Particle in Cell - ES1

Two Stream Instability

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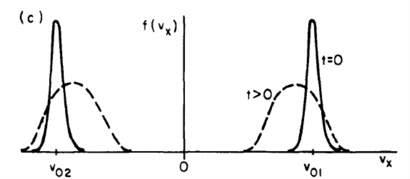
Two Stream Instability

Consist of two types of fluid which can be either cold or hot electron beam which moving through a periodic domain with the length of L.



We'll first derived the linear behavior of the instability then verified the theory by applying PIC code.

Theory

Govern Equation

Describing:

$$\textit{electrostatic wave} \longleftrightarrow \textit{electron motion}$$

Using:

Momentum Equation + Continuity Equation + Poisson's equation

$$\begin{aligned}
&\partial_t v + v\partial_x v = -\frac{e}{m}E \\
&\partial_t n + \partial_x(nv) = 0 \\
&\partial_x E = -\frac{e}{\epsilon_0}(n - n_0)
\end{aligned} \tag{1}$$

where v is the velocity for electron fluid, n is the density for electron fluid and n_0 is the density of ion backgroud.

Case 1

Consider a small perturbations of density, velocity and field in a stationary background.

$$\begin{aligned} v &= v_0 + v_1 \\ n &= n_0 + n_1 \\ E &= E_0 + E_1 \end{aligned} \tag{2}$$

First we consider the case where $E_0 = 0$ and $v_0 = 0$, combine (1) and (2) and keeping only linear terms we have

$$\begin{aligned} \partial_t v_1 &= -\frac{e}{m} E_1 \\ \partial_t n_1 + n_0 \partial_x v_1 &= 0 \\ \partial_x E_1 &= -\frac{e}{\epsilon_0} n_1 \end{aligned} \tag{3}$$

Case 1

Then we assuming the traveling wave solution which is $\propto e^{i(kx-\omega t)}$

$$\begin{aligned} i\omega v_1 &= -\frac{e}{m}E_1 \\ -i\omega n_1 + i n_0 k v_1 &= 0 \\ ikE_1 &= -\frac{e}{\epsilon_0}n_1 \end{aligned} \tag{4}$$

We have

$$\left(1 - \frac{n_0 e^2}{\epsilon_0 m \omega^2}\right)E_1 = 0 \tag{5}$$

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right)E_1 = 0 \tag{6}$$

For nontrivial solution

$$\boxed{\omega = \pm\omega_p} \tag{7}$$

which is consistence with our physics picture, the electrostatic wave has the frequency = **plasma frequency**.

Case 2

Consider another situation where an electron fluid has velocity so that $v_0 \neq 0$. Therefore the govern equation became

$$\begin{aligned} \partial_t v_1 + v_0 \partial_x v_1 &= -\frac{e}{m} E_1 \\ \partial_t n_1 + n_0 \partial_x v_1 + v_0 \partial_x n_1 &= 0 \\ \partial_x E_1 &= -\frac{e}{\epsilon_0} n_1 \end{aligned} \tag{8}$$

Again assuming plane wave solution we have

$$\left[1 - \frac{\omega_p^2}{(\omega - k v_0)^2}\right] E_1 = 0 \tag{9}$$

$$\boxed{\omega = \omega_D \pm \omega_p} \tag{10}$$

where $\omega_D \equiv k v_0$ in which the plasma oscillation frquenecy are shifted by the **Doppler frequency**.

Two stream: Case 1 + Case 2

Consist of two electron fluid species, a relative stationary to ion background, another with the velocity of v_0. For plasma neutrality we required

$$n_0 = n_{01} + n_{02} \tag{11}$$

And the momentum and continuity equation same as previous case (3) and (8). However, Poisson's equation are coupled together by two fluid.

$$\partial_x E_1 = -\frac{e}{\epsilon_0}\left(n_{11} + n_{12}\right) \tag{12}$$

Substitute n_{11} and n_{12} by momentum and continuity equation we have

$$\left[\frac{n_{01}e^2}{m\epsilon_0\omega^2} + \frac{n_{02}e^2}{m\epsilon_0\left(\omega - \omega_D\right)^2} - 1\right]E_1 = 0 \tag{13}$$

