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GAMES WITH IMPERFECTLY OBSERVABLE ACTIONS IN CONTINUOUS TIME

BY YULIY SANNIKOV[1]

This paper investigates a new class of two-player games in continuous time, in which the players' observations of each other's actions are distorted by Brownian motions. These games are analogous to repeated games with imperfect monitoring in which the players take actions frequently. Using a differential equation, we find the set $\mathcal{E}(r)$ of payoff pairs achievable by all public perfect equilibria of the continuous-time game, where r is the discount rate. The same differential equation allows us to find public perfect equilibria that achieve any value pair on the boundary of the set $\mathcal{E}(r)$. These public perfect equilibria are based on a pair of continuation values as a state variable, which moves along the boundary of $\mathcal{E}(r)$ during the course of the game. In order to give players incentives to take actions that are not static best responses, the pair of continuation values is stochastically driven by the players' observations of each other's actions along the boundary of the set $\mathcal{E}(r)$.

KEYWORDS: Repeated games, continuous time, incentives, optimal equilibria, Brownian motion, collusion, computation.

1. INTRODUCTION

THIS PAPER ANALYZES a new class of two-player games in continuous time that are analogous to repeated games with imperfect monitoring. In these games, players do not see each other's actions directly; they only see signals that are distorted by Brownian motions. Continuous time leads to a clear characterization of the set of equilibrium payoffs and strategies that achieve the extreme points of that set.

We study *public perfect equilibria*. A public perfect equilibrium (PPE) is a pair of strategies that depend only on the commonly observable public outcomes such that each player's strategy is a best response after all public histories. We denote the set of payoff pairs that the players can achieve in a PPE of a continuous-time game by $\mathcal{E}(r)$, where r is the discount rate. The purpose of this paper is *not* to prove a Folk theorem, but to precisely characterize the set $\mathcal{E}(r)$ for all r.

We describe the boundary of the set $\mathcal{E}(r)$ using an ordinary differential equation, which we call the *optimality equation*. The optimality equation also allows

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us to construct equilibria that achieve any extreme payoff pair of the set $\mathcal{E}(r)$. The dynamics of such equilibria are based on a pair of continuation values as a state variable, which moves along the boundary of the set $\mathcal{E}(r)$ during the course of the game. At any moment of time, a player's continuation value is his future expected payoff in the remaining game. The current continuation values determine the players' actions and the impact of observed signals on the motion of continuation payoffs.

The optimality equation relates incentives, the equilibrium motion of continuation values, and the geometry of the set $\mathcal{E}(r)$. In equilibrium, a player's incentives stem from the influence of the public signals on the player's future continuation values. An action of a player is optimal when it maximizes his instantaneous payoff flow plus the expected rate of change of his continuation value. Because signals are stochastic, so is the motion of continuation values. The optimality equation, shown informally in the third panel of Figure 1, ties together four measures:

(i) Inefficiency: How much continuation values v fall behind the current flow of payoffs $g(a)$.
(ii) Incentives: The sensitivity of continuation values to public signals.
(iii) The amount of noise in signals.
(iv) The curvature of the set $\mathcal{E}(r)$.

We see that noise, curvature, and the necessity to provide incentives contribute positively to inefficiency. In equilibrium, as continuation values move on the boundary of the set $\mathcal{E}(r)$, the tangent line gives the ratio at which players can instantaneously transfer future equilibrium payoffs in order to create incentives. Because of the curvature of the set $\mathcal{E}(r)$, players cannot transfer utility between each other indefinitely at the same constant rate. Curvature, together with the magnitude of noise in the public signal, quantifies the inefficiency.[2] The greater the curvature, the more costly it is to provide incentives and the greater should be the difference between the continuation values and the flow of payoffs.

The optimality equation also assigns an equilibrium action pair a to each point v on the boundary of the set $\mathcal{E}(r)$. That action pair optimally resolves the trade-off between inefficiency and incentives to stretch the boundaries of the set $\mathcal{E}(r)$ as far out as possible.

This paper contributes to the theory of repeated games with imperfect monitoring, which has been developed by Abreu, Pearce, and Stacchetti (1990), hereafter APS, and Fudenberg, Levine, and Maskin (1994), hereafter FLM.[3]

[2]As noise increases, the variance of continuation values necessary to provide incentives increases. Because of the curvature of $\mathcal{E}(r)$, that increases inefficiency.

[3]Also, see Fudenberg and Levine (1994) for complementary results to FLM when the Folk theorem fails, and the book by Mailath and Samuelson (2006) for an excellent general exposition of the current theory of discrete-time repeated games.

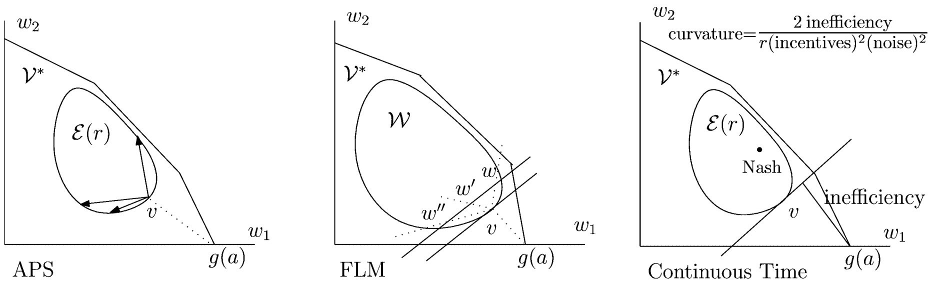

FIGURE 1.—Illustration of the methods of APS, FLM, and this paper.

Specifically, continuous-time games illustrate the pattern of equilibrium dynamics in such games and clearly outline the trade-offs involved in the choice of equilibrium actions. Figure 1, in which the horizontal and vertical axes represent the players' payoffs, illustrates the contributions of APS and FLM, and how continuous-time games improve our understanding of repeated games.

APS investigated pure strategy sequential equilibria (pure strategy PPEs) of repeated games with imperfect monitoring. These games have a great multiplicity of equilibria. APS made the problem of finding equilibrium payoffs much more manageable. They showed that any public equilibrium payoff vector can be achieved by a recursive equilibrium, in which the players' continuation values are state variables. In equilibrium, continuation values change location after every observation of the public signal. The arrows in the left panel of Figure 1 illustrate the potential jumps of continuation values after different signals. The challenge behind our understanding of discrete-time games is that it is difficult to see a pattern behind these jumps and the connection between the equilibrium dynamics and the shape of the set of equilibrium payoffs. Continuous-time games illuminate the connection between the equilibrium motion of continuation values, incentives, and the shape of the set $\mathcal{E}(r)$. In particular, the optimality equation leads naturally to a simpler computational procedure in a continuous-time setting.

FLM showed that under appropriate conditions the Folk theorem holds for repeated games with imperfect information: any smooth convex payoff set $\mathcal{W}$ inside the set $\mathcal{V}^*$ of all feasible and individually rational payoffs can be achieved in equilibrium as long as the players are sufficiently patient. The key insight behind FLM's proof of the Folk theorem is to consider a specific pattern of the motion of continuation values. Specifically, to achieve a payoff pair v on the boundary of $\mathcal{W}$ in equilibrium, they chose the future continuation values (denoted by w, w', and w'' in the middle panel of Figure 1) on a translation of the tangent line. The continuous-time setting allows us to do more: for *any* discount rate r, we can characterize the *optimal* equilibrium motion of continuation values. It turns out that this motion stays on the boundary of the set of equilibrium payoffs $\mathcal{E}(r)$, that is, it is locally tangential.

One may be surprised that the informational problem persists in our continuous-time setting. After all, if players can change their actions fast, why can they not instantaneously punish all deviations? A critical feature of our model is that while the players can adjust their actions as quickly as they want, the faster they react, the less information they observe. This feature is in sharp contrast to the model adopted in FLM, where as the duration of a period is shrunken to 0, the amount of information that the players learn per period nevertheless remains the same. This issue was addressed by Abreu, Milgrom, and Pearce (1991) in a model with Poisson arrival of signals.[4]

Holmstrom and Milgrom (1987) is the first paper to apply Brownian motion to study dynamic incentive provision. Their paper is a good example that in some situations a continuous-time formulation allows us to better recognize patterns and prove clean results. The information flow in our paper is similar to Holmstrom and Milgrom (1987) in the sense that players learn about each other's actions from a continuous process with independent and identically distributed increments.[5]

Simon and Stinchcombe (1989) illustrated many difficulties associated with the modeling of games in continuous time. For example, a simple description of a strategy in discrete time often has no equivalent in continuous time. These difficulties arise when the actions of one player instantaneously create information available to his opponent. This issue is not a problem in our framework. In our continuous-time games, information is defined exogenously in terms of all possible signals, and a strategy of a player simply defines a probability measure over all possible signals.

Recently, a number of authors have derived new insights about optimal incentive provision in a dynamic setting using the mathematical tools of optimal control of diffusion processes. Sannikov (2004) and Williams (2004) both introduced a new method of analyzing the informational problem in a dynamic principal–agent relationship. In both models, the agent drives a stochastic state X with his choice of controls, but the agent's choice is not directly observable.[6] Both papers analyze models with one-sided imperfect information, where only the agent takes hidden actions. This paper extends the continuous-time method to a two-sided setting, where both players take hidden actions.

This paper is organized as follows. Section 2 provides an example of a prisoners' dilemma in continuous time. Section 3 formally describes the class of

[4]Also, Kandori (1992) shows that the set of payoffs achieved in PPE increases in the accuracy of monitoring.

[5]We do not allow statistically meaningful jumps in the players' observations, as in the Poisson model of Abreu, Milgrom, and Pearce (1991). As a result, the noise has the form of a Brownian motion.

[6]Williams (2004) characterized the optimal contract with a partial differential equation based on the following state variables: time, state X, the agent's value, and possibly other variables in an enriched formulation. In Sannikov (2004), the optimal contract can be derived using an ordinary differential equation based on a single state variable, the agent's continuation value.

continuous-time games that we analyze. Section 4 describes several standard game-theoretic concepts in our setting: the stage game, the minmax payoff, and the sets of unconstrained payoffs and Nash equilibrium payoffs. Section 5 identifies incentive compatibility conditions, discusses the concept of a continuation value, and describes PPE in terms of the stochastic motion of continuation values. Section 6 interprets this description of PPE as a stochastic control problem and characterizes the set $\mathcal{E}(r)$ as well as the PPEs that achieve its extreme points. Section 7 summarizes the main results and provides an intuitive discussion of public perfect equilibria. Section 8 presents computational techniques and an additional example of duopoly with differentiated products. Section 9 concludes the paper.

2. AN EXAMPLE

This section presents an example: a dynamic partnership game in continuous time. Two players participate in a joint venture. They continuously take actions from the sets $\mathcal{A}_1 = \mathcal{A}_2 = \{0, 1\}$ at each moment of time $t \in [0, \infty)$, where action 0 means "no effort" and 1 means "effort." Players do not directly observe each other's past actions. Instead, they get imperfect information about each others' actions through publicly observable random processes

$$X_t^1 = \int_0^t A_s^1\,ds + Z_t^1, \quad X_t^2 = \int_0^t A_s^2\,ds + Z_t^2,$$

where Z^1 and Z^2 are independent standard Brownian motions and A_t^i is the action of player i at time t. A public strategy for player i is a stochastic process $\{A_t^i\}_{t\geq 0}$ that is progressively measurable with respect to the history of public information that contains X^1 and X^2 (to be defined later).

The increments of the process X^i reflect how much the actions of player i contribute to the success of the joint venture. Players enjoy their joint success, but dislike effort. The actual payoffs of players 1 and 2 are given by

$$r\int_0^\infty e^{-rt}(2\,dX_t^1 + 2\,dX_t^2 - 3A_t^1\,dt) \quad \text{and}$$
$$r\int_0^\infty e^{-rt}(2\,dX_t^1 + 2\,dX_t^2 - 3A_t^2\,dt).$$

Note that the instantaneous payoff of player i depends on A_t^i, dX_t^1, and dX_t^2. The expected payoffs can be written as

$$E\left[r\int_0^\infty e^{-rt}g_1(A_t^1, A_t^2)\,dt\right] \quad \text{and} \quad E\left[r\int_0^\infty e^{-rt}g_2(A_t^1, A_t^2)\,dt\right],$$

where

$$g_1(a_1, a_2) = 2a_2 - a_1 \quad \text{and} \quad g_2(a_1, a_2) = 2a_1 - a_2.$$

	0	1
0	0, 0	2, -1
1	-1, 2	1, 1

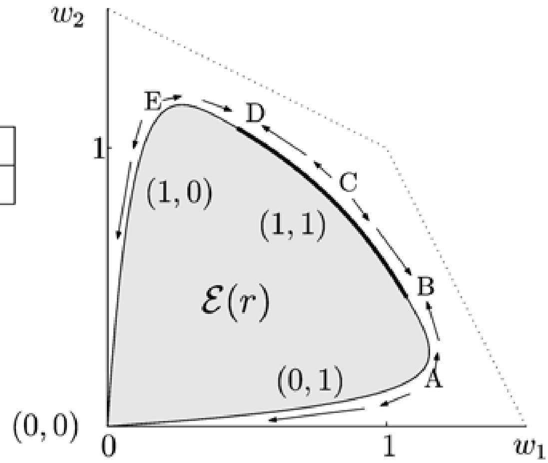

FIGURE 2.—Matrix of static payoffs and set $\mathcal{E}(r)$ in partnership.

Static payoff functions g_1 and g_2 give the expectation of the rates at which the players receive their payoffs for any pair of actions. The matrix of expected payoffs of the stage game, shown in Figure 2, is that of a prisoners' dilemma.

To give a taste of our results, Figure 2 also shows a computed set $\mathcal{E}(r)$ for $r = 0.2$. In the figure, players can achieve payoffs much better than those of a static Nash equilibrium, but cannot achieve full efficiency due to noise.

Let us describe the equilibrium that achieves the largest sum of payoffs, corresponding to point C on the boundary of the set $\mathcal{E}(r)$. During the equilibrium play, the pair of continuation values follows a diffusion process on the boundary of $\mathcal{E}(r)$, driven by the realizations of X. The pair of continuation values has a drift and a volatility. The tangential component of the drift is shown in Figure 2: it is directed away from points A, C, and E, toward points B, D, and the origin. Players choose their effort levels depending on the current pair of continuation values as shown in Figure 2. Both players put forth effort on the thick portion of the boundary of $\mathcal{E}(r)$.

Figure 3 gives three sample paths of the players' continuation values in the equilibrium that achieves payoff pair C. The vertical axis represents the boundary of $\mathcal{E}(r)$, denoted by $\partial\mathcal{E}(r)$, with points A, B, C, D, and E clearly marked.

In Figure 3, the drift of continuation values is directed away from the dashed horizontal lines, toward the solid lines. The solid lines represent the boundaries, where one of the players switches from effort to no effort. Because of the drift pattern, players typically spend considerable amounts of time in "unequal" regimes between the dashed lines, where one player puts forth effort and the other alternates between effort and no effort.

The realizations of X cause players to switch from one unequal regime to another until they become absorbed in the static Nash equilibrium, in which players stop putting forth effort. We see from Figure 3 that the collapse into

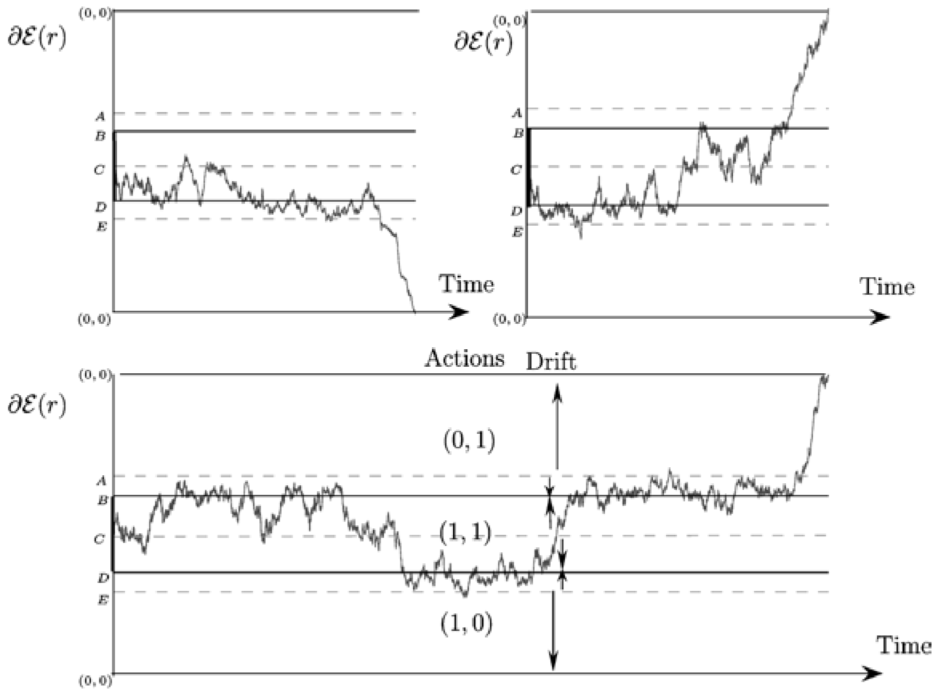

FIGURE 3.—Sample paths of continuation values.

