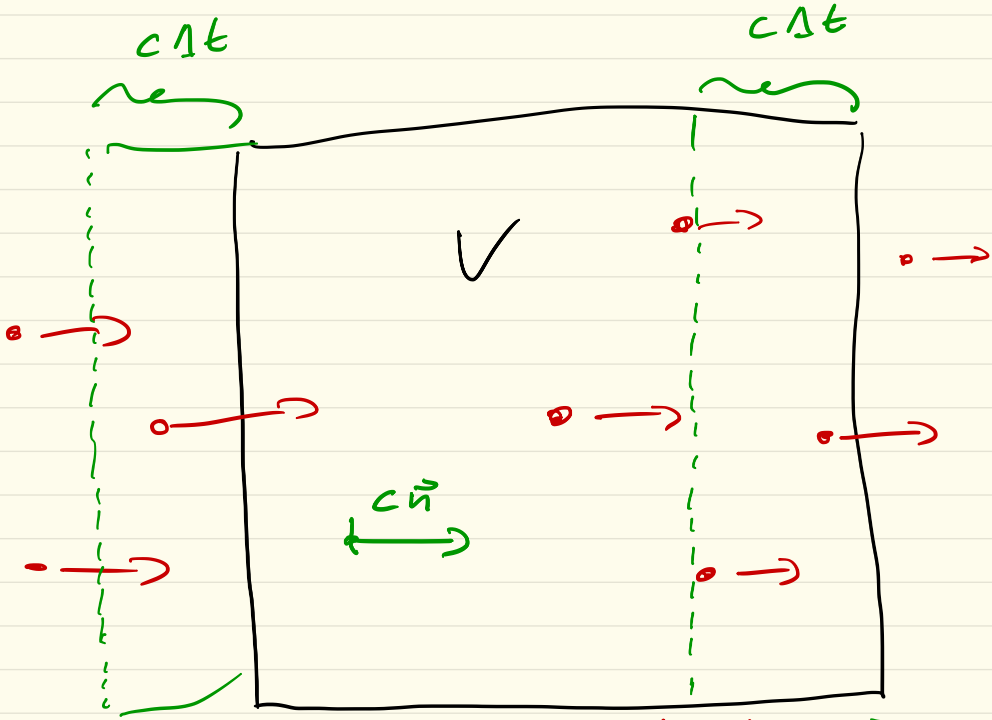
Reynolds transport theorem



photons traveling with velocity $c\vec{n}$

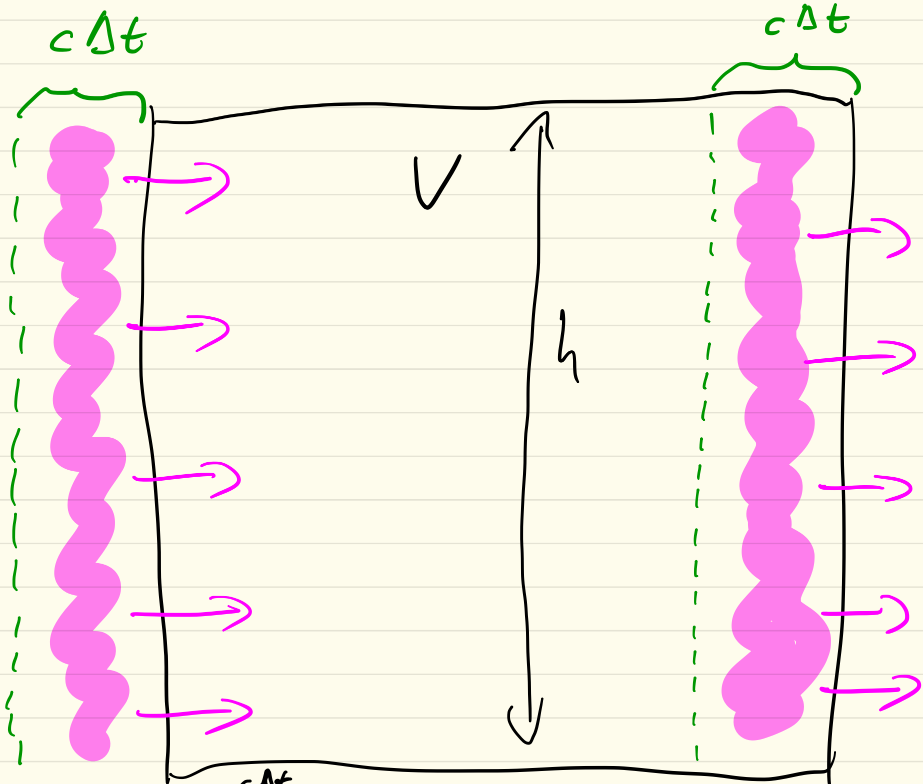
Change of number of photons in V over time interval $[t, t+\Delta t]$

