1 Motion in one dimension

Proposition 1.1. *Let .... Then, it is true that*

$$\vec{F}^{ext} = m\vec{a} + (\vec{v} - \vec{u})\dot{m} \tag{1}$$

or which is equivalent,

$$\vec{F}^{ext} + \dot{m}\vec{u} = \dot{\vec{p}} \tag{2}$$

2 Oscillations

Proposition 2.1. *Let be the following differential equation*

$$\ddot{x} + \omega_0^2 x = 0, \tag{3}$$

with the initial value condition of $x(0) = x_0$ *and* $v(0) = v_0$*. Then, the general solution is*

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t, \tag{4}$$

or, which is equivalent,

$$x(t) = A \cos [\omega_0 t + \phi_0], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \phi_0 = -\arctan \frac{v_0}{\omega_0 x_0}. \tag{5}$$

Definition 2.1. Let $U(x)$ be a potential function of class $C^2(\mathbb{R})$. Then, we say x_0 *is a point of stable equilibrium* if U has a maxima in x_0.

Definition 2.2. Let $U(x)$ be a potential function of class $C^2(\mathbb{R})$. Then, we say x_0 *is a point of unstable equilibrium* if U has a minima in x_0.

Proposition 2.2. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{6}$$

with the initial value conditions of $x(0) = x_0$ *and* $v(0) = v_0$*. Then, the general solution is*

$$x(t) = e^{-\beta t}\left[x_0 \cos \tilde{\omega} t + \frac{v_0 + \beta x_0}{\tilde{\omega}} \sin \tilde{\omega} t\right], \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \beta^2} \tag{7}$$

if $\beta < \omega_0$,

$$x(t) = e^{-\omega_0 t}\left[x_0 + (x_0\omega_0 + v_0)t\right] \tag{8}$$

if $\beta = \omega_0$, *and*

$$x(t) = \frac{x_0(\bar{\omega} - \beta) - v_0}{2\bar{\omega}} e^{-(\beta+\bar{\omega})t} + \frac{x_0(\bar{\omega} + \beta) + v_0}{2\bar{\omega}} e^{-(\beta-\bar{\omega})t}, \qquad \bar{\omega} = \sqrt{\beta^2 - \omega_0^2} \tag{9}$$

if $\beta > \omega_0$.

Definition 2.3. Amplitude as a function of time.

$$A(t) = Ae^{-\beta t} \tag{10}$$

Definition 2.4. The quality factor Q is defined as

$$Q := \frac{\omega_0}{2\beta} \tag{11}$$

Proposition 2.3. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0 \cos[\omega t + \psi_0], \tag{12}$$

with the initial value conditions of $x(0) = x_0$ *and* $v(0) = v_0$*. Then, the particular solution is*

$$x_p(t) = A \cos[\omega t + \psi_0 - \phi_0], \qquad A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}, \qquad \phi \tag{13}$$

3 Central forces

Definition 3.1. Central force

$$\vec{F}(\vec{r}) = f(r)\vec{e}_\rho \tag{14}$$

Definition 3.2. The origin $\vec{r} = \vec{0}$ is the *center of forces.*

Proposition 3.1. *All central forces are conservatives.*

Proposition 3.2. *The angular momentum with respect the origin is conserved.*

$$\dot{\vec{L}} = \vec{0} \tag{15}$$

Proposition 3.3. *The areal velocity is constant.*

$$\frac{dA}{dt} = \frac{L}{2m} = ctt \tag{16}$$

Theorem 3.4 (Bertrand's Theorem)**.** *The only central potentials where every bounded orbit is closed are:*

$$U(r) = -\frac{k}{r}, \qquad U(r) = \frac{k}{2}r^2, \qquad k > 0 \tag{17}$$

Definition 3.3. The pericenter is the minimum value of $r(\theta)$.

Definition 3.4. The apocenter is the maximum value of $r(\theta)$.

Definition 3.5. Apsides are the turning points.

Definition 3.6. Latus rectum

$$\alpha := \frac{LL'}{2} = r(\pi/2) = \epsilon d \tag{18}$$

Law 1. *Planets move in elliptical orbits about the Sun with the Sun at one focus.*

Law 2. *The area per unit time swept out by a radius vector from the Sun to a planet is constant.*

Law 3. *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.*

4 Coupled oscillations 1

5 Coupled oscillations 2

6 Rotations

7 Dynamics of rigid body

Proposition 7.1. *The vector* $\mathbf{\Omega}$ *is independent on the origin of the system* S.

