# 1 Arithmetic and topology

**Definition 1.1.** Let $\mathbb{R}^2 = \{(x.y) \mid x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the sum "+" by components
$$(a, b) + (c, d) \coloneqq (a + b, c + d). \tag{1}$$
- Product: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the product by
$$(a, b)(c, d) = (ac - bd, ad + bc). \tag{2}$$

We define the set $\mathbb{C}$ as $(\mathbb{R}^2, +, )$.

**Proposition 1.1.** *The set $\mathbb{C}$ of complex numbers is an abelian field.*

**Proposition 1.2.** *Let $\mathbb{C}$ be defined in the second way. Then,*

1. *$\mathbb{C}$ is an abelian ring.*
2. *If we define $f$ as*
$$\begin{aligned} f : (\mathbb{C}, +, ) &\longrightarrow (\mathbb{R}^2, +, ) \\ (x, y) &\longmapsto x + yi \end{aligned}, \tag{3}$$
*then $f$ is a morphism of rings.*
3. *The function $f$ is, in fact, an isomorphism and $\mathbb{C}$ is an abelian field.*

**Proposition 1.3.** *The subset of $\mathbb{C}$ generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.*

**Theorem 1.4.** *$\mathbb{C}$ is not an ordered field.*

**Definition 1.2.** Let $z = a + bi \in \mathbb{C}$. We define the *conjugate of $z$* as
$$\bar{z} \coloneqq a - bi. \tag{4}$$

**Proposition 1.5.** *For all $z, w \in \mathbb{C}$, we have:*

1. *$\bar{\bar{z}} = z$.*
2. *$\overline{z + w} = \bar{z} + \bar{w}$.*
3. *$\overline{zw} = \bar{z}\bar{w}$.*
4. *$z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.*
5. *$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.*
6. *The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.*

**Definition 1.3.** Let $z = a + bi \in \mathbb{C}$. We define the *real part of $z$ and imaginary part of $z$* respectively as
$$\operatorname{Re}\{z\} \coloneqq a, \qquad \operatorname{Im}\{z\} \coloneqq b. \tag{5}$$

**Proposition 1.6.** *Let $z \in \mathbb{C}$. Then,*
$$\operatorname{Re}\{z\} = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im}\{z\} = \frac{z - \bar{z}}{2i} \tag{6}$$

**Proposition 1.7.** *Let $z, w \in \mathbb{C}$ and the following distance function.*
$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto \tilde{d}(z, w) \coloneqq |z - w| \end{aligned} \tag{7}$$
*Then, $(\mathbb{C}, \tilde{d})$ is a metric space.*

**Definition 1.4.** Let $z = a + bi \in \mathbb{C}$. We define the *modulus of $z$* as
$$|z| \coloneqq \tilde{d}(z, 0), \tag{8}$$
which is equivalent to $\sqrt{z\bar{z}}$.

**Definition 1.5.** Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define an *open disc of radius $r$ and center $z_0$* as follows
$$B_r(z_0) \coloneqq \{z \in \mathbb{C} \mid |z - z_0| < r\}. \tag{9}$$

**Definition 1.6.** Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *punctured disc of radius $r$ and center $z_0$* as follows
$$B_r^*(z_0) \coloneqq \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}. \tag{10}$$

**Definition 1.7.** Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a *closed disc of radius $r$ and center $z_0$* as follows
$$\overline{B_r(z_0)} \coloneqq \{z \in \mathbb{C} \mid |z - z_0| \leq r\}. \tag{11}$$

**Definition 1.8.** We denote by $\mathbb{D}$ the unitary disc of center 0 and radius 1. Besides, we denote by $\mathbb{T} \subseteq \mathbb{C}$ the unitary circumference, that is,
$$\mathbb{T} \coloneqq \{z \in \mathbb{C} \mid |z| = 1\}. \tag{12}$$
We also denote it by $\mathbb{S}^1$.

**Lemma 1.8.** *The set $B = \left\{B_r(z_0) \,\middle|\, r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\right\}$ is a basis of the topology of $\mathbb{R}^2$ as a metric space. The set $D = \{D_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of $\mathbb{C}$ as a metric space.*

**Proposition 1.9.** *The sets $\mathbb{C}$ and $\mathbb{R}^2$ with the topology of metric space are homeomorphs.*

**Corollary 1.10.** *There is a bijection between $B$ and $D$, that is, between balls of $\mathbb{R}^2$ and discs of $\mathbb{C}$.*

**Proposition 1.11.** *Let $z, w \in \mathbb{C}$. Then,*

1. *$|z| \geq 0$.*
2. *$|z| = 0 \Leftrightarrow z = 0$.*
3. *$-|z| \leq \operatorname{Re}\{z\} \leq |z|$ and $-|z| \leq \operatorname{Im}\{z\} \leq |z|$.*
4. *$|zw| = |z||w|$.*
5. *If $w \neq 0$, $|z/w| = |z|/|w|$.*
6. *$|z + w| \leq |z| + |w|$.*
7. *$|z + w| \geq ||z| - |w||$.*
8. *$|\operatorname{Re}\{zw\}| \leq |z||w|$ and $|\operatorname{Im}\{z\}| \leq |z||w|$.*
9. *$|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$.*

10. $|z^n| = |z|^n$

**Corollary 1.12.** *Let $z_1, \dots, z_n \in \mathbb{C}$. Then,*

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|, \qquad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \qquad |\operatorname{Re}\{z_1 \cdots z_n\}| \leq |z_1| \cdots |z_n|. \tag{13}$$

**Definition 1.9.** Let $z \in \mathbb{C}^*$. We define the *argument of $z$*, denoted by $\arg z$, as the real number $\theta$ such that $z = |z|(\cos\theta + i\sin\theta)$. Let us observe that $\arg z$ is not a function but a multivalued application. We define the *principal argument of $z$* as

$$\operatorname{Arg} z \coloneqq \theta_0 \in [0, 2\pi) \mid z = |z|(\cos\theta + i\sin\theta). \tag{14}$$

In general, to make $\theta$ to be unique, it is enough to impose it to belong to a certain semiopen interval of length $2\pi$. Choosing the interval $I$ is called by *taking a determination of the argument*.

**Definition 1.10.** Given a complex number $z$ that we can express by $z = |z|(\cos\theta + i\sin\theta)$ for some $\theta \in \mathbb{R}$, we use the notation $r = |z|$ to write

$$z = r_\theta^z = r(\cos\theta + i\sin\theta) \tag{15}$$

or simply $r_\theta$ when it is obvious which complex number are we referring to. We call it *polar form of $z$*.

**Proposition 1.13.** *Let $z \in \mathbb{C}$ and $r_\theta$ its polar form. Then,*

$$z^n = (r^n)_{n\theta}. \tag{16}$$

**Corollary 1.14** (De Moivre's Formula)**.** *Let $\theta \in \mathbb{R}$. Then,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{17}$$

**Proposition 1.15.** *Let $z, w \in \mathbb{C}$. Then,*

1. $\arg zw = \arg z + \arg w + 2\pi k$.
2. $\arg z^n = n \arg z + 2\pi k$.

**Definition 1.11.** We denote the complex numbers $z$ generated by moving the point $z_0 = 1$ around $\mathbb{T}$ a length $t$ in a counter-clockwise direction by $1_t$. In other words, $1_t$ are the complex numbers $z = \cos t + i\sin t$.

