1 Arithmetic and topology

Definition 1.1. Let $\mathbb{R}^2 = \{(x.y) \mid x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the sum "+" by components
$$(a, b) + (c, d) := (a + b, c + d). \tag{1}$$
- Product: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the product by
$$(a, b)(c, d) = (ac - bd, ad + bc). \tag{2}$$

We define the set $\mathbb{C}$ as $(\mathbb{R}^2, +,)$.

Proposition 1.1. *The set $\mathbb{C}$ of complex numbers is an abelian field.*

Proposition 1.2. *Let $\mathbb{C}$ be defined in the second way. Then,*

1. *$\mathbb{C}$ is an abelian ring.*
2. *If we define f as*
$$\begin{aligned} f : (\mathbb{C}, +,) &\longrightarrow (\mathbb{R}^2, +,) \\ (x, y) &\longmapsto x + yi \end{aligned}, \tag{3}$$
then f is a morphism of rings.
3. *The function f is, in fact, an isomorphism and $\mathbb{C}$ is an abelian field.*

Proposition 1.3. *The subset of $\mathbb{C}$ generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.*

Theorem 1.4. *$\mathbb{C}$ is not an ordered field.*

Definition 1.2. Let $z = a + bi \in \mathbb{C}$. We define the *conjugate of z* as
$$\bar{z} := a - bi. \tag{4}$$

Proposition 1.5. *For all $z, w \in \mathbb{C}$, we have:*

1. *$\bar{\bar{z}} = z$.*
2. *$\overline{z + w} = \bar{z} + \bar{w}$.*
3. *$\overline{zw} = \bar{z}\bar{w}$.*
4. *$z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.*
5. *$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.*
6. *The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.*

Definition 1.3. Let $z = a + bi \in \mathbb{C}$. We define the *real part of z and imaginary part of z* respectively as
$$\operatorname{Re}\{z\} := a, \qquad \operatorname{Im}\{z\} := b. \tag{5}$$

Proposition 1.6. *Let $z \in \mathbb{C}$. Then,*
$$\operatorname{Re}\{z\} = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im}\{z\} = \frac{z - \bar{z}}{2i} \tag{6}$$

Proposition 1.7. *Let $z, w \in \mathbb{C}$ and the following distance function.*
$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto \tilde{d}(z, w) := |z - w| \end{aligned} \tag{7}$$
Then, $(\mathbb{C}, \tilde{d})$ is a metric space.

Definition 1.4. Let $z = a + bi \in \mathbb{C}$. We define the *modulus of z* as
$$|z| := \tilde{d}(z, 0), \tag{8}$$
which is equivalent to $\sqrt{z\bar{z}}$.

Definition 1.5. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define an *open disc of radius r and center z_0* as follows
$$B_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}. \tag{9}$$

Definition 1.6. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *punctured disc of radius r and center z_0* as follows
$$B_r^*(z_0) := \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}. \tag{10}$$

Definition 1.7. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *closed disc of radius r and center z_0* as follows
$$\overline{B_r(z_0)} := \{z \in \mathbb{C} \mid |z - z_0| \leq r\}. \tag{11}$$

Definition 1.8. We denote by $\mathbb{D}$ the unitary disc of center 0 and radius 1. Besides, we denote by $\mathbb{T} \subseteq \mathbb{C}$ the unitary circumference, that is,
$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}. \tag{12}$$
We also denote it by $\mathbb{S}^1$.

Lemma 1.8. *The set $B = \left\{B_r(z_0) \,\middle|\, r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\right\}$ is a basis of the topology of $\mathbb{R}^2$ as a metric space. The set $D = \{D_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of $\mathbb{C}$ as a metric space.*

Proposition 1.9. *The sets $\mathbb{C}$ and $\mathbb{R}^2$ with the topology of metric space are homeomorphs.*

Corollary 1.10. *There is a bijection between B and D, that is, between balls of $\mathbb{R}^2$ and discs of $\mathbb{C}$.*

Proposition 1.11. *Let $z, w \in \mathbb{C}$. Then,*

1. *$|z| \geq 0$.*
2. *$|z| = 0 \Leftrightarrow z = 0$.*
3. *$-|z| \leq \operatorname{Re}\{z\} \leq |z|$ and $-|z| \leq \operatorname{Im}\{z\} \leq |z|$.*
4. *$|zw| = |z||w|$.*
5. *If $w \neq 0$, $|z/w| = |z|/|w|$.*
6. *$|z + w| \leq |z| + |w|$.*
7. *$|z + w| \geq ||z| - |w||$.*
8. *$|\operatorname{Re}\{zw\}| \leq |z||w|$ and $|\operatorname{Im}\{z\}| \leq |z||w|$.*
9. *$|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$.*

10. $|z^n| = |z|^n$

Corollary 1.12. *Let $z_1, \ldots, z_n \in \mathbb{C}$. Then,*

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|, \qquad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \qquad |\operatorname{Re}\{z_1 \cdots z_n\}| \leq |z_1| \cdots |z_n|. \tag{13}$$

Definition 1.9. Let $z \in \mathbb{C}^*$. We define the *argument of z*, denoted by $\arg z$, as the real number θ such that $z = |z|(\cos\theta + i\sin\theta)$. Let us observe that $\arg z$ is not a function but a multivalued application. We define the *principal argument of z* as

$$\operatorname{Arg} z \coloneqq \theta_0 \in [0, 2\pi) \mid z = |z|(\cos\theta + i\sin\theta). \tag{14}$$

In general, to make θ to be unique, it is enough to impose it to belong to a certain semiopen interval of length 2π. Choosing the interval I is called by *taking a determination of the argument.*

Definition 1.10. Given a complex number z that we can express by $z = |z|(\cos\theta + i\sin\theta)$ for some $\theta \in \mathbb{R}$, we use the notation $r = |z|$ to write

$$z = r_\theta^z = r(\cos\theta + i\sin\theta) \tag{15}$$

or simply r_θ when it is obvious which complex number are we referring to. We call it *polar form of z*.

Proposition 1.13. *Let $z \in \mathbb{C}$ and r_θ its polar form. Then,*

$$z^n = (r^n)_{n\theta}. \tag{16}$$

Corollary 1.14 (De Moivre's Formula)**.** *Let $\theta \in \mathbb{R}$. Then,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{17}$$

Proposition 1.15. *Let $z, w \in \mathbb{C}$. Then,*

1. $\arg zw = \arg z + \arg w + 2\pi k$.
2. $\arg z^n = n \arg z + 2\pi k$.

Definition 1.11. We denote the complex numbers z generated by moving the point $z_0 = 1$ around $\mathbb{T}$ a length t in a counter-clockwise direction by 1_t. In other words, 1_t are the complex numbers $z = \cos t + i\sin t$.

Proposition 1.16. *Let $f : t \longrightarrow 1_t$. Then, f is a morphism from $(\mathbb{R}, +)$ to $(\mathbb{T}, \cdot)$, with $\ker f = 2\pi\mathbb{Z}$.*

Definition 1.12. Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $w \in \mathbb{C}$ is an *n-th root of z* if and only if

$$w^n = z. \tag{18}$$

Theorem 1.17. *Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \ldots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \ldots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \ldots, n\}$.*

Theorem 1.18. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a continuous curve such that $\gamma(t) \neq 0 \forall t \in [a, b]$. Then, there exists a continuous determination ϕ of the argument of γ. Then, $\phi(t) + 2\pi k$ with $k \in \mathbb{Z}$ is the general expression of all the argument determinations of γ. If γ is differentiable, then ϕ is differentiable and $\phi' = \operatorname{Im}\{\gamma'/\gamma\}$.*

Definition 1.13. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a regular curve. We define the *variation of the argument along* γ as

$$\Delta_\gamma \arg \coloneqq \operatorname{Im}\left\{\int_a^b \frac{\gamma'(t)}{\gamma(t)}\, \mathrm{d}t\right\}. \tag{19}$$

Definition 1.14. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve such that $\gamma(t) \neq 0 \forall t \in [a, b]$. Then, we define the *index of γ with respect to the origin* or *the number of revolutions of γ around the origin*

$$\operatorname{Ind}(\gamma, 0) \coloneqq \frac{1}{2\pi} \Delta_\gamma \arg. \tag{20}$$

Proposition 1.19. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piece-wise regular curve. Then,*

$$\operatorname{Ind}(\gamma, 0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, \mathrm{d}t \tag{21}$$

Definition 1.15. Let γ be a closed curve and $z \notin \Gamma$. We define the *index of γ with respect to z* as

$$\operatorname{Ind}(\gamma, z) \coloneqq \operatorname{Ind}(\gamma - z, 0). \tag{22}$$

Proposition 1.20. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve piece-wise of class $C^1([a, b])$. Then,*

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z}\, \mathrm{d}t\,. \tag{23}$$

Proposition 1.21. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piece-wise of class $C^1([a, b])$. Then,* $\operatorname{Ind}(-\gamma, z) = -\operatorname{Ind}(\gamma, z)$.

2 Sequences and limits

Definition 2.1. A *sequence of complex numbers* is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned}. \tag{24}$$

We denote it by $\{z_n\}_{n=m}^\infty$

Definition 2.2. Let $\{z_n\}_{n=0}^\infty$ be a sequence. We say *the sequence has limit L* or *it converges to the limit L* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \tag{25}$$

We denote it by

$$\lim_{h \to \infty} z_n = L, \qquad \lim \{z_n\}_{n=0}^\infty = L, \qquad \{z_n\}_{n=0}^\infty \to L. \tag{26}$$

Theorem 2.1. *Let $z_n = z_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^\infty$ and $L = L_x + iL_y \in \mathbb{C}$. Then,*

$$\{z_n\}_{n=0}^\infty \to L \Leftrightarrow \{x_n\}_{n=0}^\infty \to L_x \wedge \{y_n\}_{n=0}^\infty \to L_z. \tag{27}$$

Definition 2.3. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *it tends to infinity* and denote it by $\lim z_n = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \tag{28}$$

Definition 2.4. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \tag{29}$$

Theorem 2.2. *Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.*

Theorem 2.3. *Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence } \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \text{ are Cauchy sequences.} \tag{30}$$

Theorem 2.4. *The field $\mathbb{C}$ of complex numbers is complete.*

Definition 2.5. The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, together with two charts.

3 Functions

Definition 3.1. A *topology* is an ordered pair $(\mathbb{X}, \tau)$, where X is a set and τ a collection of subsets of X satisfying the following properties:

1. The empty set and X belong to τ.
2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ.
3. The intersection of any finite number of members of τ still belongs to τ.

The elements of τ are called *open sets* and the collection τ is called the *topology on X*.

Definition 3.2. Let $(\mathbb{X}, d)$ be a metric space. A *topology on the metric space by the metric d* is the set τ of all open sets of M.

