Metric Space

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1 Metric Space

1.1 Definition

A metric space is a pair (X,d), where X is a set and d is a metric on X (or *distance function* on X), that is, a function defined on $X \times X$ such that for all x,y,z $\in X$ we have:

1. d is real-valued, finite and nonnegative.
2. d(x,y) = 0 **if and only if** x = y.
3. d(x,y) = d(y,x)
4. d(x,y) $\leq$ d(x,z) + d(z,y)

1.2 Subspace

A **subspace** $(Y,\widetilde{d})$ of (X,d) is obtained if we take a subset $Y \subset X$ and restrict d to $Y \times Y$ is the restriction

$$\widetilde{d} = d|_{Y \times Y}$$

$\widetilde{d}$ is called the metric induced on Y by d.

1.3 Examples

Real line R	$d(x,y) = \lvert x - y \rvert$
Euclidean space R^n, unitary space C^n, complex C	$d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + \cdots + (\xi_n - \eta_n)^2}$
Sequence space l^∞(bounded)	$d(x, y) = \sup\limits_{j \in N} \lvert \xi_j - \eta_j \rvert$
Function space C[a,b].(continuous)	$d(x, y) = \max\limits_{t \in J} \lvert x(t) - y(t) \rvert$
Discrete metric space	$d(x, x) = 0$, $d(x, y) = 1$ $(x \neq y)$
Sequence space s	$d(x, y) = \sum\limits_{j=1}^{\infty} \frac{1}{2^j} \frac{\lvert \xi_j - \eta_j \rvert}{1 + \lvert \xi_j - \eta_j \rvert}$
Space B(A) of bounded functions	$d(x, y) = \sup\limits_{t \in A} \lvert x(t) - y(t) \rvert$
Space l^p	$d(x, y) = (\sum\limits_{j=1}^{\infty} \lvert \xi_j - \eta_j \rvert^p)^{1/p}$

2 Open set, Closed set, Neighborhood

2.1 Ball and Sphere

Definition: Given a point $x_0 \in X$ and a real number $r > 0$, we define three types of set:

(a) $B(x_0; r) = \{x \in X | d(x, x_0) < r\}$ (Open shell)

(b) $\widetilde{B}(x_0; r) = \{x \in X | d(x, x_0) \leq r\}$ (Closed shell)

(c) $S(x_0; r) = \{x \in X | d(x, x_0) = r\}$ (Sphere)

We see that an open ball of radius r is the set of all points in X whose distance from the center of the ball is less than r. Furthermore, the definition immediately implies that

$$S(x_0; r) = \widetilde{B}(x_0; r) - B(x_0; r)$$

2.2 Open set and Closed set

A subset M of a metric space X is said to be ***open*** if it contains a ball about each of its points.
A subset K of X is said to ***closed*** if its complement (in X) is open, that is, $K^C = X - K$ is open.
An open ball $B(x_0; \epsilon)$ of radius ϵ is often called an ϵ ***- neighborhood*** of x_0.