# 1 Arithmetic and topology

**Definition 1.1.** Let $\mathbb{R}^2 = \{(x.y) \,|\, x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the sum "+" by components

$$(a, b) + (c, d) := (a + b, c + d). \qquad (1)$$

- Product: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the product by

$$(a, b)(c, d) = (ac - bd, ad + bc). \qquad (2)$$

We define the set $\mathbb{C}$ as $(\mathbb{R}^2, +, )$.

**Proposition 1.1.** *The set $\mathbb{C}$ of complex numbers is an abelian field.*

**Proposition 1.2.** *Let $\mathbb{C}$ be defined in the second way. Then,*

1. *$\mathbb{C}$ is an abelian ring.*

2. *If we define $f$ as*

$$f : (\mathbb{C}, +, ) \longrightarrow (\mathbb{R}^2, +, ), \atop (x, y) \longmapsto x + yi \qquad (3)$$

   *then $f$ is a morphism of rings.*

3. *The function $f$ is, in fact, an isomorphism and $\mathbb{C}$ is an abelian field.*

**Proposition 1.3.** *The subset of $\mathbb{C}$ generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.*

**Theorem 1.4.** *$\mathbb{C}$ is not an ordered field.*

**Definition 1.2.** Let $z = a + bi \in \mathbb{C}$. We define the *conjugate of $z$* as

$$\bar{z} := a - bi. \qquad (4)$$

**Proposition 1.5.** *For all $z, w \in \mathbb{C}$, we have:*

1. *$\bar{\bar{z}} = z$.*

2. *$\overline{z + w} = \bar{z} + \bar{w}$.*

3. *$\overline{zw} = \bar{z}\bar{w}$.*

4. *$z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.*

5. *$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.*

6. *The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.*

**Definition 1.3.** Let $z = a + bi \in \mathbb{C}$. We define the *real part of $z$ and imaginary part of $z$* respectively as

$$\mathrm{Re}\{z\} := a, \qquad \mathrm{Im}\{z\} := b. \qquad (5)$$

**Proposition 1.6.** *Let $z \in \mathbb{C}$. Then,*

$$\mathrm{Re}\{z\} = \frac{z + \bar{z}}{2}, \qquad \mathrm{Im}\{z\} = \frac{z - \bar{z}}{2i} \qquad (6)$$

**Proposition 1.7.** *Let $z, w \in \mathbb{C}$ and the following distance function.*

$$\tilde{d} : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R} \atop (z, w) \longmapsto \tilde{d}(z, w) := |z - w| \qquad (7)$$

*Then, $(\mathbb{C}, \tilde{d})$ is a metric space.*

**Definition 1.4.** Let $z = a + bi \in \mathbb{C}$. We define the *modulus of $z$* as

$$|z| := \tilde{d}(z, 0), \qquad (8)$$

which is equivalent to $\sqrt{z\bar{z}}$.

**Definition 1.5.** Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define an *open disc of radius $r$ and center $z_0$* as follows

$$B_r(z_0) := \{z \in \mathbb{C} \,|\, |z - z_0| < r\} . \qquad (9)$$

**Definition 1.6.** Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *punctured disc of radius $r$ and center $z_0$* as follows

$$B_r^*(z_0) := \{z \in \mathbb{C} \,|\, 0 < |z - z_0| < r\} . \qquad (10)$$

**Definition 1.7.** Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *closed disc of radius $r$ and center $z_0$* as follows

$$\overline{B_r(z_0)} := \{z \in \mathbb{C} \,|\, |z - z_0| \le r\} . \qquad (11)$$

**Definition 1.8.** We denote by $\mathbb{D}$ the unitary disc of center 0 and radius 1. Besides, we denote by $\mathbb{T} \subseteq \mathbb{C}$ the unitary circumference, that is,

$$\mathbb{T} := \{z \in \mathbb{C} \,|\, |z| = 1\} . \qquad (12)$$

We also denote it by $\mathbb{S}^1$.

**Lemma 1.8.** *The set $B = \left\{ B_r(z_0) \,\middle|\, r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2 \right\}$ is a basis of the topology of $\mathbb{R}^2$ as a metric space. The set $D = \{D_r(z_0) \,|\, r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of $\mathbb{C}$ as a metric space.*

**Proposition 1.9.** *The sets $\mathbb{C}$ and $\mathbb{R}^2$ with the topology of metric space are homeomorphs.*

**Corollary 1.10.** *There is a bijection between $B$ and $D$, that is, between balls of $\mathbb{R}^2$ and discs of $\mathbb{C}$.*

**Proposition 1.11.** *Let $z, w \in \mathbb{C}$. Then,*

1. *$|z| \ge 0$.*

2. *$|z| = 0 \Leftrightarrow z = 0$.*

3. *$-|z| \le \mathrm{Re}\{z\} \le |z|$ and $-|z| \le \mathrm{Im}\{z\} \le |z|$.*

4. *$|zw| = |z||w|$.*

5. *If $w \neq 0$, $|z/w| = |z|/|w|$.*

6. *$|z + w| \le |z| + |w|$.*

7. *$|z + w| \ge ||z| - |w||$.*

8. *$|\mathrm{Re}\{zw\}| \le |z||w|$ and $|\mathrm{Im}\{z\}| \le |z||w|$.*

9. *$|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\,\mathrm{Re}\{z\bar{w}\}$.*

10. $|z^n| = |z|^n$

**Corollary 1.12.** *Let* $z_1, \ldots, z_n \in \mathbb{C}$. *Then,*

$$\left| \sum_{i=1}^n z_i \right| \leq \sum_{i=1}^n |z_i|, \qquad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \qquad |\operatorname{Re}\{z_1 \cdots z_n\}| \leq |z_1| \cdots |z_n|. \tag{13}$$

**Definition 1.9.** *Let* $z \in \mathbb{C}^*$. *We define the* argument *of* $z$, *denoted by* $\arg z$, *as the real number* $\theta$ *such that* $z = |z|(\cos\theta + i\sin\theta)$. *Let us observe that* $\arg z$ *is not a function but a multivalued application. We define the* principal argument of $z$ *as*

$$\operatorname{Arg} z := \theta_0 \in [0, 2\pi) \mid z = |z|(\cos\theta + i\sin\theta). \tag{14}$$

In general, to make $\theta$ to be unique, it is enough to impose it to belong to a certain semiopen interval of length $2\pi$. Choosing the interval $I$ is called by *taking a determination of the argument*.

**Definition 1.10.** *Given a complex number* $z$ *that we can express by* $z = |z|(\cos\theta + i\sin\theta)$ *for some* $\theta \in \mathbb{R}$, *we use the notation* $r = |z|$ *to write*

$$z = r_\theta^z = r(\cos\theta + i\sin\theta) \tag{15}$$

or simply $r_\theta$ when it is obvious which complex number are we referring to. We call it *polar form of* $z$.

**Proposition 1.13.** *Let* $z \in \mathbb{C}$ *and* $r_\theta$ *its polar form. Then,*

$$z^n = (r^n)_{n\theta}. \tag{16}$$

**Corollary 1.14** (De Moivre's Formula). *Let* $\theta \in \mathbb{R}$. *Then,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{17}$$

**Proposition 1.15.** *Let* $z, w \in \mathbb{C}$. *Then,*

1. $\arg zw = \arg z + \arg w + 2\pi k$.

2. $\arg z^n = n \arg z + 2\pi k$.

**Definition 1.11.** *We denote the complex numbers* $z$ *generated by moving the point* $z_0 = 1$ *around* $\mathbb{T}$ *a length* $t$ *in a counter-clockwise direction by* $1_t$. *In other words,* $1_t$ *are the complex numbers* $z = \cos t + i \sin t$.

**Proposition 1.16.** *Let* $f : t \longrightarrow 1_t$. *Then,* $f$ *is a morphism from* $(\mathbb{R}, +)$ *to* $(\mathbb{T}, \cdot)$, *with* $\ker f = 2\pi\mathbb{Z}$.

**Definition 1.12.** *Let* $z \in \mathbb{C}$ *and* $n \in \mathbb{N}$. *We say* $w \in \mathbb{C}$ *is an* n-th root of $z$ *if and only if*

$$w^n = z. \tag{18}$$

**Theorem 1.17.** *Let* $n \in \mathbb{N}^*$ *and* $z \in \mathbb{C}$. *Then, there exist* $w_1, \ldots, w_n \in \mathbb{C}$ *such that* $w_i^n = z$ *for all* $i \in \{1, \ldots, n\}$, *and* $w_i \neq w_j$ *for all* $i \neq j$. *Besides, if* $\omega \in \mathbb{C}$ *satisfies* $\omega^n = z$, *then* $\omega = w_k$ *for some* $k \in \{1, \ldots, n\}$.

**Theorem 1.18.** *Let* $\gamma : [a, b] \longrightarrow \mathbb{C}$ *be a continuous curve such that* $\gamma(t) \neq 0 \forall t \in [a, b]$. *Then, there exists a continuous determination* $\phi$ *of the argument of* $\gamma$. *Then,* $\phi(t) + 2\pi k$ *with* $k \in \mathbb{Z}$ *is the general expression of all the argument determinations of* $\gamma$. *If* $\gamma$ *is differentiable, then* $\phi$ *is differentiable and* $\phi' = \operatorname{Im}\{\gamma'/\gamma\}$.

**Definition 1.13.** *Let* $\gamma : [a, b] \longrightarrow \mathbb{C}$ *be a regular curve. We define the* variation of the argument along $\gamma$ *as*

$$\Delta_\gamma \arg := \operatorname{Im}\left\{ \int_a^b \frac{\gamma'(t)}{\gamma(t)} \, dt \right\}. \tag{19}$$

**Definition 1.14.** *Let* $\gamma : [a, b] \longrightarrow \mathbb{C}$ *be a curve such that* $\gamma(t) \neq 0 \forall t \in [a, b]$. *Then, we define the* index of $\gamma$ with respect to the origin *or* the number of revolutions of $\gamma$ around the origin

$$\operatorname{Ind}(\gamma, 0) := \frac{1}{2\pi} \Delta_\gamma \arg. \tag{20}$$

**Proposition 1.19.** *Let* $\gamma : [a, b] \longrightarrow \mathbb{C}$ *be a piece-wise regular curve. Then,*

$$\operatorname{Ind}(\gamma, 0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} \, dt \tag{21}$$

**Definition 1.15.** *Let* $\gamma$ *be a closed curve and* $z \notin \Gamma$. *We define the* index of $\gamma$ with respect to $z$ *as*

$$\operatorname{Ind}(\gamma, z) := \operatorname{Ind}(\gamma - z, 0). \tag{22}$$

**Proposition 1.20.** *Let* $\gamma : [a, b] \longrightarrow \mathbb{C}$ *be a curve piece-wise of class* $C^1([a, b])$. *Then,*

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} \, dt. \tag{23}$$

**Proposition 1.21.** *Let* $\gamma : [a, b] \longrightarrow \mathbb{C}$ *be a piece-wise of class* $C^1([a, b])$. *Then,* $\operatorname{Ind}(-\gamma, z) = -\operatorname{Ind}(\gamma, z)$.