Proposition 7.2. *The energy of the rigid body is an invariant scalar under change of basis.*

8 Special relativity

Theorem 8.1. *If Maxwell's equations*

$$\langle \mathbf{\nabla}, \mathbf{E} \rangle_I = \frac{\rho}{\epsilon_0}, \tag{19}$$

$$\langle \mathbf{\nabla}, \mathbf{B} \rangle_I = 0, \tag{20}$$

$$\mathbf{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{21}$$

$$\mathbf{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \tag{22}$$

are invariant under Galileo transformations, then $\mathbf{E} = \mathbf{B} = \mathbf{0}$.

Definition 8.1 (Reference system)**.** We define a *reference system* S as a set of three axis and one origin over which we have determined an orientation. We will suppose we have selected a unit of length and that in each point a in the immobile space with respect the axis there is a clock q_a such that the clocks q_a and q_b corresponding to two different points a and b immobile with respect these axis are synchronized

Definition 8.2. Let S and S' two reference systems. We say S' *is moving in a uniform straight motion* if and only if there exists a vector $\mathbf{v} \in \mathbb{R}^3$ such that every particle a immobile with respect to S' moves in each instant t of the time of S to the point of coordinates $(a^1(t), a^2(t), a^3(t))$ with respect the axis of S, such that $\mathrm{d}a^i \,/\, \mathrm{d}t = v^i$ for $i = 1, 2, 3$

Definition 8.3. Let S, S' two reference systems. We say they are *equivalent* if and only if they are immobile one with respect the other and use the same unit of length and scale of time, or satisfy the following conditions

1. Each one of them is moving with respect to the other with a uniform line motion with $\mathbf{v} \neq \mathbf{0}$.

2. Two events occur in an immobile point with respect to s are seen from S' with the same temporal order as that seen from S. The same happens exchanging the roles of S and S'.

3. We make the following experiment: from S' two events are observed in an immobile point a with respect to S and separated by an interval of temps of S of length 1. Since S is moving with respect to S', these events will be seen in two different positions from S'. From S' it is measured the distance between these two points. The condition we are enunciating is the result is the same exchanging the roles of S and S'.

Axiom 1 (Relativity principle)**.** *There exists a privileged class of reference systems, each one called inertial system, that satisfies 4 properties*

1. *If S is inertial, it is inertial every other system S' immobile with respect to S obtained from S by a displacement (in a mathematical sense) of the axis and a change of origin of time.*

2. *If S is inertial, and P is a point that moves with respect to S in a uniform line motion and velocity $\mathbf{v}$, there exists a reference system that has P as the origin.*

3. *Two inertial systems S and S' are always equivalent.*

4. *The observers that travel in any inertial reference system can state the physical laws in the same way, independently of the system where we are working.*

Axiom 2 (Invariance of light velocity)**.** *Let S be a reference system Then,*

1. *If an immobile observer with respect to an inertial reference system S contemplates the propagation of light in the vacuum emitted by any focus, the observer will see it propagates with a uniform line motion.*

2. *The speed of such ray of light is constant, independent on the inertial reference system S. We denote by c this speed.*

Lemma 8.2. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms lines that are not contained in hyperplanes of the form $t = \mathrm{ctt}$ to lines that are not contained in hyperplanes of the form $t' = \mathrm{ctt}$.*

Lemma 8.3. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms planes that are not contained in hyperplanes of the form $t = \mathrm{ctt}$ to planes that are not contained in hyperplanes of the form $t' = \mathrm{ctt}$.*

Lemma 8.4. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms hyperplanes that are not of the form $t = \mathrm{ctt}$ to hyperplanes that are not of the form $t' = \mathrm{ctt}$.*

Theorem 8.5. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f is an affine transformation.*

Corollary 8.6. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation and H an hyperplane of the form $t = \mathrm{ctt}$. Then, $f(H)$ is an hyperplane of the form $t' = \mathrm{ctt}$.*

Theorem 8.7. *Let S, S' be two inertial reference systems. We can make orthogonal changes (isometries) of axis to S and S' and a change of origin of time such thate the Lorentz transformation has the form of the equation* **??**.

