Mission d'occulatation lunaire

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1 introduction

2 zone d'observation

2.1 calcul de la zone exacte

during the study of these problems, we will take the earth as the orgin in cartesian coordinate. The axis Earth-Sun at time t0 will be defined as the X axis and the axis X and Y define the earth ecliptic plane.

for now we fix the time at (0,0)

a point is in the observation zone if the sun is totally hidden and the crown is visible around the occulting body. This zone has the shape of a double cone that we can represent by the rotation of a triangle.

the triangle could be defined by its three points P_1,P_2 et P_3, with as coordinates $(P_{1x}, 0, 0)$, $(P_{2x}, P_{2y}, 0)$,$(P_{3x}, 0, 0)$

the value of P_{1x} and P_{3x} can be easily computed usingt the Thales theorem.

$$P_{1x} = \frac{\bar{D}R_l}{R_s - R_l}$$

$$P_{3x} = \frac{\bar{D}R_l}{R_s(\alpha + 1) - R_l}$$

where $\bar{D}$ is the Solar-Moon distance, R_s the Solar Radius ,R_l the Lunar radius and α is the solar crown's added size in solar radius (the more *alpha* is near of zero the more the solar crown's radius is near of the solar raidus meaning that we will observe part of the crown near of the solar surface)

we can then compute the position of P_2 by computing the intersection of the lines linking the surface of the Moon and the points P_1 and P_3

$$P_{2x} = \frac{P_{1x}tan(\theta_1) + P_{3x}tan(\theta_3)}{tan(\theta_1) + tan(\theta_3)}$$

$$P_{2y} = tan(\theta_1)(P_{1x} - P_{2x})$$

with

$$\theta_1 = \sin^{-1}\left(\frac{R_l}{P_{1x}}\right)$$

$$\theta_3 = \sin^{-1}\left(\frac{R_l}{P_{3x}}\right)$$

with $\alpha = 0.05$

2.2 approximation of the zone

the Position of the points P_1,P_2 and P_3 relative to the moon will change depending of the moon position around the earth, hovewer the shape of the the zone in itself will not change much (with a dimension error smaller than 0.5%)

we can obtain a good approximation of the observation zone at any position of the moon around the Earth by computing the shape of the observation zone with the moon at the origin and then move it and rotate it to place it at the right position.

We name $\hat{P}_1$, $\hat{P}_2$ and $\hat{P}_3$ the points of the observation zone when the moon is at the origin. the points P_1 and P_3 are collinear with the vector R_{ls} witch are the position of the Moon relative to the Sun.

Also the points $\hat{P}_1$ $\hat{P}_3$ are colinear with the X axis. We can conclude of the following formula of the points P_3 and P_1

$$P_3 = R_{lt} + \hat{P_{3x}} D^{-1} R_{ls} \tag{1}$$

$$P_1 = R_{lt} + \hat{P_{1x}} D^{-1} R_{ls} \tag{2}$$

a point D is consider in the observation zone if the following inequality are verified :

$$\begin{aligned} ||a|| &< ||b||p_1 \\ ||a|| &< O - ||b||p_2 \end{aligned} \tag{3}$$

with b being the projection of $S - P_3$ on the axis Sun$_M oon$,a=S-P$_3 - b$. p_1, p_2 and O are real value: here this parameter will be approximated by their value when the Moon is at the origin :

$$\begin{aligned} p_1 &= \frac{P_2 y}{P_2 x - P_3 x} \\ p_2 &= \frac{P_2 y}{P_1 x - P_2 x} \\ O &= P_2 y + (P_2 x - P_3 x) p_2 \end{aligned}$$

if we want to increase the precision of the observation zone we can take in account the variation in length of the observation zone by scaling the parameter b by