# 2 Sequences and limits

**Definition 2.1.** *A* sequence of complex numbers *is an application of the form*

$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned}. \tag{24}$$

We denote it by $\{z_n\}_{n=m}^\infty$

**Definition 2.2.** *Let* $\{z_n\}_{n=0}^\infty$ *be a sequence. We say the sequence has limit* $L$ *or* it converges to the limit $L$ *if and only if*

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \tag{25}$$

We denote it by

$$\lim_{h \to \infty} z_n = L, \qquad \lim \{z_n\}_{n=0}^\infty = L, \qquad \{z_n\}_{n=0}^\infty \to L. \tag{26}$$

**Theorem 2.1.** *Let* $z_n = z_n + iy_n$ *be the general term of a sequence* $\{z_n\}_{n=0}^\infty$ *and* $L = L_x + iL_y \in \mathbb{C}$. *Then,*

$$\{z_n\}_{n=0}^\infty \to L \Leftrightarrow \{x_n\}_{n=0}^\infty \to L_x \wedge \{y_n\}_{n=0}^\infty \to L_z. \tag{27}$$

**Definition 2.3.** Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *it tends to infinity* and denote it by $\lim z_n = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \qquad (28)$$

**Definition 2.4.** Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \qquad (29)$$

**Theorem 2.2.** *Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.*

**Theorem 2.3.** *Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence } \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \text{ are Cauchy sequences.} \qquad (30)$$

**Theorem 2.4.** *The field $\mathbb{C}$ of complex numbers is complete.*

**Definition 2.5.** The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, together with two charts.

# 3   Functions

**Definition 3.1.** A *topology* is an ordered pair $(\mathbb{X}, \tau)$, where $X$ is a set and $\tau$ a collection of subsets of $X$ satisfying the following properties:

1. The empty set and $X$ belong to $\tau$.

2. Any arbitrary (finite or infinite) union of members of $\tau$ still belongs to $\tau$.

3. The intersection of any finite number of members of $\tau$ still belongs to $\tau$.

The elements of $\tau$ are called *open sets* and the collection $\tau$ is called the *topology on $X$*.

**Definition 3.2.** Let $(\mathbb{X}, d)$ be a metric space. A *topology on the metric space by the metric d* is the set $\tau$ of all open sets of $M$.

**Definition 3.3.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *interior point of $A$* if there is a ball $B_{(\mathbb{M},d)}(a, r) \subset A$.

**Definition 3.4.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *exterior point of $A$* if there is a ball such that $B_{(\mathbb{M},d)}(a, r) \cup A = \emptyset$.

**Definition 3.5.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is a *boundary point of $A$* if it is not interior or exterior or, which is equivalent, if every ball $B_{(\mathbb{M},d)}(a, r)$ contains elements of $A$ and $A^c$.

**Definition 3.6.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *accumulation point of $A$* if every ball with center $a$ contains points of $A$ different to $a$. In other words, every punctured ball satisfies $B_{(\mathbb{M},d)}^*(a, r) \cup A \neq \emptyset$.

**Definition 3.7.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the interior of $A$* as the set of all interior points of $A$, and we denote it by $\text{int}(A)$.

**Definition 3.8.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the exterior of $A$* as the set of all exterior points of $A$, and we denote it by $\text{ext}(A)$.

**Definition 3.9.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the boundary of $A$* as the set of all boundary points of $A$, and we denote it by $\partial A$.

**Definition 3.10.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the closure of $A$* as the set of all accumulation points of $A$, and we denote it by $\bar{A}$.

**Definition 3.11.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *an open set* if it contains none of its boundary points, that is, if $\partial A \cap A = \emptyset$.

**Definition 3.12.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a closed set* if it contains all its boundary points, that is, if $\partial A \subseteq A$.

**Definition 3.13.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a bounded set* if there exist a point $a \in \mathbb{M}$ and a positive real number $r$ such that the ball $B_{(\mathbb{M},d)}(a, r)$ contains $A$.

**Definition 3.14.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a compact set* if it a bounded and closed set.

**Proposition 3.1.** *Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. Then, $A$ is open if and only if $A^c$ is closed.*

**Definition 3.15.** Let $\Omega \subseteq \mathbb{C}$ be a set. We say $\Omega$ *is connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally, $\Omega$ is connected if there are not two open sets $U, V \subseteq \mathbb{C}$ such that

$$U_1 = U \cap \Omega, \qquad V_1 = V \cap \Omega, \qquad U_1 \cap V_1 = \varnothing, \qquad U_1 \cup V_1 = \Omega. \qquad (31)$$

Otherwise, we say $\Omega$ *is disconnected*.

**Definition 3.16.** Let $\Omega \subseteq \mathbb{C}$ be a set. We say $\Omega$ *is simply connected* if and only if every circuit is homotopic in $\Omega$ to a point in $\Omega$. Equivalently, is simply connected if and only if every pair of curves with the same extremes are homotopic.

**Definition 3.17.** Let $\Omega \subseteq \mathbb{C}$ be a set. We say $\Omega$ *is convex* if and only if for all pair of point $a, b \in \Omega$, the segment defined by

$$[a, b] = \{z \mid z = (1 - t)a + tb, 0 \leq t \leq 1\} \qquad (32)$$

is contained in $\Omega$, that is, if every pair of points can be connected by a straight line that belongs to the set.

**Definition 3.18.** Let $\Omega \in \mathbb{C}$ be a set. We say $\Omega$ *is a star-convex set* if and only if there exists $z_0 \in \mathbb{C}$ such that for all $z \in \Omega$ the segment $[z_0, z]$ is contained by $\Omega$.

**Definition 3.19.** Let $(\mathbb{M}, d)$ be a metric space and $S \subseteq \mathbb{M}$ a set. We say $S$ *is path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

**Definition 3.20.** Let $\Omega \in \mathbb{C}$ be a set. We say $\Omega$ *is a region or domain* if and only if it is open, non-empty, and connected.

**Definition 3.21.** Let $\Omega \subseteq \mathbb{C}$ be a non-empty set. We say $\Omega_1 \subseteq \Omega$ *is a connected component of* $\Omega$ if and only if it is a maximal connected subset, that is, if $z_0 \in \Omega_1$ and $W$ is a connected subset of $\mathbb{C}$ that contains $z_0$, then $W \subseteq \Omega_1$.

**Definition 3.22.** Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function $f$* as the application

$$f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto w = f(z) \tag{33}$$

**Definition 3.23.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say it *tends to infinity at the point $z_0$* and denote it by $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \tag{34}$$

**Definition 3.24.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We write $\lim_{z \to \infty} f(z) = L$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \tag{35}$$

**Proposition 3.2.** *Let $f_1, f_2 : \Omega \longrightarrow \mathbb{C}$ be two functions and $z_0$ a point such that $\lim_{z \to z_0} f_1 = w_1, \lim_{z \to z_0} f_2 = w_2$. Then,*

1. *$f_1 + f_2$ has also a limit and $\lim_{z \to z_0} f + g = w_1 + w_2$.*

2. *$f_1 f_2$ has also a limit and $\lim_{z \to z_0} fg = w_1 w_2$.*

3. *If $w_2 \neq 0$, then $f/g$ has also a limit and $\lim_{z \to z_0} f/g = w_1/w_2$.*

4. *If $h(z)$ is a continuous function defined on a neighborhood of $w_1$, then $\lim_{z \to z_0} h(f_1(z)) = h(w_1)$.*

**Definition 3.25.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. We say $f$ *is continuous in $z_0$* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \tag{36}$$

**Proposition 3.3.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \mathrm{Re}\{f\} + i \mathrm{Im}\{f\}$ is continuous at $z_0$ if and only if $\mathrm{Re}\{f\}$ and $\mathrm{Im}\{f\}$ are continuous at $z_0$.*

**Proposition 3.4.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f$ is continuous at $z_0$ if and only if for all sequence $\{z_n\}_{n=1}^{\infty}$ of $\Omega$ convergent at $z_0$ it is true that the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$.*

**Proposition 3.5.** *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, $\lambda f$, $f + g$, and $fg$ are continuous at $z_0$. The function $f/g$ is continuous at $z_0$ if $g(z_0) \neq 0$.*

**Definition 3.26.** For all $z \in \mathbb{C}$, we define the *complex exponential function* as the following series

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{37}$$

**Proposition 3.6.** *The radius of convergence of $\mathrm{e}^z$ is infinite.*

**Proposition 3.7.** *$\mathrm{e}^z = \mathrm{e}^x$ for all $x \in \mathbb{R}$.*

**Proposition 3.8.** *$\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$ for all $z, w \in \mathbb{C}$.*

**Proposition 3.9.** *For all $z \in \mathbb{C}$ we have $\mathrm{e}^z \neq 0$.*

**Proposition 3.10.** *The image of $\mathrm{e}^z$ is $\mathbb{C}^*$.*

**Proposition 3.11.** *The derivative of $\mathrm{e}^z$ is $\mathrm{e}^z$.*

**Proposition 3.12.** *$\overline{\mathrm{e}^z} = \mathrm{e}^{\bar{z}}$.*

**Proposition 3.13.** *$|\mathrm{e}^z| = \mathrm{e}^{\mathrm{Re}\{z\}}$.*

**Proposition 3.14** (Euler's Formula)**.** *If $\theta \in \mathbb{R}$, then $\mathrm{e}^{xi}$ has modulus one and we have that*

$$\boxed{\mathrm{e}^{xi} = \cos x + i \sin x.} \tag{38}$$

**Corollary 3.15.** *Let $z \in \mathbb{C}^*$. Then,*

$$z = |z| \mathrm{e}^{i\theta}, \tag{39}$$

*with $\theta \in [0, 2\pi)$.*

**Proposition 3.16.** *The following function*

$$\exp : (\mathbb{R}, +) \longrightarrow (\mathbb{C}^*, \cdot)$$
$$x \longmapsto \mathrm{e}^{xi} \tag{40}$$

*is a morphism of groups and its image is $\mathbb{T}$.*

**Proposition 3.17.** *The complex exponential function is a periodic function with period $2\pi i$.*

**Proposition 3.18.** *Let $a \in \mathbb{C}^*$. Then, $\mathrm{e}^z = a$ has infinite solutions.*

**Proposition 3.19.** *The equation $\mathrm{e}^z = 0$ does not have solutions.*

**Proposition 3.20.** *Let $y_0 \in \mathbb{C}$ be a numbers, $B := \{z \in \mathbb{C} \mid y_0 < \mathrm{Im}\{z\} < y_0 + 2\pi\}$ a set, and $f : B \longrightarrow \mathbb{C}^*$ be the exponential function. Then, $f$ is bijective in $B$ ?.*

**Proposition 3.21.** *Let $x_0, y_0, m \in \mathbb{C}$ be two numbers with $m \neq 0$ and $f$ the exponential function ?. Then,*

1. *$f$ transforms the line $y = y_0$ to a line that starts at $z = 0$ and continues with an argument $y_0$ from the real positive axis.*

2. *$f$ transforms the line $x = x_0$ to a circle centered at the origin and radius $r = \mathrm{e}^{x_0}$.*

3. *f transforms the line $y = mx$ to the parametric curve $z = e^x e^{imx}$ (a spiral).*

**Definition 3.27.** Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric functions* as

$$\cos z := \frac{e^{zi} + e^{-zi}}{2}, \tag{41}$$

$$\sin z := \frac{e^{zi} - e^{-zi}}{2}, \tag{42}$$

$$\tan z := \frac{e^{zi} - e^{-zi}}{e^{zi} + e^{-zi}}. \tag{43}$$

**Proposition 3.22.** *For all $z \in \mathbb{C}$,*

$$\sin^2 z + \cos^2 z = 1. \tag{44}$$

**Proposition 3.23.** *For all $z \in \mathbb{C}$,*

$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z). \tag{45}$$

**Proposition 3.24.** *For all $z, w \in \mathbb{C}$,*

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \qquad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \tag{46}$$

**Proposition 3.25.** *The functions $\cos z, \sin z$ have period of $2\pi$.*

**Proposition 3.26.** *Let $z_0 \in \mathbb{C}$. Then, $z_0$ is root of $\sin z$ ($\cos z$) if and only if it is a root of $\sin x$ ($\cos x$).*

