The game of the name

1) Galerkin: weighted residuals, subspaces
2) discontinuous: the integral is not the same anymore, but contains interface terms, the functions are discontinuous

Why DG instead of FV?

More flexibility of spaces.

E.g. higher order polynomials instead of constant.

Suitable Solution Spaces

[for the mathematically inclined]

What are minimal requirements on ρ and φ such that the weak formulation makes sense?

Answer: $\int_D (n_e \cdot \nabla)\rho \, \varphi \, dx$ must exist

Necessary conditions: $\rho \in V = \{ v \in L^2(D) \mid (\rho \cdot \nabla) v \in L^2(D) \}$

$\varphi \in L^2(D)$

Integrate by parts $\int_D \rho \, (n_e \cdot \nabla)\varphi \, dx$ must exist

Necessary: $\rho \in L^2(D)$, $\varphi \in L^2(D)$

Solution theory possible in these spaces, but cumbersome

Sufficient: $u, v \in V$

Boundary conditions

Is $\int_{\Gamma_-} \rho v \, |n_\Omega \cdot n_D| \, ds$ well-defined on V?

Answer: Yes. [Dautray / Lions] volumes 5 and 6

Back to DG (and Galerkin)

We write the Galerkin form of the differential equation as $a(\rho, \varphi) \equiv A(\rho)(\varphi) = f(\varphi)$

Notation: mesh $\mathcal{M} = \{V_i\}$ mesh cells V

F: face of a cell, F_{ij}: face between V_i and V_j

$\mathcal{F}^i$: set of interior faces

$\mathcal{F}^-$: faces on inflow boundary

$\mathcal{F}^+$: " " outflow "

The bilinear form

$$b_h(\rho,\varphi) \equiv \sum_{V\in\mathcal{M}} \int_V (n_\Omega\cdot\nabla)\rho\varphi\,dx + \sum_{F\in\mathcal{E}^-}\int_F \rho\varphi\,|n_\Omega\cdot n_F|\,ds + \sum_{F\in\mathcal{E}^i}(\rho^\downarrow-\rho^\uparrow)\varphi\,|n_\Omega\cdot n_F|\,ds$$

$$= \sum_{V\in\mathcal{M}}\int_V q\varphi|n_\Omega\cdot n_F|\,dx + \sum_{F\in\mathcal{E}^-}\int_F \rho^{inc}\varphi\,|n_\Omega\cdot n_F|\,ds$$

$$\equiv f_h(\varphi)$$

This must hold for all test functions φ.

The linear system

1) Choose a discrete space

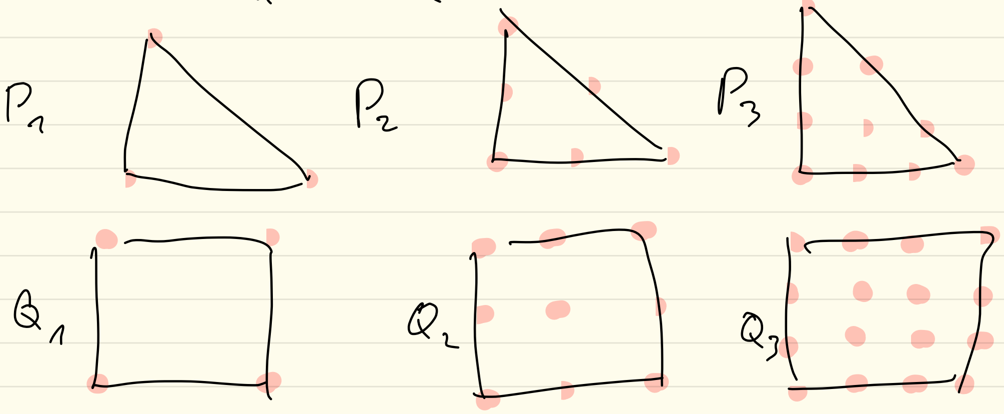
Finite Volumes: piewise constant

DGFEM: piecewise polynomial

$$V_h = \{\varphi\in L^2(D)\;|\;\varphi|_V \in \underset{Q_k}{P_k}\}$$

P_k: multivariate polynomials of degree k

2) Choose a basis. Typically: interpolating basis for P_k or Q_k



$$P_k = \operatorname{span}\{\tilde\varphi_1 \dots \tilde\varphi_{n_k}\} \qquad Q_k = \operatorname{span}\{\tilde\varphi_1 \dots \tilde\varphi_{n_k}\}$$

We obtain n_k basis functions on each cell V_i.

Using double indices, we obtain the basis

$$\varphi_{ml}$$

cell polynomial

The dimension of X_h is $\#\text{cells} \cdot n_k$

The basis can be renumbered to a single index using

$$\varphi_i = \varphi_{ml} \quad \text{for} \quad i = n_k \cdot (m-1) + l$$

3) Since $b_h(\cdot,\cdot)$ is linear in its second argument, it is sufficient to test with basis functions:

$$b_h(g,\varphi) = f_h(\varphi) \qquad \forall \varphi \in X_h$$

$$\Leftrightarrow \quad b_h(g,\varphi_i) = f_h(\varphi_i) \qquad i = 1 \ldots \dim X_h$$

4) Write $g(x) = \sum u_j \varphi_j(x)$ and use linearity in the first argument

$$b_h(g,\varphi_i) = b_h\left(\sum u_j \varphi_j, \varphi_i\right)$$

$$= \sum u_j \, b_h(\varphi_j, \varphi_i)$$

5) Collect: solve for $u \in \mathbb{R}^n$

$$Bu = f$$

$$b_{ij} = b_h(\varphi_j, \varphi_i) \qquad f_i = f_h(\varphi_i)$$

Existence of solutions

$B : \mathbb{R}^n \to \mathbb{R}^n$

$\Rightarrow$ a solution exists iff it is unique

The solution is unique if

$Bu = 0$ implies $u = 0$

$Bu = 0 \iff b_h(\varrho, \varphi) = 0 \quad \forall \varphi \in X_h$

The linear system is uniquely solvable, iff for every $\varrho \in X_h$ there exists $\varphi \in X_h$ such that

$$b_h(\varrho, \varphi) \neq 0$$

Stronger condition, but useful later

The linear system is uniquely solvable, if for every $\varrho \in X_h$ there exists $\varphi \in X_h$ such that

$$\|\varphi\| \leq \|\varrho\| \quad \text{and} \quad \beta \|\varrho\|^2 \leq b_h(\varrho, \varphi)$$

for some norm $\|.\|$ on X_h and $\beta > 0$