$\frac{||\hat{P}_1 - \hat{P}_3||}{||P_1 - P_3||}$

2.3 numeric application

we have the following distances :

$$R_l = 1.7374 \times 10^6 m$$
$$R_s = 6.955 \times 10^8 m$$
$$D_{sl} = 1.496 \times 10^{11} m$$
$$\Delta_D = 2D_{tl} = 7.69496 \times 10^8 m$$

on obtient les valeur suivantes pour les points de la zone pour $\alpha = 0.05$

$$P_{3x} = 3.577 \times 10^8 m$$
$$P_{1x} = 3.758 \times 10^8 m$$
$$P_{1x} - P_{3x} = 1.7928 \times 10^7 m$$
$$P_{2x} = 3.655 \times 10^8 m$$
$$P_{2y} = 4.248 \times 10^4 m$$

avec

$$\Delta_{P_{3x}} = 1.835 \times 10^6 m$$
$$\Delta_{P_{1x}} = 1.927 \times 10^6 m$$
$$\Delta_{P_{1x}-P_{3x}} = 9.198 \times 10^4 m$$
$$\Delta_{P_{2x}-P_{3x}} = 4.487 \times 10^4 m$$
$$\Delta_{P_{2x}} = 1.880 \times 10^6 m$$
$$\Delta_{P_{2y}} = 4.935 \times 10^{-3} m$$

on observe qu'en utilisant l'approximation définie plus tôt, on néglige les variation de l'ordre du millier de kilomètres ($10^6 m$). et en ajoutant l'optimisation supplémentaire sur la valeur de b on néglige également les variation de l'ordre de la dizaine de kilomètre ($10^4 m$) ne laissant que les erreurs de l'ordre de quelques kilomètre sur la position de $P_2 x$ ce qui ne change pas grand chose compte tenu de la grande longueur de la zone par rapport à son épaisseur (environ $20000 km$ contre $100 km$)

3 problème

on va considérer le problème suivant:

la lune suit une orbite circulaire autour de la Terre de rayon $a = 384000 km$.

la forme de la zone d'observation de la Lune est considérée comme étant égale à la zone d'observation de la lune si elle se trouvait à l'origine (la position de la terre). La position du point P_3 est déterminé par la formule suivante: avec

$$P_3(R_l) = R_{lt} + \hat{P_{3x}} D^{-1} R_{ls}$$

avec $\hat{P_3}$ étant la position du point P_3 quand la lune est à l'origine ,R_{lt} est la position de la lune relativement à la Terre. et R_{ls} est la position de la Lune relativement au Soleil.

(on a $R_{ls} = R_{lt} + D\hat{x}$);

le but est de trouver des orbites Kepleriennes qui effectue des observations répété et les plus longues possibles.

Les temps d'observation peuvent beaucoup varier allant d'une durée de quelques minutes à plusieurs heures. dans la suite on va donc se concentrer sur une seule observation.

afin de pouvoir reproduire les observations, il vaut mieux prendre une période d'orbite qui est un multiple de celle de la Lune.

de ce fait on peut déterminer le demi grand axe du satellite avec la formule suivante:

$$a_s = a_l k^{\frac{2}{3}}$$

avec

$$P_s = kP_l$$

étant donné que l'objectif est de faire une observation, on peut faire partir le satellite directement de la zone d'observation.

De plus la dimension de la zone étant très étirée (environ $10000km \times 100km \times 100km$) on peut considérer que le satellite coupera forcement le segment $[P_3, P_1]$, on peut donc décrire la position initiale du satellite à l'aide de l'anomalie vraie de la lune ν et un scalaire λ entre 0 et 1. la position initiale du satellite devient :

$$S_0 = \lambda(P_1 - P_3) + P_3(R_l(\nu))$$

avec

$$R_l(\nu) = \begin{bmatrix} r(\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos\ i) \\ r(\sin\Omega\cos\theta - \cos\Omega\sin\theta\cos\ i) \\ r\sin\theta\sin\ i \end{bmatrix}$$

avec $\theta = \nu + \omega$

pour l'instant on est en 2D donc l'équation se simplifie par:

$$X_l(\nu) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{bmatrix}$$

Maintenant que l'on connait la position du satellite on peut déterminer sa vitesse à l'aide de la formule suivante:

$$||\dot{S}(0)|| = \sqrt{\frac{2\mu}{||S_0||} - \frac{\mu}{a_s}}$$

on peut ensuite determiner l'orientation de la vitesse initiale avec deux angles θ_s et ϕ_s.