**Definition 3.28.** Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic functions* as

$$\cosh z := \frac{e^z + e^{-z}}{2}, \tag{47}$$

$$\sinh z := \frac{e^z - e^{-z}}{2}, \tag{48}$$

$$\tanh z := \frac{e^z - e^{-z}}{e^z + e^{-z}}. \tag{49}$$

**Proposition 3.27.** *For all $z \in \mathbb{C}$,*

$$\cosh^2 z - \sinh^2 z = 1. \tag{50}$$

**Proposition 3.28.** *For all $z \in \mathbb{C}$,*

$$\cosh(-z) = \cosh(z), \qquad \sinh(-z) = -\sinh(z). \tag{51}$$

**Proposition 3.29.** *For all $z, w \in \mathbb{C}$,*

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \tag{52}$$

$$\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \tag{53}$$

**Proposition 3.30.** *For all $z \in \mathbb{C}$,*

$$\cosh z = \cos(iz), \cos z = \cosh(iz) \tag{54}$$

$$\sinh z = -i\sin(iz), \sin z = -i\sinh(iz) \tag{55}$$

**Proposition 3.31.** *For all $z = x + iy \in \mathbb{C}$,*

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \tag{56}$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \tag{57}$$

$$\tan(x + iy) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i \frac{\sinh y}{\cos(2x) + \cosh(2y)}. \tag{58}$$

**Proposition 3.32.** *For all $z = x + iy \in \mathbb{C}$,*

$$\tanh(x + iy) = \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)} + i \frac{\sin(2y)}{\cosh(2x) + \cos(2y)}. \tag{59}$$

**Proposition 3.33.** *For all $z = x + iy$,*

$$|\cos z| = \sqrt{\cosh^2 y - \sin^2 x} = \sqrt{\cos^2 x + \sinh^2 y}, \tag{60}$$

$$|\sin z| = \sqrt{\sinh^2 x + \sin^2 y} = \sqrt{\cosh^2 x - \cos^2 y}. \tag{61}$$

**Corollary 3.34.** *For all $z = x + iy$,*

$$|\sinh y| \leq |\cos z| \leq \cosh y, \qquad |\sinh y| \leq |\sin z| \leq \cosh y. \tag{62}$$

**Proposition 3.35.** *The roots of the function $\sinh z$ are of the form $z_n = n\pi$ and those for the function $\cosh z$ are of the form $w_n = (2n + 1)\pi/2i$.*

**Definition 3.29.** Let $D \subseteq \mathbb{C}$ be a set. We define a *multivalued function* from $D$ to $\mathbb{C}$ as a subset of $D \times \mathbb{C}$ such that for every $z \in D$ there exists a number $y \in \mathbb{C}$ such that $(z, w) \in f$.

**Definition 3.30.** For $z \in \mathbb{C}^*$, we call the *natural logarithm of $z$* every number $w$ such that $e^w = z$, that is,

$$\ln z := \{w \in \mathbb{C} \,|\, e^w = z\}. \tag{63}$$

**Proposition 3.36.** *Given $z \in \mathbb{C}$ we can define $\ln z$ from the natural logarithm of a real number as*

$$\ln z = \ln |z| + i \arg z = \ln |z| + i\text{Arg}z + 2\pi ki. \tag{64}$$

**Definition 3.31.** We define the *principal natural logarithm of $z$* as the value defined by the principal argument of $z$, that is,

$$\text{Log}z = \ln |z| + i\text{Arg}z. \tag{65}$$

**Definition 3.32.** We define the determination $I$ (with $I$ being a semiopen interval) of the logarithm as

$$\log_I z := \ln |z| + i \arg_I z. \tag{66}$$

**Definition 3.33.** Let $E \subseteq \mathbb{C}^*$ be a connected set. We define the *continuous determination of the logarithm in $E$* as the continuous function $g : E \longrightarrow \mathbb{C}$ such that $e^{g(z)} = z$. More generally, if $f : E \longrightarrow \mathbb{C}$ is a function such that $f(z) \neq 0$ for all $z \in E$, then we define the *continuous determination of $\ln f$* as a function $g : E \longrightarrow \mathbb{C}$ such that $e^{g(z)} = f(z)$.

**Proposition 3.37.** *Let $z, w \in \mathbb{C}$ two numbers. Then,*

*1. $\ln(zw) = \ln z + \ln w + 2\pi ki, k \in \mathbb{Z}$.*

*2. If we want to stay in the principal argument,*

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \text{if } \text{Arg}z + \text{Arg}w < 2\pi \\ \ln z + \ln w - 2\pi i, & \text{if } \text{Arg}z + \text{Arg}w \geq 2\pi \end{cases}. \tag{67}$$

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**Definition 3.34.** Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric inverse functions* as

$$\arcsin z := -i \ln\left(iz + \sqrt{1-z^2}\right), \qquad (68)$$

$$\arccos z := -i \ln\left(z + \sqrt{z^2-1}\right), \qquad (69)$$

$$\arctan z := -\frac{i}{2} \ln\frac{1+iz}{1-iz}. \qquad (70)$$

**Definition 3.35.** Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic inverse functions* as

$$\operatorname{arcsinh} z := \ln\left(z + \sqrt{1+z^2}\right), \qquad (71)$$

$$\operatorname{arccosh} z := \ln\left(z + \sqrt{z^2-1}\right), \qquad (72)$$

$$\operatorname{arctanh} z := \frac{1}{2} \ln\frac{1+z}{1-z}. \qquad (73)$$

**Definition 3.36.** Let $z, a \in \mathbb{C}$ with $z \neq 0$. Then, we define the *complex power function* as

$$z^a := e^{a \ln z}. \qquad (74)$$

If $E \subseteq \mathbb{C}^*$ is a connected set and $f : E \longrightarrow \mathbb{C}$ a functions such that $f(z) \neq 0$ for all $z \in E$, and $w \in \mathbb{C}$ a number, we define a *continuous determination of* $f^w$ as a continuous function $g : E \longrightarrow \mathbb{C}$ such that $g(z) \in [f(z)]^w$.

**Proposition 3.38.** *If $a = \alpha + \beta i$ and $z = re^{\theta i}$, then*

$$z^a = e^{\alpha \ln r - \beta(\theta + 2\pi k)} e^{(\beta \ln r + \alpha(\theta + 2\pi k))i}, \qquad (75)$$

$$|z^a| = e^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \qquad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k), \qquad (76)$$

**Proposition 3.39.** *Let $a, z \in \mathbb{C}$ be two numbers. Then,*

1. *If $a = n \in \mathbb{Z}$, the complex power is a function and*

$$z^n = r^n e^{n\theta i}. \qquad (77)$$

2. *If $a = n/m \in \mathbb{Q}$, there are $n$ values and*

$$z^a = \sqrt[m]{r^n} e^{(\theta + 2\pi k)n/mi}. \qquad (78)$$

3. *If $a$ is irrational, the norm is uniquely determined but the argument has infinite values.*

4. *IF $a \in \mathbb{C} \backslash \mathbb{R}$, the argument is uniquely determined and the norm has infinite values.*

**Proposition 3.40.** *Let $z, w \in \mathbb{C}$. Then,*

1. $\left(e^b\right)^a = e^{a(b+2\pi ki)}$

**Definition 3.37.** A *Riemann surface $X$* is a connected complex 1-manifold.

**Definition 3.38.** We define a *sheet* as each of the complex planes of the Riemann surface.

**Definition 3.39.** We define a *cut* as the line (not necessarly straight) of union between sheets.

**Definition 3.40.** We define a *branch point* as a point where start or finish a cut.

# 4 Derivatives

**Definition 4.1.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. We define the *derivative of $f$ at $z_0$* as

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \qquad (79)$$

in case the limit exists. If $f$ has derivative, we say *$f$ is derivable at $z_0$*.

**Definition 4.2.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *$f$ is holomorphic at $\Omega$* if and only if it is $\mathbb{C}$-derivable at every point of $\Omega$. In that case, it is defined the function $f' : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ that associates each point $z$ of $\Omega$ with $f'(z)$.

**Definition 4.3.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We define the *domain of holomphism* as the region where $f$ is derivable. We say *$f$ is entire* if and only if the domain of holomorphism is $\mathbb{C}$.

**Definition 4.4.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *$f$ is holomorphic at $z_0$* if and only if it is holomorphic at some neighborhood of $z_0$.

**Proposition 4.1.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. If $f$ is derivable at $z_0$, then it is continuous at $z_0$.*

**Theorem 4.2.** *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two functions and $z_0 \in \Omega$ a point. Then, the following statements are true.*

1. *If $f$ is constant at $\Omega$, then $f$ is derivable at $z_0$ and $f'(z_0) = 0$.*

2. *If $f(z) = z$ in every point of $\Omega$, then $f$ is derivable at $z_0$ and $f'(z_0) = 1$.*

3. *If $f, g$ are derivable at $z_0$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ are derivable at $z_0$ and $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.*

4. *If $f, g$ are derivable at $z_0$, then $fg$ is derivable at $z_0$ and*

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \qquad (80)$$

5. *If $f, g$ are derivable at $z_0$ and $g(z_0) \neq 0$, then $f/g$ is derivable at $z_0$ and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \qquad (81)$$

**Theorem 4.3.** *Let $f : \Omega_1 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a derivable function at a point $z_0 \in \mathbb{C}$ and $g : \Omega_2 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be another derivable function at a point $f(z_0) \in \Omega_2$. Then, $g \circ f$ is derivable at $z_0$ and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \qquad (82)$$

**Definition 4.5.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ *is of class* $C^1(\Omega)$ *or simply* $f \in c^1(\Omega)$ if and only if, using $f = u + iv$ with $u = \operatorname{Re}\{f\}, v = \operatorname{Im}\{f\}$, the partial derivatives of $u$ and $v$ as a two variable real functions exist and are continuous. In other words, $f \in C^1(\Omega)$ if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \tag{83}$$

exist and are continuous.

**Theorem 4.4** (Cauchy-Riemann conditions)**.** *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a function and* $z_0 \in \Omega$ *an interior point. Then,* $f$ *is derivable at* $z_0$ *if and only if is differentiable at* $z_0$ *and* $df(z_0)$ *is* $\mathbb{C}$-*linear, that is,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{84}$$

*which are known as Cauchy-Riemann conditions.*

**Theorem 4.5.** *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a function and* $z_0 \in \Omega$ *an interior point. If* $u, v$ *satisfy the Cauchy-Riemann equation and their partial derivatives are continuous, then* $f$ *is derivable.*

**Theorem 4.6.** *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *a function of class* $C^1(\Omega)$ *with* $\Omega$ *an open set, injective, and derivable at every point of* $\Omega$ *with non-zero derivative. Then,*

1. $\Omega' = f(\Omega)$ *is an open subset of* $\mathbb{C}$.

2. *The inverse function* $f^{-1}$ *exist, it is well defined and it is derivable at* $\Omega'$.

3. *If* $z \in \Omega$ *and* $z' = f(z)$, *then*

$$(f^{-1})'(z') = \frac{1}{f'(z)}. \tag{85}$$

**Proposition 4.7.** *A determination of* $\ln z$ *with* $z \in \mathbb{C}$ *is continuous except in a semiline.*

**Theorem 4.8.** *Let* $\Omega \subseteq \mathbb{C}$ *be an open set and* $\phi \in C(\Omega)$, *such that* $e^{\phi(z)} = z$ *for all* $z \in \Omega$. *Then, we have* $\phi \in H(\Omega)$ *and*

$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \tag{86}$$

**Proposition 4.9.** *A determination of* $\ln z$ *with* $z \in \mathbb{C}$ *is holomorphic except in a semiline.*

**Proposition 4.10.** *Let* $I = [\theta, \theta + 2\pi)$ *a determination of the logarithm,* $\ln_I z$. *Then,* $\ln_I z$ *is holomorphic except in the semiline* $L_\theta = \left\{ re^{\theta i} \in \mathbb{C} \,\middle|\, r \geq 0 \right\}$.

**Theorem 4.11.** *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be an analytic function in* $\Omega$. *Then,* $f$ *is holomorphic in* $\Omega$. *In particular, given a point* $z_0 \in \mathbb{C}$, $f'(z_0) = a_1$ *where* $a_1$ *is the first coefficient of the power series that represents* $f$ *in a neighborhood of* $z_0$.

**Definition 4.6.** We define the operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{87}$$

that act over the functions such that the real and imaginary part $u, v$ have partial derivatives.

