Contents

Chapter 1

Introduction

Definition 1.0.1. Let $\mathbb{R}^2 = \{(x.y) \,|\, x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the sum "+" by components
$$(a, b) + (c, d) := (a + b, c + d). \tag{1.1}$$
- Product: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the product by
$$(a, b)(c, d) = (ac - bd, ad + bc). \tag{1.2}$$

We define the set $\mathbb{C}$ as $(\mathbb{R}^2, +,)$.

Proposition 1.0.1. *The set $\mathbb{C}$ of complex numbers is an abelian field.*

This is only one possible formulation, but we will use another that, as we will prove now, it is completely equivalent. Now, we define $\mathbb{C} = \left\{a + bi \,\middle|\, a, b \in \mathbb{R}, i^2 = -1\right\}$, and we use the addition and subtraction as before:

$$\begin{gathered}(a + bi) + (c + di) = (a + c) + (c + d)i \\ (a + bi)(c + di) = \cdots = (ac - bd) + (ad + bc)i.\end{gathered}$$

To see both formulations are equivalent, we can express the second in terms of the first only adding the definition $i = (0, 1)$. This way, every element (x, y) can be expressed ad $x + iy$ as follows.

$$z = (x, y) = (x, 0)(1, 0) + (y, 0)(0, 1) = x \cdot 1 + y \cdot i = x`iy \tag{1.3}$$

Proposition 1.0.2. *Let $\mathbb{C}$ be defined in the second way. Then,*

1. *$\mathbb{C}$ is an abelian ring.*
2. *If we define f as*
$$\begin{aligned} f : (\mathbb{C}, +,) &\longrightarrow (\mathbb{R}^2, +,) \\ (x, y) &\longmapsto x + yi \end{aligned}, \tag{1.4}$$
then f is a morphism of rings.
3. *The function f is, in fact, an isomorphism and $\mathbb{C}$ is an abelian field.*

From this isomorphism, we see two complex numbers are equal if and only if $a = a'$ and $b = b'$. Besides, $(\mathbb{C}^*,)$ is an abelian group. In order to simplify the expressions, the following notation will be used.

$$zw = z \cdot w, \qquad z - w = z + (-w), \qquad 1/z = z^{-1}, \qquad z/w = zw^{-1} \tag{1.5}$$

Proposition 1.0.3. *The subset of $\mathbb{C}$ generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.*

Theorem 1.0.4. *$\mathbb{C}$ is not an ordered field.*

1.1 Topology

Definition 1.1.1. Let $z = a + bi \in \mathbb{C}$. We define the *conjugate of z* as

$$\bar{z} := a - bi. \tag{1.6}$$

Proposition 1.1.1. *For all $z, w \in \mathbb{C}$, we have:*

1. *$\bar{\bar{z}} = z$.*
2. *$\overline{z + w} = \bar{z} + \bar{w}$.*
3. *$\overline{zw} = \bar{z}\bar{w}$.*

4. *$z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.*

5. *$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.*

6. *The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.*

Definition 1.1.2. Let $z = a + bi \in \mathbb{C}$. We define the *real part of z and imaginary part of z* respectively as

$$\mathrm{Re}\{z\} := a, \qquad \mathrm{Im}\{z\} := b. \tag{1.7}$$

Proposition 1.1.2. *Let $z \in \mathbb{C}$. Then,*

$$\mathrm{Re}\{z\} = \frac{z + \bar{z}}{2}, \qquad \mathrm{Im}\{z\} = \frac{z - \bar{z}}{2i} \tag{1.8}$$

Proposition 1.1.3. *Let $z, w \in \mathbb{C}$ and the following distance function.*

$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto \tilde{d}(z, w) := |z - w| \end{aligned} \tag{1.9}$$

Then, $(\mathbb{C}, \tilde{d})$ is a metric space.

Definition 1.1.3. Let $z = a + bi \in \mathbb{C}$. We define the *modulus of z* as

$$|z| := \tilde{d}(z, 0), \tag{1.10}$$

which is equivalent to $\sqrt{z\bar{z}}$.

Definition 1.1.4. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define an *open disc of radius r and center z_0* as follows

$$B_r(z_0) := \{z \in \mathbb{C} \,|\, |z - z_0| < r\}. \tag{1.11}$$

Definition 1.1.5. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *punctured disc of radius r and center z_0* as follows

$$B_r^*(z_0) := \{z \in \mathbb{C} \,|\, 0 < |z - z_0| < r\}. \tag{1.12}$$

Definition 1.1.6. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *closed disc of radius r and center z_0* as follows

$$\overline{B_r(z_0)} := \{z \in \mathbb{C} \,|\, |z - z_0| \le r\}. \tag{1.13}$$

Definition 1.1.7. We denote by $\mathbb{D}$ the unitary disc of center 0 and radius 1. Besides, we denote by $\mathbb{T} \subseteq \mathbb{C}$ the unitary circumference, that is,

$$\mathbb{T} := \{z \in \mathbb{C} \,|\, |z| = 1\}. \tag{1.14}$$

We also denote it by $\mathbb{S}^1$.

Lemma 1.1.4. *The set $B = \left\{B_r(z_0) \,\middle|\, r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\right\}$ is a basis of the topology of $\mathbb{R}^2$ as a metric space.*
The set $D = \{D_r(z_0) \,|\, r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of $\mathbb{C}$ as a metric space.

The concepts of interior, exterior, boundary, and accumulation points are the same than those presented in Multivariable Calculus Notes. The same for the rest of topological definitions.

Proposition 1.1.5. *The sets $\mathbb{C}$ and $\mathbb{R}^2$ with the topology of metric space are homeomorphs.*

Corollary 1.1.6. *There is a bijection between B and D, that is, between balls of $\mathbb{R}^2$ and discs of $\mathbb{C}$.*

Proposition 1.1.7. *Let $z, w \in \mathbb{C}$. Then,*

1. *$|z| \ge 0$.*

2. *$|z| = 0 \Leftrightarrow z = 0$.*

3. $-|z| \leq \operatorname{Re}\{z\} \leq |z|$ *and* $-|z| \leq \operatorname{Im}\{z\} \leq |z|$.

4. $|zw| = |z||w|$.

5. *If* $w \neq 0$, $|z/w| = |z|/|w|$.

6. $|z+w| \leq |z| + |w|$.

7. $|z+w| \geq ||z| - |w||$.

8. $|\operatorname{Re}\{zw\}| \leq |z||w|$ *and* $|\operatorname{Im}\{z\}| \leq |z||w|$.

9. $|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$.

10. $|z^n| = |z|^n$

Corollary 1.1.8. *Let* $z_1, \ldots, z_n \in \mathbb{C}$. *Then,*

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|, \qquad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \qquad |\operatorname{Re}\{z_1 \cdots z_n\}| \leq |z_1| \cdots |z_n|. \tag{1.15}$$

1.1.1 Representation

By the proposition 1.1.5, we can identify the elements of $\mathbb{C}$ as the elements of $\mathbb{R}^2$, so we can represent them in the same way. The plane used to represent complex numbers is called the Argand plane, where the real part the abscissa axis and the imaginary part in the ordinate axis [].