la dynamique du satellite et de la lune doivent être calculé pour calculer le temps de l'observation.

on à la dynamique suivante :

$$\begin{bmatrix} \dot{\overrightarrow{R_s}} \\ \dot{\overrightarrow{V_s}} \\ \dot{\overrightarrow{R_l}} \\ \dot{\overrightarrow{V_l}} \end{bmatrix} = \begin{bmatrix} \overrightarrow{V_s} \\ \frac{-\mu}{||R_s||^3}\overrightarrow{R_s} \\ \overrightarrow{V_l} \\ \frac{-\mu}{||R_l||^3}\overrightarrow{R_l} \end{bmatrix}$$

avec comme condition initiale :

$$\begin{bmatrix} \overrightarrow{R_{s0}} \\ \overrightarrow{V_{s0}} \\ \overrightarrow{R_{l0}} \\ \overrightarrow{V_{l0}} \end{bmatrix} = \begin{bmatrix} \lambda\left(\widehat{P_1} - \widehat{P_3}\right) + R_{lt} + \frac{\widehat{P_{3x}}}{D} R_{ls} \\ \sqrt{\mu\left(\frac{2}{||R_{s0}||} - 2\right)}\widehat{v_0}(\theta_s, \phi_s) \\ r(\nu)\begin{bmatrix} (\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos\ i) \\ (\sin\Omega\cos\theta + \cos\Omega\sin\theta\cos\ i) \\ \sin\theta\sin\ i \end{bmatrix} \\ \sqrt{\frac{\mu}{p}}\begin{bmatrix} -\cos\Omega(\sin\theta + e\ \sin\omega) - \sin\Omega(\cos\theta + e\ \cos\omega)\cos\ i \\ -\sin\Omega(\sin\theta + e\ \sin\omega) + \cos\Omega(\cos\theta + e\ \cos\omega)\cos\ i \\ (\cos\theta + e\ \cos\omega)\sin\ i \end{bmatrix} \end{bmatrix}$$

la dynamique devra être simulée après et avant l'état initial pour trouver l'instant où l'objet entre et sort de la zone.

la fonctions d'objectif est définit comme suit :

$$\int_{-\tau_d}^{\tau_u} dx = \tau_d + \tau_u$$

où τ_d est l'instant où l'objet rentre dans la zone d'observation et τ_u l'instant où l'objet en sort.

l'équation utilisé pour vérifier si l'objet est dans la zone est l'équation (2).

les paramètre de contrôle sont :

- la position de la lune ν qui définit la position de la zone d'observation.
- la position initiale λ de l'objet dans la zone d'observation qui est simplifier par un segment allant de P_3 à P_1.
- les angles θ_s et ϕ_s qui définissent l'orientation de la vitesse de l'objet.

on a donc un espace de dimension 4 : $(\nu, \lambda, \theta_s, \phi_s) = [0, 2\pi] \times [0, 1] \times [0, 2\pi] \times [0, \pi]$

l'espace est contraint mais étant donné que la plupart des dimension sont des angles et qu'il suffit de donner un score de 0 si λ est en dehors du domaine on peut considéré que l'espace est égal à $\mathbb{R}^4$ pour avoir un problème sans contrainte.

4 Result

after studying the possible optimum it is shown that random initial condition tend to give obervation time of a few minutes, however observation that happen when the angle form by the Sun, Earth and Moon aproach 60° tend to be way higher .

When we are working with a simple 2D problem with circular Moon orbit , there are an optimum near the value $(pi/3, 0.5, 38/45\pi, 0)$ that give observation time of almost 20h.

we managed to get observation time this long because the speed of the object and the Moon are equivalent. Meaning that the device can stay in the observation zone for a long time.