**Proposition 1.16.** *Let $f : t \longrightarrow 1_t$. Then, $f$ is a morphism from $(\mathbb{R}, +)$ to $(\mathbb{T}, \cdot)$, with $\ker f = 2\pi\mathbb{Z}$.*

**Definition 1.12.** Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $w \in \mathbb{C}$ is an *$n$-th root of $z$* if and only if

$$w^n = z. \tag{18}$$

**Theorem 1.17.** *Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \dots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \dots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \dots, n\}$.*

**Theorem 1.18.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a continuous curve such that $\gamma(t) \neq 0 \forall t \in [a, b]$. Then, there exists a continuous determination $\phi$ of the argument of $\gamma$. Then, $\phi(t) + 2\pi k$ with $k \in \mathbb{Z}$ is the general expression of all the argument determinations of $\gamma$. If $\gamma$ is differentiable, then $\phi$ is differentiable and $\phi' = \operatorname{Im}\{\gamma'/\gamma\}$.*

**Definition 1.13.** Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a regular curve. We define the *variation of the argument along* $\gamma$ as

$$\Delta_\gamma \arg \coloneqq \operatorname{Im}\left\{\int_a^b \frac{\gamma'(t)}{\gamma(t)}\,\mathrm{d}t\right\}. \tag{19}$$

**Definition 1.14.** Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve such that $\gamma(t) \neq 0 \forall t \in [a, b]$. Then, we define the *index of $\gamma$ with respect to the origin* or *the number of revolutions of $\gamma$ around the origin*

$$\operatorname{Ind}(\gamma, 0) \coloneqq \frac{1}{2\pi} \Delta_\gamma \arg. \tag{20}$$

**Proposition 1.19.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piece-wise regular curve. Then,*

$$\operatorname{Ind}(\gamma, 0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\,\mathrm{d}t \tag{21}$$

**Definition 1.15.** Let $\gamma$ be a closed curve and $z \notin \Gamma$. We define the *index of $\gamma$ with respect to $z$* as

$$\operatorname{Ind}(\gamma, z) \coloneqq \operatorname{Ind}(\gamma - z, 0). \tag{22}$$

**Proposition 1.20.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve piece-wise of class $C^1([a, b])$. Then,*

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z}\,\mathrm{d}t\,. \tag{23}$$

**Proposition 1.21.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piece-wise of class $C^1([a, b])$. Then, $\operatorname{Ind}(-\gamma, z) = -\operatorname{Ind}(\gamma, z)$.*

# 2 Sequences and limits

**Definition 2.1.** A *sequence of complex numbers* is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned}. \tag{24}$$

We denote it by $\{z_n\}_{n=m}^{\infty}$

**Definition 2.2.** Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *the sequence has limit $L$* or *it converges to the limit $L$* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \tag{25}$$

We denote it by

$$\lim_{h \to \infty} z_n = L, \qquad \lim\,\{z_n\}_{n=0}^{\infty} = L, \qquad \{z_n\}_{n=0}^{\infty} \to L. \tag{26}$$

**Theorem 2.1.** *Let $z_n = z_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$ and $L = L_x + iL_y \in \mathbb{C}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \to L \Leftrightarrow \{x_n\}_{n=0}^{\infty} \to L_x \wedge \{y_n\}_{n=0}^{\infty} \to L_z. \tag{27}$$

**Definition 2.3.** Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say *it tends to infinity* and denote it by $\lim z_n = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \quad (28)$$

**Definition 2.4.** Let $\{z_n\}_{n=0}^{\infty}$ be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \quad (29)$$

**Theorem 2.2.** *Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.*

**Theorem 2.3.** *Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence } \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \text{ are Cauchy sequences.} \quad (30)$$

**Theorem 2.4.** *The field $\mathbb{C}$ of complex numbers is complete.*

**Definition 2.5.** The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, together with two charts.

# 3 Functions

**Definition 3.1.** A *topology* is an ordered pair $(\mathbb{X}, \tau)$, where $X$ is a set and $\tau$ a collection of subsets of $X$ satisfying the following properties:

1. The empty set and $X$ belong to $\tau$.

2. Any arbitrary (finite or infinite) union of members of $\tau$ still belongs to $\tau$.

3. The intersection of any finite number of members of $\tau$ still belongs to $\tau$.

The elements of $\tau$ are called *open sets* and the collection $\tau$ is called the *topology on $X$*.

**Definition 3.2.** Let $(\mathbb{X}, d)$ be a metric space. A *topology on the metric space by the metric $d$* is the set $\tau$ of all open sets of $M$.

**Definition 3.3.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *interior point of $A$* if there is a ball $B_{(\mathbb{M},d)}(a, r) \subset A$.

**Definition 3.4.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *exterior point of $A$* if there is a ball such that $B_{(\mathbb{M},d)}(a, r) \cup A = \emptyset$.

**Definition 3.5.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is a *boundary point of $A$* if it is not interior or exterior or, which is equivalent, if every ball $B_{(\mathbb{M},d)}(a, r)$ contains elements of $A$ and $A^c$.

**Definition 3.6.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *accumulation point of $A$* if every ball with center $a$ contains points of $A$ different to $a$. In other words, every punctured ball satisfies $B^*_{(\mathbb{M},d)}(a, r) \cup A \neq \emptyset$.

**Definition 3.7.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the interior of $A$* as the set of all interior points of $A$, and we denote it by $\text{int}(A)$.

**Definition 3.8.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the exterior of $A$* as the set of all exterior points of $A$, and we denote it by $\text{ext}(A)$.

**Definition 3.9.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the boundary of $A$* as the set of all boundary points of $A$, and we denote it by $\partial A$.

**Definition 3.10.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the closure of $A$* as the set of all accumulation points of $A$, and we denote it by $\bar{A}$.

**Definition 3.11.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *an open set* if it contains none of its boundary points, that is, if $\partial A \cap A = \emptyset$.

**Definition 3.12.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a closed set* if it contains all its boundary points, that is, if $\partial A \subseteq A$.

**Definition 3.13.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a bounded set* if there exist a point $a \in \mathbb{M}$ and a positive real number $r$ such that the ball $B_{(\mathbb{M},d)}(a, r)$ contains $A$.

**Definition 3.14.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a compact set* if it a bounded and closed set.

**Proposition 3.1.** *Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. Then, $A$ is open if and only if $A^c$ is closed.*

**Definition 3.15.** Let $\Omega \subseteq \mathbb{C}$ be a set. We say $\Omega$ *is connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally, $\Omega$ is connected if there are not two open sets $U, V \subseteq \mathbb{C}$ such that

$$U_1 = U \cap \Omega, \qquad V_1 = V \cap \Omega, \qquad U_1 \cap V_1 = \varnothing, \qquad U_1 \cup V_1 = \Omega. \quad (31)$$

Otherwise, we say $\Omega$ *is disconnected.*

**Definition 3.16.** Let $\Omega \subseteq \mathbb{C}$ be a set. We say $\Omega$ *is simply connected* if and only if every circuit is homotopic in $\Omega$ to a point in $\Omega$. Equivalently, is simply connected if and only if every pair of curves with the same extremes are homotopic.

**Definition 3.17.** Let $\Omega \subseteq \mathbb{C}$ be a set. We say $\Omega$ *is convex* if and only if for all pair of point $a, b \in \Omega$, the segment defined by

$$[a, b] = \{z \mid z = (1-t)a + tb, 0 \leq t \leq 1\} \quad (32)$$

is contained in $\Omega$, that is, if every pair of points can be connected by a straight line that belongs to the set.

**Definition 3.18.** Let $\Omega \in \mathbb{C}$ be a set. We say $\Omega$ *is a star-convex set* if and only if there exists $z_0 \in \mathbb{C}$ such that for all $z \in \Omega$ the segment $[z_0, z]$ is contained by $\Omega$.

**Definition 3.19.** Let $(\mathbb{M}, d)$ be a metric space and $S \subseteq \mathbb{M}$ a set. We say $S$ *is path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

**Definition 3.20.** Let $\Omega \in \mathbb{C}$ be a set. We say $\Omega$ *is a region or domain* if and only if it is open, non-empty, and connected.

**Definition 3.21.** Let $\Omega \subseteq \mathbb{C}$ be a non-empty set. We say $\Omega_1 \subseteq \Omega$ *is a connected component of* $\Omega$ if and only if it is a maximal connected subset, that is, if $z_0 \in \Omega_1$ and $W$ is a connected subset of $\mathbb{C}$ that contains $z_0$, then $W \subseteq \Omega_1$.