Two stream: Case 1 + Case 2

Finally, the dispersion relation is

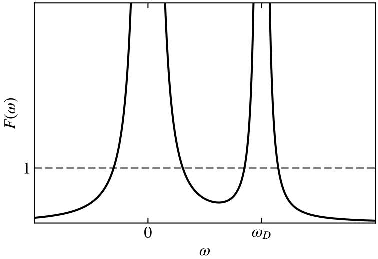
$$F(\omega) \equiv \boxed{\frac{\omega_{p1}^2}{\omega^2} + \frac{\omega_{p2}^2}{(\omega - \omega_D)^2} = 1} \tag{14}$$

where ω_{p1} and ω_{p2} are the plasma frequency of two species respectively.

Two stream: Case 1 + Case 2

$$F(\omega) = \frac{\omega_{p1}^2}{\omega^2} + \frac{\omega_{p2}^2}{(\omega - \omega_D)^2} = 1$$

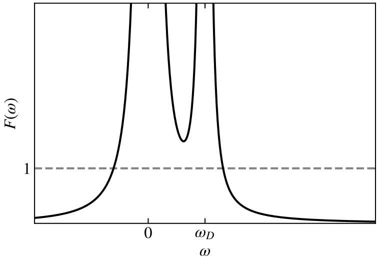
For $\omega_D >> 0$



$$\omega \approx \pm\omega_{p1} \qquad \omega \approx \omega_D \pm \omega_{p2} \tag{15}$$

Two stream: Case 1 + Case 2

For $\omega_D \sim 0$



We look for the minimum of $F(\omega)$ which is happen at

$$\omega = \omega_m = \frac{\omega_D}{1 + (\omega_{p2}/\omega_{p1})^{2/3}} \equiv \frac{\omega_D}{1 + \alpha^{2/3}} \tag{16}$$

where $\alpha \equiv \dfrac{\omega_{p2}}{\omega_{p1}}$.

Two stream: Case 1 + Case 2

For $\omega_D \sim 0$

Then the minimum of function $F(\omega)$ is

$$F_m = F(\omega_m) = \frac{\omega_{p1}^2}{\omega_D^2}\left(1 + \alpha^{2/3}\right)^3 \tag{17}$$

Let's assume that $n_{02} \ll n_{01}$, $\omega_{p2} \ll \omega_{p1}$ and $\alpha \ll 1$.

$$F_m \approx \frac{\omega_{p1}^2}{\omega_D^2}\left(1 + 3\alpha^{2/3}\right) \tag{18}$$

So then if

- $\omega_{p1}^2/\omega_D^2 < \left(1 + 3\alpha^{2/3}\right), \quad F_m < 1 \quad \Rightarrow$ 4 real roots.
- $\omega_{p1}^2/\omega_D^2 > \left(1 + 3\alpha^{2/3}\right), \quad F_m > 1 \quad \Rightarrow$ 2 real + 2 complex roots.

$\Rightarrow \omega = \omega_{re} \pm i\omega_{im}$

It will have the solution of $E \propto e^{i(kx - \omega_{re} t)} e^{\omega_{im} t}$ which grows in time.

Under what condition did the instability happened ?

We expand the $F(\omega)$ around ω_m

$$F(\omega) = F(\omega_m) + \frac{1}{2!}F''(\omega_m)(\omega - \omega_m)^2 + \dots \tag{19}$$

then

$$\omega = \omega_m \pm i\left[\frac{2F(\omega_m) - 2}{F''(\omega_m)}\right]^{1/2} \tag{20}$$

$$\omega = \omega_D\left(1 - \alpha^{2/3}\right) \pm i\omega_D\frac{\alpha^{1/3}}{\sqrt{3}}\left(1 - \frac{\omega_D^2}{\omega_{p1}^2}\right)^{1/2} \tag{21}$$

In particular $\omega_{re} = kv_0\left(1 - \alpha^{2/3}\right)$, so the phase velocity of the electrostatic wave is

$$v_f = \frac{\omega_{re}}{k} = v_0\left(1 - \alpha^{2/3}\right) \approx v_0 \tag{22}$$

TWO STREAM: CASE 1 + CASE 2

The instability happened roughly at

$$\boxed{v_f \approx v_0} \tag{23}$$

When **phase velocity are close to fluid velocity**, the coupling effect between electrons and wave became very strong. With v_0 slightly larger than phase velocity , the energy of electron is easily transfer to wave which result in positive feedback **growth of the electrostatic wave**.