The satellite describe a loop inside the observation zone, meaning that it is possible to get two fairly long observation very near to each other if the tip of the loop is outside of the observation zone. Even if there is a good chance that these solution are less efficient that a solution with the entire loop inside the observation zone, i could be usefull to improve the objective function to detect when the object reenter the observation zone after a short time.

we can easily determine that there are two point in the lunar orbit where we can obtain very long observation time:

The satellite can have the same speed vector as the moon only if the following equation is verified :

$$\sqrt{\frac{2\mu}{R_s} - \frac{\mu}{a_s}} = \sqrt{\frac{2\mu}{R_l} - \frac{\mu}{a_l}}$$

Considering that the moon's speed is constant (because of its small eccentricity) and that the period of the device is the same as the moon (ie $a_s = a_l$), we get the relation :

$$R_s = R_l$$

as the satellite is in the obsevation zone, the region in which the satelite can cross the observation zone and at the same speed as the moon is the intersection between the possible observation zone space (which is in our case simplified by an ellipse) and a sphere of radius R with $R = R_l$ in this case.

There are two point in the orbit that satisfy these condition

Using the solution that we found in 2D we can try to use it as an initial condition for 3D problems, the solution found with varying Ω value and considering the eccentricity of the Moon tend to show that there are always a solution near this point.

5 low thrust transfert

now that we know that the best configuration to make observation are with no relative speed, it mean that we can compare the different Δv needed to transfert from one observation to another.

the goal of this computing is to get an idea of which subset of observations could be done (as it is very likely that we won't be able to make the device attend all the observations).

For now we will consider the simulation over one year, with the earth at the origin, the sun moving around the earth along a perfect circle with a radius of one UA at a constant speed and. The moon moving allong a keplerian orbit with the following component :

$$\begin{aligned}\Omega &= 0\\ \omega &= 0\\ i &= 5\\ e &= 0.054\\ \nu &= 0\end{aligned}$$

The value of ν correspond to the true anomaly of the moon at time $t = 0$. the value of ω and Ω are set at 0 for now but we will test other value later.

we can achieve zero velocity observation if the velocity of the device at the observation zone ($\sqrt{\frac{2\mu}{R_s} - \frac{\mu}{a_l}}$) is the same as the velocity of the observation zone. As the position of the point P_{2x} is given by the following formula :

$$P_{2x} = R_l + \frac{D_{p2}}{D_{ts}}(R_s + R_s)$$

where D_{p2} is the distance between the moon and the point P_{2x} where the moon is at the origin. so we have:

$$\dot{P_{2x}} = \dot{R_l} + \frac{D_{p2}}{D_{ts}}(\dot{R_s} + \dot{R_s})$$

we can get from this the following payoff formula :

$$D_{obs} = \left(||\dot{P_{2x}}|| - \sqrt{\frac{2\mu}{P_{2x}} - \frac{\mu}{a_l}} \right)^2$$

ploting the value of this function over one year give the follwoing graph :

we can see that we have a almost a symetric and a periodic function, this mean that we just need to find the first optimum and then we could compute good initial value to find the other optimum of the function using the follwoing formulas :

$$\begin{aligned}O_{2ni} &= nT - O_1 + a\\ O_{2n+1i} &= nT + O_1 + a\end{aligned}$$

where T is the synodic period of the Moon, O_{ki} is the initial value used to find the k^{th} optimum of the function using a simple gradient descent method O_1 is the value of the first optimum of the function relative to a.

and a is the first time the Sun-Earth-Moon system align (when projected on the ecliptic plane of earth) in this order, for now the value a is equal to zero.

5.1 low thrust model

we want to move from a point R_i at time t_i to a point R_f at time t_f with initial velocity V_i and final velocity V_f.

the device can be control with a low thrust engine that can generate an acceleration of maximum magnitude A_{max} in any direction, as the final objective is to use a solar sail, the variation of mass is neglected.

the passive dynamic of the device is the following :

$$\begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ \sum_{\beta \in B} \frac{\mu_\beta}{\|r_\beta - r\|^3}(r_\beta - r) \end{bmatrix}$$

where B is the set of every bodies that attract the device (for now the two body are The Earth and the Moon). the dynamics of the bodies must be computed before (we suppose that the device does not influence the dynamics of the system).