Nash equilibrium is fast because the drift toward the Nash equilibrium point becomes stronger as the continuation values approach that point.

The partnership game belongs to a simple subclass of repeated games with two separate one-dimensional signals, whose drifts equal the actions of the two players. These games have a product structure (see footnote 11) and are a useful starting point to understand continuous-time games in general. The provision of incentives in these games is especially simple, as discussed in Section 7.1. In Section 8, which mostly discusses computation, we present another example of a continuous-time game from this class: a duopoly with differentiated products.

3. THE SETTING

Two players participate in a repeated game with imperfect monitoring in continuous time. At each moment of time $t \in [0, \infty)$, player i takes an action A^i_t from a finite set $\mathcal{A}^i$. Players do not see each others' actions $A_t = (A^1_t, A^2_t)$ directly, but only observe d-dimensional public signals

$$X_t = \int_0^t \mu(A_s)\, ds + Z_t, \tag{1}$$

where Z is a d-dimensional Brownian motion and $\mu : \mathcal{A}^1 \times \mathcal{A}^2 \to \mathbb{R}^d$ is a drift function.[7] The arrival of public information is captured by the filtration $\{\mathcal{F}_t\}$. It can be strictly bigger than the filtration generated by X to allow for public randomization (by both continuous and discontinuous processes). A pure public strategy of player i is a stochastic process $A^i = \{A^i_t\}$ with values in $\mathcal{A}^i$, which is progressively measurable with respect to $\{\mathcal{F}_t\}$.

Formally, the game takes place on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}$. The state space Ω of all possible paths of X and outcomes of public randomization, as well as the filtration $\{\mathcal{F}_t\}$ with $\mathcal{F}_\infty = \mathcal{F}$, are fixed for the game. However, the probability measure P is determined by the players' actions in such a way that (1) holds.[8]

Player i's random total discounted payoff for a profile of public strategies A is[9]

$$\begin{aligned} &r\int_0^\infty e^{-rt}(c_i(A^i_t)\,dt + b_i(A^i_t)\,dX_t) \\ &\quad = r\int_0^\infty e^{-rt}(c_i(A^i_t) + b_i(A^i_t)\mu(A_t))\,dt + r\int_0^\infty e^{-rt} b_i(A^i_t)\,dZ_t \end{aligned}$$

for some functions $c_i : \mathcal{A}^i \to \mathbb{R}$ and $b_i : \mathcal{A}^i \to \mathbb{R}^d$, where $r > 0$ denotes the common discount rate of the two players. Let

$$g_i(A_t) = c_i(A^i_t) + b_i(A^i_t)\mu(A_t)$$

be player i's expected payoff flow at time t.

DEFINITION: A profile of public strategies $A = (A^1, A^2)$ is a *perfect public equilibrium* (PPE) if, for $i = 1, 2$, A^i maximizes the expected discounted payoff of player i given the strategy A^j of his opponent after all public histories.

[7]Throughout the paper, d-dimensional vectors X, Z, and μ are column vectors, b, β, ϕ, ψ, and χ are row vectors, two-dimensional vectors g, v, w, and W are column vectors, and $\mathbf{T}$ and $\mathbf{N}$ are unit row vectors.

[8] If players deviate from a pair of strategies $A = (A^1, A^2)$ to a pair of strategies $\hat{A}$, then the probability measure over signals becomes altered by the relative density process ξ defined by $\xi_0 = 1$ and $d\xi_t = \xi_t(\mu(\hat{A}_t) - \mu(A_t))^\top\,dX_t$. The value of ξ_t captures the relative likelihood that a path $\{X_s, s \in [0, t]\}$ of the public signals becomes realized from strategies A' in comparison with A. Note that the players' actions affect only signals X and nothing else, as the relative density process depends only on the players' actions and the realizations of X.

[9]The proper way to writing the payoff of player i is

$$r\int_0^\infty e^{-rt} c_i(A^i_t)\,dt + r\int_0^\infty e^{-rt} b_i(A^i_t)\,dX_t,$$

where the Lebesgue integral and the stochastic integral are separated. Throughout the paper, we put them under one integral sign to shorten notation, recognizing that we are mixing two types of integrals, but believing that this cannot cause confusion.

Formally, the expected discounted payoff (a.k.a. continuation value) of player i after a public history at time t is

$$W_t^i(A) = E_t\left[r\int_t^\infty e^{-r(s-t)}(c_i(A_s^i)\,ds + b_i(A_s^i)\,dX_s)\middle| A_s, s\in[t,\infty)\right] \tag{2}$$
$$= E_t\left[r\int_t^\infty e^{-r(s-t)} g_i(A_s)\,ds\middle| A_s, s\in[t,\infty)\right].$$

This expectation, conditioned on the public information at time t, makes explicit the fact that actions affect payoffs directly through $g_i(A_s)$ and indirectly because $\{A_s; s\in[t,\infty)\}$ determines the probability distribution over the future paths of X. Denote $W_t(A) = (W_t^1(A), W_t^2(A))^\top$ and $g(a) = (g_1(a), g_2(a))^\top$.

Starting with Section 6, we make two assumptions about payoffs and signals:[10]

ASSUMPTION 1: *All action profiles $(a_1, a_2)\in\mathcal{A}_1\times\mathcal{A}_2$ of the stage game are* pairwise identifiable, *that is, the spans of the $d\times(|\mathcal{A}^1|-1)$ matrix $M_1(a)$ with columns $\mu(a_1', a_2)-\mu(a)$, $a_1'\neq a_1$, and the $d\times(|\mathcal{A}^2|-1)$ matrix $M_2(a)$ with columns $\mu(a_1, a_2')-\mu(a)$, $a_2'\neq a_2$, intersect only at the origin.*

Pairwise identifiability, adapted to our continuous-time setting from FLM, implies that deviations of different players can be statistically distinguished. Note that we do not require *individual full rank*, that is, independence of the columns of $M_i(a)$.

ASSUMPTION 2: *Either of the following statements holds*:
(i) *For all $i=1,2$ and $a_i\in\mathcal{A}^i$, the static best response to a_i is unique.*
(ii) *For all $a\in\mathcal{A}$, the spans of $M_1(a)$ and $M_2(a)$ are orthogonal.*[11]

In the next sections we characterize the set $\mathcal{E}(r)$ of payoff pairs achievable by all PPEs and pay special attention to the PPEs that achieve extreme value pairs of the set $\mathcal{E}(r)$.[12] We find that the equilibrium play in those PPEs is de-

[10]These assumptions ensure that the *optimality equation* that characterizes the set $\mathcal{E}(r)$ is Lipschitz-continuous. Without these assumptions, the optimality equation may not have a solution in the classical sense, and the set $\mathcal{E}(r)$ is characterized using viscosity solutions (as in Faingold and Sannikov (2007)). In that case, it is no longer true that continuation values must stay on the boundary of $\mathcal{E}(r)$ in a PPE that achieves an extreme point of $\mathcal{E}(r)$. Lemma 1 in Appendix A uses Assumption 1, and Lemma 3 uses Assumption 2.

[11] For example, games with product structure in which two separate independent signals reflect the actions of players 1 and 2 satisfy Assumption 2(ii).

[12]In a game with public monitoring, any pure strategy is realization equivalent to a pure public strategy. Therefore, under pure strategies, the set of PPE payoffs coincides with the set of payoffs achievable by all sequential equilibria. This is not always the case for mixed strategies: as shown in Kandori and Obara (2006), players can sometimes get higher payoffs than those achievable in any mixed-strategy PPE by the use of private strategies, in which a player's current action can depend not only on the public history of signals, but also the private history of his own past actions.

termined essentially uniquely and it does not use public randomization.[13] The equilibrium dynamics are described in terms of the stochastic motion of continuation values on the boundary of $\mathcal{E}(r)$ driven by the public signals.

4. IMPORTANT SETS

Let us review several concepts that are familiar from the theory of repeated games. A stage game G has the set of players $N = \{1, 2\}$, an action set of each player $\mathcal{A}_i$, and payoff functions g_i:

$$G = \{N, (\mathcal{A}_i)_{i\in N}, (g_i)_{i\in N}\}.$$

Denote the set of all action profiles of the stage game G by $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and denote the set of pure strategy Nash equilibria by $\mathcal{A}^N \subseteq \mathcal{A}$. Let $\mathcal{N}$ be the convex hull of all pure strategy Nash equilibrium payoff pairs of game G and let $\mathcal{V}$ be the convex hull of all feasible payoff pairs:

$$\mathcal{N} \equiv \mathrm{co}\{g(a)|a \in \mathcal{A}^N\}, \quad \mathcal{V} \equiv \mathrm{co}\{g(a)|a \in \mathcal{A}\}.$$

The pure strategy minmax payoff of player i is

$$\underline{v}_i \equiv \min_{a_j} \max_{a_i} g_i(a_i, a_j). \tag{3}$$

Player i can guarantee himself pure strategy minmax payoff for any pure strategy of his opponent. Define by

$$\mathcal{V}^* \equiv \{v \in \mathcal{V}| v_i \geq \underline{v}_i \text{ for } i = 1, 2\}$$

the subset of $\mathcal{V}$ on which each player receives at least his minmax payoff.

Due to the possibility of public randomization, the set $\mathcal{E}(r)$ of payoff pairs achievable by all PPEs is convex. Indeed, if A and $\hat{A}$ are two PPEs with expected values $W_0(A)$ and $W_0(\hat{A})$, a PPE with value $\lambda W_0(A) + (1-\lambda)W_0(\hat{A})$ for some $\lambda \in (0, 1)$ can be achieved by selecting equilibrium A or $\hat{A}$ according to the realization of a discrete random variable at time 0.

As in repeated games in discrete time, we have

$$\mathcal{N} \subseteq \mathcal{E}(r) \subseteq \mathcal{V}^* \subseteq \mathcal{V}$$

as illustrated in Figure 4.

[13]By allowing public randomization, we can conclude early in Section 4 that the set $\mathcal{E}(r)$ is convex. This conclusion greatly simplifies the derivation of other properties of $\mathcal{E}(r)$ on the way to our main result. In the end, Theorem 2 shows that public randomization is unnecessary to generate any extreme point of $\mathcal{E}(r)$. Because time is continuous, it follows that we can generate any point of $\mathcal{E}(r)$ without public randomization.

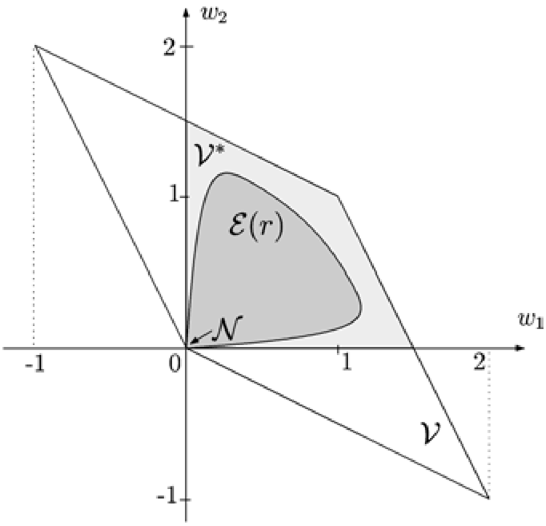

FIGURE 4.—Sets $\mathcal{N}$, $\mathcal{E}(r)$, $\mathcal{V}^*$, and $\mathcal{V}$ for the noisy partnership with $r = 0.2$.

5. CONTINUATION VALUES, INCENTIVES, AND PPES

In this section, we characterize public perfect equilibria in terms of the stochastic properties of the continuation values $W_t^1(A)$ and $W_t^2(A)$. Our analysis proceeds as follows. We start with the definition of a continuation value: it is the future expected payoff of a player from a given pair of strategies after a given public history. As time passes and the history unfolds, the continuation values change. Their motion is determined by the public information: the signals X_t and public randomization. Proposition 1 represents the relationship between public information and the motion of continuation values formally, and shows that this motion must satisfy a *promise keeping* condition. This condition relates a player's current continuation value, his current payoff flow, and the expected change of his continuation value under an arbitrary pair of strategies A.

Next, we explore conditions under which a pair of strategies A is a PPE. We find that the players' incentives are connected with the relationship between the public signals and the motion of continuation values. A player may have incentives to take an action different from a static best response because (a) actions affect public signals and (b) public signals affect future continuation values. Proposition 2 proves that a strategy of player i is optimal in response to the strategy of his opponent at all times if and only if the instantaneous *incentive compatibility* condition always holds. The analogue of Proposition 2 in discrete time is the *one-shot deviation principle.* The results of Propositions 1 and 2 are summarized in Theorem 1, which characterizes $\mathcal{E}(r)$ as the largest bounded *self-generating* set.

Recall that the continuation value of player i, the expected future payoff from a strategy profile A after a given public history, is

$$(4)\qquad W_t^i(A) = E_t\left[r\int_t^{\infty} e^{-r(s-t)} g_i(A_s)\,ds \,\middle|\, A_s, s\in[t,\infty)\right].$$

$W_t^i(A)$ has the following representation in terms of the public information in $\{\mathcal{F}_t\}$.

PROPOSITION 1—Representation and Promise Keeping: *A bounded stochastic process W_t^i is the continuation value $W_t^i(A)$ of player i under a strategy profile A if and only if there exist processes $\beta^i = (\beta^{i1}\ \cdots\ \beta^{id})$ in $\mathcal{L}^*$ and a martingale $\breve{\varepsilon}^i$ orthogonal to X with $\breve{\varepsilon}_0^i = 0$, such that for all $t \geq 0$, W_t^i satisfies*[14]

$$(5)\qquad W_t^i = W_0^i + r\int_0^t (W_s^i - g_i(A_s))\,ds + r\int_0^t \beta_s^i(dX_s - \mu(A_s)\,ds) + \breve{\varepsilon}_t^i.$$

When the filtration $\{\mathcal{F}_t\}$ is generated by X, so that there is no public randomization, then a bounded process W_t^i is the continuation value of player i if and only if (5) holds with $\breve{\varepsilon}^i = 0$.

The representation in Proposition 1 formalizes how the process $W_t^i(A)$ is determined by public information: signals X and orthogonal information $\breve{\varepsilon}_t^i$, with $\breve{\varepsilon}_t^i = 0$ when there is no public randomization. There is a simple logic behind the representation (5), whose shorthand form

$$(6)\qquad dW_t^i = r(W_t^i - g_i(A_t))\,dt + r\beta_t^i(dX_t - \mu(A_t)\,dt) + d\breve{\varepsilon}_t^i$$

can be interpreted as an instantaneous projection of dW_t^i onto a "constant" dt and $dX_t - \mu(A_t)\,dt$. Clearly, the "residual" $d\breve{\varepsilon}_t^i$ has to be orthogonal to the constant, and thus is a martingale. The residual is 0 when $\{\mathcal{F}_t\}$ is generated by X, as then 0 is the only martingale orthogonal to X. Coefficient β_t^i denotes the exposure of player i's continuation value to $dX_t - \mu(A_t)\,dt$.

The term $r(W_t^i - g_i(A_t))$ is the drift of continuation values $W_t^i(A)$ when the players are actually following strategies A so that $dX_t = \mu(A_t)\,dt + dZ_t$. This drift condition under the pair of strategies A, which we can call the *promise keeping* condition, is simply a consequence of bookkeeping. It has an analogue in discrete time: If a payoff vector W_t is decomposable by an action vector a and continuation promises $W_{t+1}(y_t)$ for different public signals y_t in period t, then

$$W_t = (1-\delta)g(A_t) + \delta E[W_{t+1}(y_t)|A_t]$$

[14] $\mathcal{L}^*$ is the space of all progressively measurable processes δ such that $E[\int_0^T \delta_t^2\,dt] < \infty$ for all $T < \infty$.

$$\Rightarrow \quad E[W_{t+1}(y_t)|A_t] - W_t = \frac{1-\delta}{\delta}(W_t - g(A_t)),$$

where δ is the discount factor. Thus, the expected movement of continuation values is proportional to $W_t - g(A_t)$.

