University of Calcutta

B.Sc Physics Honours Tutorial Examination

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Semester : 5th, Date : 03/01/2022

INTRODUCTION

Most of our knowledge about the astrophysical Universe is based on the electromagnetic radiation that reaches us from the sky. By analyzing this radiation, we infer various characteristics of the astrophysical systems from which the radiation was emitted or through which the radiation passed. Hence an understanding of how radiation interacts with matter is very vital in the study of astrophysics. Such an interaction between matter and radiation can be studied at two levels: *macroscopic and microscopic*. At the macroscopic level, we introduce suitably defined emission and absorption coefficients, and then try to solve our basic equations assuming these coefficients to be given. This subject is known as *radiative transfer*. At the microscopic level, on the other hand, we try to calculate the emission and absorption coefficients from the fundamental physics of the atom.

First we introduced the concept of Thomson scattering and a little bit of what opacity is because elaborated mathematical derivations are not required to solve the problem. Next, we calculated the necessary concepts microscopically (with the concept of probability theory including the zeroth order approximation). we could have calculated the mean free path in a simpler way (by simple geometrical approximation of a classical particle) but we haven’t done so because that calculation wouldn’t lead us to the BEER-LAMBERT equation that is important to draw relations between the two sets of parameters (microscopic and macroscopic cum astrophysical).

THE PROBLEM

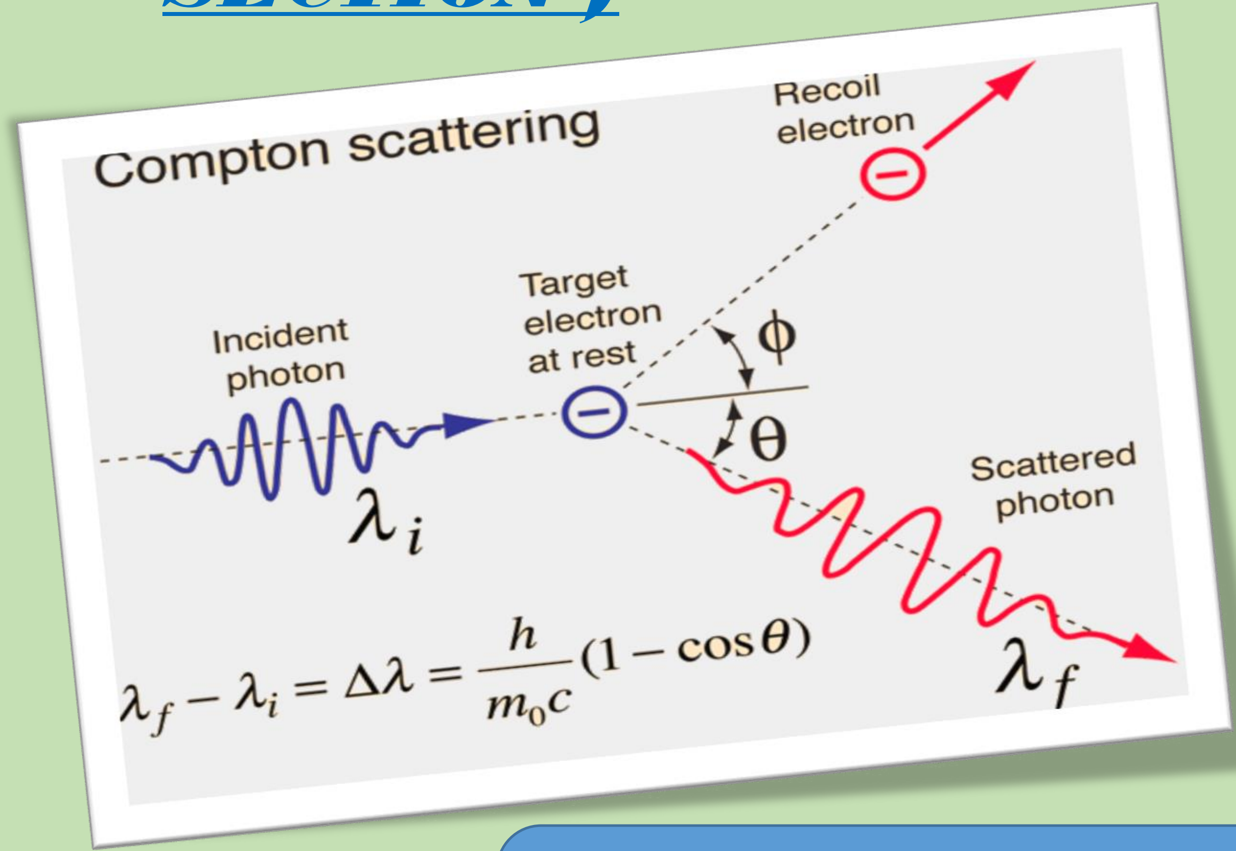
- Electron scattering is the primary source of opacity in the stellar core, with the Thompson scattering cross section being $\sigma_{th} = 6 \times 10^{-29}\ m^2$. Assuming that the solar core consists of fully ionized hydrogen of density of $10^5\ kg/m^3$, calculate the distance travelled by a photon between collisions.

