Contents

1 Arithmetic and topology **3**
1.1 Topology 4
1.1.1 Representation 6
1.1.2 Geometric interpretation of addition and multiplication 7
1.1.3 Roots of a complex number 7
1.2 Argument and index 7

2 Sequences and limits **9**
2.1 Sequences 10
2.1.1 Riemann sphere 10

3 Complex functions **11**
3.1 Introduction 12
3.2 Functions 13
3.3 Continuity 14
3.4 Complex exponential function 14
3.4.1 Exponential form 16
3.4.2 Transformations 16
3.5 Complex trigonometric functions 16
3.6 Hyperbolic functions 17
3.7 Logarithm 18
3.7.1 Inverse trigonometric and hyperbolic functions 18
3.8 Complex power 19
3.9 Riemann surfaces 19
3.9.1 Logarithm 20
3.9.2 Power 20

4 Derivatives **21**
4.1 Introduction 22
4.2 Cauchy-Riemann Equations 23
4.2.1 The operators $\boldsymbol{\partial}$ and $\overline{\boldsymbol{\partial}}$ 24

5 Series **25**
5.1 Numeric series 26
5.2 Function series 26
5.3 Power series 27
5.4 Analytic functions and Taylor series 28

6 Holomorphic functions and differential forms **31**
6.1 Complex line integrals 32
6.1.1 Curves, paths, and arcs 32
6.1.2 Integrals 34
6.2 Fundamental theorem of complex calculus 35
6.3 Cauchy's theorem 36
6.4 Holomorphic functions as vector fields and harmonic functions 37

6.4.1 Solenoidal fields . . . 37
6.4.2 Holomorphic vector fields. Harmonic functions . . . 37
6.4.3 Harmonic conjugate functions . . . 37

7 Local properties of holomorphic functions 39
7.1 Cauchy's integral formula . . . 40
7.2 Analytic functions and holomorphic functions . . . 40

8 Isolated singularities of holomorphic functions 43
8.1 Isolated singular points . . . 44
8.2 Laurent expansion . . . 44

9 Homology and holomorphic functions 45

10 Harmonic functions 47

11 Conforming representation 49

12 Riemann's theorem and Dirichlet's problem 51

13 Runge's theorem and Cauchy-Riemann equations 53

14 Zeros of holomorphic functions 55

15 Complex Fourier transform 57
15.1 Introduction . . . 58
15.2 Fourier inversion theorem in Schwartz space . . . 61
15.3 Inversion theorem in the algebra $L^1(\mathbb{R})$. Approximation theory . . . 62
15.3.1 Examples . . . 64

16 Complex Fourier transform in $L^2(\mathbb{R})$ 65
16.1 Examples . . . 66

17 Multidimensional Fourier transform 67

Chapter 1

Arithmetic and topology

Definition 1.0.1. Let $\mathbb{R}^2 = \{(x.y) \mid x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a,b),(c,d) \in \mathbb{R}^2$ we define the sum "+" by components
$$(a,b)+(c,d) \coloneqq (a+b, c+d). \tag{1.1}$$
- Product: given two $(a,b),(c,d) \in \mathbb{R}^2$ we define the product by
$$(a,b)(c,d) = (ac-bd, ad+bc). \tag{1.2}$$

We define the set $\mathbb{C}$ as $(\mathbb{R}^2, +,)$.

Proposition 1.0.1. *The set $\mathbb{C}$ of complex numbers is an abelian field.*

This is only one possible formulation, but we will use another that, as we will prove now, it is completely equivalent. Now, we define $\mathbb{C} = \left\{a+bi \,\middle|\, a,b \in \mathbb{R}, i^2 = -1\right\}$, and we use the addition and subtraction as before:

$$\begin{gathered}(a+bi)+(c+di) = (a+c)+(c+d)i \\ (a+bi)(c+di) = \cdots = (ac-bd)+(ad+bc)i.\end{gathered}$$

To see both formulations are equivalent, we can express the second in terms of the first only adding the definition $i = (0,1)$. This way, every element (x,y) can be expressed ad $x+iy$ as follows.

$$z = (x,y) = (x,0)(1,0)+(y,0)(0,1) = x\cdot 1 + y \cdot i = x`iy \tag{1.3}$$

Proposition 1.0.2. *Let $\mathbb{C}$ be defined in the second way. Then,*

1. *$\mathbb{C}$ is an abelian ring.*
2. *If we define f as*
$$\begin{aligned} f : (\mathbb{C}, +,) &\longrightarrow (\mathbb{R}^2, +,) \\ (x,y) &\longmapsto x+yi \end{aligned}, \tag{1.4}$$
then f is a morphism of rings.
3. *The function f is, in fact, an isomorphism and $\mathbb{C}$ is an abelian field.*

From this isomorphism, we see two complex numbers are equal if and only if $a = a'$ and $b = b'$. Besides, $(\mathbb{C}^*,)$ is an abelian group. In order to simplify the expressions, the following notation will be used.
$$zw = z\cdot w, \qquad z-w = z+(-w), \qquad 1/z = z^{-1}, \qquad z/w = zw^{-1} \tag{1.5}$$

Proposition 1.0.3. *The subset of $\mathbb{C}$ generated by numbers of the form $\underline{x} = (x,0)$ is isomorph to the set of real numbers.*

Theorem 1.0.4. *$\mathbb{C}$ is not an ordered field.*

1.1 Topology

Definition 1.1.1. Let $z = a+bi \in \mathbb{C}$. We define the *conjugate of z* as
$$\bar{z} \coloneqq a-bi. \tag{1.6}$$

Proposition 1.1.1. *For all $z,w \in \mathbb{C}$, we have:*

1. *$\bar{\bar{z}} = z$.*
2. *$\overline{z+w} = \bar{z}+\bar{w}$.*
3. *$\overline{zw} = \bar{z}\bar{w}$.*

4. *$z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.*

5. *$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.*

6. *The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.*

Definition 1.1.2. Let $z = a + bi \in \mathbb{C}$. We define the *real part of z and imaginary part of z* respectively as

$$\mathrm{Re}\{z\} := a, \qquad \mathrm{Im}\{z\} := b. \tag{1.7}$$

Proposition 1.1.2. *Let $z \in \mathbb{C}$. Then,*

$$\mathrm{Re}\{z\} = \frac{z + \bar{z}}{2}, \qquad \mathrm{Im}\{z\} = \frac{z - \bar{z}}{2i} \tag{1.8}$$

Proposition 1.1.3. *Let $z, w \in \mathbb{C}$ and the following distance function.*

$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto \tilde{d}(z, w) := |z - w| \end{aligned} \tag{1.9}$$

Then, $(\mathbb{C}, \tilde{d})$ is a metric space.

Definition 1.1.3. Let $z = a + bi \in \mathbb{C}$. We define the *modulus of z* as

$$|z| := \tilde{d}(z, 0), \tag{1.10}$$

which is equivalent to $\sqrt{z\bar{z}}$.

Definition 1.1.4. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define an *open disc of radius r and center z_0* as follows

$$B_r(z_0) := \{z \in \mathbb{C} \,|\, |z - z_0| < r\}. \tag{1.11}$$

Definition 1.1.5. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *punctured disc of radius r and center z_0* as follows

$$B_r^*(z_0) := \{z \in \mathbb{C} \,|\, 0 < |z - z_0| < r\}. \tag{1.12}$$

Definition 1.1.6. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *closed disc of radius r and center z_0* as follows

$$\overline{B_r(z_0)} := \{z \in \mathbb{C} \,|\, |z - z_0| \leq r\}. \tag{1.13}$$

Definition 1.1.7. We denote by $\mathbb{D}$ the unitary disc of center 0 and radius 1. Besides, we denote by $\mathbb{T} \subseteq \mathbb{C}$ the unitary circumference, that is,

$$\mathbb{T} := \{z \in \mathbb{C} \,|\, |z| = 1\}. \tag{1.14}$$

We also denote it by $\mathbb{S}^1$.

Lemma 1.1.4. *The set $B = \left\{B_r(z_0) \,\middle|\, r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\right\}$ is a basis of the topology of $\mathbb{R}^2$ as a metric space.*
The set $D = \{D_r(z_0) \,|\, r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of $\mathbb{C}$ as a metric space.

The concepts of interior, exterior, boundary, and accumulation points are the same than those presented in Multivariable Calculus Notes. The same for the rest of topological definitions.

Proposition 1.1.5. *The sets $\mathbb{C}$ and $\mathbb{R}^2$ with the topology of metric space are homeomorphs.*

Corollary 1.1.6. *There is a bijection between B and D, that is, between balls of $\mathbb{R}^2$ and discs of $\mathbb{C}$.*

Proposition 1.1.7. *Let $z, w \in \mathbb{C}$. Then,*

1. *$|z| \geq 0$.*

2. *$|z| = 0 \Leftrightarrow z = 0$.*

3. $-|z| \leq \mathrm{Re}\{z\} \leq |z|$ *and* $-|z| \leq \mathrm{Im}\{z\} \leq |z|$.

4. $|zw| = |z||w|$.

5. *If* $w \neq 0$, $|z/w| = |z|/|w|$.

6. $|z+w| \leq |z| + |w|$.

7. $|z+w| \geq ||z| - |w||$.

8. $|\mathrm{Re}\{zw\}| \leq |z||w|$ *and* $|\mathrm{Im}\{z\}| \leq |z||w|$.

9. $|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\,\mathrm{Re}\{z\bar{w}\}$.

10. $|z^n| = |z|^n$

Corollary 1.1.8. *Let* $z_1, \ldots, z_n \in \mathbb{C}$. *Then,*

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|, \qquad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \qquad |\mathrm{Re}\{z_1 \cdots z_n\}| \leq |z_1| \cdots |z_n|. \tag{1.15}$$

1.1.1 Representation

By the proposition 1.1.5, we can identify the elements of $\mathbb{C}$ as the elements of $\mathbb{R}^2$, so we can represent them in the same way. The plane used to represent complex numbers is called the Argand plane, where the real part the abscissa axis and the imaginary part in the ordinate axis [].

Definition 1.1.8. Let $z \in \mathbb{C}^*$. We define the *argument of* z, denoted by $\arg z$, as the real number θ such that $z = |z|(\cos\theta + i\sin\theta)$. Let us observe that $\arg z$ is not a function but a multivalued application.
We define the *principal argument of* z as

$$\mathrm{Arg} z \coloneqq \theta_0 \in [0, 2\pi) \mid z = |z|(\cos\theta + i\sin\theta). \tag{1.16}$$

In general, to make θ to be unique, it is enough to impose it to belong to a certain semiopen interval of length 2π. Choosing the interval I is called by *taking a determination of the argument.*

Another common convention is the interval $(-\pi, \pi]$. If we take now $\arg_I : \mathbb{C}^* \longrightarrow I$ where $\arg_I$ is the unique value of $\arg z$ such that it belongs to I, the $\arg_I(z)$ is a function but not continuous. If we have an argument determination with $I = [\varphi_0, \varphi_0 + 2\pi)$, then $\arg_I(z)$ is discontinuous at the closed semiline that forms an angle φ_0 with the real positive semiaxis. SEE BOOK FROM THE BIBLIO———-

Definition 1.1.9. Given a complex number z that we can express by $z = |z|(\cos\theta + i\sin\theta)$ for some $\theta \in \mathbb{R}$, we use the notation $r = |z|$ to write

$$z = r_\theta^z = r(\cos\theta + i\sin\theta) \tag{1.17}$$

or simply r_θ when it is obvious which complex number are we referring to. We call it *polar form of* z.

Sometimes we use the notation $\arg z$ to design the set of all arguments of z. If θ is one of the arguments of z, then

$$\arg z = \{\theta + 2\pi k \mid k \in \mathbb{Z}\}\,. \tag{1.18}$$

Formally $\arg z$ is, hence, an equivalence class of $\mathbb{R}/2\pi\mathbb{Z}$. Informally, we say $\arg z$ is determined except by multiples of 2π.

1.1.2 Geometric interpretation of addition and multiplication

Since both $\mathbb{C}$ and $\mathbb{R}^2$ can be represented in a plane and addition between complex numbers is defined like addition between vectors, the geometric visualization is the same as that from addition of vectors. Hence, complex numbers can be seen as "arrows" whose addition obeys the parallelogram rule. With respect to multiplication, let us take two complex numbers z_1, z_2 in polar form.

$$z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1) = r_{\theta_1}^{z_1}, \quad z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2) = r_{\theta_2}^{z_2}$$

If we compute now the product and use some trigonometric identities,

$$z_1 z_2 = |z_1||z_2|(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) = \\ |z_1||z_2|(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) = r_{\theta_1+\theta_2}^{z_1 z_2}.$$

Then, we multiply two complex numbers, angles are added and modules are multiplied.

Proposition 1.1.9. *Let $z \in \mathbb{C}$ and r_θ its polar form. Then,*

$$z^n = (r^n)_{n\theta}. \tag{1.19}$$

Corollary 1.1.10 (De Moivre's Formula)**.** *Let $\theta \in \mathbb{R}$. Then,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{1.20}$$

Proposition 1.1.11. *Let $z, w \in \mathbb{C}$. Then,*

1. $\arg zw = \arg z + \arg w + 2\pi k$.
2. $\arg z^n = n\arg z + 2\pi k$.

Unitary terms

Definition 1.1.10. We denote the complex numbers z generated by moving the point $z_0 = 1$ around $\mathbb{T}$ a length t in a counter-clockwise direction by 1_t. In other words, 1_t are the complex numbers $z = \cos t + i\sin t$.

