do**
 $Q(s, a) \leftarrow \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r | s, a)(r + \gamma V(s'))$
 end
 $\pi'(s) \leftarrow \arg \max_{a \in \mathcal{A}(s)} Q(s, a)$
end
return π'

Algorithm 4: Policy Iteration

Input: MDP, small positive number θ
Output: policy $\pi \approx \pi_*$
Initialize π arbitrarily (e.g., $\pi(a|s) = \frac{1}{|\mathcal{A}(s)|}$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$)
 $policy_stable \leftarrow false$
repeat
 $V \leftarrow \text{Policy_Evaluation}(\text{MDP}, \pi, \theta)$
 $\pi' \leftarrow \text{Policy_Improvement}(\text{MDP}, V)$
 if $\pi = \pi'$ **then**
 $policy_stable \leftarrow true$
 end
 $\pi \leftarrow \pi'$
until $policy_stable = true$;
return π

Algorithm 5: Truncated Policy Evaluation

Input: MDP, policy π , value function V , positive integer $max_iterations$
Output: $V \approx v_\pi$ (if $max_iterations$ is large enough)
 $counter \leftarrow 0$
while $counter < max_iterations$ **do**
 for $s \in \mathcal{S}$ **do**
 $V(s) \leftarrow \sum_{a \in \mathcal{A}(s)} \pi(a|s) \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r | s, a)(r + \gamma V(s'))$
 end
 $counter \leftarrow counter + 1$
end
return V

Algorithm 6: Truncated Policy Iteration

Input: MDP, positive integer $max_iterations$, small positive number θ

Output: policy $\pi \approx \pi_*$

Initialize V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Initialize π arbitrarily (e.g., $\pi(a|s) = \frac{1}{|\mathcal{A}(s)|}$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$)

repeat

$\pi \leftarrow \text{Policy_Improvement}(\text{MDP}, V)$

$V_{old} \leftarrow V$

$V \leftarrow \text{Truncated_Policy_Evaluation}(\text{MDP}, \pi, V, max_iterations)$

until $\max_{s \in \mathcal{S}} |V(s) - V_{old}(s)| < \theta$;

return π

Algorithm 7: Value Iteration

Input: MDP, small positive number θ

Output: policy $\pi \approx \pi_*$

Initialize V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

repeat

$\Delta \leftarrow 0$

for $s \in \mathcal{S}$ **do**

$v \leftarrow V(s)$

$V(s) \leftarrow \max_{a \in \mathcal{A}(s)} \sum_{s' \in \mathcal{S}, r \in \mathcal{R}} p(s', r|s, a)(r + \gamma V(s'))$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

end

until $\Delta < \theta$;

$\pi \leftarrow \text{Policy_Improvement}(\text{MDP}, V)$

return π

5. MONTE CARLO METHODS

Algorithm 8: First-Visit MC Prediction (*for state values*)

Input: policy π , positive integer $num_episodes$
Output: value function V ($\approx v_\pi$ if $num_episodes$ is large enough)
Initialize $N(s) = 0$ for all $s \in \mathcal{S}$
Initialize $returns_sum(s) = 0$ for all $s \in \mathcal{S}$
for $i \leftarrow 1$ **to** $num_episodes$ **do**
 Generate an episode $S_0, A_0, R_1, \dots, S_T$ using π
 for $t \leftarrow 0$ **to** $T - 1$ **do**
 if S_t is a first visit (with return G_t) **then**
 $N(S_t) \leftarrow N(S_t) + 1$
 $returns_sum(S_t) \leftarrow returns_sum(S_t) + G_t$
 end
 end
end
 $V(s) \leftarrow returns_sum(s)/N(s)$ for all $s \in \mathcal{S}$
return V

Algorithm 9: First-Visit MC Prediction (*for action values*)

Input: policy π , positive integer $num_episodes$
Output: value function Q ($\approx q_\pi$ if $num_episodes$ is large enough)
Initialize $N(s, a) = 0$ for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
Initialize $returns_sum(s, a) = 0$ for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
for $i \leftarrow 1$ **to** $num_episodes$ **do**
 Generate an episode $S_0, A_0, R_1, \dots, S_T$ using π
 for $t \leftarrow 0$ **to** $T - 1$ **do**
 if (S_t, A_t) is a first visit (with return G_t) **then**
 $N(S_t, A_t) \leftarrow N(S_t, A_t) + 1$
 $returns_sum(S_t, A_t) \leftarrow returns_sum(S_t, A_t) + G_t$
 end
 end
end
 $Q(s, a) \leftarrow returns_sum(s, a)/N(s, a)$ for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
return Q