**Proposition 4.12.** *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *a function of class* $C^1(\Omega)$. *Then, for all* $z_0 \in \Omega$

$$f(z_0 + h) = f(z_0) + \left(\frac{\partial f}{\partial z}\right)_{z_0} h + \left(\frac{\partial f}{\partial \bar{z}}\right)_{z_0} \bar{h} + o(|h|^2). \tag{88}$$

**Corollary 4.13.** *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a function of class* $C^1(\Omega)$. *Then,* $f$ *is holomorphic in* $\Omega$ *if and only if*

$$\frac{\partial f}{\partial \bar{z}} = 0 \ \text{at} \ \Omega. \tag{89}$$

**Definition 4.7.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$ such that $f = u + iv$ with $u = \operatorname{Re}\{f\}, v = \operatorname{Im}\{f\}$ and $z_0 \in \mathbb{C}$ a point. Then, we call $(\partial_{\bar{z}} f)_{z_0} = 0$ the *Cauchy-Riemann condition*, which is equivalent to

$$\left(\frac{\partial u}{\partial x}\right)_{z_0} = \left(\frac{\partial v}{\partial y}\right)_{z_0}, \qquad \left(\frac{\partial v}{\partial x}\right)_{z_0} = -\left(\frac{\partial u}{\partial y}\right)_{z_0}, \tag{90}$$

which are called the *Cauchy-Riemann equations.*

**Theorem 4.14.** *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a function and* $z_0$ *an interior point. Then, at* $z_0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}. \tag{91}$$

# 5 Series

**Definition 5.1.** We say $\sum_{n=1}^{\infty} z_n$ converges if and only if $S_n := \sum_{n=1}^{N} z_n$ has limit at $n \to \infty$.

**Proposition 5.1.** $\sum_{n=1}^{\infty} z_n$ *converges if and only if* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converge.*

**Definition 5.2.** We say $\sum_{n=1}^{\infty} z_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |z_n|$ converges.

**Proposition 5.2.** $\sum_{n=1}^{\infty} |z_n|$ *converges if and only if* $\sum_{n=1}^{\infty} |a_n|$ *and* $\sum_{n=1}^{\infty} |b_n|$ *converge.*

**Proposition 5.3.** *1. A series converges absolutely with sum* $S$ *if and only if every rearrangement is convergent with the same sum* $S$.

2. *An absolutely convergent series can be summed by blocks in an arbitrary way.*

**Proposition 5.4.** *Let* $\sum_n a_n, \sum_n b$ *be two absolutely convergent series with sums* $A$ *and* $B$ *respectively. Then, the series* $\sum_k c_k$ *with* $c_k = \sum_{n=0}^{k} a_n b_{k-n}$ *is absolutely convergent with sum* $AB$.

**Theorem 5.5** (Weierstrass M-test). *If* $|f_n(p)| < M_n$ *for all* $p \in X, n \geq 1$ *and* $\sum_{n=0}^{\infty} M_n < \infty$, *then the series* $\sum_{n=0}^{\infty} f_n(p)$ *is uniformly convergent on* $X$.

**Lemma 5.6** (Abel's summation formula). *Let* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be two sequences of complex numbers and* $A_n = a_1 + \cdots + a_n$. *Then,*

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k). \qquad (92)$$

**Theorem 5.7** (Dirichlet's criteria). *Let* $\sum_{n=1}^{\infty} f_n(p)g_n(p)$ *be a series where* $f_n(p)$ *are complex and* $g_n(p)$ *are real for all* $p \in X, n \geq 1$. *If we denote* $F_n(p) = f_1(p) + \cdots f_n(p)$, *there exists a constant* $M$ *such that* $|F_n|(p) \leq M$ *for all* $n \geq, p \in X$, $g_n(p)$ *is monotonous decreasing and converges uniformly to zero on* $X$, *then the series* $\sum_{n=1}^{\infty} f_n(p)g_n(p)$ *is uniformly convergent on* $X$.

**Theorem 5.8** (Abel's criteria). *Let* $\sum_{n=1}^{\infty} f_n(p)g_n(p)$ *be a series where* $f_n(p), g_n(p)$ *are complex. If* $\sum_{n=1}^{\infty} f_n(p)$ *is uniformly convergent on* $X$ *andd there exists a number* $M \in \mathbb{R}^+$ *such that for all* $p \in X$

$$|g_1(p)| + \sum_{n=1}^{\infty} |g_n(p) - g_{n+1}(p)| \leq M, \qquad (93)$$

*then the series* $\sum_{n=1}^{\infty} f_n(p)g_n(p)$ *is uniformly convergent on* $X$.

**Definition 5.3.** We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n, \qquad a_n, z, z_0 \in \mathbb{C}. \qquad (94)$$

We call the term $a_n$ the *n-th coefficient of the series*. In case $a_n = 0 \; \forall n \leq m$, we will start the counting directly from $m$.

**Definition 5.4.** Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. We define its *domain of convergence* as

$$E := \left\{ z \in \mathbb{C} \;\middle|\; \sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ converges} \right\}. \qquad (95)$$

**Theorem 5.9.** *Let* $\sum_n a_n(z - z_0)^n$ *be a power series and* $R = 1/\rho$, *where* $\rho = \limsup_n |a_n|^{1/n}$. *Then, the series converges uniformly on the compacts of the open disc* $D(z_0, R)$, *converges absolutely at every point* $z \in D$ *and diverges outside* $\bar{D}$. *Hence, the set of converges* $E$ *satisfies* $D \subseteq E \subseteq \bar{D}$ *and* $D = \text{int} E$.

**Definition 5.5.** Radius of convergence.

**Proposition 5.10.** *Let* $\sum_n a_n(z - z_0)^n$ *be a power series and* $R = \lim_{n \to \infty} |a_n|/|a_{n+1}|$. *If the limit exists, then* $R$ *is the radius of convergence.*

**Theorem 5.11** (Cauchy-Hadamard Theorem). *Let*

$$S = \sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad (96)$$

*be the following complex power series with* $a_n, z_0 \in \mathbb{C}$ *and* $R \in [0, +\infty) \cup \{+\infty\}$ *the radius of convergence. Then,*

1. *If* $|z - z_0| < R$ *then* $S$ *converges. In fact, for all* $r < R$ *we have* $S$ *converges uniformly at the disc* $\overline{D_r(z_0)}$.

2. *If* $|z - z_0| > R$ *then* $S$ *diverges.*

3. *The function* $f(z) = S(z)$ *is derivable at* $D_R(z_0)$ *and its formal derivative is*

$$f'(z) = \sum_{n=0}^{\infty} n a_n(z - z_0)^{n-1}, \qquad (97)$$

   *with the same radius of convergence.*

**Definition 5.6.** Let $\sum_n a_n(z - z_0)^n$ be a series, $S = E \cap C(z_0, R)$ non empty, and $m > 1$ a real number. We define

$$S_m := \{z \in \mathbb{C} \,|\, |z - z_0| < R, d(z, S) \leq m(R - |z - a|)\}. \qquad (98)$$

**Definition 5.7** (Stolz angle). Let $S$ be formed by one point $w$. We define the *Stolz angle* as the angle generated by the $S_m$.

**Theorem 5.12** (Abel's theorem). *Let* $\sum_n a_n(z - z_0)^n$ *be a series with* $S$ *non empty and such that the series converges uniformly on it. Then, the series converges uniformly on* $S_m$ *for all* $m > 1$. *In particular, the sum function is continuous on* $S_m$ *and one has*

$$\lim_{z \to w, z \in S_m} \sum_n a_n(z - z_0)^n = \sum_n a_n(w - z_0)^n, \qquad w \in S. \qquad (99)$$

**Theorem 5.13.** *Let* $\sum_n a_n(z - z_0)^n$ *be a series with radius of convergence* $R$. *Then,* $f(z) = \sum_n a_n(z - z_0)^n$ *is holomorphic on* $D(a, R)$ *and it has a derivative*

$$f'(z) = \sum_n n a_n(z - z_0)^{n-1}, \; \forall z \in D. \qquad (100)$$

**Proposition 5.14.** *Let $f : D(a, R) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. If there exists a power series $\sum_n a_n(z - z_0)^n$, convergent on $D$ such that*

$$f(z) = \sum_n a_n(z - z_0)^n, \ |z - z_0| < R, \qquad (101)$$

*then the series is unique. In fact, $f$ is infinitely holomorphic and the coefficients $a_n$ are determined by $f$ with the relation*

$$a_n = \frac{f^n(z_0)}{n!}, \ n \in \mathbb{N}. \qquad (102)$$

**Definition 5.8.** *Let $f : D(a, R) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ admits a series expansion if and only if there exists a power series $\sum_n a_n(z - z_0)^n$, convergent on $D$ such that*

$$f(z) = \sum_n a_n(z - z_0)^n, \ |z - z_0| < R. \qquad (103)$$

**Definition 5.9.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with $\Omega$ an open set. We say $f$ is analytic on $\Omega$ if and only if it admits locally a series expansion, that is, if for every point $z_0 \in \Omega$ there exists a disc $D(z_0, \delta)$ and a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \forall z \in D.$$

**Theorem 5.15.** *Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ on $D(z_0, R)$ and $w_0 \in D(z_0, R_0)$. Then, the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z - z_1)^n$ has a radius of convergence $R_1 \geq R_0 - |z_0 - z_1|$ and it satisfies*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z - z_1)^n, \qquad if \ |z - z_1| < R - |z_0 - z_1|. \qquad (104)$$

**Definition 5.10.** *Let $f(z) = \sum a_n(z - z_0)^n$ be a series with radius of convergence $R$. Then, its formal derivative is*

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z}. \qquad (105)$$

**Corollary 5.16.** *Let $f(z) = \sum a_n(z - z_0)^n$ be a series with radius of convergence $R$. Then, $f$ is infinitely derivable at $D_R(z_0)$.*

**Corollary 5.17.** *Let $R$ be the radius of convergence of the function*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

*Then $f$ has as Taylor polynomial of degree $m$ around $z_0$ the following one*

$$P_{m,f,z_0}(z) = \sum_{n=0}^{m} a_n(z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}. \qquad (106)$$

**Theorem 5.18** (Abel's Theorem). *Let be the following series*

$$\sum_{n=0}^{\infty} f_n(z_0)g_n(z_0),$$

*where $f, g_n$ are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,*

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \qquad (107)$$

*then the original series converges uniformly in $\Omega$.*

**Theorem 5.19** (Weierstrass' criterion). *Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant $M_n$, satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at $\Omega$ for all $z_0 \in \Omega$.*

**Definition 5.11.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a complex function with $\Omega$ an open set. We say $f$ is complex analytic if and only if for all $z_0 \in \Omega$ exists a real number $R(z_0)$ and a sequence $\{a_n\} \subseteq \mathbb{C}$ (that can also depend on $z_0$) such that is $z \in D_R(z_0)$, then formally it is true that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n. \qquad (108)$$

*We denote the set of complex analytic functions with domain $\Omega$ by $C^{\omega}(\Omega)$.*

**Corollary 5.20.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. If $f \in C^{\omega}(\Omega)$, then $f \in C^{\infty}(\Omega)$.*

**Corollary 5.21.** *Let $z_0$. Then, the coefficients $z_0$ of the local expression of $f$ given by the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ are determined by*

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \qquad (109)$$

**Proposition 5.22.** *Let $\varnothing \neq \Omega \subseteq \mathbb{C}$. Then,*

*1. Every connected component of $\Omega$ is a closed of $\Omega$ with a subspace topology.*

*2. Two connected components are the same or are disjoint.*

*3. Every connected of $\Omega$ is one and only one connected component.*

*4. $\Omega$ is the disjoint union of its connected components.*

**Theorem 5.23** (Analytic prolongation Principle). *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0 = 0)$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of $\Omega$ that contains $z_0$ (the function can be zero also out $\Omega$, but we are not studying that).*

**Corollary 5.24.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function with $\Omega$ a region. Then, the following statements are equivalent:*

1. $f(z) = 0$ for all $z \in \Omega$.