PROOF OF PROPOSITION 1: First, let us prove that $W_t^i(A)$ has a representation (5). In order to identify how $W_t^i(A)$ depends on the public information $\{\mathcal{F}_t\}$, suppose that the players are actually following strategies A. Then $X_t - \int_0^t \mu(A_s)\,ds$ is a Brownian motion and the process $\{V_t^i(A)\}$ defined by

$$V_t^i(A) = r\int_0^t e^{-rs} g_i(A_s)\,ds + e^{-rt} W_t^i(A) = E_t\left[r\int_0^\infty e^{-rs} g_i(A_s)\,ds \middle| A\right] \tag{7}$$

is a martingale. By Karatzas and Shreve (1991, Proposition 3.4.14), we can choose processes $\beta^i = (\beta^{i1} \ \cdots \ \beta^{id})$ in $\mathcal{L}^*$ such that

$$V_t^i(A) = r\int_0^t e^{-rs}\beta_s^i (dX_s - \mu(A_s)\,ds) + V_t^X, \tag{8}$$

where V_t^X is a martingale orthogonal to X.[15] If the filtration $\{\mathcal{F}_t\}$ is generated by X, then by the martingale representation theorem (whose proof in Karatzas and Shreve (1991), incidentally, relies on Proposition 3.4.14), we get the representation (8) with $V_t^X = V_0^i(A)$ for all $t \geq 0$.

Putting together (7) and (8), and differentiating with respect to t, we get

$$\begin{aligned} &re^{-rt} g_i(A_t)\,dt - re^{-rt} W_t^i(A)\,dt + e^{-rt}\,dW_t^i(A) \\ &\quad = re^{-rt}\beta_t^i(dX_t - \mu(A_t)\,dt) + dV_t^X, \end{aligned}$$

which implies (5) with $\breve{\varepsilon}_t^i = \int_0^t e^{rs}\,dV_t^X$. If the filtration $\{\mathcal{F}_t\}$ is generated by X, then $\breve{\varepsilon}_t^i = 0$ for all $t \geq 0$. Note that $W_t^i(A)$ is a bounded process because g_i is a bounded function.

Now, let us prove the converse: if a bounded process W_t^i satisfies (5) (with $\breve{\varepsilon}^i = 0$ or not), then it must be $W_t^i(A)$. The process

$$V_t^i = r\int_0^t e^{-rs} g_i(A_s)\,ds + e^{-rt} W_t^i$$

is a martingale under the strategies A because $dV_t^i = e^{-rt}(r\beta_t^i(dX_t - \mu(A_t)\,dt) + d\breve{\varepsilon}_t^i)$ from (5). Moreover, martingales V_t^i and $V_t^i(A)$ converge

[15] Formally, Karatzas and Shreve (1991, Proposition 3.4.14) only said that the processes $e^{-rt}\beta_t^i$ are in $\mathcal{L}^*$ and that V_t^X is orthogonal to the underlying Brownian motion. However, trivially, $E[\int_0^T (\beta_t^i)^2\,dt] \leq e^{2rT} E[\int_0^T e^{-2rt}(\beta_t^i)^2\,dt] < \infty$, so β_t^i are also in $\mathcal{L}^*$. Also, since V_t^X is a martingale, it must be orthogonal to X also.

because both $e^{-rt}W_t^i$ and $e^{-rt}W_t^i(A)$ converge to 0. We conclude that for all $t \geq 0$,

$$V_t^i = E_t[V_\infty^i] = E_t[V_\infty^i(A)] = V_t^i(A) \quad \Rightarrow \quad W_t^i = W_t^i(A). \qquad Q.E.D.$$

It is important that even if players are following an alternative strategy profile $\hat{A}$, the process $W_t^i(A)$ is still well defined. As emphasized by the conditioning on $\{A_s, s \geq t\}$ in the definition (4), $W_t^i(A)$ can be interpreted as the value that player i would get if the play proceeds according to the strategy profile A after time t. Because A uniquely determines the players' actions after all public histories, the value of $W_t^i(A)$ is completely determined by the public history at time t independently of the actual strategy profile $\hat{A}$ that caused this history to be realized. We conclude that the representation (5) for $W_t^i(A)$ is valid no matter which strategy profile $\hat{A}$ is being played, even though we derived (5) by assuming that A is the actual profile being followed. Moreover, from our assumptions $\breve{\varepsilon}^i$ is still a martingale under an alternative strategy profile $\hat{A}$.[16] These facts are important for Proposition 2 that deals with incentives.[17]

The process β^i represents the extent to which player i's value $W_t^i(A)$ is driven by the public signal X. Therefore, β^i is responsible for player i's incentives, as shown below:

PROPOSITION 2—Incentive Compatibility: *Strategy A^i of player i is optimal in response to strategy A^j at all times if and only if the incentive compatibility condition*

$$\text{(9)} \qquad \forall a_i' \in \mathcal{A}^i, \quad g_i(A_t) + \beta_t^i \mu(A_t) \geq g_i(a_i', A_t^j) + \beta_t^i \mu(a_i', A_t^j)$$

holds for all t. Therefore, a pair of strategies A is a PPE if and only if (9) *holds for both players.*

Let us interpret the incentive compatibility condition. Suppose that player i is contemplating a deviation to an alternative action a_i' at time t. This will change his expected instantaneous payoff flow by $g_i(a_i', A_t^j) - g_i(A_t)$. At the same time, this deviation changes the drift of X_t by $\mu(a_i', A_t^j) - \mu(A_t)$. Since β_t^i

[16]Because $\breve{\varepsilon}_t^i$ is orthogonal to X, it is also orthogonal to the relative density process $\xi_t = 1 + \int_0^t \xi_s(\mu(\hat{A}_s) - \mu(A_s))^\top (dX_s - \mu(A_s)\,ds)$ of the probability measures under $\hat{A}$ and A (see footnote 8).

[17]Note that $W_t^i(\hat{A})$, player i's continuation value when a strategy profile $\hat{A}$ is followed after time t, is different from $W_t^i(A)$. The representation of $W_t^i(\hat{A})$ involves different processes $\beta^i(\hat{A})$ and $\breve{\varepsilon}^i(\hat{A})$. To avoid confusion, in the entire paper β^i and $\breve{\varepsilon}^i$ always denote the processes that represents specifically $W_t^i(A)$, and not $W_t^i(\hat{A})$, even when we discuss a deviation to an alternative strategy in Proposition 2.

is the sensitivity of player i's continuation value to dX_t, this will change player i's continuation value at rate $\beta^i_t(\mu(a'_i, A^j_t) - \mu(A_t))$. If the incentive compatibility condition holds, such an instantaneous deviation will affect player i's expected payoff by

$$g_i(a'_i, A^j_t) - g_i(A_t) + \beta^i_t(\mu(a'_i, A^j_t) - \mu(A_t)) \leq 0. \tag{10}$$

Therefore, condition (9) states that an instantaneous deviation is not profitable. Proposition 2 is analogous to the one-shot deviation principle in discrete time.

In the proof of Proposition 2 we show that instantaneous incentive compatibility implies full incentive compatibility. Instantaneous losses from deviations integrate to a loss globally.

PROOF OF PROPOSITION 2: Since $e^{-rt}W^i_t(A)$ converges to 0 as $t \to \infty$, it follows that

$$W^i_0(A) + \int_0^\infty d(e^{-rt}W^i_t(A)) = 0,$$

where, by (6), $d(e^{-rt}W^i_t(A))$ is expressed as a function of public information X and $\breve{\varepsilon}^i$ as

$$d(e^{-rt}W^i_t(A)) = re^{-rt}\big(-g_i(A_t)\,dt + \beta^i_t(dX_t - \mu(A_t)\,dt) + d\breve{\varepsilon}^i_t\big).$$

The expected payoff to player i from deviating to a strategy $\hat{A}^i$ in response to A^j can be expressed as

$$\begin{aligned}
&W^i_0(\hat{A}^i, A^j)\\
&\quad = E\left[r\int_0^\infty g_i(\hat{A}^i_t, A^j_t)\,dt\,\middle|\,\hat{A}^i, A^j\right]\\
&\quad = E\Bigg[W^i_0(A) + \int_0^\infty \underbrace{re^{-rt}(-g_i(A_t)\,dt + \beta^i_t(dX_t - \mu(A_t)\,dt) + d\breve{\varepsilon}^i_t)}_{d(e^{-rt}W^i_t(A))}\\
&\qquad\qquad + r\int_0^\infty e^{-rt}g_i(\hat{A}^i_t, A^j_t)\,dt\,\Bigg|\,\hat{A}^i, A^j\Bigg]\\
&\quad = W^i_0(A) + E\Bigg[r\int_0^\infty e^{-rt}\big(g_i(\hat{A}^i_t, A^j_t) - g_i(A_t)\\
&\qquad\qquad\qquad + \beta^i_t(\mu(\hat{A}^i_t, A^j_t) - \mu(A_t))\big)\,dt\,\Bigg|\,\hat{A}^i, A^j\Bigg].
\end{aligned}$$

Throughout the derivation we condition on the players' strategies $(\hat{A}^i, A^j)$, which affect the probability measure over the paths of X. Under this measure, X_t has drift $\mu(\hat{A}^i_t, A^j_t)$ and $\breve{\varepsilon}^i_t$ is still a martingale.

If condition (9) holds for all t, then $W^i_0(\hat{A}^i, A^j) \leq W^i_0(A)$ and player i does not have a profitable deviation at time 0. By a similar argument, player i will not have a profitable deviation after any public history. Conversely, if (9) fails, choose a strategy $\hat{A}^i$ such that $\hat{A}^i_t$ maximizes $g_i(a_i, A^j_t) + \beta^i_t \mu(a_i, A^j_t)$ for all t. Then $W^i_0(\hat{A}^i, A^j) > W^i_0(A)$ and A^i is not an optimal response to the strategy A^j.[18] *Q.E.D.*

DEFINITION: A $2 \times d$ matrix

$$B = \begin{bmatrix} \beta^1 \\ \beta^2 \end{bmatrix} = \begin{bmatrix} \beta^{11} \cdots \beta^{1d} \\ \beta^{21} \cdots \beta^{2d} \end{bmatrix}$$

enforces action profile $a \in \mathcal{A}$ if for $i = 1, 2$,

$$\forall a'_i \in \mathcal{A}^i, \quad g_i(a) + \beta^i \mu(a) \geq g_i(a'_i, a_j) + \beta^i \mu(a'_i, a_j). \tag{11}$$

An action profile $a \in \mathcal{A}$ is *enforceable* if there exists some matrix B that enforces it.

Together Propositions 1 and 2 provide two properties that the motion of continuation values has to satisfy in any PPE: *promise keeping*, related to the drift of continuation values, and *incentive compatibility*, related to their volatility. Moreover, Propositions 1 and 2 also imply the converse: if the motion of a bounded stochastic process W associated with strategies A satisfies *promise keeping* and *incentive compatibility*, then $W = W(A)$ and A must be a PPE. We summarize this characterization of PPE in the following theorem:

THEOREM 1—Characterization of PPE: *In any PPE A, the pair of continuation values $W = W(A)$ is a process in $\mathcal{V}^*$ that satisfies*

$$W_t = W_0 + r\int_0^t (W_s - g(A_s))\,ds + r\int_0^t B_s \underbrace{(dX_s - \mu(A_s)\,ds)}_{dZ_s} + \breve{\varepsilon}_t, \tag{12}$$

where

(i) *B is $2 \times d$ matrix process in $\mathcal{L}^*$ such that B_t enforces A_t for all t and*
(ii) *$\breve{\varepsilon}$ is a two-dimensional martingale orthogonal to X with $\breve{\varepsilon}_0 = (0, 0)^\top$.*

[18]In this case the strategy $\hat{A}^i$ is better than A^i in response to A^j, but it does not have to be optimal. The reason is that $\hat{A}^i_t$ does not have to maximize $g_i(a_i, A^j_t) + \hat{\beta}^i_t \mu(a_i, A^j_t)$, where $\hat{\beta}^i_t$, the exposure of $W(\hat{A}^i, A^j)$ to X, does not have to equal β^i_t.

If $\{\mathcal{F}_t\}$ is generated by X, then $\breve{\varepsilon}_t = (0, 0)^\top$ for all $t \geq 0$.[19] *Conversely, if W is a bounded two-dimensional process that satisfies* (12) *for A, B and $\breve{\varepsilon}$ that satisfy properties* (i) *and* (ii), *then W is a pair of continuation values in public perfect equilibrium A.*

DEFINITION: A set $\mathcal{W} \subseteq \mathbb{R}^2$ is *self-generating* if and only if, for any point $W_0 \in \mathcal{W}$, there exists a process W that starts at W_0, stays in $\mathcal{W}$, and satisfies (12) for some processes A, B, and $\breve{\varepsilon}$ that satisfy conditions (i) and (ii) of Theorem 1.

COROLLARY 1: *$\mathcal{E}(r)$ is the largest bounded self-generating set.*

Corollary 1 is a continuous-time analogue of Theorems 1 and 2 (self-generation and factorization) from APS, which imply that the set of equilibrium payoff vectors is the largest bounded self-generating set. The boundedness assumption in APS is related to the boundedness assumption in Corollary 1 and Proposition 1.

Corollary 1 formulates the problem of finding the set $\mathcal{E}(r)$ as a problem from optimal stochastic control. We will use this result to characterize the set $\mathcal{E}(r)$ and the PPE that achieve extreme points of the set $\mathcal{E}(r)$ in the next section.

6. PPE WITH EXTREME VALUES: A DERIVATION

In this section, we use the characterization of PPE from Corollary 1 to derive our main result: an ordinary differential equation, called the *optimality equation*, which describes the boundary of $\mathcal{E}(r)$ and the PPE that achieve extreme points of $\mathcal{E}(r)$. The boundary of $\mathcal{E}(r)$ is denoted by $\partial\mathcal{E}(r)$. First, we discuss our results. Then in Section 6.2, we argue informally that the boundary of $\mathcal{E}(r)$ is characterized by the optimality equation—a result proved formally in Proposition 5 in Appendix B. We also prove Proposition 3, which constructs PPEs with extreme values and shows that $\mathcal{E}(r)$ is the largest bounded set whose boundary satisfies the optimality equation. Our main result, Theorem 2, follows from Propositions 3 and 5.

6.1. *Informal Discussion*

Let us review the properties of PPEs from the previous section, and then introduce the main results of this section about the geometry of the set $\mathcal{E}(r)$ and the PPE that achieve extreme value pairs of $\mathcal{E}(r)$. According to Corollary 1, $\mathcal{E}(r)$ is the largest subset of $\mathcal{V}^*$ such that for any $W_0 \in \mathcal{E}(r)$, there exist processes

[19]The martingale representation theorem implies that if $\{\mathcal{F}_t\}$ is generated by X, then $\breve{\varepsilon}_t = (0, 0)^\top$, $t \geq 0$, is the *only* martingale that satisfies (ii).

$(A, B, \breve{\varepsilon})$ that satisfy conditions (i) and (ii) of Theorem 1, so that the process $\{W_t\}$, given by

$$W_t = W_0 + r\int_0^t (W_s - g(A_s))\,ds + r\int_0^t B_s\,dZ_s + \breve{\varepsilon}_t, \tag{13}$$

remains in $\mathcal{E}(r)$. We have the freedom to choose actions A, volatilities B that enforce those actions, and public randomization $\breve{\varepsilon}$. If the initial value pair W_0 is inside the set $\mathcal{E}(r)$, this freedom gives a lot of room for very many equilibria. However, if the initial value pair W_0 is an extreme point of the set $\mathcal{E}(r)$, the choice of controls is severely restricted because continuation values cannot escape from the set $\mathcal{E}(r)$. In fact, we show that in a PPE that achieves an extreme value pair of the set $\mathcal{E}(r)$,

(a) future continuation values W_t must be extreme points of $\mathcal{E}(r)$;
(b) there is no public randomization, that is, $\breve{\varepsilon} = 0$;
(c) the span of B_t is in the tangential direction to the set $\mathcal{E}(r)$ at point W_t at all times;
(d) the choice of A_t and B_t is generically unique at all times;
(e) if there are static Nash equilibria with payoff pairs on the boundary of $\mathcal{E}(r)$, the players' actions are eventually absorbed into one of those Nash equilibria with probability 1.

We also show that the entire boundary of $\mathcal{E}(r)$ outside $\mathcal{N}$ has a strictly positive and continuous curvature and, therefore, consists of extreme points.