Proposition 1.1.12. *Let $f : t \longrightarrow 1_t$. Then, f is a morphism from $(\mathbb{R}, +)$ to $(\mathbb{T}, \cdot)$, with $\ker f = 2\pi\mathbb{Z}$.*

1.1.3 Roots of a complex number

Definition 1.1.11. Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $w \in \mathbb{C}$ is an *n-th root of z* if and only if

$$w^n = z. \tag{1.21}$$

Theorem 1.1.13. *Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \dots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \dots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \dots, n\}$.*

From that we can see

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right], \qquad k = 0, \dots, n-1. \tag{1.22}$$

1.2 Argument and index

Theorem 1.2.1. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous curve such that $\gamma(t) \neq 0 \forall t \in [a,b]$. Then, there exists a continuous determination ϕ of the argument of γ. Then, $\phi(t) + 2\pi k$ with $k \in \mathbb{Z}$ is the general expression of all the argument determinations of γ. If γ is differentiable, then ϕ is differentiable and $\phi' = \operatorname{Im}\{\gamma'/\gamma\}$.*

Definition 1.2.1. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a regular curve. We define the *variation of the argument along* γ as

$$\Delta_\gamma \arg \coloneqq \operatorname{Im}\left\{ \int_a^b \frac{\gamma'(t)}{\gamma(t)} \,\mathrm{d}t \right\}. \tag{1.23}$$

Definition 1.2.2. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve such that $\gamma(t) \neq 0 \forall t \in [a,b]$. Then, we define the *index of* γ *with respect to the origin* or *the number of revolutions of* γ *around the origin*

$$\operatorname{Ind}(\gamma, 0) \coloneqq \frac{1}{2\pi} \Delta_\gamma \arg. \tag{1.24}$$

Proposition 1.2.2. *Let* $\gamma : [a,b] \longrightarrow \mathbb{C}$ *be a piece-wise regular curve. Then,*

$$\operatorname{Ind}(\gamma, 0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} \,\mathrm{d}t \tag{1.25}$$

Definition 1.2.3. Let γ be a closed curve and $z \notin \Gamma$. We define the *index of* γ *with respect to* z as

$$\operatorname{Ind}(\gamma, z) \coloneqq \operatorname{Ind}(\gamma - z, 0). \tag{1.26}$$

Proposition 1.2.3. *Let* $\gamma : [a,b] \longrightarrow \mathbb{C}$ *be a curve piece-wise of class* $C^1([a,b])$*. Then,*

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} \,\mathrm{d}t\,. \tag{1.27}$$

Proposition 1.2.4. *Let* $\gamma : [a,b] \longrightarrow \mathbb{C}$ *be a piece-wise of class* $C^1([a,b])$*. Then,* $\operatorname{Ind}(-\gamma, z) = -\operatorname{Ind}(\gamma, z)$.

Chapter 2

Sequences and limits

2.1 Sequences

Definition 2.1.1. A *sequence of complex numbers* is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned}. \tag{2.1}$$

We denote it by $\{z_n\}_{n=m}^{\infty}$

Definition 2.1.2. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *the sequence has limit* L or *it converges to the limit* L if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \tag{2.2}$$

We denote it by

$$\lim_{h\to\infty} z_n = L, \qquad \lim \{z_n\}_{n=0}^{\infty} = L, \qquad \{z_n\}_{n=0}^{\infty} \to L. \tag{2.3}$$

Theorem 2.1.1. *Let $z_n = z_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$ and $L = L_x + iL_y \in \mathbb{C}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \to L \Leftrightarrow \{x_n\}_{n=0}^{\infty} \to L_x \wedge \{y_n\}_{n=0}^{\infty} \to L_z. \tag{2.4}$$

Definition 2.1.3. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *it tends to infinity* and denote it by $\lim z_n = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \tag{2.5}$$

Definition 2.1.4. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \tag{2.6}$$

Theorem 2.1.2. *Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.*

Theorem 2.1.3. *Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \textit{ is a Cauchy sequence } \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \textit{ are Cauchy sequences.} \tag{2.7}$$

Theorem 2.1.4. *The field $\mathbb{C}$ of complex numbers is complete.*

2.1.1 Riemann sphere

According to the definition of a sequence tending to infinity, we mean that the absolute value of z gets arbitrarily big. Another way to formulate it is by the equivalence

$$\lim z_n = \infty \Leftrightarrow \lim \frac{1}{z_n} = 0. \tag{2.8}$$

Notice that with this comparison there is a *unique infinity* in the complex plane, that we can describe as the infinitely far horizon. To see graphically the relation presented above, we can use the *Riemann sphere*.

Definition 2.1.5. The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, together with two charts.

The straight line that joint the *north pole* with a point of the plane determines a unique point of the spherical surface and establishes a bijective correspondence between points of the plane and of the sphere, except for the north pole that has no actual correspondence.
Notice that a parallel line of the sphere becomes a circle centered at the origin in the plane. The points located at the south region of that line are transformed in point at the interior of the circle, while the point in the north region are now in the exterior. Besides, the higher the line, the bigger the circle. This way, we associate the *skullcap* infinitely near to the north pol to the exterior of the circle infinitely large, that is, the infinity of the complex plane.
We could wonder why in $\mathbb{R}$ there are two infinities instead of one as in $\mathbb{C}$. The reason is that $\mathbb{R}$ is an ordered field, and unifying the infinities would break this property (intuitively, the unique infinity would be greater and lower that every real number). Since $\mathbb{C}$ is not an ordered field we only talk about one infinity.

Chapter 3

Complex functions

3.1 Introduction

Definition 3.1.1. A *topology* is an ordered pair $(\mathbb{X}, \tau)$, where X is a set and τ a collection of subsets of X satisfying the following properties:

1. The empty set and X belong to τ.
2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ.
3. The intersection of any finite number of members of τ still belongs to τ.

The elements of τ are called *open sets* and the collection τ is called the *topology on* X.

Definition 3.1.2. Let $(\mathbb{X}, d)$ be a metric space. A *topology on the metric space by the metric* d is the set τ of all open sets of M.

Since we have seen $(\mathbb{C}, d)$ is a metric space, we can induce a topological space. Hence, $\mathbb{C}$ is a topological space and we can define all topological concepts. We will expose the general definitions for arbitrary topologies.

Points

Definition 3.1.3. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *interior point of* A if there is a ball $B_{(\mathbb{M},d)}(a,r) \subset A$.

Definition 3.1.4. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *exterior point of* A if there is a ball such that $B_{(\mathbb{M},d)}(a,r) \cup A = \emptyset$.

Definition 3.1.5. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is a *boundary point of* A if it is not interior or exterior or, which is equivalent, if every ball $B_{(\mathbb{M},d)}(a,r)$ contains elements of A and A^c.

Definition 3.1.6. Let A be a subset of a metric space $(\mathbb{M}, d)$ and a a point in $\mathbb{M}$. We say that a is an *accumulation point of* A if every ball with center a contains points of A different to a. In other words, every punctured ball satisfies $B^*_{(\mathbb{M},d)}(a,r) \cup A \neq \emptyset$.

Components of a set

Definition 3.1.7. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the interior of* A as the set of all interior points of A, and we denote it by $\text{int}(A)$.

Definition 3.1.8. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the exterior of* A as the set of all exterior points of A, and we denote it by $\text{ext}(A)$.

Definition 3.1.9. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the boundary of* A as the set of all boundary points of A, and we denote it by ∂A.

Definition 3.1.10. Let A be a subset of a metric space $(\mathbb{M}, d)$. We define *the closure of* A as the set of all accumulation points of A, and we denote it by $\bar{A}$.

Definition 3.1.11. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *an open set* if it contains none of its boundary points, that is, if $\partial A \cap A = \emptyset$.

Definition 3.1.12. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a closed set* if it contains all its boundary points, that is, if $\partial A \subseteq A$.

Note that a set can be both open and closed, if it has no boundary.

Definition 3.1.13. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a bounded set* if there exist a point $a \in \mathbb{M}$ and a positive real number r such that the ball $B_{(\mathbb{M},d)}(a,r)$ contains A.

Definition 3.1.14. Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. We say A is *a compact set* if it a bounded and closed set.

Proposition 3.1.1. *Let $(\mathbb{M}, d)$ be a metric space and A a subset of $\mathbb{M}$. Then, A is open if and only if A^c is closed.*

Now we can define some more specific concepts.

Definition 3.1.15. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally, Ω is connected if there are not two open sets $U, V \subseteq \mathbb{C}$ such that

$$U_1 = U \cap \Omega, \qquad V_1 = V \cap \Omega, \qquad U_1 \cap V_1 = \varnothing, \qquad U_1 \cup V_1 = \Omega. \tag{3.1}$$

Otherwise, we say Ω *is disconnected.*

Definition 3.1.16. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is simply connected* if and only if every circuit is homotopic in Ω to a point in Ω. Equivalently, is simply connected if and only if every pair of curves with the same extremes are homotopic.

Example 3.1.1. Every disc $D_r(z_0)$ is connected, but the union $D_{r_1}(z_1) \cup D_{r_2}(z_2)$ is disconnected.

Definition 3.1.17. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is convex* if and only if for all pair of point $a, b \in \Omega$, the segment defined by

$$[a, b] = \{z \mid z = (1-t)a + tb, 0 \leq t \leq 1\} \tag{3.2}$$

is contained in Ω, that is, if every pair of points can be connected by a straight line that belongs to the set.

Definition 3.1.18. Let $\Omega \in \mathbb{C}$ be a set. We say Ω *is a star-convex set* if and only if there exists $z_0 \in \mathbb{C}$ such that for all $z \in \Omega$ the segment $[z_0, z]$ is contained by Ω.

Definition 3.1.19. Let $(\mathbb{M}, d)$ be a metric space and $S \subseteq \mathbb{M}$ a set. We say S *is path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

Definition 3.1.20. Let $\Omega \in \mathbb{C}$ be a set. We say Ω *is a region or domain* if and only if it is open, non-empty, and connected.

Definition 3.1.21. Let $\Omega \subseteq \mathbb{C}$ be a non-empty set. We say $\Omega_1 \subseteq \Omega$ *is a connected component of* Ω if and only if it is a maximal connected subset, that is, if $z_0 \in \Omega_1$ and W is a connected subset of $\mathbb{C}$ that contains z_0, then $W \subseteq \Omega_1$.

3.2 Functions

Definition 3.2.1. Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function* f as the application

$$\begin{aligned} f : D \subseteq \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto w = f(z) \end{aligned}. \tag{3.3}$$

Definition 3.2.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say it *tends to infinity at the point* z_0 and denote it by $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \tag{3.4}$$

Definition 3.2.3. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We write $\lim_{z \to \infty} f(z) = L$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \tag{3.5}$$

Proposition 3.2.1. *Let $f_1, f_2 : \Omega \longrightarrow \mathbb{C}$ be two functions and z_0 a point such that $\lim_{z \to z_0} f_1 = w_1, \lim_{z \to z_0} f_2 = w_2$. Then,*

1. *$f_1 + f_2$ has also a limit and $\lim_{z \to z_0} f + g = w_1 + w_2$.*
2. *$f_1 f_2$ has also a limit and $\lim_{z \to z_0} fg = w_1 w_2$.*
3. *If $w_2 \neq 0$, then f/g has also a limit and $\lim_{z \to z_0} f/g = w_1/w_2$.*
4. *If $h(z)$ is a continuous function defined on a neighborhood of w_1, then $\lim_{z \to z_0} h(f_1(z)) = h(w_1)$.*

3.3 Continuity

Definition 3.3.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. We say f *is continuous in* z_0 if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \tag{3.6}$$

As we have mentioned before, we can characterize the function as $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$. This way, we can establish an equivalent criterion of continuity.

Proposition 3.3.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$ is continuous at z_0 if and only if $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are continuous at z_0.*

Proposition 3.3.2. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, f is continuous at z_0 if and only if for all sequence $\{z_n\}_{n=1}^{\infty}$ of Ω convergent at z_0 it is true that the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$.*

Proposition 3.3.3. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, λf, $f + g$, and fg are continuous at z_0. The function f/g is continuous at z_0 if $g(z_0) \neq 0$.*

Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. Since the image is a set of complex elements numbers, we can represent it as $f = u + iv$, where u, v are functions with the following form.

$$\begin{aligned} u : \Omega &\longrightarrow \mathbb{R} \\ z &\longmapsto \operatorname{Re}\{f(z)\} \end{aligned}, \qquad \begin{aligned} v : \Omega &\longrightarrow \mathbb{R} \\ z &\longmapsto \operatorname{Im}\{f(z)\} \end{aligned}.$$

Example 3.3.1. One of the most fundamental kinds of functions are the polynomials of complex variables in complex coefficients. If $a_0, a_1, \dots, a_n \in \mathbb{C}$, then the general expression is

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

By the Fundamental Theorem of Algebra there are some complex values $\alpha_1, \dots, \alpha_r$ and natural numbers $m_1, \dots, m_r$ such that

$$P(z) = a_n (z - \alpha_1)^{m_1} \cdots (z - \alpha_r)^{m_r}, \qquad m_1 + \cdots m_r = n.$$

Using again the identification between $\mathbb{C}$ and $\mathbb{R}^2$, we can interpret P as a two variable function $P(x, y)$, where $z = x + iy$. Separating the function in real and imaginary part, we get $P = P_1(x, y) + iP_2(x, y)$.

