Quality factor of 1 dof nonlinear damped pendulum
(Computational Study)

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1 Problem Statement

What is the effect of velocity dependent nonlinear damping force defined as $F = -bv|v|^{n-1}$ on quality factor of the pendulum. Quality factor is defined as $Q = 2\pi \times \frac{\text{Energy stored}}{\text{Energy loss per cycle}}$.

2 Introduction

Why is nonlinear damping considered here? Because quality factor is defined for Linear systems, and this study aims to find up to what extent physical interpretation of quality factor works in nonlinear damping.

The nonlinear damping force is defined as $F = -bv|v|^{n-1}$, why is it defined this way?, Because damping in system resists the motion, and hence we model the damping force in such a way that it resists the motion considered. Why not defined as $F = -bv^n$?, because when n is odd it is fine, but when n is even then irrespective of the sign of velocity, the force will always act in one direction and will not resist the motion the way it is supposed to, and hence damping cannot be modeled as that way. Also here || is a modulus function.

Quality factor is a way to measure the energy decay, lesser the energy decay more the quality factor, we can call it as a performance parameter of the system. There are various definitions of quality factor, the one which will be used in this study is:

$$Q = 2\pi \times \frac{\text{Energy stored}}{\text{Energy loss per cycle}}$$

When we say energy stored, we are talking about the initial energy present in the cycle considered to find the energy loss. Hence one can define quality factor also as:

$$Q = 2\pi \times \frac{\text{Energy stored at the start of the cycle}}{\text{Energy loss in that same cycle}}$$

The equation considered here are nonlinear and hence elementary analytical solution is not possible, we solve the equations numerically. For numerical computations, MATLAB is used.

3 Assumptions

1) Ideal pivot: no hinge forces or resisting torque.
2) It's motion is always in one plane with a single degree of freedom which is the angle it makes with the vertical.
3) No electromagnetic forces.
4) No relativistic effects.

4 Theory

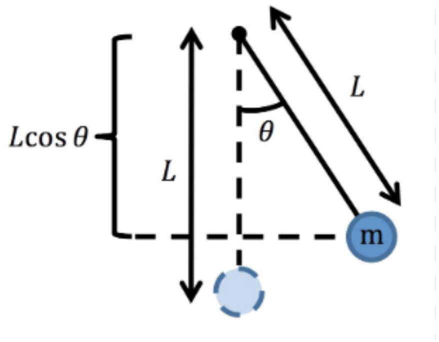

Figure 1: Pendulum with 1 dof

As we have defined $F = -b|v|^{n-1}v$, let's express it in θ

We have $v = l\dot{\theta}$, therefore $F = -bl^n|\dot{\theta}|^{n-1}\dot{\theta}$
We will use Euler Lagrangian equations for finding equations of motion.

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \quad \text{and} \quad V = mgl(1 - \cos\theta)$$

Lagrangian of the system is:

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \tag{1}$$

For generalized force we will use the fact that the work done remains same in both coordinate system.

$$F \cdot dr = Q \cdot dq, \quad F = -bl^n|\dot{\theta}|^{n-1}\dot{\theta}, \quad dr = ld\theta \text{ and } dq = d\theta \implies Q = -bl^{n+1}|\dot{\theta}|^{n-1}\dot{\theta}$$

Following is the Euler Lagrangian equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_i$$

Solving that will give us the equation of motion:

$$ml^2\ddot{\theta} + mgl\sin\theta = -bl^{n+1}|\dot{\theta}|^{n-1}\dot{\theta}$$

$$ml^2\ddot{\theta} + bl^{n+1}|\dot{\theta}|^{n-1}\dot{\theta} + mgl\sin\theta = 0$$

Rearranging it will give us:

$$\ddot{\theta} + 2\left(\frac{bl^n}{2m\sqrt{gl}}\right)\left(\sqrt{\frac{g}{l}}\right)|\dot{\theta}|^{n-1}\dot{\theta} + \left(\sqrt{\frac{g}{l}}\right)^2 \sin\theta = 0$$

And we introduce new variables here as:

$$\zeta = \frac{bl^n}{2m\sqrt{gl}} \text{ and } \omega_n = \sqrt{\frac{g}{l}}$$

The equation of motion finally we have is:

$$\ddot{\theta} + 2\zeta\omega_n|\dot{\theta}|^{n-1}\dot{\theta} + \omega_n^2\sin\theta = 0 \tag{2}$$

If we dimensionally analyze the zeta parameter, it's dimensions would be:

$$[\zeta] = \left[\frac{bl^n}{2m\sqrt{gl}}\right] = [T]^{n-1}$$

If you observe, ζ is dimensionless only when n = 1.