Definition 1.1.8. Let $z \in \mathbb{C}^*$. We define the *argument of* z, denoted by $\arg z$, as the real number θ such that $z = |z|(\cos\theta + i\sin\theta)$. Let us observe that $\arg z$ is not a function but a multivalued application.
We define the *principal argument of* z as

$$\operatorname{Arg} z \coloneqq \theta_0 \in [0, 2\pi) \mid z = |z|(\cos\theta + i\sin\theta). \tag{1.16}$$

In general, to make θ to be unique, it is enough to impose it to belong to a certain semiopen interval of length 2π. Choosing the interval I is called by *taking a determination of the argument.*

Another common convention is the interval $(-\pi, \pi]$. If we take now $\arg_I : \mathbb{C}^* \longrightarrow I$ where $\arg_I$ is the unique value of $\arg z$ such that it belongs to I, the $\arg_I(z)$ is a function but not continuous. If we have an argument determination with $I = [\varphi_0, \varphi_0 + 2\pi)$, then $\arg_I(z)$ is discontinuous at the closed semiline that forms an angle φ_0 with the real positive semiaxis. SEE BOOK FROM THE BIBLIO———-

Definition 1.1.9. Given a complex number z that we can express by $z = |z|(\cos\theta + i\sin\theta)$ for some $\theta \in \mathbb{R}$, we use the notation $r = |z|$ to write

$$z = r_\theta^z = r(\cos\theta + i\sin\theta) \tag{1.17}$$

or simply r_θ when it is obvious which complex number are we referring to. We call it *polar form of* z.

Sometimes we use the notation $\arg z$ to design the set of all arguments of z. If θ is one of the arguments of z, then

$$\arg z = \{\theta + 2\pi k \mid k \in \mathbb{Z}\}. \tag{1.18}$$

Formally $\arg z$ is, hence, an equivalence class of $\mathbb{R}/2\pi\mathbb{Z}$. Informally, we say $\arg z$ is determined except by multiples of 2π.

1.1.2 Geometric interpretation of addition and multiplication

Since both $\mathbb{C}$ and $\mathbb{R}^2$ can be represented in a plane and addition between complex numbers is defined like addition between vectors, the geometric visualization is the same as that from addition of vectors. Hence, complex numbers can be seen as "arrows" whose addition obeys the parallelogram rule. With respect to multiplication, let us take two complex numbers z_1, z_2 in polar form.

$$z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1) = r^{z_1}_{\theta_1}, \quad z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2) = r^{z_2}_{\theta_2}$$

If we compute now the product and use some trigonometric identities,

$$\begin{gathered} z_1 z_2 = |z_1||z_2|(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) = \\ |z_1||z_2|(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) = r^{z_1 z_2}_{\theta_1+\theta_2}. \end{gathered}$$

Then, we multiply two complex numbers, angles are added and modules are multiplied.

Proposition 1.1.9. *Let $z \in \mathbb{C}$ and r_θ its polar form. Then,*

$$z^n = (r^n)_{n\theta}. \tag{1.19}$$

Corollary 1.1.10 (De Moivre's Formula)**.** *Let $\theta \in \mathbb{R}$. Then,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{1.20}$$

Proposition 1.1.11. *Let $z, w \in \mathbb{C}$. Then,*

1. $\arg zw = \arg[z] + \arg[w] + 2\pi k$.
2. $\arg z^n = n\arg z + 2\pi k$.

1.1.3 Roots of a complex number

Definition 1.1.10. Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $w \in \mathbb{C}$ is an *n-th root of z* if and only if

$$w^n = z. \tag{1.21}$$

Theorem 1.1.12. *Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \ldots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \ldots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \ldots, n\}$.*

From that we can see

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right], \qquad k = 0, \ldots, n-1. \tag{1.22}$$

1.2 Series

Definition 1.2.1. Let $z_n \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $\lim_{n\to\infty} z_n = l$ if and only if

$$\forall\varepsilon \in \mathbb{R}^+ \exists n_0(\varepsilon) \in \mathbb{N}^* \mid |z_n - l| < \varepsilon, n \geq n_0. \tag{1.23}$$

Proposition 1.2.1. *Let $\{z_n\} = \{a_n + ib_n\}$ be a sequence of complex numbers. Then, it converges if and only if $\{a_n\}$ and $\{b_n\}$ converge.*

Definition 1.2.2. We say $\sum_{n=1}^{\infty} z_n$ converges if and only if $S_n := \sum_{n=1}^{N} z_n$ has limit at $n \to \infty$.

Proposition 1.2.2. *$\sum_{n=1}^{\infty} z_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.*

Definition 1.2.3. We say $\sum_{n=1}^{\infty} z_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |z_n|$ converges.

Proposition 1.2.3. *$\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge.*

Chapter 2

Continuity

2.1 Sequences

Definition 2.1.1. A *sequence of complex numbers* is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned}. \tag{2.1}$$

We denote it by $\{z_n\}_{n=m}^{\infty}$

Definition 2.1.2. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *the sequence has limit* L or *it converges to the limit* L if and only if

$$\forall \varepsilon \in \mathbb{R}^{+} \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \tag{2.2}$$

We denote it by

$$\lim_{h \to \infty} z_n = L, \qquad \lim \{z_n\}_{n=0}^{\infty} = L, \qquad \{z_n\}_{n=0}^{\infty} \to L. \tag{2.3}$$

Theorem 2.1.1. *Let $z_n = z_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$ and $L = L_x + iL_y \in \mathbb{C}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \to L \Leftrightarrow \{x_n\}_{n=0}^{\infty} \to L_x \wedge \{y_n\}_{n=0}^{\infty} \to L_z. \tag{2.4}$$

Definition 2.1.3. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *it tends to infinity* and denote it by $\lim z_n = \infty$ if and only if

$$\forall k \in \mathbb{R}^{+} \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \tag{2.5}$$

Definition 2.1.4. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^{+} \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \tag{2.6}$$

Theorem 2.1.2. *Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.*

Theorem 2.1.3. *Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence } \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \text{ are Cauchy sequences.} \tag{2.7}$$

Theorem 2.1.4. *The field $\mathbb{C}$ of complex numbers is complete.*

2.1.1 Riemann sphere

According to the definition of a sequence tending to infinity, we mean that the absolute value of z gets arbitrarily big. Another way to formulate it is by the equivalence

$$\lim z_n = \infty \Leftrightarrow \lim \frac{1}{z_n} = 0. \tag{2.8}$$

Notice that with this comparison there is a *unique infinity* in the complex plane, that we can describe as the infinitely far horizon. To see graphically the relation presented above, we can use the *Riemann sphere.*

Definition 2.1.5. The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, together with two charts.