Definition 3.3. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *interior point of A* if there is a ball $B_{(\mathbb{M},d)}(a, r) \subset A$.

Definition 3.4. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *exterior point of A* if there is a ball such that $B_{(\mathbb{M},d)}(a, r) \cup A = \emptyset$.

Definition 3.5. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is a *boundary point of A* if it is not interior or exterior or, which is equivalent, if every ball $B_{(\mathbb{M},d)}(a, r)$ contains elements of A and A^c.

Definition 3.6. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *accumulation point of A* if every ball with center a contains points of A different to a. In other words, every punctured ball satisfies $B^*_{(\mathbb{M},d)}(a, r) \cup A \neq \emptyset$.

Definition 3.7. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the interior of A* as the set of all interior points of A, and we denote it by $\text{int}(A)$.

Definition 3.8. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the exterior of A* as the set of all exterior points of A, and we denote it by $\text{ext}(A)$.

Definition 3.9. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the boundary of A* as the set of all boundary points of A, and we denote it by ∂A.

Definition 3.10. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the closure of A* as the set of all accumulation points of A, and we denote it by $\bar{A}$.

Definition 3.11. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *an open set* if it contains none of its boundary points, that is, if $\partial A \cap A = \emptyset$.

Definition 3.12. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a closed set* if it contains all its boundary points, that is, if $\partial A \subseteq A$.

Definition 3.13. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a bounded set* if there exist a point $a \in \mathbb{M}$ and a positive real number r such that the ball $B_{(\mathbb{M},d)}(a, r)$ contains A.

Definition 3.14. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a compact set* if it a bounded and closed set.

Proposition 3.1. *Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. Then, A is open if and only if A^c is closed.*

Definition 3.15. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally, Ω is connected if there are not two open sets $U, V \subseteq \mathbb{C}$ such that

$$U_1 = U \cap \Omega, \qquad V_1 = V \cap \Omega, \qquad U_1 \cap V_1 = \varnothing, \qquad U_1 \cup V_1 = \Omega. \tag{31}$$

Otherwise, we say Ω *is disconnected.*

Definition 3.16. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is simply connected* if and only if every circuit is homotopic in Ω to a point in Ω. Equivalently, is simply connected if and only if every pair of curves with the same extremes are homotopic.

Definition 3.17. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is convex* if and only if for all pair of point $a, b \in \Omega$, the segment defined by

$$[a, b] = \{z \mid z = (1 - t)a + tb, 0 \leq t \leq 1\} \tag{32}$$

is contained in Ω, that is, if every pair of points can be connected by a straight line that belongs to the set.

Definition 3.18. Let $\Omega \in \mathbb{C}$ be a set. We say Ω *is a star-convex set* if and only if there exists $z_0 \in \mathbb{C}$ such that for all $z \in \Omega$ the segment $[z_0, z]$ is contained by Ω.

Definition 3.19. Let $(\mathbb{M}, d)$ be a metric space and $S \subseteq \mathbb{M}$ a set. We say S *is path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

Definition 3.20. Let $\Omega \in \mathbb{C}$ be a set. We say Ω *is a region or domain* if and only if it is open, non-empty, and connected.

Definition 3.21. Let $\Omega \subseteq \mathbb{C}$ be a non-empty set. We say $\Omega_1 \subseteq \Omega$ *is a connected component of* Ω if and only if it is a maximal connected subset, that is, if $z_0 \in \Omega_1$ and W is a connected subset of $\mathbb{C}$ that contains z_0, then $W \subseteq \Omega_1$.

Definition 3.22. Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function* f as the application

$$\begin{aligned} f : D \subseteq \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto w = f(z) \end{aligned}. \tag{33}$$

Definition 3.23. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say it *tends to infinity at the point* z_0 and denote it by $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \tag{34}$$

Definition 3.24. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We write $\lim_{z \to \infty} f(z) = L$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \tag{35}$$

Proposition 3.2. *Let $f_1, f_2 : \Omega \longrightarrow \mathbb{C}$ be two functions and z_0 a point such that $\lim_{z \to z_0} f_1 = w_1, \lim_{z \to z_0} f_2 = w_2$. Then,*

1. *$f_1 + f_2$ has also a limit and $\lim_{z \to z_0} f + g = w_1 + w_2$.*
2. *$f_1 f_2$ has also a limit and $\lim_{z \to z_0} fg = w_1 w_2$.*
3. *If $w_2 \neq 0$, then f/g has also a limit and $\lim_{z \to z_0} f/g = w_1/w_2$.*
4. *If $h(z)$ is a continuous function defined on a neighborhood of w_1, then $\lim_{z \to z_0} h(f_1(z)) = h(w_1)$.*

Definition 3.25. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. We say f *is continuous in* z_0 if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \tag{36}$$

Proposition 3.3. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$ is continuous at z_0 if and only if $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are continuous at z_0.*

Proposition 3.4. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, f is continuous at z_0 if and only if for all sequence $\{z_n\}_{n=1}^{\infty}$ of Ω convergent at z_0 it is true that the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$.*

Proposition 3.5. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, λf, $f + g$, and fg are continuous at z_0. The function f/g is continuous at z_0 if $g(z_0) \neq 0$.*

Definition 3.26. For all $z \in \mathbb{C}$, we define the *complex exponential function* as the following series

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{37}$$

Proposition 3.6. *The radius of convergence of e^z is infinite.*

Proposition 3.7. $\mathrm{e}^z = \mathrm{e}^x$ *for all $x \in \mathbb{R}$.*

Proposition 3.8. $\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$ *for all $z, w \in \mathbb{C}$.*

Proposition 3.9. *For all $z \in \mathbb{C}$ we have $\mathrm{e}^z \neq 0$.*

Proposition 3.10. *The image of e^z is $\mathbb{C}^*$.*

Proposition 3.11. *The derivative of e^z is e^z.*

Proposition 3.12. $\overline{\mathrm{e}^z} = \mathrm{e}^{\bar{z}}$.

Proposition 3.13. $|\mathrm{e}^z| = \mathrm{e}^{\operatorname{Re}\{z\}}$.

Proposition 3.14 (Euler's Formula)**.** *If $\theta \in \mathbb{R}$, then e^{xi} has modulus one and we have that*

$$\boxed{\mathrm{e}^{xi} = \cos x + i \sin x.} \tag{38}$$

Corollary 3.15. *Let $z \in \mathbb{C}^*$. Then,*

$$z = |z| \mathrm{e}^{i\theta}, \tag{39}$$

with $\theta \in [0, 2\pi)$.

Proposition 3.16. *The following function*

$$\begin{aligned} \exp : (\mathbb{R}, +) &\longrightarrow (\mathbb{C}^*, \cdot) \\ x &\longmapsto \mathrm{e}^{xi} \end{aligned} \tag{40}$$

is a morphism of groups and its image is $\mathbb{T}$.

Proposition 3.17. *The complex exponential function is a periodic function with period $2\pi i$.*

Proposition 3.18. *Let $a \in \mathbb{C}^*$. Then, $\mathrm{e}^z = a$ has infinite solutions.*

Proposition 3.19. *The equation $\mathrm{e}^z = 0$ does not have solutions.*

Proposition 3.20. *Let $y_0 \in \mathbb{C}$ be a numbers, $B := \{z \in \mathbb{C} \,|\, y_0 < \operatorname{Im}\{z\} < y_0 + 2\pi\}$ a set, and $f : B \longrightarrow \mathbb{C}^*$ be the exponential function. Then, f is bijective in B* **?**.

Proposition 3.21. *Let $x_0, y_0, m \in \mathbb{C}$ be two numbers with $m \neq 0$ and f the exponential function* **?**. *Then,*

1. *f transforms the line $y = y_0$ to a line that starts at $z = 0$ and continues with an argument y_0 from the real positive axis.*
2. *f transforms the line $x = x_0$ to a circle centered at the origin and radius $r = \mathrm{e}^{x_0}$.*

3. *f transforms the line $y = mx$ to the parametric curve $z = \mathrm{e}^x \mathrm{e}^{imx}$ (a spiral).*

Definition 3.27. Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric functions as*

$$\cos z \coloneqq \frac{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}{2}, \tag{41}$$

$$\sin z \coloneqq \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{2}, \tag{42}$$

$$\tan z \coloneqq \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}. \tag{43}$$

Proposition 3.22. *For all $z \in \mathbb{C}$,*

$$\sin^2 z + \cos^2 z = 1. \tag{44}$$

Proposition 3.23. *For all $z \in \mathbb{C}$,*

$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z). \tag{45}$$

Proposition 3.24. *For all $z, w \in \mathbb{C}$,*

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \qquad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \tag{46}$$

Proposition 3.25. *The functions $\cos z, \sin z$ have period of 2π.*

Proposition 3.26. *Let $z_0 \in \mathbb{C}$. Then, z_0 is root of $\sin z$ ($\cos z$) if and only if it is a root of $\sin x$ ($\cos x$).*

Definition 3.28. Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic functions* as

$$\cosh z \coloneqq \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}, \tag{47}$$

$$\sinh z \coloneqq \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}, \tag{48}$$

$$\tanh z \coloneqq \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{\mathrm{e}^{z} + \mathrm{e}^{-z}}. \tag{49}$$

Proposition 3.27. *For all $z \in \mathbb{C}$,*

$$\cosh^2 z - \sinh^2 z = 1. \tag{50}$$

Proposition 3.28. *For all $z \in \mathbb{C}$,*

$$\cosh(-z) = \cosh(z), \qquad \sinh(-z) = -\sinh(z). \tag{51}$$

Proposition 3.29. *For all $z, w \in \mathbb{C}$,*

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \tag{52}$$

$$\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \tag{53}$$

Proposition 3.30. *For all $z \in \mathbb{C}$,*

$$\cosh z = \cos(iz), \cos z = \cosh(iz) \tag{54}$$

$$\sinh z = -i \sin(iz), \sin z = -i \sinh(iz) \tag{55}$$

Proposition 3.31. *For all $z = x + iy \in \mathbb{C}$,*

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \tag{56}$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \tag{57}$$

$$\tan(x + iy) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i \frac{\sinh y}{\cos(2x) + \cosh(2y)}. \tag{58}$$

Proposition 3.32. *For all $z = x + iy \in \mathbb{C}$,*

$$\tanh(x + iy) = \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)} + i \frac{\sin(2y)}{\cosh(2x) + \cos(2y)}. \tag{59}$$

Proposition 3.33. *For all $z = x + iy$,*

$$|\cos z| = \sqrt{\cosh^2 y - \sin^2 x} = \sqrt{\cos^2 x + \sinh^2 y}, \tag{60}$$

$$|\sin z| = \sqrt{\sinh^2 x + \sin^2 y} = \sqrt{\cosh^2 x - \cos^2 y}. \tag{61}$$

Corollary 3.34. *For all $z = x + iy$,*

$$|\sinh y| \leq |\cos z| \leq \cosh y, \qquad |\sinh y| \leq |\sin z| \leq \cosh y. \tag{62}$$

Proposition 3.35. *The roots of the function $\sinh z$ are of the form $z_n = n\pi$ and those for the function $\cosh z$ are of the form $w_n = (2n + 1)\pi/2i$.*

Definition 3.29. Let $D \subseteq \mathbb{C}$ be a set. We define a *multivalued function from D to $\mathbb{C}$* as a subset of $D \times \mathbb{C}$ such that for every $z \in D$ there exists a number $y \in \mathbb{C}$ such that $(z, w) \in f$.