Lemma 8.8. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation and $r \subseteq \mathbb{R}^4$ a line with a timelike direction vector. Then, f transforms r to a line with a timelike direction vector.*

Definition 8.4. Let $\pi \subseteq \mathbb{R}^4$ be a plane (hyperplane). We say it is *admissible* if and only if it has the form $a + H$ with H being a subspace of dimension 2 (3) that admits a basis of vectors of C^+.

Lemma 8.9. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation and V and admissible plane (hyperplane). Then, $f(V)$ is an admissible plane (hyperplane).*

Theorem 8.10. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f is an affine transformation.*

Theorem 8.11. *(Lorenz transformation) Let S, S' be two reference systems with the same origin such that S' moves with a constant velocity $\mathbf{v} = v\mathbf{e}_x$. Then,*

$$P_{\mathrm{s}'} = \Lambda P_{\mathrm{s}} \Leftrightarrow P^{\nu}_{\mathrm{s}'} = \Lambda^{\nu}_{\mu} P^{\mu}_{\mathrm{s}}, \qquad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

Definition 8.5. A Lorentz space is an orthogonal geometry $(L, \langle , \rangle)$ with Sylvester invariants $r_0 = 0$ and $r_- = 1$.

Definition 8.6. Let $\mathbf{R} \in L$ be a vector. We define is norm as

$$\|\mathbf{R}\|_m \coloneqq \sqrt{|\langle \mathbf{R}, \mathbf{R} \rangle_m|}. \tag{24}$$

Proposition 8.12. *Let $\mathbf{R} \in L$ be a vector and $a \in \mathbb{R}$ a scalar. Then, $\|a\mathbf{R}\|_m = |a|\|\mathbf{R}\|_m$.*

Definition 8.7. Let $\mathbf{R} \in L$ be a vector different from $\mathbf{0}_L$.

1. We say $\mathbf{R}$ *is timelike* if and only if $\langle \mathbf{R}, \mathbf{R} \rangle_m < 0$.
2. We say $\mathbf{R}$ *is spacelike* if and only if $\langle \mathbf{R}, \mathbf{R} \rangle_m > 0$.
3. We say $\mathbf{R}$ *is lightlike* if and only if $\langle \mathbf{R}, \mathbf{R} \rangle_m = 0$.

Definition 8.8. Let $W \subseteq L$ be a subspace.

1. We say W is *timelike* if and only if $(W, \langle , \rangle)$ is a Lorentz space.
2. We say W is *spacelike* if and only if $(W, \langle , \rangle)$ is an Euclidean space.
3. We say W is *lightlike* if and only if $(W, \langle , \rangle)$ is singular.

Proposition 8.13. *Every subspace W of L is either timelike, spacelike, or lightlike. Besides,*

1. *S is timelike $\Leftrightarrow W^{\perp}$ is spacelike.*
2. *S is spacelike $\Leftrightarrow W^{\perp}$ is timelike.*
3. *W is lightlike $\Leftrightarrow W^{\perp}$ is lightlike.*

Proposition 8.14. *Two orthogonal vectors different from zero and non spacelike are necessarily lightlike and collinear. In particular, there is not a subspace of dimension 2 where $\langle , \rangle$ is null.*

Corollary 8.15. *Let $W \subseteq L$ be a subspace of dimension 2. Then, the following statements are equivalent.*

1. *W is timelike.*
2. *W contains two lightlike vectors that are linearly independent.*
3. *W contains a timelike vector.*

Corollary 8.16. *Let $W \subseteq L$ be a subspace. Then, the following statements are equivalent.*

1. *W is lightlike.*
2. *W contains lightlike vectors and no timelike vectors.*
3. *If we add the vector $\mathbf{0}_L$ to the set of isotrop vectors of W, we get a one dimensional subspace.*

Proposition 8.17. *Let $\mathbf{R}_1, \mathbf{R}_2 \in T$ be two timelike vectors. Then, the following statements are true.*

