1 Arithmetic and topology

Definition 1.1. Let $\mathbb{R}^2 = \{(x.y) \mid x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the sum "+" by components
$$(a, b) + (c, d) \coloneqq (a + b, c + d). \tag{1}$$
- Product: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the product by
$$(a, b)(c, d) = (ac - bd, ad + bc). \tag{2}$$

We define the set $\mathbb{C}$ as $(\mathbb{R}^2, +,)$.

Proposition 1.1. *The set $\mathbb{C}$ of complex numbers is an abelian field.*

Proposition 1.2. *Let $\mathbb{C}$ be defined in the second way. Then,*

1. *$\mathbb{C}$ is an abelian ring.*
2. *If we define f as*
$$\begin{aligned} f : (\mathbb{C}, +,) &\longrightarrow (\mathbb{R}^2, +,) \\ (x, y) &\longmapsto x + yi \end{aligned}, \tag{3}$$
then f is a morphism of rings.
3. *The function f is, in fact, an isomorphism and $\mathbb{C}$ is an abelian field.*

Proposition 1.3. *The subset of $\mathbb{C}$ generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.*

Theorem 1.4. *$\mathbb{C}$ is not an ordered field.*

Definition 1.2. Let $z = a + bi \in \mathbb{C}$. We define the *conjugate of z* as
$$\bar{z} \coloneqq a - bi. \tag{4}$$

Proposition 1.5. *For all $z, w \in \mathbb{C}$, we have:*

1. *$\bar{\bar{z}} = z$.*
2. *$\overline{z + w} = \bar{z} + \bar{w}$.*
3. *$\overline{zw} = \bar{z}\bar{w}$.*
4. *$z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.*
5. *$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.*
6. *The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.*

Definition 1.3. Let $z = a + bi \in \mathbb{C}$. We define the *real part of z and imaginary part of z* respectively as
$$\operatorname{Re}\{z\} \coloneqq a, \qquad \operatorname{Im}\{z\} \coloneqq b. \tag{5}$$

Proposition 1.6. *Let $z \in \mathbb{C}$. Then,*
$$\operatorname{Re}\{z\} = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im}\{z\} = \frac{z - \bar{z}}{2i} \tag{6}$$

Proposition 1.7. *Let $z, w \in \mathbb{C}$ and the following distance function.*
$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto \tilde{d}(z, w) \coloneqq |z - w| \end{aligned} \tag{7}$$
Then, $(\mathbb{C}, \tilde{d})$ is a metric space.

Definition 1.4. Let $z = a + bi \in \mathbb{C}$. We define the *modulus of z* as
$$|z| \coloneqq \tilde{d}(z, 0), \tag{8}$$
which is equivalent to $\sqrt{z\bar{z}}$.

Definition 1.5. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define an *open disc of radius r and center z_0* as follows
$$B_r(z_0) \coloneqq \{z \in \mathbb{C} \mid |z - z_0| < r\}. \tag{9}$$

Definition 1.6. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *punctured disc of radius r and center z_0* as follows
$$B_r^*(z_0) \coloneqq \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}. \tag{10}$$

Definition 1.7. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *closed disc of radius r and center z_0* as follows
$$\overline{B_r(z_0)} \coloneqq \{z \in \mathbb{C} \mid |z - z_0| \leq r\}. \tag{11}$$

Definition 1.8. We denote by $\mathbb{D}$ the unitary disc of center 0 and radius 1. Besides, we denote by $\mathbb{T} \subseteq \mathbb{C}$ the unitary circumference, that is,
$$\mathbb{T} \coloneqq \{z \in \mathbb{C} \mid |z| = 1\}. \tag{12}$$
We also denote it by $\mathbb{S}^1$.

Lemma 1.8. *The set $B = \left\{B_r(z_0) \,\middle|\, r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\right\}$ is a basis of the topology of $\mathbb{R}^2$ as a metric space. The set $D = \{D_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of $\mathbb{C}$ as a metric space.*

Proposition 1.9. *The sets $\mathbb{C}$ and $\mathbb{R}^2$ with the topology of metric space are homeomorphs.*

Corollary 1.10. *There is a bijection between B and D, that is, between balls of $\mathbb{R}^2$ and discs of $\mathbb{C}$.*

Proposition 1.11. *Let $z, w \in \mathbb{C}$. Then,*

1. *$|z| \geq 0$.*
2. *$|z| = 0 \Leftrightarrow z = 0$.*
3. *$-|z| \leq \operatorname{Re}\{z\} \leq |z|$ and $-|z| \leq \operatorname{Im}\{z\} \leq |z|$.*
4. *$|zw| = |z||w|$.*
5. *If $w \neq 0$, $|z/w| = |z|/|w|$.*
6. *$|z + w| \leq |z| + |w|$.*
7. *$|z + w| \geq ||z| - |w||$.*
8. *$|\operatorname{Re}\{zw\}| \leq |z||w|$ and $|\operatorname{Im}\{z\}| \leq |z||w|$.*
9. *$|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$.*

10. $|z^n| = |z|^n$

Corollary 1.12. *Let $z_1, \dots, z_n \in \mathbb{C}$. Then,*

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|, \qquad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \qquad |\operatorname{Re}\{z_1 \cdots z_n\}| \leq |z_1| \cdots |z_n| \tag{13}$$

Definition 1.9. Let $z \in \mathbb{C}^*$. We define the *argument of z*, denoted by $\arg z$, as the real number θ such that $z = |z|(\cos\theta + i\sin\theta)$. Let us observe that $\arg z$ is not a function but a multivalued application. We define the *principal argument of z* as

$$\mathrm{Arg} z \coloneqq \theta_0 \in [0, 2\pi) \mid z = |z|(\cos\theta + i\sin\theta). \tag{14}$$

In general, to make θ to be unique, it is enough to impose it to belong to a certain semiopen interval of length 2π. Choosing the interval I is called by *taking a determination of the argument.*

Definition 1.10. Given a complex number z that we can express by $z = |z|(\cos\theta + i\sin\theta)$ for some $\theta \in \mathbb{R}$, we use the notation $r = |z|$ to write

$$z = r_\theta^z = r(\cos\theta + i\sin\theta) \tag{15}$$

or simply r_θ when it is obvious which complex number are we referring to. We call it *polar form of z*.

Proposition 1.13. *Let $z \in \mathbb{C}$ and r_θ its polar form. Then,*

$$z^n = (r^n)_{n\theta}. \tag{16}$$

Corollary 1.14 (De Moivre's Formula)**.** *Let $\theta \in \mathbb{R}$. Then,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{17}$$

Proposition 1.15. *Let $z, w \in \mathbb{C}$. Then,*

1. $\arg zw = \arg z + \arg w + 2\pi k$.
2. $\arg z^n = n \arg z + 2\pi k$.

Definition 1.11. We denote the complex numbers z generated by moving the point $z_0 = 1$ around $\mathbb{T}$ a length t in a counter-clockwise direction by 1_t. In other words, 1_t are the complex numbers $z = \cos t + i\sin t$.

Proposition 1.16. *Let $f : t \longrightarrow 1_t$. Then, f is a morphism from $(\mathbb{R}, +)$ to $(\mathbb{T}, \cdot)$, with $\ker f = 2\pi\mathbb{Z}$.*

Definition 1.12. Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $w \in \mathbb{C}$ is an *n-th root of z* if and only if

$$w^n = z. \tag{18}$$

Theorem 1.17. *Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \dots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \dots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \dots, n\}$.*

Definition 1.13. Let $z_n \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $\lim_{n\to\infty} z_n = l$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0(\varepsilon) \in \mathbb{N}^* \mid |z_n - l| < \varepsilon, n \geq n_0. \tag{19}$$

Proposition 1.18. *Let $\{z_n\} = \{a_n + ib_n\}$ be a sequence of complex numbers. Then, it converges if and only if $\{a_n\}$ and $\{b_n\}$ converge.*

Definition 1.14. We say $\sum_{n=1}^{\infty} z_n$ converges if and only if $S_n \coloneqq \sum_{n=1}^{N} z_n$ has limit at $n \to \infty$.

