Project work at Variable Energy Cyclotron Centre(VECC). Studying particle decay kinematics and reconstruction of unstable particles using Invariant mass technique.

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April 25, 2023

Contents

0.1 Introduction

The understanding of the dynamics of particles and their interactions rests on the foundation of relativistic kinematics. Kinematic calculations allow us to determine the energy, masses and momenta of elementary particles entering any kind of interaction with each other or with atoms and nuclei given certain measured or known quantities. With these kinematic techniques we do not deal with the nature of the interaction itself, its strength, range, spin dependance etc., but rather with a description of the energies and momenta of the initial and final state particles. No experiment in nuclear and particle physics could be interpreted without the techniques of relativistic kinematics. In this project these techniques will be investigated and applied to decay process of Omega Meson.
In section 1, an introduction of the project is given. In section 2, the basic ideas

of relativistic kinematics, notations and units used are introduced.
In section 3, the formulas of energies, momenta and velocities of the daughter particles are derived keeping the mother particle at rest.
In section 4, reconstruction of the mother particle in The mother particle's rest frame is done using isotropic decay of the mother.
In section 5, decay in the laboratory frame is established.
In section 6, references are given.

0.2 Notations and units

In relativity, space and time are no longer separate concepts but are interwoven in a way that makes the introduction of a four-dimensional space-time continuum very useful. In such a four-dimensional space many of the laws of classical physics can be expressed in a particularly elegant form which greatly simplifies the transformation from one inertial system to another. It is possible to derive everything in special relativity without the use of four-vectors. However, the use of this advanced concept is extremely helpful in making calculations and concepts much simpler and more transparent. Any event is described by two different things: One is the place where it occurred and another is time when it occurred. Therefore, an event is expressed by three spatial coordinates and one time coordinate. Such a space (four axes of which represent three space coordinates and one time coordinate) is called Minkowski's four-dimensional world. Any event is represented by a point, called world point in this space.

0.2.1 Four-vector:

The above discussion readily suggests that the space-time coordinates of a World point must have four components. In an inertial frame S, coordinates are $(x^0 = ct, x^1 = x, x^2 = y, x^3 = z)$. If x^0, x^1, x^2, x^3 are the coordinates of an event in frame S, we may consider these quantities as the components of a four-dimensional radius vector for this event. Thus, if $\hat{e}_\mu$ represents unit tangent vector along x^μ increasing direction, then the four-position vector of the event is given by

$$R = \sum_{i=1}^{3} \hat{e}_\mu x^\mu = (x^0 = ct, x^1 = x, x^2 = y, x^3 = z) = (ct, \hat{r})$$

(Note) : An upper case Roman letter (without any arrow) is used to denote a four-vector. If one wants to denote the component of a four-vector (say, (R)), it will be denoted as with a superscript to the four-vector symbol, i.e., as R^μ with $\mu = 0 - 3$ .Greek symbols are used as superscripts to denote all the four space-time components. The three dimensional (Galilean) vectors are denoted as lower case Roman letter with an over-head arrow (say, $\vec{r}$); the spatial components of a three vector are denoted by the roman superscript (e.g.v^i with I = 1 through 3).

0.2.2 Four-velocity

With the passage of time a particle, in general, changes its position and its trajectory in Minkowski's space is known as the World line. A suitable tangent vector to the World line may be thought of as the velocity of the particle in 4D. To gain simplicity in relativity we must cease to regard the ordinary velocity (the three-velocity, we may call it) as fundamental, and substitute a four-vector for it. The separation between two neighbouring world points in Minkowski's space is given by a four-displacement

$$dR \equiv (dR^0, dR^1, dR^2, dR^3) = (dx^0, dx^1, dx^2, dx^3) = (cdt, dx, dy, dz) \quad ----a) \tag{1}$$

Since space-time interval ds and the proper-time $d\tau = ds/c = dt/\gamma$ are both Lorentz invariant, the quantity $ds/d\tau$ is also Lorentz invariant. Here $\gamma = \frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}$. Accordingly, the velocity four-vector or four-velocity is defined as:

$$V \equiv (c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}) = (\frac{(dx^0)}{d\tau}, \frac{(dx^1)}{d\tau}, \frac{(dx^2)}{d\tau}, \frac{(dx^3)}{d\tau}) \quad -----b)$$