$=$ number of photons entering
$-$ number of photons leaving

Transition to density ρ

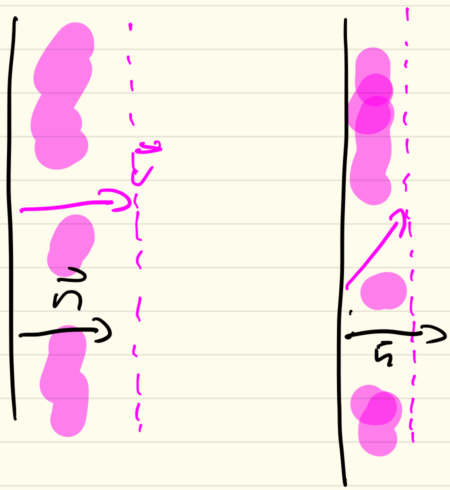
- Number of photons in volume V:

$$\# = \int_V \rho(x)\, dx$$



Entering: $\int_0^h \int_0^{c\Delta t} \rho(x)\, dx$

Exiting



Travel speed : c

Travel direction : $\vec{d}$

Outer normal to the domain : $\vec{n}$

width of the strip: $c\,\Delta t\,\vec{n}\cdot\vec{d}$

- integrate in vertical direction
- Limit for $\Delta t \to 0$
- [density and domain are suff. smooth]

$$\frac{d}{dt}\int_V \rho(x)\,dx = -\int_{\partial V} \rho(s)\,c\,\vec{n}\cdot\vec{d}\,ds$$

Physical principle

$$\frac{d}{dt}\int_V \rho(x)\,dx = -\int_{\partial V} \rho(s)\,\vec{d}\cdot\vec{n}\,ds$$

Gauss-Theorem

$$\int_V \nabla\cdot\vec{u} = \int_{\partial V} \vec{u}\cdot\vec{n}\,ds$$

$$\nabla\cdot u = \partial_x u_x + \partial_y u_y$$

$$\int_V \frac{\partial}{\partial t}\rho(x)\,dx = -c\int_{\partial V} \rho(s)\,\vec{d}\cdot\vec{n}\,ds$$

$$= -c\int_V \nabla\cdot\left(\rho(x)\,\vec{d}\right)dx$$

Differential equation

$$\frac{1}{c}\partial_t \rho + \nabla\cdot(\rho\,\vec{d}) = 0$$

Different Forms of the flux term

conservative form

$$\vec{\nabla}\cdot(\rho\vec{d}) = \partial_x(\rho\, d_x) + \partial_y(\rho\, d_y)$$

$$= \partial_x\rho\, d_x + \partial_y\rho\, d_y + \rho\,\partial_x d_x + \rho\,\partial_y d_y$$

$$= (\vec{d}\cdot\nabla)\rho + \rho\,\nabla\cdot\vec{d}$$

directional derivative

radiation: $\vec{d} \equiv \text{const}$

$$= (\vec{d}\cdot\nabla)\rho \qquad \text{advective form}$$

Transport of a density function ρ by a (divergence free) vector field $\vec{v}$

$$\partial_t\rho + (\vec{v}\cdot\nabla)\rho = 0$$

Application to radiation

1. Consider density in phase space

$$\rho(\vec{x}, \vec{n}_\Omega, \nu)$$

2. Apply transport theorem to all directions

$$\frac{1}{c}\partial_t \rho(\vec{x}, \vec{n}_\Omega, \nu) + (\vec{n}_\Omega \cdot \nabla)\rho(\vec{x}, \vec{n}_\Omega, \nu) = 0$$

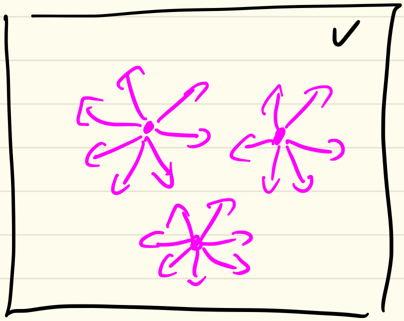
3. Convert photon density to energy density
= specific intensity

$$I_\nu(\vec{x}, \vec{n}_\Omega) = h\nu\, \rho(\vec{x}, \vec{n}_\Omega, \nu)$$

Radiation transport equation in vaccuum

$$\frac{1}{c}\partial_t I_\nu + (\vec{n}_\Omega \cdot \nabla) I_\nu = 0$$

Emission



- Let there be a source of photons releasing particles in V at a certain rate $n/\Delta t$

  Then, # photons in V at $t+\Delta t$ − # photons in V at t $= n$

- Represent this source by a source density in space time

$$q_\rho(\vec{x}, t)$$

  Then, with no movement

$$\int_V \rho(x+\Delta t)\,dx = \int_V \rho(x)\,dx + \int_V \int_t^{t+\Delta t} q_\rho(\vec{x}, s)\,ds\,dx$$

- Since this applies to every volume V, we derive the pointwise eqn.

$$\rho(x+\Delta t) = \rho(x) + \int_t^{t+\Delta t} q_\rho(\vec{x}, s)\,ds$$

Finally, we convert this to differential form

$$\partial_t \rho(\vec{x}) = q_\rho(\vec{x})$$

Application to radiation

- rescaling

$$\rho(\vec{x}, \vec{n}_\Omega, \nu) \longrightarrow I_\nu(\vec{x}, \vec{n}_\Omega)$$

$$q_\rho(\vec{x}, \vec{n}_\Omega, \nu) \longrightarrow q_\nu(\vec{x}, \vec{n}_\Omega, \nu)$$

$$\frac{1}{c} \partial_t I_\nu = q_\nu$$

Radiation transport in an emitting atmosphere (unphysical)

$$\frac{1}{c} \partial_t I_\nu + \vec{n}_\Omega \cdot \nabla I_\nu = q_\nu$$