Proposition 1.19. *$\sum_{n=1}^{\infty} z_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.*

Definition 1.15. We say $\sum_{n=1}^{\infty} z_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |z_n|$ converges.

Proposition 1.20. *$\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge.*

Theorem 1.21. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a continuous curve such that $\gamma(t) \neq 0 \forall t \in [a, b]$. Then, there exists a continuous determination ϕ of the argument of γ. Then, $\phi(t) + 2\pi k$ with $k \in \mathbb{Z}$ is the general expression of all the argument determinations of γ. If γ is differentiable, then ϕ is differentiable and $\phi' = \operatorname{Im}\{\gamma'/\gamma\}$.*

Definition 1.16. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a regular curve. We define the *variation of the argument along γ* as

$$\Delta_\gamma \arg \coloneqq \operatorname{Im}\left\{\int_a^b \frac{\gamma'(t)}{\gamma(t)}\, \mathrm{d}t\right\}. \tag{20}$$

Definition 1.17. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve such that $\gamma(t) \neq 0 \forall t \in [a, b]$. Then, we define the *index of γ with respect to the origin* or *the number of revolutions of γ around the origin*

$$\operatorname{Ind}(\gamma, 0) \coloneqq \frac{1}{2\pi} \Delta_\gamma \arg. \tag{21}$$

Proposition 1.22. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piece-wise regular curve. Then,*

$$\operatorname{Ind}(\gamma, 0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, \mathrm{d}t \tag{22}$$

Definition 1.18. Let γ be a closed curve and $z \notin \Gamma$. We define the *index of γ with respect to z* as

$$\operatorname{Ind}(\gamma, z) \coloneqq \operatorname{Ind}(\gamma - z, 0). \tag{23}$$

Proposition 1.23. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve piece-wise of class $C^1([a, b])$. Then,*

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z}\, \mathrm{d}t. \tag{24}$$

2 Sequences and limits

Definition 2.1. A *sequence of complex numbers* is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned}. \tag{25}$$

We denote it by $\{z_n\}_{n=m}^{\infty}$

Definition 2.2. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *the sequence has limit* L or *it converges to the limit* L if and only if

$$\forall \varepsilon \in \mathbb{R}^{+} \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \tag{26}$$

We denote it by

$$\lim_{h\to\infty} z_n = L, \qquad \lim \{z_n\}_{n=0}^{\infty} = L, \qquad \{z_n\}_{n=0}^{\infty} \to L. \tag{27}$$

Theorem 2.1. *Let $z_n = z_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$ and $L = L_x + iL_y \in \mathbb{C}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \to L \Leftrightarrow \{x_n\}_{n=0}^{\infty} \to L_x \wedge \{y_n\}_{n=0}^{\infty} \to L_z. \tag{28}$$

Definition 2.3. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *it tends to infinity* and denote it by $\lim z_n = \infty$ if and only if

$$\forall k \in \mathbb{R}^{+} \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \tag{29}$$

Definition 2.4. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^{+} \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \tag{30}$$

Theorem 2.2. *Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.*

Theorem 2.3. *Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence } \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \text{ are Cauchy sequences.} \tag{31}$$

Theorem 2.4. *The field $\mathbb{C}$ of complex numbers is complete.*

Definition 2.5. The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, together with two charts.

3 Functions

Definition 3.1. A *topology* is an ordered pair $(\mathbb{X}, \tau)$, where X is a set and τ a collection of subsets of X satisfying the following properties:

1. The empty set and X belong to τ.
2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ.
3. The intersection of any finite number of members of τ still belongs to τ.

The elements of τ are called *open sets* and the collection τ is called the *topology on* X.

Definition 3.2. Let $(\mathbb{X}, d)$ be a metric space. A *topology on the metric space by the metric* d is the set τ of all open sets of M.

Definition 3.3. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *interior point of* A if there is a ball $B_{(\mathbb{M},d)}(a, r) \subset A$.

Definition 3.4. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *exterior point of* A if there is a ball such that $B_{(\mathbb{M},d)}(a, r) \cup A = \emptyset$.

Definition 3.5. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is a *boundary point of* A if it is not interior or exterior or, which is equivalent, if every ball $B_{(\mathbb{M},d)}(a, r)$ contains elements of A and A^c.

Definition 3.6. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *accumulation point of* A if every ball with center a contains points of A different to a. In other words, every punctured ball satisfies $B^{*}_{(\mathbb{M},d)}(a, r) \cup A \neq \emptyset$.

Definition 3.7. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the interior of* A as the set of all interior points of A, and we denote it by $\operatorname{int}(A)$.

Definition 3.8. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the exterior of* A as the set of all exterior points of A, and we denote it by $\operatorname{ext}(A)$.

Definition 3.9. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the boundary of* A as the set of all boundary points of A, and we denote it by ∂A.

Definition 3.10. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the closure of* A as the set of all accumulation points of A, and we denote it by $\bar{A}$.

Definition 3.11. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *an open set* if it contains none of its boundary points, that is, if $\partial A \cap A = \emptyset$.

Definition 3.12. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a closed set* if it contains all its boundary points, that is, if $\partial A \subseteq A$.

Definition 3.13. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a bounded set* if there exist a point $a \in \mathbb{M}$ and a positive real number r such that the ball $B_{(\mathbb{M},d)}(a, r)$ contains A.

Definition 3.14. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a compact set* if it a bounded and closed set.

Proposition 3.1. *Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. Then, A is open if and only if A^c is closed.*

Definition 3.15. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally, Ω is connected if there are not two open sets $U, V \subseteq \mathbb{C}$ such that

$$U_1 = U \cap \Omega, \qquad V_1 = V \cap \Omega, \qquad U_1 \cap V_1 = \varnothing, \qquad U_1 \cup V_1 = \Omega. \tag{32}$$

Otherwise, we say Ω *is disconnected.*

Definition 3.16. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is simply connected* if and only if every circuit is homotopic in Ω to a point in Ω. Equivalently, is simply connected if and only if every pair of curves with the same extremes are homotopic.

Definition 3.17. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is convex* if and only if for all pair of point $a, b \in \Omega$, the segment defined by

$$[a, b] = \{z \mid z = (1-t)a + tb, 0 \leq t \leq 1\} \tag{33}$$

is contained in Ω, that is, if every pair of points can be connected by a straight line that belongs to the set.

Definition 3.18. Let $\Omega \in \mathbb{C}$ be a set. We say Ω *is a star-convex set* if and only if there exists $z_0 \in \mathbb{C}$ such that for all $z \in \Omega$ the segment $[z_0, z]$ is contained by Ω.

Definition 3.19. Let $(\mathbb{M}, d)$ be a metric space and $S \subseteq \mathbb{M}$ a set. We say S *is path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

Definition 3.20. Let $\Omega \in \mathbb{C}$ be a set. We say Ω *is a region or domain* if and only if it is open, non-empty, and connected.

Definition 3.21. Let $\Omega \subseteq \mathbb{C}$ be a non-empty set. We say $\Omega_1 \subseteq \Omega$ *is a connected component of* Ω if and only if it is a maximal connected subset, that is, if $z_0 \in \Omega_1$ and W is a connected subset of $\mathbb{C}$ that contains z_0, then $W \subseteq \Omega_1$.