Now, $\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma\frac{d}{dt} = \frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}\frac{d}{dt} \quad ----(c)$

With the help of (c), eqn.(b) becomes,

$$V \equiv (c\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = \gamma(\frac{(dx^0)}{dt}, \frac{(dx^1)}{dt}, \frac{(dx^2)}{dt}, \frac{(dx^3)}{dt})$$

Or, $V \equiv \gamma(c, v_x, v_y, v_z) = \gamma(c, v^1, v^2, v^3) = \gamma(c, \vec{v}) \quad -----(d)$

$\vec{v} \equiv (v^1, v^2, v^3)$ is the three-velocity. It is also called Galilean velocity as the magnitude of a three-velocity vector is invariant under Galilean transformation. In component form, four-velocity is written as

$$V \equiv (V^0, V^1, V^2, V^3) = (\gamma c, \gamma v^1, \gamma v^2, \gamma v^3) \quad -------(e)$$

In the rest frame of the particle (or, rather, the instantaneous inertial frame) we have $\vec{v} = 0$ giving $\gamma = 1$ and whence eqn.(e) reduces to

$$V \equiv (V^0, V^1, V^2, V^3) = (\gamma c, \gamma v^1, \gamma v^2, \gamma v^3) = (c, 0, 0, 0) \quad -------(f)$$

The contra-variant components of the four-velocity vectors are [from eqn.(e)]

$$V^0 = \gamma c \ \ and \ \ V^i = \gamma v^i; (i = 1, 2, 3) \quad -------(g)$$

The norm of the four-velocity is given by

$$(V)^2 = (V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2 = \gamma^2 c^2 - \gamma^2[(v^1)^2 + (v^2)^2 + (v^3)^2] = \gamma^2(c^2 - v^2) = +c^2 \quad -------(h)$$

Since $(V)^2 = c^2$, its Lorentz invariance is too obvious.

0.2.3 Four-momentum

Let's consider a particle of mass m_o, as measured by an observer to whom the particle is at rest, called the rest mass of the particle. Obviously, it is an invariant quantity. If we multiply the four-velocity by invariant mass m_o we get another four-vector. In Newtonian mechanics the momentum of a particle is defined as its mass times velocity. Accordingly, we define by analogy the four-momentum as

$$P = m_o V \equiv (P^0, P^1, P^2, P^3) = m_o(\gamma c, \gamma v^1, \gamma v^2, \gamma v^3) \qquad ------a)$$

In component form, the four-momentum is given by
$P^0 = m_o V^0 = m_o \gamma c = mc \;\; and \;\; P^i = m_o; V^i = m_o \gamma v^i = mv^i; (i = 1, 2, 3) \; - \; -(b)$

In writing the final forms of the eqns.(b) we have used the notation

$$m = m_o \gamma = m_o \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} \qquad --------(c)$$

Using (b), eqn.(a) can be written as

$P = m_o V \equiv (P^0, P^1, P^2, P^3) = (mc, mv^1, mv^2, mv^3) = (mc, m\vec{v}) = (mc, \vec{p}) \; - \; -(d)$

The Galilean momentum $\vec{p} = m\vec{v}$ could be defined if one identifies the mass of a moving particle as $m = m_o \gamma = m_o \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}}$. Here, m_o is the value of m when $v = 0$ and hence is called the rest mass of the particle. The mass of a particle is thus velocity dependent; its rest mass is, however, an invariant quantity. Eqn.(c) is nothing but the famous formulation for the variation of mass with velocity, which has been verified experimentally. It is seen that the expression for the moving mass comes out automatically in course of derivation of four-momentum. The norm of four-momentum is

$(P)^2 = (P^0)^2 - (P^1)^2 - (P^2)^2 - (P^3)^2 = m_o^2 \gamma^2 c^2 - m_o^2 \gamma^2 [(v^1)^2 + (v^2)^2 + (v^3)^2] = m_o^2 \gamma^2 (c^2 - v^2) = +m_o^2 c^2 \qquad -------- (e)$

