From Finite Volumes to Discontinuous Galerkin

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Galerkin's method of weighted residuals

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Problem: given equation $Ax = b$ in infinite dimensional spaces, how to find finite dimensional approximations?

1) Choose "trial function space" X_h, "test function space" Y_h.
2) Denote residual $r(x) = b - Ax \quad x \in X_h$
3) Test residual with functions in Y_h

$$r(x)(y) = b(y) - (Ax)(y)$$

4) Approximate solution is obtained as: find $x \in X_h$ such that

$$r(x)(y) = 0 \qquad \forall\, y \in Y_h$$

5) Choice of bases generates linear system with matrix A.

Example: $A = (u_\Omega \cdot \nabla)\, x \qquad r(x)(y) = \int_D \left(b\,y - (u_\Omega \cdot \nabla) x\, y\right) dx$

"weak formulation"
"virtual displacements" "Lagrange formalism"

1) Begin with weak formulation

2) Cover domain D with finite volume mesh as before

3) Choose ϱ and φ constant on each V_i

4) Integrate by parts (on each cell V_i !!)

$$\sum_{V_i}\left[-\int_{V_i}\varrho\,(u_\Omega\cdot\nabla)\varphi\,dx+\int_{\partial V_i}\varrho\,\varphi\,(u_\Omega\cdot n_{V_i})\,ds\right]$$

5) Replace ϱ on ∂V_i by its upwind value $\varrho^\uparrow$

$$\sum_{V_i}\left[-\int_{V_i}\varrho\,(u_\Omega\cdot\nabla)\varphi\,dx+\int_{\partial V_i^-}\varrho^\uparrow\varphi\,(u_\Omega\cdot n_{V_i})\,ds\right.$$
$$\left.+\int_{\partial V_i^+}\varrho\,\varphi\,(u_\Omega\cdot n_{V_i})\,ds\right]$$

6) Integrate back

$$\sum_{V_i}\left[\int_{V_i}(u_\Omega\cdot\nabla)\varrho\,\varphi\,dx+\int_{\partial V_i^-}(\varrho^\uparrow-\varrho)\,\varphi\,(u_\Omega\cdot n_{V_i})\,ds\right]$$

Relation to Finite Volume:

- Choose $\varrho|_{V_j}=\varrho_j$
- Choose a basis for test functions

$$\varphi_i(x)=\begin{cases}1 & x\in V_i\\ 0 & \text{else}\end{cases}$$