The straight line that joint the *north pole* with a point of the plane determines a unique point of the spherical surface and establishes a bijective correspondence between points of the plane and of the sphere, except for the north pole that has no actual correspondence.
Notice that a parallel line of the sphere becomes a circle centered at the origin in the plane. The points located at the south region of that line are transformed in point at the interior of the circle, while the point in the north region are now in the exterior. Besides, the higher the line, the bigger the circle. This way, we associate the *skullcap* infinitely near to the north pol to the exterior of the circle infinitely large, that is, the infinity of the complex plane.
We could wonder why in $\mathbb{R}$ there are two infinities instead of one as in $\mathbb{C}$. The reason is that $\mathbb{R}$ is an ordered field, and unifying the infinities would break this property (intuitively, the unique infinity would be greater and lower that every real number). Since $\mathbb{C}$ is not an ordered field we only talk about one infinity.

2.2 Functions

Definition 2.2.1. Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function* f as the application

$$\begin{aligned} f : D \subseteq \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto w = f(z) \end{aligned}. \tag{2.9}$$

Definition 2.2.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say it *tends to infinity at the point* z_0 and denote it by $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \tag{2.10}$$

Definition 2.2.3. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We write $\lim_{z \to \infty} f(z) = L$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \tag{2.11}$$

2.3 Continuity

Definition 2.3.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. We say f *is continuous in* z_0 if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \tag{2.12}$$

As we have mentioned before, we can characterize the function as $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$. This way, we can establish an equivalent criterion of continuity.

Proposition 2.3.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$ is continuous at z_0 if and only if $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are continuous at z_0.*

Proposition 2.3.2. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, f is continuous at z_0 if and only if for all sequence $\{z_n\}_{n=1}^{\infty}$ of Ω convergent at z_0 it is true that the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$.*

Proposition 2.3.3. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, λf, $f + g$, and fg are continuous at z_0. The function f/g is continuous at z_0 if $g(z_0) \neq 0$.*

Chapter 3

Complex functions

3.1 Introduction

Definition 3.1.1. A *topology* is an ordered pair $(\mathbb{X}, \tau)$, where X is a set and τ a collection of subsets of X satisfying the following properties:

1. The empty set and X belong to τ.
2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ.
3. The intersection of any finite number of members of τ still belongs to τ.

The elements of τ are called *open sets* and the collection τ is called the *topology on* X.

Definition 3.1.2. Let $(\mathbb{X}, d)$ be a metric space. A *topology on the metric space by the metric* d is the set τ of all open sets of M.

Since we have seen $(\mathbb{C}, d)$ is a metric space, we can induce a topological space. Hence, $\mathbb{C}$ is a topological space and we can define all topological concepts.

Definition 3.1.3. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally, Ω is connected if there are not two open sets $U, V \subseteq \mathbb{C}$ such that

$$U_1 = U \cap \Omega, \qquad V_1 = V \cap \Omega, \qquad U_1 \cap V_1 = \varnothing, \qquad U_1 \cup V_1 = \Omega. \tag{3.1}$$

Otherwise, we say Ω *is disconnected.*

Definition 3.1.4. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is simply connected* if and only if every circuit is homotopic in Ω to a point in Ω. Equivalently, is simply connected if and only if every pair of curves with the same extremes are homotopic.

Example 3.1.1. Every disc $D_r(z_0)$ is connected, but the union $D_{r_1}(z_1) \cup D_{r_2}(z_2)$ is disconnected.

Definition 3.1.5. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω *is convex* if and only if for all pair of point $a, b \in \Omega$, the segment defined by

$$[a, b] = \{z \,|\, z = (1-t)a + tb, 0 \leq t \leq 1\} \tag{3.2}$$

is contained in Ω, that is, if every pair of points can be connected by a straight line that belongs to the set.

Definition 3.1.6. Let $\Omega \in \mathbb{C}$ be a set. We say Ω *is a star-convex set* if and only if there exists $z_0 \in \mathbb{C}$ such that for all $z \in \Omega$ the segment $[z_0, z]$ is contained by Ω.

Definition 3.1.7. Let $\Omega \in \mathbb{C}$ be a set. We say Ω *is a region or domain* if and only if it is open, non-empty, and connected.

Definition 3.1.8. Let $\Omega \subseteq \mathbb{C}$ be a non-empty set. We say $\Omega_1 \subseteq \Omega$ *is a connected component of* Ω if and only if it is a maximal connected subset, that is, if $z_0 \in \Omega_1$ and W is a connected subset of $\mathbb{C}$ that contains z_0, then $W \subseteq \Omega_1$.

Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. Since the image is a set of complex elements numbers, we can represent it as $f = u + iv$, where u, v are functions with the following form.

$$\begin{aligned} u : \Omega &\longrightarrow \mathbb{R} \\ z &\longmapsto \mathrm{Re}\{f(z)\} \end{aligned}, \qquad \begin{aligned} v : \Omega &\longrightarrow \mathbb{R} \\ z &\longmapsto \mathrm{Im}\{f(z)\} \end{aligned}.$$

Example 3.1.2. One of the most fundamental kinds of functions are the polynomials of complex variables in complex coefficients. If $a_0, a_1, \ldots, a_n \in \mathbb{C}$, then the general expression is

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

By the Fundamental Theorem of Algebra there are some complex values $\alpha_1, \ldots, \alpha_r$ and natural numbers $m_1, \ldots, m_r$ such that

$$P(z) = a_n (z - \alpha_1)^{m_1} \cdots (z - \alpha_r)^{m_r}, \qquad m_1 + \cdots m_r = n.$$

Using again the identification between $\mathbb{C}$ and $\mathbb{R}^2$, we can interpret P as a two variable function $P(x, y)$, where $z = x + iy$. Separating the function in real and imaginary part, we get $P = P_1(x, y) + iP_2(x, y)$.

Definition 3.1.9. We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n, z, z_0 \in \mathbb{C}. \tag{3.3}$$

We call the term a_n the *n-th coefficient of the series.* In case $a_n = 0 \forall n \leq m$, we will start the counting directly from m.

Definition 3.1.10. Radius of convergence.

Proposition 3.1.1. *The radius of convergence can be calculated as follows*

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}. \tag{3.4}$$

Theorem 3.1.2 (Cauchy-Hadamard Theorem)**.** *Let*

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{3.5}$$

be the following complex power series with $a_n, z_0 \in \mathbb{C}$ and $R \in [0, +\infty) \cup \{+\infty\}$ the radius of convergence. Then,

1. *If $|z - z_0| < R$ then S converges. In fact, for all $r < R$ we have S converges uniformly at the disc $\overline{D_r(z_0)}$.*
2. *If $|z - z_0| > R$ then S diverges.*
3. *The function $f(z) = S(z)$ is derivable at $D_R(z_0)$ and its formal derivative is*

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \tag{3.6}$$

with the same radius of convergence.

