1 Harmonic oscillator

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space in a harmonic potential

$$V(\hat{x}) = \frac{\omega^2}{2}\hat{x}^2, \qquad \omega^2 = \frac{k}{m}. \tag{1}$$

We define the *creation* and *annihilation operators* as

$$\hat{a}^\dagger \coloneqq \frac{\alpha}{\sqrt{2}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right), \tag{2}$$

$$\hat{a} \coloneqq \frac{\alpha}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \tag{3}$$

$$\alpha \coloneqq \sqrt{\frac{m\omega}{\hbar}}. \tag{4}$$

Proposition 1.1. *Let $\mathcal{H}$ be a Hilbert space in a harmonic potential. Then,*

$$\langle x|\,\hat{a}^\dagger = \frac{\alpha}{\sqrt{2}}\left(x - \frac{1}{\alpha^2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{5}$$

$$\langle x|\,\hat{a} = \frac{\alpha}{\sqrt{2}}\left(x + \frac{1}{\alpha^2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{6}$$

$$\alpha = \frac{m\omega}{\hbar}. \tag{7}$$

Proposition 1.2. *Let $\mathcal{H}$ be a Hilberet space with a harmonic potential. Then,*

$$\hat{x} = \frac{1}{\sqrt{2}\alpha}(\hat{a}^\dagger + \hat{a}), \qquad \hat{p} = i\hbar\frac{\alpha}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}). \tag{8}$$

Proposition 1.3. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{a}, \hat{a}^\dagger$ are not hermitian.*
2. *$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$.*
3. *$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$.*

Definition 1.2. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *number operator* as

$$\hat{N} \coloneqq \hat{a}^\dagger\hat{a}. \tag{9}$$

Proposition 1.4. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{H}$ is hermitian.*
2. *$\left[\hat{N}, \hat{a}\right] = -\hat{a}$, $\left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger$,*
3. *$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\hat{I}\right)$.*

Proposition 1.5. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, $\hat{H}$ and $\hat{N}$ have a common basis of eigenvectors, which is countable, and*

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \qquad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{10}$$

$$\hat{N}\,|n\rangle = n\,|n\rangle\,, \qquad \hat{H}\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle\,, \tag{11}$$

$$n \in \mathbb{N}. \tag{12}$$

Corollary 1.6. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle\,. \tag{13}$$

Proposition 1.7. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, the eigenstates form a nondgenerate basis.*

Definition 1.3 (Fock states)**.** Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *Fock states* as the states that determine the basis $(|n\rangle)$ and have a well-defined number of excitations.

Definition 1.4. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We call the fundamental Fock state *the vaccum.*

Proposition 1.8. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, $\hat{a}, \hat{a}^\dagger$ and $\hat{N}$ have the following matrix representation in the basis $(|n\rangle)$.*

$$[\hat{N}]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{14}$$

$$[\hat{a}]_B = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ 0 & 0 & \sqrt{2} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{15}$$

$$[\hat{a}^\dagger]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{16}$$

or in coefficient representation,

$$[\hat{N}]_{ij} = (i-1)\delta_{ij}, \tag{17}$$

$$[\hat{a}]_{ij} = \sqrt{j-1}\delta_{i,j-1}, \tag{18}$$

$$[\hat{a}^\dagger]_{ij} = \sqrt{i-1}\delta_{i-1,j}. \tag{19}$$

Proposition 1.9. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\varphi_0(x) = \langle x|0\rangle = \left(\frac{\beta^2}{\pi}\right)^{1/4}\exp\left(-\frac{\alpha^2x^2}{2}\right), \tag{20}$$

$$\varphi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{\beta}{\sqrt{2}}x - \frac{1}{\sqrt{2}\beta}\frac{\mathrm{d}}{\mathrm{d}x}\right)\varphi_0(x) = \tag{21}$$

$$\frac{1}{\sqrt{2^n n!}}H_n(\beta x)\varphi_0(x). \tag{22}$$

Proposition 1.10. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $\hat{\sigma}$ a sequence formed by k $\hat{a}$ and l $\hat{a}^\dagger$. Then,*