Definition 3.30. For $z \in \mathbb{C}^*$, we call the *natural logarithm of z* every number w such that $\mathrm{e}^w = z$, that is,

$$\ln z \coloneqq \{ w \in \mathbb{C} \mid \mathrm{e}^w = z \}. \tag{63}$$

Proposition 3.36. *Given $z \in \mathbb{C}$ we can define $\ln z$ from the natural logarithm of a real number as*

$$\ln z = \ln |z| + i \arg z = \ln |z| + i \mathrm{Arg} z + 2\pi k i. \tag{64}$$

Definition 3.31. We define the *principal natural logarithm of z* as the value defined by the principal argument of z, that is,

$$\mathrm{Log} z = \ln |z| + i \mathrm{Arg} z. \tag{65}$$

Definition 3.32. We define the determination I (with I being a semiopen interval) of the logarithm as

$$\log_I z \coloneqq \ln |z| + i \arg_I z. \tag{66}$$

Definition 3.33. Let $E \subseteq \mathbb{C}^*$ be a connected set. We define the *continuous determination of the logarithm in E* as the continuous function $g : E \longrightarrow \mathbb{C}$ such that $\mathrm{e}^{g(z)} = z$. More generally, if $f : E \longrightarrow \mathbb{C}$ is a function such that $f(z) \neq 0$ for all $z \in E$, then we define the *continuous determination of* $\ln f$ as a function $g : E \longrightarrow \mathbb{C}$ such that $\mathrm{e}^{g(z)} = f(z)$.

Proposition 3.37. *Let $z, w \in \mathbb{C}$ two numbers. Then,*

1. *$\ln(zw) = \ln z + \ln w + 2\pi k i, k \in \mathbb{Z}$.*
2. *If we want to stay in the principal argument,*

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \text{if } \mathrm{Arg} z + \mathrm{Arg} w < 2\pi \\ \ln z + \ln w - 2\pi i, & \text{if } \mathrm{Arg} z + \mathrm{Arg} w \geq 2\pi \end{cases}. \tag{67}$$

3. *SEARCH MORE PROPERTIES*

Definition 3.34. Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric inverse functions* as

$$\arcsin z := -i \ln\Big(iz + \sqrt{1-z^2}\Big), \tag{68}$$

$$\arccos z := -i \ln\Big(z + \sqrt{z^2-1}\Big), \tag{69}$$

$$\arctan z := -\frac{i}{2} \ln \frac{1+iz}{1-iz}. \tag{70}$$

Definition 3.35. Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic inverse functions* as

$$\operatorname{arcsinh} z := \ln\Big(z + \sqrt{1+z^2}\Big), \tag{71}$$

$$\operatorname{arccosh} z := \ln\Big(z + \sqrt{z^2-1}\Big), \tag{72}$$

$$\operatorname{arctanh} z := \frac{1}{2} \ln \frac{1+z}{1-z}. \tag{73}$$

Definition 3.36. Let $z, a \in \mathbb{C}$ with $z \neq 0$. Then, we define the *complex power function* as

$$z^a := \mathrm{e}^{a \ln z}. \tag{74}$$

If $E \subseteq \mathbb{C}^*$ is a connected set and $f : E \longrightarrow \mathbb{C}$ a functions such that $f(z) \neq 0$ for all $z \in E$, and $w \in \mathbb{C}$ a number, we define a *continuous determination of* f^w as a continuous function $g : E \longrightarrow \mathbb{C}$ such that $g(z) \in [f(z)]^w$.

Proposition 3.38. *If $a = \alpha + \beta i$ and $z = r\mathrm{e}^{\theta i}$, then*

$$z^a = \mathrm{e}^{\alpha \ln r - \beta(\theta + 2\pi k)} \mathrm{e}^{(\beta \ln r + \alpha(\theta+2\pi k))i}, \tag{75}$$

$$|z^a| = \mathrm{e}^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \qquad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k) \tag{76}$$

Proposition 3.39. *Let $a, z \in \mathbb{C}$ be two numbers. Then,*

1. *If $a = n \in \mathbb{Z}$, the complex power is a function and*
$$z^n = r^n \mathrm{e}^{n\theta i}. \tag{77}$$
2. *If $a = n/m \in \mathbb{Q}$, there are n values and*
$$z^a = \sqrt[m]{r^n} \mathrm{e}^{(\theta+2\pi k)n/mi}. \tag{78}$$
3. *If a is irrational, the norm is uniquely determined but the argument has infinite values.*
4. *IF $a \in \mathbb{C} \backslash \mathbb{R}$, the argument is uniquely determined and the norm has infinite values.*

Proposition 3.40. *Let $z, w \in \mathbb{C}$. Then,*

1. $\left(\mathrm{e}^b\right)^a = \mathrm{e}^{a(b+2\pi k i)}$

Definition 3.37. A *Riemann surface X* is a connected complex 1-manifold.

Definition 3.38. We define a *sheet* as each of the complex planes of the Riemann surface.

Definition 3.39. We define a *cut* as the line (not necessaryly straight) of union between sheets.

Definition 3.40. We define a *branch point* as a point where start or finish a cut.

4 Derivatives

Definition 4.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. We define the *derivative of f at z_0* as

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{79}$$

in case the limit exists. If f has derivative, we say *f is derivable at z_0.*

Definition 4.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *f is holomorphic at* Ω if and only if it is $\mathbb{C}$-derivable at every point of Ω. In that case, it is defined the function $f' : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ that associates each point z of Ω with $f'(z)$.

Definition 4.3. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We define the *domain of holomphism* as the region where f is derivable. We say *f is entire* if and only if the domain of holomorphism is $\mathbb{C}$.

Definition 4.4. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *f is holomorphic at z_0* if and only if it is holomorphic at some neighborhood of z_0.

Proposition 4.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. If f is derivable at z_0, then it is continuous at z_0.*

Theorem 4.2. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two functions and $z_0 \in \Omega$ a point. Then, the following statements are true.*

1. *If f is constant at Ω, then f is derivable at z_0 and $f'(z_0) = 0$.*
2. *If $f(z) = z$ in every point of Ω, then f is derivable at z_0 and $f'(z_0) = 1$.*
3. *If f, g are derivable at z_0 and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ are derivable at z_0 and $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.*
4. *If f, g are derivable at z_0, then fg is derivable at z_0 and*
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{80}$$
5. *If f, g are derivable at z_0 and $g(z_0) \neq 0$, then f/g is derivable at z_0 and*
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{81}$$

Theorem 4.3. *Let $f : \Omega_1 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a derivable function at a point $z_0 \in \mathbb{C}$ and $g : \Omega_2 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be another derivable function at a point $f(z_0) \in \Omega_2$. Then, $g \circ f$ is derivable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \tag{82}$$

Theorem 4.4. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function of class $C^1(\Omega)$ with Ω an open set, injective, and derivable at every point of Ω with non-zero derivative. Then,*

1. *$\Omega' = f(\Omega)$ is an open subset of $\mathbb{C}$.*
2. *The inverse function f^{-1} exist, it is well defined and it is derivable at Ω'.*
3. *If $z \in \Omega$ and $z' = f(z)$, then*

$$(f^{-1})'(z') = \frac{1}{f'(z)}. \tag{83}$$

Proposition 4.5. *A determination of $\ln z$ with $z \in \mathbb{C}$ is continuous except in a semiline.*

Theorem 4.6. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $\phi \in C(\Omega)$, such that $\mathrm{e}^{\phi(z)} = z$ for all $z \in \Omega$. Then, we have $\phi \in H(\Omega)$ and*

$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \tag{84}$$

Proposition 4.7. *A determination of $\ln z$ with $z \in \mathbb{C}$ is holomorphic except in a semiline.*

Proposition 4.8. *Let $I = [\theta, \theta + 2\pi)$ a determination of the logarithm, $\ln_I z$. Then, $\ln_I z$ is holomorphic except in the semiline $L_\theta = \{r\mathrm{e}^{\theta i} \in \mathbb{C} \,|\, r \geq 0\}$.*

Definition 4.5. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f *is of class $C^1(\Omega)$ or simply $f \in c^1(\Omega)$* if and only if, using $f = u + iv$ with $u = \mathrm{Re}\{f\}, v = \mathrm{Im}\{f\}$, the partial derivatives of u and v as a two variable real functions exist and are continuous. In other words, $f \in C^1(\Omega)$ if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \tag{85}$$

exist and are continuous.

Theorem 4.9 (Cauchy-Riemann conditions)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. Then, f is derivable at z_0 if and only if is differentiable at z_0 and $df(z_0)$ is $\mathbb{C}$-linear, that is,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{86}$$

which are known as Cauchy-Riemann conditions.

Theorem 4.10. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. If u, v satisfy the Cauchy-Riemann equation and their partial derivatives are continuous, then f is derivable.*

Definition 4.6. We define the operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{87}$$

that act over the functions such that the real and imaginary part u, v have partial derivatives.

Proposition 4.11. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function of class $C^1(\Omega)$. Then, for all $z_0 \in \Omega$*

$$f(z_0 + h) = f(z_0) + \left(\frac{\partial f}{\partial z}\right)_{z_0} h + \left(\frac{\partial f}{\partial \bar{z}}\right)_{z_0} \bar{h} + o(|h|^2). \tag{88}$$

Theorem 4.12. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and z_0 an interior point. Then, at z_0*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}. \tag{89}$$

5 Series

Definition 5.1. We say $\sum_{n=1}^{\infty} z_n$ converges if and only if $S_n := \sum_{n=1}^{N} z_n$ has limit at $n \to \infty$.

Proposition 5.1. *$\sum_{n=1}^{\infty} z_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.*

Definition 5.2. We say $\sum_{n=1}^{\infty} z_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |z_n|$ converges.

Proposition 5.2. *$\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge.*

Proposition 5.3.