2. There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.

**Corollary 5.25** (Analytic Prolongation Principle). *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions with $\Omega$ a region. Then, the following statements are equivalent:*

1. $f(z) = g(z)$ for all $z \in \Omega$.

2. There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}$.

**Lemma 5.26.** *Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then*

$$\left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right). \quad (110)$$

**Corollary 5.27.** *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions. Then, $fg$ is analytic.*

**Proposition 5.28.** *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, $f$ is analytic at $\Omega = D_R(0)$.*

# 6 Holomorphic functions

**Definition 6.1.** Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{aligned} \gamma : I &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned} \quad (111)$$

**Definition 6.2.** Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval and $D \subseteq \mathbb{C}$ a domain. We define an *arc* as a continuous application of the form

$$\begin{aligned} \gamma : I &\longrightarrow D \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned} \quad (112)$$

Equivalently, we can say an arc is a curve restricted to some interval.

**Definition 6.3.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We call $\gamma(a)$ and $\gamma(b)$ *the extremes of $\gamma$.* In particular, we call $\gamma(a)$ the *initial point* and $\gamma(b)$ the *final point.*

**Definition 6.4.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We define the *route or graph of $\gamma$* as

$$\gamma^* := \{ z \in D \mid z = \gamma(t), t \in I \}. \quad (113)$$

**Definition 6.5.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ is closed if and only if $\gamma(a) = \gamma(b)$.

**Definition 6.6.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ is simple if and only if there is no two numbers $t_1, t_2 \in (a, b)$ such that $\gamma(t_1) = \gamma(t_2)$. We also call it a *Jordan curve*, and if it is closed, a *circuit*.

**Definition 6.7.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ is differentiable if for al value $t_0 \in [a, b]$ there exists the limit

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \quad (114)$$

For $t_0 = a$ or $t_0 = b$ we consider the laterals limits from the right and from the left respectively.

**Definition 6.8.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ is of class $C^1$ if and only if $\gamma'$ exists and is continuous at $[a, b]$.

**Definition 6.9.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ is regular or smooth if and only if it is differentiable and $\gamma'$ never vanishes.

**Definition 6.10.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ is piece-wise of class $C^1$ if and only if $\gamma'$ exists and is continuous in $I$ except in a finite number of points where $\gamma$ has lateral derivatives.

**Definition 6.11.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We define the *opposite arc* as

$$\begin{aligned} -\gamma : [-b, -a] &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma(-t) \end{aligned} \quad (115)$$

**Definition 6.12.** Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be an arc. We say $\Gamma(s), s \in [c, d] \subseteq \mathbb{R}$ has been obtained from $\gamma(t), t \in [a, b]$ by a change of parametrization if and only if the new parameter $s$ and the original parameter $t$ are related by a relation $t = \phi(s)$, where $\phi : [c, d] \longrightarrow [a, b]$ is an homeomorphism that satisfies $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$. We call $\Gamma$ *the reparametrization of $\gamma$.*

**Definition 6.13.** Let $\gamma_1 : I_1 \longrightarrow \mathbb{C}$ and $\gamma_2 : I_2 \longrightarrow \mathbb{C}$ be two arcs. We say they *are equivalent* if and only if there exists a bijective, monotone, and continuous function $\rho : I_2 \longrightarrow I_1$ such that $\gamma_2 = \gamma_1 \circ \rho$. If $\rho$ is an increasing function we say *$\gamma_1$ and $\gamma_2$ have the same orientation*; otherwise, we say *$\gamma_1$ and $\gamma_2$ have opposite orientations*.

**Definition 6.14.** Let $\gamma_1 [a, b] \longrightarrow \mathbb{C}$ and $\gamma_2 : [c, d] \longrightarrow \mathbb{C}$ be two arcs such that $[a, b] \cap [c, d] = \varnothing$. We define the application $\gamma_1 \cup \gamma_2$ (sometimes denoted by $\gamma_1 + \gamma_2$) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \leq t \leq b \\ \gamma_2(t - b + c), & \text{if } b \leq t \leq b + d - c \end{cases}. \quad (116)$$

We say $\gamma_1, \gamma_2$ can be joined/added or that there exists its union/sum if and only $\gamma_1(b) = \gamma_2(x)$. In this case $\gamma_1 + \gamma_2$ is an arc, and we call it *the sum arc of $\gamma_1$ plus $\gamma_2$*.

**Definition 6.15.** We define the *segment of extremes $z_1, z_2 \in \mathbb{C}$* as the arc defined by the expression

$$\begin{aligned} [z_1, z_2] : [0, 1] &\longrightarrow \mathbb{C} \\ t &\longmapsto (1 - t)z_1 + tz_2 \end{aligned} \quad (117)$$

**Definition 6.16.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ is polygonal if and only if can be expressed as a finite union of segments, that is, if there exist a natural number $n$ and points $\{z_0, \ldots, z_n\}$ such that

$$p = [z_0, z_1] \cup \cdots \cup [z_{n-1}, z_n]. \quad (118)$$

**Definition 6.17.** Let $\gamma : [a,b] \longrightarrow D$ be an arc with $a,b$ finite. We say $\gamma$ *is a basic curve* if and only if $\gamma \in C^1((a,b)) \cap C([a,b])$ and there exist $\lim_{t\to a^+} \gamma'(t), \lim_{t\to b^-} \gamma'(t)$.

**Definition 6.18.** A *path* is a function $\gamma : [a,b] \longrightarrow \mathbb{C}$ such that there exist basic curves $\gamma_j : [a_j, b_j] \longrightarrow \mathbb{C}, j \in \{1, \ldots, k\}$ such that $\gamma = \gamma_1 + \cdots + \gamma_k$ and therefore $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ and $a = a_1, b = a_k$.

**Definition 6.19.** Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous curve and $a_1, \ldots, a_l \in \mathbb{R}$ such that $a = a_0 \leq \cdots \leq a_l \leq b = a_{l+1}$. We say $\gamma$ *is piece-wise differentiable* if and only if

$$\gamma \in C^1 \left( \bigcup_{j=0}^{l} (\alpha_j, \alpha_{j+1}) \right),$$

$\forall j \in \{0, \ldots, l+1\} \exists \lim_{t\to a_j^+} \gamma'(t) (\text{except if } j = l+1), \lim_{t\to a_j^-} \gamma'(t)(\text{except if } j = 0)$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

**Theorem 6.1.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with $\Omega$ an open set and $\phi : I \longrightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curev (hence a derivable curve) and its real derivative is*

$$\psi'(t) = f'(\phi(t))\phi'(t). \tag{119}$$

**Definition 6.20.** Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two curves. We say $\gamma_1, \gamma_2$ *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.

2. $h(t,1) = \gamma_2(t), t \in [0,1]$.

3. $h(0,s) = \gamma_1(0) = \gamma_2(0), s \in [0,1]$.

4. $h(1,s) = \gamma_1(1) = \gamma_2(1), s \in [0,1]$.

**Definition 6.21.** Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two circuits. We say $\gamma_1, \gamma_2$ *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.

2. $h(t,1) = \gamma_2(t), t \in [0,1]$.

3. $h(0,s) = h(1,s), s \in [0,1]$.

**Definition 6.22.** Let $f : [a,b] \longrightarrow \mathbb{C}$ be a function with the notation $f = u + iv$. We define the integral of $f$ as

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + i \int_a^b v(t)\,\mathrm{d}t. \tag{120}$$

**Proposition 6.2.** *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ be two integrable functions and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,*

$$\int_a^b \lambda f + \mu g \,\mathrm{d}t = \lambda \int_a^b f \,\mathrm{d}t + \mu \int_a^b g \,\mathrm{d}t. \tag{121}$$

**Proposition 6.3.** *Let $f : [a,b] \longrightarrow \mathbb{C}$ be an integrable function. Then,*

$$\left| \int_a^b f(t)\,\mathrm{d}t \right| \leq \int_a^b |f(t)|\,\mathrm{d}t. \tag{122}$$

**Definition 6.23.** Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of $f$ over $\gamma$* as

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t. \tag{123}$$

**Proposition 6.4.** *The previous definition is well defined.*

**Proposition 6.5.** *If we use the notation $f = u + iv$ and $\gamma = x + iy$, then the integral has the form*

$$\int_\gamma f = \int_a^b u\frac{\mathrm{d}x}{\mathrm{d}t} + v\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t + i \int_a^b v\frac{\mathrm{d}x}{\mathrm{d}t} + u\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t. \tag{124}$$

**Definition 6.24.** Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of $f$ over $\gamma$ with respect the differential of length* as

$$\int_\gamma f(z)\,\mathrm{d}s := \int_\gamma f(z)|\mathrm{d}z| = \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t. \tag{125}$$

**Theorem 6.6.** *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$, $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two functions, and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,*

$$\int_\gamma \lambda f + \mu g \,\mathrm{d}z = \lambda \int_\gamma f \,\mathrm{d}z + \mu \int_\gamma g \,\mathrm{d}z. \tag{126}$$

**Theorem 6.7.** *Let $\gamma_1, \gamma_2$ be two equivalent curves of the same orientation and of class $C^1$ on their respective domains and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z. \tag{127}$$

**Proposition 6.8.** *Let $\gamma_1, \ldots, \gamma_n$ be $n$ curves of class $C^1$ on their respective domains and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \ldots, \Gamma_n \subseteq \Omega$. If we define $\gamma = \gamma_1 + \cdots + \gamma_n$, then*

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^n \int_{\gamma_i} f(z)\,\mathrm{d}z. \tag{128}$$

**Proposition 6.9.** *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \leq \int_\gamma |f|\,\mathrm{d}s. \tag{129}$$

**Corollary 6.10.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. If $|f(z)| \leq M$ for all $z \in \Gamma$, then,*

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| \leq ML(\gamma). \tag{130}$$

**Proposition 6.11.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\mathrm{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z}\, \mathrm{d}w. \tag{131}$$

**Proposition 6.12.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$|\mathrm{Ind}(\gamma, z)| \leq \frac{1}{2\pi} \frac{L(\gamma)}{|z - \Gamma|}. \tag{132}$$

**Proposition 6.13.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $\{f_n\}_{n=0}^\infty$ a sequence of continuous functions on $\Gamma$ such that $\sum_{n=0}^\infty f_n$ converges uniformly on $\Gamma$. Then, $\sum_{n=0}^\infty \int_\gamma f_n\, \mathrm{d}z$ converges and*

$$\int_\gamma \sum_{n=0}^\infty f_n(z)\, \mathrm{d}z = \sum_{n=0}^\infty \int_\gamma f_n(z)\, \mathrm{d}z. \tag{133}$$

**Definition 6.25.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ has a *primitive on $\Omega$* if and only if there exists a function $F : \Omega \subseteq \longrightarrow \mathbb{C}$ such that $F' = f \forall z \in \Omega$.

**Definition 6.26.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ has a *local primitive on $D$* if and only if for all $z$ there exists a neigborhood where $f$ has a primitive.