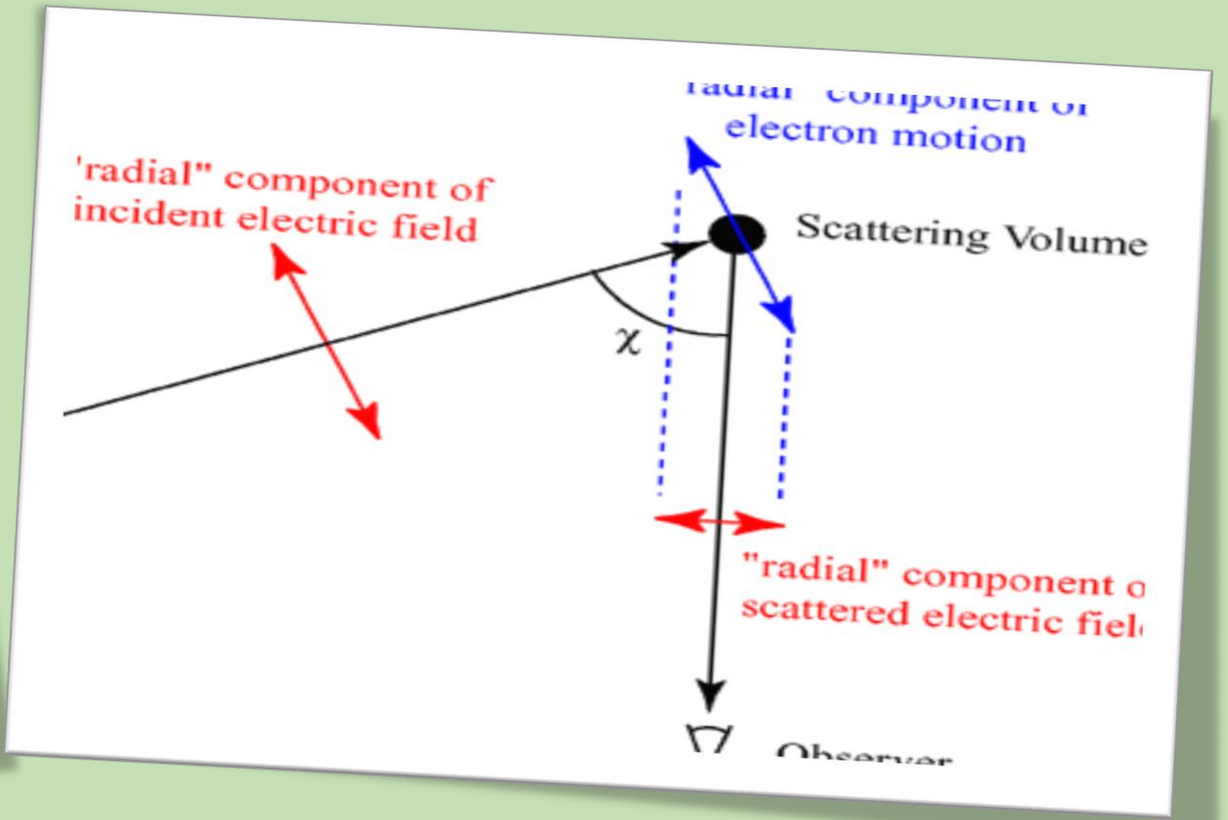
CONCEPTS NECESSARY TO SOLVE THE GIVEN PROBLEM :