This proves the Lorentz invariance of the norm of the four-momentum. If we assume, for the time being, that $E = m_o \gamma c^2 = mc^2$ as the expression for the total energy of a relativistic particle, then the temporal part of the four-momentum becomes

$$P^0 = m_o V^0 = m_o \gamma c = mc = E/c \qquad ------(f)$$

Consequently, eqn.(d) becomes

$$P = m_o V \equiv (P^0, P^1, P^2, P^3) = (mc, \vec{p}) = (E/c, \vec{p}) \qquad ------(g)$$

Hence the momentum four-vector is popularly known as energy-momentum four-vector. Equating the norm of the four-momentum of eqn.(6) with he previously determined value $+m_o^2 c^2$ [vide eqn.(e)] we get

$$E^2/c^2 - p^2 = +m_o^2 c^2$$

Or,$E^2 = +m_o^2c^4 + c^2p^2 \qquad - - - - - - - - (h)$

This is another relation for the energy of a relativistic particle in terms of its rest mass and momentum.

It is interesting to investigate the four-momentum of a particle with zero rest mass. Eqn.(e) gives the norm of four-momentum for such a particle vanishes, and as a result, we have

$(P^0)^2 = (P^1)^2 + (P^2)^2 + (P^3)^2 = p^2 \qquad - - - - - - - - -(i)$

From eqn.(h) we get

$E^2 = c^2p^2$

In the following we shall use units defined by $\mathbf{c = 1}$, where c is the speed of light. This is convenient in the kind of calculations characteristic of relativistic kinematics, because all expressions must then be homogeneous in energies, momenta and masses, which all have the same dimension. Velocities do not occur very often in these calculations, but one must remember that particle velocities are dimensionless and do not exceed 1. Thus the relativistic factor of a particle of velocity v is written as :

$\frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}$

0.3 Two boby decay of unstable particles

The simplest kind of particle reaction is the two-body decay of unstable particles. The unstable particle is the mother particle and its decay products are the daughter particles. In this project the decay kinematics of the mother particle **Omega Meson** into the daughter particles (**Muons**: $\mu^+ and \mu^-$) are discussed.

- **(Decay equation) :**

$$\omega ====> \mu^+ + \mu^-$$

Now let's consider the decay of the mother particle of mass M which is initially at rest. Then its 4-momentum is $P = (M; 0; 0; 0)$. This reference frame is called the centre-of mass frame (CMS). We denote the 4- momenta of the two daughter particles by $\vec{p}_{\mu^+}$ and $\vec{p}_{\mu^-}$: $P_{\mu^+} = (E_{\mu^+}; \vec{p}_{\mu^+}), P_{\mu^-} = (E_{\mu^-}; \vec{p}_{\mu^-})$. In the CMS frame $\vec{p}_{\mu^+} = -\vec{p}_{\mu^-}$. So, $(abs(\vec{p}_{\mu^+}) = abs(-\vec{p}_{\mu^-}) = p$ 4-momentum conservation requires that

$$P_\omega = P_{\mu^+} + P_{\mu^-} \tag{2}$$

Or, ${P_{\mu^-}}^2 = (P_\omega - P_{\mu^+})^2 = {P_\omega}^2 + {P_{\mu^+}}^2 - 2P_\omega P_{\mu^+}$

Or, $m_{0,\mu^-}^2 = m_{0,\omega}^2 + m_{0,\mu^+}^2 - 2[(E_\omega, 0, 0, 0)(E_{\mu^+}, \vec{p}_{\mu^+})]$

Or, $m^2_{0,\mu^-} = m^2_{0,\omega} + m^2_{0,\mu^+} - 2E_\omega E_{\mu^+}$

Or, $2(m_{0,\omega})E_{\mu^+} = m^2_{0,\omega} + m^2_{0,\mu^+} - m^2_{0,\mu^-}$

Or, $E_{\mu^+} = \frac{(m^2_{0,\omega}+m^2_{0,\mu^+}-m^2_{0,\mu^-})}{(2m_{0,\omega})}$

Or, $E^2_{\mu^+} = {p_\mu^+}^2 + m^2_{0,\mu^+} = \frac{(m^4_{0,\omega}+m^4_{0,\mu^+}+m^4_{0,\mu^-}-2(-m^2_{0,\omega}m^2_{0,\mu^+}+m^2_{0,\mu^+}m^2_{0,\mu^-}+m^2_{0,\mu^-}m^2_{0,\omega})}{(4m^2_{0,\omega})}$