The spirit of properties (a)–(e) is present in the existing literature on repeated games. However, in discrete time these properties hold only under special continuity assumptions or in approximation. In relation to (a) and (b) in discrete time, one can always choose extreme continuation values if there is public randomization. Without public randomization, APS showed that future continuation values can be chosen to be extreme points of the equilibrium value set if the distribution of signals is nonatomic. Moreover, under certain analyticity conditions, future continuation values have to be extreme. The property (c) that B_t must have a tangential span is related to FLM's concept of enforceability of action pairs on tangent hyperplanes that is used to prove the Folk theorem. Fudenberg and Levine (1994) showed that the Folk theorem fails when equilibrium action pairs cannot be enforced on tangent hyperplanes.[20] Also, the Folk theorem typically fails in strongly symmetric equilibria that do not use tangent hyperplanes (see Proposition 8.2.1 in Mailath and Samuelson (2006) for the proof of this fact, and Green and Porter (1984) and Abreu, Pearce, and Stacchetti (1986) for analyses of strongly symmetric equi-

[20] Sannikov and Skrzypacz (2006) showed that in the limit as the players act more and more frequently, Brownian signals cannot provide incentives in any other way than by tangential transfers of continuation values.

libria).[21] Point (d) holds only under very strict continuity assumptions (e.g., the analyticity assumptions of APS that guarantee that continuation values *must* be extreme points). For point (e), if there is a unique way to support any extreme value pair, extreme Nash equilibrium payoff pairs would absorb equilibrium play if continuation values get there. However, in discrete time it may be possible for continuation values to never reach such an absorbing state.[22]

Even though the spirit of properties (a)–(e) is present in discrete-time games, it is difficult to formalize them. However, they come out cleanly in our continuous-time setting.

In addition to proving (a)–(e), we also show that $\partial\mathcal{E}(r)\backslash\mathcal{N}$ is characterized by the optimality equation. This ordinary differential equation connects the curvature of the boundary with the equilibrium actions and the stochastic motion of continuation values, and can be used for computation.[23] To understand this equation, we must first provide an analogue of FLM's concept of enforceability on tangent hyperplanes in our setting:

DEFINITION: A vector of tangential volatilities $\phi \in \mathbb{R}^d$ *enforces* $a \in \mathcal{A}$ on tangent $\mathbf{T} = (t_1, t_2)$ if the matrix

$$B = \mathbf{T}^\top \phi = \begin{bmatrix} t_1\phi_1 & \cdots & t_1\phi_d \\ t_2\phi_1 & \cdots & t_2\phi_d \end{bmatrix}$$

enforces a. Of all vectors ϕ that enforce a on tangent $\mathbf{T}$, let $\phi(a, \mathbf{T})$ be the one of the smallest length.

The main result of this section is that $\mathcal{E}(r)$ is the largest bounded set with the curvature of the boundary outside $\mathcal{N}$ given by the *optimality equation*

$$\kappa(w) = \max_{a\in\mathcal{A}\backslash\mathcal{A}^N} \frac{2\mathbf{N}(w)(g(a) - w)}{r|\phi(a, \mathbf{T}(w))|^2}, \tag{14}$$

where $\mathbf{T}(w)$ and $\mathbf{N}(w)$ are unit tangent and outward normal vectors to $\partial\mathcal{E}(r)$ at $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$ and $|\phi|$ is the length of the vector ϕ. In the maximization problem in (14), some action profiles a may not be enforceable on $\mathbf{T}$; we substitute $|\phi(a, \mathbf{T})| = \infty$ for those a.

[21]The observation after the proof of Lemma 6 in Appendix B implies that $\mathcal{E}(r) = \mathcal{N}$ if the set $\mathcal{E}(r)$ has an empty interior. The same argument goes through if we restrict attention to symmetric PPE, implying that the set of symmetric PPE payoffs coincides with the convex hull of symmetric Nash equilibrium payoffs in continuous-time games with Brownian signals. Sannikov and Skrzypacz (2007) showed that this is also true in discrete-time games with frequent actions.

[22]See Hauser and Hopenhayn (2004) for a continuous-time example with Poisson signal arrival, in which continuation values never reach static Nash equilibria. This phenomenon can occur in discrete time, but not in our games with Brownian signals.

[23]The characterization in terms of an ordinary differential equation, as well as property (e), relies on the restriction to two players.

REMARK: Lemma 2 in Appendix A, analogous to Lemma 5.5 in FLM, shows that under pairwise identifiability, any enforceable action profile a is enforceable on all *regular* tangent vectors $\mathbf{T} = (t_1, t_2)$, such that $t_1, t_2 \neq 0$. An enforceable profile a is enforceable on a *coordinate* vector $\mathbf{T}$ with $t_i = 0$ if and only if a involves a best response of player i.

6.2. *Derivation*

In this subsection, we justify our characterization. We start with a heuristic argument that the boundary of $\mathcal{E}(r)$ satisfies the optimality equation at all points outside $\mathcal{N}$, a result proved formally in Proposition 5 in Appendix B. We proceed in four steps. First, we argue that when a current pair of continuation values is extreme, then public randomization cannot be used and the volatilities of continuation values must be tangential to the boundary of $\mathcal{E}(r)$. Second, we take a detour to explore the geometric properties of a two-dimensional diffusion, whose volatility is focused along one line, and identify a "natural curvature" of this process. Third, we argue that the curvature of the boundary of $\mathcal{E}(r)$ at any extreme point outside $\mathcal{N}$ must satisfy (14). Fourth, we show that any point $w \notin \mathcal{N}$ on the boundary of $\mathcal{E}(r)$ must be extreme.

After that we present Proposition 3, which implies that $\mathcal{E}(r)$ is the largest bounded set whose boundary satisfies (14) outside $\mathcal{N}$ and constructs PPE that achieve its extreme values.[24] At the end we comment on uniqueness and absorption in Nash equilibria.

Let us go through the details of our argument. Recall that the equilibrium motion of continuation values is described by

$$dW_t = r(W_t - g(A_t))\,dt + rB_t\,dZ_t + d\breve{\varepsilon}_t, \tag{15}$$

where B_t enforces A_t and $\breve{\varepsilon}$ is a martingale orthogonal to Z.

Public randomization and tangential volatility

Suppose that W_t is an extreme point of the set $\mathcal{E}(r)$. We can immediately make two observations about the motion of W. First, as in discrete time, public randomization by a discrete random variable should not be used at a time when W_t is an extreme payoff pair of $\mathcal{E}(r)$, so $d\breve{\varepsilon}_t = 0$. Second, matrix B_t must have span in the tangential direction to the set $\mathcal{E}(r)$ at point W_t. Indeed, a normal component of volatility would instantaneously throw future continuation values outside the set $\mathcal{E}(r)$ with positive probability. Since the matrix B_t has a tangential span, we can represent it as $B_t = \mathbf{T}^\top \phi_t$ for some $\phi_t \in \mathbb{R}^d$, where $\mathbf{T}$ is a unit tangent vector at point W_t.

Curved trajectories of continuation values

It turns out that when the span of $B_t \neq 0$ is focused along one line, the trajectories of continuation values become locally bent with a curvature that depends on the drift of W_t. This property connects the geometry of the set $\mathcal{E}(r)$ and the stochastic motion of continuation values. To formalize the fact that the trajectories of continuation values have a natural curvature, consider a diffusion process

$$dW_t = r(W_t - g(A_t))\,dt + r\mathbf{T}^\top \phi_t\,dZ_t \tag{16}$$

with W_t on a convex curve $\mathcal{C}$ at time t, where $\mathbf{T}$ is a unit tangent vector to $\mathcal{C}$ at point W_t. Let $\mathbf{N}$ be an *outward* unit normal vector to $\mathcal{C}$ at W_t, such that the tangent line separates the direction of $\mathbf{N}$ from the rest of the curve $\mathcal{C}$. Let $(x, f(x))$ be a parameterization of $\mathcal{C}$ in tangential and normal coordinates, and let

$$D_{t+\varepsilon} = \mathbf{N}W_{t+\varepsilon} - f(\mathbf{T}W_{t+\varepsilon}).$$

be a measure of the distance from $W_{t+\varepsilon}$ to the curve $\mathcal{C}$ (see Figure 5). Then in the next paragraph, using Ito's lemma, we will show that D has volatility zero and drift

$$r\mathbf{N}(W_t - g(A_t)) + \frac{\kappa}{2}r^2|\phi_t|^2 \tag{17}$$

at time t, where κ is the curvature of $\mathcal{C}$ at W_t.[25] Note that $\kappa = -f''(\mathbf{T}W_t)$ because $f'(\mathbf{T}W_t) = 0$. Then, speaking loosely, the natural curvature of the trajec-

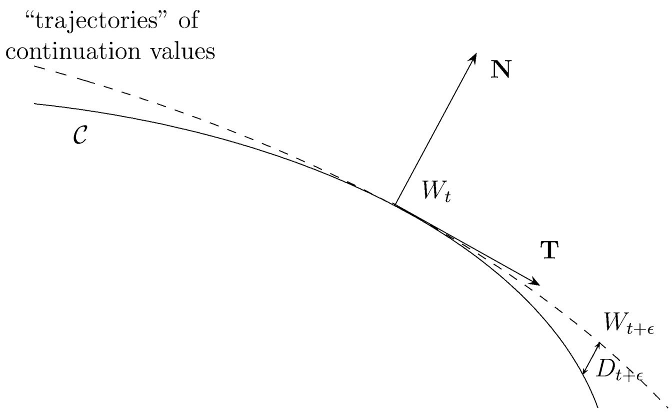

FIGURE 5.—The definition of $D_{t+\varepsilon}$.

[24] Proposition 3 is also used in the proof of Lemma 8 in Appendix B.

[25] Curvature is the rate at which the tangential angle changes with arc length.

tories of W_t coincides with the curvature of $\mathcal{C}$ if and only if the drift of D is 0 at time t, that is,

$$\kappa = \frac{2\mathbf{N}(g(A_t) - W_t)}{r|\phi_t|^2}.$$

Now let us demonstrate that (17) gives the drift of D_t. By projecting (16) onto the tangent axis, we get

$$d(\mathbf{T}W_t) = r\mathbf{T}(W_t - g(A_t))\,dt + r\phi_t\,dZ_t. \tag{18}$$

Using Ito's lemma,

$$df(\mathbf{T}W_t) = \underbrace{f'(\mathbf{T}W_t)}_{0}\,d(\mathbf{T}W_t) + \underbrace{f''(\mathbf{T}W_t)}_{-\kappa}\,\frac{r^2|\phi_t|^2}{2}\,dt. \tag{19}$$

By projecting (16) onto the normal axis, we get

$$d(\mathbf{N}W_t) = r\mathbf{N}(W_t - g(A_t))\,dt. \tag{20}$$

Combining (19) and (20), we get the desired result:

$$dD_t = d(\mathbf{N}W_t - f(\mathbf{T}W_t)) = \left(r\mathbf{N}(W_t - g(A_t)) + \kappa\frac{r^2|\phi_t|^2}{2}\right)dt.$$

Optimality equation and extreme points of $\mathcal{E}(r)$

Let us argue that the curvature $\kappa(W_t)$ of the set $\mathcal{E}(r)$ is given by (14) when $W_t \notin \mathcal{N}$ is an extreme point of $\mathcal{E}(r)$. We do this in two steps.

First, let us show that

$$\kappa(W_t) \le \frac{2\mathbf{N}(g(A_t) - W_t)}{r|\phi_t|^2}, \tag{21}$$

where $\mathbf{T}^\top\phi_t = B_t$. Note that $A_t \notin \mathcal{A}^N \Rightarrow \phi_t \neq 0$, because the drift of continuation values at time t cannot be directed outside $\mathcal{E}(r)$ as shown in the left panel of Figure 6. If (21) failed, the trajectories of continuation values, represented by a dashed curve in the right panel of Figure 6, would have a smaller curvature than the curvature of $\mathcal{E}(r)$ at W_t. Then continuation values would instantaneously escape from $\mathcal{E}(r)$, which leads to a contradiction.

Second, let us show that for any $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$ and $a \notin \mathcal{A}^N$,

$$\kappa(w) \ge \frac{2\mathbf{N}(g(a) - w)}{r|\phi(a, \mathbf{T})|^2}, \tag{22}$$

where $\mathbf{T}$ and $\mathbf{N}$ are unit tangent and normal vectors at w. If (22) failed, then the continuation values associated with the action profile a enforced by $\phi(a, \mathbf{T})$

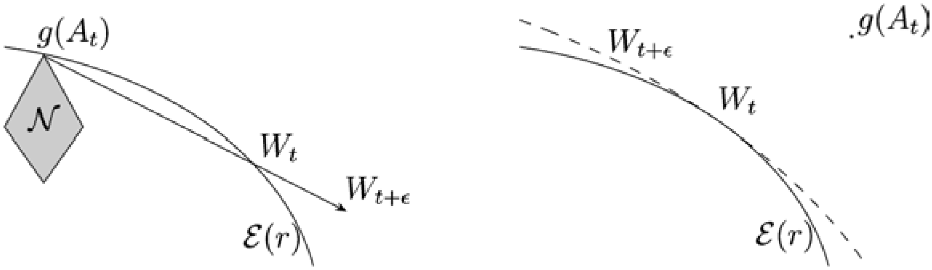

FIGURE 6.—Demonstration that (21) holds.

on tangent **T** would have trajectories with a curvature greater than $\kappa(w)$. Intuitively this means that w can be generated using continuation values strictly inside the set $\mathcal{E}(r)$, as shown in Figure 7. Then by moving continuation values in the direction of **N**, we would be able to generate a value pair $w + \varepsilon \mathbf{N}^\top$ outside the set $\mathcal{E}(r)$ with the action profile a and continuation values in the set $\mathcal{E}(r)$. This leads to a contradiction.

These two steps imply that the curvature $\kappa(W_t)$ of $\mathcal{E}(r)$ satisfies equation (14), in which the action pair A_t is the maximizer, and that $\phi_t = \phi(A_t, \mathbf{T})$.

All points of $\mathcal{E}(r)\backslash\mathcal{N}$ are extreme

See Corollary 2 of Proposition 5. Note that (22) holds for all points of $\partial\mathcal{E}(r)\backslash\mathcal{N}$, not just extreme points. Intuitively, if there was a nonextreme point $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$, then by (22) there would be no action profile enforceable on $\mathbf{T}(w)$ such that $\mathbf{N}(w)(g(a) - w) > 0$. Then in any PPE that achieves w, continuation values would have to stay on the tangent line and escape from $\mathcal{E}(r)$ due to positive volatility.

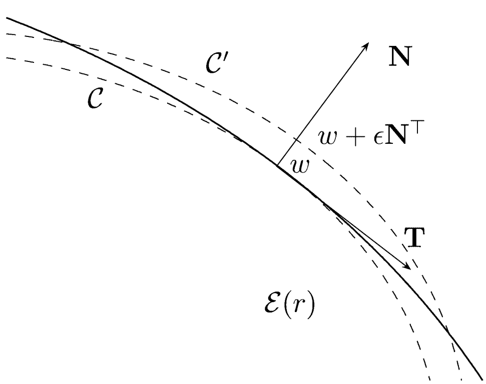

FIGURE 7.—Demonstration that (22) holds.

$\mathcal{E}(r)$ is the largest bounded set that satisfies (14) *outside $\mathcal{N}$*

The following proposition, which is also used in the formal argument in the Appendix, implies that any bounded set with a boundary that satisfies the optimality equation outside $\mathcal{N}$ must be a subset $\mathcal{E}(r)$.

PROPOSITION 3: *Suppose that the curve $\mathcal{C}$ satisfies* (14). *Furthermore, suppose that $\mathcal{C}$ either is a closed curve or has endpoints achievable by some PPE. Then $\mathcal{C} \subset \mathcal{E}(r)$.*

PROOF: By Theorem 1, to achieve $W_0 \in \mathcal{C}$ in a PPE it is sufficient to construct a bounded process W_t that satisfies

$$W_t = W_0 + r\int_0^t (W_s - g(A_s))\,ds + r\int_0^t B_s\,dZ_s, \quad B_t \text{ enforces } A_t \tag{23}$$

for all t.