Two stream: Instability

Instability occur when $F(\omega_m) > 1$, when $\omega_{p1} = \omega_{p2} = \omega_p$ it gives us

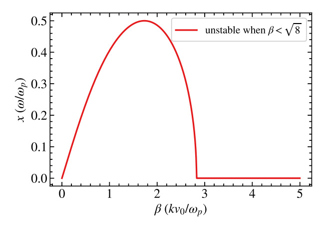
$$\omega_p^2/\omega_D^2 > 1/8 \tag{24}$$

And defining $\beta \equiv \omega_D/\omega_p$, we have following instabiility condition

$$\boxed{\beta < \sqrt{8}} \tag{25}$$

Which is verified by directly solve the dispersion relation.

$$\frac{\omega_p^2}{\omega^2} + \frac{\omega_p^2}{(\omega - \omega_D)^2} = 1$$



Simulation

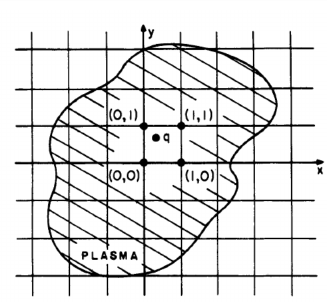
Temporal Grid

$$\begin{aligned} \frac{dr_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= -\frac{e}{m_e} E(r_i) \\ E(x) &= \sum_i \frac{1}{4\pi\epsilon_0} \frac{e}{r_i^2} \end{aligned} \tag{26}$$

In this scheme we only need two step to implement this simulation: first find the electric field and then integrate the equation of motion. **Combine this two procedure we have form the temporal grid**, however this process is **time consuming** (eg. for N particles it has the computation complexity of $\approx 2N + N(N-1)/2$).

A better way is to simplified the process of calculating electric field, since **plasma provide shielding that only the particles within few nearby Debye cube are significant**, we don't need all the information of particles to calculate the electric field.

Instead, we **divide the space into spatial grid** that store the information regarding to density(ρ), potential(ϕ) and field(E).



In this scenario we are equivalent of solving following four equation.

$$\begin{aligned} \frac{dr_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= -\frac{e}{m_e}E(r_i) \\ E(x) &= -\frac{d\phi(x)}{dx} \\ \frac{d^2\phi(x)}{dx^2} &= \frac{e}{\epsilon_0}\left(n_0 - n(x)\right) \end{aligned} \tag{27}$$

Changing the first and second order ODE into finite difference form.

$$\begin{aligned} \frac{dr_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= -\frac{e}{m_e}E(r_i) \\ E(x) &= -\frac{d\phi(x)}{dx} = \frac{\phi_{j-1} - \phi_{j+1}}{2\Delta x} \\ \frac{\phi_{j-1} - 2\phi_j + \phi_{j+1}}{(\Delta x)^2} &= \frac{d^2\phi(x)}{dx^2} = \frac{e}{\epsilon_0}\left(n_0 - n(x)\right) \end{aligned} \tag{28}$$

Weighting - First order weighting (CIC)

$$
\begin{aligned}
q_j &= q_c \frac{X_{j+1} - x_i}{\Delta x} \\
q_{j+1} &= q_c \frac{x_i - X_j}{\Delta x} \\
E(x_i) &= \left[\frac{X_{j+1} - x_i}{\Delta x}\right] E_j + \left[\frac{x_i - X_j}{\Delta x}\right] E_{j+}
\end{aligned}
\tag{29}
$$

