

1 Arithmetic and topology

Definition 1.1. Let $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the sum “+” by components

$$(a, b) + (c, d) := (a + b, c + d). \quad (1)$$

- Product: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the product by

$$(a, b)(c, d) = (ac - bd, ad + bc). \quad (2)$$

We define the set \mathbb{C} as $(\mathbb{R}^2, +, \cdot)$.

Proposition 1.1. *The set \mathbb{C} of complex numbers is an abelian field.*

Proposition 1.2. *Let \mathbb{C} be defined in the second way. Then,*

1. \mathbb{C} is an abelian ring.
2. If we define f as

$$f : (\mathbb{C}, +, \cdot) \longrightarrow (\mathbb{R}^2, +, \cdot), \quad (x, y) \longmapsto x + yi, \quad (3)$$

then f is a morphism of rings.

3. The function f is, in fact, an isomorphism and \mathbb{C} is an abelian field.

Proposition 1.3. *The subset of \mathbb{C} generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.*

Theorem 1.4. \mathbb{C} is not an ordered field.

Definition 1.2. Let $z = a + bi \in \mathbb{C}$. We define the conjugate of z as

$$\bar{z} := a - bi. \quad (4)$$

Proposition 1.5. *For all $z, w \in \mathbb{C}$, we have:*

1. $\bar{\bar{z}} = z$.
2. $\overline{z + w} = \bar{z} + \bar{w}$.
3. $\overline{zw} = \bar{z}\bar{w}$.
4. $z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.
5. $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.
6. The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.

Definition 1.3. Let $z = a + bi \in \mathbb{C}$. We define the real part of z and imaginary part of z respectively as

$$\operatorname{Re}\{z\} := a, \quad \operatorname{Im}\{z\} := b. \quad (5)$$

Proposition 1.6. *Let $z \in \mathbb{C}$. Then,*

$$\operatorname{Re}\{z\} = \frac{z + \bar{z}}{2}, \quad \operatorname{Im}\{z\} = \frac{z - \bar{z}}{2i} \quad (6)$$

Proposition 1.7. *Let $z, w \in \mathbb{C}$ and the following distance function.*

$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto \tilde{d}(z, w) := |z - w| \end{aligned} \quad (7)$$

Then, (\mathbb{C}, \tilde{d}) is a metric space.

Definition 1.4. Let $z = a + bi \in \mathbb{C}$. We define the modulus of z as

$$|z| := \tilde{d}(z, 0), \quad (8)$$

which is equivalent to $\sqrt{z\bar{z}}$.

Definition 1.5. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define an open disc of radius r and center z_0 as follows

$$B_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}. \quad (9)$$

Definition 1.6. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a punctured disc of radius r and center z_0 as follows

$$B_r^*(z_0) := \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}. \quad (10)$$

Definition 1.7. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a closed disc of radius r and center z_0 as follows

$$\overline{B_r(z_0)} := \{z \in \mathbb{C} \mid |z - z_0| \leq r\}. \quad (11)$$

Definition 1.8. We denote by \mathbb{D} the unitary disc of center 0 and radius 1. Besides, we denote by $\mathbb{T} \subseteq \mathbb{C}$ the unitary circumference, that is,

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}. \quad (12)$$

We also denote it by \mathbb{S}^1 .

Lemma 1.8. *The set $B = \{B_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\}$ is a basis of the topology of \mathbb{R}^2 as a metric space. The set $D = \{D_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of \mathbb{C} as a metric space.*

Proposition 1.9. *The sets \mathbb{C} and \mathbb{R}^2 with the topology of metric space are homeomorphs.*

Corollary 1.10. *There is a bijection between B and D , that is, between balls of \mathbb{R}^2 and discs of \mathbb{C} .*

Proposition 1.11. *Let $z, w \in \mathbb{C}$. Then,*

1. $|z| \geq 0$.
2. $|z| = 0 \Leftrightarrow z = 0$.
3. $-|z| \leq \operatorname{Re}\{z\} \leq |z|$ and $-|z| \leq \operatorname{Im}\{z\} \leq |z|$.
4. $|zw| = |z||w|$.
5. If $w \neq 0$, $|z/w| = |z|/|w|$.
6. $|z + w| \leq |z| + |w|$.
7. $|z + w| \geq ||z| - |w||$.
8. $|\operatorname{Re}\{zw\}| \leq |z||w|$ and $|\operatorname{Im}\{z\bar{w}\}| \leq |z||w|$.
9. $|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$.

$$10. |z^n| = |z|^n$$

Corollary 1.12. Let $z_1, \dots, z_n \in \mathbb{C}$. Then,

$$\left| \sum_{i=1}^n z_i \right| \leq \sum_{i=1}^n |z_i|, \quad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \quad |\operatorname{Re}\{z_1 \cdots z_n\}| \leq |z_1| \cdots |z_n| \leq \operatorname{Im} \left\{ \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \right\}. \quad (13)$$

Definition 1.9. Let $z \in \mathbb{C}^*$. We define the *argument* of z , denoted by $\arg z$, as the real number θ such that $z = |z|(\cos \theta + i \sin \theta)$. Let us observe that $\arg z$ is not a function but a multivalued application. We define the *principal argument* of z as

$$\operatorname{Arg} z := \theta_0 \in [0, 2\pi) \mid z = |z|(\cos \theta + i \sin \theta). \quad (14)$$

In general, to make θ to be unique, it is enough to impose it to belong to a certain semiopen interval of length 2π . Choosing the interval I is called by *taking a determination of the argument*.

Definition 1.10. Given a complex number z that we can express by $z = |z|(\cos \theta + i \sin \theta)$ for some $\theta \in \mathbb{R}$, we use the notation $r = |z|$ to write

$$z = r_\theta^z = r(\cos \theta + i \sin \theta) \quad (15)$$

or simply r_θ when it is obvious which complex number are we referring to. We call it *polar form* of z .

Proposition 1.13. Let $z \in \mathbb{C}$ and r_θ its polar form. Then,

$$z^n = (r^n)_{n\theta}. \quad (16)$$

Corollary 1.14 (De Moivre's Formula). Let $\theta \in \mathbb{R}$. Then,

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta). \quad (17)$$

Proposition 1.15. Let $z, w \in \mathbb{C}$. Then,

1. $\arg zw = \arg z + \arg w + 2\pi k$.
2. $\arg z^n = n \arg z + 2\pi k$.

Definition 1.11. We denote the complex numbers z generated by moving the point $z_0 = 1$ around \mathbb{T} a length t in a counter-clockwise direction by 1_t . In other words, 1_t are the complex numbers $z = \cos t + i \sin t$.

Proposition 1.16. Let $f : t \rightarrow 1_t$. Then, f is a morphism from $(\mathbb{R}, +)$ to (\mathbb{T}, \cdot) , with $\ker f = 2\pi\mathbb{Z}$.

Definition 1.12. Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $w \in \mathbb{C}$ is an n -th root of z if and only if

$$w^n = z. \quad (18)$$

Theorem 1.17. Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \dots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \dots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \dots, n\}$.

Theorem 1.18. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a continuous curve such that $\gamma(t) \neq 0 \forall t \in [a, b]$. Then, there exists a continuous determination ϕ of the argument of γ . Then, $\phi(t) + 2\pi k$ with $k \in \mathbb{Z}$ is the general expression of all the argument determinations of γ . If γ is differentiable, then ϕ is differentiable and $\phi' = \operatorname{Im}\{\gamma'/\gamma\}$.

Definition 1.13. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a regular curve. We define the *variation of the argument along γ* as

$$\Delta_\gamma \arg z = \operatorname{Im} \left\{ \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \right\}. \quad (19)$$

Definition 1.14. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve such that $\gamma(t) \neq 0 \forall t \in [a, b]$. Then, we define the *index of γ with respect to the origin* or *the number of revolutions of γ around the origin*

$$\operatorname{Ind}(\gamma, 0) := \frac{1}{2\pi} \Delta_\gamma \arg. \quad (20)$$

Proposition 1.19. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a piece-wise regular curve. Then,

$$\operatorname{Ind}(\gamma, 0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \quad (21)$$

Definition 1.15. Let γ be a closed curve and $z \notin \Gamma$. We define the *index of γ with respect to z* as

$$\operatorname{Ind}(\gamma, z) := \operatorname{Ind}(\gamma - z, 0). \quad (22)$$

Proposition 1.20. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve piece-wise of class $C^1([a, b])$. Then,

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} dt. \quad (23)$$

Proposition 1.21. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a piece-wise of class $C^1([a, b])$. Then, $\operatorname{Ind}(-\gamma, z) = -\operatorname{Ind}(\gamma, z)$.