1) *CONCEPT OF THOMSON SCATTERING STARTING FROM COMPTON SCATTERING AND FINALLY KNOWING THE SCATTERING CROSS SECTION WITH THE “KLEIN-NISHINA” THEORY.*

2) *DERIVATION OF MEAN FREE PATH FOR TRAVELLING PHOTONS.*

(COMPTON SCATTERING TO THOMSON SCATTERING AND THE KLEIN-NISHINA THEORY TO CALCULATE THE SCATTERING CROSS SECTION)





$$\epsilon_f = \frac{\epsilon_i}{1+\frac{\epsilon_i}{m_0c^2}(1-\cos\theta)},$$ (ϵ_i, ϵ_f → energies of incident photon before and after collision, respectively.)

THOMSON SCATTERING :

- *Here we use classical theory to derive the elastic scattering of electromagnetic radiation from an (unbound ≡ the binding energy is very low in compared to the incident photon) (free), charged particle. The assumptions are that:*

1) The wavelength of the radiation is much larger than the size of the particle

2) Net, average momentum transfer to the particle is negligible

3) The intensity of the radiation is low enough that the induced velocities of the particle are very small.

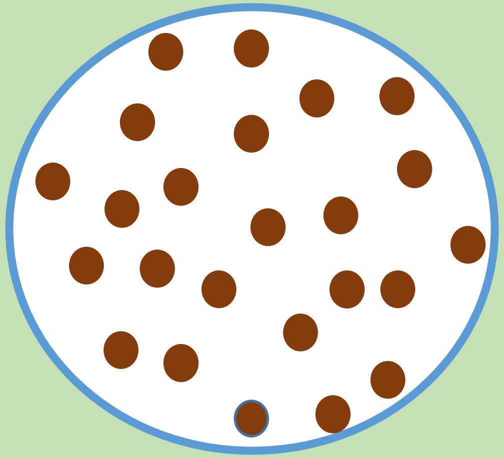
This represents the low-energy limit of Compton scattering of a single photon and a free electron, and it was first derived by J. J. Thomson, the discoverer of the electron, in a book published in 1906; subsequently this theory has been known as Thomson scattering

- *Main cause of the acceleration of the particle is due to the electric field component of the incident wave.*
- *In a first approximation, the influence of the magnetic field can be neglected.*
- *The particle will move in the direction of the oscillating electric field, resulting in electromagnetic dipole radiation.*
- *The moving particle radiates most strongly in a direction perpendicular to its acceleration and that radiation will be polarized along the direction of its motion*
- *Depending on where an observer is located, the light scattered from a small volume element may appear to be more or less polarized.*

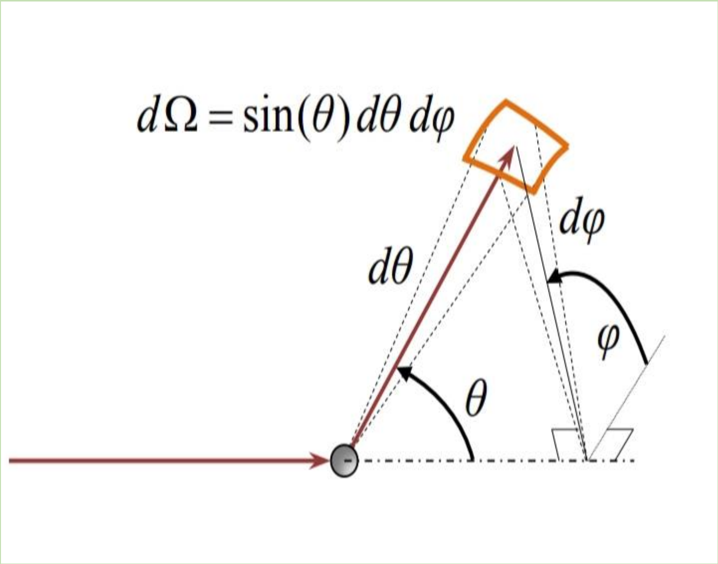
➢ (SCATTERING CROSS SECTION):

The ratio of the intensities of the induced radiation to the incident radiation would be interpreted as the probability of scattering . Scattering probability is normally expressed as a scattering cross section, which represents effective area subtended by a stationary target as seen by a stream of incident particles.

Figure



A very thin section of a volume containing *N* target particles, each with cross sectional area *σ*. If the volume's total cross sectional area is *A*, the targets obstruct a fraction *Nσ/A* of the area. On average, that fraction of a uniform beam of incident particles will encounter targets and be scattered.