$$\langle n|\,\hat{\sigma}\,|n\rangle \leftrightarrow k = l. \tag{23}$$

Proposition 1.11. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\langle \hat{x} \rangle_n = 0, \qquad \langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega}(2n+1), \tag{24}$$

$$\langle \hat{p} \rangle_n = 0, \qquad \langle \hat{p}^2 \rangle = \frac{\hbar m\omega}{2}(2n+1), \tag{25}$$

$$\Delta x \Delta p = \frac{\hbar}{2}(2n+1). \tag{26}$$

Proposition 1.12. *Let $\mathcal{H}$ a Hilbert space with a harmonic potential. Then,*

$$\langle T \rangle = \langle V \rangle . \tag{27}$$

Definition 1.5. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define a *coherent state* as a state $|\alpha\rangle \in \mathcal{H}$ such that

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle . \tag{28}$$

Definition 1.6. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *displaced state* as the state $|\psi_\alpha\rangle \in \mathcal{H}$ determined by

$$\psi_\alpha(x) = \psi_0(x - x_0). \tag{29}$$

Proposition 1.13. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and a force $F = f$. Then, the fundamental state is a displaced state with $x_0 = f/m\omega^2$.*

Proposition 1.14. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\psi_\alpha\rangle \in \mathcal{H}$ a displaced state with displacement x_0. Then, $|\psi_\alpha\rangle$ is a coherent state with eigenvalue*

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}} x_0. \tag{30}$$

Proposition 1.15. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha \hat{a}^\dagger} |0\rangle . \tag{31}$$

Proposition 1.16. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle$ a coherent state. Then,*

$$\left\langle \hat{N} \right\rangle_\alpha = |\alpha|^2, \qquad p_{|\alpha\rangle}(n) = \mathrm{e}^{-\langle \hat{N} \rangle} \frac{\left\langle \hat{N} \right\rangle^n}{n!}. \tag{32}$$

Theorem 1.17 (Baker-Campbell-Hausdorff formula)**.** *Let $\mathcal{H}$ be a Hilbert space and $\hat{A}, \hat{B} : \mathcal{H} \longrightarrow \mathcal{H}$ two operators such that $\left[\left[\hat{A}, \hat{B}\right], \hat{A}\right], \left[\left[\hat{A}, \hat{B}\right], \hat{B}\right] = 0$. Then,*

$$\exp\left(\hat{A} + \hat{B}\right) = \exp\left(-\frac{1}{2}\left[\hat{A}, \hat{B}\right]\right) \exp\left(\hat{A}\right) \exp\left(\hat{B}\right). \tag{33}$$

Proposition 1.18. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\left[\bar{\alpha}\hat{a}, \alpha\hat{a}^\dagger\right] = |\alpha|^2 \hat{I}, \tag{34}$$

$$|\alpha\rangle = \exp\left(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}\right) |0\rangle \coloneqq \hat{\mathcal{D}}(\alpha) |0\rangle . \tag{35}$$

Definition 1.7. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *displacement operator* as

$$\hat{\mathcal{D}}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}\right). \tag{36}$$

Proposition 1.19. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{\lceil}(\alpha)$ is unitary.*
2. *$\hat{\mathcal{D}}^\dagger(\alpha) = \hat{\mathcal{D}}(-\alpha)$.*
3. *$\hat{\mathcal{D}}(\alpha)\hat{\mathcal{D}}^\dagger(\alpha) = \hat{I}$.*

Proposition 1.20. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\hat{\mathcal{D}}(\alpha) = \exp\left(-i\frac{x_0\hat{p} - p_0\hat{x}}{\hbar}\right) = \tag{37}$$

$$\exp\left(-\frac{i}{2}\frac{x_0p_0}{\hbar}\right) \exp\left(i\frac{p_0\hat{x}}{\hbar}\right) \exp\left(-i\frac{x_0\hat{p}}{\hbar}\right), \tag{38}$$

$$x_0 = \sqrt{2}l \operatorname{Re}\{\alpha\}, \qquad p_0 = \sqrt{2}\frac{l}{\hbar} \operatorname{Im}\{\alpha\}, \tag{39}$$

$$l = \sqrt{\frac{\hbar}{m\omega}}. \tag{40}$$

Proposition 1.21. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\langle x|\alpha\rangle = \psi_\alpha(x) = \tag{41}$$