Definition 3.22. Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function* f as the application

$$\begin{aligned} f : D \subseteq \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto w = f(z) \end{aligned}. \tag{34}$$

Definition 3.23. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say it *tends to infinity at the point* z_0 and denote it by $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \tag{35}$$

Definition 3.24. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We write $\lim_{z \to \infty} f(z) = L$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \tag{36}$$

Proposition 3.2. *Let $f_1, f_2 : \Omega \longrightarrow \mathbb{C}$ be two functions and z_0 a point such that $\lim_{z \to z_0} f_1 = w_1, \lim_{z \to z_0} f_2 = w_2$. Then,*

1. *$f_1 + f_2$ has also a limit and $\lim_{z \to z_0} f + g = w_1 + w_2$.*
2. *$f_1 f_2$ has also a limit and $\lim_{z \to z_0} fg = w_1 w_2$.*
3. *If $w_2 \neq 0$, then f/g has also a limit and $\lim_{z \to z_0} f/g = w_1/w_2$.*
4. *If $h(z)$ is a continuous function defined on a neighborhood of w_1, then $\lim_{z \to z_0} h(f_1(z)) = h(w_1)$.*

Definition 3.25. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. We say f *is continuous in* z_0 if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \tag{37}$$

Proposition 3.3. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$ is continuous at z_0 if and only if $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are continuous at z_0.*

Proposition 3.4. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, f is continuous at z_0 if and only if for all sequence $\{z_n\}_{n=1}^{\infty}$ of Ω convergent at z_0 it is true that the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$.*

Proposition 3.5. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, λf, $f + g$, and fg are continuous at z_0. The function f/g is continuous at z_0 if $g(z_0) \neq 0$.*

Definition 3.26. We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n, z, z_0 \in \mathbb{C}. \tag{38}$$

We call the term a_n the *n-th coefficient of the series.* In case $a_n = 0 \forall n \leq m$, we will start the counting directly from m.

Definition 3.27. Radius of convergence.

Proposition 3.6. *The radius of convergence can be calculated as follows*

$$\frac{1}{R} = \lim_{n \to \infty} \sup \sqrt[n]{|a_n|}. \tag{39}$$

Theorem 3.7 (Cauchy-Hadamard Theorem)**.** *Let*

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{40}$$

be the following complex power series with $a_n, z_0 \in \mathbb{C}$ and $R \in [0, +\infty) \cup \{+\infty\}$ the radius of convergence. Then,

1. *If $|z - z_0| < R$ then S converges. In fact, for all $r < R$ we have S converges uniformly at the disc $\overline{D_r(z_0)}$.*
2. *If $|z - z_0| > R$ then S diverges.*

3. *The function $f(z) = S(z)$ is derivable at $D_R(z_0)$ and its formal derivative is*

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \tag{41}$$

with the same radius of convergence.

Definition 3.28. Let $f(z) = \sum a_n(z - z_0)^n$ be a series with radius of convergence R. Then, its formal derivative is

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z}. \tag{42}$$

Corollary 3.8. *Let $f(z) = \sum a_n(z - z_0)^n$ be a series with radius of convergence R. Then, f is infinitely derivable at $D_R(z_0)$.*

Corollary 3.9. *Let R be the radius of convergence of the function*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then f has as Taylor polynomial of degree m around z_0 the following one

$$P_{m,f,z_0}(z) = \sum_{n=0}^{m} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{43}$$

Theorem 3.10 (Abel's Theorem)**.** *Let be the following series*

$$\sum_{n=0}^{\infty} f_n(z_0) g_n(z_0),$$

where f,g_n are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \tag{44}$$

then the original series converges uniformly in Ω.

Theorem 3.11 (Weierstrass' criterion)**.** *Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant M_n, satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at Ω for all $z_0 \in \Omega$.*

Definition 3.29. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a complex function with Ω an open set. We say f *is complex analytic* if and only if for all $z_0 \in \Omega$ exists a real number $R(z_0)$ and a sequence $\{a_n\} \subseteq \mathbb{C}$ (that can also depend on z_0) such that is $z \in D_R(z_0)$, then formally it is true that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{45}$$

We denote the set of complex analytic functions with domain Ω by $C^{\omega}(\Omega)$.

Corollary 3.12. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. If $f \in C^{\omega}(\Omega)$, then $f \in C^{\infty}(\Omega)$.*

Corollary 3.13. *Let z_0. Then, the coefficients z_0 of the local expression of f given by the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ are determined by*

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{46}$$

Proposition 3.14. *Let $\varnothing \neq \Omega \subseteq \mathbb{C}$. Then,*

1. *Every connected component of Ω is a closed of Ω with a subspace topology.*

2. *Two connected components are the same or are disjoint.*

3. *Every connected of Ω is one and only one connected component.*

4. *Ω is the disjoint union of its connected components.*

Theorem 3.15 (Analytic prolongation Principle)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0 = 0)$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of Ω that contains z_0 (the function can be zero also out Ω, but we are not studying that).*

Corollary 3.16. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function with Ω a region. Then, the following statements are equivalent:*

1. *$f(z) = 0$ for all $z \in \Omega$.*

2. *There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.*

Corollary 3.17 (Analytic Prolongation Principle)**.** *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions with Ω a region. Then, the following statements are equivalent:*

1. *$f(z) = g(z)$ for all $z \in \Omega$.*

2. *There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}$.*

Lemma 3.18. *Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then*

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right). \tag{47}$$

Corollary 3.19. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions. Then, fg is analytic.*

Proposition 3.20. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, f is analytic at $\Omega = D_R(0)$.*

Definition 3.30. For all $z \in \mathbb{C}$, we define the *complex exponential function* as the following series

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{48}$$

Proposition 3.21. *The radius of convergence of* e^z *is infinite.*

Proposition 3.22. $\mathrm{e}^z = \mathrm{e}^x$ *for all* $x \in \mathbb{R}$.

Proposition 3.23. $\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$ *for all* $z, w \in \mathbb{C}$.

Proposition 3.24. *For all* $z \in \mathbb{C}$ *we have* $\mathrm{e}^z \neq 0$.

Proposition 3.25. *The image of* e^z *is* $\mathbb{C}^*$.

Proposition 3.26. *The derivative of* e^z *is* e^z.

Proposition 3.27. $\overline{\mathrm{e}^z} = \mathrm{e}^{\bar{z}}$.

Proposition 3.28. $|\mathrm{e}^z| = \mathrm{e}^{\mathrm{Re}\{z\}}$.

Proposition 3.29 (Euler's Formula)**.** *If* $\theta \in \mathbb{R}$, *then* e^{xi} *has modulus one and we have that*

$$\boxed{\mathrm{e}^{xi} = \cos x + i \sin x.} \tag{49}$$

Corollary 3.30. *Let* $z \in \mathbb{C}^*$. *Then,*

$$z = |z|\mathrm{e}^{i\theta}, \tag{50}$$

with $\theta \in [0, 2\pi)$.

Proposition 3.31. *The following function*

$$\begin{aligned} \exp : (\mathbb{R}, +) &\longrightarrow (\mathbb{C}^*, \cdot) \\ x &\longmapsto \mathrm{e}^{xi} \end{aligned} \tag{51}$$

is a morphism of groups and its image is $\mathbb{T}$.

Proposition 3.32. *The complex exponential function is a periodic function with period* $2\pi i$.

Proposition 3.33. *Let* $a \in \mathbb{C}^*$. *Then,* $\mathrm{e}^z = a$ *has infinite solutions.*

Proposition 3.34. *The equation* $\mathrm{e}^z = 0$ *does not have solutions.*

Proposition 3.35. *Let* $y_0 \in \mathbb{C}$ *be a numbers,* $B := \{z \in \mathbb{C} \,|\, y_0 < \mathrm{Im}\{z\} < y_0 + 2\pi\}$ *a set, and* $f : B \longrightarrow \mathbb{C}^*$ *be the exponential function. Then,* f *is bijective in* B **?**.