Denote by $a:\mathcal{C} \to \mathcal{A}\backslash\mathcal{A}^N$ the maximizer in (14). Let us parameterize the curve $\mathcal{C}$ by arc length l. Let l_t be a weak solution of the equation

$$dl_t = r(W_t - g(A_t))\,dt + r\phi(A_t, \mathbf{T}(W_t))\,dZ_t,$$

starting from an initial value that corresponds to the point W_0 until a stopping time τ when l_t hits an endpoint of $\mathcal{C}$, where W_t is the point on $\mathcal{C}$ that corresponds to l_t, $A_t = a(W_t)$, and $\mathbf{T}(W_t)$ is the unit tangent vector to the curve $\mathcal{C}$ at point W_t. Then W_t has tangential drift $r\mathbf{T}(W)(W_t - g(A_t))$ and volatility $r\mathbf{T}(W_t)^\top\phi(A_t, \mathbf{T}(W_t))$. Since the function $f(x)$ that represents the curve $\mathcal{C}$ in tangential and normal coordinates at W_t satisfies $f'(\mathbf{T}W_t) = 0$ and $f''(\mathbf{T}W_t) = -\kappa(W_t)$, by Ito's lemma the normal component of the drift of W_t is

$$\begin{aligned} &0 \cdot r\mathbf{T}(W)(W_t - g(A_t)) - \kappa(W_t)\frac{r^2|\phi(A_t, \mathbf{T}(W_t))|^2}{2} \\ &\quad = r\mathbf{N}(W_t)(W_t - g(A_t)) \end{aligned}$$

for any $t \le \tau$. Therefore, the process W satisfies (23) until time τ.

Let us extend process W beyond time τ by letting it follow the path of a PPE that achieves value W_τ. Then W becomes a bounded random process that satisfies the conditions of Theorem 1 until time ∞. Therefore, we have constructed a PPE that achieves W_0. *Q.E.D.*

Comments on uniqueness and absorption in a Nash equilibrium

The optimality equation (14) assigns a generically unique action profile to each point $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$, which must be enforced by the vector of tangential volatilities $\phi(a, \mathbf{T}(w))$. These action profiles and volatilities uniquely pin

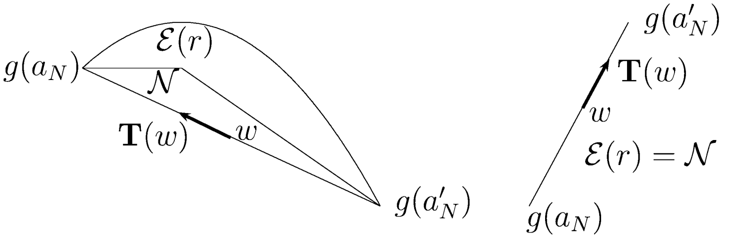

FIGURE 8.—Extreme and nonextreme points of $\partial\mathcal{E}(r)$.

down equilibrium dynamics in a PPE that achieves a point in $\partial\mathcal{E}(r)\backslash\mathcal{N}$. Since the volatilities are bounded away from 0, continuation payoffs must eventually hit the set $\mathcal{N}$ at an extreme point and become absorbed there whenever $\mathcal{N} \cap \partial\mathcal{E}(r) \neq \emptyset$.

The boundary of $\mathcal{E}(r)$ may include nonextreme points only if they are in $\mathcal{N}$. Figure 8 illustrates two such possibilities. In the left panel, the boundary of $\mathcal{E}(r)$ contains a line segment of nonextreme points, whose endpoints $g(a_N)$ and $g(a_N')$ are Nash equilibrium payoff pairs. In the right panel, the set $\mathcal{E}(r) = \mathcal{N}$ has an empty interior. For any nonextreme point $w \in \partial\mathcal{E}(r)$, there is no action profile $a \in \mathcal{A}$ enforceable on tangent $\mathbf{T}(w)$ with $\mathbf{N}(w)(g(a) - w) > 0$, so (14) would not yield a positive curvature at w. There are many PPEs that achieve any such point w, and in these PPEs players do not need to become absorbed in a static Nash equilibrium. For either panel of Figure 8, players can achieve w by alternating between a_N and a_N'.

This completes the derivation of our main result, which is summarized in the next section. The next section also provides an intuitive discussion of the set $\mathcal{E}(r)$ and the PPE that achieve extreme payoff pairs.

7. THE MAIN SECTION: SUMMARY AND DISCUSSION

The following theorem characterizes the set $\mathcal{E}(r)$ and the public perfect equilibria (PPE) that achieve extreme value pairs of $\mathcal{E}(r)$.

THEOREM 2—Characterization: *$\mathcal{E}(r)$ is the largest closed subset of $\mathcal{V}^*$ with curvature*

$$\kappa(w) = \max_{a \in \mathcal{A}\backslash\mathcal{A}^N} \frac{2\mathbf{N}(w)(g(a) - w)}{r|\phi(a, \mathbf{T}(w))|^2} \tag{24}$$

at all points $w \notin \mathcal{N}$ on the boundary of $\mathcal{E}(r)$, where $\mathbf{T}(w)$ and $\mathbf{N}(w)$ are unit tangent and outward normal vectors at w. We call (24) *the optimality equation.*[26]

[26]In our model, we normalized each component of the signal X to be independent of the others and have volatility 1. Alternatively, if the players observed signals

$$dX_t = \mu(A_t^1, A_t^2)\, dt + \Sigma\, dZ_t,$$

PPE WITH EXTREME VALUES: *Denote by* $a:\partial\mathcal{E}\backslash\mathcal{N}\to\mathcal{A}\backslash\mathcal{A}^N$ *the maximizing action pairs in* (24), *where* $\partial\mathcal{E}(r)$ *denotes the boundary of* $\mathcal{E}(r)$. *Any value pair* $W_0\in\partial\mathcal{E}(r)\backslash\mathcal{N}$ *is achieved by a PPE with the following characteristics. The pair of continuation values under this PPE satisfies the stochastic differential equation*

$$W_t = W_0 + \int_0^t \underbrace{r(W_s - g(A_s))}_{\text{drift}}\,ds + \int_0^t \underbrace{r\mathbf{T}(W_s)^\top \phi(A_s,\mathbf{T}(W_s))}_{B_s,\text{volatility}}(dX_s - A_s\,ds) \tag{25}$$

until time τ *when* W_t *hits the set* $\mathcal{N}$. *For* $t<\tau$, *the players take action pairs* $A_t = a(W_t)$. *After time* τ, *the players follow a static Nash equilibrium with value* W_τ. *When* $\partial\mathcal{E}(r)\cap\mathcal{N}=\emptyset$, *then* $\tau=\infty$. *Otherwise, players become absorbed in a static Nash equilibrium with probability* 1 *in finite time*.[27]

In the remainder of this section, we discuss the implications of this result on various questions of interest: the equilibrium dynamics, the nature of inefficiency, the choice of equilibrium actions, and the provision of incentives.

Let us describe dynamics in a PPE that achieves a value pair $W_0\in\partial\mathcal{E}(r)\backslash\mathcal{N}$. As soon as the game begins, the players' continuation values W_t start moving along the boundary of the set $\mathcal{E}(r)$.[28] This motion is a diffusion process defined by (25). Point W_t plays the role of a single state variable in this equilibrium. As a state variable, W_t determines the actions that the players take in a given instant and the law by which W_t itself evolves based on the observations of signal X. If there are Nash equilibrium payoff pairs on the boundary of $\mathcal{E}(r)$, then a pair of continuation values must eventually hit one of them with probability 1. When that happens, the players become absorbed in a static Nash equilibrium forever. Of course, if all static Nash equilibrium payoff pairs are inside the set $\mathcal{E}(r)$, then players never become absorbed in a Nash equilibrium and the motion of continuation values never stops.

At times $t<\tau$ before the players become absorbed in a static Nash equilibrium (if ever), they choose action pairs A_t and receive payoff flows $g(A_t)\notin$

where the volatility matrix Σ has full rank, then after appropriate rescaling, the optimality equation would be

$$\kappa(w) = \max_{a\in\mathcal{A}\backslash\mathcal{A}^N} \frac{2\mathbf{N}(w)(g(a)-w)}{r|\phi(a,\mathbf{T}(w))\Sigma|^2},$$

where $\phi(a,\mathbf{T})$ is defined the same way as before.

[27]There is a great multiplicity of equilibria that achieve nonextreme values. In those equilibria, players do not need to become absorbed in a static Nash equilibrium.

[28]Typically, as in all our examples, the pair of continuation values will diffuse along the entire boundary of $\mathcal{E}(r)$, not just its Pareto efficient portion.

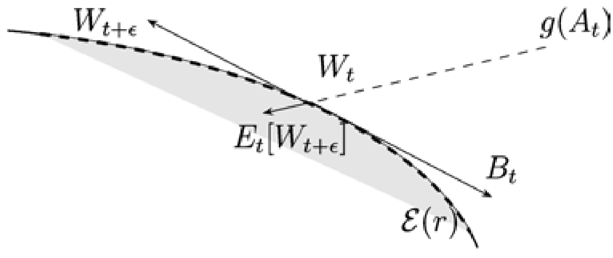

FIGURE 9.—The drift and volatility of continuation values.

$\mathcal{E}(r)$. The pair of continuation values W_t has drift directed away from point $g(A_t)$ inside the set $\mathcal{E}(r)$. This drift accounts for promise keeping: the current continuation value W_t is always a weighted average of the current payoff flow $g(A_t)$ and the expected continuation value a moment later $E_t[W_{t+\varepsilon}]$, as shown in Figure 9.

It may seem surprising that the drift of continuation values is directed inside $\mathcal{E}(r)$ even though continuation values stay on the boundary. We can reconcile these two facts as follows: because continuation values diffuse along the boundary due to tangential volatility and because the boundary has curvature, the expectation of future continuation values must be inside the set $\mathcal{E}(r)$.

The equilibrium actions pairs A_t come from the optimality equation (24). The objective of this equation is to describe the largest set of payoff pairs achievable in equilibrium. The choice among action pairs involves a trade-off between the extremity of payoffs and the incentives required to enforce them. The extremity of a payoff pair is measured by the payoff gain in the direction of the normal vector (see the numerator of (24)). The incentives are measured by the instantaneous tangential variance of continuation values (see the denominator of (24)). An optimal action pair achieves the maximum in (24). This action pair can be enforced by using continuation values on the boundary of $\mathcal{E}(r)$. If we tried to enforce a suboptimal action pair, the required drift and tangential volatility of W_t would take future continuation values outside $\mathcal{E}(r)$.

Let us discuss the provision of incentives. Before time τ, actions A_t are not static Nash equilibria, so players must have incentives to take actions that are not static best responses. These incentives arise because actions affect the drift of the public signals, which in turn affect continuation values. The volatility matrix in (25) is the sensitivity of continuation values to the signal X. From Section 5, we know that player i has incentives not to deviate from action pair A_t if his action maximizes the sum of his instantaneous payoff and the expected change of his continuation value, that is,

$$g_i(A_t) + \beta_t^i \mu(A_t) = \max_{a_i} g_i(a_i, A_t^j) + \beta_t^i \mu(a_i, A_t^j), \tag{26}$$

where β_t^i is row i of the volatility matrix at time t. In an equilibrium that achieves an extreme payoff pair, the volatility matrix must be of the form

$\mathbf{T}(w)^\top \phi_t$ to have a tangential span. Generally, there could be many ways to enforce A_t on a tangent line, but only the smallest tangential variance must be used in an equilibrium for which W_0 is extreme. Our assumptions do not guarantee that all action pairs can be enforced on all tangent lines, so the Folk theorem may fail. To guarantee enforceability of all action profiles, one would also need to assume individual full rank or to make an appropriate concavity assumption, as we elaborate for games with a special signal structure in the next section.

7.1. *Incentives in Games with Special Signal Structure*

The partnership example from Section 2 and the duopoly example from the Section 8 have a special signal structure, under which the provision of incentives is especially clear. For that class of games $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}$, the public signal is two-dimensional and has drift $\mu(a_1, a_2) = (a_1, a_2)^\top$. Therefore, there is a separate signal that is indicative of each player's actions.

For this class of games, the volatility matrix is 2×2 and condition (11) reduces to

$$g_i(A_t) + \beta_t^{ii} A_t^i = \max_{a_i} g_i(a_i, A_t^j) + \beta_t^{ii} a_i, \tag{27}$$

where β_t^{ii} for $i = 1, 2$ are the diagonal entries of the volatility matrix B_t. When $W_t \in \partial\mathcal{E}(r)\backslash\mathcal{N}$, the tangential volatility condition pins down uniquely the off-diagonal entries of the 2×2 matrix B_t given the diagonal entries β_t^{ii}. To enforce an action profile A_t on a tangent with minimal volatility, we must choose β_t^{ii} of the smallest absolute value for (27) to hold.

DEFINITION OF γ: Consider all values of β^{ii} for which a_i maximizes $g_i(a_i, a_j) + \beta^{ii} a_i$ given a_j. Of these values, define $\gamma_i(a_i, a_j)$ to be the smallest in terms of absolute value.

The values of $\gamma_i(a)$ are defined for all action pairs a if and only if g_i is concave in a_i. Figure 10 illustrates the computation of $\gamma_i(a)$ for this case. If we fix a_j and plot $g_i(\cdot, a_j)$, then $-\gamma_i(a_i, a_j)$ equals the slope of function $g_i(\cdot, a_j)$ between point a_i and the nearest profitable deviation.

The matrix that enforces an action profile A_t tangentially with minimal volatility has the form

$$B_t = \begin{bmatrix} \gamma_1(A_t) & \gamma_2(A_t)\frac{t_1}{t_2} \\ \gamma_1(A_t)\frac{t_2}{t_1} & \gamma_2(A_t) \end{bmatrix}.$$

We can make two useful observations about incentive provision for the games with special signal structure and in general:

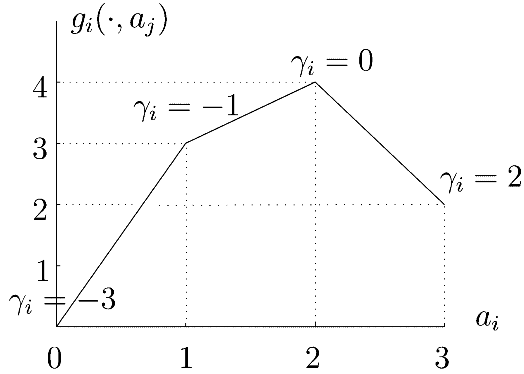

FIGURE 10.—How to find γ.

(i) Generally, because the local motion of continuation values is restricted to a tangent line, the necessity to provide incentives to one player affects the continuation value of another player. With special signal structure, player j's continuation value has sensitivity $\gamma_i(A_t)t_j/t_i$ to the signal X_i, which reflects player i's action exclusively.

(ii) In a game with special signal structure, the incentives provided to different players do not interfere; there is a separate signal and a separate column of the matrix B_t that is responsible for the actions of each player. In general, this is not true because the same signal can be affected by both players.

Let us comment on the enforceability of action pairs on tangent lines and the Folk theorem in this class of games. If g_i is concave in a_i, then $\gamma_i(a)$ is well defined for all $a \in \mathcal{A}$ and $i = 1, 2$, so that all action profiles are enforceable. Then all action profiles can be enforced on all regular tangent lines by Lemma 2, and action profiles with a best response property for player i can be enforced on coordinate lines with $t_i = 0$.[29] From the optimality equation, we can see immediately that the Folk theorem holds under these conditions. As r decreases to 0, the numerator $2\mathbf{N}(g(a) - w)$ in the optimality equation also decreases to 0, making the set $\mathcal{E}(r)$ expand toward the boundaries of $\mathcal{V}^*$. Conversely, if the function g_i is not concave in a_i for $i = 1$ or 2, then $\gamma_i(a)$ is sometimes undefined and the Folk theorem may fail, that is, the closure of $\lim_{r\to 0} \mathcal{E}(r)$ may be smaller than $\mathcal{V}^*$.

8. COMPUTATION

In this section we discuss the computation of the set $\mathcal{E}(r)$, and present the outcomes of computation for the partnership example of Section 2 and a new example of a duopoly with differentiated products.

[29] Note that the off-diagonal entries of B_t blow up when $t_i = 0$ and $\gamma_i(a) \neq 0$.

To write the optimality equation in a form suitable for computation, recall that curvature is the rate at which the tangential angle changes with arc length. Therefore, $\kappa(\theta) = d\theta/dl$, where θ is the tangential angle (so that $\mathbf{T}(\theta) = (-\sin\theta, \cos\theta)$ and $\mathbf{N}(\theta) = (\cos\theta, \sin\theta)$) and l is arc length. As the tangential angle θ changes, the corresponding point $w(\theta)$ moves along the curve in the tangential direction $\mathbf{T}(\theta)$ with speed $dl/d\theta = 1/\kappa(\theta)$, so

$$\frac{dw(\theta)}{d\theta} = \frac{\mathbf{T}(\theta)^\top}{\kappa(\theta)}. \tag{28}$$

Equation (28) is easy to solve numerically starting from any initial conditions $(\theta, w) \in [0, 2\pi) \times \mathcal{V}^*$, treating the coordinates of $w(\theta)$ as functions of θ, and using

$$\kappa(\theta) = \max_{a\in\mathcal{A}\setminus\mathcal{A}^N} \frac{2\mathbf{N}(\theta)(g(a) - w(\theta))}{r|\phi(a, \mathbf{T}(\theta))|^2}. \tag{29}$$

Generally, $|\phi(a, \mathbf{T})|^2$ can be found by solving a quadratic program

$$\begin{aligned}
&\left|\phi(a, (t_1, t_2))\right|^2 = \min_{\phi} |\phi|^2 \\
&\text{s. t.} \quad \forall i = 1, 2, \quad \forall a_i' \in \mathcal{A}^i, \\
&\quad g_i(a) + t_i\phi\mu(a) \geq g_i(a_i', a_j) + t_i\phi\mu(a_i', a_j).
\end{aligned}$$

For our examples and other games with a special structure,

$$\left|\phi(a, (t_1, t_2))\right|^2 = \frac{\gamma_1(a)^2}{t_1^2} + \frac{\gamma_2(a)^2}{t_2^2}.$$

We present computational examples in an increasing order of difficulty.