**Definition 3.22.** Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function* $f$ as the application

$$\begin{aligned} f : D \subseteq \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto w = f(z) \end{aligned}. \tag{33}$$

**Definition 3.23.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say it *tends to infinity at the point* $z_0$ and denote it by $\lim_{z\to z_0} f(z) = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \tag{34}$$

**Definition 3.24.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We write $\lim_{z\to\infty} f(z) = L$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \tag{35}$$

**Proposition 3.2.** *Let $f_1, f_2 : \Omega \longrightarrow \mathbb{C}$ be two functions and $z_0$ a point such that $\lim_{z\to z_0} f_1 = w_1, \lim_{z\to z_0} f_2 = w_2$. Then,*

1. *$f_1 + f_2$ has also a limit and $\lim_{z\to z_0} f + g = w_1 + w_2$.*
2. *$f_1 f_2$ has also a limit and $\lim_{z\to z_0} fg = w_1 w_2$.*
3. *If $w_2 \neq 0$, then $f/g$ has also a limit and $\lim_{z\to z_0} f/g = w_1/w_2$.*
4. *If $h(z)$ is a continuous function defined on a neighborhood of $w_1$, then $\lim_{z\to z_0} h(f_1(z)) = h(w_1)$.*

**Definition 3.25.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. We say $f$ *is continuous in* $z_0$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \tag{36}$$

**Proposition 3.3.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$ is continuous at $z_0$ if and only if $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are continuous at $z_0$.*

**Proposition 3.4.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f$ is continuous at $z_0$ if and only if for all sequence $\{z_n\}_{n=1}^{\infty}$ of $\Omega$ convergent at $z_0$ it is true that the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$.*

**Proposition 3.5.** *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, $\lambda f$, $f + g$, and $fg$ are continuous at $z_0$. The function $f/g$ is continuous at $z_0$ if $g(z_0) \neq 0$.*

**Definition 3.26.** For all $z \in \mathbb{C}$, we define the *complex exponential function* as the following series

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{37}$$

**Proposition 3.6.** *The radius of convergence of $\mathrm{e}^z$ is infinite.*

**Proposition 3.7.** $\mathrm{e}^z = \mathrm{e}^x$ *for all $x \in \mathbb{R}$.*

**Proposition 3.8.** $\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$ *for all $z, w \in \mathbb{C}$.*

**Proposition 3.9.** *For all $z \in \mathbb{C}$ we have $\mathrm{e}^z \neq 0$.*

**Proposition 3.10.** *The image of $\mathrm{e}^z$ is $\mathbb{C}^*$.*

**Proposition 3.11.** *The derivative of $\mathrm{e}^z$ is $\mathrm{e}^z$.*

**Proposition 3.12.** $\overline{\mathrm{e}^z} = \mathrm{e}^{\bar{z}}$.

**Proposition 3.13.** $|\mathrm{e}^z| = \mathrm{e}^{\operatorname{Re}\{z\}}$.

**Proposition 3.14** (Euler's Formula)**.** *If $\theta \in \mathbb{R}$, then $\mathrm{e}^{xi}$ has modulus one and we have that*

$$\boxed{\mathrm{e}^{xi} = \cos x + i \sin x.} \tag{38}$$

**Corollary 3.15.** *Let $z \in \mathbb{C}^*$. Then,*

$$z = |z| \mathrm{e}^{i\theta}, \tag{39}$$

*with $\theta \in [0, 2\pi)$.*

**Proposition 3.16.** *The following function*

$$\begin{aligned} \exp : (\mathbb{R}, +) &\longrightarrow (\mathbb{C}^*, \cdot) \\ x &\longmapsto \mathrm{e}^{xi} \end{aligned} \tag{40}$$

*is a morphism of groups and its image is $\mathbb{T}$.*

**Proposition 3.17.** *The complex exponential function is a periodic function with period $2\pi i$.*

**Proposition 3.18.** *Let $a \in \mathbb{C}^*$. Then, $\mathrm{e}^z = a$ has infinite solutions.*

**Proposition 3.19.** *The equation $\mathrm{e}^z = 0$ does not have solutions.*

**Proposition 3.20.** *Let $y_0 \in \mathbb{C}$ be a numbers, $B := \{z \in \mathbb{C} \mid y_0 < \operatorname{Im}\{z\} < y_0 + 2\pi\}$ a set, and $f : B \longrightarrow \mathbb{C}^*$ be the exponential function. Then, $f$ is bijective in $B$* **?***.*

**Proposition 3.21.** *Let $x_0, y_0, m \in \mathbb{C}$ be two numbers with $m \neq 0$ and $f$ the exponential function* **?***. Then,*

1. *$f$ transforms the line $y = y_0$ to a line that starts at $z = 0$ and continues with an argument $y_0$ from the real positive axis.*
2. *$f$ transforms the line $x = x_0$ to a circle centered at the origin and radius $r = \mathrm{e}^{x_0}$.*

*3. $f$ transforms the line $y = mx$ to the parametric curve $z = \mathrm{e}^x \mathrm{e}^{imx}$ (a spiral).*

**Definition 3.27.** Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric functions as*

$$\cos z \coloneqq \frac{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}{2}, \tag{41}$$
$$\sin z \coloneqq \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{2}, \tag{42}$$
$$\tan z \coloneqq \frac{\mathrm{e}^{zi} - \mathrm{e}^{-zi}}{\mathrm{e}^{zi} + \mathrm{e}^{-zi}}. \tag{43}$$

**Proposition 3.22.** *For all $z \in \mathbb{C}$,*

$$\sin^2 z + \cos^2 z = 1. \tag{44}$$

**Proposition 3.23.** *For all $z \in \mathbb{C}$,*

$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z). \tag{45}$$

**Proposition 3.24.** *For all $z, w \in \mathbb{C}$,*

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \qquad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \tag{46}$$

**Proposition 3.25.** *The functions $\cos z, \sin z$ have period of $2\pi$.*

**Proposition 3.26.** *Let $z_0 \in \mathbb{C}$. Then, $z_0$ is root of $\sin z$ ($\cos z$) if and only if it is a root of $\sin x$ ($\cos x$).*

**Definition 3.28.** Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic functions* as

$$\cosh z \coloneqq \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}, \tag{47}$$
$$\sinh z \coloneqq \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}, \tag{48}$$
$$\tanh z \coloneqq \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{\mathrm{e}^{z} + \mathrm{e}^{-z}}. \tag{49}$$

**Proposition 3.27.** *For all $z \in \mathbb{C}$,*

$$\cosh^2 z - \sinh^2 z = 1. \tag{50}$$

**Proposition 3.28.** *For all $z \in \mathbb{C}$,*

$$\cosh(-z) = \cosh(z), \qquad \sinh(-z) = -\sinh(z). \tag{51}$$

**Proposition 3.29.** *For all $z, w \in \mathbb{C}$,*

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \tag{52}$$
$$\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \tag{53}$$

**Proposition 3.30.** *For all $z \in \mathbb{C}$,*

$$\cosh z = \cos(iz), \cos z = \cosh(iz) \tag{54}$$
$$\sinh z = -i \sin(iz), \sin z = -i \sinh(iz) \tag{55}$$

**Proposition 3.31.** *For all $z = x + iy \in \mathbb{C}$,*

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \tag{56}$$
$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \tag{57}$$
$$\tan(x + iy) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i \frac{\sinh y}{\cos(2x) + \cosh(2y)}. \tag{58}$$

**Proposition 3.32.** *For all $z = x + iy \in \mathbb{C}$,*

$$\tanh(x + iy) = \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)} + i \frac{\sin(2y)}{\cosh(2x) + \cos(2y)}. \tag{59}$$

**Proposition 3.33.** *For all $z = x + iy$,*

$$|\cos z| = \sqrt{\cosh^2 y - \sin^2 x} = \sqrt{\cos^2 x + \sinh^2 y}, \tag{60}$$
$$|\sin z| = \sqrt{\sinh^2 x + \sin^2 y} = \sqrt{\cosh^2 x - \cos^2 y}. \tag{61}$$

**Corollary 3.34.** *For all $z = x + iy$,*

$$|\sinh y| \leq |\cos z| \leq \cosh y, \qquad |\sinh y| \leq |\sin z| \leq \cosh y. \tag{62}$$

**Proposition 3.35.** *The roots of the function $\sinh z$ are of the form $z_n = n\pi$ and those for the function $\cosh z$ are of the form $w_n = (2n + 1)\pi/2i$.*

**Definition 3.29.** Let $D \subseteq \mathbb{C}$ be a set. We define a *multivalued function from $D$ to $\mathbb{C}$* as a subset of $D \times \mathbb{C}$ such that for every $z \in D$ there exists a number $y \in \mathbb{C}$ such that $(z, w) \in f$.

