# Exercise D-3
# Computational Methods for the Interaction of Light and Matter (WS2019/20)

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## Exoplanetary Atmosphere

Since the nobel-prize-winning discovery of the first planet around a sun-like star (Mayor & Queloz, 1995, Nature 378, 355), over 5000 "extrasolar planets" have been discovered. In fact, statistically speaking, each star in the Milky Way has at least one planet (Cassan et al. 2012, Nature 481, 167). Initially these extrasolar planets could only be detected indirectly, by their gravitational pull on their host star ("radial velocity detection"), by the passage of the planet in front of the star ("transit detection"), or by "gravitational microlensing".

In 2002 Charbonneau et al. (Astrophysical Journal 568, 377) were the first to actually obtain a spectrum of the atmosphere of an extrasolar planet. This planet, called HD 209458 b, is a so-called "Hot Jupiter". It is a gaseous giant planet similar to Jupiter (a mass of 0.71 Jupiter mass, and a radius of 1.35 Jupiter radius). It orbits very close to the star (orbital radius of 0.047 astronomical units), with an orbital period of only a few days. As a result, the atmosphere of the planet is about 1500 K hot, hence the name "Hot Jupiter". The host star has a luminosity $L_* = 1.77\,L_\odot$.

The planet HD 209458 b has since been studied intensively, and nowadays we know that its atmosphere contains water vapor, carbon dioxide and methane, among many other atomic and molecular species such as vanadium and titanium oxides as well as sodium and potassium.

Let us make a simplified radiative transfer model of this atmosphere. We follow the discussion in the paper by Guillot (2010, Astronomy and Astrophysics 520, 27), but we will use our own numerical tools, that we develop in this lecture.

We start from the plane-parallel radiative transfer code we developed in exercise 2. However, instead of making our vertical grid in units of length (the coordinate $z$), we transform our system of equations to the coordinate of pressure $p$. The pressure and the density of an atmosphere are related by the hydrostatic equilibrium equation:

$$\frac{dP(z)}{dz} = -g\rho(z) \tag{1}$$

where $g = GM_{\mathrm{planet}}/R^2_{\mathrm{planet}}$ is the gravitational acceleration at the planet's surface. Using this equation we can see that

$$d\tau_\nu = \rho\kappa_\nu dz = -\kappa_\nu dP/g \tag{2}$$

The advantage is that we then do not need to know (or solve) the vertical density structure $\rho(z)$. In fact, at any given depth in the atmosphere, the pressure $P(z)$ depends only on the total column of mass above it:

$$P(z) = g\int_z^\infty \rho(z')dz' \tag{3}$$

which means that we do not need to have any knowledge of the equation of state of the gas.

1. Reformulate the FTE using $P$ as a coordinate, replacing $z$.
2. Set up a logarithmic grid $\{P_i\}_{i=0,N-1}$ ranging from $P = 1\,\text{bar}$ at the bottom[1] of the atmosphere down to $P = 5 \times 10^{-5}\,\text{bar}$ at the top of the atmosphere. For consistency with your previous code, it might be helpful to go from high $P$ (first array element) to low $P$ (last array element).
3. Rewrite your computer program to use this $P$-grid as a coordinate instead of $z$.

Next we include the irradiation of the atmosphere by the star (like the Sun irradiates the Earth's atmosphere). For simplicity we consider the star as a point source of light at a large distance from the atmosphere. The light from a point source is best described by a flux $F^*_\nu$ (instead of an intensity $I^*_\nu$). To include it into our plane-parallel model, we make use of the *superposition principle* of light: We can treat the starlight completely independently from the radiation emitted by the atmosphere itself (which we treat using intensity $I_{\nu,\mu}(P)$). At the top of the atmosphere ($P = P_{\text{min}}$) the irradiative flux of the star is $F^*_\nu = L^*_\nu/(4\pi d^2)$, where $L^*_\nu$ is the frequency-dependent luminosity of the star, and $d$ is the distance from the star to the planet. The flux enters the atmosphere under an angle $i$ with respect to the $z$-axis. Due to the opacity of the atmosphere, this flux is attenuated as follows:

$$\frac{dF^*_\nu(P)}{dP} = -\frac{\kappa_\nu}{g\cos(i)}F^*_\nu(P) \tag{4}$$

4. Derive Eq. (4) from the FTE along a ray entering the atmosphere at inclination angle $i$ with respect to the $z$-axis.

Next we compute the atmospheric temperature structure, by assuming that the atmosphere is everywhere in radiative equilibrium. Since the atmosphere "feels" both the stellar radiation field $F^*_\nu(P)$ as well as the atmospheric thermal radiation field $I_{\nu,\mu}(P)$, we obtain

$$\int_0^\infty \kappa_\nu \left[B_\nu(T(P)) - J_\nu(P) - \frac{1}{4\pi}F^*_\nu(P)\right] d\nu = 0 \tag{5}$$

which we solve for $T(P)$ for each vertical location in the atmosphere $P$, and where $J_\nu(P)$ is the mean intensity computed from $I_{\nu,\mu}(P)$ and, as usual, $B_\nu(T)$ is the Planck function

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/k_BT} - 1} \tag{6}$$

where $h$ is the Planck constant, $c$ the speed of light and $k_B$ the Boltzmann constant. Solving Eq. (5) requires knowledge of the full spectral shape of $J_\nu(P)$ and $F^*_\nu(P)$. As we will learn later in the course, this can be extraordinarily complex, requiring literally millions of frequency grid points $\{\nu_i\}$.