8.1. *Partnership*

From symmetry considerations, the boundary of the set $\mathcal{E}(r)$ must contain a point on the 45-degree line with an outward unit normal $\mathbf{N} = (\cos(45°), \sin(45°))$. Also, point $(0, 0)$ will be on the boundary as well. For all points w on the line segment between the origin and point $(1, 1)$, consider the curve $\mathcal{C}(w)$ that solves the optimality equation from initial conditions $(w, \mathbf{N})$. To compute the set $\mathcal{E}(r)$, we search along the 45-degree line and find point w, removed farthest from the origin, such that the curve $\mathcal{C}(w)$ reaches the origin. First, we do a grid search to identify an interval where the desired point w is located. After that, we do a binary search within the interval to compute w exactly. Figure 11 illustrates the computational procedure for $r = 0.2$.

From the grid search on Figure 11(a), we know that there are two symmetric closed curves that satisfy the optimality equation everywhere except in the

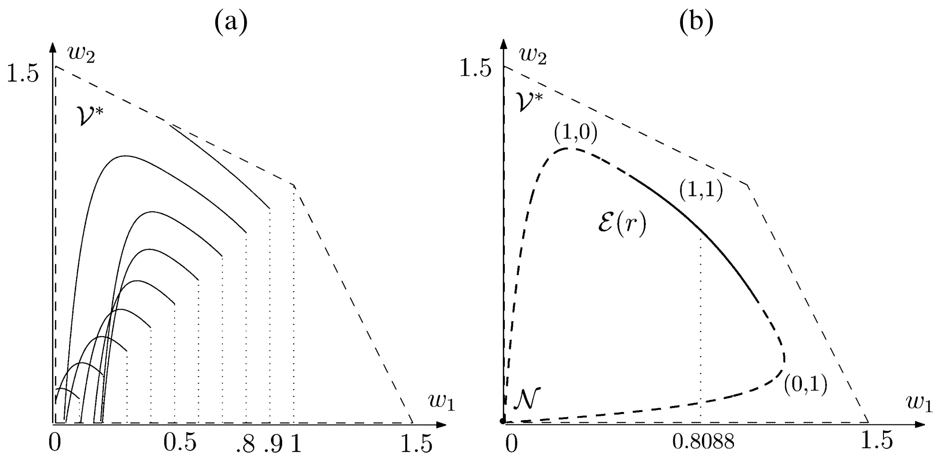

FIGURE 11.—Computation of $\mathcal{E}(r)$ in noisy partnership.

origin: one in the interval $(0.2, 0.3)$ and one in the interval $(0.8, 0.9)$. We are interested in the latter curve, because it is larger. That curve can be found by means of a binary search in the interval $(0.8, 0.9)$.

The computed boundary of $\mathcal{E}(r)$, along with recommended action pairs at every point, is shown in Figure 11(b).

There is another easy method to compute the set $\mathcal{E}(r)$ in this example, using the fact that $\mathcal{N}$ must be on the boundary of $\mathcal{E}(r)$. This method involves a one-dimensional search over the slopes of the boundary of $\mathcal{E}(r)$ at point $\mathcal{N}$. In the next subsection, we see how computation can be performed on an asymmetric example in which $\mathcal{N}$ is in the interior of the set $\mathcal{V}^*$.

8.2. *Duopoly with Differentiated Products*

Consider the following example. There are two firms whose products are imperfect substitutes. The private actions of firm i are supply rates from the set $\mathcal{A}^i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The instantaneous prices of firms 1 and 2 are given by the increments of the processes

$$dP_t^1 = (25 - 2A_t^1 - A_t^2)\,dt + \text{noise} \quad \text{and}$$
$$dP_t^2 = (30 - 2A_t^2 - 2A_t^1)\,dt + \text{noise}.$$

Prices are publicly observable and the noise structure is such that firms can isolate signals about each firm's quantity from the prices

$$dX_t^1 = \frac{1}{2}dP_t^2 - dP_t^1 + 10\,dt = A_t^1\,dt + dZ_t^1 \quad \text{and}$$
$$dX_t^2 = dP_t^1 - dP_t^2 + 5\,dt = A_t^2\,dt + dZ_t^2,$$

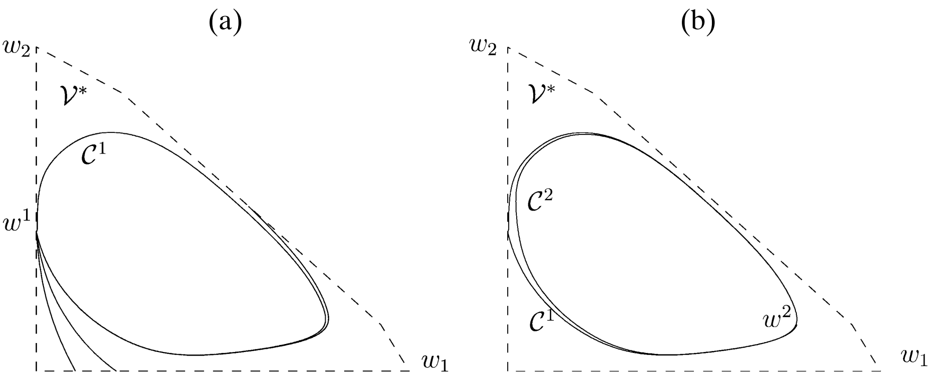

FIGURE 12.—Computation of the asymmetric game.

where Z^1 and Z^2 are independent standard Brownian motions. The payoffs of firms 1 and 2 are given by

$$r\int_0^\infty e^{-rt}A_t^1\,dP_t^1 \quad \text{and} \quad r\int_0^\infty e^{-rt}A_t^2\,dP_t^2.$$

The payoff functions can be identified as

$$g_1(a_1,a_2)=a_1(25-2a_1-a_2) \quad \text{and}$$
$$g_2(a_1,a_2)=a_2(30-2a_2-2a_1).$$

This stage game has a unique Nash equilibrium $(5,5)$, but ideally firms could collude by producing $(4,4)$.

For this game, the computational procedure is illustrated in Figure 12. We start at an arbitrary point w^1 on the boundary of the set $\mathcal{V}^*$ and compute the solutions of the optimality equation from initial conditions (w^1,θ) for $\theta\geq\pi$. We raise θ continuously until the corresponding solution $\mathcal{C}^1$ (for some angle $\hat{\theta}^1$) hits point w^1 after making a loop, as shown in Figure 12(a). We claim that the resulting solution must enclose the set $\mathcal{E}(r)$. If not, as we vary θ continuously between π to $\hat{\theta}^1$, some solution would have to be tangent to $\mathcal{E}(r)$. However, this is impossible, because then the solution would have to coincide with the boundary of $\mathcal{E}(r)$ (from the uniqueness of solutions given the initial conditions at the point of tangency).

Next, take point w^2 on the curve $\mathcal{C}^1$ with an outward unit normal $(1,0)$. Again, we compute the solutions of the optimality equation from initial conditions (w^2,θ) for $\theta\geq 0$. We raise θ continuously until the corresponding solution $\mathcal{C}^2$ (for some angle $\hat{\theta}^2$) hits point w^2 after making a loop, as shown in Figure 12(b). Then the curve $\mathcal{C}^2$ must enclose the set $\mathcal{E}(r)$ inside. By continuing this procedure iteratively, we will converge to the set $\mathcal{E}(r)$.

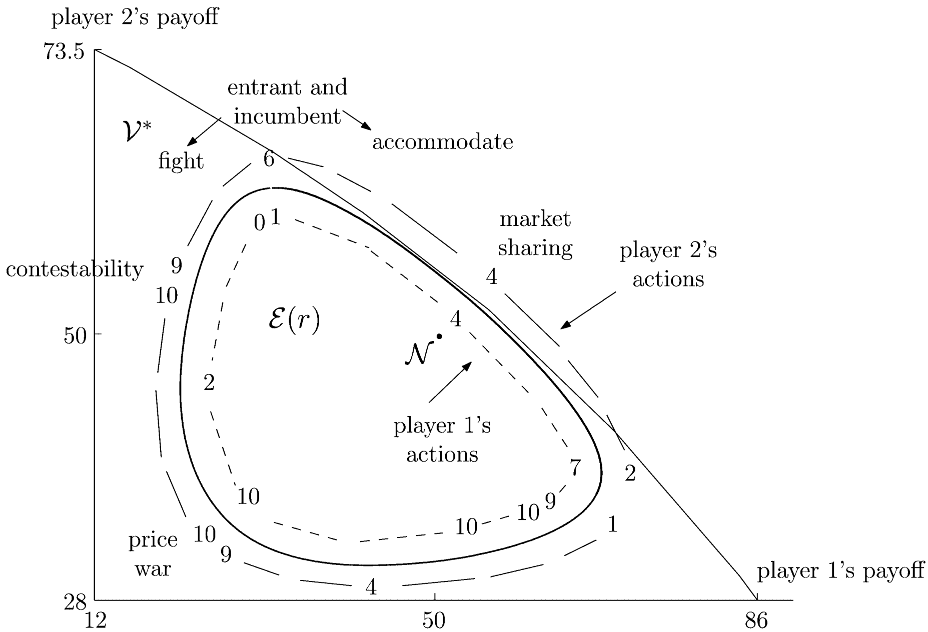

FIGURE 13.—Set $\mathcal{E}(r)$ in duopoly.

Figure 13 illustrates the outcome of computation for discount rate $r = 1.5$. The boundary of $\mathcal{E}(r)$ is divided into many segments on which players keep their actions constant. Figure 13 illustrates the general pattern of actions, as well as an interpretation of each portion on the boundary of the set $\mathcal{E}(r)$.[30] For comparison, recall that a static Nash equilibrium is $(5, 5)$. Along the Pareto frontier of $\mathcal{E}(r)$, players collude by producing less than their static best responses. We call this regime *market sharing*. In this regime, when a player's continuation value increases, his market share also increases. Therefore, players are rewarded for underproducing by an increased future market share. On top of the set $\mathcal{E}(r)$, player 2 receives the maximal payoff that he possibly could in a PPE. At that point, player 1 produces very little, while player 2 produces close to his monopoly quantity. While player 2 chooses a static best response, player 1 needs strong incentives to "stay out." To reward player 1 for "staying out," player 2 accommodates, and to punish player 1 for cheating, player 2 fights. We call this regime *entrant and incumbent*. On the left side of $\mathcal{E}(r)$, player 1 is acting passively by producing a static best response, while player 2 is overproducing aggressively. At this point, player 2 is rewarded for overproducing by being able to drive player 1 out of the market. We call this regime *contestability*. At the bottom left portion of $\mathcal{E}(r)$, players are fighting a "price

[30]I am thankful to William Fuchs for helping me find these interpretations.

war" by overproducing. They have incentives to do so because the player who looks more aggressive will come out as the winner of the price war. The winner gets his reward by becoming a monopolist for some period of time.

9. CONCLUSION

This paper introduces a new class of games in continuous time, in which the players' observations of each other's actions are distorted by Brownian motion. In these games, the set of value pairs that are achievable in public perfect equilibria has a clean characterization. The form of public perfect equilibria that achieve values on the boundary of the set $\mathcal{E}(r)$ and the manner in which the players organize the provision of incentives are intuitive. We saw examples of various economic interactions that can be modeled as continuous-time games. Besides our examples of a partnership and a duopoly, our model can be applied to principal–agent problems, risk-sharing models, and so forth. One is hopeful that the simplicity of characterizations in continuous-time models will allow deeper analysis of applications to various dynamic incentive problems with imperfect information.

Let us discuss several questions for development of future theory. First, it is necessary to illustrate the connection between discrete-time repeated games and continuous-time games, and to understand how continuous-time games can be used to approximate repeated games in discrete time. Second, it is beneficial to extend the continuous-time approach to games with private information. DeMarzo and Sannikov (2006) showed how to attack the issue of private information in a setting with one-sided imperfect information. Third, one has to extend the continuous-time approach to settings where more than one state variable is required. Finally, it would be interesting to explore other computational procedures to find the set $\mathcal{E}(r)$.

Dept. of Economics, University of California, Berkeley, 593 Evans Hall 3880, Berkeley, CA 94720-3880, U.S.A.; sannikov@gmail.com.

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APPENDIX A: THE OPTIMALITY EQUATION

This appendix explores the enforceability of action pairs on tangent lines and the regularity properties of the optimality equation. Our first lemma formalizes the notion that pairwise identifiability allows the provision of incentives to one player without interfering with the incentives of the other player. Recall that the $d \times (|\mathcal{A}^i| - 1)$ matrix $M_i(a)$ is composed of columns $\mu(a_i', a_j) - \mu(a)$, $a_i' \neq a_i$. Similarly denote by $G_i(a)$ the row vector with $|\mathcal{A}^i| - 1$ components $g_i(a_i', a_j) - g_i(a)$, $a_i' \neq a_i$. Then a matrix B enforces an action profile a if and only if

$$G_i(a) \leq \beta^i M_i(a) \quad \text{for} \quad i = 1, 2,$$

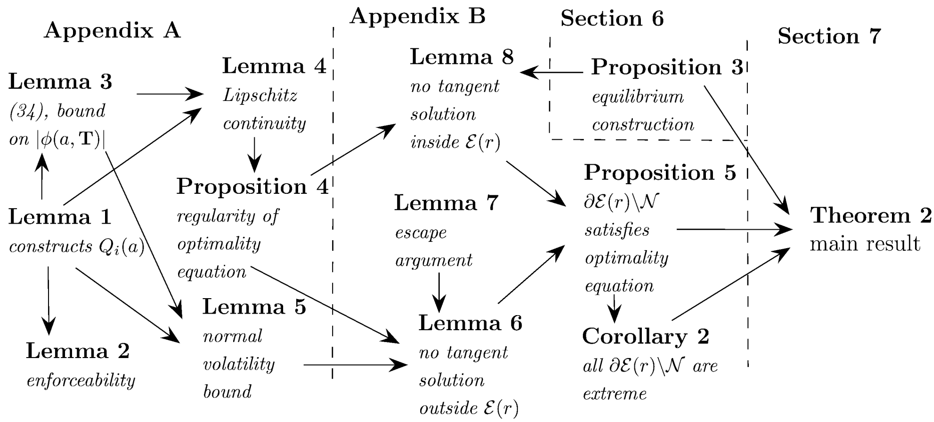

FIGURE 14.—Roadmap of the appendixes.

where β^i is the ith row of B. Also, a vector β enforces an action profile a on tangent $\mathbf{T}$ if and only if

$$G_i(a) \leq t_i \beta M_i(a) \quad \text{for} \quad i = 1, 2.$$

LEMMA 1: *For each action profile $a \in \mathcal{A}$ and $i = 1, 2$ there exists a $d \times d$ idempotent matrix $Q_i(a)$ such that $Q_i(a)M_i(a) = M_i(a)$ and $Q_i(a)M_j(a) = 0$.*

PROOF: For $i = 1, 2$, let us construct a matrix $M_i'(a)$ whose columns form the basis of the column space of $M_i(a)$. Pairwise identifiability of a implies that there is no linear dependence among the columns of $[M_1'(a), M_2'(a)]$. Let us choose a matrix $L(a)$ that makes

$$[M_1'(a), M_2'(a), L(a)]$$

into a $d \times d$ invertible matrix. Then an idempotent matrix that satisfies the required properties can be defined by

$$Q_i(a) = [M_i'(a), 0, 0][M_i'(a), M_j'(a), L(a)]^{-1}$$
$$\Leftrightarrow \quad Q_i(a)[M_i'(a), M_j'(a), L(a)] = [M_i'(a), 0, 0]. \qquad \textit{Q.E.D.}$$

The matrix $Q_i(a)$ isolates the incentives of player i from a given vector β, since $\beta Q_i(a)M_i(a) = \beta M_i(a)$ and $\beta Q_i(a)M_j(a) = 0$. The following bound will be useful:

$$\bar{Q} = \max_{a, i, |\beta|=1} |\beta Q_i(a)|.$$

We can use $Q_i(a)$ to prove an analogue of Lemma 5.5 from FLM.