**Theorem 6.14** (Fundamental theorem of complex calculus)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with $\Omega$ a domain. Then, the line integral of $f$ is independent on the path on $\Omega$ if and only if $f$ has an holomorphic primitive $F$ such that $F' = f$ on $\Omega$. In that case,*

$$\int_\gamma f(z)\, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a)). \tag{134}$$

**Theorem 6.15.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function on a star domain $S \subseteq \Omega$. Then, $f$ has an holomorphic primitive $F$ on $S$ if and only if*

$$\int_{\partial \triangle} f(z)\, \mathrm{d}z = 0 \tag{135}$$

*for all triangle $\triangle \subseteq \Omega$.*

**Proposition 6.16.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with no roots on a domain $D \subseteq \Omega$. Then, there is a determination of the logarithm of $f$ on $D$ if and only if $f'/f$ has an holomorphic primitive on $D$.*

**Proposition 6.17.** *Let $K \subseteq \mathbb{C}$ be a compact set. Then,*

1. *If $\alpha \in V_\infty$, then the non-bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log(z - \alpha)$ in a neighborhood of $K$.*

2. *If $\alpha, \beta$ belong to the same bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log\left(\frac{z-\alpha}{z-\beta}\right)$ in a neighborhood of $K$.*

**Theorem 6.18** (Green's theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary. Let $\mathbf{F} = (P, Q)$ be a vector field with $P, Q$ being differentiable functions on a neighborhood of $\bar{\Omega}$ such that $\partial_x P - \partial_y Q$ is continuous on $\bar{\Omega}$. Then,*

$$\int_{\partial \Omega} \langle \mathbf{F}, \mathrm{d}\mathbf{s} \rangle_I = \int_{\partial \Omega} P\, \mathrm{d}x + Q\, \mathrm{d}y = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\, \mathrm{d}x\, \mathrm{d}y. \tag{136}$$

**Theorem 6.19** (Cauchy's integral theorem)**.** *Let $\Omega$ be a bounded domain with piece-wise regular and positively oriented boundary and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an holomorphic function in a neighborhood of $\bar{\Omega}$. Then,*

$$\int_{\partial \Omega} f(z)\, \mathrm{d}z = 0. \tag{137}$$

**Corollary 6.20.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function in a domain $D \subseteq \Omega$. Then, $f$ ha local primitive on $D$. If $D$ is a star domain, $f$ has a global holomorphic primitive.*

**Corollary 6.21.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with no roots in a domain $D \subseteq \Omega$. Then, $f$ has a local determination of the logarithm on $D$. If $D$ is a star domain, $f$ has a global determination of the logarithm.*

**Theorem 6.22** (Cauchy's integral theorem for homotopic curves)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with $\Omega$ a domain and $\gamma_1, \gamma_2$ two homotopic curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z)\, \mathrm{d}z = \int_{\gamma_2} f(z)\, \mathrm{d}z. \tag{138}$$

**Theorem 6.23** (Cauchy's general integral theorem)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a regular function on $\Omega$ except a finite numbers of points where $f$ is continuous. If $\gamma$ is a constant curve, then*

$$\oint_\gamma f(z)\, \mathrm{d}z = 0. \tag{139}$$

**Theorem 6.24** (Morera's theorem)**.** *Let $f$ be a continuous function in a region $\Omega$. If*

$$\oint_\gamma f(z)\, \mathrm{d}z = 0 \tag{140}$$

*for all simple and closed curve $\gamma$ such that $\Gamma \subseteq \Omega$, then $f$ is analytic on $\Omega$.*

**Theorem 6.25.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a differentiable function on a domain $D$. Then, $f = u + iv$ is holomorphic if and only if the field $\bar{f} = (u, -v)$ is locally conservative and locally solenoidal.*

**Definition 6.27.** Let $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable vector field on a domain $D \subseteq \mathbb{R}^n$. We say the field is *holomorphic* if and only if is locally conservative and locally solenoidal, that is, it satisfies

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}, \; \forall i, j; \qquad \operatorname{div} \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = 0, \text{ on } D. \tag{141}$$

**Definition 6.28.** Let $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field two times differentiable on an open set $\Omega \subseteq \mathbb{R}^n$. We say the field is *harmonic* if and only if $\nabla^2 \Phi = 0$ on $\Omega$.

**Theorem 6.26.** *Holomorphic vector fields are the fields that are locally the gradient of an harmonic function. Holomorphic functions are the functions $f$ that, locally, satisfy $\bar{f} = \Phi_x + i\Phi_y$ with $\Phi$ harmonic.*

**Definition 6.29.** Let $u$ be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$. We say a differentiable function $\tilde{u}$ on $\Omega$ is *the harmonic conjugate of $u$* if and only if $d\tilde{u} = d^*u$, that is, if the function $f = u + i\tilde{u}$ is holomorphic on $\Omega$.

**Theorem 6.27.** *Let $u$ be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$ and $f = \overline{\boldsymbol{\nabla}} u$. Then, $u$ is has an harmonic conjugate on $\Omega$, $\tilde{u}$, if and only if $f$ has an holomorphic primitive $F$ on $\Omega$. In that case, $F = u + i\tilde{u}$.*

**Proposition 6.28.** *Let $u$ be an harmonic function on a domain $\Omega$. Then, it has an harmonic conjugate if an only if the closed form $d^*u$ is exact on $\Omega$, that is, if $\int_\gamma d^*u = 0$ for all closed curve $\gamma$ such that $\Gamma \subseteq \Omega$, condition that is is always locally completed. If $\Omega$ is a star domain, every harmonic function on $\Omega$ has a harmonic conjugate function on $\Omega$.*

# 7 Local properties of holomorphic functions

**Lemma 7.1.** *Let $a \in \mathbb{C}$ be a number and $f = 1/|z - a|$. Then, $f$ is Lebesgue-integrable on every subset of $\mathbb{C}$ of finite measure.*

**Theorem 7.2** (Cauchy-Green formula). *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and $f$ a differentiable function on a neighborhood of $\bar{\Omega}$ such that $\bar{\partial} f$ is continuous on $\bar{\Omega}$. Then, for all $z_0 \in \Omega$,*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{z - z_0} \, dz - \frac{1}{\pi} \int_\Omega \frac{\overline{\partial} f(z)}{z - z_0} \, dm(z). \tag{142}$$

**Corollary 7.3** (Cauchy's integral formula). *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and $f$ an holomorphic function on a neighborhood of $\bar{\Omega}$. Then,*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{z - z_0} \, dz. \tag{143}$$

**Corollary 7.4.** *Let $f$ be a differentiable function on $\mathbb{C}$ with compact support and $\overline{\partial} f$ continuous on $\mathbb{C}$. Then,*

$$f(z_0) = -\frac{1}{\pi} \int_\Omega \frac{\overline{\partial} f(z)}{z - z_0} \, dm(z). \tag{144}$$

**Proposition 7.5.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and $\gamma$ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z_0) f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, dz. \tag{145}$$

**Corollary 7.6.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and $\gamma_1, \gamma_2$ two homotopic, piece-wise regular, and positively oriented curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z - z_0} \, dz - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z - z_0} \, dz. \tag{146}$$

**Theorem 7.7.** *Let $f$ be an holomorphic function on a disc $D(z_0, R)$. Then, there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence greater or equal to $R$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall z \in D(z_0, R). \tag{147}$$

**Theorem 7.8.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with $\Omega$ an open set. Then, $f$ is holomorphic on $\Omega$ if and only if $f$ is analytic on $\Omega$. More precisely, every holomorphic function $f$ on $\Omega$ is indefinitely holomorphic on $\Omega$, and for all $z_0 \in \Omega$ the Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \tag{148}$$

*is valid on the greatest disc centered at $z_0$ and contained on $\Omega$, which is $D(z_0, \delta(z_0))$, where $\delta(z_0) = \inf\{|z_0 - w|, w \notin \Omega\}$.*

**Theorem 7.9.** *The assignation $f \longrightarrow \left( \frac{f^{(n)}(0)}{n!} \right)_{n=0}^{\infty}$ is a bijection between the space of entire functions and the space formed by the sequences $\{a_n\}_{n=0}^{\infty}$ such that the series $\sum_{n=0}^{\infty} a_n z^n$ has an infinite radius of convergence, that is, $\lim_{n \to \infty} |a_n|^{1/n} = 0$.*

**Theorem 7.10** (Morera's theorem). *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C(\Omega)$ with $\Omega$ an open set. Then, $f$ is holomorphic on $\Omega$ if and only if*

$$\int_{\partial \triangle} f(z) \, dz = 0 \tag{149}$$

*for all triangle $\triangle \subseteq \Omega$.*

**Theorem 7.11.** *Let $f$ be a function continuous on an open set $\Omega$ and holomorphic on $\Omega \setminus E$, where $E$ is a finite collection of points and segments. Then, $f$ is holomorphic on $\Omega$.*

**Proposition 7.12.** *Let $f$ be a function and $\Omega$ a bounded domain with piece-wise regular and positively oriented boundary. If $f$ is holomorphic on a neighborhood of $\bar{\Omega}$, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z-z_0)^{n+1}} \, dz. \qquad (150)$$

**Proposition 7.13.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and $\gamma$ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\mathrm{Ind}(\gamma, z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} \, dz. \quad (151)$$

**Lemma 7.14.** *let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in H(\Omega)$ a function, and $z_0 \in \Omega$ a number. Then, the following statements are equivalent.*

1. *$f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.*

2. *$f(z) = 0$ for all $z$ in a neighborhood of $z_0$.*

3. *$f$ is identically null on $\Omega$.*

**Definition 7.1.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0$ a number. We say $z_0$ is a zero of order $n$ of $f$ if and only if $f^{(k)}(z_0) = 0$ for all $0 \leq k \leq n$. We call $k$ the order of $z_0$ as a zero of $f$.*

**Proposition 7.15.** *The zeros of finite order of an holomorphic function are isolated points.*

**Proposition 7.16.** *All the zeros of an non null analytic function are isolated points and of finite order.*

**Definition 7.2.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with $\Omega$ a domain. Then, we denote the set of zeros of $f$ as*

$$Z(f) := \{w \in \Omega \,|\, f(w) = 0\}. \qquad (152)$$

**Theorem 7.17.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with $\Omega$ a domain such that $f \not\equiv 0$. Then, $Z(f) \subseteq \Omega$ is a closed set without accumulation points. In particular, $Z(f)$ is a finite or countable set and and on every compact of $\Omega$ there is a finite number of zeros of $f$.*

**Theorem 7.18** (Principle of analytic continuation). *Let $f, g$ be two holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$. Then, $f(z) = g(z)$ for all $z \in \Omega$ if and only if they satisfy one of the following conditions.*

1. *There exists a point $w \in \Omega$ such that $f^{(n)}(w) = g^{(n)}(w)$ for all $n \in \mathbb{N}$, that is, $|f(z) - g(z)| = o(|z - a|^n)$, if $z \to a$, for all $n \in \mathbb{N}$.*

2. *There exists a set $\Psi \subseteq \Omega$ that contains an accumulation point on $\Omega$ and $f(z) = g(z)$ for all $z \in \Psi$.*

3. *There exists an open set $\Psi \subseteq \Omega$ such that $f(z) = g(z)$ for all $z \in \Psi$.*

**Theorem 7.19** (Schwarz reflection principle). *Let $\Omega \subseteq \mathbb{C}$ be a symmetric domain and $f \in H(\Omega)$ such that $f(x) \in \mathbb{R}$ for all $x \in \Omega \cap \mathbb{R}$. Then, $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Omega$.*

**Theorem 7.20.** *Every analytic function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is the restriction on $\mathbb{R}$ of an holomorphic function $F : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ defined in a symmetric domain $\Omega$, that is, $\mathbb{R} \subseteq \Omega$ and $f = F|_{\mathbb{R}}$.*

**Theorem 7.21.** *Let $f, g$ be two analytic functions on a domain $\Omega \subseteq \mathbb{R}^2$. Then, $f(x,y) = g(x,y)$ for all $(x,y) \in \Omega$ if and only if they satisfy one of the following conditions.*

1. *There exists a point $(x_0, y_0) \in \Omega$ such that*

$$\frac{\partial^{n+m}f}{\partial x^n \partial y^m}(x_0, y_0) = \frac{\partial^{n+m}g}{\partial x^n \partial y^m}(x_0, y_0) \qquad (153)$$

*for all $n, m \in \mathbb{N}$, that is, $|f(x,y) - g(x,y)| = o\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$, if $(x,y) \to (x_0, y_0)$ for all $n \in \mathbb{N}$.*

2. *There exists an open set $\Psi$ such that $f(x,y) = g(x,y)$ fir all $(x,y) \in \Psi$.*

**Theorem 7.22** (Maximum modulus principle). *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with $\Omega$ a domain. If $f$ is not constant, then $|f|$ does not have any local maxima on $\Omega$.*

**Corollary 7.23.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $f$ an holomorphic function on a neighborhood of $\bar{\Omega}$ or, more generally, $f \in C(\bar{\Omega}) \cap H(\Omega)$. Let $M$ be the maxima of $|f|$ on $\partial\Omega$. Then, one has*

$$|f(z)| \leq M, \qquad \text{for all } z \in \Omega. \qquad (154)$$

*In other words, $\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|$.*

**Theorem 7.24** (Cauchy's inequality). *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on a neighborhood of the disc $\bar{D}(z_0, R)$ and $|f(z)| \leq M$ for $z \in C(z_0, R)$. Then,*

$$\left|f^{(n)}(z_0)\right| \leq M \frac{n!}{R^n}. \qquad (155)$$

**Corollary 7.25.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with $\Omega$ a domain such that $|f(z)| \leq M, z \in \Omega$. Then,*

$$\left|f^{(n)}(z)\right| \leq M \frac{n!}{d(z, U^c)^n}, \qquad z \in U, n \in \mathbb{N}. \quad (156)$$

**Theorem 7.26** (Liouville's theorem). *Let $f$ be a bounded entire function. Then, $f$ is constant. Also, a function $u$ harmonic and bounded on $\mathbb{C}$ is constant.*

**Theorem 7.27** (Fundamental theorem of algebra). *Let $P(<) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree $n$ of complex coefficients and $n \geq 1$. Then, $P$ has exactly $n$ roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ (some of which can be counted with their multiplicity) and*

$$P(z) = a_n \prod_{i=1}^{n}(z - \alpha_i). \qquad (157)$$

# 8 Isolated singularities of holomorphic functions

**Definition 8.1.** We say $f$ *has an isolated singularity at $z_0$* if and only if $f$ is holomorphic on $D_r^*(z_0)$ for some $r \in \mathbb{R}^+$. We say the singularity is *removable* if and only if $f$ can be extended to an holomorphic function on $D_r(z_0)$.