1. *$|\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m| \geq \|\mathbf{R}_1\|_m \|\mathbf{R}_2\|_m$, and the equality is equivalent to both vectors being collinear.*
2. *$\mathbf{R}_1, \mathbf{R}_2$ are in the same time cone (C_+ or C_-) if and only if $\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m < 0$. In this case,*
   (a) *There is a unique $\varphi \in \mathbb{R}$ such that*
   $$\cosh \varphi = -\frac{\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m}{\|\mathbf{R}_1\|_m \|\mathbf{R}_2\|_m}. \tag{25}$$
   We call this φ the hyperbolic angle.
   (b) *$\|\mathbf{R}_1\|_m + \|\mathbf{R}_2\|_m \leq \|\mathbf{R}_1 + \mathbf{R}_2\|_m$.*

Definition 8.9. As a model of space-time, we consider the affine space $\mathbb{M} = \mathbb{R}^4$ over the Lorenz vector space L.

Definition 8.10. We define and *event* as a point that has location in $\mathbb{M}$.

Definition 8.11. Let $P \in \mathbb{M}$ be an event and C_+, C_- be the past an future cones. Then, we define the *future cone of P* and the *past cone of P* as the sets $P + C_+$ and $P + C_-$.

Proposition 8.18. *The Lorentz-Minkowski metric (using the proper orthonormal basis) can be expressed by the bilinear form η*

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Proposition 8.19. *The transformation in $\mathbb{M} = \mathbb{R}^4$ (using the proper orthonormal basis) can be expressed by the matrix Λ*

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

Proposition 8.20. *The Lorenz-Minkowski metric is invariant under Lorentz transformations.*

Corollary 8.21. *The norm of the vector of the Minkowski space are independent on the inertial reference system.*

Definition 8.12. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a parametrization $\gamma(t) : I \longrightarrow \Gamma$ of class $C^2(I)$ and a tangent vector $u \in C^+$ in an inertial reference system S. We define the *4-position vector* as

$$\mathbf{X} = (ct, x, y, z). \tag{28}$$

Definition 8.13. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a piece-wise differentiable parametrization $\gamma(t) : I \longrightarrow \Gamma$ and tangent vector $u \in C^+$ in an inertial reference system S. We define the *space-time interval* as

$$\Delta s^2 \coloneqq -\Delta\mathbf{R}^t \boldsymbol{\eta} \Delta\mathbf{R} = -\langle \Delta\mathbf{R}, \Delta\mathbf{R} \rangle_m = -\langle \Delta\mathbf{R}', \Delta\mathbf{R}' \rangle_m, \tag{29}$$

or as a differential form,

$$\mathrm{d}s^2 \coloneqq \mathrm{d}x^\mu \, \eta_{\mu\nu} \, \mathrm{d}x^\nu = c^2 \, \mathrm{d}t^2 - \langle \mathrm{d}\mathbf{r} \,, \mathrm{d}\mathbf{r} \rangle_I. \tag{30}$$

Axiom 3. *Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a piece-wise differentiable parametrization $\gamma(t) : I \longrightarrow \Gamma$ and tangent vector $u \in C^+$ in an inertial reference system S. If the motion starts at an event A and ends at an event B, then the time interval observed by a system S' that moves along p is given by*

$$\tau = \frac{1}{c} \int_\Gamma \sqrt{-\,\mathrm{d}x^\mu \, \eta_{\mu\nu} \, \mathrm{d}x^\nu} = \int_{t_A}^{t_B} \frac{1}{\gamma(t)} \, \mathrm{d}t \Leftrightarrow c \, \mathrm{d}\tau = \mathrm{d}s \,. \tag{31}$$

Definition 8.14. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a piece-wise differentiable parametrization $\gamma(t) : I \longrightarrow \Gamma$ and tangent vector $u \in C^+$ in an inertial reference system S. We define the *proper time* of p as

$$\mathrm{d}\tau \coloneqq \frac{1}{c} \, \mathrm{d}s = \frac{1}{\gamma} \, \mathrm{d}t \,. \tag{32}$$

Proposition 8.22. *Let S, S' be two inertial reference systems such that the velocity of S' is $\mathbf{w} = w\mathbf{e}_x$ with respect to S. Then,*

$$v^1_{\mathrm{s}'} = \frac{v^1_{\mathrm{s}} - w}{1 - \beta_v \beta_w}, \qquad v^2_{\mathrm{s}'} = \frac{1}{\gamma_w} \frac{v^2_{\mathrm{s}}}{1 - \beta_v \beta_w}, \qquad v^3_{\mathrm{s}'} = \frac{1}{\gamma_w} \frac{v^3_{\mathrm{s}}}{1 - \beta_v \beta_w}, \tag{33}$$