2 Sequences and limits

Definition 2.1. A *sequence of complex numbers* is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\rightarrow \mathbb{C} \\ n &\mapsto z_n \end{aligned} \quad (24)$$

We denote it by $\{z_n\}_{n=m}^\infty$

Definition 2.2. Let $\{z_n\}_{n=0}^\infty$ be a sequence. We say *the sequence has limit L* or *it converges to the limit L* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \quad (25)$$

We denote it by

$$\lim_{n \rightarrow \infty} z_n = L, \quad \lim_{n \rightarrow \infty} \{z_n\}_{n=0}^\infty = L, \quad \{z_n\}_{n=0}^\infty \rightarrow L. \quad (26)$$

Theorem 2.1. Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^\infty$ and $L = L_x + iL_y \in \mathbb{C}$. Then,

$$\{z_n\}_{n=0}^\infty \rightarrow L \Leftrightarrow \{x_n\}_{n=0}^\infty \rightarrow L_x \wedge \{y_n\}_{n=0}^\infty \rightarrow L_y. \quad (27)$$

Definition 2.3. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say it *tends to infinity* and denote it by $\lim z_n = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \quad (28)$$

Definition 2.4. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \quad (29)$$

Theorem 2.2. Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.

Theorem 2.3. Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,

$$\{z_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence} \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \text{ are Cauchy sequences.} \quad (30)$$

Theorem 2.4. The field \mathbb{C} of complex numbers is complete.

Definition 2.5. The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, together with two charts.

3 Functions

Definition 3.1. A *topology* is an ordered pair (X, τ) , where X is a set and τ a collection of subsets of X satisfying the following properties:

1. The empty set and X belong to τ .
2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ .
3. The intersection of any finite number of members of τ still belongs to τ .

The elements of τ are called *open sets* and the collection τ is called the *topology on X* .

Definition 3.2. Let (X, d) be a metric space. A *topology on the metric space by the metric d* is the set τ of all open sets of M .

Definition 3.3. Let A be a subset of a metric space (\mathbb{M}, d) and a a point in \mathbb{M} . We say that a is an *interior point of A* if there is a ball $B_{(\mathbb{M}, d)}(a, r) \subset A$.

Definition 3.4. Let A be a subset of a metric space (\mathbb{M}, d) and a a point in \mathbb{M} . We say that a is an *exterior point of A* if there is a ball such that $B_{(\mathbb{M}, d)}(a, r) \cup A = \emptyset$.

Definition 3.5. Let A be a subset of a metric space (\mathbb{M}, d) and a a point in \mathbb{M} . We say that a is a *boundary point of A* if it is not interior or exterior or, which is equivalent, if every ball $B_{(\mathbb{M}, d)}(a, r)$ contains elements of A and A^c .

Definition 3.6. Let A be a subset of a metric space (\mathbb{M}, d) and a a point in \mathbb{M} . We say that a is an *accumulation point of A* if every ball with center a contains points of A different to a . In other words, every punctured ball satisfies $B_{(\mathbb{M}, d)}^*(a, r) \cup A \neq \emptyset$.

Definition 3.7. Let A be a subset of a metric space (\mathbb{M}, d) . We define the *interior of A* as the set of all interior points of A , and we denote it by $\text{int}(A)$.

Definition 3.8. Let A be a subset of a metric space (\mathbb{M}, d) . We define the *exterior of A* as the set of all exterior points of A , and we denote it by $\text{ext}(A)$.

Definition 3.9. Let A be a subset of a metric space (\mathbb{M}, d) . We define the *boundary of A* as the set of all boundary points of A , and we denote it by ∂A .

Definition 3.10. Let A be a subset of a metric space (\mathbb{M}, d) . We define the *closure of A* as the set of all accumulation points of A , and we denote it by \bar{A} .

Definition 3.11. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . We say A is an *open set* if it contains none of its boundary points, that is, if $\partial A \cap A = \emptyset$.

Definition 3.12. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . We say A is a *closed set* if it contains all its boundary points, that is, if $\partial A \subseteq A$.

Definition 3.13. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . We say A is a *bounded set* if there exist a point $a \in \mathbb{M}$ and a positive real number r such that the ball $B_{(\mathbb{M}, d)}(a, r)$ contains A .

Definition 3.14. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . We say A is a *compact set* if it is a bounded and closed set.

Proposition 3.1. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . Then, A is open if and only if A^c is closed.

Definition 3.15. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is *connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally, Ω is connected if there are not two open sets $U, V \subseteq \mathbb{C}$ such that

$$U_1 = U \cap \Omega, \quad V_1 = V \cap \Omega, \quad U_1 \cap V_1 = \emptyset, \quad U_1 \cup V_1 = \Omega. \quad (31)$$

Otherwise, we say Ω is *disconnected*.

Definition 3.16. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is *simply connected* if and only if every circuit is homotopic in Ω to a point in Ω . Equivalently, Ω is simply connected if and only if every pair of curves with the same extremes are homotopic.

Definition 3.17. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is *convex* if and only if for all pair of point $a, b \in \Omega$, the segment defined by

$$[a, b] = \{z \mid z = (1 - t)a + tb, 0 \leq t \leq 1\} \quad (32)$$

is contained in Ω , that is, if every pair of points can be connected by a straight line that belongs to the set.

Definition 3.18. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is a *star-convex set* if and only if there exists $z_0 \in \mathbb{C}$ such that for all $z \in \Omega$ the segment $[z_0, z]$ is contained by Ω .

Definition 3.19. Let (\mathbb{M}, d) be a metric space and $S \subseteq \mathbb{M}$ a set. We say S is *path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

Definition 3.20. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is a *region or domain* if and only if it is open, non-empty, and connected.

Definition 3.21. Let $\Omega \subseteq \mathbb{C}$ be a non-empty set. We say $\Omega_1 \subseteq \Omega$ is a *connected component* of Ω if and only if it is a maximal connected subset, that is, if $z_0 \in \Omega_1$ and W is a connected subset of \mathbb{C} that contains z_0 , then $W \subseteq \Omega_1$.

Definition 3.22. Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function* f as the application

$$f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C} \\ z \longmapsto w = f(z). \quad (33)$$

Definition 3.23. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say it *tends to infinity at the point* z_0 and denote it by $\lim_{z \rightarrow z_0} f(z) = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \quad (34)$$

Definition 3.24. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We write $\lim_{z \rightarrow \infty} f(z) = L$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \quad (35)$$

Proposition 3.2. Let $f_1, f_2 : \Omega \longrightarrow \mathbb{C}$ be two functions and z_0 a point such that $\lim_{z \rightarrow z_0} f_1 = w_1, \lim_{z \rightarrow z_0} f_2 = w_2$. Then,

1. $f_1 + f_2$ has also a limit and $\lim_{z \rightarrow z_0} f + g = w_1 + w_2$.
2. $f_1 f_2$ has also a limit and $\lim_{z \rightarrow z_0} f g = w_1 w_2$.
3. If $w_2 \neq 0$, then f/g has also a limit and $\lim_{z \rightarrow z_0} f/g = w_1/w_2$.
4. If $h(z)$ is a continuous function defined on a neighborhood of w_1 , then $\lim_{z \rightarrow z_0} h(f_1(z)) = h(w_1)$.

Definition 3.25. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. We say f is *continuous in* z_0 if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon. \quad (36)$$

Proposition 3.3. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$ is continuous at z_0 if and only if $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are continuous at z_0 .

Proposition 3.4. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, f is continuous at z_0 if and only if for all sequence $\{z_n\}_{n=1}^\infty$ of Ω convergent at z_0 it is true that the sequence $\{f(z_n)\}_{n=1}^\infty$ converges to $f(z_0)$.

Proposition 3.5. Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, λf , $f + g$, and fg are continuous at z_0 . The function f/g is continuous at z_0 if $g(z_0) \neq 0$.

Definition 3.26. We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n, z, z_0 \in \mathbb{C}. \quad (37)$$

We call the term a_n the *n-th coefficient of the series*. In case $a_n = 0 \forall n \leq m$, we will start the counting directly from m .

Definition 3.27. Radius of convergence.

Proposition 3.6. The radius of convergence can be calculated as follows

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \sup \sqrt[n]{|a_n|}. \quad (38)$$

Theorem 3.7 (Cauchy-Hadamard Theorem). Let

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (39)$$

be the following complex power series with $a_n, z_0 \in \mathbb{C}$ and $R \in [0, +\infty) \cup \{+\infty\}$ the radius of convergence. Then,

1. If $|z - z_0| < R$ then S converges. In fact, for all $r < R$ we have S converges uniformly at the disc $D_r(z_0)$.
2. If $|z - z_0| > R$ then S diverges.
3. The function $f(z) = S(z)$ is derivable at $D_R(z_0)$ and its formal derivative is

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \quad (40)$$

with the same radius of convergence.

