Exercise D-2
Computational Methods for the Interaction of Light and Matter (WS2019/20)

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Plane-parallel 1-D radiative transfer problem

We now consider the atmosphere of a star. The goal is to compute how bright this atmosphere is, when we observe it under a given angle θ_{obs} (where $\theta_{\text{obs}} = 0$ means we look straight down).

We describe the atmosphere with a vertical coordinate z. The gas density $\rho(z)$, gas temperature $T(\rho)$ and all radiative quantities will only be a function of this vertical coordinate. We assume that all these quantities will be *independent* of the two horizontal coordinates x and y. The problem therefore becomes mathematically 1-D, although it remains physically 3-D. Also, the radiation still "feels" all three dimensions: light can not only go vertically up and down, but also under any given angle θ.

This radiative transfer problem is called a *Plane-parallel 1-D radiative transfer problem*. We will consider a *semi-infinite space*, where for simplicity the gas density of the atmosphere obeys:

$$\rho(z) = \begin{cases} \rho_0 & \text{for } z <= 0 \\ 0 & \text{for } z > 0 \end{cases} \tag{1}$$

The extinction function will be

$$\alpha(z) = \rho(z)\kappa \tag{2}$$

where κ is called the *opacity*.

Consider now a ray going through this atmosphere under an angle θ with the vertical normal vector. Convince yourself that the Formal Transfer Equation for this ray can be written in the following form:

$$\mu \frac{dI(z)}{dz} = \alpha(z)\big(S(z) - I(z)\big) \tag{3}$$

with $\mu = \cos(\theta)$.

Now let us introduce scattering. Instead of just κ, we now have $\kappa = \kappa^{\text{abs}} + \kappa^{\text{scat}}$, and consequently $\alpha = \alpha^{\text{abs}} + \alpha^{\text{scat}}$. We define the *albedo* as

$$\eta \equiv \frac{\kappa^{\text{scat}}}{\kappa^{\text{abs}} + \kappa^{\text{scat}}} \tag{4}$$

The source function $S(z)$ now becomes

$$S(z) = (1 - \eta)B(z) + \eta J(z) \tag{5}$$

where $B(z)$ is the Planck function and $J(z)$ is the mean intensity, both functions of z. For now, we will again keep things easy, and use non-physical units. In this case we set the Planck function B to 1.

The mean intensity at every vertical location z is an integral over μ:

$$J(z) = \int_{4\pi} I(z, \mu) d\Omega = \frac{1}{2} \int_{-1}^{+1} I(z, \mu) d\mu \tag{6}$$

Convince yourself of the above expression, whereby one should keep in mind that $d\mu = d\cos\theta = -\sin\theta\, d\theta$, and that integration over a sphere involves the integration over $\sin\theta\, d\theta d\phi$.

Given that we initially do not know what the functional form of $J(z)$ is, we do not know what $S(z)$ is. This means, we cannot integrate the FTE (Eq. 3). But if we cannot integrate the FTE, then we do not know the values of $I(z,\mu)$, so we cannot perform the integration procedure in Eq. (6). This "chicken or egg" problem is typical for radiative transfer problems.

We will solve this problem using a procedure called "Lambda Iteration". It is, from its principle, rather simple: We start with assuming that $J(z) = 0$, so that $S(z) = (1-\eta)B(z)$. We then integrate the FTE (Eq. 3). This yields $I(z,\mu)$ (in the computer this is a 2-D array). We can then compute $J(z)$ using Eq. (6), using a numerical integration procedure. Then we update the source function, redo the FTE, again compute the new $J(z)$, etc etc, until we converge.

Now let's get to work. Let us set up a vertical grid between $z = -z_0$ and $z = 0$, with $z_0 = -10$ (so yes, z is always negative). We use $N_z = 100$ grid points. For the μ-grid we use N_μ grid points, and we set up a grid in μ equally spaced between $\mu = -1$ and $\mu = +1$. We use $N_\mu = 20$ (so that there is no $\mu_i = 0$! why is that important?).

At the bottom of the atmosphere ($z = z_0$) we set as our boundary condition for the incoming radiation (i.e. for $\mu > 0$): $I(z_0, \mu > 0) = B(z_0)$. At the top of the atmosphere we assume that there is *no irradiation*, so that at $z = 0$ we set as a boundary condition $I(0, \mu < 0) = 0$.

For the test problem we set $B = 1$, $\rho = 1$, $\kappa = 1$ (therefore $\alpha = 1$) everywhere. For the moment, let us for simplicity set $\eta = 0$, so that the FTE can be directly integrated.

1. Write a computer program that can, for a given grid in z and μ (i.e. $\{z_i\}$ and $\{\mu_k\}$), and for a given source function $S(z)$ (i.e. $\{S_i\}$), extinction function $\alpha(z)$ (i.e. $\{\alpha_i\}$) compute the 2-D array of intensity values $I_{i,k} = I(z_i, \mu_k)$ by integrating numerically the FTE (Eq. 3). Use the techniques of last time for that. Write the numerical integration of the FTE as a function, i.e. not in the main program.

2. Now write a function that computes the mean intensity $J_i = J(z_i)$ at each grid point z_i.

Now let us set $\eta = 0.5$. Now we need Lambda Iteration.

3. Try out the Lambda Iteration procedure, and plot the source function $S(z)$ for each iteration step, and describe how (if) it converges.

Finally, let us compute what the outcoming intensity is.

3. Plot $I(z = 0, \mu)$ for all $\mu > 0$, and show that it declines with increasing θ (i.e. decreasing μ). This means that, when we observe an atmosphere under an angle, it will look darker. This is a well-known phenomenon for the Sun, called "limb darkening".