[In spherical polar coordinate]

- ***(Klein-Nishina scattering cross section):***

Dirac analyzed Compton scattering in 1926 while still a graduate student and derived the following differential cross section for the process:

"Spinless" electron scattering:

$$\frac{d\sigma}{d\Omega} = \frac{r_e^{\ 2}}{2} \left(\frac{\epsilon_f}{\epsilon_i} \right)^2 \ (1 + \cos^2 \theta)$$

[r_e : classical electron radius,

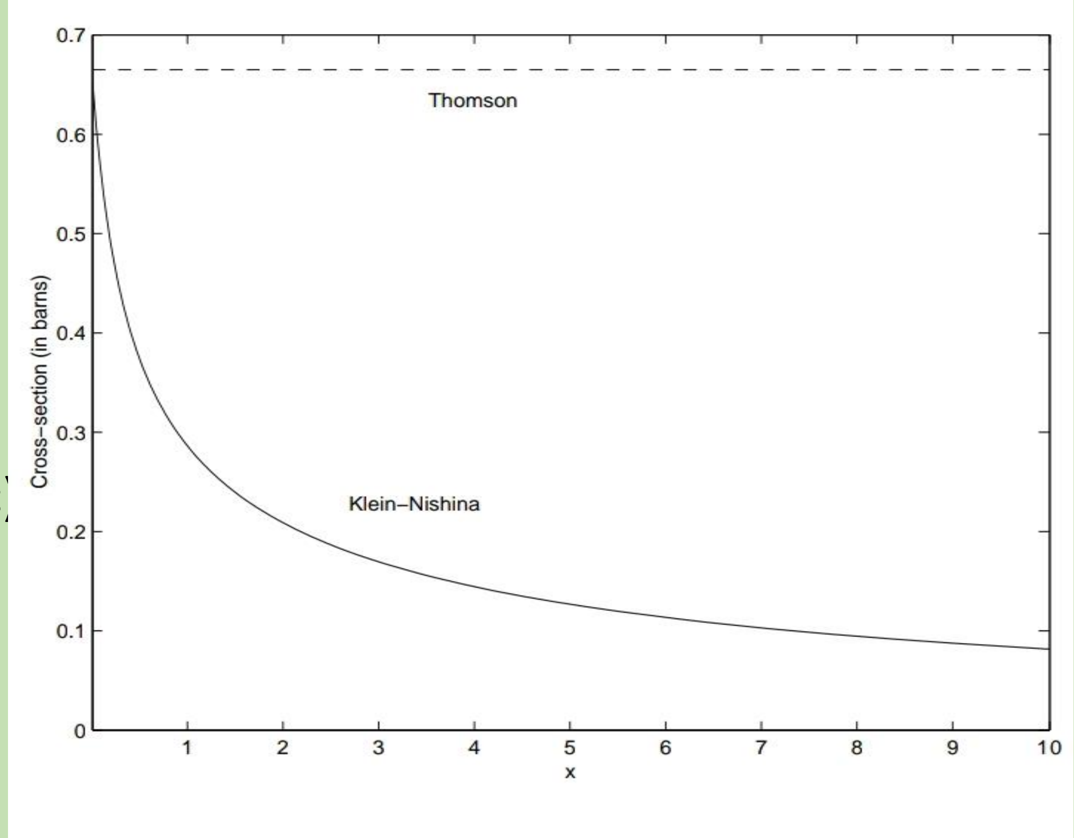
(ϵ_f , ϵ_i) : defined previously, $d\Omega$ = differential solid angle

$d\Omega = \sin\theta\, d\theta\, d\phi$]

Unfortunately, Dirac's equation neglects the fact that an electron has a magnetic moment generated by its intrinsic angular momentum (spin 1/2). This magnetic moment also interacts with the incident photons' electromagnetic fields and serves as an additional scattering mechanism. Dirac's 1928 papers provided the theoretical framework to properly calculate the additional effects of the electron's spin, but that theory is more complicated, subtle, and difficult to apply.