Definition 3.28. Let $f(z) = \sum a_n (z - z_0)^n$ be a series with radius of convergence R . Then, its formal derivative is

$$f'(z) = \frac{df}{dz}. \quad (41)$$

Corollary 3.8. Let $f(z) = \sum a_n (z - z_0)^n$ be a series with radius of convergence R . Then, f is infinitely derivable at $D_R(z_0)$.

Corollary 3.9. Let R be the radius of convergence of the function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then f has as Taylor polynomial of degree m around z_0 the following one

$$P_{m,f,z_0}(z) = \sum_{n=0}^m a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}. \quad (42)$$

Theorem 3.10 (Abel's Theorem). *Let be the following series*

$$\sum_{n=0}^{\infty} f_n(z_0)g_n(z_0),$$

where f, g_n are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \quad (43)$$

then the original series converges uniformly in Ω .

Theorem 3.11 (Weierstrass' criterion). *Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant M_n , satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at Ω for all $z_0 \in \Omega$.*

Definition 3.29. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a complex function with Ω an open set. We say f is *complex analytic* if and only if for all $z_0 \in \Omega$ exists a real number $R(z_0)$ and a sequence $\{a_n\} \subseteq \mathbb{C}$ (that can also depend on z_0) such that is $z \in D_R(z_0)$, then formally it is true that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n. \quad (44)$$

We denote the set of complex analytic functions with domain Ω by $C^\omega(\Omega)$.

Corollary 3.12. *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. If $f \in C^\omega(\Omega)$, then $f \in C^\infty(\Omega)$.*

Corollary 3.13. *Let z_0 . Then, the coefficients z_0 of the local expression of f given by the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ are determined by*

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \quad (45)$$

Proposition 3.14. *Let $\emptyset \neq \Omega \subseteq \mathbb{C}$. Then,*

1. *Every connected component of Ω is a closed of Ω with a subspace topology.*
2. *Two connected components are the same or are disjoint.*
3. *Every connected of Ω is one and only one connected component.*
4. *Ω is the disjoint union of its connected components.*

Theorem 3.15 (Analytic prolongation Principle). *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of Ω that contains z_0 (the function can be zero also out Ω , but we are not studying that).*

Corollary 3.16. *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a function with Ω a region. Then, the following statements are equivalent:*

1. $f(z) = 0$ for all $z \in \Omega$.
2. *There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.*

Corollary 3.17 (Analytic Prolongation Principle). *Let $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ two analytic functions with Ω a region. Then, the following statements are equivalent:*

1. $f(z) = g(z)$ for all $z \in \Omega$.
2. *There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}$.*

Lemma 3.18. *Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then*

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right). \quad (46)$$

Corollary 3.19. *Let $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ two analytic functions. Then, fg is analytic.*

Proposition 3.20. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, f is analytic at $\Omega = D_R(0)$.*

Definition 3.30. For all $z \in \mathbb{C}$, we define the *complex exponential function* as the following series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \quad (47)$$

Proposition 3.21. *The radius of convergence of e^z is infinite.*

Proposition 3.22. $e^z = e^x$ for all $x \in \mathbb{R}$.

Proposition 3.23. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

Proposition 3.24. For all $z \in \mathbb{C}$ we have $e^z \neq 0$.

Proposition 3.25. The image of e^z is \mathbb{C}^* .

Proposition 3.26. The derivative of e^z is e^z .

Proposition 3.27. $\overline{e^z} = e^{\bar{z}}$.

Proposition 3.28. $|e^z| = e^{\operatorname{Re}\{z\}}$.

Proposition 3.29 (Euler's Formula). *If $\theta \in \mathbb{R}$, then e^{xi} has modulus one and we have that*

$$\boxed{e^{xi} = \cos x + i \sin x.} \quad (48)$$

Corollary 3.30. *Let $z \in \mathbb{C}^*$. Then,*

$$z = |z|e^{i\theta}, \quad (49)$$

with $\theta \in [0, 2\pi)$.

Proposition 3.31. *The following function*

$$\exp : (\mathbb{R}, +) \rightarrow (\mathbb{C}^*, \cdot) \\ x \mapsto e^{xi} \quad (50)$$

is a morphism of groups and its image is \mathbb{T} .

Proposition 3.32. The complex exponential function is a periodic function with period $2\pi i$.

Proposition 3.33. Let $a \in \mathbb{C}^*$. Then, $e^z = a$ has infinite solutions.

Proposition 3.34. The equation $e^z = 0$ does not have solutions.

Proposition 3.35. Let $y_0 \in \mathbb{C}$ be a numbers, $B := \{z \in \mathbb{C} \mid y_0 < \operatorname{Im}\{z\} < y_0 + 2\pi\}$ a set, and $f : B \rightarrow \mathbb{C}^*$ be the exponential function. Then, f is bijective in B ?.

Proposition 3.36. Let $x_0, y_0, m \in \mathbb{C}$ be two numbers with $m \neq 0$ and f the exponential function ? . Then,

1. f transforms the line $y = y_0$ to a line that starts at $z = 0$ and continues with an argument y_0 from the real positive axis.
2. f transforms the line $x = x_0$ to a circle centered at the origin and radius $r = e^{x_0}$.
3. f transforms the line $y = mx$ to the parametric curve $z = e^x e^{imx}$ (a spiral).

Definition 3.31. Let $z \in \mathbb{C}$ be a number. We define the complex trigonometric functions as

$$\cos z := \frac{e^{zi} + e^{-zi}}{2}, \quad (51)$$

$$\sin z := \frac{e^{zi} - e^{-zi}}{2i}, \quad (52)$$

$$\tan z := \frac{e^{zi} - e^{-zi}}{e^{zi} + e^{-zi}}. \quad (53)$$

Proposition 3.37. For all $z \in \mathbb{C}$,

$$\sin^2 z + \cos^2 z = 1. \quad (54)$$

Proposition 3.38. For all $z \in \mathbb{C}$,

$$\cos(-z) = \cos(z), \quad \sin(-z) = -\sin(z). \quad (55)$$

Proposition 3.39. For all $z, w \in \mathbb{C}$,

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \quad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \quad (56)$$

Proposition 3.40. The functions $\cos z, \sin z$ have period of 2π .

Proposition 3.41. Let $z_0 \in \mathbb{C}$. Then, z_0 is root of $\sin z$ ($\cos z$) if and only if it is a root of $\sin x$ ($\cos x$).

Definition 3.32. Let $z \in \mathbb{C}$ be a number. We define the complex hyperbolic functions as

$$\cosh z := \frac{e^z + e^{-z}}{2}, \quad (57)$$

$$\sinh z := \frac{e^z - e^{-z}}{2}, \quad (58)$$

$$\tanh z := \frac{e^z - e^{-z}}{e^z + e^{-z}}. \quad (59)$$

Proposition 3.42. For all $z \in \mathbb{C}$,

$$\cosh^2 z - \sinh^2 z = 1. \quad (60)$$

Proposition 3.43. For all $z \in \mathbb{C}$,

$$\cosh(-z) = \cosh(z), \quad \sinh(-z) = -\sinh(z). \quad (61)$$

Proposition 3.44. For all $z, w \in \mathbb{C}$,

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \quad (62)$$

$$\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \quad (63)$$

Proposition 3.45. For all $z \in \mathbb{C}$,

$$\cosh z = \cos(iz), \quad \cos z = \cosh(iz) \quad (64)$$

$$\sinh z = -i \sin(iz), \quad \sin z = -i \sinh(iz) \quad (65)$$

Proposition 3.46. For all $z = x + iy \in \mathbb{C}$,

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \quad (66)$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \quad (67)$$

$$\tan(x + iy) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i \frac{\sinh y}{\cos(2x) + \cosh(2y)}. \quad (68)$$

Proposition 3.47. For all $z = x + iy \in \mathbb{C}$,

$$\tanh(x + iy) = \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)} + i \frac{\sin(2y)}{\cosh(2x) + \cos(2y)}. \quad (69)$$

Proposition 3.48. For all $z = x + iy$,

$$|\cos z| = \sqrt{\cosh^2 y - \sin^2 x} = \sqrt{\cos^2 x + \sinh^2 y}, \quad (70)$$

$$|\sin z| = \sqrt{\sinh^2 x + \sin^2 y} = \sqrt{\cosh^2 x - \cos^2 y}. \quad (71)$$

Corollary 3.49. For all $z = x + iy$,

$$|\sinh y| \leq |\cos z| \leq \cosh y, \quad |\sinh y| \leq |\sin z| \leq \cosh y. \quad (72)$$

Proposition 3.50. The roots of the function $\sinh z$ are of the form $z_n = n\pi$ and those for the function $\cosh z$ are of the form $w_n = (2n + 1)\pi/2i$.