Proposition 8.23. *If the system has a general velocity $\mathbf{w}$, then*

$$\mathbf{v}' = \frac{1}{1 - \langle \boldsymbol{\beta}_v, \boldsymbol{\beta}_2 \rangle_I} \left[\frac{\mathbf{v}}{\gamma_w} - \mathbf{w} + \frac{1}{c^2} \frac{\gamma_w}{1 + \gamma_w} \langle \mathbf{v}, \mathbf{w} \rangle_I \mathbf{w} \right], \tag{34}$$

$$\mathbf{v} = \frac{1}{1 + \langle \boldsymbol{\beta}_{v'}, \boldsymbol{\beta}_w \rangle_I} \left[\frac{\mathbf{v}'}{\gamma_w} + \mathbf{w} + \frac{1}{c^2} \frac{\gamma_w}{1 + \gamma_w} \langle \mathbf{w}', \mathbf{w} \rangle_I \mathbf{v} \right]. \tag{35}$$

Definition 8.15. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a parametrization $\gamma(t) : I \longrightarrow \Gamma$ of class $C^2(I)$ and a tangent vector $u \in C^+$ in an inertial reference system S. We define the *4-velocity* as

$$\mathbf{U} \coloneqq \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau}. \tag{36}$$

Proposition 8.24. *Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a 4-velocity $\mathbf{U}$. Then,*

$$\mathbf{U} = (\gamma c, \gamma \mathbf{v}), \qquad -\langle \mathbf{U}, \mathbf{U} \rangle_m = c^2. \tag{37}$$

Definition 8.16. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a parametrization $\gamma(t) : I \longrightarrow \Gamma$ of class $C^2(I)$ and a tangent vector $u \in C^+$ in an inertial reference system S. We define the *4-acceleration* as

$$\mathbf{A} \coloneqq \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}\tau^2}. \tag{38}$$

Proposition 8.25. *Let p be a particle of velocity $\mathbf{U}$ and acceleration $\mathbf{A}$. Then, $\langle \mathbf{A}, \mathbf{U} \rangle_m = 0$.*

Definition 8.17. Let p be a particle of mass m moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a parametrization $\gamma(t) : I \longrightarrow \Gamma$ of class $C^2(I)$ and a tangent vector $u \in C^+$ in an inertial reference system S. We define the *4-momentum* as

$$\mathbf{P} \coloneqq m\mathbf{U}. \tag{39}$$

Proposition 8.26. *Let p be a particle of 4-momentum $\mathbf{P}$. Then,*

$$\mathbf{P} = (E/c, \mathbf{p}) = (\gamma mc, \gamma m \mathbf{v}). \tag{40}$$

Theorem 8.27.

$$E^2 = p^2c^2 + m^2c^4. \tag{41}$$

Proposition 8.28. *There are three possible cases: stationary particle with mass, moving particle with mass, particle with no mass.*

$$E = mc^2, \qquad E^2 = m^2c^4 + \|p\|^2 c^2, \qquad E = pc \tag{42}$$

Theorem 8.29. *(Work-Energy theorem)*

$$W = \Delta E. \tag{43}$$

Theorem 8.30. *Let p be a particle of velocity $\mathbf{v}$. Then, the kinetic energy is obtained by the expression*

$$T = \int_\Gamma \langle \mathbf{F}, \mathrm{d}\mathbf{r} \rangle_I = (\gamma(\dot{\mathbf{r}}) - 1)mc^2 \tag{44}$$

Theorem 8.31 (Compton scattering)**.**

$$\Delta\lambda = \frac{h}{mc}(1-\cos\theta). \tag{45}$$

Theorem 8.32 (Center of momentum)**.** *Let be a system of particles $p_1, \dots, p_n$ with energies $E_1, \dots, E_n$ and momentum $\mathbf{p}_1, \dots, \mathbf{p}_n$. Then, the center of momentum system has a velocity determined by the expression*

$$\mathbf{v}_{\text{cp}} = \frac{1}{E_t}\sum_{i=1}^{n} \|\mathbf{p}_i\|^2 c^2. \tag{46}$$