Definition 3.1.11. Let $f(z) = \sum a_n (z - z_0)^n$ be a series with radius of convergence R. Then, its formal derivative is

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z}. \tag{3.7}$$

Corollary 3.1.3. *Let $f(z) = \sum a_n (z - z_0)^n$ be a series with radius of convergence R. Then, f is infinitely derivable at $D_R(z_0)$.*

Corollary 3.1.4. *Let R be the radius of convergence of the function*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then f has as Taylor polynomial of degree m around z_0 the following one

$$P_{m,f,z_0}(z) = \sum_{n=0}^{m} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{3.8}$$

Theorem 3.1.5 (Abel's Theorem)**.** *Let be the following series*

$$\sum_{n=0}^{\infty} f_n(z_0) g_n(z_0),$$

where f, g_n are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \tag{3.9}$$

then the original series converges uniformly in Ω.

Theorem 3.1.6 (Weierstrass' criterion). *Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant M_n, satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at Ω for all $z_0 \in \Omega$.*

Definition 3.1.12. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a complex function with Ω an open set. We say f *is complex analytic* if and only if for all $z_0 \in \Omega$ exists a real number $R(z_0)$ and a sequence $\{a_n\} \subseteq \mathbb{C}$ (that can also depend on z_0) such that is $z \in D_R(z_0)$, then formally it is true that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{3.10}$$

We denote the set of complex analytic functions with domain Ω by $C^\omega(\Omega)$.

Corollary 3.1.7. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. If $f \in C^\omega(\Omega)$, then $f \in C^\infty(\Omega)$.*

Corollary 3.1.8. *Let z_0. Then, the coefficients z_0 of the local expression of f given by the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ are determined by*

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{3.11}$$

Proposition 3.1.9. *Let $\varnothing \neq \Omega \subseteq \mathbb{C}$. Then,*

1. *Every connected component of Ω is a closed of Ω with a subspace topology.*
2. *Two connected components are the same or are disjoint.*
3. *Every connected of Ω is one and only one connected component.*
4. *Ω is the disjoint union of its connected components.*

Theorem 3.1.10 (Analytic prolongation Principle). *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0 = 0)$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of Ω that contains z_0 (the function can be zero also out Ω, but we are not studying that).*

Corollary 3.1.11. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function with Ω a region. Then, the following statements are equivalent:*

1. *$f(z) = 0$ for all $z \in \Omega$.*
2. *There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.*

Corollary 3.1.12 (Analytic Prolongation Principle). *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions with Ω a region. Then, the following statements are equivalent:*

1. *$f(z) = g(z)$ for all $z \in \Omega$.*
2. *There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}$.*

Lemma 3.1.13. *Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then*

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right). \tag{3.12}$$

Corollary 3.1.14. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions. Then, fg is analytic.*

Proposition 3.1.15. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, f is analytic at $\Omega = D_R(0)$.*

3.2 Complex exponential function

Definition 3.2.1. For all $z \in \mathbb{C}$, we define the *complex exponential function* as the following series

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{3.13}$$

Proposition 3.2.1. *The radius of convergence of* e^z *is infinite.*

Properties

Proposition 3.2.2. $\mathrm{e}^z = \mathrm{e}^x$ *for all* $x \in \mathbb{R}$.

Proposition 3.2.3. $\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$ *for all* $z, w \in \mathbb{C}$.

Proof.

$$\mathrm{e}^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} z^k w^{n-k} =$$
$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right) = \mathrm{e}^z \mathrm{e}^w.$$

■

Proposition 3.2.4. *For all* $z \in \mathbb{C}$ *we have* $\mathrm{e}^z \neq 0$.

Proposition 3.2.5. *The image of* e^z *is* $\mathbb{C}^*$.

Proposition 3.2.6. *The derivative of* e^z *is* e^z.

Proposition 3.2.7. $\overline{\mathrm{e}^z} = \mathrm{e}^{\bar{z}}$.

Proposition 3.2.8. $|\mathrm{e}^z| = \mathrm{e}^{\mathrm{Re}\{z\}}$.

Proposition 3.2.9 (Euler's Formula)**.** *If* $\theta \in \mathbb{R}$, *then* e^{xi} *has modulus one and we have that*

$$\boxed{\mathrm{e}^{xi} = \cos x + i \sin x.} \tag{3.14}$$

Proof. To see that its modulus is one we can make the following calculation.

$$\left|\mathrm{e}^{xi}\right| = \sqrt{\mathrm{e}^{xi}\mathrm{e}^{-xi}} = \sqrt{\mathrm{e}^{xi-xi}} = \sqrt{\mathrm{e}^0} = \sqrt{1} = 1 \tag{3.15}$$

To see now that $\mathrm{e}^{xi} = \cos x + i \sin x$ for all $x \in \mathbb{R}$ we only need to use the power series of $\sin x$ and $\cos x$,

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{3.16}$$

Besides, by definition,

$$\mathrm{e}^{xi} = \sum_{n=0}^{\infty} \frac{(xi)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{y^n}{n!}. \tag{3.17}$$

It is not difficult to prove that

$$i^n = \begin{cases} (-1)^n, & \text{if } n \in 2\mathbb{Z} \\ i(-1)^n, & \text{if } n \notin 2\mathbb{Z} \end{cases}. \tag{3.18}$$

Since the series of sine and cosine are absolutely convergent in $\mathbb{R}$, we can separate the sum without modifying the result, getting

$$\mathrm{e}^{xi} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} =$$
$$\cos x + i \sin y.$$

■

Corollary 3.2.10. *Let $z \in \mathbb{C}^*$. Then,*

$$z = |z|\mathrm{e}^{i\theta}, \tag{3.19}$$

with $\theta \in [0, 2\pi)$.

Proposition 3.2.11. *The following function*

$$\begin{aligned} \exp : (\mathbb{R}, +) &\longrightarrow (\mathbb{C}^*, \cdot) \\ x &\longmapsto \mathrm{e}^{xi} \end{aligned} \tag{3.20}$$

is a morphism of groups and its image is $\mathbb{T}$.

Proposition 3.2.12. *The complex exponential function is a periodic function with period $2\pi i$.*

Proposition 3.2.13. *Let $a \in \mathbb{C}^*$. Then, $\mathrm{e}^z = a$ has infinite solutions.*

Then, the exponential function is not injective.

3.2.1 Exponential form

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
This allows us to compute exponents in a faster way. If we have some number $a + ib \in \mathbb{C}$, then we can represent it in exponential form and finally pass $r^n e^{in\theta}$ to the original form.

3.3 Complex trigonometric functions

$$\cos z = \frac{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}{2}, \qquad \sin\theta = \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{2}, \qquad \tan z = \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}. \tag{3.21}$$

Properties

Proposition 3.3.1. *For all $z \in \mathbb{C}$,*

$$\sin^2 z + \cos^2 z = 1. \tag{3.22}$$

Proposition 3.3.2. *For all $z \in CC$,*

$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z). \tag{3.23}$$

Proposition 3.3.3. *For all $z, w \in \mathbb{C}$,*

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \qquad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \tag{3.24}$$

Proposition 3.3.4. *The functions $\cos z, \sin z$ have period of 2π.*

An important difference of trigonometric functions in complex numbers is that they are no more bounded.