Oscar Klein and Yoshio Nishina worked feverishly in the months following Dirac's publication to understand his ideas and apply them to Compton scattering, publishing their calculation only six months later. By successfully including the effects of the electron's spin, their result provided a slight modification (at first glance) : the addition of another term to its final factor, shown in the following equation:

$$\frac{d\sigma}{d\Omega} = \frac{r_e^{\,2}}{2}\left(\frac{\epsilon_f}{\epsilon_i}\right)^2 \left(\frac{\epsilon_f}{\epsilon_i} + \frac{\epsilon_i}{\epsilon_f} - \sin^2\theta\right)$$



r_e : classical electron radius ; $(\epsilon_f , \epsilon_i)$: defined prev.

For Thomson scattering , $(\epsilon_f = \epsilon_i)$. So, the above eqn

Takes the form : $\frac{d\sigma}{d\Omega} = \frac{r_e^{\,2}}{2}(1 + \cos^2\theta)$

The solution : $\sigma = \sigma_T\left(1 - 2x + \frac{26x^2}{5} + \cdots\right)$ (Non-relativistic)

Now, as $\left(x = \frac{h\nu}{m_0c^2}\right) \rightarrow 0$; $\sigma \rightarrow \sigma_T$

Thus, we get the Thomson scattering cross section,

$$\sigma_{th} = \frac{8\pi}{3} r_e^{\,2}$$

(INCREMENT OF THE EFFECTIVE AREA DUE TO OSCILLATION)

❖ OPACITY :

Opacity is the measurement of how opaque a stellar medium is. The gas in the interior of a star exists under such conditions of temperature and pressure which cannot be easily reproduced in the laboratory. Hence we cannot experimentally find out χ for conditions appropriate for the stellar interior. The opacity χ, therefore, has to be calculated theoretically. This is a fairly complicated calculation. With improvements in stellar models, more and more accurate computations of opacity are demanded. This has become a highly specialized and technical subject, with very few groups in the world who have the right expertise for calculating opacity accurately. Other scientists who need values of opacity for their research almost never try to calculate the opacity themselves, but use the values computed by the groups who specialize in these computations. For several decades, the so-called Los Alamos opacity tables (Cox and Stewart, 1970) remained the last word on this subject.

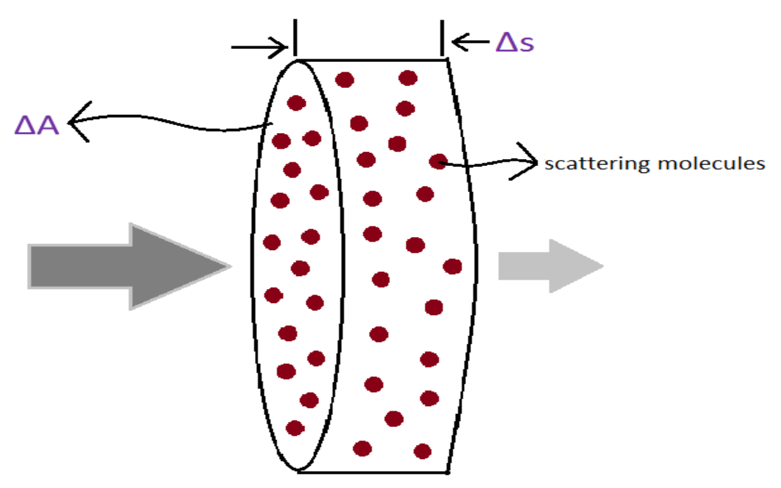
Mean free path

Let's consider a uniform beam of identical particles (photons) are travelling through a slab with face area ΔA and (infinitesimal) thickness Δs, containing an array of nearly motionless scattering particles (all identical to one another and with number density 'n').

Some small fraction of the incoming particles will be scattered by the targets in the volume.

The scattered fraction is:

$$P = n\sigma\Delta A.\Delta s/\Delta A = n\sigma\Delta s$$



Because the incident particles in the beam are identical to each other, the scattered fraction must be the probability that any particular incident particle will be scattered when passing through the target volume.

Therefore, the probability that individual photon interacts is:

$$\boxed{P = n\sigma\Delta s}$$

Where 'n' is number of particles per unit volume. In the following calculation, we consider it as a constant.

Important result: probability of interaction per unit length is $P/\Delta s = n\sigma$.

We assume that If N_p photons in the incident beam have penetrated a distance s into the target volume without being scattered. So, on exiting PN_P interacts.