**Definition 3.30.** For $z \in \mathbb{C}^*$, we call the *natural logarithm of $z$* every number $w$ such that $\mathrm{e}^w = z$, that is,

$$\ln z \coloneqq \{ w \in \mathbb{C} \mid \mathrm{e}^w = z \}. \tag{63}$$

**Proposition 3.36.** *Given $z \in \mathbb{C}$ we can define $\ln z$ from the natural logarithm of a real number as*

$$\ln z = \ln |z| + i \arg z = \ln |z| + i \mathrm{Arg} z + 2\pi k i. \tag{64}$$

**Definition 3.31.** We define the *principal natural logarithm of $z$* as the value defined by the principal argument of $z$, that is,

$$\mathrm{Log} z = \ln |z| + i \mathrm{Arg} z. \tag{65}$$

**Definition 3.32.** We define the determination $I$ (with $I$ being a semiopen interval) of the logarithm as

$$\log_I z \coloneqq \ln |z| + i \arg_I z. \tag{66}$$

**Definition 3.33.** Let $E \subseteq \mathbb{C}^*$ be a connected set. We define the *continuous determination of the logarithm in $E$* as the continuous function $g : E \longrightarrow \mathbb{C}$ such that $\mathrm{e}^{g(z)} = z$. More generally, if $f : E \longrightarrow \mathbb{C}$ is a function such that $f(z) \neq 0$ for all $z \in E$, then we define the *continuous determination of* $\ln f$ as a function $g : E \longrightarrow \mathbb{C}$ such that $\mathrm{e}^{g(z)} = f(z)$.

**Proposition 3.37.** *Let $z, w \in \mathbb{C}$ two numbers. Then,*

1. *$\ln(zw) = \ln z + \ln w + 2\pi k i, k \in \mathbb{Z}$.*
2. *If we want to stay in the principal argument,*

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \text{if } \mathrm{Arg} z + \mathrm{Arg} w < 2\pi \\ \ln z + \ln w - 2\pi i, & \text{if } \mathrm{Arg} z + \mathrm{Arg} w \geq 2\pi \end{cases}. \tag{67}$$

3. *SEARCH MORE PROPERTIES*

**Definition 3.34.** Let $z \in \mathbb{C}$ be a number. We define the *complex trigonometric inverse functions* as

$$\arcsin z := -i \ln\left(iz + \sqrt{1-z^2}\right), \tag{68}$$

$$\arccos z := -i \ln\left(z + \sqrt{z^2-1}\right), \tag{69}$$

$$\arctan z := -\frac{i}{2} \ln \frac{1+iz}{1-iz}. \tag{70}$$

**Definition 3.35.** Let $z \in \mathbb{C}$ be a number. We define the *complex hyperbolic inverse functions* as

$$\operatorname{arcsinh} z := \ln\left(z + \sqrt{1+z^2}\right), \tag{71}$$

$$\operatorname{arccosh} z := \ln\left(z + \sqrt{z^2-1}\right), \tag{72}$$

$$\operatorname{arctanh} z := \frac{1}{2} \ln \frac{1+z}{1-z}. \tag{73}$$

**Definition 3.36.** Let $z, a \in \mathbb{C}$ with $z \neq 0$. Then, we define the *complex power function* as

$$z^a := \mathrm{e}^{a \ln z}. \tag{74}$$

If $E \subseteq \mathbb{C}^*$ is a connected set and $f : E \longrightarrow \mathbb{C}$ a functions such that $f(z) \neq 0$ for all $z \in E$, and $w \in \mathbb{C}$ a number, we define a *continuous determination of* $f^w$ as a continuous function $g : E \longrightarrow \mathbb{C}$ such that $g(z) \in [f(z)]^w$.

**Proposition 3.38.** *If* $a = \alpha + \beta i$ *and* $z = r\mathrm{e}^{\theta i}$, *then*

$$z^a = \mathrm{e}^{\alpha \ln r - \beta(\theta + 2\pi k)} \mathrm{e}^{(\beta \ln r + \alpha(\theta + 2\pi k))i}, \tag{75}$$

$$|z^a| = \mathrm{e}^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \qquad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k) \tag{76}$$

**Proposition 3.39.** *Let* $a, z \in \mathbb{C}$ *be two numbers. Then,*

1. *If* $a = n \in \mathbb{Z}$, *the complex power is a function and*
$$z^n = r^n \mathrm{e}^{n\theta i}. \tag{77}$$

2. *If* $a = n/m \in \mathbb{Q}$, *there are* $n$ *values and*
$$z^a = \sqrt[m]{r^n} \mathrm{e}^{(\theta + 2\pi k)n/mi}. \tag{78}$$

3. *If* $a$ *is irrational, the norm is uniquely determined but the argument has infinite values.*

4. *IF* $a \in \mathbb{C} \backslash \mathbb{R}$, *the argument is uniquely determined and the norm has infinite values.*

**Proposition 3.40.** *Let* $z, w \in \mathbb{C}$. *Then,*

1. $\left(\mathrm{e}^b\right)^a = \mathrm{e}^{a(b + 2\pi k i)}$

**Definition 3.37.** A *Riemann surface* $X$ is a connected complex 1-manifold.

**Definition 3.38.** We define a *sheet* as each of the complex planes of the Riemann surface.

**Definition 3.39.** We define a *cut* as the line (not necessaryly straight) of union between sheets.

**Definition 3.40.** We define a *branch point* as a point where start or finish a cut.

# 4 Derivatives

**Definition 4.1.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. We define the *derivative of* $f$ *at* $z_0$ as

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{79}$$

in case the limit exists. If $f$ has derivative, we say $f$ *is derivable at* $z_0$.

**Definition 4.2.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say $f$ *is holomorphic at* $\Omega$ if and only if it is $\mathbb{C}$-derivable at every point of $\Omega$. In that case, it is defined the function $f' : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ that associates each point $z$ of $\Omega$ with $f'(z)$.

**Definition 4.3.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We define the *domain of holomphism* as the region where $f$ is derivable. We say $f$ *is entire* if and only if the domain of holomorphism is $\mathbb{C}$.

**Definition 4.4.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say $f$ *is holomorphic at* $z_0$ if and only if it is holomorphic at some neighborhood of $z_0$.

**Proposition 4.1.** *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a function and* $z_0 \in \mathbb{C}$ *a point. If* $f$ *is derivable at* $z_0$, *then it is continuous at* $z_0$.

**Theorem 4.2.** *Let* $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be two functions and* $z_0 \in \Omega$ *a point. Then, the following statements are true.*

1. *If* $f$ *is constant at* $\Omega$, *then* $f$ *is derivable at* $z_0$ *and* $f'(z_0) = 0$.

2. *If* $f(z) = z$ *in every point of* $\Omega$, *then* $f$ *is derivable at* $z_0$ *and* $f'(z_0) = 1$.

3. *If* $f, g$ *are derivable at* $z_0$ *and* $\alpha, \beta \in \mathbb{C}$, *then* $\alpha f + \beta g$ *are derivable at* $z_0$ *and* $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.