3.4 Complex exponential function

Definition 3.4.1. For all $z \in \mathbb{C}$, we define the *complex exponential function* as the following series

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{3.7}$$

Proposition 3.4.1. *The radius of convergence of* e^z *is infinite.*

Properties

Proposition 3.4.2. $\mathrm{e}^z = \mathrm{e}^x$ *for all* $x \in \mathbb{R}$.

Proposition 3.4.3. $\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$ *for all* $z, w \in \mathbb{C}$.

Proof.

$$\mathrm{e}^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} z^k w^{n-k} =$$
$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right) = \mathrm{e}^z \mathrm{e}^w.$$

∎

Proposition 3.4.4. *For all $z \in \mathbb{C}$ we have $\mathrm{e}^z \neq 0$.*

Proposition 3.4.5. *The image of e^z is $\mathbb{C}^*$.*

Proposition 3.4.6. *The derivative of e^z is e^z.*

Proposition 3.4.7. $\overline{\mathrm{e}^z} = \mathrm{e}^{\bar{z}}$.

Proposition 3.4.8. $|\mathrm{e}^z| = \mathrm{e}^{\mathrm{Re}\{z\}}$.

Proposition 3.4.9 (Euler's Formula)**.** *If $\theta \in \mathbb{R}$, then e^{xi} has modulus one and we have that*

$$\boxed{\mathrm{e}^{xi} = \cos x + i \sin x.} \tag{3.8}$$

Proof. To see that its modulus is one we can make the following calculation.

$$\left|\mathrm{e}^{xi}\right| = \sqrt{\mathrm{e}^{xi}\mathrm{e}^{-xi}} = \sqrt{\mathrm{e}^{xi-xi}} = \sqrt{\mathrm{e}^0} = \sqrt{1} = 1 \tag{3.9}$$

To see now that $\mathrm{e}^{xi} = \cos x + i \sin x$ for all $x \in \mathbb{R}$ we only need to use the power series of $\sin x$ and $\cos x$,

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{3.10}$$

Besides, by definition,

$$\mathrm{e}^{xi} = \sum_{n=0}^{\infty} \frac{(xi)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{y^n}{n!}. \tag{3.11}$$

It is not difficult to prove that

$$i^n = \begin{cases} (-1)^n, & \text{if } n \in 2\mathbb{Z} \\ i(-1)^n, & \text{if } n \notin 2\mathbb{Z} \end{cases}. \tag{3.12}$$

Since the series of sine and cosine are absolutely convergent in $\mathbb{R}$, we can separate the sum without modifying the result, getting

$$\mathrm{e}^{xi} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \cos x + i \sin y.$$

■

Corollary 3.4.10. *Let $z \in \mathbb{C}^*$. Then,*

$$z = |z|\mathrm{e}^{i\theta}, \tag{3.13}$$

with $\theta \in [0, 2\pi)$.

Proposition 3.4.11. *The following function*

$$\begin{aligned} \exp : (\mathbb{R}, +) &\longrightarrow (\mathbb{C}^*, \cdot) \\ x &\longmapsto \mathrm{e}^{xi} \end{aligned} \tag{3.14}$$

is a morphism of groups and its image is $\mathbb{T}$.

Proposition 3.4.12. *The complex exponential function is a periodic function with period $2\pi i$.*

Proposition 3.4.13. *Let $a \in \mathbb{C}^*$. Then, $\mathrm{e}^z = a$ has infinite solutions.*

Then, the exponential function is not injective.

Proposition 3.4.14. *The equation $\mathrm{e}^z = 0$ does not have solutions.*

3.4.1 Exponential form

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
This allows us to compute exponents in a faster way. If we have some number $a+ib \in \mathbb{C}$, then we can represent it in exponential form and finally pass $r^n e^{in\theta}$ to the original form.

3.4.2 Transformations

Proposition 3.4.15. *Let $y_0 \in \mathbb{C}$ be a numbers, $B := \{z \in \mathbb{C} \,|\, y_0 < \mathrm{Im}\{z\} < y_0 + 2\pi\}$ a set, and $f : B \longrightarrow \mathbb{C}^*$ be the exponential function. Then, f is bijective in B [1].*

Proposition 3.4.16. *Let $x_0, y_0, m \in \mathbb{C}$ be two numbers with $m \neq 0$ and f the exponential function [1]. Then,*

1. *f transforms the line $y = y_0$ to a line that starts at $z = 0$ and continues with an argument y_0 from the real positive axis.*

2. *f transforms the line $x = x_0$ to a circle centered at the origin and radius $r = \mathrm{e}^{x_0}$.*

3. *f transforms the line $y = mx$ to the parametric curve $z = \mathrm{e}^x \mathrm{e}^{imx}$ (a spiral).*

3.5 Complex trigonometric functions

Definition 3.5.1. Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric functions as*

$$\cos z := \frac{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}{2}, \tag{3.15}$$

$$\sin z := \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{2}, \tag{3.16}$$

$$\tan z := \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}. \tag{3.17}$$

Properties

Proposition 3.5.1. *For all $z \in \mathbb{C}$,*

$$\sin^2 z + \cos^2 z = 1. \tag{3.18}$$

Proposition 3.5.2. *For all $z \in \mathbb{C}$,*

$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z). \tag{3.19}$$

Proposition 3.5.3. *For all $z, w \in \mathbb{C}$,*

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \qquad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \tag{3.20}$$

Proposition 3.5.4. *The functions $\cos z, \sin z$ have period of 2π.*

An important difference of trigonometric functions in complex numbers is that they are no more bounded.

Proposition 3.5.5. *Let $z_0 \in \mathbb{C}$. Then, z_0 is root of $\sin z$ ($\cos z$) if and only if it is a root of $\sin x$ ($\cos x$).*

With that, we conclude $\tan z$ is defined always than $\tan x$ has no discontinuity.

3.6 Hyperbolic functions

Definition 3.6.1. Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic functions* as

$$\cosh z := \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2}, \tag{3.21}$$

$$\sinh z := \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2}, \tag{3.22}$$

$$\tanh z := \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{\mathrm{e}^z + \mathrm{e}^{-z}}. \tag{3.23}$$

Properties

Proposition 3.6.1. *For all $z \in \mathbb{C}$,*

$$\cosh^2 z - \sinh^2 z = 1. \tag{3.24}$$

Proposition 3.6.2. *For all $z \in \mathbb{C}$,*

$$\cosh(-z) = \cosh(z), \qquad \sinh(-z) = -\sinh(z). \tag{3.25}$$

Proposition 3.6.3. *For all $z, w \in \mathbb{C}$,*

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \tag{3.26}$$

$$\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \tag{3.27}$$

Proposition 3.6.4. *For all $z \in \mathbb{C}$,*

$$\cosh z = \cos(iz), \cos z = \cosh(iz) \tag{3.28}$$

$$\sinh z = -i \sin(iz), \sin z = -i \sinh(iz) \tag{3.29}$$

Proposition 3.6.5. *For all $z = x + iy \in \mathbb{C}$,*

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \tag{3.30}$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \tag{3.31}$$

$$\tan(x + iy) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i \frac{\sinh y}{\cos(2x) + \cosh(2y)}. \tag{3.32}$$

Proposition 3.6.6. *For all $z = x + iy \in \mathbb{C}$,*

$$\tanh(x + iy) = \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)} + i \frac{\sin(2y)}{\cosh(2x) + \cos(2y)}. \tag{3.33}$$

Proposition 3.6.7. *For all $z = x + iy$,*

$$|\cos z| = \sqrt{\cosh^2 y - \sin^2 x} = \sqrt{\cos^2 x + \sinh^2 y}, \tag{3.34}$$

$$|\sin z| = \sqrt{\sinh^2 x + \sin^2 y} = \sqrt{\cosh^2 x - \cos^2 y}. \tag{3.35}$$

Corollary 3.6.8. *For all $z = x + iy$,*

$$|\sinh y| \le |\cos z| \le \cosh y, \qquad |\sinh y| \le |\sin z| \le \cosh y. \tag{3.36}$$

Proposition 3.6.9. *The roots of the function $\sinh z$ are of the form $z_n = n\pi$ and those for the function $\cosh z$ are of the form $w_n = (2n + 1)\pi/2i$.*

3.7 Logarithm

Definition 3.7.1. Let $D \subseteq \mathbb{C}$ be a set. We define a *multivalued function from D to $\mathbb{C}$* as a subset of $D \times \mathbb{C}$ such that for every $z \in D$ there exists a number $y \in \mathbb{C}$ such that $(z, w) \in f$.

Definition 3.7.2. For $z \in \mathbb{C}^*$, we call the *natural logarithm of z* every number w such that $\mathrm{e}^w = z$, that is,

$$\ln z \coloneqq \{w \in \mathbb{C} \,|\, \mathrm{e}^w = z\}. \tag{3.37}$$

Proposition 3.7.1. *Given $z \in \mathbb{C}$ we can define $\ln z$ from the natural logarithm of a real number as*

$$\ln z = \ln |z| + i \arg z = \ln |z| + i\mathrm{Arg}z + 2\pi k i. \tag{3.38}$$

Definition 3.7.3. We define the *principal natural logarithm of z* as the value defined by the principal argument of z, that is,

$$\mathrm{Log}z = \ln |z| + i\mathrm{Arg}z. \tag{3.39}$$

It is important to note that, although we define the principal natural logarithm as that value with an argument $\theta \in [0, 2\pi)$, symbolic programs use as the default argument $\theta \in (-\pi, \pi]$.

Definition 3.7.4. We define the determination I (with I being a semiopen interval) of the logarithm as

$$\log_I z \coloneqq \ln |z| + i \arg_I z. \tag{3.40}$$

Definition 3.7.5. Let $E \subseteq \mathbb{C}^*$ be a connected set. We define the *continuous determination of the logarithm in E* as the continuous function $g : E \longrightarrow \mathbb{C}$ such that $\mathrm{e}^{g(z)} = z$.
More generally, if $f : E \longrightarrow \mathbb{C}$ is a function such that $f(z) \neq 0$ for all $z \in E$, then we define the *continuous determination of* $\ln f$ as a function $g : E \longrightarrow \mathbb{C}$ such that $\mathrm{e}^{g(z)} = f(z)$.

Proposition 3.7.2. *Let $z, w \in \mathbb{C}$ two numbers. Then,*

1. $\ln(zw) = \ln z + \ln w + 2\pi k i, k \in \mathbb{Z}$.
2. *If we want to stay in the principal argument,*

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \textit{if } \mathrm{Arg}z + \mathrm{Arg}w < 2\pi \\ \ln z + \ln w - 2\pi i, & \textit{if } \mathrm{Arg}z + \mathrm{Arg}w \geq 2\pi \end{cases}. \tag{3.41}$$

3. *SEARCH MORE PROPERTIES*

3.7.1 Inverse trigonometric and hyperbolic functions

The logarithm function allows us to define the multivalued inverses of trigonometric and hyperbolic function. Solving a quadratic equation, we get two possible expressions. For convention, we get that expression such that $\operatorname{arcsinh} 0 = 0$.

Definition 3.7.6. Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric inverse functions* as

$$\arcsin z \coloneqq -i \ln\left(iz + \sqrt{1 - z^2}\right), \tag{3.42}$$

$$\arccos z \coloneqq -i \ln\left(z + \sqrt{z^2 - 1}\right), \tag{3.43}$$

$$\arctan z \coloneqq -\frac{i}{2} \ln \frac{1 + iz}{1 - iz}. \tag{3.44}$$

Definition 3.7.7. Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic inverse functions* as

$$\operatorname{arcsinh} z \coloneqq \ln\left(z + \sqrt{1 + z^2}\right), \tag{3.45}$$

$$\operatorname{arccosh} z \coloneqq \ln\left(z + \sqrt{z^2 - 1}\right), \tag{3.46}$$

$$\operatorname{arctanh} z \coloneqq \frac{1}{2} \ln \frac{1 + z}{1 - z}. \tag{3.47}$$

3.8 Complex power

Definition 3.8.1. Let $z, a \in \mathbb{C}$ with $z \neq 0$. Then, we define the *complex power function* as

$$z^a := \mathrm{e}^{a \ln z}. \tag{3.48}$$

If $E \subseteq \mathbb{C}^*$ is a connected set and $f : E \longrightarrow \mathbb{C}$ a functions such that $f(z) \neq 0$ for all $z \in E$, and $w \in \mathbb{C}$ a number, we define a *continuous determination of* f^w as a continuous function $g : E \longrightarrow \mathbb{C}$ such that $g(z) \in [f(z)]^w$.

Proposition 3.8.1. *If $a = \alpha + \beta i$ and $z = r\mathrm{e}^{\theta i}$, then*

$$z^a = \mathrm{e}^{\alpha \ln r - \beta(\theta + 2\pi k)} \mathrm{e}^{(\beta \ln r + \alpha(\theta + 2\pi k))i}, \tag{3.49}$$

$$|z^a| = \mathrm{e}^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \qquad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k) \tag{3.50}$$

Proposition 3.8.2. *Let $a, z \in \mathbb{C}$ be two numbers. Then,*

1. *If $a = n \in \mathbb{Z}$, the complex power is a function and*

$$z^n = r^n \mathrm{e}^{n\theta i}. \tag{3.51}$$

2. *If $a = n/m \in \mathbb{Q}$, there are n values and*

$$z^a = \sqrt[m]{r^n} \mathrm{e}^{(\theta + 2\pi k)n/mi}. \tag{3.52}$$

3. *If a is irrational, the norm is uniquely determined but the argument has infinite values.*

4. *IF $a \in \mathbb{C} \setminus \mathbb{R}$, the argument is uniquely determined and the norm has infinite values.*

Proposition 3.8.3. *Let $z, w \in \mathbb{C}$. Then,*

1. $\left(\mathrm{e}^b\right)^a = \mathrm{e}^{a(b + 2\pi k i)}$

3.9 Riemann surfaces

Definition 3.9.1. A *Riemann surface X* is a connected complex 1-manifold.