So number of remaining photons (that avoided scattering) in the beam is

$$N_P(s+\Delta s) = N_P(s) - PN_P(s)$$

Or,

$$\boxed{N_P(s+\Delta s) = N_P(s)(1 - P)} \quad \text{-----2)}$$

Rearranging :

$$N_P(s+\Delta s) - N_P(s) = -PN_P(s) = N_P(s)n\sigma\Delta s$$

In the limit , $ds \rightarrow 0$

$$(N_P(s + \Delta s) - N_P(s) / \Delta s) = dN_P/ds = -N_P n\sigma \quad \text{----3)}$$

❖ ***PHOTON TRANSPORT EQUATION*** :

The differential equation we got is,

$$dN_P/ds = -(N_P)n\sigma$$

Solution: (Initial condition : At $s = 0$; $N_P = N_{P,0}$)

Therefore,

$$N_P(s) = N_{P,0} \, . \, e^{(-n\sigma s)} \quad \text{----4)}$$

Here $N_{P,0}$ is the no. of photons when the beam just started entering the slab.

❖ ***COROLLARY*** :

Probability that a photon travels macroscopic distance s without interaction :

$$Q(s) = (N_P(s)/N_{P,0}) = e^{(-n\sigma s)} \quad \text{---- 5)}$$

Also, probability that an interaction takes place in the interval (s, s+ds) is the probability that a photon travels distance s without interaction , times probability it interacts in subsequent ds:

$$P(s)ds = Q(s)n\sigma ds = n\sigma e^{(-n\sigma s)}ds \quad \text{----- 6)}$$

- Most probable distance before interaction :

$$\langle s \rangle = \int_0^\infty n\sigma s e(-n\sigma s)ds = 1/n\sigma$$

Or,

$$\boxed{\langle s \rangle = 1/n\sigma} \quad \text{---- 7)}$$

This is the *MEAN FREE PATH* of photons.

- *The mean free path formula is correct for incident particles moving at speeds much greater than those of the target particles (thus, the target particles are relatively stationary).*

❖ For medium of density ρ, number density n, and molecular mass μ

$$n = \rho/\mu \qquad \text{----- 8)}$$

So, the mean free path becomes :

$$\langle s \rangle = \mu/\rho\sigma \qquad \text{------ 9)}$$

This is the relation , we need to solve the given problem.

❖ ***THE RELEVANCY OF MEAN FREE PATH IN THE FIELD OF ASTROPHYSICS:***

The parameters we were concerned of while calculating the mean free path for a photon were ***[THE NUMBER OF PHOTONS IN THE MOVING BEAM, THE SCATTERING CROSS SECTION and THE NUBER DENSITY OF THE SCATTERING MOLECULES].***

In astrophysics, we are rather interested in absorption of photons than the scattering phenomena of them. Therefore, in astrophysics, our concern leads us to the radiative transport equation that contains the parameters such as ***[THE ABSORPTIVE COEFFICIENT, OPTICAL DEPTH and THE SPECIFIC INTENSITY].***

We can easily establish the relation between the two aforesaid sets of different parameters.

In the previous section, we got the relation,

$$\boxed{N_P(s) = N_{P,0} \,.\, e^{(-n\sigma s)}} \qquad \text{----- 10)}$$

Now, the energy of the beam when it enters the cluster is

$$E_{p,0,\nu}(s) = h\nu . N_{P,0}(s)$$

The energy of the beam as it travels a distance s through the cluster is

$$E_{p,\nu}(s) = h\nu . N_P(s)$$

Now, the definition of specific intensity in astrophysics is,

[*SPECIFIC INTENSITY IS THE AMOUNT OF ENERGY FLOW THROUGH A UNIT SURFACE , AT UNIT TIME, THROUGH UNIT SOLID ANGLE.*]

So we can write,

$$\frac{E_{p,0,\nu}(s) \,/dA.dt.d\Omega}{E_{p,\nu}(s) \,/dA.dt.d\Omega} = \frac{I_{\nu,0}}{I_{\nu}} = e^{(n\sigma s)} \qquad \text{[simplified form]}$$

$$\boxed{I_{\nu,0} = I_{\nu} \,.\, e^{(n\sigma s)}} \qquad \text{------ 11)}$$

[BEER-LAMBERT LAW]

We previously have seen that the ratio $[\frac{N_P(s)}{N_{P,0}}]$ represents the probability that a photon travels macroscopic distance s without interaction . So, $\frac{I_{\nu,0}}{I_\nu}$ also represents the same probability as $[\frac{I_{\nu,0}}{I_\nu}] = [\frac{N_{P,0}}{N_P}]$, N_P is a function of s.