Definition 8.18. The relativistic *4-force*

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau}, \qquad \mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{47}$$

Proposition 8.33. *Let p be a particle of mass m moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Then,*

$$\langle \mathbf{f}, \mathbf{v}\rangle_I = \frac{\mathrm{d}E}{\mathrm{d}t} = m\gamma^3\langle \mathbf{v}, \mathbf{a}\rangle_I, \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\gamma\mathbf{a} + m\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v}, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\gamma^3}{c^2}\langle \mathbf{v}, \mathbf{a}\rangle_I \tag{48}$$

$$\mathbf{F} = \left(\frac{\gamma}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, \gamma\mathbf{f}\right) = \left(\frac{m\gamma^4}{c}\langle \mathbf{v}, \mathbf{a}\rangle_I, m\gamma^2\mathbf{a} + m\frac{\gamma^4}{c^2}\langle \mathbf{v}, \mathbf{a}\rangle_I\mathbf{v}\right) \tag{49}$$

Proposition 8.34. *Let p be a particle of mass m moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Then,*

1. *If $\mathbf{f} \parallel \mathbf{v}$, then*

$$f = m\gamma^3 a, \qquad m\gamma^3 := \textit{longitudinal mass}. \tag{50}$$

2. *If $\mathbf{f} \perp \mathbf{v}$, then*

$$f = m\gamma a, \qquad m\gamma := \textit{transverse mass}. \tag{51}$$

Theorem 8.35. *Let p be a particle of mass m with $v_0 = x_0 = t_0 = 0$ on which a constant force F acts. If we denote $a_0 = \gamma^3 a = F/m$ (which is constant), then*

$$x(t) = \frac{c^2}{a_0}\left[\sqrt{1 + \frac{a_0^2 t^2}{c^2}} - 1\right], \qquad v(t) = \frac{a_0 t}{\sqrt{1 + a_0^2 t^2/c^2}}, \tag{52}$$

and in the limit cases,

$$t \to \infty : v(t) \approx c, x(t) \approx ct - \frac{c^2}{a_0}, \tag{53}$$

$$a_0 t \ll c : v(t) \approx a_0 t, x(t) \approx \frac{a_0}{2}t^2. \tag{54}$$

Theorem 8.36. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$ and $x_0 = -k/mc^2, t_0 = 0$, then*

$$E = \gamma mc^2 + \frac{k}{x} = 0, \qquad \omega^2 x^2 + v^2 = c^2, \tag{55}$$

$$\omega = \frac{c}{x_0}, \qquad T = \frac{\pi}{2}\frac{x_0}{c}, \tag{56}$$

$$x(t) = x_0\cos\omega t, \qquad v(t) = -c\sin\omega t, \tag{57}$$

while n classical formalism the period would be $\frac{\pi}{2\sqrt{2}}\frac{x_0}{c} < T_{\text{rel}}$, which is normal since in the classical framework the mass can have an infinite velocity while in the relativistic one is bounded by c.

Theorem 8.37. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$ and $x_0 = -k/mc^2, t_0 = 0$, then with respect to the particle reference,*

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\gamma} = \cos\omega t = \frac{x}{x_0}, \qquad \tau = \int \gamma\,\mathrm{d}t = \frac{1}{\omega}\sin\omega t, \tag{58}$$

$$x^2 = x_0^2 - c^2\tau^2, \qquad T' = \frac{x_0}{c}, \qquad \frac{L'}{x_0} = \frac{T}{T'}. \tag{59}$$

Theorem 8.38. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$, then*

$$\mathbf{F} \perp \mathbf{v}, \qquad \gamma m\beta^2 c^2 = -\frac{k}{r},$$

$$-1 < \frac{T}{U} = -\frac{\gamma}{\gamma+1} < -\frac{1}{2}, \qquad E = \frac{mc^2}{\gamma}.$$

Theorem 8.39. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k > 0$, then it is not possible falling to the origin, and if $k < 0$, then it is possible if $Lc \leq k$ (in this case $p_r \to \infty$).*

Theorem 8.40. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k > 0$, then it is always possible to escape to the infinity is always possible, and if $k < 0$, it is possible if $E > mc^2$.*