1. *A series converges absolutely with sum S if and only if every rearrangement is convergent with the same sum S.*
2. *An absolutely convergent series can be summed by blocks in an arbitrary way.*

Proposition 5.4. *Let $\sum_n a_n, \sum_n b$ be two absolutely convergent series with sums A and B respectively. Then, the series $\sum_k c_k$ with $c_k = \sum_{n=0}^{k} a_n b_{k-n}$ is absolutely convergent with sum AB.*

Theorem 5.5 (Weierstrass M-test)**.** *If $|f_n(p)| < M_n$ for all $p \in X, n \geq 1$ and $\sum_{n=0}^{\infty} M_n < \infty$, then the series $\sum_{n=0}^{\infty} f_n(p)$ is uniformly convergent on X.*

Lemma 5.6 (Abel's summation formula)**.** *Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ be two sequences of complex numbers and $A_n = a_1 + \cdots + a_n$. Then,*

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k). \tag{90}$$

Theorem 5.7 (Dirichlet's criteria)**.** *Let* $\sum_{n=1}^{\infty} f_n(p)g_n(p)$ *be a series where* $f_n(p)$ *are complex and* $g_n(p)$ *are real for all* $p \in X, n \geq 1$*. If we denote* $F_n(p) = f_1(p) + \cdots f_n(p)$*, there exists a constant* M *such that* $|F_n|(p) \leq M$ *for all* $n \geq, p \in X$*,* $g_n(p)$ *is monotonous decreasing and converges uniformly to zero on* X*, then the series* $\sum_{n=1}^{\infty} f_n(p)g_n(p)$ *is uniformly convergent on* X*.*

Theorem 5.8 (Abel's criteria)**.** *Let* $\sum_{n=1}^{\infty} f_n(p)g_n(p)$ *be a series where* $f_n(p), g_n(p)$ *are complex. If* $\sum_{n=1}^{\infty} f_n(p)$ *is uniformly convergent on* X *andd there exists a number* $M \in \mathbb{R}^+$ *such that for all* $p \in X$

$$|g_1(p)| + \sum_{n=1}^{\infty} |g_n(p) - g_{n+1}(p)| \leq M, \tag{91}$$

then the series $\sum_{n=1}^{\infty} f_n(p)g_n(p)$ *is uniformly convergent on* X*.*

Definition 5.3. We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n, \qquad a_n, z, z_0 \in \mathbb{C}. \tag{92}$$

We call the term a_n the *n-th coefficient of the series.* In case $a_n = 0$ $\forall n \leq m$, we will start the counting directly from m.

Definition 5.4. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. We define its *domain of convergence* as

$$E := \left\{ z \in \mathbb{C} \,\middle|\, \sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ converges} \right\}. \tag{93}$$

Theorem 5.9. *Let* $\sum_n a_n(z - z_0)^n$ *be a power series and* $R = 1/\rho$*, where* $\rho = \limsup_n |a_n|^{1/n}$*. Then, the series converges uniformly on the compacts of the open disc* $D(z_0, R)$*, converges absolutely at every point* $z \in D$ *and diverges outside* $\bar{D}$*. Hence, the set of converges* E *satisfies* $D \subseteq E \subseteq \bar{D}$ *and* $D = \text{int}E$*.*

Definition 5.5. Radius of convergence.

Proposition 5.10. *Let* $\sum_n a_n(z - z_0)^n$ *be a power series and* $R = \lim_{n\to\infty} |a_n|/|a_{n+1}|$*. If the limit exists, then* R *is the radius of convergence.*

Theorem 5.11 (Cauchy-Hadamard Theorem)**.** *Let*

$$S = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{94}$$

be the following complex power series with $a_n, z_0 \in \mathbb{C}$ *and* $R \in [0, +\infty) \cup \{+\infty\}$ *the radius of convergence. Then,*

1. *If* $|z - z_0| < R$ *then* S *converges. In fact, for all* $\underline{r < R}$ *we have* S *converges uniformly at the disc* $\overline{D_r(z_0)}$*.*
2. *If* $|z - z_0| > R$ *then* S *diverges.*
3. *The function* $f(z) = S(z)$ *is derivable at* $D_R(z_0)$ *and its formal derivative is*

$$f'(z) = \sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}, \tag{95}$$

with the same radius of convergence.

Definition 5.6. Let $\sum_n a_n(z - z_0)^n$ be a series, $S = E \cap C(z_0, R)$ non empty, and $m > 1$ a real number. We define

$$S_m := \{z \in \mathbb{C} \,|\, |z - z_0| < R, d(z, S) \leq m(R - |z - a|)\}. \tag{96}$$

Definition 5.7 (Stolz angle)**.** Let S be formed by one point w. We define the *Stolz angle* as the angle generated by the S_m.

Theorem 5.12 (Abel's theorem)**.** *Let* $\sum_n a_n(z - z_0)^n$ *be a series with* S *non empty and such that the series converges uniformly on it. Then, the series converges uniformly on* S_m *for all* $m > 1$*. In particular, the sum function is continuous on* S_m *and one has*

$$\lim_{z\to w, z\in S_m} \sum_n a_n(z - z_0)^n = \sum_n a_n(w - z_0)^n, \qquad w \in S. \tag{97}$$

Theorem 5.13. *Let* $\sum_n a_n(z - z_0)^n$ *be a series with radius of convergence* R*. Then,* $f(z) = \sum_n a_n(z - z_0)^n$ *is holomorphic on* $D(a, R)$ *and it has a derivative*

$$f'(z) = \sum_n na_n(z - z_0)^{n-1}, \ \forall z \in D. \tag{98}$$

Proposition 5.14. *Let* $f : D(a, R) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a function. If there exists a power series* $\sum_n a_n(z - z_0)^n$*, convergent on* D *such that*

$$f(z) = \sum_n a_n(z - z_0)^n, \ |z - z_0| < R, \tag{99}$$

then the series is unique. In fact, f *is infinitely holomorphic and the coefficients* a_n *are determined by* f *with the relation*

$$a_n = \frac{f^n(z_0)}{n!}, \ n \in \mathbb{N}. \tag{100}$$

Definition 5.8. Let $f : D(a, R) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f *admits a series expansion* if and only if there exists a power series $\sum_n a_n(z - z_0)^n$, convergent on D such that

$$f(z) = \sum_n a_n(z - z_0)^n, \ |z - z_0| < R. \tag{101}$$

Definition 5.9. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with Ω an open set. We say f *is analytic on* Ω if and only if it admits locally a series expansion, that is, if for every point $z_0 \in \Omega$ there exists a disc $D(z_0, \delta)$ and a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ such that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \forall z \in D$.

Theorem 5.15. *Let* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *on* $D(z_0, R)$ *and* $w_0 \in D(z_0, R_0)$. *Then, the series* $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n$ *has a radius of convergence* $R_1 \geq R_0 - |z_0 - z_1|$ *and it satisfies*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n, \qquad \text{if } |z-z_1| < R - |z_0 - z_1| \tag{102}$$

Corollary 5.16. *Let* R *be the radius of convergence of the function*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

Then f *has as Taylor polynomial of degree* m *around* z_0 *the following one*

$$P_{m,f,z_0}(z) = \sum_{n=0}^{m} a_n(z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{103}$$

Proposition 5.17. *Let* $\varnothing \neq \Omega \subseteq \mathbb{C}$. *Then,*

1. *Every connected component of* Ω *is a closed of* Ω *with a subspace topology.*
2. *Two connected components are the same or are disjoint.*
3. *Every connected of* Ω *is one and only one connected component.*
4. Ω *is the disjoint union of its connected components.*

6 Holomorphic functions

Definition 6.1. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{aligned} \gamma : I &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned}. \tag{104}$$

Definition 6.2. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval and $D \subseteq \mathbb{C}$ a domain. We define an *arc* as a continuous application of the form

$$\begin{aligned} \gamma : I &\longrightarrow D \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned}. \tag{105}$$

Equivalently, we can say an arc is a curve restricted to some interval.

Definition 6.3. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We call $\gamma(a)$ and $\gamma(b)$ *the extremes of* γ. In particular, we call $\gamma(a)$ the *initial point* and $\gamma(b)$ the *final point*.

Definition 6.4. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We define the *route or graph of* γ as

$$\gamma^* \coloneqq \{z \in D \,|\, z = \gamma(t), t \in I\}. \tag{106}$$

Definition 6.5. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is closed* if and only if $\gamma(a) = \gamma(b)$.

Definition 6.6. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is simple* if and only if there is no two numbers $t_1, t_2 \in (a, b)$ such that $\gamma(t_1) = \gamma(t_2)$. We also call it a *Jordan curve*, and if it is closed, a *circuit*.

Definition 6.7. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is differentiable* if for al value $t_0 \in [a, b]$ there exists the limit

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \tag{107}$$

For $t_0 = a$ or $t_0 = b$ we consider the laterals limits from the right and from the left respectively.

Definition 6.8. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is of class* C^1 if and only if γ' exists and is continuous at $[a, b]$.

Definition 6.9. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is regular or smooth* if and only if it is differentiable and γ' never vanishes.

Definition 6.10. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is piece-wise of class* C^1 if and only if γ' exists and is continuous in I except in a finite number of points where γ has lateral derivatives.

Definition 6.11. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We define the *opposite arc* as

$$\begin{aligned} -\gamma : [-b, -a] &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma(-t) \end{aligned}. \tag{108}$$

Definition 6.12. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be an arc. We say $\Gamma(s), s \in [c, d] \subseteq \mathbb{R}$ *has been obtained from* $\gamma(t), t \in [a, b]$ *by a change of parametrization* if and only if the new parameter s and the original parameter t are related by a relation $t = \phi(s)$, where $\phi : [c, d] \longrightarrow [a, b]$ is an homeomorphism that satisfies $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$. We call Γ *the reparametrization of* γ.

Definition 6.13. Let $\gamma_1 : I_1 \longrightarrow \mathbb{C}$ and $\gamma_2 : I_2 \longrightarrow \mathbb{C}$ be two arcs. We say they *are equivalent* if and only if there exists a bijective, monotone, and continuous function $\rho : I_2 \longrightarrow I_1$ such that $\gamma_2 = \gamma_1 \circ \rho$. If ρ is an increasing function we say γ_1 *and* γ_2 *have the same orientation*; otherwise, we say γ_1 *and* γ_2 *have opposite orientations*.

Definition 6.14. Let $\gamma_1[a,b] \longrightarrow \mathbb{C}$ and $\gamma_2 : [c,d] \longrightarrow \mathbb{C}$ be two arcs such that $[a,b] \cap [c,d] = \varnothing$. We define the application $\gamma_1 \cup \gamma_2$ (sometimes denoted by $\gamma_1 + \gamma_2$) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \leq t \leq b \\ \gamma_2(t-b+c), & \text{if } b \leq t \leq b+d-c \end{cases}. \tag{109}$$

We say γ_1, γ_2 can be joined/added or that there exists its union/sum if and only $\gamma_1(b) = \gamma_2(x)$. In this case $\gamma_1 + \gamma_2$ is an arc, and we call it *the sum arc of* γ_1 *plus* γ_2.