For now let us greatly simplify the problem: we will assume the opacity to be independent of $\nu$. But to not oversimplify things, we allow the opacity for the starlight $F^*_\nu(P)$ to be a different constant than the opacity for the atmospheric thermal radiation $I_{\nu,\mu}(P)$. We will write the former as $\kappa_*$ and the latter as $\kappa_{\text{th}}$. We "justify" this be assuming that

[1] It should be noted that there is no real "bottom", since gas giant planets are made up of gas and have no "surface"; so this value is just a choice.

the starlight is much hotter (and therefore at a much shorter wavelength) than the atmospheric thermal radiation. In other words: stellar photons and atmospheric thermal photons are assumed to be at vastly different wavelength ranges, and can thus be assumed to each "feel" a different part of the opacity function $\kappa_\nu$. For hot Jupiters this assumption is only marginally valid, but it turns out not to be an all to bad assumption. Using this assumption, we can rewrite Eq. (5) as

$$\frac{\sigma_{\rm SB}}{\pi}T(P)^4 = J(P) + \frac{\kappa_*}{\kappa_{\rm th}}\frac{1}{4\pi}F^*(P) \tag{7}$$

where $\sigma_{\rm SB}$ is the Stefan-Boltzmann constant, $J = \int_0^\infty J_\nu d\nu$ is the frequency-integrated mean intensity and $F^* = \int_0^\infty F_\nu^* d\nu$ is the frequency-integrated stellar flux.

From here onward, we will only work with these frequency-integrated quantities, and also solve the FTE in frequency-integrated form. We thus avoid having to do millions of FTE calculations: instead we just solve, for each $\mu$, just one FTE for $I_\mu(P)$. Since the program we wrote for exercise 2 is anyway written for a single wavelength, we can simply use this code.

5. For a star with frequency-integrated luminosity $L_*$ and a planet at a distance $d$ from that star, give an expression for $F^*$ at the top of the atmosphere.

6. Implement the irradiation into the code, for a given irradiation inclination $i$, using the method described above. Allow your code have a $\kappa_* \neq \kappa_{\rm th}$.

7. Write a method that computes the atmosphere temperature at all gridpoints $\{P_i\}$, given the mean intensity of the atmospheric radiation field $J_i \equiv J(P_i)$ and the stellar radiation field $F_i^* \equiv F^*(P_i)$.

Now we are all set to use Lambda Iteration to compute the temperature structure of the atmosphere $T_i \equiv T(P_i)$. The only thing still mising is the boundary condition at the bottom of the atmosphere. This is not trivial, and for now let us simply specify the temperature of a blackbody radiation field at the bottom $T_{\rm bot}$, or in other words: set the upward pointing intensity to $I_{\mu>0}(P = P_{\rm max}) = (\sigma_{\rm SB}/\pi)T_{\rm bot}^4$.

A small tip to enhance the speed of your calculation: It turns out that a $\mu$-grid of only two well-chosen values, $\mu = \pm 1/\sqrt{3}$, is usually sufficiently accurate for the atmospheric thermal radiation field. This is called the *two-stream approximation.*

8. Implement this bottom boundary condition into your code. Set it to $T_{\rm bot} = 1500\,\mathrm{K}$ for now.

9. Now set all the parameters to the case of HD 209458 b. Set $\kappa_* = 4 \times 10^{-3}$ cm$^2$/g and $\kappa_{\rm th} = 10^{-2}$ cm$^2$/g. Perform the Lambda Iteration procedure for $i = 0°$ (the star at zenith). As convergence criterion you can choose $|\Delta T|/T < 10^{-4}$. Be generous with the max numer of iterations: It can easily be of order thousand!

10. Compare the result to the case of $i = 89°$ (the star near "sunset").

11. Compare your results to Fig. 2 of Guillot (2010, Astronomy and Astrophysics 520, 27). If all went well, you should get similar results for the top part of the atmosphere. The bottom part is, in our case, strongly affected by our ad-hoc choice of $T_{\rm bot} = 1500\,\mathrm{K}$.

Apparently we need a better boundary condition at the bottom of the atmosphere. Let us do some experiments.

12. Let us try to set $P_{\rm max} = 10$ (i.e. a factor of 10 deeper into the atmosphere). You will find that the convergence of the Lambda Iteration is much slower. Maybe it has not even converged yet for $|\Delta T|/T < 10^{-4}$?

13. Set $P_{\rm max}$ back to the original value of 1. Now let us assume that the is *zero flux* at the boundary. The way to implement that is to set up a *mirror boundary condition.* The idea is to set $I_{\mu>0} = I_{-\mu}$ at the base of the atmosphere: all radiation going down (out of the grid), will return as upward radiation, as if the bottom of the grid is a mirror. You will find that the convergence is also slow, but the results look much more like Guillot (2010, Astronomy and Astrophysics 520, 27).

If you want to get a "feel" for the vertical extent of this atmosphere in units of kilometers, you can now (with the computed temperature profile $T(P)$) convert the $P$-coordinate back to a $z$-coordinate. You can assume the ideal gas law $P = \rho k_B T/\mu m_p$ with $k_B$ the Boltzmann constant, $m_p$ the proton mass and $\mu \simeq 2.3$ the mean molecular weight of the gas in this atmosphere, which is mostly made up of molecular hydrogen $H_2$ and helium He. Since a gas planet such as this one does not have a "surface", you have to choose a $z = 0$ point, which you can choose at will.