Or, $p^2_{\mu^+} = \frac{(m^4_{0,\omega}+m^4_{0,\mu^+}+m^4_{0,\mu^-}-2(m^2_{0,\omega}m^2_{0,\mu^+}+m^2_{0,\mu^+}m^2_{0,\mu^-}+m^2_{0,\mu^-}m^2_{0,\omega})}{(4m^2_{0,\omega})}$

Or, $p^2_{\mu^+} = \frac{[m^2_{0,\omega}-(m_{0,\mu^+}+m_{0,\mu^-})^2][m^2_{0,\omega}-(m_{0,\mu^+}-m_{0,\mu^-})^2]}{(4m^2_{0,\omega})}$

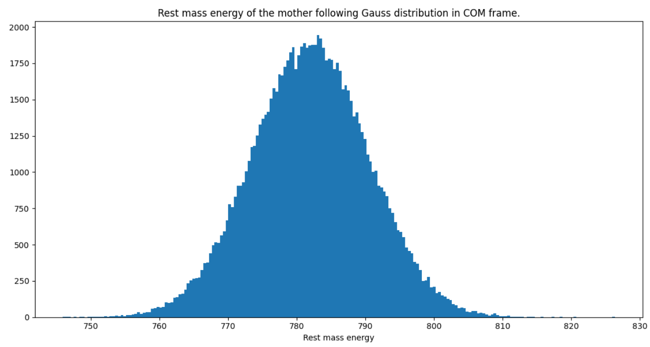
Or, $p_{\mu^+} = \frac{\sqrt{[m^2_{0,\omega}-(m_{0,\mu^+}+m_{0,\mu^-})^2][m^2_{0,\omega}-(m_{0,\mu^+}-m_{0,\mu^-})^2]}}{(2m_{0,\omega})}$

Now, as $\mu^+$and μ^- are antiparticles of each other, $m_{0,\mu^+} = m_{0,\mu^-} = m_{0,\mu}$. So, eqns (a) and (b) reduces to :
$E_{\mu^+} = \frac{m_0,_\omega}{2}$ $, p_{\mu^+} = \frac{\sqrt{[m_{0,\omega^2} - 4m^2_{0,\mu}]}}{2}$ *and* $v_{\mu^+} = \frac{\sqrt{[m_{0,\omega^2} - 4m^2_{0,\mu}]}}{2}$ $[using\, v = p/E]$

So, for the other daughter particle, $E_{\mu^-} = m_{0,\omega} - \frac{m_{0,\omega}}{2} = \frac{m_{0,\omega}}{2}, p_{\mu^-} = \frac{\sqrt{[m^2_{0,\omega} - 4m^2_{0,\mu}]}}{2}, v_{\mu^-} = \frac{\sqrt{[m^2_{0,\omega} - 4m^2_{0,\mu}]}}{m_{0,\omega}}$ In the following plots, the rest masses of omega meson and muons are considered to be 782.00 Mev (Using particle data guide) and 105.66 Mev (From Wikipedia), respectively.

0.4 Reconstruction of the mother particle in the Mother particle's rest frame:

First we generate a Gaussian distribution for the rest mass of the mother particle. This distribution has its **mean value at the x value of 782 Mev and a standard deviation of 8.5** .In the following, the reconstruction of the mother particle is established by using the invariant mass technique.



0.4.1 (Establishing isotropic decay in mother's rest frame):

The condition for an isotropic decay is $\frac{dN}{d\omega} = constant$

Or, $\frac{dN}{sin\theta d\theta d\phi} = \frac{dN}{d(cos\theta)d\phi} = constant$

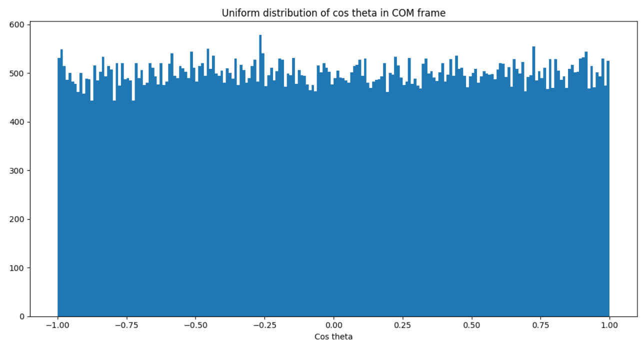
Or, $\frac{dN}{d(cos\theta)} = constant; \frac{dN}{d\phi} = constant$ [as θ and ϕ are independent of each other]