4. *If* $f, g$ *are derivable at* $z_0$, *then* $fg$ *is derivable at* $z_0$ *and*
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{80}$$

5. *If* $f, g$ *are derivable at* $z_0$ *and* $g(z_0) \neq 0$, *then* $f/g$ *is derivable at* $z_0$ *and*
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{81}$$

**Theorem 4.3.** *Let* $f : \Omega_1 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a derivable function at a point* $z_0 \in \mathbb{C}$ *and* $g : \Omega_2 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be another derivable function at a point* $f(z_0) \in \Omega_2$. *Then,* $g \circ f$ *is derivable at* $z_0$ *and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \tag{82}$$

**Definition 4.5.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ *is of class* $C^1(\Omega)$ *or simply* $f \in c^1(\Omega)$ if and only if, using $f = u + iv$ with $u = \mathrm{Re}\{f\}, v = \mathrm{Im}\{f\}$, the partial derivatives of $u$ and $v$ as a two variable real functions exist and are continuous. In other words, $f \in C^1(\Omega)$ if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \tag{83}$$

exist and are continuous.

**Theorem 4.4** (Cauchy-Riemann conditions)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. Then, $f$ is derivable at $z_0$ if and only if is differentiable at $z_0$ and $df(z_0)$ is $\mathbb{C}$-linear, that is,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{84}$$

*which are known as Cauchy-Riemann conditions.*

**Theorem 4.5.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ an interior point. If $u, v$ satisfy the Cauchy-Riemann equation and their partial derivatives are continuous, then $f$ is derivable.*

**Theorem 4.6.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function of class $C^1(\Omega)$ with $\Omega$ an open set, injective, and derivable at every point of $\Omega$ with non-zero derivative. Then,*

1. *$\Omega' = f(\Omega)$ is an open subset of $\mathbb{C}$.*
2. *The inverse function $f^{-1}$ exist, it is well defined and it is derivable at $\Omega'$.*
3. *If $z \in \Omega$ and $z' = f(z)$, then*

$$\left(f^{-1}\right)'(z') = \frac{1}{f'(z)}. \tag{85}$$

**Proposition 4.7.** *A determination of $\ln z$ with $z \in \mathbb{C}$ is continuous except in a semiline.*

**Theorem 4.8.** *Let $\Omega \subseteq \mathbb{C}$ be an open set and $\phi \in C(\Omega)$, such that $\mathrm{e}^{\phi(z)} = z$ for all $z \in \Omega$. Then, we have $\phi \in H(\Omega)$ and*

$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \tag{86}$$

**Proposition 4.9.** *A determination of $\ln z$ with $z \in \mathbb{C}$ is holomorphic except in a semiline.*

**Proposition 4.10.** *Let $I = [\theta, \theta + 2\pi)$ a determination of the logarithm, $\ln_I z$. Then, $\ln_I z$ is holomorphic except in the semiline $L_\theta = \left\{r\mathrm{e}^{\theta i} \in \mathbb{C} \,\middle|\, r \geq 0\right\}$.*

**Theorem 4.11.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function in $\Omega$. Then, $f$ is holomorphic in $\Omega$. In particular, given a point $z_0 \in \mathbb{C}$, $f'(z_0) = a_1$ where $a_1$ is the first coefficient of the power series that represents $f$ in a neighborhood of $z_0$.*

**Definition 4.6.** We define the operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{87}$$

that act over the functions such that the real and imaginary part $u, v$ have partial derivatives.

**Proposition 4.12.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function of class $C^1(\Omega)$. Then, for all $z_0 \in \Omega$*

$$f(z_0 + h) = f(z_0) + \left(\frac{\partial f}{\partial z}\right)_{z_0} h + \left(\frac{\partial f}{\partial \bar{z}}\right)_{z_0} \bar{h} + o(|h|^2). \tag{88}$$

**Corollary 4.13.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$. Then, $f$ is holomorphic in $\Omega$ if and only if*

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ at } \Omega. \tag{89}$$

**Definition 4.7.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$ such that $f = u + iv$ with $u = \mathrm{Re}\{f\}, v = \mathrm{Im}\{f\}$ and $z_0 \in \mathbb{C}$ a point. Then, we call $\left(\partial_{\bar{z}} f\right)_{z_0} = 0$ the *Cauchy-Riemann condition*, which is equivalent to

$$\left(\frac{\partial u}{\partial x}\right)_{z_0} = \left(\frac{\partial v}{\partial y}\right)_{z_0}, \qquad \left(\frac{\partial v}{\partial x}\right)_{z_0} = -\left(\frac{\partial u}{\partial y}\right)_{z_0}, \tag{90}$$

which are called the *Cauchy-Riemann equations.*

**Theorem 4.14.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0$ an interior point. Then, at $z_0$*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}. \tag{91}$$

# 5 Series

**Definition 5.1.** We say $\displaystyle\sum_{n=1}^{\infty} z_n$ converges if and only if $S_n := \displaystyle\sum_{n=1}^{N} z_n$ has limit at $n \to \infty$.

**Proposition 5.1.** *$\displaystyle\sum_{n=1}^{\infty} z_n$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converge.*

**Definition 5.2.** We say $\displaystyle\sum_{n=1}^{\infty} z_n$ converges absolutely if and only if $\displaystyle\sum_{n=1}^{\infty} |z_n|$ converges.

**Proposition 5.2.** *$\displaystyle\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ and $\displaystyle\sum_{n=1}^{\infty} |b_n|$ converge.*

**Proposition 5.3.**

1. *A series converges absolutely with sum $S$ if and only if every rearrangement is convergent with the same sum $S$.*
2. *An absolutely convergent series can be summed by blocks in an arbitrary way.*

**Proposition 5.4.** *Let* $\sum_n a_n, \sum_n b$ *be two absolutely convergent series with sums* $A$ *and* $B$ *respectively. Then, the series* $\sum_k c_k$ *with* $c_k = \sum_{n=0}^{k} a_n b_{k-n}$ *is absolutely convergent with sum* $AB$.

**Theorem 5.5** (Weierstrass M-test)**.** *If* $|f_n(p)| < M_n$ *for all* $p \in X, n \geq 1$ *and* $\sum_{n=0}^{\infty} M_n < \infty$*, then the series* $\sum_{n=0}^{\infty} f_n(p)$ *is uniformly convergent on* $X$.

**Lemma 5.6** (Abel's summation formula)**.** *Let* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be two sequences of complex numbers and* $A_n = a_1 + \cdots + a_n$*. Then,*

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k). \tag{92}$$

**Theorem 5.7** (Dirichlet's criteria)**.** *Let* $\sum_{n=1}^{\infty} f_n(p) g_n(p)$ *be a series where* $f_n(p)$ *are complex and* $g_n(p)$ *are real for all* $p \in X, n \geq 1$*. If we denote* $F_n(p) = f_1(p) + \cdots f_n(p)$*, there exists a constant* $M$ *such that* $|F_n|(p) \leq M$ *for all* $n \geq, p \in X$*,* $g_n(p)$ *is monotonous decreasing and converges uniformly to zero on* $X$*, then the series* $\sum_{n=1}^{\infty} f_n(p) g_n(p)$ *is uniformly convergent on* $X$.

**Theorem 5.8** (Abel's criteria)**.** *Let* $\sum_{n=1}^{\infty} f_n(p) g_n(p)$ *be a series where* $f_n(p), g_n(p)$ *are complex. If* $\sum_{n=1}^{\infty} f_n(p)$ *is uniformly convergent on* $X$ *andd there exists a number* $M \in \mathbb{R}^+$ *such that for all* $p \in X$

$$|g_1(p)| + \sum_{n=1}^{\infty} |g_n(p) - g_{n+1}(p)| \leq M, \tag{93}$$

*then the series* $\sum_{n=1}^{\infty} f_n(p) g_n(p)$ *is uniformly convergent on* $X$.