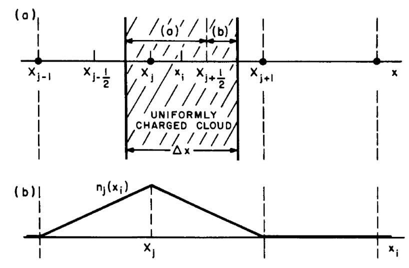
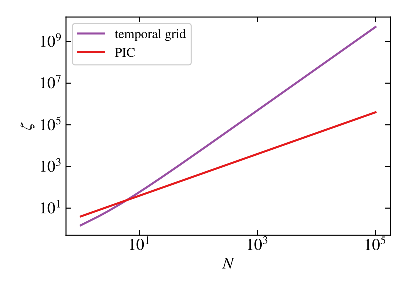


Table 1: Computation Complexity

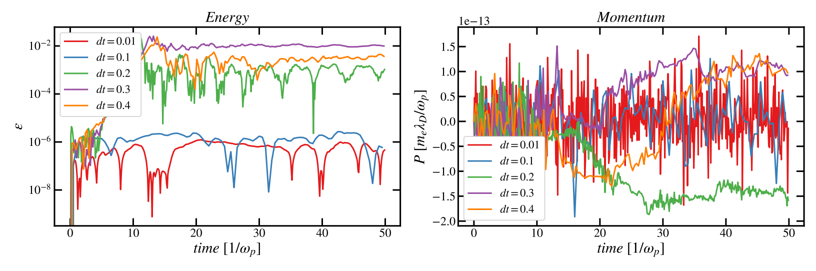
Method	Complexity
Temporal Grid	$\approx N^2/2 + N$
PIC	$\approx 4N$



Simulation Parameters & Convergence Test

Table 2: Parameters

Variables	Quantity
L	$2\pi/0.612\ (\lambda_D)$
Cell length	$0.7\ (\lambda_D)$
Particle per Cell	$10,000$
beam velocity	$1\ (\lambda_D\omega_p)$



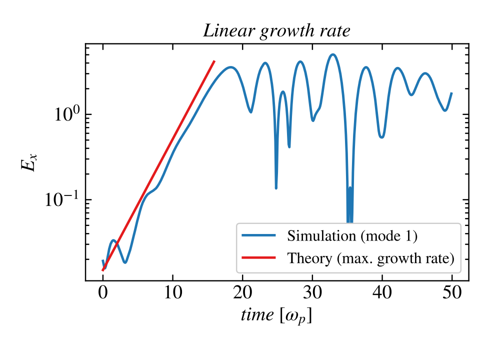
Verification

Growth Rate

For two opposing moving electron beams with immobile ions as a background the dispersion relation gives:

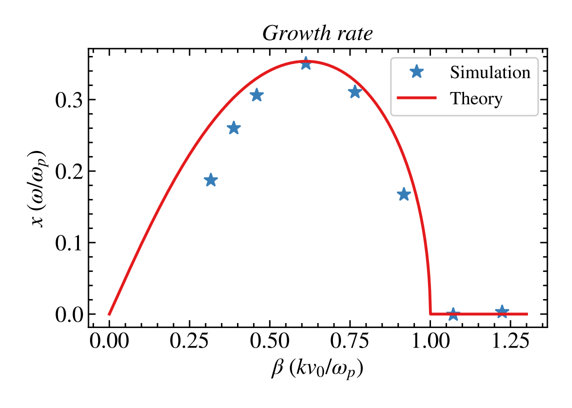
$$1 = \frac{\omega_p^2/2}{(\omega + \omega_D)^2} + \frac{\omega_p^2/2}{(\omega - \omega_D)^2}$$

We can then verified that the growth rates of electric field in particular mode accord the theory we derive earlier.



Growth Rate

Finally we found out that in PIC simulation, mode with larger wave number are more consistence with theory, compare to small wave number mode which are more prone to competed with different modes in consequence of smaller growth rate than theory predict.



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- 1985 Berkely Plasma Physics via Computer Simulation
- Nonlinear Space Plasma Physics (I) [SS-841] Chapter 4 Chapter 4. Two-Stream Instabilities