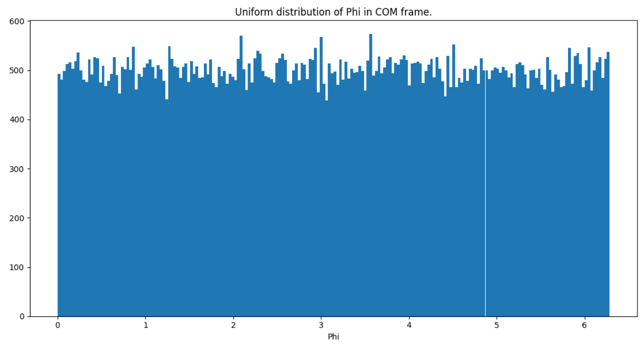
Now to establish the isotropic decay in python,

i) ϕ is uniformly generated in the range (0 , 2π)

ii) cosθ is uniformly generated in the range (-1 , +1)

Finally considering N to be large enough (a large number of mother particle decays) so that a proper distribution can be obtained, we plot histograms corresponding to the uniform distributions.





Histograms corresponding to the cartesian components of the momentum of the daughter particles $p_{x,1} = p_1 sin\theta cos\phi; p_{y,1} = p_1 sin\theta sin\phi; p_{z,1} = p_1 cos\theta\ and\ p_{x,2} = -p_{x,1}; p_{y,2} = -p_{y,1}; p_{z,1} = -p_{z,1}$ are also plotted.

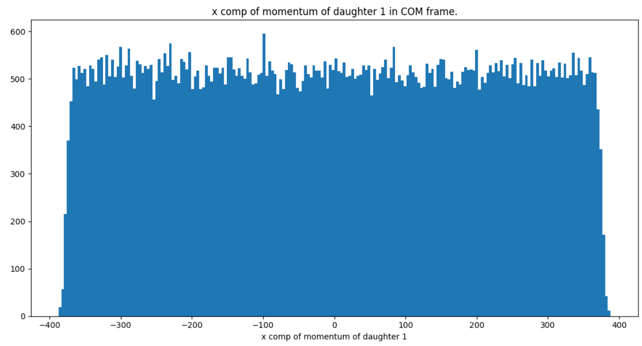
Now, the energies of the corresponding daughter particles are

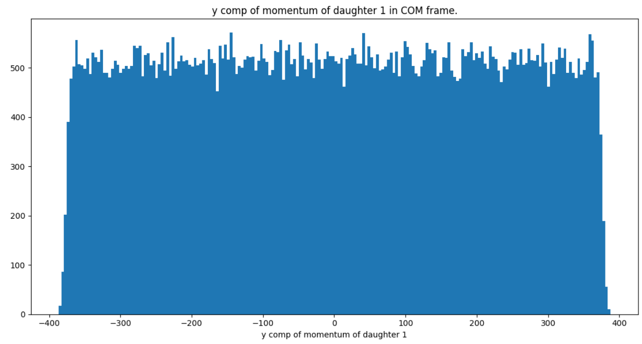
$E_1 = \sqrt{{p_1}^2 + {m_{0,\mu}}^2}$

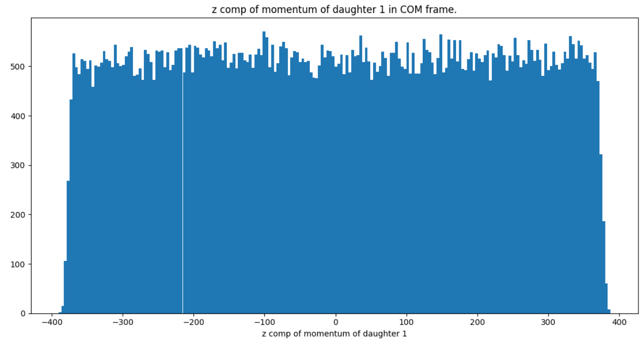
$E_2 = \sqrt{{p_2}^2 + {m_{0,\mu}}^2}$

Finally, using the formula : $m_{0,\omega} = 2{m_{0,\mu}}^2 + 2(E_1E_2 - \vec{p}_1.\vec{p}_2)^{1/2}$ the rest mass of the mother particle is reconstructed. (Here $\vec{p}_1, \vec{p}_2$ are three vectors.)