Definition 6.15. We define the *segment of extremes* $z_1, z_2 \in \mathbb{C}$ as the arc defined by the expression

$$\begin{array}{l} [z_1, z_2] : [0,1] \longrightarrow \mathbb{C} \\ t \longmapsto (1-t)z_1 + tz_2 \end{array}. \tag{110}$$

Definition 6.16. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f *is polygonal* if and only if can be expressed as a finite union of segments, that is, if there exist a natural number n and points $\{z_0, \ldots, z_n\}$ such that

$$p = [z_0, z_1] \cup \cdots \cup [z_{n-1}, z_n]. \tag{111}$$

Definition 6.17. Let $\gamma : [a,b] \longrightarrow D$ be an arc with a, b finite. We say γ *is a basic curve* if and only if $\gamma \in C^1((a,b)) \cap C([a,b])$ and there exist $\lim_{t\to a^+} \gamma'(t), \lim_{t\to b^-} \gamma'(t)$.

Definition 6.18. A *path* is a function $\gamma : [a,b] \longrightarrow \mathbb{C}$ such that there exist basic curves $\gamma_j : [a_j, b_j] \longrightarrow \mathbb{C}, j \in \{1, \ldots, k\}$ such that $\gamma = \gamma_1 + \cdots + \gamma_k$ and therefore $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ and $a = a_1, b = a_k$.

Definition 6.19. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous curve and $a_1, \ldots, a_l \in \mathbb{R}$ such that $a = a_0 \leq \cdots \leq a_l \leq b = a_{l+1}$. We say γ *is piece-wise differentiable* if and only if

$$\gamma \in C^1\left(\bigcup_{j=0}^{l} (\alpha_j, \alpha_{j+1})\right),$$

$\forall j \in \{0, \ldots, l+1\} \exists \lim\limits_{t\to a_j^+} \gamma'(t)(\text{except if } j = l+1), \lim\limits_{t\to a_j^-} \gamma'(t)(\text{except if } j = 0).$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

Theorem 6.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with Ω an open set and $\phi : I \longrightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curev (hence a derivable curve) and its real derivative is*

$$\psi'(t) = f'(\phi(t))\phi'(t). \tag{112}$$

Definition 6.20. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two curves. We say γ_1, γ_2 *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = \gamma_1(0) = \gamma_2(0), s \in [0,1]$.
4. $h(1,s) = \gamma_1(1) = \gamma_2(1), s \in [0,1]$.

Definition 6.21. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two circuits. We say γ_1, γ_2 *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = h(1,s), s \in [0,1]$.

Definition 6.22. Let $f : [a,b] \longrightarrow \mathbb{C}$ be a function with the notation $f = u + iv$. We define the integral of f as

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t\,. \tag{113}$$

Proposition 6.2. *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ be two integrable functions and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,*

$$\int_a^b \lambda f + \mu g\,\mathrm{d}t = \lambda\int_a^b f\,\mathrm{d}t + \mu\int_a^b g\,\mathrm{d}t\,. \tag{114}$$

Proposition 6.3. *Let $f : [a,b] \longrightarrow \mathbb{C}$ be an integrable function. Then,*

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t\,. \tag{115}$$

Definition 6.23. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of* f *over* γ as

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\,. \tag{116}$$

Proposition 6.4. *The previous definition is well defined.*

Proposition 6.5. *If we use the notation $f = u + iv$ and $\gamma = x + iy$, then the integral has the form*

$$\int_\gamma f = \int_a^b u\frac{\mathrm{d}x}{\mathrm{d}t} + v\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t + i\int_a^b v\frac{\mathrm{d}x}{\mathrm{d}t} + u\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t\,. \tag{117}$$

Definition 6.24. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of* f *over* γ *with respect the differential of length* as

$$\int_\gamma f(z)\,\mathrm{d}s := \int_\gamma f(z)|\mathrm{d}z| = \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t\,. \tag{118}$$

Theorem 6.6. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$, $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two functions, and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,*

$$\int_\gamma \lambda f + \mu g \,\mathrm{d}z = \lambda \int_\gamma f \,\mathrm{d}z + \mu \int_\gamma g \,\mathrm{d}z \,. \tag{119}$$

Theorem 6.7. *Let γ_1, γ_2 be two equivalent curves of the same orientation and of class C^1 on their respective domains and$f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z) \,\mathrm{d}z = \int_{\gamma_2} f(z) \,\mathrm{d}z \,. \tag{120}$$

Proposition 6.8. *Let $\gamma_1, \dots, \gamma_n$ be n curves of class C^1 on their respective domains and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \dots, \Gamma_n \subseteq \Omega$. If we define $\gamma = \gamma_1 + \dots + \gamma_n$, then*

$$\int_\gamma f(z) \,\mathrm{d}z = \sum_{i=1}^{n} \int_{\gamma_i} f(z) \,\mathrm{d}z \,. \tag{121}$$

Proposition 6.9. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\left| \int_\gamma f(z) \,\mathrm{d}z \right| \leq \int_\gamma |f| \,\mathrm{d}s \,. \tag{122}$$

Corollary 6.10. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. If $|f(z)| \leq M$ for all $z \in \Gamma$, then,*

$$\left| \int_\gamma f(z) \,\mathrm{d}z \right| \leq M L(\gamma). \tag{123}$$

Proposition 6.11. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\mathrm{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} \,\mathrm{d}w \,. \tag{124}$$

Proposition 6.12. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$|\mathrm{Ind}(\gamma, z)| \leq \frac{1}{2\pi} \frac{L(\gamma)}{|z - \Gamma|}. \tag{125}$$

Proposition 6.13. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $\{f_n\}_{n=0}^{\infty}$ a sequence of continuous functions on Γ such that $\sum_{n=0}^{\infty} f_n$ converges uniformly on Γ. Then, $\sum_{n=0}^{\infty} \int_\gamma f_n \,\mathrm{d}z$ converges and*

$$\int_\gamma \sum_{n=0}^{\infty} f_n(z) \,\mathrm{d}z = \sum_{n=0}^{\infty} \int_\gamma f_n(z) \,\mathrm{d}z \,. \tag{126}$$

Definition 6.25. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f has a *primitive on* Ω if and only if there exists a function $F : \Omega \subseteq\longrightarrow \mathbb{C}$ such that $F' = f \forall z \in \Omega$.

Definition 6.26. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f has a *local primitive on* D if and only if for all z there exists a neigborhood where f has a primitive.

Theorem 6.14 (Fundamental theorem of complex calculus)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with Ω a domain. Then, the line integral of f is independent on the path on Ω if and only if f has an holomorphic primitive F such that $F' = f$ on Ω. In that case,*

$$\int_\gamma f(z) \,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)). \tag{127}$$

Theorem 6.15. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function on a star domain $S \subseteq \Omega$. Then, f has an holomorphic primitive F on S if and only if*

$$\int_{\partial\triangle} f(z) \,\mathrm{d}z = 0 \tag{128}$$

for all triangle $\triangle \subseteq \Omega$.

Proposition 6.16. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with no roots on a domain $D \subseteq \Omega$. Then, there is a determination of the logarithm of f on D if and only if f'/f has an holomorphic primitive on D.*

Proposition 6.17. *Let $K \subseteq \mathbb{C}$ be a compact set. Then,*

1. *If $\alpha \in V_\infty$, then the non-bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log(z - \alpha)$ in a neighborhood of K.*
2. *If α, β belong to the same bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log\left(\frac{z-\alpha}{z-\beta}\right)$ in a neighborhood of K.*

Theorem 6.18 (Green's theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary. Let $\mathbf{F} = (P, Q)$ be a vector field with P, Q being differentiable functions on a neighborhood of $\bar{\Omega}$ such that $\partial_x P - \partial_y Q$ is continuous on $\bar{\Omega}$. Then,*

$$\int_{\partial\Omega} \langle \mathbf{F}, \mathrm{d}\mathbf{s} \rangle_I = \int_{\partial\Omega} P \,\mathrm{d}x + Q \,\mathrm{d}y = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y} \,\mathrm{d}x \,\mathrm{d}y \,. \tag{129}$$

Theorem 6.19 (Cauchy's integral theorem)**.** *Let Ω be a bounded domain with piece-wise regular and positively oriented boundary and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an holomorphic function in a neighborhood of $\bar{\Omega}$. Then,*

$$\int_{\partial\Omega} f(z) \,\mathrm{d}z = 0. \tag{130}$$

Corollary 6.20. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function in a domain $D \subseteq \Omega$. Then, f ha local primitive on D. If D is a star domain, f has a global holomorphic primitive.*

Corollary 6.21. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with no roots in a domain $D \subseteq \Omega$. Then, f has a local determination of the logarithm on D. If D is a star domain, f has a global determination of the logarithm.*

Theorem 6.22 (Cauchy's integral theorem for homotopic curves)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with Ω a domain and γ_1, γ_2 two homotopic curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z\,. \tag{131}$$

Theorem 6.23 (Cauchy's general integral theorem)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a regular function on Ω except a finite numbers of points where f is continuous. If γ is a constant curve, then*

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 0. \tag{132}$$

Theorem 6.24 (Morera's theorem)**.** *Let f be a continuous function in a region Ω. If*

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 0 \tag{133}$$

for all simple and closed curve γ such that $\Gamma \subseteq \Omega$, then f is analytic on Ω.

Theorem 6.25. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a differentiable function on a domain D. Then, $f = u + iv$ is holomorphic if and only if the field $\bar{f} = (u, -v)$ is locally conservative and locally solenoidal.*

Definition 6.27. Let $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable vector field on a domain $D \subseteq \mathbb{R}^n$. We say the field is *holomorphic* if and only if is locally conservative and locally solenoidal, that is, it satisfies

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},\ \forall i, j; \qquad \operatorname{div}\mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = 0,\ \text{on } D. \tag{134}$$

Definition 6.28. Let $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field two times differentiable on an open set $\Omega \subseteq \mathbb{R}^n$. We say the field is *harmonic* if and only if $\nabla^2\Phi = 0$ on Ω.

Theorem 6.26. *Holomorphic vector fields are the fields that are locally the gradient of an harmonic function. Holomorphic functions are the functions f that, locally, satisfy $\bar{f} = \Phi_x + i\Phi_y$ with Φ harmonic.*

Definition 6.29. Let u be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$. We say a differentiable function $\tilde{u}$ on Ω is *the harmonic conjugate of* u if and only if $\mathrm{d}\tilde{u} = \mathrm{d}^*u$, that is, if the function $f = u + i\tilde{u}$ is holomorphic on Ω.