Quality factor calculation for a limiting case:

We will find a quality factor for very special case, whose formula is generally known to everyone:

$$Q = \frac{1}{2\zeta}$$

Now this is valid in very special case when:
1) n = 1
2) θ is very small
3) ζ is very small

Using this assumptions we will derive the formula:

Energy expression:

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta) \tag{3}$$

Since θ is very small, so $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$E = \frac{ml}{2}(l\dot{\theta}^2 + g\theta^2)$$

$$E = \frac{ml^2}{2}(\dot{\theta}^2 + \omega_n^2\theta^2) \tag{4}$$

Now for the assumptions made, the equation of motion will be:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \tag{5}$$

The solution of the above equation when $\zeta < 1$ would be:

$$\theta(t) = Ae^{-\zeta\omega_n t}\sin(\omega_n t\sqrt{1-\zeta^2} + \phi) \tag{6}$$

differentiating it will give:

$$\dot{\theta}(t) = (-\zeta\omega_n)Ae^{-\zeta\omega_n t}\sin(\omega_n t\sqrt{1-\zeta^2} + \phi) + Ae^{-\zeta\omega_n t}(\omega_n\sqrt{1-\zeta^2})\cos(\omega_n t\sqrt{1-\zeta^2} + \phi) \tag{7}$$

We also have:

$$\omega_n^2\theta^2 = A^2\omega_n^2 e^{-2\zeta\omega_n t}\sin^2(\omega_n t\sqrt{1-\zeta^2} + \phi) \tag{8}$$

$$\begin{aligned}\dot{\theta}^2 = A^2e^{-2\zeta\omega_n t}\zeta^2\omega_n^2\sin^2(\omega_n t\sqrt{1-\zeta^2} + \phi) + A^2e^{-2\zeta\omega_n t}(\omega_n^2(1-\zeta^2))\cos^2(\omega_n t\sqrt{1-\zeta^2} + \phi)\\ -2\omega_n^2\zeta\sqrt{1-\zeta^2}A^2e^{-2\zeta\omega_n t}\sin(\omega_n t\sqrt{1-\zeta^2} + \phi)\cos(\omega_n t\sqrt{1-\zeta^2} + \phi)\end{aligned} \tag{9}$$

Using (8) and (9) in (4) we get the following expression for energy:

$$E = \frac{ml^2A^2\omega_n^2e^{-2\zeta\omega_n t}}{2}(1 - \zeta^2\cos(2\omega_n t\sqrt{1-\zeta^2} + 2\phi) - \zeta\sqrt{1-\zeta^2}\sin(2\omega_n t\sqrt{1-\zeta^2} + 2\phi)) \tag{10}$$

For an under-damped system, the time period for a cycle is $T_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$

Quality factor can be defined as $Q = 2\pi \times \frac{E(nT_d)}{E(nT_d)-E((n+1)T_d)}$, where n is some integer, That gives us:

$$Q = \frac{2\pi}{1-e^{-2\zeta\omega_n T_d}} \tag{11}$$

$$e^{-2\zeta\omega_n T_d} = 1 - 2\zeta\omega_n T_d + \frac{(2\zeta\omega_n T_d)^2}{2!} - \frac{(2\zeta\omega_n T_d)^3}{3!} + ...$$

Since $\zeta << 1$ then we can ignore higher order terms and $e^{-2\zeta\omega_n T_d} \approx 1 - 2\zeta\omega_n T_d$ Whch gives us:

$$Q = \frac{\sqrt{1-\zeta^2}}{2\zeta}, \text{since } \zeta << 1 \implies Q = \frac{1}{2\zeta} \tag{12}$$

Since the ζ definition is very limited, we are not going to use it, we are going to use the energy definition.

So now let's see how will we implement the governing equation in MATLAB. So back to our governing equation and we won't linearize, we will take the exact equation:

$$\ddot{\theta} + 2\zeta\omega_n|\dot{\theta}|^{n-1}\dot{\theta} + \omega_n^2 \sin\theta = 0$$

5 Implementation

To solve the given equation numerically, ode113 solver is used, to use this solver, we need to write the equation in state space form:

The governing equation along with the initial conditions is:

$$\ddot{\theta} + 2\zeta\omega_n|\dot{\theta}|^{n-1}\dot{\theta} + \omega_n^2 \sin\theta = 0,\ \theta(0) = \frac{\pi}{3}\ \text{rad},\ \dot{\theta}(0) = 0\ \text{rad/s} \tag{13}$$

Here we take $m = 1\ kg$, $g = 9.8\ m/s^2$ and $l = 1\ m$
The motion is computed from $t = 0\ s$ to $t = 15\ s$

we will see more about errors in Limitations section.