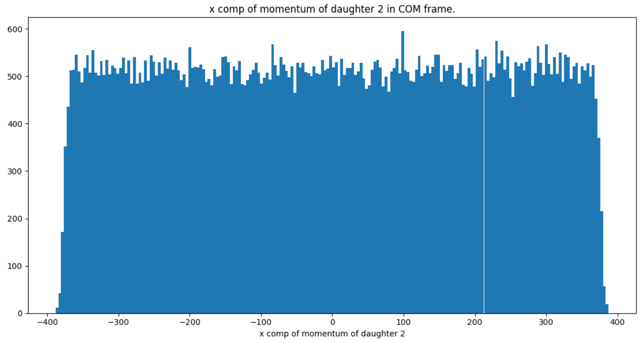
0.4.2 (Momentum components for one of the daughter particles in mother's rest frame):

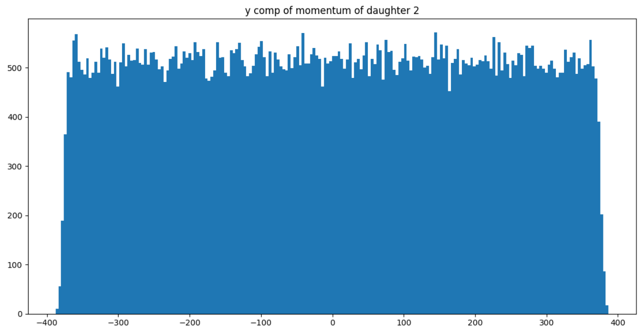


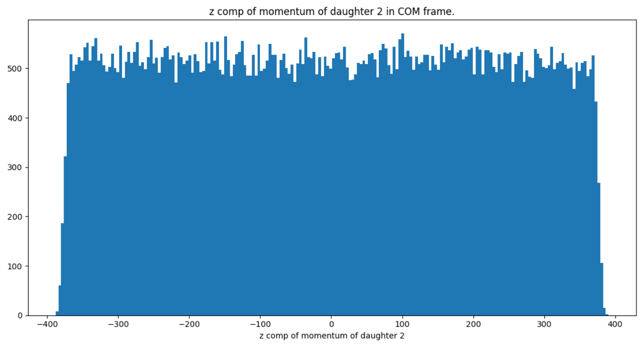




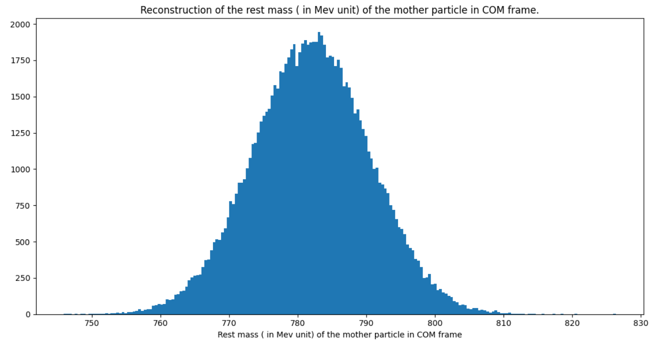
0.4.3 (Momentum components for the other daughter particle in mother's rest frame):







0.4.4 (Plot of the reconstructed rest mass of the mother particle):



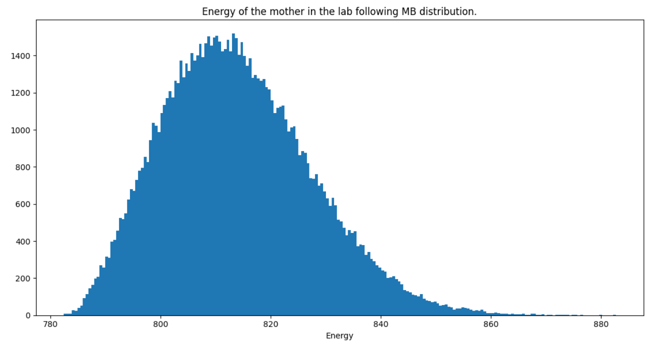
0.4.5 (Results):

1)The histograms corresponding to $(cos\theta, \phi, p_{x,1}, p_{y,1}, p_{z,1}, p_{x,2}, p_{y,2}, p_{z,2})$are uniform in nature.