Definition 3.33. Let $D \subseteq \mathbb{C}$ be a set. We define a multivalued function from D to \mathbb{C} as a subset of $D \times \mathbb{C}$ such that for every $z \in D$ there exists a number $y \in \mathbb{C}$ such that $(z, y) \in f$.

Definition 3.34. For $z \in \mathbb{C}^*$, we call the natural logarithm of z every number w such that $e^w = z$, that is,

$$\ln z := \{w \in \mathbb{C} \mid e^w = z\}. \quad (73)$$

Proposition 3.51. Given $z \in \mathbb{C}$ we can define $\ln z$ from the natural logarithm of a real number as

$$\ln z = \ln |z| + i \arg z = \ln |z| + i \operatorname{Arg} z + 2\pi ki. \quad (74)$$

Definition 3.35. We define the principal natural logarithm of z as the value defined by the principal argument of z , that is,

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z. \quad (75)$$

Definition 3.36. We define the determination I (with I being a semiopen interval) of the logarithm as

$$\log_I z := \ln |z| + i \arg_I z. \quad (76)$$

Definition 3.37. Let $E \subseteq \mathbb{C}^*$ be a connected set. We define the *continuous determination of the logarithm in E* as the continuous function $g : E \rightarrow \mathbb{C}$ such that $e^{g(z)} = z$. More generally, if $f : E \rightarrow \mathbb{C}$ is a function such that $f(z) \neq 0$ for all $z \in E$, then we define the *continuous determination of $\ln f$* as a function $g : E \rightarrow \mathbb{C}$ such that $e^{g(z)} = f(z)$.

Proposition 3.52. Let $z, w \in \mathbb{C}$ two numbers. Then,

$$1. \ln(zw) = \ln z + \ln w + 2\pi ki, k \in \mathbb{Z}.$$

2. If we want to stay in the principal argument,

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \text{if } \text{Arg} z + \text{Arg} w < 2\pi \\ \ln z + \ln w - 2\pi i, & \text{if } \text{Arg} z + \text{Arg} w \geq 2\pi \end{cases} \quad (77)$$

3. SEARCH MORE PROPERTIES

Definition 3.38. Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric inverse functions* as

$$\arcsin z := -i \ln \left(iz + \sqrt{1 - z^2} \right), \quad (78)$$

$$\arccos z := -i \ln \left(z + \sqrt{z^2 - 1} \right), \quad (79)$$

$$\arctan z := -\frac{i}{2} \ln \frac{1 + iz}{1 - iz}. \quad (80)$$

Definition 3.39. Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic inverse functions* as

$$\operatorname{arcsinh} z := \ln \left(z + \sqrt{1 + z^2} \right), \quad (81)$$

$$\operatorname{arccosh} z := \ln \left(z + \sqrt{z^2 - 1} \right), \quad (82)$$

$$\operatorname{arctanh} z := \frac{1}{2} \ln \frac{1 + z}{1 - z}. \quad (83)$$

Definition 3.40. Let $z, a \in \mathbb{C}$ with $z \neq 0$. Then, we define the *complex power function* as

$$z^a := e^{a \ln z}. \quad (84)$$

If $E \subseteq \mathbb{C}^*$ is a connected set and $f : E \rightarrow \mathbb{C}$ a functions such that $f(z) \neq 0$ for all $z \in E$, and $w \in \mathbb{C}$ a number, we define a *continuous determination of f^w* as a continuous function $g : E \rightarrow \mathbb{C}$ such that $g(z) \in [f(z)]^w$.

Proposition 3.53. If $a = \alpha + \beta i$ and $z = re^{\theta i}$, then

$$z^a = e^{\alpha \ln r - \beta(\theta + 2\pi k)} e^{(\beta \ln r + \alpha(\theta + 2\pi k))i}, \quad (85)$$

$$|z^a| = e^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \quad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k) \quad (86)$$

Proposition 3.54. Let $a, z \in \mathbb{C}$ be two numbers. Then,

1. If $a = n \in \mathbb{Z}$, the complex power is a function and

$$z^n = r^n e^{n\theta i}. \quad (87)$$

2. If $a = n/m \in \mathbb{Q}$, there are n values and

$$z^a = \sqrt[m]{r^n} e^{(\theta + 2\pi k)n/mi}. \quad (88)$$

3. If a is irrational, the norm is uniquely determined but the argument has infinite values.

4. If $a \in \mathbb{C} \setminus \mathbb{R}$, the argument is uniquely determined and the norm has infinite values.

Proposition 3.55. Let $z, w \in \mathbb{C}$. Then,

$$1. (e^b)^a = e^{a(b + 2\pi ki)}$$

Definition 3.41. A *Riemann surface* X is a connected complex 1-manifold.

Definition 3.42. We define a *sheet* as each of the complex planes of the Riemann surface.

Definition 3.43. We define a *cut* as the line (not necessarily straight) of union between sheets.

Definition 3.44. We define a *branch point* as a point where start or finish a cut.

4 Series

Definition 4.1. We say $\sum_{n=1}^{\infty} z_n$ converges if and only

if $S_n := \sum_{n=1}^N z_n$ has limit at $n \rightarrow \infty$.

Proposition 4.1. $\sum_{n=1}^{\infty} z_n$ converges if and only if

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

Definition 4.2. We say $\sum_{n=1}^{\infty} z_n$ converges absolutely if

and only if $\sum_{n=1}^{\infty} |z_n|$ converges.

Proposition 4.2. $\sum_{n=1}^{\infty} |z_n|$ converges if and only if

$\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge.

Proposition 4.3. 1. A series converges absolutely with sum S if and only if every rearrangement is convergent with the same sum S .

2. An absolutely convergent series can be summed by blocks in an arbitrary way.

Proposition 4.4. Let $\sum_n a_n, \sum_n b_n$ be two absolutely convergent series with sums A and B respectively.

Then, the series $\sum_k c_k$ with $c_k = \sum_{n=0}^k a_n b_{k-n}$ is absolutely convergent with sum AB .

Theorem 4.5. Let $\sum_n c_n(z - z_0)^n$ be a power series and $R = 1/\rho$, where $\rho = \limsup_n |c_n|^{1/n}$. Then, the series converges uniformly on the compacts of the open disc $D(z_0, R)$, converges absolutely at every point $z \in D$ and diverges outside \bar{D} . Hence, the set of converges E satisfies $D \subseteq E \subseteq \bar{D}$ and $D = \text{int} E$.

5 Derivatives

Definition 5.1. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. We define the *derivative* of f at z_0 as

$$f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (89)$$

in case the limit exists. If f has derivative, we say f is *derivable* at z_0 .

Definition 5.2. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say f is *holomorphic* at Ω if and only if it is \mathbb{C} -derivable at every point of Ω . In that case, it is defined the function $f' : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ that associates each point z of Ω with $f'(z)$.

Definition 5.3. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. We define the *domain of holomorphism* as the region where f is derivable. We say f is *entire* if and only if the domain of holomorphism is \mathbb{C} .

Definition 5.4. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say f is *holomorphic* at z_0 if and only if it is holomorphic at some neighborhood of z_0 .

Proposition 5.1. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. If f is derivable at z_0 , then it is continuous at z_0 .

Theorem 5.2. Let $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be two functions and $z_0 \in \Omega$ a point. Then, the following statements are true.

1. If f is constant at Ω , then f is derivable at z_0 and $f'(z_0) = 0$.
2. If $f(z) = z$ in every point of Ω , then f is derivable at z_0 and $f'(z_0) = 1$.
3. If f, g are derivable at z_0 and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ are derivable at z_0 and $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.
4. If f, g are derivable at z_0 , then fg is derivable at z_0 and

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \quad (90)$$

5. If f, g are derivable at z_0 and $g(z_0) \neq 0$, then f/g is derivable at z_0 and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \quad (91)$$

Theorem 5.3. Let $f : \Omega_1 \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a derivable function at a point $z_0 \in \mathbb{C}$ and $g : \Omega_2 \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be another derivable function at a point $f(z_0) \in \Omega_2$. Then, $g \circ f$ is derivable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \quad (92)$$

Definition 5.5. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. We say f is of class $C^1(\Omega)$ or simply $f \in C^1(\Omega)$ if and only if, using $f = u + iv$ with $u = \text{Re}\{f\}, v = \text{Im}\{f\}$, the partial derivatives of u and v as a two variable real functions exist and are continuous. In other words, $f \in C^1(\Omega)$ if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \quad (93)$$

exist and are continuous.