**Definition 8.2.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on a disc $D_r^*(z_0)$. We say $f$ has a *pole of order $k$ at $z_0$* if and only if there exist $\alpha \in \mathbb{C}, k \in \mathbb{N}_{\geq 1}$ such that $f(z) \propto \alpha(z - z_0)^k$ when $z \to z_0$. We call $k$ the *multiplicity of the pole* or *order of the pole*.

**Definition 8.3.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with $\Omega$ a domain. We say $f$ is *meromorphic on $\Omega$* if and only if there exists a set $A \subseteq \Omega$, discrete and closed on $\Omega$, such that $f$ is defined and holomorphic on $\Omega \setminus A$ and has a pole on every point $z \in A$.

**Proposition 8.1.** $f$ has a pole of order $k$ at $z_0$ if and only if there exists an holomorphic function $g(z)$ in a neighborhood of $z_0$ such that $g(z_0) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - z_0)^k}. \tag{158}$$

**Theorem 8.2.** *Every holomorphic function on an annulus admits a Laurent expansion.*

**Proposition 8.3.** *Let $f$ be an holomorphic function on an annulus $C(z_0, R_2, R_1)$. If $f$ has an isolated singularity at $z_0$, then its Laurent expansion is uniquely determined by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \qquad a_n = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \tag{159}$$

*where $a_n$ is independent of $r$, $r \in (R_2, R_1)$.*

**Definition 8.4.** Let $f \in H(D_\epsilon^*(z_0))$ be an holomorphic function with a Laurent expansion $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ around $z_0$. We define the *residue of $f$ at $z_0$* as

$$\operatorname{Res}(f, z_0) := a_{-1} = \frac{1}{2\pi i} \int_{C(z_0, r)} f(z) \, dz. \tag{160}$$

**Theorem 8.4.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary. Let $\Psi$ be an open set such that $\bar{\Omega} \subseteq \Psi$, $X \subseteq \Psi$ a closed set formed by isolated points (the accumulation points of $X$, if there are, must be in $\partial \Psi$) such that $X \cap \partial \Omega = \varnothing$, and $f$ an holomorphic function on the open set $\Psi \setminus X$. Then,*

$$\frac{1}{2\pi i} \int_{\partial \Omega} f(z) \, dz = \sum_{w \in X \cap \Omega} \operatorname{Res}(f, w). \tag{161}$$

**Theorem 8.5.** *For a general curve,*

$$\frac{1}{2\pi i} \int_\gamma f(z) \, dz = \sum_{i=1}^n \operatorname{Ind}(\gamma, z_i) \operatorname{Res}(f, z_i). \tag{162}$$

**Proposition 8.6.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on a neighborhood of $z_0$ with $z_0$ a pole. Then,*

1. *If $z_0$ is a removable singularity, $\operatorname{Res}(f, z_0) = 0$.*

2. *If $z_0$ is a simple singularity,*

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{163}$$

3. *If $z_0$ is a singularity of order $k$,*

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} [(z - z_0) f(z)]. \tag{164}$$

4. *If $z_0$ is an essential singularity, the residue $a_{-1}$ must be obtained directly from the Laurent series.*

**Proposition 8.7.** *If $f = g/h$, with $f, g$ holomorphic in a neighborhood of $z_0$, $g(z_0) \neq 0, h(z_0) = 0, h'(z_0) \neq 0$, then*

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}. \tag{165}$$

**Proposition 8.8.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a meromorphic function in a neighborhood of $z_0 \in \mathbb{C}$. If we denote $f(z) = (z - z_0)^m g(z)$ with $w \in \mathbb{Z}^*$ (depending on the sign $z_0$ can be a zero or a pole), then $z_0$ is a single singularity of $f'/f$ and $\operatorname{Res}(f'/f, z_0) = m$.*

**Proposition 8.9.** *Let $f(z) = g(\frac{1}{z - z_0})$ be a function with $g(w)$ an entire function that admits an expansion $g(w) = \sum_{n=0}^{\infty} b_n w^n$. If $g$ is not a polynomial, then $f$ has an essential singularity at $z_0$ and $\operatorname{Res}(f, z_0) = g'(0) = b_1$.*

**Proposition 8.10.** *Let $f$ be a function with a simple pole at $z_0$ and $g$ an holomorphic function in a neighborhood of $z_0$. Then, $fg$ has a simple singularity at $z_0$ and $\operatorname{Res}(fg, z_0) = g(z_0) \operatorname{Res}(f, z_0)$.*

**Definition 8.5.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ *is holomorphic at infinity* if and only if $g(w) = f(1/w)$ is holomorphic at the origin.

**Proposition 8.11.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on $D_\epsilon^*(\infty)$. Then,*

$$\operatorname{Res}(f, \infty) = -a_{-1} = -\frac{1}{2\pi i} \int_{C(z_0, r)} f(z) \, dz. \tag{166}$$

**Proposition 8.12.** *Let $f$ be a meromorphic function on the Riemann sphere. Then, $f$ is a rational function. Besides, if $X$ is the sent formed by the poles of $F$ and the infinite point, then $X$ is finite and*

$$\sum_{w \in X} \operatorname{Res}(f, w) = 0 \Leftrightarrow \operatorname{Res}(f, \infty) = -\sum_{w \in X \setminus \{\infty\}} \operatorname{Res}(f, w). \tag{167}$$

**Theorem 8.13.** *Let $\Omega$ be a bounded domain with a piece-wise regular and positively oriented boundary. Let $\Psi$ be an open set such that $\bar{\Omega} \subseteq \Psi$, $f$ a meromorphic function on $\Psi$ and $h$ an holomorphic function on $\Psi$. Let $\{a_j\}$ be the zeros of $f$ on $\Psi$ and $n_j$ the multiplicities of $a_j$, and let $\{b_j\}$ be the poles of $f$ on $\Psi$ and $m_j$ the multiplicities of $b_j$. If there is no zeros or poles on $\partial\Omega$, then*

$$\frac{1}{2\pi i}\int_{\partial\Omega} h(z)\frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{a_j\in\Omega} h(a_j)n_j - \sum_{b_j\in\Omega} h(b_j)m_j. \tag{168}$$

**Corollary 8.14.** *Let $\Omega$ be a bounded domain with a piece-wise regular and positively oriented boundary and $f$ a meromorphic function on a neighborhood of $\bar{\Omega}$ that does not have zeros or poles on $\partial\Omega$. Let $N$ the total number of zeros of $f$ on $\Omega$ and $P$ the total number of poles on $\Omega$ (counting multiplicities). If we denote $\Gamma = f(\partial\Omega)$, then*

$$\mathrm{Ind}(\Gamma, 0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(z)}{f(z)}\,\mathrm{d}z = N - P. \tag{169}$$

# 9 Homology

# 10 Harmonic functions

**Theorem 10.1.** *Let $f \in H(\Omega), C^1(\Omega)$ be a function. If $f = u + iv$, then $u, v$ are harmonic functions on $\Omega$.*

# 11 Conforming representation

# 12 Riemann theorem

# 13 Runge theorem

# 14 Zeros of holomorphic functions

**Theorem 14.1** (Weierstrass Factorization Theorem). *content...*

# 15 Fourier transform

**Definition 15.1.** Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform of $f$ at the point $\xi$* as

$$\boxed{\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x\,.} \tag{170}$$

**Proposition 15.1.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, the application*

$$\begin{aligned} \mathscr{F}\{f\} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \hat{f}(\xi) \end{aligned} \tag{171}$$

*is a well defined application.*

**Definition 15.2.** Let $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \le p \le \infty$. We say *the functions $f_n$ converge to $f$ with a norm $\|\cdot\|_p$* or *converge in $L^p(\mathbb{R})$* if and only if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0. \tag{172}$$

**Theorem 15.2.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.*

1. *The Fourier transform of $f$ satisfies*

$$\mathscr{F}\{f\} \in L^\infty(\mathbb{R}), \qquad \|\mathscr{F}\{f\}\|_\infty \le \frac{1}{\sqrt{2\pi}}\|f\|_1. \tag{173}$$

2. *$\mathscr{F}\{f\}$ is $\mathbb{C}$ linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,*

$$\mathscr{F}\{\alpha f + \beta g\} = \alpha\mathscr{F}\{f\} + \beta\mathscr{F}\{g\}. \tag{174}$$

3. *If $g(x) = \bar{f}(x)$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \tag{175}$$

4. *If $g(x) = g(\lambda x)$ and $\lambda \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \frac{1}{|\lambda|}\hat{f}\left(\frac{\xi}{\lambda}\right). \tag{176}$$

5. *If $g(x) = f(x - a)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \mathrm{e}^{-ia\xi}\hat{f}(\xi). \tag{177}$$

6. *If $g(x) = \mathrm{e}^{iax}f(x)$ with $\alpha \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \hat{f}(\xi - a) \tag{178}$$

7. *If $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$ when $n \to \infty$, then $\mathscr{F}\{f_n\} \to \mathscr{F}\{f\}$ uniformly in $\mathbb{R}$.*

8. *The Fourier transform $\mathscr{F}\{f\}$ is a continuous function in $\mathbb{R}$, $\mathscr{F}\{f\} \in C(\mathbb{R})$.*

**Proposition 15.3.** *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty} |f(x)| = 0$. Then,*

$$\hat{f}'(\xi) = i\xi\hat{f}(\xi). \tag{179}$$

**Corollary 15.4.** *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its $n$-th derivative $f^{(n)} \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty} |f(x)| = 0$. Then,*

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n\hat{f}(\xi). \tag{180}$$

**Definition 15.3.** Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We define the support of $f$ as

$$\mathrm{supp}\,f := \overline{\{x \in I \mid f(x) \ne 0\}}. \tag{181}$$

**Definition 15.4.** We define the set $\mathscr{D}(\mathbb{R})$ as

$$\mathscr{D}(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}) \mid \mathrm{supp}\,\varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \tag{182}$$

**Theorem 15.5.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_{\in} \mathscr{D}(\mathbb{R})$ such that*

$$\lim_{h \to \infty} \int_{\mathbb{R}} |f - \phi_n| \, \mathrm{d}x = 0, \qquad (183)$$