**Definition 5.3.** We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n, z, z_0 \in \mathbb{C}. \tag{94}$$

We call the term $a_n$ the *n-th coefficient of the series.* In case $a_n = 0\ \forall n \leq m$, we will start the counting directly from $m$.

**Definition 5.4.** Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series. We define its *domain of convergence* as

$$E := \left\{ z \in \mathbb{C} \,\middle|\, \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges} \right\}. \tag{95}$$

**Theorem 5.9.** *Let* $\sum_n a_n (z - z_0)^n$ *be a power series and* $R = 1/\rho$*, where* $\rho = \limsup_n |a_n|^{1/n}$*. Then, the series converges uniformly on the compacts of the open disc* $D(z_0, R)$*, converges absolutely at every point* $z \in D$ *and diverges outside* $\bar{D}$*. Hence, the set of converges* $E$ *satisfies* $D \subseteq E \subseteq \bar{D}$ *and* $D = \text{int}E$.

**Definition 5.5.** Radius of convergence.

**Proposition 5.10.** *Let* $\sum_n a_n (z - z_0)^n$ *be a power series and* $R = \lim_{n \to \infty} |a_n|/|a_{n+1}|$*. If the limit exists, then* $R$ *is the radius of convergence.*

**Theorem 5.11** (Cauchy-Hadamard Theorem)**.** *Let*

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{96}$$

*be the following complex power series with* $a_n, z_0 \in \mathbb{C}$ *and* $R \in [0, +\infty) \cup \{+\infty\}$ *the radius of convergence. Then,*

1. *If* $|z - z_0| < R$ *then* $S$ *converges. In fact, for all* $r < R$ *we have* $S$ *converges uniformly at the disc* $\overline{D_r(z_0)}$.
2. *If* $|z - z_0| > R$ *then* $S$ *diverges.*
3. *The function* $f(z) = S(z)$ *is derivable at* $D_R(z_0)$ *and its formal derivative is*

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \tag{97}$$

*with the same radius of convergence.*

**Definition 5.6.** Let $\sum_n a_n (z - z_0)^n$ be a series, $S = E \cap C(z_0, R)$ non empty, and $m > 1$ a real number. We define

$$S_m := \{ z \in \mathbb{C} \mid |z - z_0| < R, d(z, S) \leq m(R - |z - a|) \}. \tag{98}$$

**Definition 5.7** (Stolz angle)**.** Let $S$ be formed by one point $w$. We define the *Stolz angle* as the angle generated by the $S_m$.

**Theorem 5.12** (Abel's theorem)**.** *Let* $\sum_n a_n (z - z_0)^n$ *be a series with* $S$ *non empty and such that the series converges uniformly on it. Then, the series converges uniformly on* $S_m$ *for all* $m > 1$*. In particular, the sum function is continuous on* $S_m$ *and one has*

$$\lim_{z \to w, z \in S_m} \sum_n a_n (z - z_0)^n = \sum_n a_n (w - z_0)^n, \qquad w \in S. \tag{99}$$

**Theorem 5.13.** *Let* $\sum_n a_n (z - z_0)^n$ *be a series with radius of convergence* $R$*. Then,* $f(z) = \sum_n a_n (z - z_0)^n$ *is holomorphic on* $D(a, R)$ *and it has a derivative*

$$f'(z) = \sum_n n a_n (z - z_0)^{n-1}, \ \forall z \in D. \tag{100}$$

**Proposition 5.14.** *Let $f : D(a, R) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. If there exists a power series $\sum_n a_n(z-z_0)^n$, convergent on $D$ such that*

$$f(z) = \sum_n a_n(z-z_0)^n,\ |z-z_0| < R, \tag{101}$$

*then the series is unique. In fact, $f$ is infinitely holomorphic and the coefficients $a_n$ are determined by $f$ with the relation*

$$a_n = \frac{f^n(z_0)}{n!},\ n \in \mathbb{N}. \tag{102}$$

**Definition 5.8.** Let $f : D(a, R) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ *admits a series expansion* if and only if there exists a power series $\sum_n a_n(z-z_0)^n$, convergent on $D$ such that

$$f(z) = \sum_n a_n(z-z_0)^n,\ |z-z_0| < R. \tag{103}$$

**Definition 5.9.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with $\Omega$ an open set. We say $f$ *is analytic on* $\Omega$ if and only if it admits locally a series expansion, that is, if for every point $z_0 \in \Omega$ there exists a disc $D(z_0, \delta)$ and a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ such that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \forall z \in D$.

**Theorem 5.15.** *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $D(z_0, R)$ and $w_0 \in D(z_0, R_0)$. Then, the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n$ has a radius of convergence $R_1 \geq R_0 - |z_0 - z_1|$ and it satisfies*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n, \qquad \text{if } |z-z_1| < R - |z_0 - z_1|. \tag{104}$$

**Definition 5.10.** Let $f(z) = \sum a_n(z-z_0)^n$ be a series with radius of convergence $R$. Then, its formal derivative is

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z}. \tag{105}$$

**Corollary 5.16.** *Let $f(z) = \sum a_n(z-z_0)^n$ be a series with radius of convergence $R$. Then, $f$ is infinitely derivable at $D_R(z_0)$.*

**Corollary 5.17.** *Let $R$ be the radius of convergence of the function*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

*Then $f$ has as Taylor polynomial of degree $m$ around $z_0$ the following one*

$$P_{m,f,z_0}(z) = \sum_{n=0}^{m} a_n(z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{106}$$

**Theorem 5.18** (Abel's Theorem)**.** *Let be the following series*

$$\sum_{n=0}^{\infty} f_n(z_0) g_n(z_0),$$

*where $f$,$g_n$ are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,*

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \tag{107}$$

*then the original series converges uniformly in $\Omega$.*

**Theorem 5.19** (Weierstrass' criterion)**.** *Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant $M_n$, satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at $\Omega$ for all $z_0 \in \Omega$.*

**Definition 5.11.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a complex function with $\Omega$ an open set. We say $f$ *is complex analytic* if and only if for all $z_0 \in \Omega$ exists a real number $R(z_0)$ and a sequence $\{a_n\} \subseteq \mathbb{C}$ (that can also depend on $z_0$) such that is $z \in D_R(z_0)$, then formally it is true that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n. \tag{108}$$

We denote the set of complex analytic functions with domain $\Omega$ by $C^\omega(\Omega)$.

**Corollary 5.20.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. If $f \in C^\omega(\Omega)$, then $f \in C^\infty(\Omega)$.*

**Corollary 5.21.** *Let $z_0$. Then, the coefficients $z_0$ of the local expression of $f$ given by the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ are determined by*

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{109}$$

**Proposition 5.22.** *Let $\varnothing \neq \Omega \subseteq \mathbb{C}$. Then,*

1. *Every connected component of $\Omega$ is a closed of $\Omega$ with a subspace topology.*
2. *Two connected components are the same or are disjoint.*
3. *Every connected of $\Omega$ is one and only one connected component.*
4. *$\Omega$ is the disjoint union of its connected components.*

**Theorem 5.23** (Analytic prolongation Principle)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0 = 0)$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of $\Omega$ that contains $z_0$ (the function can be zero also out $\Omega$, but we are not studying that).*

**Corollary 5.24.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a function with $\Omega$ a region. Then, the following statements are equivalent:*

1. *$f(z) = 0$ for all $z \in \Omega$.*

2. *There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.*

**Corollary 5.25** (Analytic Prolongation Principle). *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions with $\Omega$ a region. Then, the following statements are equivalent:*

1. *$f(z) = g(z)$ for all $z \in \Omega$.*

2. *There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}$.*

**Lemma 5.26.** *Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then*

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right). \tag{110}$$

**Corollary 5.27.** *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two analytic functions. Then, $fg$ is analytic.*

**Proposition 5.28.** *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, $f$ is analytic at $\Omega = D_R(0)$.*

# 6 Holomorphic functions

**Definition 6.1.** Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{aligned} \gamma : I &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned}. \tag{111}$$

**Definition 6.2.** Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval and $D \subseteq \mathbb{C}$ a domain. We define an *arc* as a continuous application of the form

$$\begin{aligned} \gamma : I &\longrightarrow D \\ t &\longmapsto \gamma_1(t) + i\gamma_2(t) \end{aligned}. \tag{112}$$

Equivalently, we can say an arc is a curve restricted to some interval.