Theorem 5.4 (Cauchy-Riemann conditions). Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. Then, f is derivable at z_0 if and only if it is differentiable at z_0 and $df(z_0)$ is \mathbb{C} -linear, that is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad (94)$$

which are known as *Cauchy-Riemann conditions*.

Theorem 5.5. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. If u, v satisfy the Cauchy-Riemann equation and their partial derivatives are continuous, then f is derivable.

Theorem 5.6. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a function of class $C^1(\Omega)$ with Ω an open set, injective, and derivable at every point of Ω with non-zero derivative. Then,

1. $\Omega' = f(\Omega)$ is an open subset of \mathbb{C} .
2. The inverse function f^{-1} exist, it is well defined and it is derivable at Ω' .
3. If $z \in \Omega$ and $z' = f(z)$, then

$$(f^{-1})'(z') = \frac{1}{f'(z)}. \quad (95)$$

Proposition 5.7. A determination of $\ln z$ with $z \in \mathbb{C}$ is continuous except in a semiline.

Theorem 5.8. Let $\Omega \subseteq \mathbb{C}$ be an open set and $\phi \in C(\Omega)$, such that $e^{\phi(z)} = z$ for all $z \in \Omega$. Then, we have $\phi \in H(\Omega)$ and

$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \quad (96)$$

Proposition 5.9. A determination of $\ln z$ with $z \in \mathbb{C}$ is holomorphic except in a semiline.

Proposition 5.10. Let $I = [\theta, \theta + 2\pi)$ a determination of the logarithm, $\ln_I z$. Then, $\ln_I z$ is holomorphic except in the semiline $L_\theta = \{re^{i\theta} \in \mathbb{C} \mid r \geq 0\}$.

Theorem 5.11. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function in Ω . Then, f is holomorphic in Ω . In particular, given a point $z_0 \in \mathbb{C}$, $f'(z_0) = a_1$ where a_1 is the first coefficient of the power series that represents f in a neighborhood of z_0 .

Definition 5.6. We define the operators

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad (97)$$

that act over the functions such that the real and imaginary part u, v have partial derivatives.

Proposition 5.12. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a function of class $C^1(\Omega)$. Then, for all $z_0 \in \Omega$

$$f(z_0 + h) = f(z_0) + \left(\frac{\partial f}{\partial z} \right)_{z_0} h + \left(\frac{\partial f}{\partial \bar{z}} \right)_{z_0} \bar{h} + o(|h|^2). \quad (98)$$

Corollary 5.13. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$. Then, f is holomorphic in Ω if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ at } \Omega. \quad (99)$$

Definition 5.7. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$ such that $f = u + iv$ with $u = \operatorname{Re}\{f\}, v = \operatorname{Im}\{f\}$ and $z_0 \in \mathbb{C}$ a point. Then, we call $(\partial_{\bar{z}} f)_{z_0} = 0$ the *Cauchy-Riemann condition*, which is equivalent to

$$\left(\frac{\partial u}{\partial x} \right)_{z_0} = \left(\frac{\partial v}{\partial y} \right)_{z_0}, \quad \left(\frac{\partial v}{\partial x} \right)_{z_0} = - \left(\frac{\partial u}{\partial y} \right)_{z_0}, \quad (100)$$

which are called the *Cauchy-Riemann equations*.

Theorem 5.14. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and z_0 an interior point. Then, at z_0

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}. \quad (101)$$

6 Holomorphic functions

Definition 6.1. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{aligned} \gamma : I &\rightarrow \mathbb{C} \\ t &\mapsto \gamma_1(t) + i\gamma_2(t). \end{aligned} \quad (102)$$

Definition 6.2. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval and $D \subseteq \mathbb{C}$ a domain. We define an *arc* as a continuous application of the form

$$\begin{aligned} \gamma : I &\rightarrow D \\ t &\mapsto \gamma_1(t) + i\gamma_2(t). \end{aligned} \quad (103)$$

Equivalently, we can say an arc is a curve restricted to some interval.

Definition 6.3. Let $\gamma : [a, b] \rightarrow D$ be an arc. We call $\gamma(a)$ and $\gamma(b)$ the *extremes of γ* . In particular, we call $\gamma(a)$ the *initial point* and $\gamma(b)$ the *final point*.

Definition 6.4. Let $\gamma : [a, b] \rightarrow D$ be an arc. We define the *route or graph of γ* as

$$\gamma^* := \{z \in D \mid z = \gamma(t), t \in I\}. \quad (104)$$

Definition 6.5. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *closed* if and only if $\gamma(a) = \gamma(b)$.

Definition 6.6. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *simple* if and only if there is no two numbers $t_1, t_2 \in (a, b)$ such that $\gamma(t_1) = \gamma(t_2)$. We also call it a *Jordan curve*, and if it is closed, a *circuit*.

Definition 6.7. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *differentiable* if for all value $t_0 \in [a, b]$ there exists the limit

$$\gamma'(t_0) = \lim_{t \rightarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \quad (105)$$

For $t_0 = a$ or $t_0 = b$ we consider the lateral limits from the right and from the left respectively.

Definition 6.8. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is of class C^1 if and only if γ' exists and is continuous at $[a, b]$.

Definition 6.9. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *regular or smooth* if and only if it is differentiable and γ' never vanishes.

Definition 6.10. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *piece-wise of class C^1* if and only if γ' exists and is continuous in I except in a finite number of points where γ has lateral derivatives.

Definition 6.11. Let $\gamma : [a, b] \rightarrow D$ be an arc. We define the *opposite arc* as

$$\begin{aligned} -\gamma : [-b, -a] &\rightarrow \mathbb{C} \\ t &\mapsto \gamma(-t). \end{aligned} \quad (106)$$

Definition 6.12. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be an arc. We say $\Gamma(s), s \in [c, d] \subseteq \mathbb{R}$ has been obtained from $\gamma(t), t \in [a, b]$ by a *change of parametrization* if and only if the new parameter s and the original parameter t are related by a relation $t = \phi(s)$, where $\phi : [c, d] \rightarrow [a, b]$ is a homeomorphism that satisfies $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$. We call Γ the *reparametrization of γ* .

Definition 6.13. Let $\gamma_1 : I_1 \rightarrow \mathbb{C}$ and $\gamma_2 : I_2 \rightarrow \mathbb{C}$ be two arcs. We say they are *equivalent* if and only if there exists a bijective, monotone, and continuous function $\rho : I_2 \rightarrow I_1$ such that $\gamma_2 = \gamma_1 \circ \rho$. If ρ is an increasing function we say γ_1 and γ_2 have the *same orientation*; otherwise, we say γ_1 and γ_2 have *opposite orientations*.

Definition 6.14. Let $\gamma_1 : [a, b] \rightarrow \mathbb{C}$ and $\gamma_2 : [c, d] \rightarrow \mathbb{C}$ be two arcs such that $[a, b] \cap [c, d] = \emptyset$. We define the application $\gamma_1 \cup \gamma_2$ (sometimes denoted by $\gamma_1 + \gamma_2$) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \leq t \leq b \\ \gamma_2(t - b + c), & \text{if } b \leq t \leq b + d - c \end{cases}. \quad (107)$$

We say γ_1, γ_2 can be joined/added or that there exists its union/sum if and only if $\gamma_1(b) = \gamma_2(c)$. In this case $\gamma_1 + \gamma_2$ is an arc, and we call it the *sum arc of γ_1 plus γ_2* .

Definition 6.15. We define the *segment of extremes* $z_1, z_2 \in \mathbb{C}$ as the arc defined by the expression

$$\begin{aligned} [z_1, z_2] : [0, 1] &\longrightarrow \mathbb{C} \\ t &\longmapsto (1-t)z_1 + tz_2. \end{aligned} \quad (108)$$

Definition 6.16. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say f is *polygonal* if and only if can be expressed as a finite union of segments, that is, if there exist a natural number n and points $\{z_0, \dots, z_n\}$ such that

$$p = [z_0, z_1] \cup \dots \cup [z_{n-1}, z_n]. \quad (109)$$

Definition 6.17. Let $\gamma : [a, b] \longrightarrow D$ be an arc with a, b finite. We say γ is a *basic curve* if and only if $\gamma \in C^1((a, b)) \cap C([a, b])$ and there exist $\lim_{t \rightarrow a^+} \gamma'(t), \lim_{t \rightarrow b^-} \gamma'(t)$.

Definition 6.18. A *path* is a function $\gamma : [a, b] \longrightarrow \mathbb{C}$ such that there exist basic curves $\gamma_j : [a_j, b_j] \longrightarrow \mathbb{C}, j \in \{1, \dots, k\}$ such that $\gamma = \gamma_1 + \dots + \gamma_k$ and therefore $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ and $a = a_1, b = a_k$.

