Ising Model Cheatsheet

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Overview

The **Ising model** is a mathematical model of ferromagnetism in statistical mechanics, consisting of discrete variables representing magnetic dipole moments of atomic spins that can be in one of two states (+1 or -1).

Basic Definitions

Symbol	Meaning
s_i	Spin at site i (± 1)
J	Coupling constant (interaction strength)
h	External magnetic field
k_B	Boltzmann constant
T	Temperature
β	Inverse temperature $= 1/(k_B T)$
N	Number of spins

Hamiltonian

General Form

Equation	Description
$H = -J\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i$	Full Hamiltonian with nearest-neighbor interactions and external field
$H = -J\sum_{\langle i,j\rangle} s_i s_j$	Zero-field Hamiltonian

Key points: - $\langle i,j\rangle$ denotes nearest-neighbor pairs - $J > 0$: ferromagnetic (parallel spins favored) - $J < 0$: antiferromagnetic (antiparallel spins favored)

Statistical Mechanics

Partition Function and Energy

Quantity	Formula	Description
Partition function	$Z = \sum_{\{s\}} e^{-\beta H}$	Sum over all possible spin configurations
Free energy	$F = -k_B T \ln Z$	Thermodynamic potential
Energy	$\langle E\rangle = -\frac{\partial \ln Z}{\partial \beta}$	Average energy
Probability	$P(\{s\}) = \frac{e^{-\beta H(\{s\})}}{Z}$	Boltzmann distribution

Order Parameters and Observables

Observable	Formula	Physical Meaning
Magnetization	$M = \frac{1}{N}\sum_i s_i$	Average spin per site
Susceptibility	$\chi = \beta N(\langle M^2\rangle - \langle M\rangle^2)$	Response to external field
Specific heat	$C = k_B\beta^2 N(\langle E^2\rangle - \langle E\rangle^2)$	Heat capacity fluctuations
Correlation function	$G(r) = \langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle$	Spin correlations at distance r

Critical Behavior (2D Square Lattice)

Exact Results (Onsager Solution)

Property	Value/Formula	Notes
Critical temperature	$T_c = \frac{2J}{k_B \ln(1+\sqrt{2})} \approx 2.269J/k_B$	Phase transition point
Critical exponents	$\alpha = 0$ (log), $\beta = 1/8$, $\gamma = 7/4$, $\nu = 1$	Exact values for 2D
Spontaneous magnetization	$M(T) \sim (T_c - T)^\beta$ for $T < T_c$	Order parameter near T_c
Susceptibility	$\chi(T) \sim \|T - T_c\|^{-\gamma}$	Diverges at T_c

Monte Carlo Simulation

Metropolis Algorithm

Step	Action	Formula
1	Select random spin s_i	-
2	Calculate energy change	$\Delta E = 2s_i(J\sum_{\text{neighbors}} s_j + h)$
3	Accept flip if	$\Delta E < 0$ or $\text{rand}(0,1) < e^{-\beta\Delta E}$
4	Update configuration	Flip spin if accepted

Note: One Monte Carlo step (MCS) = N attempted flips

Other Algorithms

Algorithm	Key Feature
Wolff/Swendsen-Wang	Cluster flipping; reduces critical slowing down
Heat bath	Direct sampling from conditional probability

Dimensionality Effects

Dimension	T_c	Phase Transition
1D	0	No finite-T transition (Peierls argument)
2D	> 0	Continuous transition (Onsager)
3D	$\approx 4.511 J/k_B$	Continuous transition
Mean field ($d \to \infty$)	zJ/k_B	First-order approximation

Mean Field Approximation

Equation	Description
$M = \tanh(\beta z J M + \beta h)$	Self-consistency equation (z = coordination number)
$T_c^{MF} = zJ/k_B$	Mean field critical temperature
$M(T) \sim (T_c - T)^{1/2}$	Mean field magnetization (incorrect exponent)

Limitation: Overestimates T_c and gives incorrect critical exponents

Finite Size Effects

Effect	Scaling	Notes
Rounded transition	Width $\sim L^{-1/\nu}$	Transition smears out
Shifted T_c	$T_c(L) - T_c(\infty) \sim L^{-1/\nu}$	Apparent T_c moves
Finite magnetization	$M(T_c, L) \sim L^{-\beta/\nu}$	Non-zero at pseudo-T_c

L = linear system size, ν = correlation length exponent

Quick Reference Formulas

Energy per Spin (2D, $h = 0$)

$$\langle E \rangle / N = -J \langle s_i s_j \rangle_{\mathrm{nn}}$$

Magnetization (with field)

$$M = \frac{\partial F}{\partial h}\bigg|_T$$

Correlation Length

$$\xi(T) \sim |T - T_c|^{-\nu}$$

Near T_c, correlations extend to $\xi \to \infty$