Integration by parts

$$\int_V (n_\Omega\cdot\nabla)\varrho\varphi\,dx = -\int_V \varrho\,(n_\Omega\cdot\nabla)\varphi\,dx + \int_{\partial V}\varrho\varphi\, n_\Omega\cdot n_V\,ds$$

Abbreviate: $d_\Omega s = (n_\Omega\cdot n_V)\,ds$ no absolute value!

ignore ∂D^-

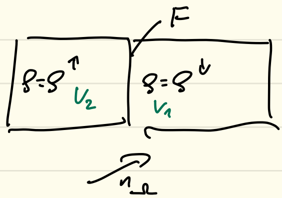
$$\Rightarrow b_h(\varrho,\varphi) = \sum_{V\in\mathcal{M}}\left[\int_V (n_\Omega\cdot\nabla)\varrho\varphi\,dx - \int_{\partial V^-}(\varrho-\varrho^\uparrow)\varphi\,d_\Omega s\right]$$

$$= \sum_{V\in\mathcal{M}}\left[-\int_V\varrho\,(n_\Omega\cdot\nabla)\varphi\,dx + \int_{\partial V^+}\varrho\varphi\,d_\Omega s + \int_{\partial V^-}\varrho^\uparrow\varphi\,d_\Omega s\right]$$

In particular,

$$2b_h(\varrho,\varrho) = \sum_{V\in\mathcal{M}}\left[0 + \int_{\partial V^-}(2\varrho^\uparrow-\varrho)\varrho\,d_\Omega s + \int_{\partial V^+}\varrho^2\,d_\Omega s\right]$$

At interface F between V_2 and V_1:
$F\subset\partial V_2^+$ $\quad F\subset\partial V_1^-$



$$\int_{\partial V_1^-}(2\varrho^\uparrow-\varrho^\downarrow)\varrho^\downarrow\,d_\Omega s + \int_{\partial V_2^+}\varrho^{\uparrow 2}\,d_\Omega s$$

$\boxed{n_1 = -n_2 \;!!}$

$$= \int_F\left(\varrho^{\uparrow 2} - 2\varrho^\uparrow\varrho^\downarrow + \varrho^{\downarrow 2}\right)|n_\Omega\cdot n_F|\,ds$$

$$= \int_F\left(\varrho^\uparrow-\varrho^\downarrow\right)^2|n_\Omega\cdot n_F|\,ds$$

On inflow boundary: $\rho^{\uparrow}$ is data on right hand side, not part of $b_h(.,.)$

For any piecewise smooth function ρ, there holds

$$2 b_h(\rho,\rho) = \sum_{F\in\mathcal{F}^i} \int_F (\rho^{\uparrow}-\rho^{\downarrow})^2 |n_\beta\cdot n_F|\,ds + \sum_{F\in\mathcal{F}^+\cup\mathcal{F}^-} \int \rho^2 |n_\beta\cdot n_F|\,ds$$

In particular, $b_h(\rho,\rho)\geq 0$.

We obtain that $b_h(\rho,\rho)=0$ implies continuity of ρ across edges not parallel to n_β.

Furthermore, $\sqrt{b_h(.,.)}$ is a norm on piecewise constant functions

It is not for any higher order than 0 since all continuous finite element spaces with zero boundary are in its kernel.

The form $b_h(.,.)$ is not coercive on higher order DG spaces

A new test function

Needed: a norm of derivatives

$$b_h\left(v, v+\delta(n_\Omega\cdot\nabla)v\right) = b_h(v,v) + \delta\, b_h\left(v, (n_\Omega\cdot\nabla)v\right)$$

$$b_h\left(v,(n_\Omega\cdot\nabla)v\right) = \sum_{V\in\mathcal{M}} \int_V \left[(n_\Omega\cdot\nabla)v\right]^2 dx$$

$$+ \sum_{F\in\mathcal{E}^i} \int_F (\rho^\downarrow-\rho^\uparrow)(n_\Omega\cdot\nabla)\rho^\downarrow\, |n_\Omega\cdot n_F|\, ds$$

$$+ \sum_{F\in\mathcal{E}^-} \int_F \rho\, (n_\Omega\cdot\nabla)\rho\, |n_\Omega\cdot n_F|\, ds$$

The boundary terms are indefinite and even involve derivatives. How to get rid of them?

Answer: Get out the finite element toolbox (next page)

$$\int_F \rho\,(n_\Omega\cdot\nabla)\rho\, |n_\Omega\cdot n_F|\, ds \le \underbrace{\sqrt{\int_F \rho^2\, ds_{|n_\Omega\cdot n_F|}}}_{\text{part of } b_h(\rho,\rho)} \sqrt{\int_F \left[(n_\Omega\cdot\nabla)\rho\right]^2 ds_{|n\cdot n|}}$$

Use inverse trace estimate and Young's inequality to determine δ such that the ugly term can be absorbed in the good ones

Finite Element Toolbox

Reminder: For a mesh cell V of diameter h and polynomial shape functions φ on V holds

1) $\| \varphi - \frac{1}{|V|} \int_V \varphi \|_V \le c\, h\, \| \nabla \varphi \|_V$ Poincaré

2) $\| \varphi \|_V^2 \le c \left[h^2 \| \nabla \varphi \|_V^2 + h \| \varphi \|_{\partial V}^2 \right]$

Friedrichs - DG

3) $\| \nabla \varphi \|_V \le c\, \| \varphi \|_V$ inverse estimate

4) $\| \varphi \|_F \le c \left[h^{-\frac{1}{2}} \| \varphi \|_V + h^{\frac{1}{2}} \| \nabla \varphi \|_V \right]$

trace estimate

4a) $\| \varphi \|_F \le c\, h^{-\frac{1}{2}} \| \varphi \|_V$ inverse trace estimate

Note: 3 and 4 hold for polynomials only, others for H^1-functions

The Swiss Pocket Knife of FEM

(Scaling Lemma)

V: mesh cell of diameter h | $\hat{V}$: reference cell of diameter 1

$\hat{V}$ maps to V and φ corresponds to $\hat{\varphi}$ under this mapping

$$\varphi(x) = \hat{\varphi}(\hat{x})$$

Then:

$$\|\partial^\alpha \varphi\|_V \sim h^{\frac{d}{2} - |\alpha|} \|\hat{\partial}^\alpha \hat{\varphi}\|_{\hat{V}}$$

d = space dimension

$$\|\partial^\alpha \varphi\|_F \sim h^{\frac{(d-1)}{2} - |\alpha|} \|\hat{\partial}^\alpha \hat{\varphi}\|_{\hat{F}}$$

Requirement: shape regularity

Finite Element Approximation

(Bramble-Hilbert Lemma)

For a mesh cell V of diameter h holds that for any smooth function φ on V there exists $\varphi_h \in P_k$ such that

$$\|\partial^\alpha(\varphi - \varphi_h)\|_V \leq c\, h^{m-|\alpha|} \|D^m \varphi\|_V$$

for $|\alpha| \leq m$