Theorem 8.41. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. Then,*

$$L = \gamma m r^2\dot\theta = \text{ctt}, \qquad E = \gamma mc^2 + \frac{k}{r} = \text{ctt}, \qquad \mathbf{p} = \gamma m(\dot r\mathbf{e}_r + r\dot\theta\mathbf{e} \tag{60}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\dot r) - \frac{L^2}{\gamma m r^3} = \frac{k}{r^2}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\gamma m k}{L^2}, \tag{61}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + (1-\alpha^2)\frac{1}{r} = -\frac{kE}{L^2c^2}, \ \alpha^2 = \frac{k^2}{L^2c^2}. \tag{62}$$

Proposition 8.42. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If the variation of r is negligible, then $\alpha^2 < 1$.*

Proposition 8.43. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If $\alpha^2 <$, $E < mc^2$, and $E > 0$, then the trajectory of p is bounded.*

Theorem 8.44. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If $\alpha^2 < 1$, $E < mc^2$, and $E > mc^2\sqrt{1-\alpha^2}$, then the trajectory of p is determined by the expression*

$$r = \frac{a(1-e^2)}{1 + e\cos\left(\sqrt{1-\alpha^2}\theta\right)}, \tag{63}$$

$$\frac{1}{a} = \frac{E}{k}\left[1 - \frac{m^2c^4}{E^2}\right], \ e = \frac{1}{\alpha}\sqrt{1 + (\alpha^2-1)\frac{m^2c^4}{E^2}}, \tag{64}$$

which is an ellipse with a precession $2\pi(1/\sqrt{1-\alpha^2} - 1)$ per revolution (and $\pi\alpha^2$ if $\alpha^2 \ll 1$).

Theorem 8.45. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If p has a closed bounded trajectory, then the average of $\frac{\mathrm{d}\langle \mathbf{r},\mathbf{p}\rangle_I}{\mathrm{d}t} = 0$ on an interval of nT.*

Proposition 8.46. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If p has a closed bounded trajectory, then*

$$E = \left\langle \frac{1}{\gamma} \right\rangle mc^2. \tag{65}$$

9 Generalized coordinates

Definition 9.1. Let S be a system of particles $p_1, \ldots, p_n$ with masses $m_1, \cdots, m_n$. Then, we say the system has *non stationary holonomic constraints* or *rheonomic constraints* if and only if there is a function $\mathbf{f} : \mathbb{R}^{3n} \times \mathbb{R} \longrightarrow \mathbb{R}^k$ such that

$$\mathbf{f}(\mathbf{r}_1, \ldots, \mathbf{r}_n, t) = \mathbf{0}. \tag{66}$$

Definition 9.2. Let S be a system of particles $p_1, \ldots, p_n$ with masses $m_1, \cdots, m_n$. Then, we say the system has *stationary holonomic constraints* or *scleronomic constraints* if and only if there is a function $\mathbf{f} : \mathbb{R}^{3n} \longrightarrow \mathbb{R}^k$ such that

$$\mathbf{f}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \mathbf{0}. \tag{67}$$

Definition 9.3. Let S be a system of n particles. We say the system is *free* if and only if there is no constraints.

Definition 9.4. Let S be a system of n particles and an holonomic constraint $\mathbf{f} = \mathbf{0}$, where $\mathbf{f}$ is a differentiable function. Then, we define the *system positions manifold* as the manifold M determined by the constraint.

Definition 9.5. Let M be the system positions manifold of dimension r of a system S. Then, we define the *generalized coordinates* $q_1, \ldots, q_r$ as a set of independent variables such that $\mathbf{x}(q_1, \ldots, q_r) \in M$.

Definition 9.6. Let S be a system of n particles and M be the system positions manifold of dimension r. We define the *set of possible velocities at the instant t* as the set V that contains $\dot{\mathbf{x}}$, where $\mathbf{x} \in M$.