The Riemann surface is a way to transform a multivalued function to a function. This process consists of separating the different values in several complex planes according the the argument.

Definition 3.9.2. We define a *sheet* as each of the complex planes of the Riemann surface.

Definition 3.9.3. We define a *cut* as the line (not necessaryly straight) of union between sheets.

Definition 3.9.4. We define a *branch point* as a point where start or finish a cut.

Note that, since a number goes from one interval of the argument to another continuously, there must be a way to go from one sheet to another. This can be achieved by the cuts. These cuts allow us to join one edge of one sheet to another and hence connect them in a continuous way. Each sheet must be connected to the next one (since one interval is connected to the next one) and, in case of a finite valued function, the last one to the first one. We call the sheet associated to the principal argument *principal sheet*.
Now we know how to connect sheets, we shall determine where do the cuts start or finish, that is, where are located the branch points. To achieve that, we must follow the path of a circle (although it could be any other closed curve) centered at a point z_0 and see if the value at the beginning and the end of the curve is the same or has changed. If the value of f changes, then z_0 is a branch point.

Example 3.9.1. Let us study the function $\sqrt{z}$. If we follow a path where the argument is always in $[0, 2\pi)$, then $f(z) = \sqrt{r}\mathrm{e}^{\theta/2i}$ always and the result does not change. However, if we make a complete revolution around $z = 0$ and we take now the value of $f(z') = f(r^{z'}_{\theta+2\pi})$, then

$$\sqrt{z'} = \sqrt{r}\mathrm{e}^{(\theta+2\pi)/2i} = \sqrt{r}\mathrm{e}^{\theta/2i}\mathrm{e}^{\pi i} = -\sqrt{r}\mathrm{e}^{\theta/2i} \neq f(z). \tag{3.53}$$

Therefore, we conclude $z = 0$ is a branch point. The cut could be done with a straight line from $z = 0$ to the infinity along the positive real axis.

3.9.1 Logarithm

Since $\ln z$ function is infinite valued, its associated Riemann surfaces has infinite sheets. Each sheet is connected to the next one through a cut done in the real axis, beginning at the branch point at $z = 0$. Note that, contrary to finite valued functions, there is not a last sheet that connects to the first one. Since each sheet is associated to a value of $k \in \mathbb{Z}$, and the set of integers if an ordered set, there are two infinites (like real numbers which are also ordered). For this reason there is no connection between $k = -\infty$ and $k = \infty$.
By the logarithm leads the expression $\ln |z| + (\arg z + 2\pi k)i$, we see the unique variation is in the imaginary part.

3.9.2 Power

Note that two last cases of exponents have infinite values, so their Riemann surfaces will have a similar aspect of that from the logarithm function.

Bibliography

[1] J. Cufí J. Bruna. *Anàlisi complexa.*

Chapter 4

Derivatives

4.1 Introduction

Definition 4.1.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. We define the *derivative of f at z_0* as

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{4.1}$$

in case the limit exists. If f has derivative, we say *f is derivable at z_0*.

Definition 4.1.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *f is holomorphic at Ω* if and only if it is $\mathbb{C}$-derivable at every point of Ω. In that case, it is defined the function $f' : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ that associates each point z of Ω with $f'(z)$.

We denote the set of all holomorphic functions at Ω by $H(\Omega)$.

Example 4.1.1. The function $f(z) = z^n$ with $n \in \mathbb{N}$ is derivable everywhere and its derivative is $f'(z) = nz^{n-1}$.

Example 4.1.2. The function $f(z) = \bar{z}$ is not derivable anywhere.

Example 4.1.3. The function $f(z) = \mathrm{e}^z$ is derivable everywhere and its derivative is $f'(z) = \mathrm{e}^z$.

Definition 4.1.3. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We define the *domain of holomphism* as the region where f is derivable. We say *f is entire* if and only if the domain of holomorphism is $\mathbb{C}$.

Definition 4.1.4. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *f is holomorphic at z_0* if and only if it is holomorphic at some neighborhood of z_0.

Proposition 4.1.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. If f is derivable at z_0, then it is continuous at z_0.*

Theorem 4.1.2. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two functions and $z_0 \in \Omega$ a point. Then, the following statements are true.*

1. *If f is constant at Ω, then f is derivable at z_0 and $f'(z_0) = 0$.*

2. *If $f(z) = z$ in every point of Ω, then f is derivable at z_0 and $f'(z_0) = 1$.*

3. *If f, g are derivable at z_0 and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ are derivable at z_0 and $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.*

4. *If f, g are derivable at z_0, then fg is derivable at z_0 and*
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{4.2}$$

5. *If f, g are derivable at z_0 and $g(z_0) \neq 0$, then f/g is derivable at z_0 and*
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{4.3}$$

Theorem 4.1.3. *Let $f : \Omega_1 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a derivable function at a point $z_0 \in \mathbb{C}$ and $g : \Omega_2 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be another derivable function at a point $f(z_0) \in \Omega_2$. Then, $g \circ f$ is derivable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \tag{4.4}$$

Definition 4.1.5. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say *f is of class $C^1(\Omega)$ or simply* $f \in c^1(\Omega)$ if and only if, using $f = u + iv$ with $u = \mathrm{Re}\{f\}, v = \mathrm{Im}\{f\}$, the partial derivatives of u and v as a two variable real functions exist and are continuous. In other words, $f \in C^1(\Omega)$ if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \tag{4.5}$$

exist and are continuous.

Theorem 4.1.4 (Cauchy-Riemann conditions). *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be a function and $z_0\in\Omega$ an interior point. Then, f is derivable at z_0 if and only if is differentiable at z_0 and $df(z_0)$ is $\mathbb{C}$-linear, that is,*

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}, \tag{4.6}$$

which are known as Cauchy-Riemann conditions.

Theorem 4.1.5. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be a function and $z_0\in\Omega$ an interior point. If u,v satisfy the Cauchy-Riemann equation and their partial derivatives are continuous, then f is derivable.*

Theorem 4.1.6. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ a function of class $C^1(\Omega)$ with Ω an open set, injective, and derivable at every point of Ω with non-zero derivative. Then,*

1. *$\Omega'=f(\Omega)$ is an open subset of $\mathbb{C}$.*
2. *The inverse function f^{-1} exist, it is well defined and it is derivable at Ω'.*
3. *If $z\in\Omega$ and $z'=f(z)$, then*

$$(f^{-1})'(z')=\frac{1}{f'(z)}. \tag{4.7}$$

Example 4.1.4. The functions $\mathrm{e}^z,\sin z,\cos z$ are holomorphic at $\mathbb{C}$ (hence they are entire) and their derivatives are respectively $\mathrm{e}^z,\cos z,-\sin z$.

Proposition 4.1.7. *A determination of $\ln z$ with $z\in\mathbb{C}$ is continuous except in a semiline.*

Theorem 4.1.8. *Let $\Omega\subseteq\mathbb{C}$ be an open set and $\phi\in C(\Omega)$, such that $\mathrm{e}^{\phi(z)}=z$ for all $z\in\Omega$. Then, we have $\phi\in H(\Omega)$ and*

$$\phi'(z)=\frac{1}{z},\forall z\in\Omega. \tag{4.8}$$

Proposition 4.1.9. *A determination of $\ln z$ with $z\in\mathbb{C}$ is holomorphic except in a semiline.*

Proposition 4.1.10. *Let $I=[\theta,\theta+2\pi)$ a determination of the logarithm, $\ln_I z$. Then, $\ln_I z$ is holomorphic except in the semiline $L_\theta=\left\{r\mathrm{e}^{\theta i}\in\mathbb{C}\,\middle|\,r\geq 0\right\}$.*

Theorem 4.1.11. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be an analytic function in Ω. Then, f is holomorphic in Ω. In particular, given a point $z_0\in\mathbb{C}$, $f'(z_0)=a_1$ where a_1 is the first coefficient of the power series that represents f in a neighborhood of z_0.*

From the previous theorem we see that if f is analytic then $f'(z_0)$ coincides with the formal derivative of the power series that represents f in a neighborhood of z_0.

4.2 Cauchy-Riemann Equations

Definition 4.2.1. We define the operators

$$\frac{\partial}{\partial z}:=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\qquad \frac{\partial}{\partial \bar z}:=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right), \tag{4.9}$$

that act over the functions such that the real and imaginary part u,v have partial derivatives.

Proposition 4.2.1. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ a function of class $C^1(\Omega)$. Then, for all $z_0\in\Omega$*

$$f(z_0+h)=f(z_0)+\left(\frac{\partial f}{\partial z}\right)_{z_0}h+\left(\frac{\partial f}{\partial \bar z}\right)_{z_0}\bar h+o(|h|^2). \tag{4.10}$$

Corollary 4.2.2. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be a function of class $C^1(\Omega)$. Then, f is holomorphic in Ω if and only if*

$$\frac{\partial f}{\partial \bar z}=0 \text{ at } \Omega. \tag{4.11}$$

Definition 4.2.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$ such that $f = u + iv$ with $u = \text{Re}\{f\}, v = \text{Im}\{f\}$ and $z_0 \in \mathbb{C}$ a point. Then, we call $(\partial_{\bar{z}} f)_{z_0} = 0$ the *Cauchy-Riemann condition*, which is equivalent to

$$\left(\frac{\partial u}{\partial x}\right)_{z_0} = \left(\frac{\partial v}{\partial y}\right)_{z_0}, \qquad \left(\frac{\partial v}{\partial x}\right)_{z_0} = -\left(\frac{\partial u}{\partial y}\right)_{z_0}, \tag{4.12}$$

which are called the *Cauchy-Riemann equations.*

4.2.1 The operators ∂ and $\overline{\partial}$

Theorem 4.2.3. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and z_0 an interior point. Then, at z_0*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}. \tag{4.13}$$

Chapter 5

Series

5.1 Numeric series

Definition 5.1.1. We say $\sum_{n=1}^{\infty} z_n$ converges if and only if $S_n := \sum_{n=1}^{N} z_n$ has limit at $n \to \infty$.

Proposition 5.1.1. *$\sum_{n=1}^{\infty} z_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.*

Definition 5.1.2. We say $\sum_{n=1}^{\infty} z_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |z_n|$ converges.

Proposition 5.1.2. *$\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge.*

Proposition 5.1.3.

1. *A series converges absolutely with sum S if and only if every rearrangement is convergent with the same sum S.*
2. *An absolutely convergent series can be summed by blocks in an arbitrary way.*

Proposition 5.1.4. *Let $\sum_{n} a_n, \sum_{n} b$ be two absolutely convergent series with sums A and B respectively. Then, the series $\sum_k c_k$ with $c_k = \sum_{n=0}^{k} a_n b_{k-n}$ is absolutely convergent with sum AB.*

5.2 Function series

Theorem 5.2.1 (Weierstrass M-test)**.** *If $|f_n(p)| < M_n$ for all $p \in X, n \geq 1$ and $\sum_{n=0}^{\infty} M_n < \infty$, then the series $\sum_{n=0}^{\infty} f_n(p)$ is uniformly convergent on X.*

Lemma 5.2.2 (Abel's summation formula)**.** *Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ be two sequences of complex numbers and $A_n = a_1 + \cdots + a_n$. Then,*

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k). \tag{5.1}$$

Another way to express the relation is

$$\sum_{k=1}^{n} (A_k - A_{k-1}) b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k), \tag{5.2}$$

which is analogous to the integration by parts formula.

Theorem 5.2.3 (Dirichlet's criteria)**.** *Let $\sum_{n=1}^{\infty} f_n(p) g_n(p)$ be a series where $f_n(p)$ are complex and $g_n(p)$ are real for all $p \in X, n \geq 1$. If we denote $F_n(p) = f_1(p) + \cdots f_n(p)$, there exists a constant M such that $|F_n|(p) \leq M$ for all $n \geq, p \in X$, $g_n(p)$ is monotonous decreasing and converges uniformly to zero on X, then the series $\sum_{n=1}^{\infty} f_n(p) g_n(p)$ is uniformly convergent on X.*

Theorem 5.2.4 (Abel's criteria)**.** *Let $\sum_{n=1}^{\infty} f_n(p) g_n(p)$ be a series where $f_n(p), g_n(p)$ are complex. If $\sum_{n=1}^{\infty} f_n(p)$ is uniformly convergent on X andd there exists a number $M \in \mathbb{R}^+$ such that for all*

$p \in X$

$$|g_1(p)| + \sum_{n=1}^{\infty} |g_n(p) - g_{n+1}(p)| \leq M, \tag{5.3}$$

then the series $\sum_{n=1}^{\infty} f_n(p) g_n(p)$ *is uniformly convergent on* X.

5.3 Power series

Definition 5.3.1. We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n, z, z_0 \in \mathbb{C}. \tag{5.4}$$

We call the term a_n the *n-th coefficient of the series*. In case $a_n = 0$ $\forall n \leq m$, we will start the counting directly from m.