Proposition 3.3.5. *Let $z_0 \in \mathbb{C}$. Then, z_0 is root of $\sin z$ ($\cos z$) if and only if it is a root of $\sin x$ ($\cos x$).*

With that, we conclude $\tan z$ is defined always than $\tan x$ has no discontinuity.

3.4 Hyperbolic functions

$$\cosh z = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}, \qquad \sinh\theta = \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}, \qquad \tanh z = \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{\mathrm{e}^{z} + \mathrm{e}^{-z}}. \tag{3.25}$$

Properties

Proposition 3.4.1. *For all $z \in \mathbb{C}$,*

$$\sinh^2 z - \cosh^2 z = 1. \tag{3.26}$$

Proposition 3.4.2. *For all $z \in CC$,*

$$\cosh(-z) = \cosh(z), \qquad \sinh(-z) = -\sinh(z). \tag{3.27}$$

Proposition 3.4.3. *For all $z, w \in \mathbb{C}$,*

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \qquad \sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \tag{3.28}$$

Proposition 3.4.4. *For all $z \in \mathbb{C}$,*

$$\cosh z = \cos(iz), \cos z = \cosh(iz) \qquad \sinh z = -i\sin(iz), \sin z = -i\sinh(iz) \tag{3.29}$$

Proposition 3.4.5. *The roots of the function $\sinh z$ are of the form $z_n = n\pi$ and those for the function $\cosh z$ are of the form $w_n = (2n+1)\pi/2i$.*

3.5 Logarithm

Definition 3.5.1. Let $D \subseteq \mathbb{C}$ be a set. We define a *multivalued function from D to $\mathbb{C}$* as a subset of $D \times \mathbb{C}$ such that for every $z \in D$ there exists a number $y \in \mathbb{C}$ such that $(z, w) \in f$.

Definition 3.5.2. For $z \in \mathbb{C}^*$, we call the *natural logarithm of z* every number w such that $\mathrm{e}^w = z$, that is,

$$\ln z \coloneqq \{w \in \mathbb{C} \mid \mathrm{e}^w = z\}. \tag{3.30}$$

Proposition 3.5.1. *Given $z \in \mathbb{C}$ we can define $\ln z$ from the natural logarithm of a real number as*

$$\ln z = \ln|z| + i\arg z = \ln|z| + i\mathrm{Arg}z + 2\pi k i. \tag{3.31}$$

Definition 3.5.3. We define the *principal natural logarithm of z* as the value defined by the principal argument of z, that is,

$$\mathrm{Log}z = \ln|z| + i\mathrm{Arg}z. \tag{3.32}$$

It is important to note that, although we define the principal natural logarithm as that value with an argument $\theta \in [0, 2\pi)$, symbolic programs use as the default argument $\theta \in (-\pi, \pi]$.

Definition 3.5.4. We define the determination I (with I being a semiopen interval) of the logarithm as

$$\log_I z \coloneqq \ln|z| + i\arg_I z. \tag{3.33}$$

Proposition 3.5.2. *Let $z, w \in \mathbb{C}$ two numbers. Then,*

1. $\ln(zw) = \ln z + \ln w + 2\pi k i, k \in \mathbb{Z}$.
2. *If we want to stay in the principal argument,*

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \textit{if } \mathrm{Arg}z + \mathrm{Arg}w < 2\pi \\ \ln z + \ln w - 2\pi i, & \textit{if } \mathrm{Arg}z + \mathrm{Arg}w \geq 2\pi \end{cases}. \tag{3.34}$$

3. *SEARCH MORE PROPERTIES*

3.5.1 Inverse trigonometric and hyperbolic functions

The logarithm function allows us to define the multivalued inverses of trigonometric and hyperbolic function. Solving a quadratic equation, we get two possible expressions. For convention, we get that expression such that $\operatorname{arcsinh} 0 = 0$.

$$\begin{aligned} \arcsin z &= -i\ln\left(iz + \sqrt{1-z^2}\right), & \operatorname{arcsinh} z &= \ln\left(z + \sqrt{1+z^2}\right) \\ \arccos z &= -i\ln\left(z + \sqrt{z^2-1}\right), & \operatorname{arccosh} z &= \ln\left(z + \sqrt{z^2-1}\right) \\ \arctan z &= -\frac{i}{2}\ln\frac{1+iz}{1-iz}, & \operatorname{arctanh} z &= \frac{1}{2}\ln\frac{1+z}{1-z} \end{aligned} \tag{3.35}$$

3.6 Complex power

Definition 3.6.1. Let $z, a \in \mathbb{C}$. Then,

$$z^a := \mathrm{e}^{a \ln z} \tag{3.36}$$

Proposition 3.6.1. *If $a = \alpha + \beta i$ and $z = r\mathrm{e}^{\theta i}$, then*

$$z^a = \mathrm{e}^{\alpha \ln r - \beta(\theta + 2\pi k)} \mathrm{e}^{\beta \ln r + \alpha(\theta + 2\pi k)}, \tag{3.37}$$

$$|z^a| = \mathrm{e}^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \qquad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k) \tag{3.38}$$

We can see four possible cases.

1. If $a = n \in \mathbb{Z}$, the complex power is a function and
$$z^n = r^n \mathrm{e}^{n\theta i}. \tag{3.39}$$
2. If $a = n/m \in \mathbb{Q}$, there are n values and
$$z^a = \sqrt[m]{r^n} \mathrm{e}^{(\theta + 2\pi k)n/mi}. \tag{3.40}$$
3. If a is irrational, the norm is uniquely determined but the argument has infinite values.
4. IF $a \in \mathbb{C} \setminus \mathbb{R}$, the argument is uniquely determined and the norm has infinite values.

Proposition 3.6.2. *Let $z, w \in \mathbb{C}$. Then,*

1. $\left(\mathrm{e}^b\right)^a = \mathrm{e}^{a(b + 2\pi k i)}$

3.7 Riemann surfaces

Definition 3.7.1. A *Riemann surface* X is a connected complex 1-manifold.

The Riemann surface is a way to transform a multivalued function to a function. This process consists of separating the different values in several complex planes according the the argument.

Definition 3.7.2. We define a *sheet* as each of the complex planes of the Riemann surface.

Definition 3.7.3. We define a *cut* as the line (not necessaryly straight) of union between sheets.

Definition 3.7.4. We define a *branch point* as a point where start or finish a cut.

Note that, since a number goes from one interval of the argument to another continuously, there must be a way to go from one sheet to another. This can be achieved by the cuts. These cuts allow us to join one edge of one sheet to another and hence connect them in a continuous way. Each sheet must be connected to the next one (since one interval is connected to the next one) and, in case of a finite valued function, the last one to the first one. We call the sheet associated to the principal argument *principal sheet.*
Now we know how to connect sheets, we shall determine where do the cuts start or finish, that is, where are located the branch points. To achieve that, we must follow the path of a circle (although it could be any other closed curve) centered at a point z_0 and see if the value at the beginning and the end of the curve is the same or has changed. If the value of f changes, then z_0 is a branch point.