Proposition 3.36. *Let* $x_0, y_0, m \in \mathbb{C}$ *be two numbers with* $m \neq 0$ *and* f *the exponential function* **?**. *Then,*

1. *f transforms the line $y = y_0$ to a line that starts at $z = 0$ and continues with an argument y_0 from the real positive axis.*
2. *f transforms the line $x = x_0$ to a circle centered at the origin and radius $r = \mathrm{e}^{x_0}$.*
3. *f transforms the line $y = mx$ to the parametric curve $z = \mathrm{e}^x \mathrm{e}^{imx}$ (a spiral).*

Definition 3.31. Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric functions as*

$$\cos z := \frac{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}{2}, \tag{52}$$

$$\sin z := \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{2}, \tag{53}$$

$$\tan z := \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}. \tag{54}$$

Proposition 3.37. *For all* $z \in \mathbb{C}$,

$$\sin^2 z + \cos^2 z = 1. \tag{55}$$

Proposition 3.38. *For all* $z \in \mathbb{C}$,

$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z). \tag{56}$$

Proposition 3.39. *For all* $z, w \in \mathbb{C}$,

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \qquad \sin(z \pm w) = \sin z \cos w \pm \mathrm{co} \tag{57}$$

Proposition 3.40. *The functions* $\cos z, \sin z$ *have period of* 2π.

Proposition 3.41. *Let* $z_0 \in \mathbb{C}$. *Then,* z_0 *is root of* $\sin z$ $(\cos z)$ *if and only if it is a root of* $\sin x$ $(\cos x)$.

Definition 3.32. Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic functions* as

$$\cosh z := \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}, \tag{58}$$

$$\sinh z := \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}, \tag{59}$$

$$\tanh z := \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{\mathrm{e}^{z} + \mathrm{e}^{-z}}. \tag{60}$$

Proposition 3.42. *For all* $z \in \mathbb{C}$,

$$\cosh^2 z - \sinh^2 z = 1. \tag{61}$$

Proposition 3.43. *For all* $z \in \mathbb{C}$,

$$\cosh(-z) = \cosh(z), \qquad \sinh(-z) = -\sinh(z). \tag{62}$$

Proposition 3.44. *For all* $z, w \in \mathbb{C}$,

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \tag{63}$$

$$\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \tag{64}$$

Proposition 3.45. *For all* $z \in \mathbb{C}$,

$$\cosh z = \cos(iz), \cos z = \cosh(iz) \tag{65}$$

$$\sinh z = -i \sin(iz), \sin z = -i \sinh(iz) \tag{66}$$

Proposition 3.46. *For all* $z = x + iy \in \mathbb{C}$,

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \tag{67}$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \tag{68}$$

$$\tan(x + iy) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i\frac{\sinh y}{\cos(2x) + \cosh(2y)}. \tag{69}$$

Proposition 3.47. *For all* $z = x + iy \in \mathbb{C}$,

$$\tanh(x + iy) = \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)} + i\frac{\sin(2y)}{\cosh(2x) + \cos(2y)}. \tag{70}$$

Proposition 3.48. *For all* $z = x + iy$,

$$|\cos z| = \sqrt{\cosh^2 y - \sin^2 x} = \sqrt{\cos^2 x + \sinh^2 y}, \tag{71}$$

$$|\sin z| = \sqrt{\sinh^2 x + \sin^2 y} = \sqrt{\cosh^2 x - \cos^2 y}. \tag{72}$$

Corollary 3.49. *For all $z = x + iy$,*

$$|\sinh y| \leq |\cos z| \leq \cosh y, \qquad |\sinh y| \leq |\sin z| \leq \cosh y. \tag{73}$$

Proposition 3.50. *The roots of the function* $\sinh z$ *are of the form $z_n = n\pi$ and those for the function* $\cosh z$ *are of the form $w_n = (2n+1)\pi/2i$.*

Definition 3.33. Let $D \subseteq \mathbb{C}$ be a set. We define a *multivalued function from D to $\mathbb{C}$* as a subset of $D \times \mathbb{C}$ such that for every $z \in D$ there exists a number $y \in \mathbb{C}$ such that $(z, w) \in f$.

Definition 3.34. For $z \in \mathbb{C}^*$, we call the *natural logarithm of z* every number w such that $\mathrm{e}^w = z$, that is,

$$\ln z := \{w \in \mathbb{C} \,|\, \mathrm{e}^w = z\}. \tag{74}$$

Proposition 3.51. *Given $z \in \mathbb{C}$ we can define* $\ln z$ *from the natural logarithm of a real number as*

$$\ln z = \ln|z| + i \arg z = \ln|z| + i\mathrm{Arg}z + 2\pi k i. \tag{75}$$

Definition 3.35. We define the *principal natural logarithm of z* as the value defined by the principal argument of z, that is,

$$\mathrm{Log}z = \ln|z| + i\mathrm{Arg}z. \tag{76}$$

Definition 3.36. We define the determination I (with I being a semiopen interval) of the logarithm as

$$\log_I z := \ln|z| + i \arg_I z. \tag{77}$$

Definition 3.37. Let $E \subseteq \mathbb{C}^*$ be a connected set. We define the *continuous determination of the logarithm in E* as the continuous function $g : E \longrightarrow \mathbb{C}$ such that $\mathrm{e}^{g(z)} = z$. More generally, if $f : E \longrightarrow \mathbb{C}$ is a function such that $f(z) \neq 0$ for all $z \in E$, then we define the *continuous determination of* $\ln f$ as a function $g : E \longrightarrow \mathbb{C}$ such that $\mathrm{e}^{g(z)} = f(z)$.

Proposition 3.52. *Let $z, w \in \mathbb{C}$ two numbers. Then,*

1. $\ln(zw) = \ln z + \ln w + 2\pi k i, k \in \mathbb{Z}$.
2. *If we want to stay in the principal argument,*

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \textit{if } \mathrm{Arg}z + \mathrm{Arg}w < 2\pi \\ \ln z + \ln w - 2\pi i, & \textit{if } \mathrm{Arg}z + \mathrm{Arg}w \geq 2\pi \end{cases}. \tag{78}$$

3. *SEARCH MORE PROPERTIES*

Definition 3.38. Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric inverse functions* as

$$\arcsin z := -i \ln\left(iz + \sqrt{1 - z^2}\right), \tag{79}$$

$$\arccos z := -i \ln\left(z + \sqrt{z^2 - 1}\right), \tag{80}$$

$$\arctan z := -\frac{i}{2} \ln \frac{1 + iz}{1 - iz}. \tag{81}$$

Definition 3.39. Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic inverse functions* as

$$\mathrm{arcsinh}\, z := \ln\left(z + \sqrt{1 + z^2}\right), \tag{82}$$

$$\mathrm{arccosh}\, z := \ln\left(z + \sqrt{z^2 - 1}\right), \tag{83}$$

$$\mathrm{arctanh}\, z := \frac{1}{2} \ln \frac{1 + z}{1 - z}. \tag{84}$$

Definition 3.40. Let $z, a \in \mathbb{C}$ with $z \neq 0$. Then, we define the *complex power function* as

$$z^a := \mathrm{e}^{a \ln z}. \tag{85}$$

If $E \subseteq \mathbb{C}^*$ is a connected set and $f : E \longrightarrow \mathbb{C}$ a functions such that $f(z) \neq 0$ for all $z \in E$, and $w \in \mathbb{C}$ a number, we define a *continuous determination of f^w* as a continuous function $g : E \longrightarrow \mathbb{C}$ such that $g(z) \in [f(z)]^w$.