2)In the final plot the Gaussian distribution of the rest mass of the mother particle is reconstructed.

0.5 (Establishing decay in the laboratory frame):

After the decay is implemented in the mother's rest frame, it is needed to boost the daughter particles in the lab frame because we get all the experimental data in the laboratory frame. To do so, we need to know the velocity of the mother in the lab frame.
The energy of the mother particle in the lab frame is generated using the Maxwell-Boltzmann distribution. In this distribution the minimum relativistic energy of the mother particle is 782 Mev (the rest mass energy of the omega meson particle).

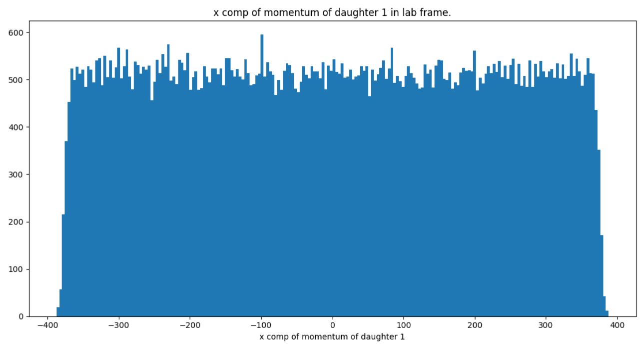


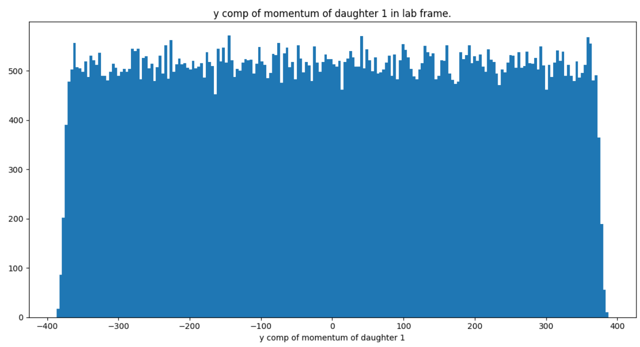
The mother can move in any arbitrary direction in the lab frame. In this section it is considered that the mother particle moves in the direction along the z-axis of the lab frame. Now, from the energy distribution, the velocity of the mother particle along the z direction is calculated using the formula: $v = \frac{p_M}{E_M}$. Next, we calculate $\gamma = \frac{1}{\sqrt{(1-v^2)}}$ to get the transformation relations of momentum in the lab frame. Hence, the transformation of the momentum components of the daughter particles from the mother's rest frame to the lab frame will be according to the Lorentz transformation relations: $(p_{x,1,lab} = p_{x,1}; \quad p_{y,1,lab} = p_{y,1}; \quad p_{z,1,lab} = \gamma p_{z,1} - vE_1)$ and

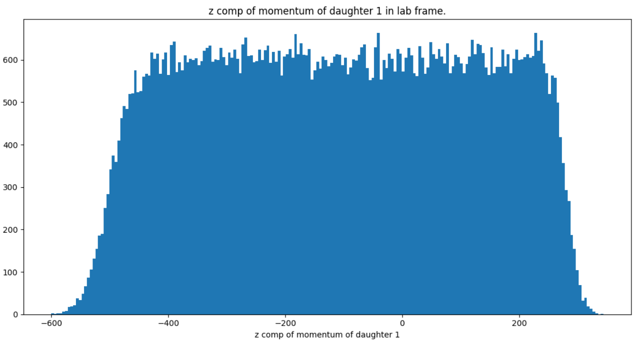
$(p_{x,2,lab} = p_{x,2}; \quad p_{y,2,lab} = p_{y,2}; \quad p_{z,2,lab} = \gamma p_{z,2} - vE_2)$

Now again by taking a large number of mother particles the distributions are plotted.