Definition 5.3.2. Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series. We define its *domain of convergence* as

$$E \coloneqq \left\{ z \in \mathbb{C} \,\middle|\, \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges} \right\}. \tag{5.5}$$

Theorem 5.3.1. *Let* $\sum_n a_n (z - z_0)^n$ *be a power series and* $R = 1/\rho$, *where* $\rho = \limsup_n |a_n|^{1/n}$. *Then, the series converges uniformly on the compacts of the open disc* $D(z_0, R)$, *converges absolutely at every point* $z \in D$ *and diverges outside* $\bar{D}$. *Hence, the set of converges* E *satisfies* $D \subseteq E \subseteq \bar{D}$ *and* $D = \operatorname{int} E$.

Definition 5.3.3. Radius of convergence.

Proposition 5.3.2. *Let* $\sum_n a_n (z - z_0)^n$ *be a power series and* $R = \lim_{n \to \infty} |a_n|/|a_{n+1}|$. *If the limit exists, then* R *is the radius of convergence.*

Theorem 5.3.3 (Cauchy-Hadamard Theorem)**.** *Let*

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{5.6}$$

be the following complex power series with $a_n, z_0 \in \mathbb{C}$ *and* $R \in [0, +\infty) \cup \{+\infty\}$ *the radius of convergence. Then,*

1. *If* $|z - z_0| < R$ *then* S *converges. In fact, for all* $r < R$ *we have* S *converges uniformly at the disc* $\overline{D_r(z_0)}$.
2. *If* $|z - z_0| > R$ *then* S *diverges.*
3. *The function* $f(z) = S(z)$ *is derivable at* $D_R(z_0)$ *and its formal derivative is*

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \tag{5.7}$$

with the same radius of convergence.

Definition 5.3.4. Let $\sum_n a_n (z - z_0)^n$ be a series, $S = E \cap C(z_0, R)$ non empty, and $m > 1$ a real number. We define

$$S_m \coloneqq \{ z \in \mathbb{C} \mid |z - z_0| < R, d(z, S) \leq m(R - |z - a|) \}. \tag{5.8}$$

Definition 5.3.5 (Stolz angle)**.** Let S be formed by one point w. We define the *Stolz angle* as the angle generated by the S_m.

Theorem 5.3.4 (Abel's theorem)**.** *Let $\sum_n a_n(z-z_0)^n$ be a series with S non empty and such that the series converges uniformly on it. Then, the series converges uniformly on S_m for all $m>1$. In particular, the sum function is continuous on S_m and one has*

$$\lim_{z\to w, z\in S_m} \sum_n a_n(z-z_0)^n = \sum_n a_n(w-z_0)^n, \qquad w\in S. \tag{5.9}$$

5.4 Analytic functions and Taylor series

Theorem 5.4.1. *Let $\sum_n a_n(z-z_0)^n$ be a series with radius of convergence R. Then, $f(z)=\sum_n a_n(z-z_0)^n$ is holomorphic on $D(a,R)$ and it has a derivative*

$$f'(z)=\sum_n n a_n(z-z_0)^{n-1},\ \forall z\in D. \tag{5.10}$$

Proposition 5.4.2. *Let $f: D(a,R)\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be a function. If there exists a power series $\sum_n a_n(z-z_0)^n$, convergent on D such that*

$$f(z)=\sum_n a_n(z-z_0)^n,\ |z-z_0|<R, \tag{5.11}$$

then the series is unique. In fact, f is infinitely holomorphic and the coefficients a_n are determined by f with the relation

$$a_n=\frac{f^n(z_0)}{n!},\ n\in\mathbb{N}. \tag{5.12}$$

Definition 5.4.1. Let $f: D(a,R)\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be a function. We say f *admits a series expansion* if and only if there exists a power series $\sum_n a_n(z-z_0)^n$, convergent on D such that

$$f(z)=\sum_n a_n(z-z_0)^n,\ |z-z_0|<R. \tag{5.13}$$

Definition 5.4.2. Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be a function with Ω an open set. We say f *is analytic on* Ω if and only if it admits locally a series expansion, that is, if for every point $z_0\in\Omega$ there exists a disc $D(z_0,\delta)$ and a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ such that $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n, \forall z\in D$.

Theorem 5.4.3. *Let $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $D(z_0,R)$ and $w_0\in D(z_0,R_0)$. Then, the series $\sum_{n=0}^{\infty}\frac{f^{(n)}(z_1)}{n!}(z-z_1)^n$ has a radius of convergence $R_1\geq R_0-|z_0-z_1|$ and it satisfies*

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_1)}{n!}(z-z_1)^n, \qquad \text{if } |z-z_1|<R-|z_0-z_1|. \tag{5.14}$$

Definition 5.4.3. Let $f(z)=\sum a_n(z-z_0)^n$ be a series with radius of convergence R. Then, its formal derivative is

$$f'(z)=\frac{\mathrm{d}f}{\mathrm{d}z}. \tag{5.15}$$

Corollary 5.4.4. *Let $f(z) = \sum a_n(z-z_0)^n$ be a series with radius of convergence R. Then, f is infinitely derivable at $D_R(z_0)$.*

Corollary 5.4.5. *Let R be the radius of convergence of the function*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

Then f has as Taylor polynomial of degree m around z_0 the following one

$$P_{m,f,z_0}(z) = \sum_{n=0}^{m} a_n(z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{5.16}$$

Theorem 5.4.6 (Abel's Theorem)**.** *Let be the following series*

$$\sum_{n=0}^{\infty} f_n(z_0)g_n(z_0),$$

where f,g_n are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \tag{5.17}$$

then the original series converges uniformly in Ω.

Theorem 5.4.7 (Weierstrass' criterion)**.** *Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant M_n, satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at Ω for all $z_0 \in \Omega$.*

Definition 5.4.4. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a complex function with Ω an open set. We say f *is complex analytic* if and only if for all $z_0 \in \Omega$ exists a real number $R(z_0)$ and a sequence $\{a_n\} \subseteq \mathbb{C}$ (that can also depend on z_0) such that is $z \in D_R(z_0)$, then formally it is true that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n. \tag{5.18}$$

We denote the set of complex analytic functions with domain Ω by $C^\omega(\Omega)$.

Corollary 5.4.8. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. If $f \in C^\omega(\Omega)$, then $f \in C^\infty(\Omega)$.*

Corollary 5.4.9. *Let z_0. Then, the coefficients z_0 of the local expression of f given by the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ are determined by*

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{5.19}$$

Proposition 5.4.10. *Let $\varnothing \neq \Omega \subseteq \mathbb{C}$. Then,*

1. *Every connected component of Ω is a closed of Ω with a subspace topology.*
2. *Two connected components are the same or are disjoint.*
3. *Every connected of Ω is one and only one connected component.*
4. *Ω is the disjoint union of its connected components.*

Theorem 5.4.11 (Analytic prolongation Principle)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0 = 0)$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of Ω that contains z_0 (the function can be zero also out Ω, but we are not studying that).*

Corollary 5.4.12. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function with Ω a region. Then, the following statements are equivalent:*

1. $f(z) = 0$ *for all* $z \in \Omega$.
2. *There exists a* $z_0 \in \Omega$ *such that* $f^{(n)}(z_0) = 0$ *for all* $n \in \mathbb{N}$.

Corollary 5.4.13 (Analytic Prolongation Principle)**.** *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions with Ω a region. Then, the following statements are equivalent:*

1. $f(z) = g(z)$ *for all* $z \in \Omega$.
2. *There exists a* $z_0 \in \Omega$ *such that* $f^{(n)}(z_0) = g^{(n)}(z_0)$ *for all* $n \in \mathbb{N}$.

Lemma 5.4.14. *Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then*

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right). \tag{5.20}$$

Corollary 5.4.15. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions. Then, fg is analytic.*

Proposition 5.4.16. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, f is analytic at $\Omega = D_R(0)$.*

Chapter 6

Holomorphic functions and differential forms

6.1 Complex line integrals

6.1.1 Curves, paths, and arcs

Definition 6.1.1. Let $I = [a,b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{aligned} \gamma : I &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned}. \tag{6.1}$$

Definition 6.1.2. Let $I = [a,b] \subseteq \mathbb{R}$ be a closed interval and $D \subseteq \mathbb{C}$ a domain. We define an *arc* as a continuous application of the form

$$\begin{aligned} \gamma : I &\longrightarrow D \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned}. \tag{6.2}$$

Equivalently, we can say an arc is a curve restricted to some interval.

Definition 6.1.3. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We call $\gamma(a)$ and $\gamma(b)$ *the extremes of* γ. In particular, we call $\gamma(a)$ the *initial point* and $\gamma(b)$ the *final point*.

Definition 6.1.4. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We define the *route or graph of* γ as

$$\gamma^* \coloneqq \{z \in D \,|\, z = \gamma(t), t \in I\}\,. \tag{6.3}$$

Definition 6.1.5. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is closed* if and only if $\gamma(a) = \gamma(b)$.

Definition 6.1.6. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is simple* if and only if there is no two numbers $t_1, t_2 \in (a,b)$ such that $\gamma(t_1) = \gamma(t_2)$. We also call it a *Jordan curve*, and if it is closed, a *circuit*.

Definition 6.1.7. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is differentiable* if for al value $t_0 \in [a,b]$ there exists the limit

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \tag{6.4}$$

For $t_0 = a$ or $t_0 = b$ we consider the laterals limits from the right and from the left respectively.

If $\gamma'(t_0) \neq 0$ and we identify the complex value as a vector, the vector is tangent to γ at $t = t_0$.

Definition 6.1.8. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is of class* C^1 if and only if γ' exists and is continuous at $[a,b]$.

Definition 6.1.9. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is regular or smooth* if and only if it is differentiable and γ' never vanishes.

Definition 6.1.10. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We say γ *is piece-wise of class* C^1 if and only if γ' exists and is continuous in I except in a finite number of points where γ has lateral derivatives.

Definition 6.1.11. Let $\gamma : [a,b] \longrightarrow D$ be an arc. We define the *opposite arc* as

$$\begin{aligned} -\gamma : [-b,-a] &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma(-t) \end{aligned}. \tag{6.5}$$

Definition 6.1.12. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be an arc. We say $\Gamma(s), s \in [c,d] \subseteq \mathbb{R}$ *has been obtained from* $\gamma(t), t \in [a,b]$ *by a change of parametrization* if and only if the new parameter s and the original parameter t are related by a relation $t = \phi(s)$, where $\phi : [c,d] \longrightarrow [a,b]$ is an homeomorphism that satisfies $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$. We call Γ *the reparametrization of* γ.

Definition 6.1.13. Let $\gamma_1 : I_1 \longrightarrow \mathbb{C}$ and $\gamma_2 : I_2 \longrightarrow \mathbb{C}$ be two arcs. We say they *are equivalent* if and only if there exists a bijective, monotone, and continuous function $\rho : I_2 \longrightarrow I_1$ such that $\gamma_2 = \gamma_1 \circ \rho$. If ρ is an increasing function we say γ_1 *and* γ_2 *have the same orientation*; otherwise, we say γ_1 *and* γ_2 *have opposite orientations*.

Definition 6.1.14. Let $\gamma_1[a,b] \longrightarrow \mathbb{C}$ and $\gamma_2 : [c,d] \longrightarrow \mathbb{C}$ be two arcs such that $[a,b] \cap [c,d] = \varnothing$. We define the application $\gamma_1 \cup \gamma_2$ (sometimes denoted by $\gamma_1 + \gamma_2$) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \le t \le b \\ \gamma_2(t - b + c), & \text{if } b \le t \le b + d - c \end{cases}. \tag{6.6}$$

We say γ_1, γ_2 can be joined/added or that there exists its union/sum if and only $\gamma_1(b) = \gamma_2(x)$. In this case $\gamma_1 + \gamma_2$ is an arc, and we call it *the sum arc of* γ_1 *plus* γ_2.

Notice that the property of the intervals of being disjoints is not restrictive since we can make changes of variables to make the intervals satisfy the condition.

Definition 6.1.15. We define the *segment of extremes* $z_1, z_2 \in \mathbb{C}$ as the arc defined by the expression

$$\begin{aligned} [z_1, z_2] : [0,1] &\longrightarrow \mathbb{C} \\ t &\longmapsto (1-t)z_1 + tz_2 \end{aligned}. \tag{6.7}$$

Definition 6.1.16. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f *is polygonal* if and only if can be expressed as a finite union of segments, that is, if there exist a natural number n and points $\{z_0, \dots, z_n\}$ such that

$$p = [z_0, z_1] \cup \cdots \cup [z_{n-1}, z_n]. \tag{6.8}$$

Definition 6.1.17. Let $\gamma : [a,b] \longrightarrow D$ be an arc with a, b finite. We say γ *is a basic curve* if and only if $\gamma \in C^1((a,b)) \cap C([a,b])$ and there exist $\lim_{t \to a^+} \gamma'(t), \lim_{t \to b^-} \gamma'(t)$.

Definition 6.1.18. A *path* is a function $\gamma : [a,b] \longrightarrow \mathbb{C}$ such that there exist basic curves $\gamma_j : [a_j, b_j] \longrightarrow \mathbb{C}, j \in \{1, \dots, k\}$ such that $\gamma = \gamma_1 + \cdots + \gamma_k$ and therefore $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ and $a = a_1, b = a_k$.