Proposition 3.53. *If $a = \alpha + \beta i$ and $z = r\mathrm{e}^{\theta i}$, then*

$$z^a = \mathrm{e}^{\alpha \ln r - \beta(\theta + 2\pi k)} \mathrm{e}^{(\beta \ln r + \alpha(\theta + 2\pi k))i}, \tag{86}$$

$$|z^a| = \mathrm{e}^{\alpha \ln|z| - \beta(\arg z + 2\pi k)}, \qquad \arg(z^a) = \beta \ln|z| + \alpha(\arg z + 2\pi k) \tag{87}$$

Proposition 3.54. *Let $a, z \in \mathbb{C}$ be two numbers. Then,*

1. *If $a = n \in \mathbb{Z}$, the complex power is a function and*

$$z^n = r^n \mathrm{e}^{n\theta i}. \tag{88}$$

2. *If $a = n/m \in \mathbb{Q}$, there are n values and*

$$z^a = \sqrt[m]{r^n} \mathrm{e}^{(\theta + 2\pi k)n/mi}. \tag{89}$$

3. *If a is irrational, the norm is uniquely determined but the argument has infinite values.*
4. *IF $a \in \mathbb{C} \backslash \mathbb{R}$, the argument is uniquely determined and the norm has infinite values.*

Proposition 3.55. *Let $z, w \in \mathbb{C}$. Then,*

1. $\left(\mathrm{e}^b\right)^a = \mathrm{e}^{a(b + 2\pi k i)}$

Definition 3.41. A *Riemann surface X* is a connected complex 1-manifold.

Definition 3.42. We define a *sheet* as each of the complex planes of the Riemann surface.

Definition 3.43. We define a *cut* as the line (not necessaryly straight) of union between sheets.

Definition 3.44. We define a *branch point* as a point where start or finish a cut.

4 Derivatives

Definition 4.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. We define the *derivative of f at z_0* as

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{90}$$

in case the limit exists. If f has derivative, we say *f is derivable at z_0*.

Definition 4.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *f is holomorphic at Ω* if and only if it is $\mathbb{C}$-derivable at every point of Ω. In that case, it is defined the function $f' : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ that associates each point z of Ω with $f'(z)$.

Definition 4.3. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We define the *domain of holomphism* as the region where f is derivable. We say *f is entire* if and only if the domain of holomorphism is $\mathbb{C}$.

Definition 4.4. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *f is holomorphic at z_0* if and only if it is holomorphic at some neighborhood of z_0.

Proposition 4.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. If f is derivable at z_0, then it is continuous at z_0.*

Theorem 4.2. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two functions and $z_0 \in \Omega$ a point. Then, the following statements are true.*

1. *If f is constant at Ω, then f is derivable at z_0 and $f'(z_0) = 0$.*

2. *If $f(z) = z$ in every point of Ω, then f is derivable at z_0 and $f'(z_0) = 1$.*

3. *If f, g are derivable at z_0 and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ are derivable at z_0 and $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.*

4. *If f, g are derivable at z_0, then fg is derivable at z_0 and*

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{91}$$

5. *If f, g are derivable at z_0 and $g(z_0) \neq 0$, then f/g is derivable at z_0 and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{92}$$

Theorem 4.3. *Let $f : \Omega_1 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a derivable function at a point $z_0 \in \mathbb{C}$ and $g : \Omega_2 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be another derivable function at a point $f(z_0) \in \Omega_2$. Then, $g \circ f$ is derivable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \tag{93}$$

Definition 4.5. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say *f is of class $C^1(\Omega)$ or simply $f \in c^1(\Omega)$* if and only if, using $f = u + iv$ with $u = \mathrm{Re}\{f\}, v = \mathrm{Im}\{f\}$, the partial derivatives of u and v as a two variable real functions exist and are continuous. In other words, $f \in C^1(\Omega)$ if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \tag{94}$$

exist and are continuous.

Theorem 4.4 (Cauchy-Riemann conditions)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. Then, f is derivable at z_0 if and only if is differentiable at z_0 and $df(z_0)$ is $\mathbb{C}$-linear, that is,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{95}$$

which are known as Cauchy-Riemann conditions.

Theorem 4.5. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. If u, v satisfy the Cauchy-Riemann equation and their partial derivatives are continuous, then f is derivable.*

Theorem 4.6. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function of class $C^1(\Omega)$ with Ω an open set, injective, and derivable at every point of Ω with non-zero derivative. Then,*

1. *$\Omega' = f(\Omega)$ is an open subset of $\mathbb{C}$.*

2. *The inverse function f^{-1} exist, it is well defined and it is derivable at Ω'.*

3. *If $z \in \Omega$ and $z' = f(z)$, then*

$$(f^{-1})'(z') = \frac{1}{f'(z)}. \tag{96}$$

Proposition 4.7. *A determination of $\ln z$ with $z \in \mathbb{C}$ is continuous except in a semiline.*

Theorem 4.8. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $\phi \in C(\Omega)$, such that $\mathrm{e}^{\phi(z)} = z$ for all $z \in \Omega$. Then, we have $\phi \in H(\Omega)$ and*

$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \tag{97}$$

Proposition 4.9. *A determination of $\ln z$ with $z \in \mathbb{C}$ is holomorphic except in a semiline.*

Proposition 4.10. *Let $I = [\theta, \theta + 2\pi)$ a determination of the logarithm, $\ln_I z$. Then, $\ln_I z$ is holomorphic except in the semiline $L_\theta = \left\{ r\mathrm{e}^{\theta i} \in \mathbb{C} \,\middle|\, r \geq 0 \right\}$.*

Theorem 4.11. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function in Ω. Then, f is holomorphic in Ω. In particular, given a point $z_0 \in \mathbb{C}$, $f'(z_0) = a_1$ where a_1 is the first coefficient of the power series that represents f in a neighborhood of z_0.*

Definition 4.6. We define the operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{98}$$

that act over the functions such that the real and imaginary part u, v have partial derivatives.

Proposition 4.12. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function of class $C^1(\Omega)$. Then, for all $z_0 \in \Omega$*

$$f(z_0+h) = f(z_0) + \left(\frac{\partial f}{\partial z}\right)_{z_0} h + \left(\frac{\partial f}{\partial \bar{z}}\right)_{z_0} \bar{h} + o(|h|^2). \tag{99}$$

Corollary 4.13. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$. Then, f is holomorphic in Ω if and only if*

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ at } \Omega. \tag{100}$$

Definition 4.7. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$ such that $f = u + iv$ with $u = \operatorname{Re}\{f\}, v = \operatorname{Im}\{f\}$ and $z_0 \in \mathbb{C}$ a point. Then, we call $(\partial_{\bar{z}} f)_{z_0} = 0$ the *Cauchy-Riemann condition*, which is equivalent to

$$\left(\frac{\partial u}{\partial x}\right)_{z_0} = \left(\frac{\partial v}{\partial y}\right)_{z_0}, \qquad \left(\frac{\partial v}{\partial x}\right)_{z_0} = -\left(\frac{\partial u}{\partial y}\right)_{z_0}, \tag{101}$$

which are called the *Cauchy-Riemann equations*.

Theorem 4.14. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and z_0 an interior point. Then, at z_0*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}. \tag{102}$$

5 Holomorphic functions

Definition 5.1. Let $I = [a,b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{array}{c} \gamma : I \longrightarrow \mathbb{C} \\ t \longmapsto \gamma_1(t) + i\gamma_2(t) \end{array}. \tag{103}$$

Definition 5.2. Let $I = [a,b] \subseteq \mathbb{R}$ be a closed interval and $D \subseteq \mathbb{C}$ a domain. We define an *arc* as a continuous application of the form

$$\begin{array}{c} \gamma : I \longrightarrow D \\ t \longmapsto \gamma_1(t) + i\gamma_2(t) \end{array}. \tag{104}$$

Equivalently, we can say an arc is a curve restricted to some interval.