We define two variables as, $\Theta_1 = \theta$ and $\Theta_2 = \dot{\theta} = \dot{\Theta}_1$, and hence $\dot{\Theta}_2 = \ddot{\theta}$
Therefore the state space form would be:

$$\dot{\Theta}_1 = \Theta_2 \tag{14}$$

$$\dot{\Theta}_2 = -2\zeta\omega_n|\Theta_2|^{n-1}\Theta_2 - \omega_n^2 \sin\Theta_1 \tag{15}$$

Now this state space form is used in MATLAB as follows:

```
[t,theta] = ode113(@(t,theta) [theta(2); (-(omega_n^2).*sin(theta(1)))], [0, 15], [pi/3, 0]);
plot(t, theta(:, 1))
```

Figure 2: Numerical solution using ode113 solver

After this a plot for θ vs t, $\dot{\theta}$ vs t, $\dot{\theta}$ vs θ , E vs t are made, and Quality factor is calculated for first 4 cycles, why calculation is done for different cycles?, because as you will see the quality factor changes with the cycle.

Here one cycle is defined as a complete motion from one zero crossing to the zero crossing after it's next consecutive zero crossing. Say from time at which there is n_{th} zero crossing of θ to time at which there is $n+2_{th}$ zero crossing of θ

Quality factor numerically is calculated as:

$$Q = 2\pi \times \frac{E(t = n_{th}\text{zero crossing of } \theta)}{E(t = n_{th}\text{zero crossing of } \theta) - E(t = n+2_{th}\text{zero crossing of } \theta)} \tag{16}$$

It's implemented in the code as follows:

```
Changing_Damping_power.mlx
for n = [1 2 3]
    [t,theta] = ode113(@(t,theta) [theta(2); (-(omega_n^2).*sin(theta(1)) -(2*zeta*omega_n).*(((abs(theta(2))^(n - 1))*(theta(2)))))], [0, 15], [pi/3, 0])
    E = (0.5*m*(l^2)).*(theta(:, 2).^2) + (m*g*l).*(1 - cos(theta(:, 1)));
    plot(t, E)
    xlabel("time")
    ylabel("Energy")
    axis equal
    grid on
    hold on
    A = [];
    for i = 2:length(theta(:, 1))
        if theta(i-1, 1) > 0 && theta(i, 1) < 0
            A = [A; i];
        end
        if theta(i-1, 1) < 0 && theta(i, 1) > 0
            A = [A; i];
        end
    end
    Quality_factor1 = 2*pi*((E(A(1)))/(E(A(1)) - E(A(3))));
    Q_1 = [Q_1; Quality_factor1];
    Quality_factor2 = 2*pi*((E(A(2)))/(E(A(2)) - E(A(4))));
    Q_2 = [Q_2; Quality_factor2];
    Quality_factor3 = 2*pi*((E(A(3)))/(E(A(3)) - E(A(5))));
    Q_3 = [Q_3; Quality_factor3];
end
legend("0", "1", "2", "3")
hold off
Q_1
Q_2
Q_3
```

Figure 3: Quality factor code implementation

Detailed MATLAB codes for the problem is recorded in my following GitHub link: https://github.com/Physicist197/Quality-factor-of-1-dof-nonlinear-damped-pendulum

6 Results

Analysis was done for 4 cases:
1) keeping $\zeta = 0.16$, n was varied from 1 to 3.
2) keeping $n = 1$, ζ is varied as $\zeta = 0.16$, $\zeta = 0.32$ and $\zeta = 0.48$.
3) keeping $n = 2$, ζ is varied as $\zeta = 0.16\ s$, $\zeta = 0.32\ s$ and $\zeta = 0.48\ s$.
4) keeping $n = 1$, ζ is varied as $\zeta = 0.16\ s^2$, $\zeta = 0.32\ s^2$ and $\zeta = 0.48\ s^2$.