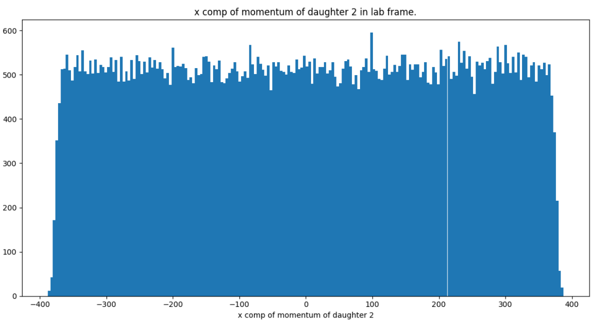
0.5.1 (Momentum components for one of the daughter particles in lab frame):

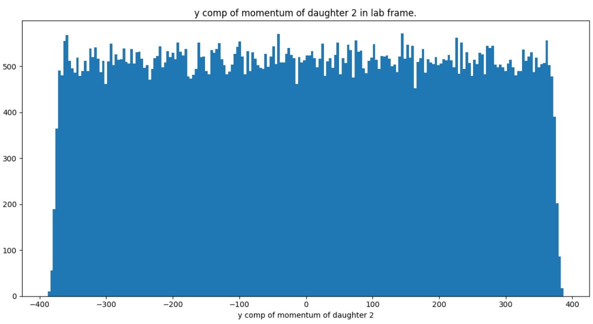


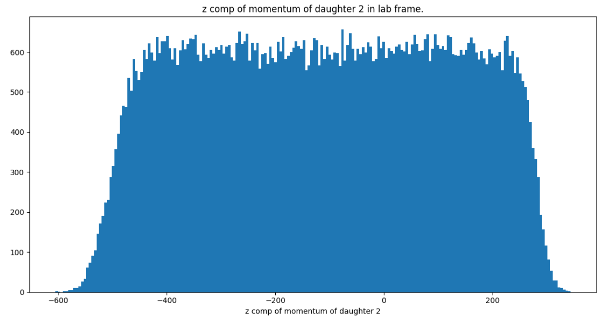




0.5.2 (Momentum components for the other daughter particle in lab frame):

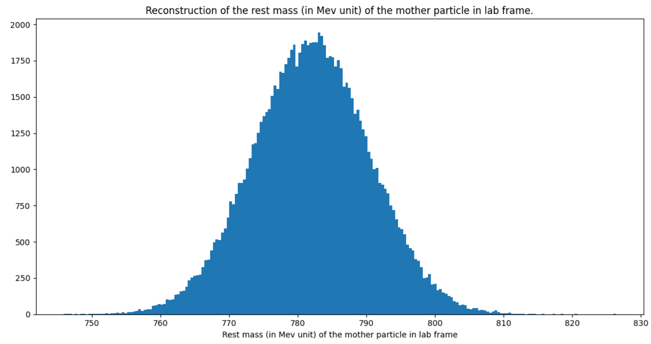






0.5.3 (Plot of the reconstructed rest mass of the mother particle):

Now, the energies of the corresponding daughter particles are $E_{1,lab} = \sqrt{{p_{1,lab}}^2 + {m_{0,\mu}}^2}$ $E_{2,lab} = \sqrt{{p_{2,lab}}^2 + {m_{0,\mu}}^2}$
Finally, using the formula : $m_{0,\omega} = 2{m_{0,\mu}}^2 + 2(E_{1,lab}E_{2,lab} - \vec{p}_{1,lab}.\vec{p}_{2,lab})^{1/2}$ the rest mass of the mother particle is reconstructed. (Here $\vec{p}_1, \vec{p}_2$ are three vectors.)



0.5.4 Results:

1)The histograms corresponding to ($p_{x,1,lab}$, $p_{y,1,lab}$; $p_{x,2,lab}$, $p_{y,2,lab}$) are uniform in nature.

2)The histograms corresponding to $p_{z,1,lab}$, $p_{z,2,lab}$) are also plotted.

3)Finally, the Gaussian distribution of the rest mass of the mother particle is reconstructed.

0.6 References:

1)Relativistic Kinematics of Particle Interactions by W Von Schlippe, March 2002

2)Relativistic Kinematics II; Paul Avery, PHZ4390, Aug. 26, 2015