Definition 6.19. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a continuous curve and $a_1, \dots, a_l \in \mathbb{R}$ such that $a = a_0 \leq \dots \leq a_l \leq b = a_{l+1}$. We say γ is *piece-wise differentiable* if and only if

$$\gamma \in C^1 \left(\bigcup_{j=0}^l (\alpha_j, \alpha_{j+1}) \right),$$

$\forall j \in \{0, \dots, l+1\} \exists \lim_{t \rightarrow a_j^+} \gamma'(t) \text{ (except if } j = l+1), \lim_{t \rightarrow a_j^-} \gamma'(t) \text{ (except if } j = 0)$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

Theorem 6.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with Ω an open set and $\phi : I \longrightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curve (hence a derivable curve) and its real derivative is

$$\psi'(t) = f'(\phi(t))\phi'(t). \quad (110)$$

Definition 6.20. Let $\gamma_1, \gamma_2 : [0, 1] \longrightarrow \mathbb{C}$ be two curves. We say γ_1, γ_2 are *homotopic* if and only if there exists a continuous function $h(t, s) : [0, 1] \times [0, 1] \longrightarrow \mathbb{C}$ such that

1. $h(t, 0) = \gamma_1(t), t \in [0, 1]$.
2. $h(t, 1) = \gamma_2(t), t \in [0, 1]$.
3. $h(0, s) = \gamma_1(0) = \gamma_2(0), s \in [0, 1]$.
4. $h(1, s) = \gamma_1(1) = \gamma_2(1), s \in [0, 1]$.

Definition 6.21. Let $\gamma_1, \gamma_2 : [0, 1] \longrightarrow \mathbb{C}$ be two circuits. We say γ_1, γ_2 are *homotopic* if and only if there exists a continuous function $h(t, s) : [0, 1] \times [0, 1] \longrightarrow \mathbb{C}$ such that

1. $h(t, 0) = \gamma_1(t), t \in [0, 1]$.
2. $h(t, 1) = \gamma_2(t), t \in [0, 1]$.
3. $h(0, s) = h(1, s), s \in [0, 1]$.

Definition 6.22. Let $f : [a, b] \longrightarrow \mathbb{C}$ be a function with the notation $f = u + iv$. We define the integral of f as

$$\int_a^b f(t) dt := \int_a^b u(t) dt + i \int_a^b v(t) dt. \quad (111)$$

Proposition 6.2. Let $f, g : [a, b] \longrightarrow \mathbb{C}$ be two integrable functions and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,

$$\int_a^b \lambda f + \mu g dt = \lambda \int_a^b f dt + \mu \int_a^b g dt. \quad (112)$$

Proposition 6.3. Let $f : [a, b] \longrightarrow \mathbb{C}$ be an integrable function. Then,

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt. \quad (113)$$

Definition 6.23. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of f over γ* as

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t))\gamma'(t) dt. \quad (114)$$

Proposition 6.4. The previous definition is well defined.

Proposition 6.5. If we use the notation $f = u + iv$ and $\gamma = x + iy$, then the integral has the form

$$\int_{\gamma} f = \int_a^b u \frac{dx}{dt} + v \frac{dy}{dt} dt + i \int_a^b v \frac{dx}{dt} + u \frac{dy}{dt} dt. \quad (115)$$

Definition 6.24. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of f over γ with respect the differential of length* as

$$\int_{\gamma} f(z) ds := \int_{\gamma} f(z) |dz| = \int_a^b f(\gamma(t)) |\gamma'(t)| dt. \quad (116)$$

Theorem 6.6. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$, $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two functions, and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,

$$\int_{\gamma} \lambda f + \mu g dz = \lambda \int_{\gamma} f dz + \mu \int_{\gamma} g dz. \quad (117)$$

Theorem 6.7. Let γ_1, γ_2 be two equivalent curves of the same orientation and of class C^1 on their respective domains and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz. \quad (118)$$

Proposition 6.8. Let $\gamma_1, \dots, \gamma_n$ be n curves of class C^1 on their respective domains and $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \dots, \Gamma_n \subseteq \Omega$. If we define $\gamma = \gamma_1 + \dots + \gamma_n$, then

$$\int_{\gamma} f(z) dz = \sum_{i=1}^n \int_{\gamma_i} f(z) dz. \quad (119)$$

Proposition 6.9. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f| ds. \quad (120)$$

Corollary 6.10. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. If $|f(z)| \leq M$ for all $z \in \Gamma$, then,

$$\left| \int_{\gamma} f(z) dz \right| \leq ML(\gamma). \quad (121)$$

Proposition 6.11. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,

$$\text{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} dw. \quad (122)$$

Proposition 6.12. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,

$$|\text{Ind}(\gamma, z)| \leq \frac{1}{2\pi} \frac{L(\gamma)}{|z - \Gamma|}. \quad (123)$$

Proposition 6.13. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $\{f_n\}_{n=0}^{\infty}$ a sequence of continuous functions on Γ such that $\sum_{n=0}^{\infty} f_n$ converges uniformly on Γ . Then, $\sum_{n=0}^{\infty} \int_{\gamma} f_n dz$ converges and

$$\int_{\gamma} \sum_{n=0}^{\infty} f_n(z) dz = \sum_{n=0}^{\infty} \int_{\gamma} f_n(z) dz. \quad (124)$$

Definition 6.25. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. We say f has a *primitive* on Ω if and only if there exists a function $F : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ such that $F' = f \forall z \in \Omega$.

Definition 6.26. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. We say f has a *local primitive* on D if and only if for all z there exists a neighborhood where f has a primitive.

Theorem 6.14 (Fundamental theorem of complex calculus). Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function with Ω a domain. Then, the line integral of f is independent on the path on Ω if and only if f has an holomorphic primitive F such that $F' = f$ on Ω . In that case,

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a)). \quad (125)$$

Theorem 6.15. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function on a star domain $S \subseteq \Omega$. Then, f has an holomorphic primitive F on S if and only if

$$\int_{\partial \Delta} f(z) dz = 0 \quad (126)$$

for all triangle $\Delta \subseteq \Omega$.

Proposition 6.16. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function with no roots on a domain $D \subseteq \Omega$. Then, there is a determination of the logarithm of f on D if and only if f'/f has an holomorphic primitive on D .

Proposition 6.17. Let $K \subseteq \mathbb{C}$ be a compact set. Then,

1. If $\alpha \in V_{\infty}$, then the non-bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log(z - \alpha)$ in a neighborhood of K .
2. If α, β belong to the same bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log\left(\frac{z-\alpha}{z-\beta}\right)$ in a neighborhood of K .

Theorem 6.18 (Green's theorem). Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary. Let $\mathbf{F} = (P, Q)$ be a vector field with P, Q being differentiable functions on a neighborhood of Ω such that $\partial_x P - \partial_y Q$ is continuous on Ω . Then,

$$\int_{\partial \Omega} \langle \mathbf{F}, ds \rangle_I = \int_{\partial \Omega} P dx + Q dy = \iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy. \quad (127)$$

Theorem 6.19 (Cauchy's integral theorem). Let Ω be a bounded domain with piece-wise regular and positively oriented boundary and $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ an holomorphic function in a neighborhood of $\bar{\Omega}$. Then,

$$\int_{\partial \Omega} f(z) dz = 0. \quad (128)$$

Corollary 6.20. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function in a domain $D \subseteq \Omega$. Then, f has local primitive on D . If D is a star domain, f has a global holomorphic primitive.

Corollary 6.21. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function with no roots in a domain $D \subseteq \Omega$. Then, f has a local determination of the logarithm on D . If D is a star domain, f has a global determination of the logarithm.

Theorem 6.22 (Cauchy's integral theorem for homotopic curves). Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function with Ω a domain and γ_1, γ_2 two homotopic curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz. \quad (129)$$

Theorem 6.23 (Cauchy's general integral theorem). *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a regular function on Ω except a finite numbers of points where f is continuous. If γ is a constant curve, then*

$$\oint_{\gamma} f(z) dz = 0. \quad (130)$$

Theorem 6.24 (Morera's theorem). *Let f be a continuous function in a region Ω . If*

$$\oint_{\gamma} f(z) dz = 0 \quad (131)$$

for all simple and closed curve γ such that $\Gamma \subseteq \Omega$, then f is analytic on Ω .