**Definition 6.3.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We call $\gamma(a)$ and $\gamma(b)$ *the extremes of* $\gamma$. In particular, we call $\gamma(a)$ the *initial point* and $\gamma(b)$ the *final point*.

**Definition 6.4.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We define the *route or graph of* $\gamma$ as

$$\gamma^* := \{z \in D \,|\, z = \gamma(t), t \in I\}. \tag{113}$$

**Definition 6.5.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ *is closed* if and only if $\gamma(a) = \gamma(b)$.

**Definition 6.6.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ *is simple* if and only if there is no two numbers $t_1, t_2 \in (a, b)$ such that $\gamma(t_1) = \gamma(t_2)$. We also call it a *Jordan curve*, and if it is closed, a *circuit*.

**Definition 6.7.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ *is differentiable* if for al value $t_0 \in [a, b]$ there exists the limit

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \tag{114}$$

For $t_0 = a$ or $t_0 = b$ we consider the laterals limits from the right and from the left respectively.

**Definition 6.8.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ *is of class* $C^1$ if and only if $\gamma'$ exists and is continuous at $[a, b]$.

**Definition 6.9.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ *is regular or smooth* if and only if it is differentiable and $\gamma'$ never vanishes.

**Definition 6.10.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We say $\gamma$ *is piece-wise of class* $C^1$ if and only if $\gamma'$ exists and is continuous in $I$ except in a finite number of points where $\gamma$ has lateral derivatives.

**Definition 6.11.** Let $\gamma : [a, b] \longrightarrow D$ be an arc. We define the *opposite arc* as

$$\begin{aligned} -\gamma : [-b, -a] &\longrightarrow \mathbb{C} \\ t &\longmapsto \gamma(-t) \end{aligned}. \tag{115}$$

**Definition 6.12.** Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be an arc. We say $\Gamma(s), s \in [c, d] \subseteq \mathbb{R}$ *has been obtained from* $\gamma(t), t \in [a, b]$ *by a change of parametrization* if and only if the new parameter $s$ and the original parameter $t$ are related by a relation $t = \phi(s)$, where $\phi : [c, d] \longrightarrow [a, b]$ is an homeomorphism that satisfies $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$. We call $\Gamma$ *the reparametrization of* $\gamma$.

**Definition 6.13.** Let $\gamma_1 : I_1 \longrightarrow \mathbb{C}$ and $\gamma_2 : I_2 \longrightarrow \mathbb{C}$ be two arcs. We say they *are equivalent* if and only if there exists a bijective, monotone, and continuous function $\rho : I_2 \longrightarrow I_1$ such that $\gamma_2 = \gamma_1 \circ \rho$. If $\rho$ is an increasing function we say *$\gamma_1$ and $\gamma_2$ have the same orientation*; otherwise, we say *$\gamma_1$ and $\gamma_2$ have opposite orientations.*

**Definition 6.14.** Let $\gamma_1[a, b] \longrightarrow \mathbb{C}$ and $\gamma_2 : [c, d] \longrightarrow \mathbb{C}$ be two arcs such that $[a, b] \cap [c, d] = \varnothing$. We define the application $\gamma_1 \cup \gamma_2$ (sometimes denoted by $\gamma_1 + \gamma_2$) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \leq t \leq b \\ \gamma_2(t - b + c), & \text{if } b \leq t \leq b + d - c \end{cases}. \tag{116}$$

We say $\gamma_1, \gamma_2$ can be joined/added or that there exists its union/sum if and only $\gamma_1(b) = \gamma_2(x)$. In this case $\gamma_1 + \gamma_2$ is an arc, and we call it *the sum arc of* $\gamma_1$ *plus* $\gamma_2$.

**Definition 6.15.** We define the *segment of extremes* $z_1, z_2 \in \mathbb{C}$ as the arc defined by the expression

$$\begin{aligned} [z_1, z_2] : [0, 1] &\longrightarrow \mathbb{C} \\ t &\longmapsto (1 - t)z_1 + tz_2 \end{aligned}. \tag{117}$$

**Definition 6.16.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ *is polygonal* if and only if can be expressed as a finite union of segments, that is, if there exist a natural number $n$ and points $\{z_0, \dots, z_n\}$ such that

$$p = [z_0, z_1] \cup \dots \cup [z_{n-1}, z_n]. \tag{118}$$

**Definition 6.17.** Let $\gamma : [a,b] \longrightarrow D$ be an arc with $a, b$ finite. We say $\gamma$ *is a basic curve* if and only if $\gamma \in C^1((a,b)) \cap C([a,b])$ and there exist $\lim_{t\to a^+} \gamma'(t), \lim_{t\to b^-} \gamma'(t)$.

**Definition 6.18.** A *path* is a function $\gamma : [a,b] \longrightarrow \mathbb{C}$ such that there exist basic curves $\gamma_j : [a_j, b_j] \longrightarrow \mathbb{C}, j \in \{1, \ldots, k\}$ such that $\gamma = \gamma_1 + \cdots + \gamma_k$ and therefore $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ and $a = a_1, b = a_k$.

**Definition 6.19.** Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous curve and $a_1, \ldots, a_l \in \mathbb{R}$ such that $a = a_0 \leq \cdots \leq a_l \leq b = a_{l+1}$. We say $\gamma$ *is piece-wise differentiable* if and only if

$$\gamma \in C^1\left(\bigcup_{j=0}^{l} (\alpha_j, \alpha_{j+1})\right),$$

$\forall j \in \{0, \ldots, l+1\} \exists \lim_{t\to a_j^+} \gamma'(t)(\text{except if } j = l+1), \lim_{t\to a_j^-} \gamma'(t)(\text{except if } j = 0)$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

**Theorem 6.1.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with $\Omega$ an open set and $\phi : I \longrightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curev (hence a derivable curve) and its real derivative is*

$$\psi'(t) = f'(\phi(t))\phi'(t). \tag{119}$$

**Definition 6.20.** Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two curves. We say $\gamma_1, \gamma_2$ *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = \gamma_1(0) = \gamma_2(0), s \in [0,1]$.
4. $h(1,s) = \gamma_1(1) = \gamma_2(1), s \in [0,1]$.

**Definition 6.21.** Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ be two circuits. We say $\gamma_1, \gamma_2$ *are homotopic* if and only if there exists a continuous function $h(t,s) : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that

1. $h(t,0) = \gamma_1(t), t \in [0,1]$.
2. $h(t,1) = \gamma_2(t), t \in [0,1]$.
3. $h(0,s) = h(1,s), s \in [0,1]$.