Let's see one by one:

1) keeping $\zeta = 0.16$, n was varied from 1 to 3:

n	First Cycle	Second cycle	Third cycle
$n = 1$	7.2003	7.1329	6.9942
$n = 2$	7.9302	9.1231	10.2370
$n = 3$	9.3702	12.5695	14.4865

Table 1: QF for $\zeta = 0.16$ with different n

In MATLAB $b = 1$, which automatically gives us $\zeta = 0.16$

Angular displacement as a function of time plot for this case was as follows:

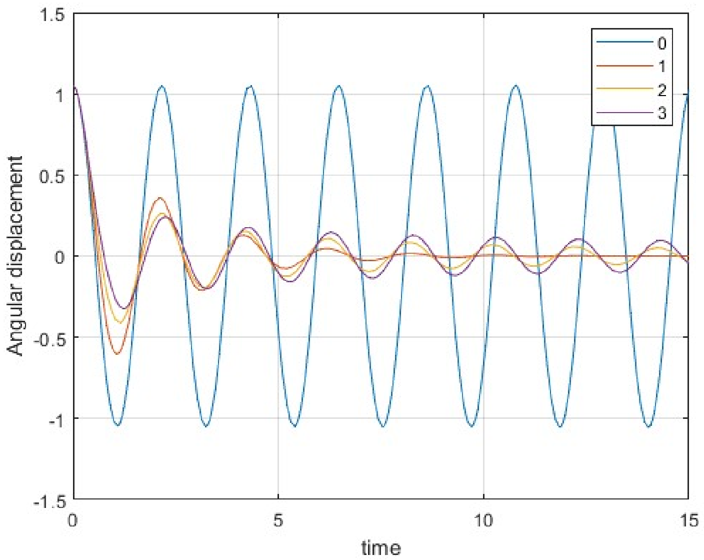

Figure 4: Angular displacement time plot for $\zeta = 0.16$ with different n

Now as you can see, as we are increasing n, the quality factor is increasing. It's because if we see the angular displacement plot, as we increase n, the motion dying rate is decreasing. Also as you go further in time for $n = 2$ and $n = 3$, Attenuation is decreasing, and amplitude remains almost constant in later cycles, although it's still decreasing but the decrease is very less, therefore we are having high quality factor. Why is not that in the case of $n = 1$, because as you can see the motion damps out quickly in later cycles for $n = 1$.

Similar arguments can be made for the following cases, where motion of the pendulum, or say the angular displacement time plot justifies the quality factor data, as you can see for the following cases, for $n = 2$ and $n = 3$, Quality factor is increases with time, because as you can see in the plot, the decrease rate of amplitude in later cycles is decreasing, And hence we see that Quality factor even in nonlinear damping retains it's physical significance.

2) keeping $n = 1$, ζ is varied as $\zeta = 0.16$, $\zeta = 0.32$ and $\zeta = 0.48$:

ζ	First Cycle	Second cycle	Third cycle
$\zeta = 0.16$	7.2003	7.1329	6.9942
$\zeta = 0.32$	6.3565	6.3670	6.3580
$\zeta = 0.48$	6.2895	6.2881	6.2905

Table 2: QF for $n = 1$ with different ζ

In MATLAB, b was varied from 0(for undamped case) to 4, $b = 1$, $b = 2$ and $b = 3$, which automatically gave us gave us calculated ζ values:

Angular displacement as a function of time plot for this case was as follows:

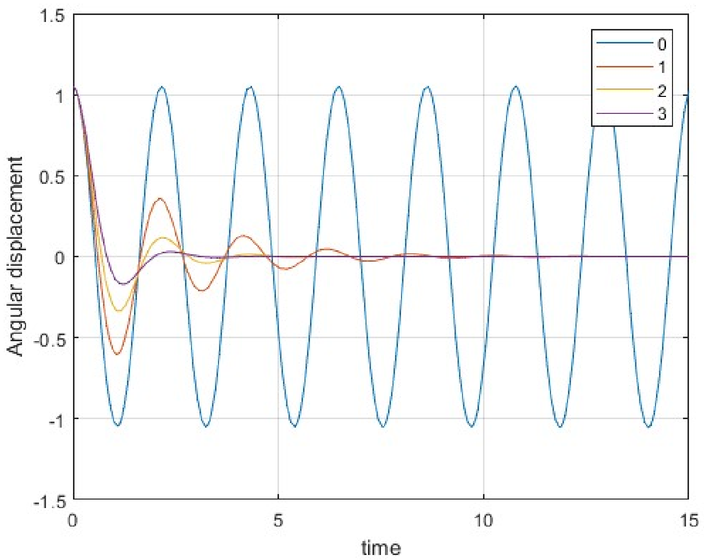

Figure 5: Angular displacement time plot for $n = 1$ with different ζ

3) keeping $n = 2$, ζ is varied as $\zeta = 0.16$, $\zeta = 0.32$ and $\zeta = 0.48$:

ζ	First Cycle	Second cycle	Third cycle
$\zeta = 0.16$	7.9302	9.1231	10.2370
$\zeta = 0.32$	7.5030	8.8036	10.2293
$\zeta = 0.48$	7.2667	8.7379	10.2377

Table 3: QF for $n = 2$ with different ζ

Angular displacement as a function of time plot for this case was as follows:

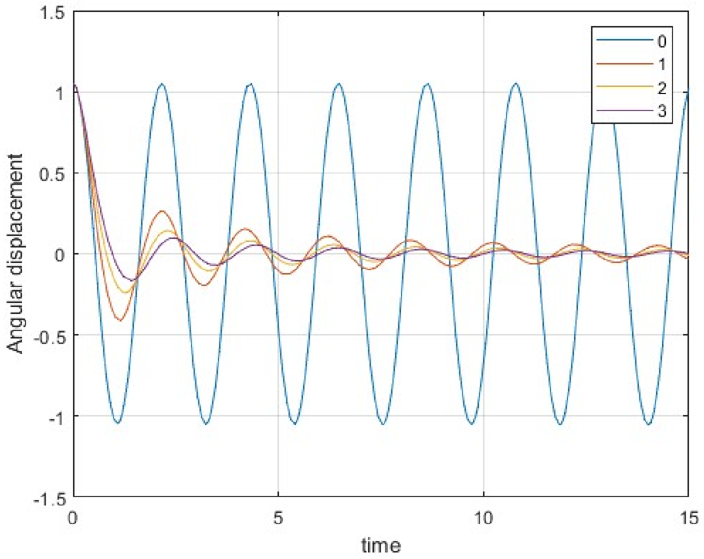

Figure 6: Angular displacement time plot for $n = 2$ with different ζ

4) keeping $n = 3$, ζ is varied as $\zeta = 0.16$, $\zeta = 0.32$ and $\zeta = 0.48$:

ζ	First Cycle	Second cycle	Third cycle
$\zeta = 0.16$	9.3702	12.5695	14.4865
$\zeta = 0.32$	9.3762	12.4588	15.2422
$\zeta = 0.48$	9.3202	11.4329	15.2553

Table 4: QF for $n = 3$ with different ζ

Angular displacement as a function of time plot for this case was as follows:

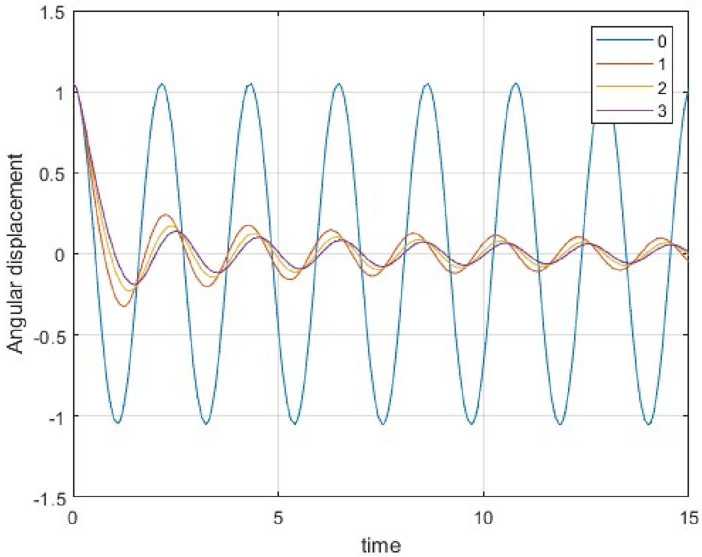

Figure 7: Angular displacement time plot for $n = 3$ with different ζ

7 Limitations

Quality factor definition: Even though Quality factor retains it's physical significance even in nonlinear damping, but still as we can see that it is coming different for different cycles, therefore the definition needs some modification so that it is consistent with the cycles for the given situation. So there is a need of modification in the definition of quality factor for higher order damping.

Conclusions

We saw that how in nonlinear damping Quality factor retains it's physical meaning, and hence can be used as performance parameter, but for the consistency within the cycles for a given situation there's a need in the modification of definition.