Algorithm 10: First-Visit GLIE MC Control

Input: positive integer $num_episodes$, GLIE $\{\epsilon_i\}$
Output: policy π ($\approx \pi_*$ if $num_episodes$ is large enough)
Initialize $Q(s, a) = 0$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$
Initialize $N(s, a) = 0$ for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
for $i \leftarrow 1$ **to** $num_episodes$ **do**
 $\epsilon \leftarrow \epsilon_i$
 $\pi \leftarrow \epsilon\text{-greedy}(Q)$
 Generate an episode $S_0, A_0, R_1, \dots, S_T$ using π
 for $t \leftarrow 0$ **to** $T - 1$ **do**
 if (S_t, A_t) is a first visit (with return G_t) **then**
 $N(S_t, A_t) \leftarrow N(S_t, A_t) + 1$
 $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{1}{N(S_t, A_t)}(G_t - Q(S_t, A_t))$
 end
 end
end
return π

Algorithm 11: First-Visit Constant- α (GLIE) MC Control

Input: positive integer $num_episodes$, small positive fraction α , GLIE $\{\epsilon_i\}$
Output: policy π ($\approx \pi_*$ if $num_episodes$ is large enough)
Initialize Q arbitrarily (e.g., $Q(s, a) = 0$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$)
for $i \leftarrow 1$ **to** $num_episodes$ **do**
 $\epsilon \leftarrow \epsilon_i$
 $\pi \leftarrow \epsilon\text{-greedy}(Q)$
 Generate an episode $S_0, A_0, R_1, \dots, S_T$ using π
 for $t \leftarrow 0$ **to** $T - 1$ **do**
 if (S_t, A_t) is a first visit (with return G_t) **then**
 $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(G_t - Q(S_t, A_t))$
 end
 end
end
return π

6. TEMPORAL-DIFFERENCE METHODS

Algorithm 12: TD(0)

Input: policy π , positive integer $num_episodes$
Output: value function V ($\approx v_\pi$ if $num_episodes$ is large enough)
Initialize V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)
for $i \leftarrow 1$ **to** $num_episodes$ **do**
 Observe S_0
 $t \leftarrow 0$
 repeat
 Choose action A_t using policy π
 Take action A_t and observe R_{t+1}, S_{t+1}
 $V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$
 $t \leftarrow t + 1$
 until S_t is terminal;
end
return V

Algorithm 13: Sarsa

Input: policy π , positive integer $num_episodes$, small positive fraction α , GLIE $\{\epsilon_i\}$
Output: value function Q ($\approx q_\pi$ if $num_episodes$ is large enough)
Initialize Q arbitrarily (e.g., $Q(s, a) = 0$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$, and $Q(\text{terminal-state}, \cdot) = 0$)
for $i \leftarrow 1$ **to** $num_episodes$ **do**
 $\epsilon \leftarrow \epsilon_i$
 Observe S_0
 Choose action A_0 using policy derived from Q (e.g., ϵ -greedy)
 $t \leftarrow 0$
 repeat
 Take action A_t and observe R_{t+1}, S_{t+1}
 Choose action A_{t+1} using policy derived from Q (e.g., ϵ -greedy)
 $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t))$
 $t \leftarrow t + 1$
 until S_t is terminal;
end
return Q

Algorithm 14: Sarsamax (Q-Learning)

Input: policy π , positive integer $num_episodes$, small positive fraction α , GLIE $\{\epsilon_i\}$
Output: value function Q ($\approx q_\pi$ if $num_episodes$ is large enough)
Initialize Q arbitrarily (e.g., $Q(s, a) = 0$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$, and $Q(terminal_state, \cdot) = 0$)
for $i \leftarrow 1$ **to** $num_episodes$ **do**
 $\epsilon \leftarrow \epsilon_i$
 Observe S_0
 $t \leftarrow 0$
 repeat
 Choose action A_t using policy derived from Q (e.g., ϵ -greedy)
 Take action A_t and observe R_{t+1}, S_{t+1}
 $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t))$
 $t \leftarrow t + 1$
 until S_t is terminal;
end
return Q

Algorithm 15: Expected Sarsa

Input: policy π , positive integer $num_episodes$, small positive fraction α , GLIE $\{\epsilon_i\}$
Output: value function Q ($\approx q_\pi$ if $num_episodes$ is large enough)
Initialize Q arbitrarily (e.g., $Q(s, a) = 0$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$, and $Q(terminal_state, \cdot) = 0$)
for $i \leftarrow 1$ **to** $num_episodes$ **do**
 $\epsilon \leftarrow \epsilon_i$
 Observe S_0
 $t \leftarrow 0$
 repeat
 Choose action A_t using policy derived from Q (e.g., ϵ -greedy)
 Take action A_t and observe R_{t+1}, S_{t+1}
 $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma \sum_a \pi(a|S_{t+1})Q(S_{t+1}, a) - Q(S_t, A_t))$
 $t \leftarrow t + 1$
 until S_t is terminal;
end
return Q