LEMMA 2: *Any enforceable action profile is enforceable on all* $\mathbf{T} = (t_1, t_2)$ *with* $t_1, t_2 \neq 0$.

PROOF: If a matrix B with rows β^1 and β^2 enforces an action profile a, then $\beta^1 Q_1(a)/t_1 + \beta^2 Q_2(a)/t_2$ enforces a on the tangent $\mathbf{T} = (t_1, t_2)$. Indeed,

$$\big(\beta^1 Q_1(a)/t_1 + \beta^2 Q_2(a)/t_2\big)[t_1 M_1(a), t_2 M_2(a)] = [\beta^1 M_1(a), \beta^2 M_2(a)] \geq [G_1(a), G_2(a)].$$

Q.E.D.

Next, our task is to show that the right-hand side of the optimality equation is Lipschitz-continuous in w and $\mathbf{T}$. This property guarantees the existence of solutions to the optimality equation from any initial conditions and helps us characterize the set $\mathcal{E}(r)$. The following lemma provides a bound for $\phi(a, \mathbf{T})$.

LEMMA 3: *Let* $\psi_i(a)$ *be the vector of minimal length such that* $G_i(a) \leq \psi_i(a) M_i(a)$. *Then*

$$|\phi(a, \mathbf{T})| \geq |\psi_i(a)/t_i|. \tag{30}$$

Moreover, if a has a best response property for player j and $|t_j|$ *is sufficiently small, then*

$$\phi(a, \mathbf{T}) = \psi_i(a)/t_i. \tag{31}$$

PROOF: Recall that $\phi(a, \mathbf{T})$ is defined as the shortest vector such that $G_i(a) \leq t_i \phi(a, \mathbf{T}) M_i(a)$ for $i = 1, 2$. Thus, (30) follows because $\psi_i(a)/t_i$ is the shortest vector such that only $G_i(a) \leq t_i(\psi_i(a)/t_i) M_i(a)$ holds.

Suppose also that a has a best response property for player j and that $|t_j|$ is sufficiently small. If we show that $G_j(a) \leq t_j(\psi_i(a)/t_i) M_j(a)$ holds, we can conclude (31).

We must rely on Assumption 2 from Section 3. When part (i) of Assumption 2 holds, that is, a_j is a unique best response to a_i, then $G_j(a) < 0$ and $G_j(a) < t_j(\psi_i(a)/t_i) M_j(a)$ when $|t_j|$ is sufficiently small. Suppose part (ii) of Assumption 2 holds, that is, the game has a product structure. Because $\psi_i(a)$ is the shortest vector that satisfies $G_i(a) \leq \psi_i(a) M_i(a)$, $\psi_i(a)$ is in the column space of $M_i(a)$, which is orthogonal to the column space of $M_j(a)$ due to the product structure. Again, we have $G_j(a) \leq t_j(\psi_i(a)/t_i) M_j(a) = 0$. *Q.E.D.*

Lemma 3 implies a useful lower bound (34) for $\phi(a, \mathbf{T})$. Let $\varepsilon_\psi \in (0, 1)$ be a constant such that (31) holds whenever a has a best response property for player j and $|t_j| < \varepsilon_\psi$. Denote

$$\bar{\Psi} = \varepsilon_\psi \min_{a,i} |\psi_i(a)|,$$

where the minimization is over $i = 1, 2$ and all action profiles a without best response property for player i. If $a \notin \mathcal{A}^N$ has a best response property for player i but $|t_i| > \varepsilon_\psi$, then (30) implies that

$$|\phi(a, \mathbf{T})| \geq |\psi_j(a)/t_j| \geq \bar{\Psi}/|t_i|. \tag{32}$$

If a does not have a best response property for player i, then

$$|\phi(a, \mathbf{T})| \geq |\psi_i(a)/t_i| > \bar{\Psi}/|t_i|. \tag{33}$$

We conclude that for all $a \notin \mathcal{A}^N$,

$$|\phi(a, \mathbf{T})| \geq \bar{\Psi}/|t_i| \tag{34}$$

unless $|t_i| < \varepsilon_\psi$ *and* a involves a best response of player i.

The following lemma is a key ingredient in the proof of Proposition 4, which deals with the regularity properties of the optimality equation.

LEMMA 4: *Denote* $\mathbf{T}(\theta) = (-\sin\theta, \cos\theta)$ *and* $\mathbf{N}(\theta) = (\cos\theta, \sin\theta)$. *Then for all* $a \notin \mathcal{A}^N$,

$$H_a(w, \theta) = \frac{2\mathbf{N}(\theta)(g(a) - w)}{r|\phi(a, \mathbf{T}(\theta))|^2} \tag{35}$$

is Lipschitz-continuous in w *and* θ *when* $\mathbf{N}(\theta)(g(a) - w) \geq 0$, *where we interpret* $H_a(w, \theta)$ *to be* 0 *when* a *is not enforceable on* $\mathbf{T}(\theta)$.

PROOF: First, $H_a(w, \theta)$ is Lipschitz-continuous in w since $|\phi(a, \mathbf{T}(\theta))|$ is bounded away from 0 (by Lemma 3) for all $a \notin \mathcal{A}^N$.

To prove Lipschitz continuity in θ, first consider the case when, for some $i = 1, 2$, we have $|t_i(\theta)| < \varepsilon_\psi$ and a involves a best response of player i. Then (31) implies that

$$H_a(w, \theta) = \frac{2t_j(\theta)^2\mathbf{N}(\theta)(g(a) - w)}{r|\psi_j(a)|^2},$$

which is continuously differentiable (and thus Lipschitz-continuous) in θ.

Otherwise, the bound (34) holds for $i = 1, 2$. Let $\beta^i = \phi(a, \mathbf{T}(\theta))Q_i(a)$. Then

$$\beta(\theta') = \phi(a, \mathbf{T}(\theta)) + \left(\frac{t_1(\theta)}{t_1(\theta')} - 1\right)\beta^1 + \left(\frac{t_2(\theta)}{t_2(\theta')} - 1\right)\beta^2$$

enforces a on tangent $\mathbf{T}(\theta')$, so $|\beta(\theta')| \geq |\phi(a, \mathbf{T}(\theta'))|$. As long as $\mathbf{N}(\theta')(g(a) - w) \geq 0$,

$$F(\theta') = \frac{2\mathbf{N}(\theta')(g(a) - w)}{r|\beta(\theta')|^2} \leq H_a(w, \theta')$$

with equality when $\theta = \theta'$. For θ' near θ, we have

$$\begin{aligned}\left|\frac{dF(\theta')}{d\theta'}\right| &= \left|\frac{2\mathbf{T}(\theta')(g(a)-w)}{r|\beta(\theta')|^2} - \frac{4\mathbf{N}(\theta')(g(a)-w)}{r|\beta(\theta')|^4}\right. \\ &\quad \left. \times \left(\frac{n_1(\theta')t_1(\theta)}{t_1(\theta')^2}\beta^1 + \frac{n_2(\theta')t_2(\theta)}{t_2(\theta')^2}\beta^2\right)^\top \beta(\theta')\right| \\ &\le \frac{2|\mathcal{V}|}{r\bar{\Psi}^2} + 4|\mathcal{V}|\frac{\left(\frac{1}{t_1(\theta')^2} + \frac{1}{t_2(\theta')^2}\right)\bar{Q}|\beta(\theta)||\beta(\theta')|}{r\left(\frac{1}{t_1(\theta')^2} + \frac{1}{t_2(\theta')^2}\right)\frac{\bar{\Psi}^2}{2}|\beta(\theta')|^2} \\ &\le \frac{2|\mathcal{V}|}{r\bar{\Psi}^2} + 4|\mathcal{V}|\frac{4\bar{Q}}{r\bar{\Psi}^2} = \bar{K},\end{aligned}$$

where we used $|\beta(\theta')|^2 \ge |\phi(a, T(\theta'))|^2$ and (34). It follows that

$$H_a(w,\theta) - H_a(w,\theta') \le F(\theta) - F(\theta') \le |\theta - \theta'|\bar{K},$$

so H_a is Lipschitz-continuous in θ. *Q.E.D.*

The consequence of Lipschitz continuity is summarized in the following proposition.

PROPOSITION 4: *Consider the following version of the optimality equation*:

$$\kappa(w,\theta) = \max\Bigl(0, \max_{a\notin\mathcal{A}^N} H_a(w,\theta)\Bigr). \tag{36}$$

Solutions to (36) *exist and are unique and continuous in initial conditions. Moreover, if the curvature is positive at initial conditions, then it stays positive along the solution.*

PROOF: Lemma 4 implies that the right-hand side of (36) is Lipschitz-continuous in w and θ. Thus, the solutions to (36) exist and are unique and continuous in initial conditions.

For the sign of the curvature, note that if $\kappa(w,\theta) = 0$, then the line that passes through w parallel to $\mathbf{T}(\theta)$ is a unique solution from the initial conditions (w,θ). It has zero curvature throughout. Therefore, if the initial curvature is positive, then it must stay positive along the solution. Had it reached 0, the entire solution would have been a straight line by uniqueness. *Q.E.D.*

The following lemma provides an important bound, which is used in Appendix B.

LEMMA 5: *There exists a constant $K > 0$ such that for any $a \notin \mathcal{A}^N$ and a matrix $B = \mathbf{T}^\top \phi + \mathbf{N}^\top \chi$ that enforces a, where $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors,*

$$\frac{4\bar{Q}}{\bar{\Psi}}|\chi| \geq 1 - \frac{|\phi|^2}{|\phi(a, \mathbf{T})|^2}. \tag{37}$$

PROOF: Suppose that $|\phi| \leq |\phi(a, \mathbf{T})|$, since otherwise (37) obviously holds. First, consider the case when (34) holds for $i = 1$ and 2. Then

$$\beta = \phi + \frac{n_1}{t_1}\chi Q_1(a) + \frac{n_2}{t_2}\chi Q_2(a)$$

enforces a on tangent $\mathbf{T} = (t_1, t_2)$. Therefore,

$$\begin{aligned}
&|\phi| + \left(\frac{1}{|t_1|} + \frac{1}{|t_2|}\right)\bar{Q}|\chi| \geq |\beta| \geq |\phi(a, \mathbf{T})| \\
&\quad \Rightarrow \quad |\phi(a, \mathbf{T})| - |\phi| \leq \left(\frac{1}{|t_1|} + \frac{1}{|t_2|}\right)\bar{Q}|\chi| \leq \frac{2|\phi(a, \mathbf{T})|}{\bar{\Psi}}\bar{Q}|\chi| \\
&\quad \Rightarrow \quad \frac{2\bar{Q}}{\bar{\Psi}}|\chi| \geq \frac{|\phi(a, \mathbf{T})| - |\phi|}{|\phi(a, \mathbf{T})|} \geq \frac{|\phi(a, \mathbf{T})|^2 - |\phi|^2}{2|\phi(a, \mathbf{T})|^2}.
\end{aligned}$$

Second, consider the case when (34) may fail, that is, $|t_i| < \varepsilon_\psi$ and a involves a best response of player i for $i = 1$ or 2. Then (31) implies that

$$\begin{aligned}
&(t_j\phi + n_j\chi)M_j(a) \geq G_j(a) \\
&\quad \Rightarrow \quad |t_j\phi| + |\chi| \geq |t_j\phi + n_j\chi| \geq |\psi_j(a)| = |t_j\phi(a, \mathbf{T})| \\
&\quad \Rightarrow \quad |\phi(a, \mathbf{T})| - |\phi| \leq \frac{|\chi|}{|t_j|} \leq \frac{|\chi|}{\bar{\Psi}}\frac{|\psi_j(a)|}{|t_j|} = \frac{|\chi|}{\bar{\Psi}}|\phi(a, \mathbf{T})| \\
&\quad \Rightarrow \quad \frac{|\chi|}{\bar{\Psi}} \geq \frac{|\phi(a, \mathbf{T})| - |\phi|}{|\phi(a, \mathbf{T})|} \geq \frac{|\phi(a, \mathbf{T})|^2 - |\phi|^2}{2|\phi(a, \mathbf{T})|^2}
\end{aligned}$$

and (37) follows since $\bar{Q} \geq 1$.[31] *Q.E.D.*

APPENDIX B: THE BOUNDARY OF THE SET $\mathcal{E}(r)$

In Section 6 we argue heuristically that the boundary of the set $\mathcal{E}(r)$ satisfies the optimality equation at all points outside $\mathcal{N}$. To prove this result formally, consider an arbitrary point $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$. We will show that a tangent solution

[31]We have $\bar{Q} \geq 1$ because the matrices $Q_i(a)$ are idempotent and some of them are nonzero.

to the optimality equation through point w coincides with the boundary of $\mathcal{E}(r)$. Therefore, the curvature of the boundary of $\mathcal{E}(r)$ outside $\mathcal{N}$ is continuous and must satisfy the optimality equation.

PROPOSITION 5—Tangent Curves: *There is a unique tangent vector* $\mathbf{T}(w)$ *at any point* $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$. *Also, the curve* $\mathcal{C}$ *that solves* (36) *from initial conditions* $(w, \mathbf{T}(w))$ *coincides with the boundary of* $\mathcal{E}(r)$ *in a neighborhood of* w.

PROOF: The proof goes in two steps. First, we show that the curve $\mathcal{C}$ must not go outside the boundary of the set $\mathcal{E}(r)$ in a neighborhood of w. Otherwise, by altering initial conditions slightly, we would be able to find a curve $\mathcal{C}'$ that solves (36) and cuts through the boundary of the set $\mathcal{E}(r)$ as shown in Figure 15. Lemma 6 shows that this leads to a contradiction. It follows that the tangent vector is unique at any point $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$ since otherwise a tangent solution would go outside the boundary of $\mathcal{E}(r)$. Second, we show that the curve $\mathcal{C}$ does not enter the interior $\mathcal{E}(r)$. Otherwise, we would be able construct a PPE that achieves a value pair outside $\mathcal{E}(r)$, as shown in Lemma 8. The analysis relies heavily on Proposition 4, which shows that the solutions of (36) change continuously with initial conditions.

Suppose that there are points $v \in \mathcal{C}$ arbitrarily close to w that satisfy

$$\{v + x\mathbf{N}(w)^\top, x \geq 0\} \cap \mathrm{cl}\mathcal{E}(r) = \emptyset,$$

where $\mathrm{cl}\mathcal{E}(r)$ denotes the closure of $\mathcal{E}(r)$. Then, by adjusting initial conditions slightly, we can draw a curve $\mathcal{C}'$ that also solves (36) and cuts through a small portion of the boundary of $\mathcal{E}(r)$, as shown in Figure 15. The left panel shows that when the set $\mathcal{E}(r)$ has a kink at w, we can find $\mathcal{C}'$ by moving initial conditions inside the set. The right panel shows that when the set $\mathcal{E}(r)$ has a unique tangent at w, we can draw $\mathcal{C}'$ from the same point w but with a rotated angle. Moreover, we can perform these adjustments to guarantee that the curve $\mathcal{C}'$ satisfies all conditions of Lemma 6.[32] Lemma 6 shows that it is impossible to have such a curve $\mathcal{C}'$.

LEMMA 6: *It is impossible for a solution* $\mathcal{C}'$ *of* (36) *with endpoints* v_L *and* v_H *to satisfy the following properties simultaneously*

(i) *There is a unit vector* $\hat{\mathbf{N}}$ *such that* $\forall x > 0$, $v^L + x\hat{\mathbf{N}}^\top \notin \mathcal{E}(r)$ *and* $v^H + x\hat{\mathbf{N}}^\top \notin \mathcal{E}(r)$.

[32] Clearly, (i) must hold for the vector $\mathbf{N}(w)$ and (iii) holds by construction. Condition (ii) is easy to satisfy by the continuity of solutions in initial conditions if $\mathbf{N}(w)w > \max_{v_N \in \mathcal{N}} \mathbf{N}(w)v_N$. For the case when there is no kink at w, it may occur that $\mathbf{N}(w)w = \mathbf{N}(w)v_N$ for some $v_N \in \mathcal{N}$. Suppose that v_N is to the left of w. Then $\mathcal{C}$ cannot go above the boundary of $\mathcal{E}(r)$ to the left of w, so it must do so to the right of w. But then, by rotating the initial condition clockwise slightly, we get a curve $\mathcal{C}'$ as in the right panel of Figure 15, for which (ii) holds.