Proposition 9.1. *Let S be a system of n particles with a constraint $\mathbf{f} : \mathbb{R}^{3n} \times \mathbb{R} \longrightarrow \mathbb{R}^k$ and V the set of possible velocities at an instant t. If $\dot{\mathbf{x}} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, then*

$$\sum_{i=1}^{n} \langle \boldsymbol{\nabla}_i f_j, \mathbf{v}_i \rangle_I + \frac{\partial f_j}{\partial t} = 0, \; j = 1, \ldots, k. \tag{68}$$

Definition 9.7. Let S be a system of n particles. Then, we define the *constraint for $\boldsymbol{\phi}_i$ over the particle p_i* as

$$\boldsymbol{\phi}_i = m_i \mathbf{a}_i - \mathbf{F}_i. \tag{69}$$

Theorem 9.2 (D'Alembert's principle)**.** *Let S be a system of particles. Then,*

$$\boxed{\sum_{i=1}^{n} \langle \mathbf{F}_i - m\mathbf{a}_i, \delta\mathbf{r}_i \rangle_I = 0, \; \forall\, \delta\mathbf{r}_i \,.} \tag{70}$$

Theorem 9.3. *Let S be a system of n particles with generalized coordinates $q^1, \ldots, q^r$. Then,*

$$\boxed{\sum_{i=1}^{n} \left\langle \mathbf{F}_i, \frac{\partial \mathbf{p}_i}{\partial q^j} \right\rangle_I = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}^j} - \frac{\partial T}{\partial q^j}, \; j = 1, \ldots, r.} \tag{71}$$

And if $\mathbf{F}$ is derived from a potential $\Phi(\mathbf{r})$, then

$$\boxed{\frac{\partial L}{\partial q^j} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^j} = 0, \; j = 1, \ldots, r.} \tag{72}$$

Theorem 9.4. *Let $J : C^2[x_0, x_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')\,\mathrm{d}x\,,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_1 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are given real numbers. If $y \in S$ is an extremal for J, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{73}$$

for all $x \in [a_0, x_1]$.

Theorem 9.5 (Lagrange multipliers method for non-holonomic constraints)**.** *If we want to find an extrema having a set of m non-holonomic constraints*

$$\begin{aligned}
\overline{\delta f_1} &= A_{11}\delta u_1 + \cdots + A_{1n}\delta u_n &&= 0, \\
\vdots &= \ddots &&= \vdots \\
\overline{\delta f_m} &= A_{m1}\delta u_1 + \cdots + A_{mn}\delta u_n &&= 0,
\end{aligned} \tag{74}$$

Then we can proceed as the original multipliers method by considering every variation as independent and solving the following equation.

$$\delta F + \lambda_1 \overline{\delta f_1} + \cdots + \lambda_m \overline{\delta f_m} = 0 \tag{75}$$

Theorem 9.6. *Let f be a continuous functions with a variation $\delta f = \epsilon\phi$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta y = \delta \frac{\mathrm{d}}{\mathrm{d}x} y, \qquad \delta \int_a^b f(x)\,\mathrm{d}x = \int_a^b \delta f(x)\,\mathrm{d}x\,. \tag{76}$$

Theorem 9.7. *Let $J : \mathbf{C}^2[t_0, t_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$J(\mathbf{q}) = \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}, t)\,\mathrm{d}t\,, \tag{77}$$

where $\mathbf{q} = (q_1, \ldots, q_n)$, and L has continuous second-order partial derivatives with respect to t, q_k, and $\dot{q}_k$, $k = 1, \ldots, n$. Let

$$S = \left\{\mathbf{q} \in \mathbf{C}^2[t_0, t_1] \,\middle|\, \mathbf{q}(t_0) = \mathbf{q}_0, \mathbf{q}(t_1) = \mathbf{q}_1\right\}, \tag{78}$$

where $\mathbf{q}_0, \mathbf{q}_1 \in \mathbb{R}^n$ are given vectors. If $\mathbf{q}$ is an extremal for J in S then for $k = 1, \ldots, n$

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_k} = 0.} \tag{79}$$

Theorem 9.8. *If we have a set of holonomic constraints*

$$\begin{aligned} f_1(q_1, \ldots, q_n, t) &= 0, \\ \ldots &= \vdots, \\ f_m(q_1, \ldots, q_n, t) &= 0, \end{aligned} \tag{80}$$

then we can treat each variable as independent and search the stationary value of

$$J' = \int_{t_1}^{t_2} L + \sum_{k=1}^{m} \lambda_m f_m \, \mathrm{d}t \,, \tag{81}$$

which leads to the equation

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_k} + \lambda_1 \frac{\partial f_1}{\partial q_k} + \cdots + \lambda_m \frac{\partial f_m}{\partial q_k} = 0.} \tag{82}$$