Theorem 6.25. *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a differentiable function on a domain D . Then, $f = u + iv$ is holomorphic if and only if the field $\bar{f} = (u, -v)$ is locally conservative and locally solenoidal.*

Definition 6.27. Let $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a differentiable vector field on a domain $D \subseteq \mathbb{R}^n$. We say the field is *holomorphic* if and only if is locally conservative and locally solenoidal, that is, it satisfies

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}, \quad \forall i, j; \quad \operatorname{div} \mathbf{F} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} = 0, \quad \text{on } D. \quad (132)$$

Definition 6.28. Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a scalar field two times differentiable on an open set $\Omega \subseteq \mathbb{R}^n$. We say the field is *harmonic* if and only if $\nabla^2 \Phi = 0$ on Ω .

Theorem 6.26. *Holomorphic vector fields are the fields that are locally the gradient of an harmonic function. Holomorphic functions are the functions f that, locally, satisfy $\bar{f} = \Phi_x + i\Phi_y$ with Φ harmonic.*

Definition 6.29. Let u be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$. We say a differentiable function \tilde{u} on Ω is the *harmonic conjugate* of u if and only if $d\tilde{u} = d^*u$, that is, if the function $f = u + i\tilde{u}$ is holomorphic on Ω .

Theorem 6.27. *Let u be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$ and $f = \bar{\nabla}u$. Then, u has an harmonic conjugate on Ω , \tilde{u} , if and only if f has an holomorphic primitive F on Ω . In that case, $F = u + i\tilde{u}$.*

Proposition 6.28. *Let u be an harmonic function on a domain Ω . Then, it has an harmonic conjugate if and only if the closed form d^*u is exact on Ω , that is, if $\int_{\gamma} d^*u = 0$ for all closed curve γ such that $\Gamma \subseteq \Omega$, condition that is always locally completed. If Ω is a star domain, every harmonic function on Ω has a harmonic conjugate function on Ω .*

7 Local properties of holomorphic functions

Lemma 7.1. *Let $a \in \mathbb{C}$ be a number and $f = 1/|z - a|$. Then, f is Lebesgue-integrable on every subset of \mathbb{C} of finite measure.*

Theorem 7.2 (Cauchy-Green formula). *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and f a differentiable function on a neighborhood of $\bar{\Omega}$ such that $\bar{\partial}f$ is continuous on $\bar{\Omega}$. Then, for all $z_0 \in \Omega$,*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0} dz - \frac{1}{\pi} \int_{\Omega} \frac{\bar{\partial}f(z)}{z - z_0} dm(z). \quad (133)$$

Corollary 7.3 (Cauchy's integral formula). *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and f an holomorphic function on a neighborhood of $\bar{\Omega}$. Then,*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0} dz. \quad (134)$$

Corollary 7.4. *Let f be a differentiable function on \mathbb{C} with compact support and $\bar{\partial}f$ continuous on \mathbb{C} . Then,*

$$f(z_0) = -\frac{1}{\pi} \int_{\Omega} \frac{\bar{\partial}f(z)}{z - z_0} dm(z). \quad (135)$$

Proposition 7.5. *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function and γ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z_0)f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz. \quad (136)$$

Corollary 7.6. *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function and γ_1, γ_2 two homotopic, piece-wise regular, and positively oriented curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z - z_0} dz. \quad (137)$$

Proposition 7.7. *Let f be a function and Ω a bounded domain with piece-wise regular and positively oriented boundary. If f is holomorphic on a neighborhood of $\bar{\Omega}$, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (138)$$

Proposition 7.8. *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function and γ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z_0)f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (139)$$

8 Isolated singularities of holomorphic functions

9 Homology

10 Harmonic functions

Theorem 10.1. *Let $f \in H(\Omega), C^1(\Omega)$ be a function. If $f = u + iv$, then u, v are harmonic functions on Ω .*

11 Conforming representation

12 Riemann theorem

13 Runge theorem

14 Zeros of holomorphic functions

Theorem 14.1 (Weierstrass Factorization Theorem).
content...

15 Fourier transform

Definition 15.1. Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform* of f at the point ξ as

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx. \quad (140)$$

Proposition 15.1. Let $f \in L^1(\mathbb{R})$ be a function. Then, the application

$$\begin{aligned} \mathcal{F}\{f\} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \hat{f}(\xi) \end{aligned} \quad (141)$$

is a well defined application.

Definition 15.2. Let $\{f_n\}_{n \in \mathbb{N}} \subseteq L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. We say the functions f_n converge to f with a norm $\|\cdot\|_p$ or converge in $L^p(\mathbb{R})$ if and only if

$$\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0. \quad (142)$$

Theorem 15.2. Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.

1. The Fourier transform of f satisfies

$$\mathcal{F}\{f\} \in L^\infty(\mathbb{R}), \quad \|\mathcal{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|f\|_1. \quad (143)$$

2. $\mathcal{F}\{f\}$ is \mathbb{C} linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,

$$\mathcal{F}\{\alpha f + \beta g\} = \alpha \mathcal{F}\{f\} + \beta \mathcal{F}\{g\}. \quad (144)$$

3. If $g(x) = \bar{f}(x)$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \quad (145)$$

4. If $g(x) = g(\lambda x)$ and $\lambda \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \frac{1}{|\lambda|} \hat{f}\left(\frac{\xi}{\lambda}\right). \quad (146)$$

5. If $g(x) = f(x - a)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = e^{-ia\xi} \hat{f}(\xi). \quad (147)$$

6. If $g(x) = e^{iax} f(x)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \hat{f}(\xi - a) \quad (148)$$

7. If $\{f_n\}_{n \in \mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \rightarrow f$ in $L^1(\mathbb{R})$ when $n \rightarrow \infty$, then $\mathcal{F}\{f_n\} \rightarrow \mathcal{F}\{f\}$ uniformly in \mathbb{R} .

8. The Fourier transform $\mathcal{F}\{f\}$ is a continuous function in \mathbb{R} , $\mathcal{F}\{f\} \in C(\mathbb{R})$.

Proposition 15.3. Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and $\lim_{|x| \rightarrow \infty} |f(x)| = 0$. Then,

$$\hat{f}'(\xi) = i\xi \hat{f}(\xi). \quad (149)$$

Corollary 15.4. Let $f \in L^1(\mathbb{R})$ be a function such that there exists its n -th derivative $f^{(n)} \in L^1(\mathbb{R})$ and $\lim_{|x| \rightarrow \infty} |f(x)| = 0$. Then,

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi). \quad (150)$$

Definition 15.3. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We define the support of f as

$$\text{supp } f := \overline{\{x \in I \mid f(x) \neq 0\}}. \quad (151)$$

Definition 15.4. We define the set $\mathcal{D}(\mathbb{R})$ as

$$\mathcal{D}(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}) \mid \text{supp } \varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \quad (152)$$

Theorem 15.5. Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_n \in \mathcal{D}(\mathbb{R})$ such that

$$\lim_{h \rightarrow \infty} \int_{\mathbb{R}} |f - \phi_n| dx = 0, \quad (153)$$

that is, we have convergence of ϕ_n to f with norm $\|\cdot\|_1$.

Proposition 15.6. Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.

Proposition 15.7. Let $f \in L^1(\mathbb{R})$ be a function. Then, $|\hat{f}(\xi)| \leq \|f\|_1$.

Theorem 15.8. Let $f \in L^1(\mathbb{R})$ be a function. Then,

$$\lim_{|x| \rightarrow \infty} \hat{f}(x) = 0. \quad (154)$$

Theorem 15.9. The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathcal{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.

Definition 15.5. We define the Schwartz space as

$$S(\mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{C} \mid f \in C^\infty(\mathbb{R}) \wedge \forall n, m \in \mathbb{N} \exists c_{n,m} < \infty \text{ such that } (1 + |x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}.$$

Proposition 15.10. Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \longrightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,

1. $f + g \in S(\mathbb{R})$.

2. $\lambda f \in S(\mathbb{R})$.
3. $fg \in S(\mathbb{R})$.
4. $Pf \in S(\mathbb{R})$.

Theorem 15.11. Let $I, J \subseteq \mathbb{R}$ be two intervals with I compact and J open. Let $f : I \times J \longrightarrow \mathbb{R}$ be a function such that

1. $f(\cdot, \lambda)$ is Riemann-integrable in I for all $\lambda \in J$,
2. $f(x, \cdot)$ is derivable in J for all $x \in I$.

If $\partial_\lambda f$ is continuous in $I \times J$, then

1. $\partial_\lambda f(\cdot, \lambda)$ is Riemann-integrable for all $\lambda \in J$.
2. $F(\lambda) = \int_I f(x, \lambda) dx$ is derivable with continuous derivative in J for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.

$$F'(\lambda) = \frac{d}{d\lambda} \int_I f(x, \lambda_0) dx = \int_I \frac{\partial f}{\partial \lambda}(x, \lambda_0) dx, \forall \lambda_0 \in J \quad (155)$$

Proposition 15.12. Let $f \in S(\mathbb{R})$. Then,

1. $S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.
2. $\widehat{xf}(\xi) = (iD_\xi \hat{f})(\xi)$ for all $\xi \in \mathbb{R}$.