Definition 5.3. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We call $\gamma(a)$ and $\gamma(b)$ *the extremes of* γ. In particular, we call $\gamma(a)$ the *initial point* and $\gamma(b)$ the *final point*.

Definition 5.4. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We define the *route or graph of* γ as

$$\gamma^* := \{z \in D \mid z = \gamma(t), t \in I\}. \tag{105}$$

Definition 5.5. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is closed* if and only if $\gamma(a) = \gamma(b)$.

Definition 5.6. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is simple* if and only if there is no two numbers $t_1, t_2 \in (a,b)$ such that $\gamma(t_1) = \gamma(t_2)$. We also call it a *Jordan curve*, and if it is closed, a *circuit*.

Definition 5.7. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is differentiable* if for al value $t_0 \in [a,b]$ there exists the limit

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \tag{106}$$

For $t_0 = a$ or $t_0 = b$ we consider the laterals limits from the right and from the left respectively.

Definition 5.8. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is of class* C^1 if and only if γ' exists and is continuous at $[a,b]$.

Definition 5.9. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is regular or smooth* if and only if it is differentiable and γ' never vanishes.

Definition 5.10. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is piece-wise of class* C^1 if and only if γ' exists and is continuous in I except in a finite number of points where γ has lateral derivatives.

Definition 5.11. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We define the *opposite arc* as

$$\begin{array}{c} -\gamma : [-b,-a] \longrightarrow \mathbb{C} \\ t \longmapsto \gamma(-t) \end{array}. \tag{107}$$

Definition 5.12. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be an arc. We say $\Gamma(s), s \in [c,d] \subseteq \mathbb{R}$ *has been obtained from* $\gamma(t), t \in [a,b]$ *by a change of parametrization* if and only if the new parameter s and the original parameter t are related by a relation $t = \phi(s)$, where $\phi : [c,d] \longrightarrow [a,b]$ is an homeomorphism that satisfies $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$. We call Γ *the reparametrization of* γ.

Definition 5.13. Let $\gamma_1 : I_1 \longrightarrow \mathbb{C}$ and $\gamma_2 : I_2 \longrightarrow \mathbb{C}$ be two arcs. We say they *are equivalent* if and only if there exists a bijective, monotone, and continuous function $\rho : I_2 \longrightarrow I_1$ such that $\gamma_2 = \gamma_1 \circ \rho$. If ρ is an increasing function we say γ_1 *and* γ_2 *have the same orientation*; otherwise, we say γ_1 *and* γ_2 *have opposite orientations.*

Definition 5.14. Let $\gamma_1[a,b] \longrightarrow \mathbb{C}$ and $\gamma_2 : [c,d] \longrightarrow \mathbb{C}$ be two arcs such that $[a,b] \cap [c,d] = \varnothing$. We define the application $\gamma_1 \cup \gamma_2$ (sometimes denoted by $\gamma_1 + \gamma_2$) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \le t \le b \\ \gamma_2(t - b + c), & \text{if } b \le t \le b + d - c \end{cases}. \tag{108}$$

We say γ_1, γ_2 can be joined/added or that there exists its union/sum if and only $\gamma_1(b) = \gamma_2(x)$. In this case $\gamma_1 + \gamma_2$ is an arc, and we call it *the sum arc of* γ_1 *plus* γ_2.

Definition 5.15. We define the *segment of extremes* $z_1, z_2 \in \mathbb{C}$ as the arc defined by the expression

$$\begin{array}{c} [z_1, z_2] : [0,1] \longrightarrow \mathbb{C} \\ t \longmapsto (1-t)z_1 + tz_2 \end{array}. \tag{109}$$

Definition 5.16. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f *is polygonal* if and only if can be expressed as a finite union of segments, that is, if there exist a natural number n and points $\{z_0, \ldots, z_n\}$ such that

$$p = [z_0, z_1] \cup \cdots \cup [z_{n-1}, z_n]. \tag{110}$$

Definition 5.17. Let $\gamma : [a,b] \longrightarrow D$ be an arc with a, b finite. We say γ *is a basic curve* if and only if $\gamma \in C^1((a,b)) \cap C([a,b])$ and there exist $\lim_{t\to a^+} \gamma'(t), \lim_{t\to b^-} \gamma'(t)$.

Definition 5.18. A *path* is a function $\gamma : [a,b] \longrightarrow \mathbb{C}$ such that there exist basic curves $\gamma_j : [a_j, b_j] \longrightarrow \mathbb{C}, j \in \{1, \ldots, k\}$ such that $\gamma = \gamma_1 + \cdots + \gamma_k$ and therefore $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ and $a = a_1, b = a_k$.

Definition 5.19. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous curve and $a_1, \ldots, a_l \in \mathbb{R}$ such that $a = a_0 \leq \cdots \leq a_l \leq b = a_{l+1}$. We say γ *is piece-wise differentiable* if and only if

$$\gamma \in C^1\left(\bigcup_{j=0}^{l} (\alpha_j, \alpha_{j+1})\right),$$

$\forall j \in \{0, \ldots, l+1\} \exists \lim_{t\to a_j^+} \gamma'(t)(\text{except if } j = l+1), \lim_{t\to a_j^-} \gamma'(t)(\text{except if } j = 0)$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

Theorem 5.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with Ω an open set and $\phi : I \longrightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curev (hence a derivable curve) and its real derivative is*

$$\psi'(t) = f'(\phi(t))\phi'(t). \tag{111}$$

Definition 5.20. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two curves. We say γ_1, γ_2 *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = \gamma_1(0) = \gamma_2(0), s \in [0,1]$.
4. $h(1,s) = \gamma_1(1) = \gamma_2(1), s \in [0,1]$.

Definition 5.21. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two circuits. We say γ_1, γ_2 *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = h(1,s), s \in [0,1]$.

Definition 5.22. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of f over γ* as

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\,. \tag{112}$$

Proposition 5.2. *The previous definition is well defined.*

Definition 5.23. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of f over γ with respect the differential of length* as

$$\int_\gamma f(z)\,\mathrm{d}s := \int_\gamma f(z)|\mathrm{d}z| = \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t\,. \tag{113}$$

Theorem 5.3. *Let γ_1, γ_2 be two equivalent curves of the same orientation and of class C^1 on their respective domains and$f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z\,. \tag{114}$$

Proposition 5.4. *Let $\gamma_1, \ldots, \gamma_n$ be n curves of class C^1 on their respective domains and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \ldots, \Gamma_n \subseteq \Omega$. If we define $\gamma = \gamma_1 + \cdots + \gamma_n$, then*

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,\mathrm{d}z\,. \tag{115}$$

Proposition 5.5. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \int_\gamma |f|\,\mathrm{d}s\,. \tag{116}$$

Corollary 5.6. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. If $|f(z)| \leq M$ for all $z \in \Gamma$, then,*

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq ML(\gamma). \tag{117}$$

Proposition 5.7. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\mathrm{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z}\,\mathrm{d}w\,. \tag{118}$$

Proposition 5.8. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$|\mathrm{Ind}(\gamma, z)| \leq \frac{1}{2\pi} \frac{L(\gamma)}{|z - \Gamma|}. \tag{119}$$

Proposition 5.9. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $\{f_n\}_{n=0}^{\infty}$ a sequence of continuous functions on Γ such that $\sum_{n=0}^{\infty} f_n$ converges uniformly on Γ. Then, $\sum_{n=0}^{\infty} \int_\gamma f_n\,\mathrm{d}z$ converges and*

$$\int_\gamma \sum_{n=0}^{\infty} f_n(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} \int_\gamma f_n(z)\,\mathrm{d}z\,. \tag{120}$$

Theorem 5.10. *Let Ω be a bounded domain with piece-wise regular boundary positively oriented and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an holomorphic function in a neighborhood of $\bar{\Omega}$. Then,*

$$\int\limits_{\partial\Omega} f(z)\,\mathrm{d}z = 0. \tag{121}$$

Theorem 5.11 (Morera's theorem)**.** *Let f be a continuous function in a region Ω. If*

$$\oint\limits_{\gamma} f(z)\,\mathrm{d}z = 0 \tag{122}$$

for all simple and closed curve γ such that $\Gamma \subseteq \Omega$, then f is analytic on Ω.