Definition 6.1.19. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous curve and $a_1, \dots, a_l \in \mathbb{R}$ such that $a = a_0 \le \cdots \le a_l \le b = a_{l+1}$. We say γ *is piece-wise differentiable* if and only if

$$\gamma \in C^1\left(\bigcup_{j=0}^{l} (\alpha_j, \alpha_{j+1})\right),$$
$$\forall j \in \{0, \dots, l+1\} \exists \lim_{t \to a_j^+} \gamma'(t) (\text{except if } j = l+1), \lim_{t \to a_j^-} \gamma'(t) (\text{except if } j = 0).$$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

Theorem 6.1.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with Ω an open set and $\phi : I \longrightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curev (hence a derivable curve) and its real derivative is*

$$\psi'(t) = f'(\phi(t))\phi'(t). \tag{6.9}$$

Definition 6.1.20. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two curves. We say γ_1, γ_2 *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = \gamma_1(0) = \gamma_2(0), s \in [0,1]$.
4. $h(1,s) = \gamma_1(1) = \gamma_2(1), s \in [0,1]$.

Definition 6.1.21. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two circuits. We say γ_1, γ_2 *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = h(1,s), s \in [0,1]$.

6.1.2 Integrals

Integrals on the real axis.

Definition 6.1.22. Let $f : [a, b] \longrightarrow \mathbb{C}$ be a function with the notation $f = u + iv$. We define the integral of f as

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t\,. \tag{6.10}$$

Proposition 6.1.2. *Let $f, g : [a, b] \longrightarrow \mathbb{C}$ be two integrable functions and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,*

$$\int_a^b \lambda f + \mu g\,\mathrm{d}t = \lambda\int_a^b f\,\mathrm{d}t + \mu\int_a^b g\,\mathrm{d}t\,. \tag{6.11}$$

Proposition 6.1.3. *Let $f : [a, b] \longrightarrow \mathbb{C}$ be an integrable function. Then,*

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \le \int_a^b |f(t)|\,\mathrm{d}t\,. \tag{6.12}$$

Line integrals

Definition 6.1.23. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of f over γ* as

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\,. \tag{6.13}$$

Proposition 6.1.4. *The previous definition is well defined.*

Proposition 6.1.5. *If we use the notation $f = u + iv$ and $\gamma = x + iy$, then the integral has the form*

$$\int_\gamma f = \int_a^b u\frac{\mathrm{d}x}{\mathrm{d}t} + v\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t + i\int_a^b v\frac{\mathrm{d}x}{\mathrm{d}t} + u\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t\,. \tag{6.14}$$

Definition 6.1.24. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of f over γ with respect the differential of length* as

$$\int_\gamma f(z)\,\mathrm{d}s := \int_\gamma f(z)|\mathrm{d}z| = \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t\,. \tag{6.15}$$

Theorem 6.1.6. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$, $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two functions, and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,*

$$\int_\gamma \lambda f + \mu g\,\mathrm{d}z = \lambda\int_\gamma f\,\mathrm{d}z + \mu\int_\gamma g\,\mathrm{d}z\,. \tag{6.16}$$

Theorem 6.1.7. *Let γ_1, γ_2 be two equivalent curves of the same orientation and of class C^1 on their respective domains and$f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z\,. \tag{6.17}$$

Proposition 6.1.8. *Let $\gamma_1, \ldots, \gamma_n$ be n curves of class C^1 on their respective domains and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \ldots, \Gamma_n \subseteq \Omega$. If we define $\gamma = \gamma_1 + \cdots + \gamma_n$, then*

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,\mathrm{d}z\,. \tag{6.18}$$

Proposition 6.1.9. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \leq \int_\gamma |f|\,\mathrm{d}s\,. \tag{6.19}$$

Corollary 6.1.10. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. If $|f(z)| \leq M$ for all $z \in \Gamma$, then,*

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \leq ML(\gamma). \tag{6.20}$$

Proposition 6.1.11. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z}\,\mathrm{d}w\,. \tag{6.21}$$

Proposition 6.1.12. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$|\operatorname{Ind}(\gamma, z)| \leq \frac{1}{2\pi} \frac{L(\gamma)}{|z - \Gamma|}. \tag{6.22}$$

Proposition 6.1.13. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $\{f_n\}_{n=0}^{\infty}$ a sequence of continuous functions on Γ such that $\sum_{n=0}^{\infty} f_n$ converges uniformly on Γ. Then, $\sum_{n=0}^{\infty} \int_\gamma f_n\,\mathrm{d}z$ converges and*

$$\int_\gamma \sum_{n=0}^{\infty} f_n(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} \int_\gamma f_n(z)\,\mathrm{d}z\,. \tag{6.23}$$

6.2 Fundamental theorem of complex calculus

Definition 6.2.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f has a *primitive on* Ω if and only if there exists a function $F : \Omega \subseteq\longrightarrow \mathbb{C}$ such that $F' = f \forall z \in \Omega$.

Definition 6.2.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f has a *local primitive on* D if and only if for all z there exists a neigborhood where f has a primitive.

Theorem 6.2.1 (Fundamental theorem of complex calculus)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with Ω a domain. Then, the line integral of f is independent on the path on Ω if and only if f has an holomorphic primitive F such that $F' = f$ on Ω. In that case,*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)). \tag{6.24}$$

Theorem 6.2.2. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function on a star domain $S \subseteq \Omega$. Then, f has an holomorphic primitive F on S if and only if*

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0 \tag{6.25}$$

for all triangle $\triangle \subseteq \Omega$.

Proposition 6.2.3. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with no roots on a domain $D \subseteq \Omega$. Then, there is a determination of the logarithm of f on D if and only if f'/f has an holomorphic primitive on D.*

Proposition 6.2.4. *Let $K \subseteq \mathbb{C}$ be a compact set. Then,*

1. *If $\alpha \in V_\infty$, then the non-bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log(z-\alpha)$ in a neighborhood of K.*

2. *If α, β belong to the same bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log\left(\frac{z-\alpha}{z-\beta}\right)$ in a neighborhood of K.*

6.3 Cauchy's theorem

Theorem 6.3.1 (Green's theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary. Let $\mathbf{F} = (P, Q)$ be a vector field with P, Q being differentiable functions on a neighborhood of $\bar{\Omega}$ such that $\partial_x P - \partial_y Q$ is continuous on $\bar{\Omega}$. Then,*

$$\int_{\partial\Omega} \langle \mathbf{F}, \mathrm{d}\mathbf{s} \rangle_I = \int_{\partial\Omega} P \,\mathrm{d}x + Q \,\mathrm{d}y = \iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y} \,\mathrm{d}x \,\mathrm{d}y \,. \tag{6.26}$$

Theorem 6.3.2 (Cauchy's integral theorem)**.** *Let Ω be a bounded domain with piece-wise regular and positively oriented boundary and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an holomorphic function in a neighborhood of $\bar{\Omega}$. Then,*

$$\int_{\partial\Omega} f(z) \,\mathrm{d}z = 0. \tag{6.27}$$

Corollary 6.3.3. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function in a domain $D \subseteq \Omega$. Then, f ha local primitive on D. If D is a star domain, f has a global holomorphic primitive.*

Corollary 6.3.4. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with no roots in a domain $D \subseteq \Omega$. Then, f has a local determination of the logarithm on D. If D is a star domain, f has a global determination of the logarithm.*

Theorem 6.3.5 (Cauchy's integral theorem for homotopic curves)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with Ω a domain and γ_1, γ_2 two homotopic curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z) \,\mathrm{d}z = \int_{\gamma_2} f(z) \,\mathrm{d}z \,. \tag{6.28}$$

Theorem 6.3.6 (Cauchy's general integral theorem)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a regular function on Ω except a finite numbers of points where f is continuous. If γ is a constant curve, then*

$$\oint_{\gamma} f(z) \,\mathrm{d}z = 0. \tag{6.29}$$

Theorem 6.3.7 (Morera's theorem)**.** *Let f be a continuous function in a region Ω. If*

$$\oint_{\gamma} f(z) \,\mathrm{d}z = 0 \tag{6.30}$$

for all simple and closed curve γ such that $\Gamma \subseteq \Omega$, then f is analytic on Ω.

6.4 Holomorphic functions as vector fields and harmonic functions

6.4.1 Solenoidal fields

6.4.2 Holomorphic vector fields. Harmonic functions

Theorem 6.4.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a differentiable function on a domain D. Then, $f = u + iv$ is holomorphic if and only if the field $\bar{f} = (u, -v)$ is locally conservative and locally solenoidal.*

Definition 6.4.1. Let $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable vector field on a domain $D \subseteq \mathbb{R}^n$. We say the field is *holomorphic* if and only if is locally conservative and locally solenoidal, that is, it satisfies

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},\ \forall i, j; \qquad \operatorname{div} \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = 0,\ \text{on } D. \tag{6.31}$$

Definition 6.4.2. Let $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field two times differentiable on an open set $\Omega \subseteq \mathbb{R}^n$. We say the field is *harmonic* if and only if $\nabla^2\Phi = 0$ on Ω.

Theorem 6.4.2. *Holomorphic vector fields are the fields that are locally the gradient of an harmonic function. Holomorphic functions are the functions f that, locally, satisfy $\bar{f} = \Phi_x + i\Phi_y$ with Φ harmonic.*

6.4.3 Harmonic conjugate functions

Definition 6.4.3. Let u be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$. We say a differentiable function $\tilde{u}$ on Ω is *the harmonic conjugate of* u if and only if $\mathrm{d}\tilde{u} = \mathrm{d}^* u$, that is, if the function $f = u + i\tilde{u}$ is holomorphic on Ω.

Theorem 6.4.3. *Let u be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$ and $f = \overline{\boldsymbol{\nabla}} u$. Then, u is has an harmonic conjugate on Ω, $\tilde{u}$, if and only if f has an holomorphic primitive F on Ω. In that case, $F = u + i\tilde{u}$.*

Proposition 6.4.4. *Let u be an harmonic function on a domain Ω. Then, it has an harmonic conjugate if an only if the closed form $\mathrm{d}^* u$ is exact on Ω, that is, if $\int_\gamma \mathrm{d}^* u = 0$ for all closed curve γ such that $\Gamma \subseteq \Omega$, condition that is is always locally completed. If Ω is a star domain, every harmonic function on Ω has a harmonic conjugate function on Ω.*

Chapter 7

Local properties of holomorphic functions

7.1 Cauchy's integral formula

Lemma 7.1.1. *Let $a \in \mathbb{C}$ be a number and $f = 1/|z-a|$. Then, f is Lebesgue-integrable on every subset of $\mathbb{C}$ of finite measure.*

Theorem 7.1.2 (Cauchy-Green formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and f a differentiable function on a neighborhood of $\bar{\Omega}$ such that $\overline{\partial} f$ is continuous on $\bar{\Omega}$. Then, for all $z_0 \in \Omega$,*

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(z)}{z - z_0} \, \mathrm{d}z - \frac{1}{\pi} \int\limits_{\Omega} \frac{\overline{\partial} f(z)}{z - z_0} \, \mathrm{d}m(z) \,. \tag{7.1}$$

Corollary 7.1.3 (Cauchy's integral formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and f an holomorphic function on a neighborhood of $\bar{\Omega}$. Then,*

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(z)}{z - z_0} \, \mathrm{d}z \,. \tag{7.2}$$

Corollary 7.1.4. *Let f be a differentiable function on $\mathbb{C}$ with compact support and $\overline{\partial} f$ continuous on $\mathbb{C}$. Then,*

$$f(z_0) = -\frac{1}{\pi} \int\limits_{\Omega} \frac{\overline{\partial} f(z)}{z - z_0} \, \mathrm{d}m(z) \,. \tag{7.3}$$

Proposition 7.1.5. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and γ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z_0) f(z_0) = \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(z)}{z - z_0} \, \mathrm{d}z \,. \tag{7.4}$$

Corollary 7.1.6. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and γ_1, γ_2 two homotopic, piece-wise regular, and positively oriented curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$f(z_0) = \frac{1}{2\pi i} \oint\limits_{\gamma_1} \frac{f(z)}{z - z_0} \, \mathrm{d}z - \frac{1}{2\pi i} \oint\limits_{\gamma_2} \frac{f(z)}{z - z_0} \, \mathrm{d}z \,. \tag{7.5}$$

7.2 Analytic functions and holomorphic functions

Theorem 7.2.1. *Let f be an holomorphic function on a disc $D(z_0, R)$. Then, there exists a power series $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence greater or equal to R such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall z \in D(z_0, R). \tag{7.6}$$

Theorem 7.2.2. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with Ω an open set. Then, f is holomorphic on Ω if and only if f is analytic on Ω. More precisely, every holomorphic function f on Ω is indefinitely holomorphic on Ω, and for all $z_0 \in \Omega$ the Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \tag{7.7}$$

is valid on the greatest disc centered on z_0 and contain on Ω, which is $D(z_0, \delta(z_0))$, where $\delta(z_0) =$ $\inf\{|z_0 - w|, w \notin \Omega\}$.

Proposition 7.2.3. *Let f be a function and Ω a bounded domain with piece-wise regular and positively oriented boundary. If f is holomorphic on a neighborhood of $\bar{\Omega}$, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \, . \tag{7.8}$$

Proposition 7.2.4. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and γ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \, . \tag{7.9}$$

Theorem 7.2.5 (Maximum modulus principle)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with Ω a domain. If f is not constant, then $|f|$ does not have any local maxima on Ω.*

Corollary 7.2.6. *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and f an holomorphic function on a neighborhood of $\bar{\Omega}$ or, more generally, $f \in C(\bar{\Omega}) \cap H(\Omega)$. Let M be the maxima of $|f|$ on $\partial\Omega$. Then, one has*

$$|f(z)| \le M, \qquad \text{for all } z \in \Omega. \tag{7.10}$$

In other words, $\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|$.