Theorem 6.27. *Let u be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$ and $f = \overline{\boldsymbol{\nabla}}u$. Then, u is has an harmonic conjugate on Ω, $\tilde{u}$, if and only if f has an holomorphic primitive F on Ω. In that case, $F = u + i\tilde{u}$.*

Proposition 6.28. *Let u be an harmonic function on a domain Ω. Then, it has an harmonic conjugate if an only if the closed form d^*u is exact on Ω, that is, if $\int_\gamma \mathrm{d}^*u = 0$ for all closed curve γ such that $\Gamma \subseteq \Omega$, condition that is is always locally completed. If Ω is a star domain, every harmonic function on Ω has a harmonic conjugate function on Ω.*

7 Local properties of holomorphic functions

Lemma 7.1. *Let $a \in \mathbb{C}$ be a number and $f = 1/|z - a|$. Then, f is Lebesgue-integrable on every subset of $\mathbb{C}$ of finite measure.*

Theorem 7.2 (Cauchy-Green formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and f a differentiable function on a neighborhood of $\bar{\Omega}$ such that $\overline{\partial} f$ is continuous on $\bar{\Omega}$. Then, for all $z_0 \in \Omega$,*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z - z_0}\,\mathrm{d}z - \frac{1}{\pi}\int_{\Omega} \frac{\overline{\partial} f(z)}{z - z_0}\,\mathrm{d}m(z)\,. \tag{135}$$

Corollary 7.3 (Cauchy's integral formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and f an holomorphic function on a neighborhood of $\bar{\Omega}$. Then,*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z - z_0}\,\mathrm{d}z\,. \tag{136}$$

Corollary 7.4. *Let f be a differentiable function on $\mathbb{C}$ with compact support and $\overline{\partial} f$ continuous on $\mathbb{C}$. Then,*

$$f(z_0) = -\frac{1}{\pi}\int_{\Omega} \frac{\overline{\partial} f(z)}{z - z_0}\,\mathrm{d}m(z)\,. \tag{137}$$

Proposition 7.5. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and γ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z_0) f(z_0) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(z)}{z - z_0}\,\mathrm{d}z\,. \tag{138}$$

Corollary 7.6. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and γ_1, γ_2 two homotopic, piece-wise regular, and positively oriented curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$f(z_0) = \frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(z)}{z - z_0}\,\mathrm{d}z - \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(z)}{z - z_0}\,\mathrm{d}z\,. \tag{139}$$

Theorem 7.7. *Let f be an holomorphic function on a disc $D(z_0, R)$. Then, there exists a power series $\sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence greater or equal to R such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall z \in D(z_0, R). \tag{140}$$

Theorem 7.8. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with Ω an open set. Then, f is holomorphic on Ω if and only if f is analytic on Ω. More precisely, every holomorphic function f on Ω is indefinitely holomorphic on Ω, and for all $z_0 \in \Omega$ the Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{141}$$

is valid on the greatest disc centered at z_0 and contained on Ω, which is $D(z_0, \delta(z_0))$, where $\delta(z_0) = \inf\{|z_0 - w|, w \notin \Omega\}$.

Theorem 7.9. *The assignation $f \longrightarrow \left(\frac{f^{(n)}(0)}{n!}\right)_{n=0}^{\infty}$ is a bijection between the space of entire functions and the space formed by the sequences $\{a_n\}_{n=0}^{\infty}$ such that the series $\sum_{n=0}^{\infty} a_n z^n$ has an infinite radius of convergence, that is, $\lim_{n\to\infty} |a_n|^{1/n} = 0$.*

Theorem 7.10 (Morera's theorem)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C(\Omega)$ with Ω an open set. Then, f is holomorphic on Ω if and only if*

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0 \tag{142}$$

for all triangle $\triangle \subseteq \Omega$.

Theorem 7.11. *Let f be a function continuous on an open set Ω and holomorphic on $\Omega \setminus E$, where E is a finite collection of points and segments. Then, f is holomorphic on Ω.*

Proposition 7.12. *Let f be a function and Ω a bounded domain with piece-wise regular and positively oriented boundary. If f is holomorphic on a neighborhood of $\bar{\Omega}$, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z\,. \tag{143}$$

Proposition 7.13. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and γ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\mathrm{Ind}(\gamma, z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z\,. \tag{144}$$

Lemma 7.14. *let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in H(\Omega)$ a function, and $z_0 \in \Omega$ a number. Then, the following statements are equivalent.*

1. *$f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.*
2. *$f(z) = 0$ for all z in a neighborhood of z_0.*
3. *f is identically null on Ω.*

Definition 7.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and z_0 a number. We say z_0 *is a zero of order* n *of* f if and only if $f^{(k)}(z_0) = 0$ for all $0 \leq k \leq n$. We call k the *order of* z_0 *as a zero of* f.

Proposition 7.15. *The zeros of finite order of an holomorphic function are isolated points.*

Proposition 7.16. *All the zeros of an non null analytic function are isolated points and of finite order.*

Definition 7.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with Ω a domain. Then, we denote the set of zeros of f as

$$Z(f) \coloneqq \{w \in \Omega \,|\, f(w) = 0\}\,. \tag{145}$$

Theorem 7.17. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with Ω a domain such that $f \not\equiv 0$. Then, $Z(f) \subseteq \Omega$ is a closed set without accumulation points. In particular, $Z(f)$ is a finite or countable set and and on every compact of Ω there is a finite number of zeros of f.*

Theorem 7.18 (Principle of analytic continuation)**.** *Let f, g be two holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$. Then, $f(z) = g(z)$ for all $z \in \Omega$ if and only if they satisfy one of the following conditions.*

1. *There exists a point $w \in \Omega$ such that $f^{(n)}(w) = g^{(n)}(w)$ for all $n \in \mathbb{N}$, that is, $|f(z) - g(z)| = o(|z - a|^n)$, if $z \to a$, for all $n \in \mathbb{N}$.*
2. *There exists a set $\Psi \subseteq \Omega$ that contains an accumulation point on Ω and $f(z) = g(z)$ for all $z \in \Psi$.*
3. *There exists an open set $\Psi \subseteq \Omega$ such that $f(z) = g(z)$ for all $z \in \Psi$.*

Theorem 7.19 (Schwarz reflection principle)**.** *Let $\Omega \subseteq \mathbb{C}$ be a symmetric domain and $f \in H(\Omega)$ such that $f(x) \in \mathbb{R}$ for all $x \in \Omega \cap \mathbb{R}$. Then, $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Omega$.*

Theorem 7.20. *Every analytic function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is the restriction on $\mathbb{R}$ of an holomorphic function $F : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ defined in a symmetric domain Ω, that is, $\mathbb{R} \subseteq \Omega$ and $f = F|_{\mathbb{R}}$.*

Theorem 7.21. *Let f, g be two analytic functions on a domain $\Omega \subseteq \mathbb{R}^2$. Then, $f(x, y) = g(x, y)$ for all $(x, y) \in \Omega$ if and only if they satisfy one of the following conditions.*

1. *There exists a point $(x_0, y_0) \in \Omega$ such that*
$$\frac{\partial^{n+m} f}{\partial x^n \partial y^m}(x_0, y_0) = \frac{\partial^{n+m} g}{\partial x^n \partial y^m}(x_0, y_0) \tag{146}$$
for all $n, m \in \mathbb{N}$, that is, $|f(x, y) - g(x, y)| = o\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right)$, if $(x, y) \to (x_0, y_0)$ for all $n \in \mathbb{N}$.
2. *There exists an open set Ψ such that $f(x, y) = g(x, y)$ fir all $(x, y) \in \Psi$.*

Theorem 7.22 (Maximum modulus principle)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with Ω a domain. If f is not constant, then $|f|$ does not have any local maxima on Ω.*

Corollary 7.23. *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and f an holomorphic function on a neighborhood of $\bar{\Omega}$ or, more generally, $f \in C(\bar{\Omega}) \cap H(\Omega)$. Let M be the maxima of $|f|$ on $\partial\Omega$. Then, one has*

$$|f(z)| \leq M, \qquad \text{for all } z \in \Omega. \tag{147}$$

In other words, $\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|$.

Theorem 7.24 (Cauchy's inequality)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on a neighborhood of the disc $\bar{D}(z_0, R)$ and $|f(z)| \leq M$ for $z \in C(z_0, R)$. Then,*

$$\left| f^{(n)}(z_0) \right| \leq M \frac{n!}{R^n}. \tag{148}$$

Corollary 7.25. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with Ω a domain such that $|f(z)| \leq M, z \in \Omega$. Then,*

$$\left| f^{(n)}(z) \right| \leq M \frac{n!}{d(z, U^c)^n}, \qquad z \in U, n \in \mathbb{N}. \tag{149}$$

Theorem 7.26 (Liouville's theorem)**.** *Let f be a bounded entire function. Then, f is constant. Also, a function u harmonic and bounded on $\mathbb{C}$ is constant.*

Theorem 7.27 (Fundamental theorem of algebra)**.** *Let $P(<) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n of complex coefficients and $n \geq 1$. Then, P has exactly n roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ (some of which can be counted with their multiplicity) and*

$$P(z) = a_n \prod_{i=1}^{n} (z - \alpha_i). \tag{150}$$

8 Isolated singularities of holomorphic functions

Definition 8.1. We say f *has an isolated singularity at* z_0 if and only if f is holomorphic on $D_r^*(z_0)$ for some $r \in \mathbb{R}^+$. We say the singularity is *removable* if and only if f can be extended to an holomorphic function on $D_r(z_0)$.

Definition 8.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on a disc $D_r^*(z_0)$. We say f has a pole of order k at z_0 if and only if there exist $\alpha \in \mathbb{C}, k \in \mathbb{N}_{\geq 1}$ such that $f(z) \propto \alpha (z - z_0)^k$ when $z \to z_0$. We call k the *multiplicity of the pole* or *order of the pole.*

Definition 8.3. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with Ω a domain. We say f is *meromorphic on* Ω if and only if there exists a set $A \subseteq \Omega$, discrete and closed on Ω, such that f is defined and holomorphic on $\Omega \setminus A$ and has a pole on every point $z \in A$.

Proposition 8.1. *f has a pole of order k at z_0 if and only if there exists an holomorphic function $g(z)$ in a neighborhood of z_0 such that $g(z_0) \neq 0$ and*

$$f(z) = \frac{g(z)}{(z - z_0)^k}. \tag{151}$$

Theorem 8.2. *Every holomorphic function on an annulus admits a Laurent expansion.*

Proposition 8.3. *Let f be an holomorphic function on an annulus $C(z_0, R_2, R_1)$. If f has an isolated singularity at z_0, then its Laurent expansion is uniquely determined by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{2\pi i} \int\limits_{C(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \,, \tag{152}$$

where a_n is independent of r, $r \in (R_2, R_1)$.