6 Local properties of holomorphic functions

7 Isolated singularities of holomorphic functions

8 Homology

9 Harmonic functions

Theorem 9.1. *Let $f \in H(\Omega), C^1(\Omega)$ be a function. If $f = u + iv$, then u, v are harmonic functions on Ω.*

10 Conforming representation

11 Riemann theorem

12 Runge theorem

13 Zeros of holomorphic functions

Theorem 13.1 (Weierstrass Factorization Theorem)**.** *content...*

14 Fourier transform

Definition 14.1. Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform of f at the point ξ* as

$$\boxed{\hat{f}(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x\,.} \tag{123}$$

Proposition 14.1. *Let $f \in L^1(\mathbb{R})$ be a function. Then, the application*

$$\begin{aligned} \mathscr{F}\{f\} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \hat{f}(\xi) \end{aligned} \tag{124}$$

is a well defined application.

Definition 14.2. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \le p \le \infty$. We say *the functions f_n converge to f with a norm $\|\cdot\|_p$* or *converge in $L^p(\mathbb{R})$* if and only if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0. \tag{125}$$

Theorem 14.2. *Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.*

1. *The Fourier transform of f satisfies*
$$\mathscr{F}\{f\} \in L^\infty(\mathbb{R}), \qquad \|\mathscr{F}\{f\}\|_\infty \le \frac{1}{\sqrt{2\pi}}\|f\|_1. \tag{126}$$
2. *$\mathscr{F}\{f\}$ is $\mathbb{C}$ linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,*
$$\mathscr{F}\{\alpha f + \beta g\} = \alpha\mathscr{F}\{f\} + \beta\mathscr{F}\{g\}. \tag{127}$$
3. *If $g(x) = \bar{f}(x)$, then for all $\xi \in \mathbb{R}$*
$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \tag{128}$$
4. *If $g(x) = g(\lambda x)$ and $\lambda \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*
$$\hat{g}(\xi) = \frac{1}{|\lambda|}\hat{f}\left(\frac{\xi}{\lambda}\right). \tag{129}$$
5. *If $g(x) = f(x - a)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*
$$\hat{g}(\xi) = \mathrm{e}^{-ia\xi}\hat{f}(\xi). \tag{130}$$
6. *If $g(x) = \mathrm{e}^{iax}f(x)$ with $\alpha \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*
$$\hat{g}(\xi) = \hat{f}(\xi - a) \tag{131}$$
7. *If $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$ when $n \to \infty$, then $\mathscr{F}\{f_n\} \to \mathscr{F}\{f\}$ uniformly in $\mathbb{R}$.*
8. *The Fourier transform $\mathscr{F}\{f\}$ is a continuous function in $\mathbb{R}$, $\mathscr{F}\{f\} \in C(\mathbb{R})$.*

Proposition 14.3. *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and* $\lim_{|x|\to\infty} |f(x)| = 0$. *Then,*

$$\hat{f'}(\xi) = i\xi\hat{f}(\xi). \tag{132}$$

Corollary 14.4. *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its n-th derivative $f^{(n)} \in L^1(\mathbb{R})$ and* $\lim_{|x|\to\infty} |f(x)| = 0$. *Then,*

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n\hat{f}(\xi). \tag{133}$$

Definition 14.3. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We define the support of f as

$$\operatorname{supp} f \coloneqq \overline{\{x \in I \mid f(x) \neq 0\}}. \tag{134}$$

Definition 14.4. We define the set $\mathscr{D}(\mathbb{R})$ as

$$\mathscr{D}(\mathbb{R}) := \{\varphi \in C^{\infty}(\mathbb{R}) \mid \operatorname{supp}\varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \tag{135}$$

Theorem 14.5. *Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_{\in}\mathscr{D}(\mathbb{R})$ such that*

$$\lim_{h\to\infty} \int_{\mathbb{R}} |f - \phi_n| \, \mathrm{d}x = 0, \tag{136}$$

that is, we have convergence of ϕ_n to f with norm $\|\cdot\|_1$.

Proposition 14.6. *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.*

Proposition 14.7. *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\left|\hat{f}(\xi)\right| \leq \|f\|_1$.*

Theorem 14.8. *Let $f \in L^1(\mathbb{R})$ be a function. Then,*

$$\lim_{|x|\to\infty} \hat{f}(x) = 0. \tag{137}$$

Theorem 14.9. *The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.*

Definition 14.5. We define the *Schwartz space* as

$$S(\mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{C} \mid f \in C^{\infty}(\mathbb{R}) \wedge \forall n, m \in \mathbb{N}\, \exists c_{n,m} < \infty \text{ such that } (1+|x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}.$$

Proposition 14.10. *Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \longrightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,*

1. *$f + g \in S(\mathbb{R})$.*
2. *$\lambda f \in S(\mathbb{R})$.*
3. *$fg \in S(\mathbb{R})$.*
4. *$Pf \in S(\mathbb{R})$.*

Theorem 14.11. *Let $I, J \subseteq \mathbb{R}$ be two intervals with I compact and J open. Let $f : I \times J \longrightarrow \mathbb{R}$ be a function such that*

1. *$f(\cdot, \lambda)$ is Riemann-integrable in I for all $\lambda \in J$,*
2. *$f(x, \cdot)$ is derivable in J for all $x \in I$.*

If $\partial_\lambda f$ is continuous in $I \times J$, then

1. *$\partial_\lambda f(\cdot, \lambda)$ is Riemann-integrable for all $\lambda \in J$.*
2. *$F(\lambda) = \int_I f(x, \lambda)\, \mathrm{d}x$ is derivable with continuous derivative in J for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.*

$$F'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int_I f(x, \lambda_0)\, \mathrm{d}x = \int_I \frac{\partial f}{\partial \lambda}(x, \lambda_0)\, \mathrm{d}x\,, \forall \lambda_0 \in J. \tag{138}$$

Proposition 14.12. *Let $f \in S(\mathbb{R})$. Then,*

1. *$S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.*
2. *$\widehat{xf}(\xi) = (iD_\xi \hat{f})(\xi)$ for all $\xi \in \mathbb{R}$.*

Corollary 14.13. *Let $f \in s(\mathbb{R})$. Then,*

$$\widehat{x^n f}(\xi) = (i^n D^n \hat{f})(\xi), \forall n \in \mathbb{N}. \tag{139}$$

Proposition 14.14. *The Fourier transform $\mathscr{F}$ restricted to $S(\mathbb{R})$ is an automorphism, that is, if $f \in S(\mathbb{R})$ then $\mathscr{F}\{f\} = \hat{f} \in S(\mathbb{R})$.*

Lemma 14.15. *If $G(x) = \mathrm{e}^{-x^2/2}$, then $\hat{G}(\xi) = e^{-\xi^2/2}$. We observe hence that G is a fixed point of $\mathscr{F}$.*

Lemma 14.16. *If $f, g \in S(\mathbb{R})$, then*

$$\int_{\mathbb{R}} f(\xi)\hat{g}(\xi)\, \mathrm{d}\xi = \int_{\mathbb{R}} \hat{f}(\tau) g(\tau)\, \mathrm{d}\tau\,. \tag{140}$$

Lemma 14.17. *Let $f, g \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,*

1. *$g(\lambda x)\hat{f}(x)$ converges to $g(0)\hat{f}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*
2. *$f(\lambda x)\hat{g}(x)$ converges to $f(0)\hat{g}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*