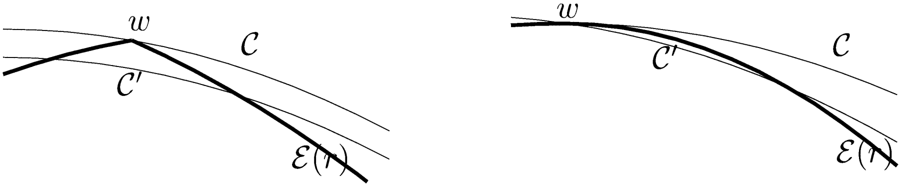

FIGURE 15.—Constructing a curve that cuts through $\mathcal{E}(r)$.

(ii) *For all* $w \in \mathcal{C}'$ *with an outward unit normal* $\mathbf{N}$, *we have*

$$\max_{v_N \in \mathcal{N}} \mathbf{N} v_N < \mathbf{N} w.$$

(iii) $\mathcal{C}'$ "*cuts through*" $\mathcal{E}(r)$, *that is*, *there exists a point* $v \in \mathcal{C}'$ *such that* $W_0 = v + x\hat{\mathbf{N}}^\top \in \mathcal{E}(r)$ *for some* $x > 0$.

PROOF: Suppose such a curve $\mathcal{C}'$ existed. Then there must be a PPE that achieves point $W_0 = v + x\hat{\mathbf{N}}^\top \in \mathcal{E}(r)$. Denote by W_t the continuation values in this PPE. We will show that such as PPE is impossible.

Consider the region $\mathcal{R}$ spanned by the "level curves" $\mathcal{C}'(x) = \mathcal{C}' + x\hat{\mathbf{N}}^\top$, $x \geq 0$, and let $f : \mathcal{R} \to \mathbb{R}$ be the *level function* defined by

$$v \in \mathcal{C}'(f(v)).$$

Let us show that $f(W_t)$ increases indefinitely with positive probability and, thus, W_t must escape from the set $\mathcal{V}$, leading to a contradiction. Note that (i) implies that W_t cannot escape from R through the sides $\{v_L + x\hat{\mathbf{N}}^\top\}$ and $\{v_H + x\hat{\mathbf{N}}^\top\}$.

To construct an escape argument, consider any moment of time t while W_t is still in $\mathcal{R}$, and let us find the drift and volatility of $f(W_t)$. Let us introduce a coordinate system (u_t, u_n) with the origin at W_t and axes parallel to the unit tangent and normal vectors $\mathbf{T}$ and $\mathbf{N}$ to the curve $\mathcal{C}'(f(W_t))$ at W_t. Then at W_t in these coordinates, f has the derivative vector

$$[\partial f/\partial u_t, \partial f/\partial u_n] = [0, (\mathbf{T}\hat{\mathbf{T}}^\top)^{-1}]$$

and the Hessian

$$\begin{bmatrix} \partial^2 f/\partial u_t^2 & \partial^2 f/(\partial u_t \, \partial u_n) \\ \partial^2 f/(\partial u_t \, \partial u_n) & \partial^2 f/\partial u_n^2 \end{bmatrix} = \begin{bmatrix} \kappa(\mathbf{T}\hat{\mathbf{T}}^\top)^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

where κ is the curvature of $\mathcal{C}'(f(W_t))$ at point W_t, which equals the curvature of $\mathcal{C}'$ at point $v = W_t - f(W_t)\hat{\mathbf{N}}^\top$.

If there is no public randomization, then the evolution of W in the **T–N** coordinates is given by

$$\begin{bmatrix} d(\mathbf{T}W_t) \\ d(\mathbf{N}W_t) \end{bmatrix} = r \begin{bmatrix} \mathbf{T}(W_t - g(A_t)) \\ \mathbf{N}(W_t - g(A_t)) \end{bmatrix} dt + r \begin{bmatrix} \mathbf{T}B_t \\ \mathbf{N}B_t \end{bmatrix} dZ_t.$$

Using Ito's lemma

$$\begin{aligned} df(W_t) &= \left(r \begin{bmatrix} 0 & \dfrac{1}{\mathbf{T}\hat{\mathbf{T}}^\top} \end{bmatrix} \begin{bmatrix} \mathbf{T}(W_t - g(A_t)) \\ \mathbf{N}(W_t - g(A_t)) \end{bmatrix} + \frac{1}{2} \frac{\kappa r^2}{\mathbf{T}\hat{\mathbf{T}}^\top} |\mathbf{T}B_t|^2 \right) dt \\ &\quad + r \begin{bmatrix} 0 & \dfrac{1}{\mathbf{T}\hat{\mathbf{T}}^\top} \end{bmatrix} \begin{bmatrix} \mathbf{T}B_t \\ \mathbf{N}B_t \end{bmatrix} dZ_t \\ &= \frac{r}{\mathbf{T}\hat{\mathbf{T}}^\top} \left(\mathbf{N}(W_t - g(A_t))\, dt + \frac{\kappa}{2} r |\mathbf{T}B_t|^2\, dt + \mathbf{N}B_t\, dZ_t \right). \end{aligned}$$

Let us show that the drift of $f(W_t)$ is greater than or equal to $rf(W_t) - Kr(\mathbf{T}\hat{\mathbf{T}}^\top)^{-1}|\mathbf{N}B_t|$ for an appropriate constant $K > 0$. We have

$$\mathbf{N}(W_t - g(A_t)) = \mathbf{N}\hat{\mathbf{N}}^\top f(W_t) - \mathbf{N}(g(A_t) - v). \tag{38}$$

If $\mathbf{N}(g(A_t) - v) \le 0$ (and by (ii), this is always the case when $a \in \mathcal{A}^N$), then the drift of $f(W_t)$ is greater than or equal to $rf(W_t)$. If $\mathbf{N}(g(A_t) - v) > 0$ (and thus $a \notin \mathcal{A}^N$), we have

$$\kappa \ge \frac{2\mathbf{N}(g(A_t) - v)}{r|\phi(A_t, \mathbf{T})|^2} \tag{39}$$

because $\mathcal{C}'$ solves (36). Therefore, the drift of $f(W_t)$ is greater than or equal to

$$\begin{aligned} &\frac{r}{\mathbf{T}\hat{\mathbf{T}}^\top} \left(\mathbf{N}\hat{\mathbf{N}}^\top f(W_t) - \mathbf{N}(g(A_t) - v) \left(1 - \frac{|\mathbf{T}B_t|^2}{|\phi(A_t, \mathbf{T})|^2} \right) \right) \\ &\quad \ge rf(W_t) - \frac{r}{\mathbf{T}\hat{\mathbf{T}}^\top} |\mathcal{V}| \frac{4\bar{Q}}{\bar{\Psi}} |\mathbf{N}B_t|, \end{aligned}$$

where we used Lemma 5.

Therefore, by Lemma 7 there exists a probability measure equivalent to the measure induced by A, under which the process $f(W_t)$ has drift greater than $rf(W_t)$. It follows that $f(W_t)$ would escape from $\mathcal{V}$ with positive probability, since $f(W_0) > 0$.

With public randomization, the expression for $df(W_t)$ would include an extra term—a submartingale orthogonal to Z_t—since f is a convex function. Again, Lemma 7 implies that $f(W_t)$ escapes from $\mathcal{V}$ with positive probability. *Q.E.D.*

If $\mathcal{E}(r) \neq \mathcal{N}$, then Lemma 6 implies that $\mathcal{E}(r)$ has nonempty interior. Indeed, if $\mathcal{E}(r)$ were an interval (or a point), its endpoints would have to be in $\mathcal{N}$ because the boundary of $\mathcal{E}(r)$ cannot have kinks outside $\mathcal{N}$.

LEMMA 7: *Let F be a process that satisfies*

$$F_0 > 0 \quad \text{and} \quad F_t = F_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dZ_s + G_t,$$

where $\mu_t \geq rF_t - K|\sigma_t|$ while $F_t \geq 0$ for some constant $K > 0$, Z is a Brownian motion, and G is a submartingale orthogonal to Z. Then there is an equivalent measure under which

$$F_t = F_0 + \int_0^t \mu_s'\, ds + \int_0^t \sigma_s\, dZ_s' + G_t,$$

where $\mu_t' \geq rF_t$ while $F_t \geq 0$ and, under the new measure, Z' is a Brownian motion and G is still a submartingale.

PROOF: By Girsanov's theorem, the density process

$$\zeta_0 = 1, \quad d\zeta_t = K\zeta_t \frac{\sigma_t}{|\sigma_t|}\, dZ_t$$

defines an equivalent probability measure Q' with the Brownian motion

$$Z_t' = Z_t - K\int_0^t \frac{\sigma_t}{|\sigma_t|}\, dt.$$

Novikov's condition holds since the process $K\sigma_t/|\sigma_t|$ is bounded. Thus,

$$F_t = F_0 + \int_0^t (\mu_s + K|\sigma_s|)\, ds + \int_0^t \sigma_s\, dZ_s' + dG_t,$$

where $\mu_t + K|\sigma_t| \geq rF_t$ while $F_t \geq 0$. The process G_t is still a submartingale because the density process ζ is orthogonal to G. *Q.E.D.*

Next, we need to prove that a solution to the optimality equation that is tangent to the boundary of $\mathcal{E}(r)$ at an arbitrary point $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$ does not enter the interior of $\mathcal{E}(r)$, denoted by $\mathcal{E}(r)^\circ$.

LEMMA 8—Tangent Curves Do Not Enter $\mathcal{E}(r)$: *Consider point $w \in \partial\mathcal{E}(r)\backslash\mathcal{N}$ with an outward unit normal vector $\mathbf{N}$. Then the curve $\mathcal{C}$, which solves (24) from initial conditions $(w, \mathbf{N})$, lies completely outside or on the boundary of the set $\mathcal{E}(r)$. It does not enter the interior of $\mathcal{E}(r)$.*

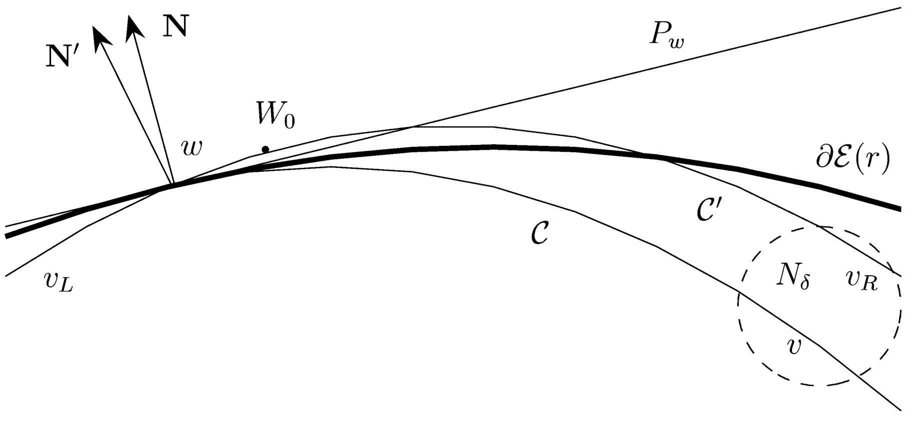

FIGURE 16.—Proof of Lemma 8.

PROOF: Suppose there is $v \in \mathcal{C} \cap \mathcal{E}(r)^\circ$ as shown in Figure 16. We will show how to construct a curve $\mathcal{C}'$ with two endpoints $v_L, v_R \in \mathcal{E}(r)$ and a point $W_0 \notin \mathcal{E}(r)$ between them.

Take a neighborhood N_δ around point v in the interior of $\mathcal{E}(r)$. Without loss of generality, assume that point v is found by moving in the clockwise direction from point w along the curve $\mathcal{C}$, as shown in Figure 16. Let us choose a normal vector $\mathbf{N}'$ by rotating $\mathbf{N}$ in the counterclockwise direction. Consider the curve $\mathcal{C}'$ that solves the optimality equation from initial conditions $(w, \mathbf{N}')$. From the continuity of solutions of the optimality equation in initial conditions, if $\mathbf{N}'$ is sufficiently close to $\mathbf{N}$, then the curve $\mathcal{C}'$ will enter the neighborhood N_δ of v. Because $\mathbf{N}'$ is rotated counterclockwise relative to $\mathbf{N}$, the curve $\mathcal{C}'$ will pass above the line P_w tangent to $\mathcal{E}(r)$ at w between w and N_δ. Because there is a unique tangent line P_w at point w, as argued earlier, the curve $\mathcal{C}'$ will enter the interior of $\mathcal{E}(r)$ in the counterclockwise direction from w. Therefore, we can choose $W_0 \notin \mathcal{E}(r)$ that is between points v_L and $v_R \in \mathcal{E}(r)^\circ$ on the curve $\mathcal{C}'$, as shown in Figure 16. By Proposition 3, there is a PPE that achieves the value pair W_0, so $W_0 \in \mathcal{E}(r)$, a contradiction. We conclude that the curve $\mathcal{C}$ cannot enter the interior of the set $\mathcal{E}(r)$. *Q.E.D.*

This concludes the proof of the Proposition 5. *Q.E.D.*

COROLLARY 2: *The set $\mathcal{E}(r)$ has a strictly positive curvature at all points $w \in \partial\mathcal{E}(r) \backslash \mathcal{N}$.*

PROOF: If a tangent solution of (36) had zero curvature at point $w \in \partial\mathcal{E}(r) \backslash \mathcal{N}$, then it must be a straight line. Since $\mathcal{E}(r)$ is a bounded set, Proposition 5 implies that this solution must reach the set $\mathcal{N}$ on both sides of w. But then $w \in \mathcal{N}$, a contradiction. *Q.E.D.*

REFERENCES

ABREU, D., P. MILGROM, AND D. PEARCE (1991): "Information and Timing in Repeated Partnerships," *Econometrica*, 59, 1713–1733. [1288]

ABREU, D., D. PEARCE, AND E. STACCHETTI (1986): "Optimal Cartel Equilibria with Imperfect Monitoring," *Journal of Economic Theory*, 39, 251–269. [1302]

——— (1990): "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica*, 58, 1041–1063. [1286]

DEMARZO, P., AND Y. SANNIKOV (2006): "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *Journal of Finance*, 51, 2681–2724. [1318]

FAINGOLD, E., AND Y. SANNIKOV (2007): "Reputation Effects and Degenerate Equilibria in Continuous-Time Games," Unpublished Manuscript, Yale University. [1293]

FUDENBERG, D., AND D. LEVINE (1994): "Efficiency and Observability with Long-Run and Short-Run Players," *Journal of Economic Theory*, 62, 103–135. [1286,1302]

FUDENBERG, D., D. LEVINE, AND E. MASKIN (1994): "The Folk Theorem with Imperfect Public Information," *Econometrica*, 62, 997–1039. [1286]

GREEN, E., AND R. PORTER (1984): "Noncooperative Collusion under Imperfect Price Information," *Econometrica*, 52, 87–100. [1302]

HAUSER, C., AND H. HOPENHAYN (2004): "Trading Favors: Optimal Exchange and Forgiveness," Unpublished Manuscript, University of California, Los Angeles. [1303]

HOLMSTROM, B., AND P. MILGROM (1987): "Aggregation and Linearity in the Provision of Intertemporal Incentives," *Econometrica*, 55, 303–328. [1288]

KANDORI, M. (1992): "The Use of Information in Repeated Games with Imperfect Monitoring," *Econometrica*, 62, 997–1040. [1288]

KANDORI, M., AND I. OBARA (2006): "Efficiency in Repeated Games Revisited: The Role of Private Strategies," *Econometrica*, 74, 499–519. [1293]

KARATZAS, I., AND S. SHREVE (1991): *Brownian Motion and Stochastic Calculus*. New York: Springer-Verlag. [1297]

MAILATH, G. J., AND L. SAMUELSON (2006): *Repeated Games and Reputations: Long-Run Relationships*. New York: Oxford University Press. [1286,1302]

SANNIKOV, Y. (2004): "A Continuous-Time Version of the Principal–Agent Problem," Unpublished Manuscript, University of California, Berkeley. [1288]

SANNIKOV, Y., AND A. SKRZYPACZ (2006): "The Role of Information in Repeated Games with Frequent Actions," Unpublished Manuscript, Graduate School of Business, Stanford. [1302]

——— (2007): "Impossibility of Collusion under Imperfect Monitoring with Flexible Production," *American Economic Review* (forthcoming). [1303]

SIMON, L., AND M. STINCHCOMBE (1989): "Extensive Form Games in Continuous Time: Pure Strategies," *Econometrica*, 57, 1171–1214. [1288]

WILLIAMS, N. (2004): "On Dynamic Principal–Agent Problems in Continuous Time," Unpublished Manuscript, Princeton University. [1288]