Theorem 7.2.7 (Cauchy's inequality)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on a neighborhood of the disc $\bar{D}(z_0, R)$ and $|f(z)| \le M$ for $z \in C(z_0, R)$. Then,*

$$\left| f^{(n)}(z_0) \right| \le M \frac{n!}{R^n}. \tag{7.11}$$

Corollary 7.2.8. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with Ω a domain such that $|f(z)| \le M, z \in \Omega$. Then,*

$$\left| f^{(n)}(z) \right| \le M \frac{n!}{d(z, U^c)^n}, \qquad z \in U, n \in \mathbb{N}. \tag{7.12}$$

Theorem 7.2.9 (Liouville's theorem)**.** *Let f be a bounded entire function. Then, f is constant. Also, a function u harmonic and bounded on $\mathbb{C}$ is constant.*

Theorem 7.2.10 (Fundamental theorem of algebra)**.** *Let $P(<) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n of complex coefficients and $n \ge 1$. Then, P has exactly n roots $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ (some of which can be counted with their multiplicity) and*

$$P(z) = a_n \prod_{i=1}^{n} (z - \alpha_i). \tag{7.13}$$

Chapter 8

Isolated singularities of holomorphic functions

8.1 Isolated singular points

8.2 Laurent expansion

Theorem 8.2.1. *Every holomorphic function on an annulus admits a Laurent expansion.*

Proposition 8.2.2. *Let f be an holomorphic function on an annulus $C(z_0, R_2, R_1)$. If f has an isolated singularity at z_0, then its Laurent expansion is uniquely determined by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{2\pi i} \int\limits_{C(z_0,r)} \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z\,, \tag{8.1}$$

where a_n is independent of r, $r \in (R_2, R_1)$.

Chapter 9

Homology and holomorphic functions

Chapter 10

Harmonic functions

Theorem 10.0.1. *Let $f \in H(\Omega), C^1(\Omega)$ be a function. If $f = u + iv$, then u, v are harmonic functions on Ω.*

Chapter 11

Conforming representation

Chapter 12

Riemann's theorem and Dirichlet's problem

Chapter 13

Runge's theorem and Cauchy-Riemann equations

Chapter 14

Zeros of holomorphic functions

From physics

Definition 14.0.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and z_0 a number. We say z_0 *is a zero of order* n *of* f if and only if $f^{(k)}(z_0) = 0$ for all $0 \leq k \leq n$.

Proposition 14.0.1. *The zeros of finite order of an holomorphic function are isolated points.*

Proposition 14.0.2. *All the zeros of an non null analytic function are isolated points and of finite order.*

Theorem 14.0.3 (Weierstrass Factorization Theorem)**.** *content...*

Chapter 15

Complex Fourier transform

15.1 Introduction

Definition 15.1.1. Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform of f at the point ξ* as

$$\boxed{\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(x) \mathrm{e}^{-i\xi x}\, \mathrm{d}x\,.} \tag{15.1}$$

Proposition 15.1.1. *Let $f \in L^1(\mathbb{R})$ be a function. Then, the application*

$$\begin{aligned} \mathscr{F}\{f\} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \hat{f}(\xi) \end{aligned} \tag{15.2}$$

is a well defined application.

When we want to talk about the Fourier transform as an operator that acts over functions of $L^1(\mathbb{R})$ we write $\mathscr{F}\{f\}$, that satisfies $\mathscr{F}\{f\}(\xi) = \hat{f}(\xi)$.

Definition 15.1.2. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. We say *the functions f_n converge to f with a norm $\|\cdot\|_p$* or *converge in $L^p(\mathbb{R})$* if and only if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0. \tag{15.3}$$

Theorem 15.1.2. *Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.*

1. *The Fourier transform of f satisfies*

$$\mathscr{F}\{f\} \in L^\infty(\mathbb{R}), \qquad \|\mathscr{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|f\|_1. \tag{15.4}$$

2. *$\mathscr{F}\{f\}$ is $\mathbb{C}$ linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,*

$$\mathscr{F}\{\alpha f + \beta g\} = \alpha\mathscr{F}\{f\} + \beta\mathscr{F}\{g\}. \tag{15.5}$$

3. *If $g(x) = \bar{f}(x)$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \tag{15.6}$$

4. *If $g(x) = g(\lambda x)$ and $\lambda \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \frac{1}{|\lambda|} \hat{f}\left(\frac{\xi}{\lambda}\right). \tag{15.7}$$

5. *If $g(x) = f(x - a)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \mathrm{e}^{-ia\xi} \hat{f}(\xi). \tag{15.8}$$

6. *If $g(x) = \mathrm{e}^{iax} f(x)$ with $\alpha \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \hat{f}(\xi - a) \tag{15.9}$$

7. *If $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$ when $n \to \infty$, then $\mathscr{F}\{f_n\} \to \mathscr{F}\{f\}$ uniformly in $\mathbb{R}$.*

8. *The Fourier transform $\mathscr{F}\{f\}$ is a continuous function in $\mathbb{R}$, $\mathscr{F}\{f\} \in C(\mathbb{R})$.*

Proof. We will numerate each proof

1. Since $\left|\int f\right| \leq \int |f|$, we have

$$\left|\hat{f}(\xi)\right| = \frac{1}{\sqrt{2\pi}}\left|\int\limits_{\mathbb{R}} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x\right| \leq \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} |f(x)|\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\|f\|_1,$$

and since this occurs for all $\xi \in \mathbb{R}$, we conclude

$$\|\mathscr{F}\{f\}\|_\infty = \sup_{x\in\mathbb{R}}\left|\hat{f}(\xi)\right| \leq \frac{1}{\sqrt{2}}\|f\|_1.$$

2. Let $\alpha, \beta \in \mathbb{C}$ two numbers and $f, g \in L^1(\mathbb{R})$ two functions. For all $\xi \in \mathbb{R}$ we have

$$\mathscr{F}\{\alpha f + \beta g\}(\xi) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} (\alpha f(x) + \beta g(x))\mathrm{e}^{-i\xi x}\,\mathrm{d}x = \alpha\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x +$$
$$\beta\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} g(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x = \alpha\hat{f}(\xi) + \beta\hat{g}(\xi) = \alpha\mathscr{F}\{f\} + \beta\mathscr{F}\{g\},$$

where we have used the property of linearity of the integral to separate both terms, since their convergence is uniform.

3. If we make the change of variable $t = -x$ and $\mathrm{d}t = -\,\mathrm{d}x$, we have

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} \overline{f(x)}\mathrm{e}^{i\xi x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} \overline{f(x)\mathrm{e}^{i\xi x}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\overline{\int\limits_{\mathbb{R}} f(x)\mathrm{e}^{-i(-\xi)x}\,\mathrm{d}x} =$$
$$\overline{\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(x)\mathrm{e}^{-i(-\xi)x}\,\mathrm{d}x} = \overline{\hat{f}(-\xi)}.$$

4. We make the change of variable $t = \lambda x$ and $\mathrm{d}t = \lambda\,\mathrm{d}x$ and get

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(\lambda x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x = \frac{1}{\lambda}\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(t)\mathrm{e}^{-i\xi t/\lambda}\,\mathrm{d}t = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right).$$

5. Now we make the change of variable $t = x - a$ and $\mathrm{d}t = \mathrm{d}x$ and get

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(x-a)\mathrm{e}^{-i\xi x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(t)\mathrm{e}^{i\xi(t+a)}\,\mathrm{d}t = \mathrm{e}^{-i\xi a}\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(t)\mathrm{e}^{-i\xi t}\,\mathrm{d}t =$$
$$\mathrm{e}^{-i\xi a}\hat{f}(\xi).$$

6. Computing directly the integral,

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(x)\mathrm{e}^{iax}\mathrm{e}^{-i\xi x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} f(x)\mathrm{e}^{-i(\xi-a)x}\,\mathrm{d}x = \hat{f}(\xi - a).$$

7. Let us observe the norm $\|\cdot\|_\infty$ is the supremum norm, which gives us the notion of uniform convergence. To see $\mathscr{F}\{f_n\} \to \mathscr{F}\{f\}$ uniformly on $\mathbb{R}$, we need then to prove that $\|\mathscr{F}\{f_n\} - \mathscr{F}\{f\}\|_\infty$ equals to $\sup_{\xi\in\mathbb{R}}|\mathscr{F}\{f_n\}(\xi) - \mathscr{F}\{f\}(\xi)| \to 0$ when n tends to infinity. By the part 1 we have

$$\|\mathscr{F}\{f_n\} - \mathscr{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}}\|f_n - f\|_1 \to 0, \text{ when } n \to \infty, \tag{15.10}$$

since f_n converges uniformly to f with norm $\|\cdot\|_1$ by hypothesis.

8. Proven in proposition 15.1.6

∎

Proposition 15.1.3. *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty} |f(x)| = 0$. Then,*

$$\hat{f}'(\xi) = i\xi \hat{f}(\xi). \tag{15.11}$$

Corollary 15.1.4. *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its n-th derivative $f^{(n)} \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty} |f(x)| = 0$. Then,*

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi). \tag{15.12}$$

We will see that the application of the Fourier transform moves functions from $L^1(\mathbb{R})$ to functions of $C_0(\mathbb{R}) = \left\{f \in C(\mathbb{R}) \,\middle|\, \lim_{|x|\to\infty} f(x) = 0\right\}$, that is, $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.

Definition 15.1.3. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We define the support of f as

$$\operatorname{supp} f \coloneqq \overline{\{x \in I \,|\, f(x) \neq 0\}}. \tag{15.13}$$

Definition 15.1.4. We define the set $\mathscr{D}(\mathbb{R})$ as

$$\mathscr{D}(\mathbb{R}) \coloneqq \{\varphi \in C^\infty(\mathbb{R}) \,|\, \operatorname{supp} \varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \tag{15.14}$$

Theorem 15.1.5. *Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_\in \mathscr{D}(\mathbb{R})$ such that*

$$\lim_{h\to\infty} \int_{\mathbb{R}} |f - \phi_n| \, \mathrm{d}x = 0, \tag{15.15}$$

that is, we have convergence of ϕ_n to f with norm $\|\cdot\|_1$.

Proposition 15.1.6. *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.*

Proposition 15.1.7. *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\left|\hat{f}(\xi)\right| \leq \|f\|_1$.*

Theorem 15.1.8. *Let $f \in L^1(\mathbb{R})$ be a function. Then,*

$$\lim_{|x|\to\infty} \hat{f}(x) = 0. \tag{15.16}$$

Theorem 15.1.9. *The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.*

Definition 15.1.5. We define the *Schwartz space* as

$$\begin{gathered} S(\mathbb{R}) \coloneqq \{f : \mathbb{R} \longrightarrow \mathbb{C} \,|\, f \in C^\infty(\mathbb{R}) \wedge \forall n, m \in \mathbb{N}\, \exists c_{n,m} < \infty \\ \text{such that } (1 + |x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}. \end{gathered}$$

From the previous definition we deduce that if $f \in S(\mathbb{R})$, then $f^{(n)} \in S(\mathbb{R})$ for all $n \in \mathbb{N}$. Besides, the condition of $(1 + |x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}$ is equivalent to say that $\|f\|_{n,m} < \infty$, where $\|\cdot\|_{n,m}$ is the norm define from $\|\cdot\|_\infty$ by

$$\|f\|_{n,m} \coloneqq \|x^m D^n f\|_\infty = \sup_{x\in\mathbb{R}} \left| x^m \frac{\mathrm{d}^n f}{\mathrm{d}x^n} \right|. \tag{15.17}$$

Finally, we see that $\mathscr{D}(\mathbb{R}) \subseteq S(\mathbb{R})$.

Proposition 15.1.10. *Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \longrightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,*

1. *$f + g \in S(\mathbb{R})$.*

2. $\lambda f \in S(\mathbb{R})$.
3. $fg \in S(\mathbb{R})$.
4. $Pf \in S(\mathbb{R})$.