Example 3.7.1. Let us study the function $\sqrt{z}$. If we follow a path where the argument is always in $[0, 2\pi)$, then $f(z) = \sqrt{r}\mathrm{e}^{\theta/2i}$ always and the result does not change. However, if we make a complete revolution around $z = 0$ and we take now the value of $f(z') = f(r_{\theta+2\pi}^{z'})$, then

$$\sqrt{z'} = \sqrt{r}\mathrm{e}^{(\theta+2\pi)/2i} = \sqrt{r}\mathrm{e}^{\theta/2i}\mathrm{e}^{\pi i} = -\sqrt{r}\mathrm{e}^{\theta/2i} \neq f(z). \tag{3.41}$$

Therefore, we conclude $z = 0$ is a branch point. The cut could be done with a straight line from $z = 0$ to the infinity along the positive real axis.

3.7.1 Logarithm

Since $\ln z$ function is infinite valued, its associated Riemann surfaces has infinite sheets. Each sheet is connected to the next one through a cut done in the real axis, beginning at the branch point at $z = 0$. Note that, contrary to finite valued functions, there is not a last sheet that connects to the first one. Since each sheet is associated to a value of $k \in \mathbb{Z}$, and the set of integers if an ordered set, there are two infinites (like real numbers which are also ordered). For this reason there is no connection between $k = -\infty$ and $k = \infty$.
By the logarithm leads the expression $\ln|z| + (\arg z + 2\pi k)i$, we see the unique variation is in the imaginary part.

3.7.2 Power

Note that two last cases of exponents have infinite values, so their Riemann surfaces will have a similar aspect of that from the logarithm function.

Chapter 4

Derivatives

4.1 Introduction

Definition 4.1.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We define the *derivative of f at z_0* as

$$f'(z_0) \coloneqq \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{4.1}$$

in case the limit exists. If f has derivative, we say *f is $\mathbb{C}$-derivable at z_0*.

Definition 4.1.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say *f is holomorphic at Ω* if and only if it is $\mathbb{C}$-derivable at every point of Ω. In that case, it is defined the function $f' : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ that associates each point z of Ω with $f'(z)$.

We denote the set of all holomorphic functions at Ω by $H(\Omega)$.

Definition 4.1.3. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We define the *domain of holomphism* as the region where f is derivable. We say *f is entire* if and only if the domain of holomorphism is $\mathbb{C}$.

Definition 4.1.4. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in CC$ a point. We say *f is holomorphic at z_0* if and only if it is holomorphic at some neighborhood of z_0.

Proposition 4.1.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in CC$ a point. If f is derivable at z_0, then it is continuous at z_0.*

Theorem 4.1.2. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two functions and $z_0 \in \Omega$ a point. Then, the following statements are true.*

1. *If f is constant at Ω, then f is derivable at z_0 and $f'(z_0) = 0$.*
2. *If $f(z) = z$ in every point of Ω, then f is derivable at z_0 and $f'(z_0) = 1$.*
3. *If f, g are derivable at z_0 and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ are derivable at z_0 and $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.*
4. *If f, g are derivable at z_0, then fg is derivable at z_0 and*
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{4.2}$$
5. *If f, g are derivable at z_0 and $g(z_0) \neq 0$, then f/g is derivable at z_0 and*
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{4.3}$$

Theorem 4.1.3. *Let $f : \Omega_1 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a derivable function at a point $z_0 \in \mathbb{C}$ and $g : \Omega_2 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be another derivable function at a point $f(z_0) \in \Omega_2$. Then, $g \circ f$ is derivable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \tag{4.4}$$

Definition 4.1.5. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say *f is of class $C^1(\Omega)$ or simply* $f \in c^1(\Omega)$ if and only if, using $f = u + iv$ with $u = \operatorname{Re}\{f\}, v = \operatorname{Im}\{f\}$, the partial derivatives of u and v as a two variable real functions exist and are continuous. In other words, $f \in C^1(\Omega)$ if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \tag{4.5}$$

exist and are continuous.

Theorem 4.1.4. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function of class $C^1(\Omega)$ with Ω an open set, injective, and derivable at every point of Ω with non-zero derivative. Then,*

1. *$\Omega' = f(\Omega)$ is an open subset of $\mathbb{C}$.*

2. *The inverse function* f^{-1} *exist, it is well defined and it is derivable at* Ω'*.*
3. *If* $z \in \Omega$ *and* $z' = f(z)$*, then*
$$(f^{-1})'(z') = \frac{1}{f'(z)}. \tag{4.6}$$

Example 4.1.1. The functions $e^z, \sin z, \cos z$ are holomorphic at $\mathbb{C}$ (hence they are entire) and their derivatives are respectively $e^z, \cos z, -\sin z$.

Proposition 4.1.5. *A determination of* $\ln z$ *with* $z \in \mathbb{C}$ *is continuous except in a semiline.*

Theorem 4.1.6. *Let* $\Omega \subseteq \mathbb{C}$ *be an open set and* $\phi \in C(\Omega)$*, such that* $e^{\phi(z)} = z$ *for all* $z \in \Omega$*. Then, we have* $\phi \in H(\Omega)$ *and*
$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \tag{4.7}$$

Proposition 4.1.7. *A determination of* $\ln z$ *with* $z \in \mathbb{C}$ *is holomorphic except in a semiline.*

Proposition 4.1.8. *Let* $I = [\theta, \theta + 2\pi)$ *a determination of the logarithm,* $\ln_I z$*. Then,* $\ln_I z$ *is holomorphic except in the semiline* $L_\theta = \left\{ re^{\theta i} \in \mathbb{C} \,\middle|\, r \geq 0 \right\}$*.*

Theorem 4.1.9. *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be an analytic function in* Ω*. Then,* f *is holomorphic in* Ω*. In particular, given a point* $z_0 \in \mathbb{C}$*,* $f'(z_0) = a_1$ *where* a_1 *is the first coefficient of the power series that represents* f *in a neighborhood of* z_0*.*

From the previous theorem we see that if f is analytic then $f'(z_0)$ coincides with the formal derivative of the power series that represents f in a neighborhood of z_0.

4.2 Cauchy-Riemann Equations

Definition 4.2.1. We define the operators
$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{4.8}$$
that act over the functions such that the real and imaginary part u, v have partial derivatives.