Definition 8.4. Let $f \in H(D_\epsilon^*(z_0))$ be an holomorphic function with a Laurent expansion $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ around z_0. We define the *residue of f at z_0* as

$$\operatorname{Res}(f, z_0) \coloneqq a_{-1} = \frac{1}{2\pi i} \int\limits_{C(z_0, r)} f(z) \, \mathrm{d}z \,. \tag{153}$$

Theorem 8.4. *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary. Let Ψ be an open set such that $\bar{\Omega} \subseteq \Psi$, $X \subseteq \Psi$ a closed set formed by isolated points (the accumulation points of X, if there are, must be in $\partial\Psi$) such that $X \cap \partial\Omega = \varnothing$, and f an holomorphic function on the open set $\Psi \setminus X$. Then,*

$$\frac{1}{2\pi i} \int\limits_{\partial\Omega} f(z) \, \mathrm{d}z = \sum_{w \in X \cap \Omega} \operatorname{Res}(f, w). \tag{154}$$

Theorem 8.5. *For a general curve,*

$$\frac{1}{2\pi i} \int\limits_{\gamma} f(z) \, \mathrm{d}z = \sum_{i=1}^{n} \operatorname{Ind}(\gamma, z_i) \operatorname{Res}(f, z_i). \tag{155}$$

Proposition 8.6. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on a neighborhood of z_0 with z_0 a pole. Then,*

1. *If z_0 is a removable singularity,* $\operatorname{Res}(f, z_0) = 0$.
2. *If z_0 is a simple singularity,*
$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{156}$$
3. *If z_0 is a singularity of order k,*
$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} [(z - z_0) f(z)]. \tag{157}$$
4. *If z_0 is an essential singularity, the residue a_{-1} must be obtained directly from the Laurent series.*

Proposition 8.7. *If $f = g/h$, with f, g holomorphic in a neighborhood of z_0, $g(z_0) \neq 0, h(z_0) = 0, h'(z_0) \neq 0$, then*

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}. \tag{158}$$

Proposition 8.8. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a meromorphic function in a neighborhood of $z_0 \in \mathbb{C}$. If we denote $f(z) = (z - z_0)^m g(z)$ with $w \in \mathbb{Z}^*$ (depending on the sign z_0 can be a zero or a pole), then z_0 is a single singularity of f'/f and $\operatorname{Res}(f'/f, z_0) = m$.*

Proposition 8.9. *Let $f(z) = g(\frac{1}{z-z_0})$ be a function with $g(w)$ an entire function that admits an expansion $g(w) = \sum_{n=0}^{\infty} b_n w^n$. If g is not a polynomial, then f has an essential singularity at z_0 and $\operatorname{Res}(f, z_0) = g'(0) = b_1$.*

Proposition 8.10. *Let f be a function with a simple pole at z_0 and g an holomorphic function in a neighborhood of z_0. Then, fg has a simple singularity at z_0 and $\operatorname{Res}(fg, z_0) = g(z_0) \operatorname{Res}(f, z_0)$.*

Definition 8.5. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f *is holomorphic at infinity* if and only if $g(w) = f(1/w)$ is holomorphic at the origin.

Proposition 8.11. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on $D^*_\epsilon(\infty)$. Then,*

$$\operatorname{Res}(f, \infty) = -a_{-1} = -\frac{1}{2\pi i} \int\limits_{C(z_0,r)} f(z)\,\mathrm{d}z\,. \tag{159}$$

Proposition 8.12. *Let f be a meromorphic function on the Riemann sphere. Then, f is a rational function. Besides, if X is the sent formed by the poles of F and the infinite point, then X is finite and*

$$\sum_{w \in X} \operatorname{Res}(f, w) = 0 \Leftrightarrow \operatorname{Res}(f, \infty) = -\sum_{w \in X \setminus \{\infty\}} \operatorname{Res}(f, w). \tag{160}$$

Theorem 8.13. *Let Ω be a bounded domain with a piece-wise regular and positively oriented boundary. Let Ψ be an open set such that $\bar{\Omega} \subseteq \Psi$, f a meromorphic function on Ψ and h an holomorphic function on Ψ. Let $\{a_j\}$ be the zeros of f on Ψ and n_j the multiplicities of a_j, and let $\{b_j\}$ be the poles of f on Ψ and m_j the multiplicities of b_j. If there is no zeros or poles on $\partial\Omega$, then*

$$\frac{1}{2\pi i} \int\limits_{\partial\Omega} h(z) \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{a_j \in \Omega} h(a_j) n_j - \sum_{b_j \in \Omega} h(b_j) m_j. \tag{161}$$

Corollary 8.14. *Let Ω be a bounded domain with a piece-wise regular and positively oriented boundary and f a meromorphic function on a neighborhood of $\bar{\Omega}$ that does not have zeros or poles on $\partial\Omega$. Let N the total number of zeros of f on Ω and P the total number of poles on Ω (counting multiplicities). If we denote $\Gamma = f(\partial\Omega)$, then*

$$\operatorname{Ind}(\Gamma, 0) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f'(z)}{f(z)}\,\mathrm{d}z = N - P. \tag{162}$$

9 Homology

10 Harmonic functions

Theorem 10.1. *Let $f \in H(\Omega), C^1(\Omega)$ be a function. If $f = u + iv$, then u, v are harmonic functions on Ω.*

11 Conforming representation

12 Riemann theorem

13 Runge theorem

14 Zeros of holomorphic functions

Theorem 14.1 (Weierstrass Factorization Theorem)**.** *content...*

15 Fourier transform

Definition 15.1. Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform of f at the point ξ* as

$$\boxed{\hat{f}(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(x) \mathrm{e}^{-i\xi x}\,\mathrm{d}x\,.} \tag{163}$$

Proposition 15.1. *Let $f \in L^1(\mathbb{R})$ be a function. Then, the application*

$$\begin{aligned} \mathscr{F}\{f\} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \hat{f}(\xi) \end{aligned} \tag{164}$$

is a well defined application.

Definition 15.2. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. We say *the functions f_n converge to f with a norm $\|\cdot\|_p$* or *converge in $L^p(\mathbb{R})$* if and only if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0. \tag{165}$$

Theorem 15.2. *Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.*

1. *The Fourier transform of f satisfies*
$$\mathscr{F}\{f\} \in L^\infty(\mathbb{R}), \qquad \|\mathscr{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|f\|_1. \tag{166}$$
2. *$\mathscr{F}\{f\}$ is $\mathbb{C}$ linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,*
$$\mathscr{F}\{\alpha f + \beta g\} = \alpha \mathscr{F}\{f\} + \beta \mathscr{F}\{g\}. \tag{167}$$
3. *If $g(x) = \bar{f}(x)$, then for all $\xi \in \mathbb{R}$*
$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \tag{168}$$
4. *If $g(x) = g(\lambda x)$ and $\lambda \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*
$$\hat{g}(\xi) = \frac{1}{|\lambda|} \hat{f}\left(\frac{\xi}{\lambda}\right). \tag{169}$$

5. *If* $g(x) = f(x-a)$ *with* $a \in \mathbb{R}$, *then for all* $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \mathrm{e}^{-ia\xi} \hat{f}(\xi). \tag{170}$$

6. *If* $g(x) = \mathrm{e}^{iax} f(x)$ *with* $\alpha \in \mathbb{R}$, *then for all* $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \hat{f}(\xi - a) \tag{171}$$

7. *If* $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ *and* $f_n \to f$ *in* $L^1(\mathbb{R})$ *when* $n \to \infty$, *then* $\mathscr{F}\{f_n\} \to \mathscr{F}\{f\}$ *uniformly in* $\mathbb{R}$.

8. *The Fourier transform* $\mathscr{F}\{f\}$ *is a continuous function in* $\mathbb{R}$, $\mathscr{F}\{f\} \in C(\mathbb{R})$.

Proposition 15.3. *Let* $f \in L^1(\mathbb{R})$ *be a function such that there exists its derivative* $f' \in L^1(\mathbb{R})$ *and* $\lim_{|x|\to\infty} |f(x)| = 0$. *Then,*

$$\hat{f}'(\xi) = i\xi \hat{f}(\xi). \tag{172}$$

Corollary 15.4. *Let* $f \in L^1(\mathbb{R})$ *be a function such that there exists its* n*-th derivative* $f^{(n)} \in L^1(\mathbb{R})$ *and* $\lim_{|x|\to\infty} |f(x)| = 0$. *Then,*

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi). \tag{173}$$

Definition 15.3. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We define the support of f as

$$\operatorname{supp} f \coloneqq \overline{\{x \in I \mid f(x) \neq 0\}}. \tag{174}$$

Definition 15.4. We define the set $\mathscr{D}(\mathbb{R})$ as

$$\mathscr{D}(\mathbb{R}) \coloneqq \{\varphi \in C^\infty(\mathbb{R}) \mid \operatorname{supp}\varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \tag{175}$$

Theorem 15.5. *Let* $f \in L^1(\mathbb{R})$ *be a function. Then, there exists a sequence of functions* $\phi_\in \mathscr{D}(\mathbb{R})$ *such that*

$$\lim_{h\to\infty} \int_{\mathbb{R}} |f - \phi_n| \,\mathrm{d}x = 0, \tag{176}$$

that is, we have convergence of ϕ_n *to* f *with norm* $\|\cdot\|_1$.

Proposition 15.6. *Let* $f \in L^1(\mathbb{R})$ *be a function. Then,* $\hat{f} \in C(\mathbb{R})$.

Proposition 15.7. *Let* $f \in L^1(\mathbb{R})$ *be a function. Then,* $\left|\hat{f}(\xi)\right| \leq \|f\|_1$.

Theorem 15.8. *Let* $f \in L^1(\mathbb{R})$ *be a function. Then,*

$$\lim_{|x|\to\infty} \hat{f}(x) = 0. \tag{177}$$

Theorem 15.9. *The application Fourier transform goes from* $L^1(\mathbb{R})$ *to* $C_0(\mathbb{R})$, *that is,* $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.

Definition 15.5. We define the *Schwartz space* as

$$\begin{aligned} S(\mathbb{R}) \coloneqq \{ & f : \mathbb{R} \longrightarrow \mathbb{C} \mid f \in C^\infty(\mathbb{R}) \wedge \forall n, m \in \mathbb{N}\, \exists c_{n,m} < \infty \\ & \text{such that } (1+|x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}. \end{aligned}$$

Proposition 15.10. *Let* $f, g \in S(\mathbb{R})$ *be two functions,* $\lambda \in \mathbb{C}$ *a number, and* $P : \mathbb{R} \longrightarrow \mathbb{C}$ *a polynomial of complex coefficients. Then,*

1. $f + g \in S(\mathbb{R})$.
2. $\lambda f \in S(\mathbb{R})$.
3. $fg \in S(\mathbb{R})$.
4. $Pf \in S(\mathbb{R})$.