the acceleration of the thruster can be take in account by adding a vector the equations of speed :

$$f(x,u) = \dot{x} = \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ u + \sum_{\beta \in B} \frac{\mu_\beta}{\|r_\beta - r\|^3}(r_\beta - r) \end{bmatrix}$$

we want to minimize the Δ_v of the transfert, so the running payoff is :

$$r(x,u) = -\|u\|$$

we also want to be at the right position and velocity at the time t_f so the terminal payoff is :

$$g(x(t_f)) = -(r(t_f) - R_f)^2 - (v(t_f) - V_f)^2$$

we get the following hamiltonian :

$$H(x,p,u) = \begin{bmatrix} v \\ u + \sum_{\beta \in B} \frac{\mu_\beta}{\|r_\beta - r\|^3}(r_\beta - r) \end{bmatrix}^t \cdot p - \|u\|$$

using the Pontryagin maximum principle theorem we can compute the differential of $p = \begin{bmatrix} p_r \\ p_v \end{bmatrix}$

$$\begin{aligned} \dot{p} &= -\nabla_x H \\ &= -\begin{bmatrix} 0 & I_3 \\ \sum_{\beta \in B} \frac{\mu_\beta}{\|r_\beta - r\|^3}\left(\frac{3(r_\beta - r)\cdot(r_\beta - r)^t}{\|r_\beta - r\|^2} - I_3\right) & 0 \end{bmatrix}^t \cdot p \\ &= \begin{bmatrix} \sum_{\beta \in B} \frac{\mu_\beta}{\|r_\beta - r\|^3}\left(I_3 - \frac{3(r_\beta - r)\cdot(r_\beta - r)^t}{\|r_\beta - r\|^2}\right)\cdot p_v \\ -p_r \end{bmatrix} \end{aligned}$$

we know that $\forall t\ ,\ u = \arg\max_{u \in B(0,A_{max})} H(x,p,u)$. We can rewrite the hamiltonian as follow :

$$\begin{aligned} H(x,p,u) =& C(x,p) + u \cdot p_v - \|u\| \\ =& C(x,p) + \|u\|(\|p_v\| \cos(\alpha_{\widehat{up_v}}) - 1) \end{aligned}$$

with this form we can easily conclude that

$$\begin{cases} \|u\| = A_{max} & \text{if } \|p_v\| \cos(\alpha_{\widehat{up_v}}) \geq 1 \\ \|u\| = 0 & \text{else} \end{cases}$$

we can then observe that $\|p_v\| \cos(\alpha_{\widehat{ap_v}})$ cannot be greater than 1 if $\|p_v\| < 1$. Also if $\|p_v\| \geq 1$, it is obvious that taking $\alpha_{\widehat{up_v}} = 0$ (i.e. u and p_v are collinear) maximize the factor $\|p_v\| \cos(\alpha_{\widehat{uap_v}})$. So we can conclude the following formula :

$$\begin{cases} u = A_{max}\widehat{p_v} & \text{if } \|p_v\| \geq 1 \\ u = 0 & \text{else} \end{cases}$$

to solve the problem numerically, we then just have to find the correct value p_i so the terminal value of x correspond to the target value.

6 result low thrust

to compute the solution, i've used the julia library : OptimalControl,

the starting point of the device is the point where the starting observation end and the ending point is the point where the ending observation start.

To simplify the next explication i will separate the observations into two category : the even observations and the odd observation. The odd observations are the first observation that happen on the moon orbit and the even one are the second one.

The algorithm had a lot of struggle to find an optimum for moving to two adjacent observations, which was excpected as the two orbit of adjacent optimum are very different and the time interval is small meaning that important Δ_v are needed to move from the starting point to the ending point. hovewer the algorithm find solution for transition between even observation and between odd observation. This was also excpected as the orbit are generally very similar and the device as a time of around one synodic period (around 29,5 days) to change its trajectory.

the maximum acceleration found in such transition are around $180\mu m \cdot s^{-2}$

About the trajectiory in itself, the maximum alowed acceleration was around 5×10^{-4}.