Proposition 4.2.1. *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *a function of class* $C^1(\Omega)$*. Then, for all* $z_0 \in \Omega$
$$f(z_0 + h) = f(z_0) + \left(\frac{\partial f}{\partial z}\right)_{z_0} h + \left(\frac{\partial f}{\partial \bar{z}}\right)_{z_0} \bar{h} + o(|h|^2). \tag{4.9}$$

Corollary 4.2.2. *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a function of class* $C^1(\Omega)$*. Then,* f *is holomorphic in* Ω *if and only if*
$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ at } \Omega. \tag{4.10}$$

Definition 4.2.2. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$ such that $f = u + iv$ with $u = \mathrm{Re}\{f\}, v = \mathrm{Im}\{f\}$ and $z_0 \in \mathbb{C}$ a point. Then, we call $(\partial_{\bar{z}} f)_{z_0} = 0$ the *Cauchy-Riemann condition*, which is equivalent to
$$\left(\frac{\partial u}{\partial x}\right)_{z_0} = \left(\frac{\partial v}{\partial y}\right)_{z_0}, \qquad \left(\frac{\partial v}{\partial x}\right)_{z_0} = -\left(\frac{\partial u}{\partial y}\right)_{z_0}, \tag{4.11}$$
which are called the *Cauchy-Riemann equations*.

Chapter 5

Line integrals

Definition 5.0.1. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{aligned} \gamma : I &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned}. \tag{5.1}$$

Definition 5.0.2. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval and $D \subseteq \mathbb{C}$ a domain. We define an *arc* as a continuous application of the form

$$\begin{aligned} \gamma : I &\longrightarrow D \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned}. \tag{5.2}$$

Equivalently, we can say an arc is a curve restricted to some interval.

Definition 5.0.3. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We call $\gamma(a)$ and $\gamma(b)$ *the extremes of* γ. In particular, we call $\gamma(a)$ the *initial point* and $\gamma(b)$ the *final point*.

Definition 5.0.4. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We define the *route or graph of* γ as

$$\gamma^* := \{z \in D \,|\, z = \gamma(t), t \in I\}. \tag{5.3}$$

Definition 5.0.5. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is closed* if and only if $\gamma(a) = \gamma(b)$.

Definition 5.0.6. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is simple* if and only if there is no two numbers $t_1, t_2 \in (a, b)$ such that $\gamma(t_1) = \gamma(t_2)$. We also call it a *Jordan curve*, and if it is closed, a *circuit*.

Definition 5.0.7. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is differentiable* if for al value $t_0 \in [a, b]$ there exists the limit

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \tag{5.4}$$

For $t_0 = a$ or $t_0 = b$ we consider the laterals limits from the right and from the left respectively.

If $\gamma'(t_0) \neq 0$ and we identify the complex value as a vector, the vector is tangent to γ at $t = t_0$.

Definition 5.0.8. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is of class* C^1 if and only if γ' exists and is continuous at $[a, b]$.

Definition 5.0.9. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is regular or smooth* if and only if it is differentiable and γ' never vanishes.

Definition 5.0.10. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say γ *is piece-wise of class* C^1 if and only if γ' exists and is continuous in I except in a finite number of points where γ has lateral derivatives.

Definition 5.0.11. Let $\gamma : [a, b] \longrightarrow D$ be an arc. We define the *opposite arc* as

$$\begin{aligned} -\gamma : [-b, -a] &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma(-t) \end{aligned}. \tag{5.5}$$

Definition 5.0.12. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be an arc. We say $\Gamma(s), s \in [c, d] \subseteq \mathbb{R}$ *has been obtained from* $\gamma(t), t \in [a, b]$ *by a change of parametrization* if and only if the new parameter s and the original parameter t are related by a relation $t = \phi(s)$, where $\phi : [c, d] \longrightarrow [a, b]$ is an homeomorphism that satisfies $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$. We call Γ *the reparametrization of* γ.

Definition 5.0.13. Let $\gamma_1 : I_1 \longrightarrow \mathbb{C}$ and $\gamma_2 : I_2 \longrightarrow \mathbb{C}$ be two arcs. We say they *are equivalent* if and only if there exists a bijective, monotone, and continuous function $\rho : I_2 \longrightarrow I_1$ such that $\gamma_2 = \gamma_1 \circ \rho$. If ρ is an increasing function we say γ_1 *and* γ_2 *have the same orientation*; otherwise, we say γ_1 *and* γ_2 *have opposite orientations.*

Definition 5.0.14. Let $\gamma_1[a, b] \longrightarrow \mathbb{C}$ and $\gamma_2 : [c, d] \longrightarrow \mathbb{C}$ be two arcs such that $[a, b] \cap [c, d] = \varnothing$. We define the application $\gamma_1 \cup \gamma_2$ (sometimes denoted by $\gamma_1 + \gamma_2$) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \leq t \leq b \\ \gamma_2(t - b + c), & \text{if } b \leq t \leq b + d - c \end{cases}. \tag{5.6}$$

We say γ_1, γ_2 can be joined/added or that there exists its union/sum if and only $\gamma_1(b) = \gamma_2(x)$. In this case $\gamma_1 + \gamma_2$ is an arc, and we call it *the sum arc of* γ_1 *plus* γ_2.

Notice that the property of the intervals of being disjoints is not restrictive since we can make changes of variables to make the intervals satisfy the condition.

Definition 5.0.15. We define the *segment of extremes* $z_1, z_2 \in \mathbb{C}$ as the arc defined by the expression

$$\begin{aligned} [z_1, z_2] : [0,1] &\longrightarrow \mathbb{C} \\ t &\longmapsto (1-t)z_1 + tz_2 \end{aligned}. \tag{5.7}$$

Definition 5.0.16. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f *is polygonal* if and only if can be expressed as a finite union of segments, that is, if there exist a natural number n and points $\{z_0, \ldots, z_n\}$ such that

$$p = [z_0, z_1] \cup \cdots \cup [z_{n-1}, z_n]. \tag{5.8}$$

Definition 5.0.17. Let $\gamma : [a,b] \longrightarrow D$ be an arc with a, b finite. We say γ *is a basic curve* if and only if $\gamma \in C^1((a,b)) \cap C([a,b])$ and there exist $\lim_{t \to a^+} \gamma'(t), \lim_{t \to b^-} \gamma'(t)$.

Definition 5.0.18. A *path* is a function $\gamma : [a,b] \longrightarrow \mathbb{C}$ such that there exist basic curves $\gamma_j : [a_j, b_j] \longrightarrow \mathbb{C}, j \in \{1, \ldots, k\}$ such that $\gamma = \gamma_1 + \cdots + \gamma_k$ and therefore $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ and $a = a_1, b = a_k$.

Definition 5.0.19. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous curve and $a_1, \ldots, a_l \in \mathbb{R}$ such that $a = a_0 \leq \cdots \leq a_l \leq b = a_{l+1}$. We say γ *is piece-wise differentiable* if and only if

$$\gamma \in C^1\left(\bigcup_{j=0}^{l} (\alpha_j, \alpha_{j+1})\right),$$
$$\forall j \in \{0, \ldots, l+1\} \exists \lim_{t \to a_j^+} \gamma'(t)(\text{except if } j = l+1), \lim_{t \to a_j^-} \gamma'(t)(\text{except if } j = 0).$$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

Theorem 5.0.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with Ω an open set and $\phi : I \longrightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curev (hence a derivable curve) and its real derivative is*

$$\psi'(t) = f'(\phi(t))\phi'(t). \tag{5.9}$$

Definition 5.0.20. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two curves. We say γ_1, γ_2 *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = \gamma_1(0) = \gamma_2(0), s \in [0,1]$.
4. $h(1,s) = \gamma_1(1) = \gamma_2(1), s \in [0,1]$.