Corollary 15.13. Let $f \in s(\mathbb{R})$. Then,

$$\widehat{x^n f}(\xi) = (i^n D_\xi^n \hat{f})(\xi), \forall n \in \mathbb{N}. \quad (156)$$

Proposition 15.14. The Fourier transform \mathcal{F} restricted to $S(\mathbb{R})$ is an automorphism, that is, if $f \in S(\mathbb{R})$ then $\mathcal{F}\{f\} = \hat{f} \in S(\mathbb{R})$.

Lemma 15.15. If $G(x) = e^{-x^2/2}$, then $\hat{G}(\xi) = e^{-\xi^2/2}$. We observe hence that G is a fixed point of \mathcal{F} .

Lemma 15.16. If $f, g \in S(\mathbb{R})$, then

$$\int_{\mathbb{R}} f(\xi) \hat{g}(\xi) d\xi = \int_{\mathbb{R}} \hat{f}(\tau) g(\tau) d\tau. \quad (157)$$

Lemma 15.17. Let $f, g \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,

1. $g(\lambda x) \hat{f}(x)$ converges to $g(0) \hat{f}(x)$ uniformly in \mathbb{R} when $\lambda \rightarrow \infty$.
2. $f(\lambda x) \hat{g}(x)$ converges to $f(0) \hat{g}(x)$ uniformly in \mathbb{R} when $\lambda \rightarrow \infty$.

Lemma 15.18. Let $f, g \in s(\mathbb{R})$. Then,

$$f(0) \int_{\mathbb{R}} \hat{g}(\xi) d\xi = g(0) \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad (158)$$

Lemma 15.19. Let $f \in s(\mathbb{R})$ be a function. Then,

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad (159)$$

Corollary 15.20 (Inversion formula). Let $f \in S(\mathbb{R})$. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi, \forall x \in \mathbb{R}. \quad (160)$$

Theorem 15.21 (Inversion of \mathcal{F} in $S(\mathbb{R})$). Let $\mathcal{F} : S(\mathbb{R}) \longrightarrow S(\mathbb{R})$, defined by $\mathcal{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, \mathcal{F} is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathcal{F}^4 = \text{Id}$. In particular, $\mathcal{F}^{-1} = \mathcal{F}^3$ and if $f \in S(\mathbb{R})$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}\{f\}(\xi) e^{ix\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} e^{ix\xi} d\xi. \quad (161)$$

In fact, \mathcal{F} is an homomorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(S(\mathbb{R}), \|\cdot\|_{n,m})$.

Theorem 15.22 (Inversion of \mathcal{F} for discontinuities). Let f be a absolutely Riemann-integrable function in \mathbb{R} with f and f' piece-wise continuous. Then,

$$\frac{f(x^-) + f(x^+)}{2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} e^{ix\xi} d\xi. \quad (162)$$

Definition 15.6. Let f be a Riemann-integrable function in \mathbb{R} . We define the *Fourier transform of cosine kind* as

$$\hat{f}_c(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\xi x) f(x) dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(\xi x) f_e(x) dx, \quad (163)$$

and the *Fourier transform of sine kind* as

$$\hat{f}_s(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\xi x) f(x) dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(\xi x) f_o(x) dx. \quad (164)$$

Proposition 15.23. Let \hat{f}_c, \hat{f}_s be the Fourier transform of cosine and sine kinds of f . Then, $\hat{f}_c(\xi)$ is even, $\hat{f}_s(\xi)$ is odd, and $\hat{f}(\xi) = \hat{f}_c(\xi) - i\hat{f}_s(\xi)$.

Theorem 15.24. Let f be a absolutely Riemann-integrable function in \mathbb{R} with f and f' piece-wise continuous. Then,

$$\frac{f_e(x^-) + f_e(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c \cos(\xi x) d\xi, \quad (165)$$

$$\frac{f_o(x^-) + f_o(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s \sin(\xi x) d\xi. \quad (166)$$

Theorem 15.25 (Tonelli's Theorem). Let $f : I \times J \longrightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x, y) \geq 0$ for all $(x, y) \in I \times J$. Then,

$$\int_{I \times J} f dx dy = \int_I \int_J f(x, y) dy dx = \int_J \int_I f(x, y) dx dy. \quad (167)$$

Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.

Corollary 15.26. Let $f, g \in L^1(\mathbb{R})$. Then, $F(x, t) = f(t)g(x - t) \in L^1(\mathbb{R}^2)$.

Definition 15.7. Let $f, g \in L^1(\mathbb{R})$ two function. We define the *convolution* of f and g as

$$(f * g) : \mathbb{R} \longrightarrow \mathbb{C} \\ x \longmapsto \int_{\mathbb{R}} f(t)g(x - t) dt, \quad (168)$$

which is from $L^1(\mathbb{R})$.

Proposition 15.27. Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi} \hat{f} \hat{g}$.

Proposition 15.28. Let $f \in L^1(\mathbb{R})$ be a function and $g = f^2$. Then,

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \hat{f} * \hat{f} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) \hat{f}(\xi - t) dt. \quad (169)$$

Theorem 15.29. Let $f \in L^p(\mathbb{R})$, $1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^\infty(\mathbb{R})$.

Theorem 15.30. Let $f \in L^p(\mathbb{R})$, $1 \leq p < +\infty$ with $\text{supp } f$ compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and $\text{supp } \{f * \phi\} \subseteq \text{supp } f + \text{supp } \phi$.

Definition 15.8. We say the functions $\phi_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ continuous in a compact support are an *approximation of the unity* if and only if

1. $\phi_\epsilon \geq 0$ for all ϵ .
2. $\int_{\mathbb{R}} \phi_\epsilon(x) dx = 1$.
3. For all $\delta > 0$ it is satisfied that

$$\lim_{\epsilon \rightarrow 0} \left\{ \sup_{|t| > \delta} \phi_\epsilon(t) \right\} = 0. \quad (170)$$

Theorem 15.31. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \rightarrow 0$ $f * \phi_\epsilon$ converges uniformly in \mathbb{R} to f .

Corollary 15.32. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \rightarrow 0$ $f * \phi_\epsilon$ converges uniformly in \mathbb{R} to f .

Theorem 15.33 (Weierstrass polynomial approximation). Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials P_n with $n \in \mathbb{N}$ such that P_n converge uniformly to f in $[a, b]$.

Theorem 15.34. Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \rightarrow f$ with norm $\|\cdot\|_p$ (that is, convergence in L^p), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \quad (171)$$

with $\|f\|_{C^k(\mathbb{R})} = \max_{0 \leq l \leq k} \left(\sup_{x \in \mathbb{R}} |D^l f(x)| \right)$ being a norm.

Lemma 15.35. Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\int_{\mathbb{R}} f(x)\phi(x) dx = 0$.

Then, $f \equiv 0$.

Corollary 15.36. The Fourier transform \mathcal{F} is injective since $\mathcal{F}\{f\} = \hat{f} = 0 \Leftrightarrow f = 0$ in $L^1(\mathbb{R})$ (the zero function class) and \mathcal{F} is a linear application.

Theorem 15.37 (Inversion theorem in $L^1(\mathbb{R})$). Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi, \quad \forall x \in \mathbb{R}. \quad (172)$$

16 Fourier transform 2

Theorem 16.1 (Parseval formula). Let $f, g \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be two functions. Then,

$$\int_{\mathbb{R}} f(x) \overline{g(x)} dx = \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi. \quad (173)$$

Theorem 16.2 (Plancherel Theorem). Let $f \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be a function. Then,

$$\boxed{\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi}, \quad (174)$$

that is, $\|f\|_2 = \|\hat{f}\|_2$ and \mathcal{F} is an isometry between vector spaces.

Definition 16.1. Let $f \in S(\mathbb{R})$ be a function. We define the following quantities

$$E(f) := \int_{\mathbb{R}} |f(x)|^2 dx, \quad (175)$$

$$\sigma(f)^2 := \int_{\mathbb{R}} |xf(x)|^2 dx. \quad (176)$$

Theorem 16.3. Let $f \in S(\mathbb{R})$ be a function. Then,

$$\sigma(f)\sigma(\hat{f}) \geq \frac{E(f)}{2}. \quad (177)$$

17 Multidimensional fourier transform

Theorem 17.1. For several variables

$$\mathcal{F}\{f(x_1, \dots, x_n)\} = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) e^{-i(x_1\xi_1 + \cdots + x_n\xi_n)} dx_1 \cdots dx_n \quad (178)$$

or simpler,

$$\boxed{\mathcal{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} d\mathbf{x}} \quad (179)$$