Theorem 15.1.11. *Let $I, J \subseteq \mathbb{R}$ be two intervals with I compact and J open. Let $f : I \times J \longrightarrow \mathbb{R}$ be a function such that*

1. *$f(\cdot, \lambda)$ is Riemann-integrable in I for all $\lambda \in J$,*
2. *$f(x, \cdot)$ is derivable in J for all $x \in I$.*

If $\partial_\lambda f$ is continuous in $I \times J$, then

1. *$\partial_\lambda f(\cdot, \lambda)$ is Riemann-integrable for all $\lambda \in J$.*
2. *$F(\lambda) = \int\limits_I f(x, \lambda)\, \mathrm{d}x$ is derivable with continuous derivative in J for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.*

$$F'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int\limits_I f(x, \lambda_0)\, \mathrm{d}x = \int\limits_I \frac{\partial f}{\partial \lambda}(x, \lambda_0)\, \mathrm{d}x\,, \forall \lambda_0 \in J \tag{15.18}$$

Proposition 15.1.12. *Let $f \in S(\mathbb{R})$. Then,*

1. *$S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.*
2. *$\widehat{xf}(\xi) = (iD_\xi \hat{f})(\xi)$ for all $\xi \in \mathbb{R}$.*

Corollary 15.1.13. *Let $f \in s(\mathbb{R})$. Then,*

$$\widehat{x^n f}(\xi) = (i^n D^n \hat{f})(\xi), \forall n \in \mathbb{N}. \tag{15.19}$$

Proposition 15.1.14. *The Fourier transform $\mathscr{F}$ restricted to $S(\mathbb{R})$ is an automorphism, that is, if $f \in S(\mathbb{R})$ then $\mathscr{F}\{f\} = \hat{f} \in S(\mathbb{R})$.*

15.2 Fourier inversion theorem in Schwartz space

Lemma 15.2.1. *If $G(x) = \mathrm{e}^{-x^2/2}$, then $\hat{G}(\xi) = e^{-\xi^2/2}$. We observe hence that G is a fixed point of $\mathscr{F}$.*

Lemma 15.2.2. *If $f, g \in S(\mathbb{R})$, then*

$$\int\limits_{\mathbb{R}} f(\xi)\hat{g}(\xi)\, \mathrm{d}\xi = \int\limits_{\mathbb{R}} \hat{f}(\tau) g(\tau)\, \mathrm{d}\tau\,. \tag{15.20}$$

Lemma 15.2.3. *Let $f, g \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,*

1. *$g(\lambda x)\hat{f}(x)$ converges to $g(0)\hat{f}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*
2. *$f(\lambda x)\hat{g}(x)$ converges to $f(0)\hat{g}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*

Lemma 15.2.4. *Let $f, g \in s(\mathbb{R})$. Then,*

$$f(0) \int\limits_{\mathbb{R}} \hat{g}(\xi)\, \mathrm{d}\xi = g(0) \int\limits_{\mathbb{R}} \hat{f}(\xi)\, \mathrm{d}\xi\,. \tag{15.21}$$

Lemma 15.2.5. *Let $f \in s(\mathbb{R})$ be a function. Then,*

$$f(0) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \hat{f}(\xi)\, \mathrm{d}\xi\,. \tag{15.22}$$

Corollary 15.2.6 (Inversion formula). *Let $f \in S(\mathbb{R})$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) \mathrm{e}^{ix\xi} \, \mathrm{d}\xi \,, \forall x \in \mathbb{R}. \tag{15.23}$$

Theorem 15.2.7 (Inversion of $\mathscr{F}$ in $S(\mathbb{R})$). *Let $\mathscr{F} : S(\mathbb{R}) \longrightarrow S(\mathbb{R})$, defined by $\mathscr{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, $\mathscr{F}$ is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathscr{F}^4 = Id$. In particular, $\mathscr{F}^{-1} = \mathscr{F}^3$ and if $f \in S(\mathbb{R})$, then*

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{F}\{f\}(\xi) \mathrm{e}^{ix\xi} \, \mathrm{d}\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} \mathrm{e}^{ix\xi} \, \mathrm{d}\xi \,.} \tag{15.24}$$

In fact, $\mathscr{F}$ is an homemorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(S(\mathbb{R}), \|\cdot\|_{n,m})$.

Theorem 15.2.8 (Inversion of $\mathscr{F}$ for discontinuities). *Let f be a absolutely Riemann-integrable function in $\mathbb{R}$ with f and f' piece-wise continuous. Then,*

$$\frac{f(x^-) + f(x^+)}{2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} \mathrm{e}^{ix\xi} \, \mathrm{d}\xi \,. \tag{15.25}$$

Even and odd components

Definition 15.2.1. Let f be a Riemann-integrable function in $\mathbb{R}$. We define the *Fourier transform of cosine kind* as

$$\hat{f}_c(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\xi x) f(x) \, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(\xi x) f_e(x) \, \mathrm{d}x \,, \tag{15.26}$$

and the *Fourier transform of sine kind* as

$$\hat{f}_s(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\xi x) f(x) \, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(\xi x) f_o(x) \, \mathrm{d}x \,. \tag{15.27}$$

Proposition 15.2.9. *Let $\hat{f}_c, \hat{f}_s$ be the Fourier transform of cosine and sine kinds of f. Then, $\hat{f}_c(\xi)$ is even, $\hat{f}_s(\xi)$ is odd, and $\hat{f}(\xi) = \hat{f}_c(\xi) - i\hat{f}_s(\xi)$.*

Theorem 15.2.10. *Let f be a absolutely Riemann-integrable function in $\mathbb{R}$ with f and f' piece-wise continuous. Then,*

$$\frac{f_e(x^-) + f_e(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_c \cos(\xi x) \, \mathrm{d}\xi \,, \tag{15.28}$$

$$\frac{f_o(x^-) + f_o(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_s \sin(\xi x) \, \mathrm{d}\xi \,. \tag{15.29}$$

15.3 Inversion theorem in the algebra $L^1(\mathbb{R})$. Approximation theory

Theorem 15.3.1 (Tonelli's Theorem). *Let $f : I \times J \longrightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x, y) \geq 0$ for all $(x, y) \in I \times J$. Then,*

$$\int_{I \times J} f \, \mathrm{d}x \, \mathrm{d}y = \int_{I} \int_{J} f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_{J} \int_{I} f(x, y) \, \mathrm{d}x \, \mathrm{d}y \,. \tag{15.30}$$

Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.

Notice that Tonelli's Theorem is similar to Fubini's Theorem but we do not impose the function to be Riemann-integrable, only to be positively defined.

Corollary 15.3.2. *Let $f, g \in L^1(\mathbb{R})$. Then, $F(x,t) = f(t)g(x-t) \in L^1(\mathbb{R}^2)$.*

Definition 15.3.1. Let $f, g \in L^1(\mathbb{R})$ two function. We define the *convolution of f and g* as

$$\begin{aligned} (f * g) : \mathbb{R} &\longrightarrow \mathbb{C} \\ x &\longmapsto \int_{\mathbb{R}} f(t)g(x-t)\,\mathrm{d}t \end{aligned} \tag{15.31}$$

which is from $L^1(\mathbb{R})$.

Proposition 15.3.3. *Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$.*

Proof. By direct integration,

$$\widehat{f * g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (f * g)(x) \mathrm{e}^{-i\xi x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)g(x-t)\,\mathrm{d}t\, \mathrm{e}^{-i\xi x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)\mathrm{e}^{-i\xi t} g(x-t)\mathrm{e}^{-i\xi(x-t)}\,\mathrm{d}t\,\mathrm{d}x\,.$$

Now, by Fubini's theorem,

$$\begin{aligned} \widehat{f * g}(\xi) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x-t)\mathrm{e}^{-i\xi(x-t)}\,\mathrm{d}x\, f(t)\mathrm{e}^{-i\xi t}\,\mathrm{d}t\,, \qquad s = x - t \\ \widehat{f * g}(\xi) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s)\mathrm{e}^{-i\xi x}\,\mathrm{d}s\, f(t)\mathrm{e}^{-i\xi t}\,\mathrm{d}t = \int_{\mathbb{R}} \hat{g}(\xi) f(t)\mathrm{e}^{-i\xi t}\,\mathrm{d}t = \\ &\hat{g}(\xi)\sqrt{2\pi}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\mathrm{e}^{-i\xi t}\,\mathrm{d}t = \sqrt{2\pi}\hat{g}(\xi)\hat{f}(\xi). \end{aligned}$$

∎

Proposition 15.3.4. *Let $f \in L^1(\mathbb{R})$ be a function and $g = f^2$. Then,*

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \hat{f} * \hat{f} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t)\hat{f}(\xi - t)\,\mathrm{d}t\,. \tag{15.32}$$

Theorem 15.3.5. *Let $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^\infty(\mathbb{R})$.*

Theorem 15.3.6. *Let $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$ with $\operatorname{supp} f$ compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and $\operatorname{supp}\{f * \phi\} \subseteq \operatorname{supp} f + \operatorname{supp} \phi$.*

Definition 15.3.2. We say the functions $\phi_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ continuous in a compact support are *an approximation of the unity* if and only if

1. $\phi_\epsilon \geq 0$ for all ϵ.
2. $\displaystyle\int_{\mathbb{R}} \phi_\epsilon(x)\,\mathrm{d}x = 1$.
3. For all $\delta > 0$ it is satisfied that

$$\lim_{\epsilon \to 0} \left\{ \sup_{|t| > \delta} \phi_\epsilon(t) \right\} = 0. \tag{15.33}$$

Theorem 15.3.7. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to f.*

Corollary 15.3.8. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to f.*

Theorem 15.3.9 (Weierstrass polynomial approximation)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials P_n with $n \in \mathbb{N}$ such that P_n converge uniformly to f in $[a, b]$.*

Theorem 15.3.10. *Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \to f$ with norm $\|\cdot\|_p$ (that is, convergence in L^p), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then*

$$\lim_{n\to\infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \tag{15.34}$$

with $\|f\|_{C^k(\mathbb{R})} = \max_{0\leq l\leq k} \left(\sup_{x\in\mathbb{R}} \left|D^l f(x)\right|\right)$ being a norm.

Lemma 15.3.11. *Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\int\limits_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x = 0$. Then, $f \equiv 0$.*

Corollary 15.3.12. *The Fourier transform $\mathscr{F}$ is injective since $\mathscr{F}\{f\} = \hat{f} = 0 \Leftrightarrow f = 0$ in $L^1(\mathbb{R})$ (the zero function class) and $\mathscr{F}$ is a linear application.*

Theorem 15.3.13 (Inversion theorem in $L^1(\mathbb{R})$)**.** *Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \hat{f}(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,, \forall x \in \mathbb{R}. \tag{15.35}$$

15.3.1 Examples

$f(x-a)$	$\mathrm{e}^{-ia\xi}\hat{f}(\xi)$
$f(ax)$	$\frac{1}{\lvert a\rvert}\hat{f}\left(\frac{\xi}{a}\right)$
$\hat{f}(x)$	$f(-\xi)$
$D^n f(x)$	$(i\xi)^n\hat{f}(\xi)$
$x^n f(x)$	$i^n D^n\hat{f}(\xi)$
$(f * g)(x)$	$\sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$\frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(\xi)$
$\overline{f(x)}$	$\overline{\hat{f}(-\xi)}$
$f(x)\cos(ax)$	$\frac{\hat{f}(\xi-a)+\hat{f}(\xi+a)}{2}$
$f(x)\sin(ax)$	$\frac{\hat{f}(\xi-a)-\hat{f}(\xi+a)}{2i}$

Table 15.1: Caption

Chapter 16

Complex Fourier transform in $L^2(\mathbb{R})$

Theorem 16.0.1 (Parseval formula)**.** *Let $f, g \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be two functions. Then,*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi\,. \tag{16.1}$$

Theorem 16.0.2 (Plancherel Theorem)**.** *Let $f \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be a function. Then,*

$$\boxed{\int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2\mathrm{d}\xi\,,} \tag{16.2}$$

that is, $\|f\|_2 = \left\|\hat{f}_2\right\|$ and $\mathscr{F}$ is an isometry between vector spaces.

Definition 16.0.1. Let $f \in S(\mathbb{R})$ be a function. We define the following quantities

$$E(f) \coloneqq \int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x\,, \tag{16.3}$$

$$\sigma(f)^2 \coloneqq \int_{\mathbb{R}} |xf(x)|^2\,\mathrm{d}x\,. \tag{16.4}$$

Theorem 16.0.3. *Let $f \in S(\mathbb{R})$ be a function. Then,*

$$\sigma(f)\sigma(\hat{f}) \geq \frac{E(f)}{2}. \tag{16.5}$$

16.1 Examples

$\mathrm{rect}(ax)$	$\frac{1}{\sqrt{2\pi}\lvert a\rvert}\,\mathrm{sinc}\left(\frac{\xi}{2\pi a}\right)$
$\mathrm{sinc}(ax)$	$\frac{1}{\sqrt{2\pi}\lvert a\rvert}\,\mathrm{rect}\left(\frac{\xi}{2\pi a}\right)$
$\mathrm{sinc}^2(ax)$	$\frac{1}{\sqrt{2\pi}\lvert a\rvert}\,\mathrm{tri}\left(\frac{\xi}{2\pi a}\right)$
$\mathrm{tri}(ax)$	$\frac{1}{\sqrt{2\pi}\lvert a\rvert}\,\mathrm{sinc}^2\left(\frac{\xi}{2\pi a}\right)$
$\theta(x)\mathrm{e}^{-ax}$	$\frac{1}{\sqrt{2\pi}}\frac{1}{a+i\xi}$
$\mathrm{e}^{-\alpha x^2}$	$\frac{1}{\sqrt{2\alpha}}\mathrm{e}^{-\xi^2/4\alpha}$
e^{-iax^2}	$\frac{1}{\sqrt{2\alpha}}\mathrm{e}^{\left(\frac{\xi^2}{4\alpha}-\frac{\pi}{4}\right)}$
$\mathrm{e}^{-a\lvert x\rvert}$	$\frac{2}{\pi}\frac{a}{a^2+\xi^2}$
$\mathrm{sech}(ax)$	$\sqrt{\frac{\pi}{2}}\frac{1}{a}\,\mathrm{sech}\left(\frac{\pi\xi}{2a}\right)$
$\mathrm{e}^{-a^2x^2/2}H_n(ax)$	$\frac{(-i)^n}{a}\mathrm{e}^{-\xi^2/2a^2}H_n\left(\frac{\xi}{a}\right)$

Table 16.1: Caption

Chapter 17

Multidimensional Fourier transform

Theorem 17.0.1. *For several variables*

$$\mathscr{F}\{f(x_1,\dots,x_n)\} = \frac{1}{(2\pi)^{n/2}} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} f(x_1,\dots,x_n) \mathrm{e}^{-i(x_1\xi_1+\cdots+x_n\xi_n)} \,\mathrm{d}x_1 \dots \mathrm{d}x_n \,, \tag{17.1}$$

or simpler,

$$\boxed{\mathscr{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{e}^{-i\langle \mathbf{x},\boldsymbol{\xi}\rangle_I} \,\mathrm{d}\hat{x} \,.} \tag{17.2}$$