Lemma 14.18. *Let $f, g \in s(\mathbb{R})$. Then,*

$$f(0) \int_{\mathbb{R}} \hat{g}(\xi)\, \mathrm{d}\xi = g(0) \int_{\mathbb{R}} \hat{f}(\xi)\, \mathrm{d}\xi\,. \tag{141}$$

Lemma 14.19. *Let $f \in s(\mathbb{R})$ be a function. Then,*

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)\, \mathrm{d}\xi\,. \tag{142}$$

Corollary 14.20 (Inversion formula)**.** *Let $f \in S(\mathbb{R})$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) \mathrm{e}^{ix\xi}\, \mathrm{d}\xi\,, \forall x \in \mathbb{R}. \tag{143}$$

Theorem 14.21 (Inversion of $\mathscr{F}$ in $S(\mathbb{R})$)**.** *Let $\mathscr{F} : S(\mathbb{R}) \longrightarrow S(\mathbb{R})$, defined by $\mathscr{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, $\mathscr{F}$ is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathscr{F}^4 = Id$. In particular, $\mathscr{F}^{-1} = \mathscr{F}^3$ and if $f \in S(\mathbb{R})$, then*

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{F}\{f\}(\xi) \mathrm{e}^{ix\xi}\, \mathrm{d}\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} \mathrm{e}^{ix\xi}\, \mathrm{d}\xi\,.} \tag{144}$$

In fact, $\mathscr{F}$ is an homemorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(S(\mathbb{R}), \|\cdot\|_{n,m})$.

Theorem 14.22 (Inversion of $\mathscr{F}$ for discontinuities)**.** *Let f be a absolutely Riemann-integrable function in $\mathbb{R}$ with f and f' piece-wise continuous. Then,*

$$\frac{f(x^-) + f(x^+)}{2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} \mathrm{e}^{ix\xi}\, \mathrm{d}\xi\,. \tag{145}$$

Definition 14.6. Let f be a Riemann-integrable function in $\mathbb{R}$. We define the *Fourier transform of cosine kind* as

$$\hat{f}_c(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\xi x) f(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(\xi x) f_e(x)\,\mathrm{d}x\,, \tag{146}$$

and the *Fourier transform of sine kind* as

$$\hat{f}_s(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\xi x) f(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(\xi x) f_o(x)\,\mathrm{d}x\,. \tag{147}$$

Proposition 14.23. *Let $\hat{f}_c, \hat{f}_s$ be the Fourier transform of cosine and sine kinds of f. Then, $\hat{f}_c(\xi)$ is even, $\hat{f}_s(\xi)$ is odd, and $\hat{f}(\xi) = \hat{f}_c(\xi) - i\hat{f}_s(\xi)$.*

Theorem 14.24. *Let f be a absolutely Riemann-integrable function in $\mathbb{R}$ with f and f' piece-wise continuous. Then,*

$$\frac{f_e(x^-) + f_e(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_c \cos(\xi x)\,\mathrm{d}\xi\,, \tag{148}$$

$$\frac{f_o(x^-) + f_o(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_s \sin(\xi x)\,\mathrm{d}\xi\,. \tag{149}$$

Theorem 14.25 (Tonelli's Theorem)**.** *Let $f : I \times J \longrightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x,y) \geq 0$ for all $(x,y) \in I \times J$. Then,*

$$\int_{I\times J} f\,\mathrm{d}x\,\mathrm{d}y = \int_I \int_J f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_J \int_I f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{150}$$

Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.

Corollary 14.26. *Let $f, g \in L^1(\mathbb{R})$. Then, $F(x,t) = f(t)g(x-t) \in L^1(\mathbb{R}^2)$.*

Definition 14.7. Let $f, g \in L^1(\mathbb{R})$ two function. We define the *convolution of f and g* as

$$\begin{aligned} (f * g) : \mathbb{R} &\longrightarrow \mathbb{C} \\ x &\longmapsto \int_{\mathbb{R}} f(t) g(x-t)\,\mathrm{d}t \end{aligned}, \tag{151}$$

which is from $L^1(\mathbb{R})$.

Proposition 14.27. *Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$.*

Proposition 14.28. *Let $f \in L^1(\mathbb{R})$ be a function and $g = f^2$. Then,*

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \hat{f} * \hat{f} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)\hat{f}(\xi - t)\,\mathrm{d}t\,. \tag{152}$$

Theorem 14.29. *Let $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^\infty(\mathbb{R})$.*

Theorem 14.30. *Let $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$ with $\operatorname{supp} f$ compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and $\operatorname{supp}\{f * \phi\} \subseteq \operatorname{supp} f + \operatorname{supp} \phi$.*

Definition 14.8. We say the functions $\phi_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ continuous in a compact support are *an approximation of the unity* if and only if

1. $\phi_\epsilon \geq 0$ for all ϵ.
2. $\displaystyle\int_{\mathbb{R}} \phi_\epsilon(x)\,\mathrm{d}x = 1$.
3. For all $\delta > 0$ it is satisfied that

$$\lim_{\epsilon \to 0} \left\{ \sup_{|t|>\delta} \phi_\epsilon(t) \right\} = 0. \tag{153}$$

Theorem 14.31. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to f.*

Corollary 14.32. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to f.*

Theorem 14.33 (Weierstrass polynomial approximation)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials P_n with $n \in \mathbb{N}$ such that P_n converge uniformly to f in $[a,b]$.*

Theorem 14.34. *Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \to f$ with norm $\|\cdot\|_p$ (that is, convergence in L^p), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then*

$$\lim_{n\to\infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \tag{154}$$

with $\|f\|_{C^k(\mathbb{R})} = \max_{0 \leq l \leq k} \left(\sup_{x\in\mathbb{R}} \left|D^l f(x)\right|\right)$ being a norm.

Lemma 14.35. *Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\displaystyle\int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x = 0$. Then, $f \equiv 0$.*

Corollary 14.36. *The Fourier transform $\mathscr{F}$ is injective since $\mathscr{F}\{f\} = \hat{f} = 0 \Leftrightarrow f = 0$ in $L^1(\mathbb{R})$ (the zero function class) and $\mathscr{F}$ is a linear application.*

Theorem 14.37 (Inversion theorem in $L^1(\mathbb{R})$)**.** *Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) \mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,, \forall x \in \mathbb{R}. \tag{155}$$

15 Fourier transform 2

Theorem 15.1 (Parseval formula)**.** *Let $f, g \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be two functions. Then,*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi\,. \tag{156}$$

Theorem 15.2 (Plancherel Theorem)**.** *Let $f \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be a function. Then,*

$$\boxed{\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \,,} \tag{157}$$

that is, $\|f\|_2 = \left\|\hat{f}_2\right\|$ and $\mathscr{F}$ is an isometry between vector spaces.

Definition 15.1. Let $f \in S(\mathbb{R})$ be a function. We define the following quantities

$$E(f) \coloneqq \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \,, \tag{158}$$

$$\sigma(f)^2 \coloneqq \int_{\mathbb{R}} |xf(x)|^2 \, \mathrm{d}x \,. \tag{159}$$

Theorem 15.3. *Let $f \in S(\mathbb{R})$ be a function. Then,*

$$\sigma(f)\sigma(\hat{f}) \geq \frac{E(f)}{2}. \tag{160}$$

16 Multidimensional fourier transform

Theorem 16.1. *For several variables*

$$\mathscr{F}\{f(x_1, \ldots, x_n)\} = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) \mathrm{e}^{-i(x_1\xi_1 + \cdots + x_n\xi} \tag{161}$$

or simpler,

$$\boxed{\mathscr{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{e}^{-i\langle \mathbf{x}, \boldsymbol{\xi} \rangle_I} \, \mathrm{d}\hat{x} \,.} \tag{162}$$