$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(\frac{ip_0}{2\hbar}(2x - x_0)\right) \exp\left(-\frac{(x - x_0)^2}{4\sigma_x^2}\right), \tag{42}$$

$$\frac{1}{4\sigma_x^2} = \frac{1}{2}\frac{m\omega}{\hbar} \tag{43}$$

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re}\{\alpha\}, \qquad p_0 = \sqrt{2\hbar m\omega} \operatorname{Im}\{\alpha\} \tag{44}$$

Proposition 1.22. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $\{|\alpha\rangle\}$ the set of coherent states. Then, they form a* overcomplete basis, *that is, not for all pair of states $|\alpha\rangle, |\alpha'\rangle$ it is satisfied $\langle\alpha'|\alpha\rangle = 0$. Hence,*

$$\hat{I} = \frac{1}{\pi} \int |\alpha\rangle\langle\alpha| \, \mathrm{d}^2\alpha \,, \qquad |\langle\alpha|\beta\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2}. \tag{45}$$

Besides,$\langle\alpha|\beta\rangle \to 0$ if and only if $|\alpha - \beta| \gg 1$.

Proposition 1.23. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$|\alpha\rangle(t) = \mathrm{e}^{i\omega t/2} |\alpha(t)\rangle = \mathrm{e}^{i\omega t/2} \left|\alpha_0 \mathrm{e}^{i\omega t}\right\rangle . \tag{46}$$

Proposition 1.24. *Let $\mathcal{H}$ be a Hilbert space witha harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\langle \hat{x} \rangle = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t), \tag{47}$$

$$\langle \hat{p} \rangle = p_0 \cos(\omega t) - m\omega x_0 \sin(\omega t). \tag{48}$$

Definition 1.8. Let $\mathcal{H}$ be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ a state. We say $|\psi\rangle$ is a *minimum uncertainty state* if and only if

$$\Delta x \Delta p = \frac{\hbar}{2}. \tag{49}$$

Proposition 1.25. *Let $\mathcal{H}$ be a Hilbert state, $|\in\rangle\,\mathcal{H}$ a state and $|\psi_x\rangle = \hat{\delta x}\,|\psi\rangle\,, |\psi_p\rangle = \hat{\delta p}\,|\psi\rangle$. Then,*

$$\langle\psi_x|\psi_x\rangle \langle\psi_p|\psi_p\rangle \geq |\langle\psi_x|\psi_p\rangle|^2. \tag{50}$$

and the equality only occures when there exists a $\lambda \in \mathbb{C}$ such that $|\psi_p\rangle = \lambda\,|\psi_x\rangle$.

Proposition 1.26. *Let $\mathcal{H}$ be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ be a state. Then,*

$$\left|\langle\psi|\hat{\delta x}\hat{\delta p}|\psi\rangle\right|^2 \geq \frac{1}{4}\left|\langle\psi|\left[\hat{\delta x}, \hat{\delta p}\right]|\psi\rangle\right|^2, \tag{51}$$

and the equality only occures when $\{\hat{\delta x}, \hat{\delta p}\} = 0$.

Proposition 1.27. *Let $\mathcal{H}$ be a Hilbert space and $|\in\rangle\,\mathcal{H}$ a minimum uncertainty state. Then,*

$$\langle x|\psi\rangle = \psi(x) = \tag{52}$$

$$C \exp\left[-\frac{|\lambda|}{2}(x - \langle x\rangle)^2\right] \exp\left[\frac{ix\,\langle p\rangle}{\hbar}\right], \tag{53}$$

for some $\lambda \in \mathbb{C}$ and with variance $\Delta x^2 = \hbar/2|\lambda|$.