Theorem 15.11. *Let* $I, J \subseteq \mathbb{R}$ *be two intervals with* I *compact and* J *open. Let* $f : I \times J \longrightarrow \mathbb{R}$ *be a function such that*

1. $f(\cdot, \lambda)$ *is Riemann-integrable in* I *for all* $\lambda \in J$,
2. $f(x, \cdot)$ *is derivable in* J *for all* $x \in I$.

If $\partial_\lambda f$ *is continuous in* $I \times J$, *then*

1. $\partial_\lambda f(\cdot, \lambda)$ *is Riemann-integrable for all* $\lambda \in J$.
2. $F(\lambda) = \int_I f(x, \lambda) \,\mathrm{d}x$ *is derivable with continuous derivative in* J *for all* $\lambda \in J$ *and it satisfies the rule of derivation over the integral sign.*

$$F'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int_I f(x, \lambda_0) \,\mathrm{d}x = \int_I \frac{\partial f}{\partial \lambda}(x, \lambda_0) \,\mathrm{d}x \,, \forall \lambda_0 \in J \tag{178}$$

Proposition 15.12. *Let* $f \in S(\mathbb{R})$. *Then,*

1. $S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.
2. $\widehat{xf}(\xi) = (iD_\xi \hat{f})(\xi)$ *for all* $\xi \in \mathbb{R}$.

Corollary 15.13. *Let* $f \in s(\mathbb{R})$. *Then,*

$$\widehat{x^n f}(\xi) = (i^n D^n \hat{f})(\xi), \forall n \in \mathbb{N}. \tag{179}$$

Proposition 15.14. *The Fourier transform* $\mathscr{F}$ *restricted to* $S(\mathbb{R})$ *is an automorphism, that is, if* $f \in S(\mathbb{R})$ *then* $\mathscr{F}\{f\} = \hat{f} \in S(\mathbb{R})$.

Lemma 15.15. *If* $G(x) = \mathrm{e}^{-x^2/2}$, *then* $\hat{G}(\xi) = e^{-\xi^2/2}$. *We observe hence that* G *is a fixed point of* $\mathscr{F}$.

Lemma 15.16. *If* $f, g \in S(\mathbb{R})$, *then*

$$\int_{\mathbb{R}} f(\xi)\hat{g}(\xi) \,\mathrm{d}\xi = \int_{\mathbb{R}} \hat{f}(\tau) g(\tau) \,\mathrm{d}\tau \,. \tag{180}$$

Lemma 15.17. *Let* $f, g \in S(\mathbb{R})$ *and* $\lambda \in \mathbb{R}$. *Then,*

1. $g(\lambda x)\hat{f}(x)$ *converges to* $g(0)\hat{f}(x)$ *uniformly in* $\mathbb{R}$ *when* $\lambda \to \infty$.
2. $f(\lambda x)\hat{g}(x)$ *converges to* $f(0)\hat{g}(x)$ *uniformly in* $\mathbb{R}$ *when* $\lambda \to \infty$.

Lemma 15.18. *Let* $f, g \in s(\mathbb{R})$. *Then,*

$$f(0) \int_{\mathbb{R}} \hat{g}(\xi) \,\mathrm{d}\xi = g(0) \int_{\mathbb{R}} \hat{f}(\xi) \,\mathrm{d}\xi \,. \tag{181}$$

Lemma 15.19. *Let $f \in s(\mathbb{R})$ be a function. Then,*

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)\,\mathrm{d}\xi\,. \qquad (182)$$

Corollary 15.20 (Inversion formula). *Let $f \in S(\mathbb{R})$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,, \forall x \in \mathbb{R}. \qquad (183)$$

Theorem 15.21 (Inversion of $\mathscr{F}$ in $S(\mathbb{R})$). *Let $\mathscr{F} : S(\mathbb{R}) \longrightarrow S(\mathbb{R})$, defined by $\mathscr{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, $\mathscr{F}$ is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathscr{F}^4 = Id$. In particular, $\mathscr{F}^{-1} = \mathscr{F}^3$ and if $f \in S(\mathbb{R})$, then*

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{F}\{f\}(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}\mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,.} \qquad (184)$$

In fact, $\mathscr{F}$ is an homemorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(S(\mathbb{R}), \|\cdot\|_{n,m})$.

Theorem 15.22 (Inversion of $\mathscr{F}$ for discontinuities). *Let f be a absolutely Riemann-integrable function in $\mathbb{R}$ with f and f' piece-wise continuous. Then,*

$$\frac{f(x^-) + f(x^+)}{2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}\mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,. \qquad (185)$$

Definition 15.6. Let f be a Riemann-integrable function in $\mathbb{R}$. We define the *Fourier transform of cosine kind* as

$$\hat{f}_c(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\xi x) f(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(\xi x) f_e(x)\,\mathrm{d}x\,, \qquad (186)$$

and the *Fourier transform of sine kind* as

$$\hat{f}_s(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\xi x) f(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(\xi x) f_o(x)\,\mathrm{d}x\,. \qquad (187)$$

Proposition 15.23. *Let $\hat{f}_c, \hat{f}_s$ be the Fourier transform of cosine and sine kinds of f. Then, $\hat{f}_c(\xi)$ is even, $\hat{f}_s(\xi)$ is odd, and $\hat{f}(\xi) = \hat{f}_c(\xi) - i\hat{f}_s(\xi)$.*

Theorem 15.24. *Let f be a absolutely Riemann-integrable function in $\mathbb{R}$ with f and f' piece-wise continuous. Then,*

$$\frac{f_e(x^-) + f_e(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_c \cos(\xi x)\,\mathrm{d}\xi\,, \qquad (188)$$

$$\frac{f_o(x^-) + f_o(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_s \sin(\xi x)\,\mathrm{d}\xi\,. \qquad (189)$$

Theorem 15.25 (Tonelli's Theorem). *Let $f : I \times J \longrightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x, y) \geq 0$ for all $(x, y) \in I \times J$. Then,*

$$\int_{I\times J} f\,\mathrm{d}x\,\mathrm{d}y = \int_I \int_J f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_J \int_I f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,. \qquad (190)$$

Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.

Corollary 15.26. *Let $f, g \in L^1(\mathbb{R})$. Then, $F(x,t) = f(t)g(x-t) \in L^1(\mathbb{R}^2)$.*

Definition 15.7. Let $f, g \in L^1(\mathbb{R})$ two function. We define the *convolution of f and g* as

$$\begin{aligned}(f * g) : \mathbb{R} &\longrightarrow \mathbb{C} \\ x &\longmapsto \int_{\mathbb{R}} f(t)g(x-t)\,\mathrm{d}t\end{aligned}, \qquad (191)$$

which is from $L^1(\mathbb{R})$.

Proposition 15.27. *Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$.*

Proposition 15.28. *Let $f \in L^1(\mathbb{R})$ be a function and $g = f^2$. Then,*

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \hat{f} * \hat{f} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)\hat{f}(\xi - t)\,\mathrm{d}t\,. \qquad (192)$$

Theorem 15.29. *Let $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^\infty(\mathbb{R})$.*

Theorem 15.30. *Let $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$ with* $\operatorname{supp} f$ *compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and* $\operatorname{supp}\{f * \phi\} \subseteq \operatorname{supp} f + \operatorname{supp} \phi$.

Definition 15.8. We say the functions $\phi_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ continuous in a compact support are *an approximation of the unity* if and only if

1. $\phi_\epsilon \geq 0$ for all ϵ.
2. $\displaystyle\int_{\mathbb{R}} \phi_\epsilon(x)\,\mathrm{d}x = 1$.
3. For all $\delta > 0$ it is satisfied that

$$\lim_{\epsilon\to 0} \left\{ \sup_{|t|>\delta} \phi_\epsilon(t) \right\} = 0. \qquad (193)$$

Theorem 15.31. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to f.*

Corollary 15.32. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to f.*

Theorem 15.33 (Weierstrass polynomial approximation). *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials P_n with $n \in \mathbb{N}$ such that P_n converge uniformly to f in $[a,b]$.*

Theorem 15.34. *Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \to f$ with norm $\|\cdot\|_p$ (that is, convergence in L^p), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then*

$$\lim_{n\to\infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \tag{194}$$

with $\|f\|_{C^k(\mathbb{R})} = \max_{0\leq l\leq k}\left(\sup_{x\in\mathbb{R}}\left|D^l f(x)\right|\right)$ being a norm.

Lemma 15.35. *Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x = 0$. Then, $f \equiv 0$.*

Corollary 15.36. *The Fourier transform $\mathscr{F}$ is injective since $\mathscr{F}\{f\} = \hat{f} = 0 \Leftrightarrow f = 0$ in $L^1(\mathbb{R})$ (the zero function class) and $\mathscr{F}$ is a linear application.*

Theorem 15.37 (Inversion theorem in $L^1(\mathbb{R})$)**.** *Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) \mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,, \forall x \in \mathbb{R}. \tag{195}$$

16 Fourier transform 2

Theorem 16.1 (Parseval formula)**.** *Let $f, g \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be two functions. Then,*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi\,. \tag{196}$$

Theorem 16.2 (Plancherel Theorem)**.** *Let $f \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be a function. Then,*

$$\boxed{\int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi\,,} \tag{197}$$

that is, $\|f\|_2 = \left\|\hat{f}_2\right\|$ and $\mathscr{F}$ is an isometry between vector spaces.

Definition 16.1. Let $f \in S(\mathbb{R})$ be a function. We define the following quantities

$$E(f) := \int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x\,, \tag{198}$$

$$\sigma(f)^2 := \int_{\mathbb{R}} |xf(x)|^2\,\mathrm{d}x\,. \tag{199}$$

Theorem 16.3. *Let $f \in S(\mathbb{R})$ be a function. Then,*

$$\sigma(f)\sigma(\hat{f}) \geq \frac{E(f)}{2}. \tag{200}$$

17 Multidimensional fourier transform

Theorem 17.1. *For several variables*

$$\mathscr{F}\{f(x_1,\dots,x_n)\} = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1,\dots,x_n)\mathrm{e}^{-i(x_1\xi_1+\cdots+x_n\xi} \tag{201}$$

or simpler,

$$\boxed{\mathscr{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{e}^{-i\langle\mathbf{x},\boldsymbol{\xi}\rangle_I}\,\mathrm{d}\hat{x}\,.} \tag{202}$$