*that is, we have convergence of $\phi_n$ to $f$ with norm $\|\cdot\|_1$.*

**Proposition 15.6.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.*

**Proposition 15.7.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\left|\hat{f}(\xi)\right| \leq \|f\|_1$.*

**Theorem 15.8.** *Let $f \in L^1(\mathbb{R})$ be a function. Then,*

$$\lim_{|x| \to \infty} \hat{f}(x) = 0. \qquad (184)$$

**Theorem 15.9.** *The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.*

**Definition 15.5.** We define the *Schwartz space* as

$$S(\mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{C} \mid f \in C^\infty(\mathbb{R}) \wedge \forall n, m \in \mathbb{N} \exists c_{n,m} < \infty$$
$$\text{such that } (1 + |x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}.$$

**Proposition 15.10.** *Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \longrightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,*

1. *$f + g \in S(\mathbb{R})$.*

2. *$\lambda f \in S(\mathbb{R})$.*

3. *$fg \in S(\mathbb{R})$.*

4. *$Pf \in S(\mathbb{R})$.*

**Theorem 15.11.** *Let $I, J \subseteq \mathbb{R}$ be two intervals with $I$ compact and $J$ open. Let $f : I \times J \longrightarrow \mathbb{R}$ be a function such that*

1. *$f(\cdot, \lambda)$ is Riemann-integrable in $I$ for all $\lambda \in J$,*

2. *$f(x, \cdot)$ is derivable in $J$ for all $x \in I$.*

*If $\partial_\lambda f$ is continuous in $I \times J$, then*

1. *$\partial_\lambda f(\cdot, \lambda)$ is Riemann-integrable for all $\lambda \in J$.*

2. *$F(\lambda) = \int_I f(x, \lambda) \, \mathrm{d}x$ is derivable with continuous derivative in $J$ for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.*

$$F'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int_I f(x, \lambda_0) \, \mathrm{d}x = \int_I \frac{\partial f}{\partial \lambda}(x, \lambda_0) \, \mathrm{d}x, \forall \lambda_0 \in J$$

$$(185)$$

**Proposition 15.12.** *Let $f \in S(\mathbb{R})$. Then,*

1. *$S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.*

2. *$\widehat{xf}(\xi) = (iD_\xi \hat{f})(\xi)$ for all $\xi \in \mathbb{R}$.*

**Corollary 15.13.** *Let $f \in s(\mathbb{R})$. Then,*

$$\widehat{x^n f}(\xi) = (i^n D^n \hat{f})(\xi), \forall n \in \mathbb{N}. \qquad (186)$$

**Proposition 15.14.** *The Fourier transform $\mathscr{F}$ restricted to $S(\mathbb{R})$ is an automorphism, that is, if $f \in S(\mathbb{R})$ then $\mathscr{F}\{f\} = \hat{f} \in S(\mathbb{R})$.*

**Lemma 15.15.** *If $G(x) = \mathrm{e}^{-x^2/2}$, then $\hat{G}(\xi) = \mathrm{e}^{-\xi^2/2}$. We observe hence that $G$ is a fixed point of $\mathscr{F}$.*

**Lemma 15.16.** *If $f, g \in S(\mathbb{R})$, then*

$$\int_{\mathbb{R}} f(\xi)\hat{g}(\xi) \, \mathrm{d}\xi = \int_{\mathbb{R}} \hat{f}(\tau)g(\tau) \, \mathrm{d}\tau. \qquad (187)$$

**Lemma 15.17.** *Let $f, g \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,*

1. *$g(\lambda x)\hat{f}(x)$ converges to $g(0)\hat{f}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*

2. *$f(\lambda x)\hat{g}(x)$ converges to $f(0)\hat{g}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*

**Lemma 15.18.** *Let $f, g \in s(\mathbb{R})$. Then,*

$$f(0) \int_{\mathbb{R}} \hat{g}(\xi) \, \mathrm{d}\xi = g(0) \int_{\mathbb{R}} \hat{f}(\xi) \, \mathrm{d}\xi. \qquad (188)$$

**Lemma 15.19.** *Let $f \in s(\mathbb{R})$ be a function. Then,*

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) \, \mathrm{d}\xi. \qquad (189)$$

**Corollary 15.20** (Inversion formula)**.** *Let $f \in S(\mathbb{R})$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)\mathrm{e}^{ix\xi} \, \mathrm{d}\xi, \forall x \in \mathbb{R}. \qquad (190)$$

**Theorem 15.21** (Inversion of $\mathscr{F}$ in $S(\mathbb{R})$)**.** *Let $\mathscr{F} : S(\mathbb{R}) \longrightarrow S(\mathbb{R})$, defined by $\mathscr{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, $\mathscr{F}$ is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathscr{F}^4 = Id$. In particular, $\mathscr{F}^{-1} = \mathscr{F}^3$ and if $f \in S(\mathbb{R})$, then*

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{F}\{f\}(\xi)\mathrm{e}^{ix\xi} \, \mathrm{d}\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}\mathrm{e}^{ix\xi} \, \mathrm{d}\xi.}$$
$$(191)$$

*In fact, $\mathscr{F}$ is an homemorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(\boldsymbol{S}(\mathbb{R}), \|\cdot\|_{n,m})$.*

**Theorem 15.22** (Inversion of $\mathscr{F}$ for discontinuities)**.** *Let $f$ be a absolutely Riemann-integrable function in $\mathbb{R}$ with $f$ and $f'$ piece-wise continuous. Then,*

$$\frac{f(x^-) + f(x^+)}{2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}\mathrm{e}^{ix\xi} \, \mathrm{d}\xi. \qquad (192)$$

**Definition 15.6.** Let $f$ be a Riemann-integrable function in $\mathbb{R}$. We define the *Fourier transform of cosine kind* as

$$\hat{f}_c(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\xi x) f(x) \, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(\xi x) f_e(x) \, \mathrm{d}x \tag{193}$$

and the *Fourier transform of sine kind* as

$$\hat{f}_s(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\xi x) f(x) \, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(\xi x) f_o(x) \, \mathrm{d}x. \tag{194}$$

**Proposition 15.23.** *Let $\hat{f}_c, \hat{f}_s$ be the Fourier transform of cosine and sine kinds of $f$. Then, $\hat{f}_c(\xi)$ is even, $\hat{f}_s(\xi)$ is odd, and $\hat{f}(\xi) = \hat{f}_c(\xi) - i\hat{f}_s(\xi)$.*

**Theorem 15.24.** *Let $f$ be a absolutely Riemann-integrable function in $\mathbb{R}$ with $f$ and $f'$ piece-wise continuous. Then,*

$$\frac{f_e(x^-) + f_e(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c \cos(\xi x) \, \mathrm{d}\xi, \tag{195}$$

$$\frac{f_o(x^-) + f_o(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s \sin(\xi x) \, \mathrm{d}\xi. \tag{196}$$

**Theorem 15.25** (Tonelli's Theorem)**.** *Let $f : I \times J \longrightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x, y) \geq 0$ for all $(x, y) \in I \times J$. Then,*

$$\int_{I \times J} f \, \mathrm{d}x \, \mathrm{d}y = \int_I \int_J f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_J \int_I f(x, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{197}$$

*Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.*

**Corollary 15.26.** *Let $f, g \in L^1(\mathbb{R})$. Then, $F(x, t) = f(t)g(x - t) \in L^1(\mathbb{R}^2)$.*

**Definition 15.7.** Let $f, g \in L^1(\mathbb{R})$ two function. We define the *convolution of $f$ and $g$* as

$$(f * g) : \mathbb{R} \longrightarrow \mathbb{C}$$
$$x \longmapsto \int_{\mathbb{R}} f(t)g(x - t) \, \mathrm{d}t, \tag{198}$$

which is from $L^1(\mathbb{R})$.

**Proposition 15.27.** *Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$.*

**Proposition 15.28.** *Let $f \in L^1(\mathbb{R})$ be a function and $g = f^2$. Then,*

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \hat{f} * \hat{f} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)\hat{f}(\xi - t) \, \mathrm{d}t. \tag{199}$$

**Theorem 15.29.** *Let $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^{\infty}(\mathbb{R})$.*

**Theorem 15.30.** *Let $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$ with $\operatorname{supp} f$ compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and $\operatorname{supp}\{f * \phi\} \subseteq \operatorname{supp} f + \operatorname{supp} \phi$.*

**Definition 15.8.** We say the functions $\phi_{\epsilon} : \mathbb{R} \longrightarrow \mathbb{R}$ continuous in a compact support are *an approximation of the unity* if and only if

1. $\phi_{\epsilon} \geq 0$ for all $\epsilon$.

2. $\displaystyle\int_{\mathbb{R}} \phi_{\epsilon}(x) \, \mathrm{d}x = 1$.

3. For all $\delta > 0$ it is satisfied that

$$\lim_{\epsilon \to 0} \left\{ \sup_{|t| > \delta} \phi_{\epsilon}(t) \right\} = 0. \tag{200}$$

**Theorem 15.31.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_{\epsilon}\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_{\epsilon}$ converges uniformly in $\mathbb{R}$ to $f$.*

**Corollary 15.32.** *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function with compact support $\{\phi_{\epsilon}\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_{\epsilon}$ converges uniformly in $\mathbb{R}$ to $f$.*

**Theorem 15.33** (Weierstrass polynomial approximation)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials $P_n$ with $n \in \mathbb{N}$ such that $P_n$ converge uniformly to $f$ in $[a, b]$.*

**Theorem 15.34.** *Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \to f$ with norm $\|\cdot\|_p$ (that is, convergence in $L^p$), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then*

$$\lim_{n \to \infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \tag{201}$$

*with $\|f\|_{C^k(\mathbb{R})} = \max_{0 \leq l \leq k} \left( \sup_{x \in \mathbb{R}} \left| D^l f(x) \right| \right)$ being a norm.*

**Lemma 15.35.** *Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\displaystyle\int_{\mathbb{R}} f(x)\phi(x) \, \mathrm{d}x = 0$. Then, $f \equiv 0$.*

**Corollary 15.36.** *The Fourier transform $\mathscr{F}$ is injective since $\mathscr{F}\{f\} = \hat{f} = 0 \Leftrightarrow f = 0$ in $L^1(\mathbb{R})$ (the zero function class) and $\mathscr{F}$ is a linear application.*

**Theorem 15.37** (Inversion theorem in $L^1(\mathbb{R})$)**.** *Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) \mathrm{e}^{ix\xi} \, \mathrm{d}\xi, \forall x \in \mathbb{R}. \tag{202}$$

# 16  Fourier transform 2

**Theorem 16.1** (Parseval formula)**.** *Let $f, g \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be two functions. Then,*

$$\int_{\mathbb{R}} f(x)\overline{g(x)} \, \mathrm{d}x = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)} \, \mathrm{d}\xi. \tag{203}$$

**Theorem 16.2** (Plancherel Theorem)**.** *Let* $f \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ *be a function. Then,*

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi, \qquad (204)$$

*that is,* $\|f\|_2 = \left\| \hat{f}_2 \right\|$ *and* $\mathscr{F}$ *is an isometry between vector spaces.*

**Definition 16.1.** Let $f \in S(\mathbb{R})$ be a function. We define the following quantities

$$E(f) := \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x, \qquad (205)$$

$$\sigma(f)^2 := \int_{\mathbb{R}} |xf(x)|^2 \, \mathrm{d}x. \qquad (206)$$

**Theorem 16.3.** *Let* $f \in S(\mathbb{R})$ *be a function. Then,*

$$\sigma(f)\sigma(\hat{f}) \geq \frac{E(f)}{2}. \qquad (207)$$

# 17 Multidimensional fourier transform

**Theorem 17.1.** *For several variables*

$$\mathscr{F}\{f(x_1, \ldots, x_n)\} = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) \mathrm{e}^{-i(x_1\xi_1 + \cdots + x_n\xi}$$

$$(208)$$

*or simpler,*

$$\mathscr{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{e}^{-i\langle \mathbf{x}, \boldsymbol{\xi} \rangle_I} \, \mathrm{d}\hat{x}. \qquad (209)$$