Now, we know that the radiative transfer equation is

$$\boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu} \qquad \text{------ 12)}$$

[without considering emission coefficient]

Solving the differential equation we get,

$$\boxed{I_{\nu,0} = I_\nu \, . \, e^{\int -\alpha_\nu ds}} \qquad \text{-------- 13)}$$

Now, we define a new parameter named *optical depth* as

$$\int -\alpha_\nu ds = \tau \qquad \text{----- 14)}$$

Now, comparing the two equations we got for specific intensity,

1) $I_{\nu,0} = I_\nu \,.\, e^{(\int -\alpha_\nu ds)}$ and 2) $I_{\nu,0} = I_\nu \,.\, e^{(-n\sigma s)}$

We get,

$$\int -\alpha_\nu ds = \tau = -n\sigma s \qquad \text{------ 15)}$$

Now if α_ν cum n is independent of s, we have →

$$\alpha_\nu = n\sigma$$

From equation 15), we get →

$$\tau = n\sigma s$$

Now, time averaging the above equation, we get →

$$n\,\sigma < s > = < \tau > = \int_0^{\infty} \tau P(\tau) d\tau \quad \text{[By definition]}$$

Or, $$n\,\sigma\,(l_m) = \int_0^{\infty} \tau.\,(1/\,e^{\tau})\; d\tau \quad \text{[Using the property of gamma integral]}$$

Finally, $$\boxed{(l_m) = 1/\,n\,\sigma}$$

[(l_m) → MEAN FREE PATH]

Now, a question may be asked about how we can relate the absorption coefficient with the scattering cross section or more generally the validation of the relations between the two sets of parameters where one set belongs to the scattering phenomena while the other one corresponds to the absorption phenomena.

The answer is that the molecules that play the role of scatterers in the scattering phenomena play the role of absorbers in the absorption phenomena. In both the cases, the entire light beam is unable to come to the observer either due to scattering or due to absorption(complete).

(FINALLY THE SOLUTION OF THE PROBLEM) :

It is said that electron scattering is the primary source of opacity in the stellar core, with the Thomson scattering cross section being

$\sigma_{th} = 6 \times 10^{-29}\ m^2$. It is also assumed that the solar core consists of fully ionized hydrogen of density of $\rho = 10^5\ kg/m^3$.

Now as we have approximated, for simplicity, that the gas is all hydrogen, then there is one scattering electron per atom of mass $m_H{}^+$.

Here, $m_H{}^+$ is the mass of an ionized hydrogen atom.

➢ *(CALCULATION OF $m_H{}^+$):*

Mass of a hydrogen atom (m_H) = 1.0079u = $1.0079 \times (1.660 \times 10^{-27})$ kg

Mass of an electron (m_e) = 9.11×10^{-31} kg.

So, mass of an ionized hydrogen atom $(m_H{}^+) = (m_H - m_e) = 1.67 \times 10^{-27}$ kg.

The density (ρ) is given. So the number density (n) is →

$n_e = \rho / m_{H^+} = (10^5 / 1.67 \times 10^{-27})$

So, the mean free path for electron scattering is →

$$\ell_m = (1/ n_e \sigma_{th}) = (m_{H^+} / \rho\, \sigma_{th}) = \frac{1.67 \times 10^{-27}}{10^5 \times 6 \times 10^{-29}}\ \text{m} = 0.00028\ \text{m}$$

$$= 0.028\ \text{cm}.$$

THE RESULT :

$$\boxed{\ell_m = 0.028\ \text{cm}}$$

conclusion

- *The calculation that is done above to evaluate the mean free path is the simplest approximation. There is one more advantage in doing this calculation that the calculation is entirely based on classical electrodynamics, no relativistic or quantum mechanical approach is associated. But in reality there are many more scatterings that include the relativity and quantum mechanics.*
- *This part of astrophysics is completely theoretical because it is very difficult to reproduce the situations of pressure and temperature under which the stellar systems actually exist. So, with rigorous mathematical calculations the necessary parameters can be calculated.*
- *Finally, it can be said that the more we expertise on this zone the more we understand the stellar systems.*


THANK YOU!!