Definition 5.0.21. Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two circuits. We say γ_1, γ_2 *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = h(1,s), s \in [0,1]$.

Chapter 6

Fourier transform

6.1 Introduction

Definition 6.1.1. Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform of f at the point ξ* as

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) \mathrm{e}^{-i\xi x} \, \mathrm{d}x \, . \tag{6.1}$$

Proposition 6.1.1. *Let $f \in L^1(\mathbb{R})$ be a function. Then, the application*

$$\begin{aligned} \mathscr{F}\{f\} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \hat{f}(\xi) \end{aligned} \tag{6.2}$$

is a well defined application.

When we want to talk about the Fourier transform as an operator that acts over functions of $L^1(\mathbb{R})$ we write $\mathscr{F}\{f\}$, that satisfies $\mathscr{F}\{f\}(\xi) = \hat{f}(\xi)$.

Definition 6.1.2. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. We say *the functions f_n converge to f with a norm $\|\cdot\|_p$* or *converge in $L^p(\mathbb{R})$* if and only if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0. \tag{6.3}$$

Theorem 6.1.2. *Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.*

1. *The Fourier transform of f satisfies*

$$\mathscr{F}\{f\} \in L^\infty(\mathbb{R}), \qquad \|\mathscr{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|f\|_1. \tag{6.4}$$

2. *$\mathscr{F}\{f\}$ is $\mathbb{C}$ linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,*

$$\mathscr{F}\{\alpha f + \beta g\} = \alpha \mathscr{F}\{f\} + \beta \mathscr{F}\{g\}. \tag{6.5}$$

3. *For all $\xi \in \mathbb{R}$,*

$$\hat{\bar{f}}(\xi) = \overline{\hat{f}(-\xi)}. \tag{6.6}$$

4. *For all $\xi \ni \mathbb{R}$,*

$$\hat{f}(\lambda\xi) = \frac{1}{\lambda} \hat{f}\left(\frac{\xi}{\lambda}\right). \tag{6.7}$$

5. *For all $a \in \mathbb{R}$,*

$$\hat{f}(\xi - a) = \mathrm{e}^{-ia\xi} \hat{f}(\xi). \tag{6.8}$$

6. *If $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$ when $n \to \infty$, then $\mathscr{F}\{f_n\} \to \mathscr{F}\{f\}$ uniformly in $\mathbb{R}$.*

7. *The Fourier transform $\mathscr{F}\{f\}$ is a continuous function in $\mathbb{R}$, $\mathscr{F}\{f\} \in C(\mathbb{R})$.*

Proposition 6.1.3. *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty} |f(x)| = 0$. Then,*

$$\hat{f}'(\xi) = i\xi \hat{f}(\xi). \tag{6.9}$$

Corollary 6.1.4. *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its n-th derivative $f^{(n)} \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty} |f(x)| = 0$. Then,*

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi). \tag{6.10}$$

We will see that the application of the Fourier transform moves functions from $L^1(\mathbb{R})$ to functions of $C_0(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) \,\middle|\, \lim_{|x|\to\infty} f(x) = 0 \right\}$, that is, $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.

Definition 6.1.3. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We define the support of f as

$$\operatorname{supp} f \coloneqq \overline{\{x \in I \mid f(x) \neq 0\}}. \tag{6.11}$$

Definition 6.1.4. We define the set $\mathscr{D}(\mathbb{R})$ as

$$\mathscr{D}(\mathbb{R}) \coloneqq \{\varphi \in C^{\infty}(\mathbb{R}) \mid \operatorname{supp} \varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \tag{6.12}$$

Theorem 6.1.5. *Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_{\in}\mathscr{D}(\mathbb{R})$ such that*

$$\lim_{h\to\infty} \int_{\mathbb{R}} |f - \phi_n| \, \mathrm{d}x = 0, \tag{6.13}$$

that is, we have convergence of ϕ_n to f with norm $\|\cdot\|_1$.

Proposition 6.1.6. *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.*

Proposition 6.1.7. *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\left|\hat{f}(\xi)\right| \leq \|f\|_1$.*

Theorem 6.1.8. *Let $f \in L^1(\mathbb{R})$ be a function. Then,*

$$\lim_{|x|\to\infty} \hat{f}(x) = 0. \tag{6.14}$$

Theorem 6.1.9. *The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.*

Definition 6.1.5. We define the *Schwartz space* as

$$\begin{aligned} S(\mathbb{R}) \coloneqq \{f : \mathbb{R} \longrightarrow \mathbb{C} \mid f \in C^{\infty}(\mathbb{R}) \wedge \forall n, m \in \mathbb{N}\, \exists c_{n,m} < \infty \\ \text{such that } (1 + |x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}\,. \end{aligned}$$

From the previous definition we deduce that if $f \in S(\mathbb{R})$, then $f^{(n)} \in S(\mathbb{R})$ for all $n \in \mathbb{N}$. Besides, the condition of $(1 + |x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}$ is equivalent to say that $\|f\|_{n,m} < \infty$, where $\|\cdot\|_{n,m}$ is the norm define from $\|\cdot\|_\infty$ by

$$\|f\|_{n,m} \coloneqq \|x^m D^n f\|_\infty = \sup_{x\in\mathbb{R}} \left| x^m \frac{\mathrm{d}^n f}{\mathrm{d}x^n} \right|. \tag{6.15}$$

Finally, we see that $\mathscr{D}(\mathbb{R}) \subseteq S(\mathbb{R})$.

Proposition 6.1.10. *Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \longrightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,*

1. *$f + g \in S(\mathbb{R})$.*
2. *$\lambda f \in S(\mathbb{R})$.*
3. *$fg \in S(\mathbb{R})$.*
4. *$Pf \in S(\mathbb{R})$.*

Theorem 6.1.11. *Let $I, J \subseteq \mathbb{R}$ be two intervals with I compact and J open. Let $f : I \times J \longrightarrow \mathbb{R}$ be a function such that*

1. *$f(\cdot, \lambda)$ is integrable in I for all $\lambda \in J$,*
2. *$f(x, \cdot)$ is derivable in J for all $x \in I$.*

If $\partial_\lambda f$ is continuous in $I \times J$, then

1. *$\partial_\lambda f(\cdot, \lambda)$ is integrable for all $\lambda \in J$.*
2. *$F(\lambda) = \int_I f(x, \lambda) \, \mathrm{d}x$ is derivable with continuous derivative in J for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.*

$$F'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int_I f(x, \lambda_0) \, \mathrm{d}x = \int_I \frac{\partial f}{\partial \lambda}(x, \lambda_0) \, \mathrm{d}x \,, \forall \lambda_0 \in J \tag{6.16}$$