**Definition 6.22.** Let $f : [a,b] \longrightarrow \mathbb{C}$ be a function with the notation $f = u + iv$. We define the integral of $f$ as

$$\int_a^b f(t)\,\mathrm{d}t \coloneqq \int_a^b u(t)\,\mathrm{d}t + i \int_a^b v(t)\,\mathrm{d}t\,. \tag{120}$$

**Proposition 6.2.** *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ be two integrable functions and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,*

$$\int_a^b \lambda f + \mu g\,\mathrm{d}t = \lambda \int_a^b f\,\mathrm{d}t + \mu \int_a^b g\,\mathrm{d}t\,. \tag{121}$$

**Proposition 6.3.** *Let $f : [a,b] \longrightarrow \mathbb{C}$ be an integrable function. Then,*

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t\,. \tag{122}$$

**Definition 6.23.** Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of $f$ over $\gamma$* as

$$\int_\gamma f(z)\,\mathrm{d}z \coloneqq \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\,. \tag{123}$$

**Proposition 6.4.** *The previous definition is well defined.*

**Proposition 6.5.** *If we use the notation $f = u + iv$ and $\gamma = x + iy$, then the integral has the form*

$$\int_\gamma f = \int_a^b u\frac{\mathrm{d}x}{\mathrm{d}t} + v\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t + i\int_a^b v\frac{\mathrm{d}x}{\mathrm{d}t} + u\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t\,. \tag{124}$$

**Definition 6.24.** Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then, we define the *line integral of $f$ over $\gamma$ with respect the differential of length* as

$$\int_\gamma f(z)\,\mathrm{d}s \coloneqq \int_\gamma f(z)|\mathrm{d}z| = \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t\,. \tag{125}$$

**Theorem 6.6.** *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$, $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ two functions, and $\lambda, \mu \in \mathbb{C}$ two numbers. Then,*

$$\int_\gamma \lambda f + \mu g\,\mathrm{d}z = \lambda \int_\gamma f\,\mathrm{d}z + \mu \int_\gamma g\,\mathrm{d}z\,. \tag{126}$$

**Theorem 6.7.** *Let $\gamma_1, \gamma_2$ be two equivalent curves of the same orientation and of class $C^1$ on their respective domains and$f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z\,. \tag{127}$$

**Proposition 6.8.** *Let $\gamma_1, \ldots, \gamma_n$ be $n$ curves of class $C^1$ on their respective domains and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma_1, \ldots, \Gamma_n \subseteq \Omega$. If we define $\gamma = \gamma_1 + \cdots + \gamma_n$, then*

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^n \int_{\gamma_i} f(z)\,\mathrm{d}z\,. \tag{128}$$

**Proposition 6.9.** *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a,b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \int_\gamma |f|\,\mathrm{d}s\,. \tag{129}$$

**Corollary 6.10.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. If $|f(z)| \leq M$ for all $z \in \Gamma$, then,*

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq ML(\gamma). \tag{130}$$

**Proposition 6.11.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$\mathrm{Ind}(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w - z}\,\mathrm{d}w\,. \tag{131}$$

**Proposition 6.12.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function in $\Gamma \subseteq \Omega$. Then,*

$$|\mathrm{Ind}(\gamma, z)| \leq \frac{1}{2\pi}\frac{L(\gamma)}{|z - \Gamma|}. \tag{132}$$

**Proposition 6.13.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve of class $C^1([a, b])$ and $\{f_n\}_{n=0}^\infty$ a sequence of continuous functions on $\Gamma$ such that $\sum_{n=0}^\infty f_n$ converges uniformly on $\Gamma$. Then, $\sum_{n=0}^\infty \int_\gamma f_n\,\mathrm{d}z$ converges and*

$$\int_\gamma \sum_{n=0}^\infty f_n(z)\,\mathrm{d}z = \sum_{n=0}^\infty \int_\gamma f_n(z)\,\mathrm{d}z\,. \tag{133}$$

**Definition 6.25.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ has a *primitive on* $\Omega$ if and only if there exists a function $F : \Omega \subseteq\longrightarrow \mathbb{C}$ such that $F' = f \forall z \in \Omega$.

**Definition 6.26.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say $f$ has a *local primitive on* $D$ if and only if for all $z$ there exists a neigborhood where $f$ has a primitive.

**Theorem 6.14** (Fundamental theorem of complex calculus)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with $\Omega$ a domain. Then, the line integral of $f$ is independent on the path on $\Omega$ if and only if $f$ has an holomorphic primitive $F$ such that $F' = f$ on $\Omega$. In that case,*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)). \tag{134}$$

**Theorem 6.15.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function on a star domain $S \subseteq \Omega$. Then, $f$ has an holomorphic primitive $F$ on $S$ if and only if*

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0 \tag{135}$$

*for all triangle $\triangle \subseteq \Omega$.*

**Proposition 6.16.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with no roots on a domain $D \subseteq \Omega$. Then, there is a determination of the logarithm of $f$ on $D$ if and only if $f'/f$ has an holomorphic primitive on $D$.*

**Proposition 6.17.** *Let $K \subseteq \mathbb{C}$ be a compact set. Then,*

1. *If $\alpha \in V_\infty$, then the non-bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log(z - \alpha)$ in a neighborhood of $K$.*
2. *If $\alpha, \beta$ belong to the same bounded component of $\mathbb{C} \setminus K$, then there exists a determination of $\log\left(\frac{z-\alpha}{z-\beta}\right)$ in a neighborhood of $K$.*

**Theorem 6.18** (Green's theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary. Let $\mathbf{F} = (P, Q)$ be a vector field with $P, Q$ being differentiable functions on a neighborhood of $\bar{\Omega}$ such that $\partial_x P - \partial_y Q$ is continuous on $\bar{\Omega}$. Then,*

$$\int_{\partial\Omega} \langle \mathbf{F}, \mathrm{d}\mathbf{s}\rangle_I = \int_{\partial\Omega} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{136}$$

**Theorem 6.19** (Cauchy's integral theorem)**.** *Let $\Omega$ be a bounded domain with piece-wise regular and positively oriented boundary and $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an holomorphic function in a neighborhood of $\bar{\Omega}$. Then,*

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z = 0. \tag{137}$$

**Corollary 6.20.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function in a domain $D \subseteq \Omega$. Then, $f$ ha local primitive on $D$. If $D$ is a star domain, $f$ has a global holomorphic primitive.*

**Corollary 6.21.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with no roots in a domain $D \subseteq \Omega$. Then, $f$ has a local determination of the logarithm on $D$. If $D$ is a star domain, $f$ has a global determination of the logarithm.*

**Theorem 6.22** (Cauchy's integral theorem for homotopic curves)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with $\Omega$ a domain and $\gamma_1, \gamma_2$ two homotopic curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z\,. \tag{138}$$

**Theorem 6.23** (Cauchy's general integral theorem)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a regular function on $\Omega$ except a finite numbers of points where $f$ is continuous. If $\gamma$ is a constant curve, then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0. \tag{139}$$

**Theorem 6.24** (Morera's theorem)**.** *Let $f$ be a continuous function in a region $\Omega$. If*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0 \tag{140}$$

*for all simple and closed curve $\gamma$ such that $\Gamma \subseteq \Omega$, then $f$ is analytic on $\Omega$.*

**Theorem 6.25.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a differentiable function on a domain $D$. Then, $f = u + iv$ is holomorphic if and only if the field $\bar{f} = (u, -v)$ is locally conservative and locally solenoidal.*

**Definition 6.27.** Let $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable vector field on a domain $D \subseteq \mathbb{R}^n$. We say the field is *holomorphic* if and only if is locally conservative and locally solenoidal, that is, it satisfies

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},\ \forall i, j; \qquad \operatorname{div} \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = 0, \text{ on } D. \tag{141}$$

**Definition 6.28.** Let $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field two times differentiable on an open set $\Omega \subseteq \mathbb{R}^n$. We say the field is *harmonic* if and only if $\nabla^2\Phi = 0$ on $\Omega$.

**Theorem 6.26.** *Holomorphic vector fields are the fields that are locally the gradient of an harmonic function. Holomorphic functions are the functions $f$ that, locally, satisfy $\bar{f} = \Phi_x + i\Phi_y$ with $\Phi$ harmonic.*

**Definition 6.29.** Let $u$ be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$. We say a differentiable function $\tilde{u}$ on $\Omega$ is *the harmonic conjugate of* $u$ if and only if $\mathrm{d}\tilde{u} = \mathrm{d}^* u$, that is, if the function $f = u + i\tilde{u}$ is holomorphic on $\Omega$.

**Theorem 6.27.** *Let $u$ be an harmonic real function on a domain $\Omega \subseteq \mathbb{C}$ and $f = \overline{\boldsymbol{\nabla}} u$. Then, $u$ is has an harmonic conjugate on $\Omega$, $\tilde{u}$, if and only if $f$ has an holomorphic primitive $F$ on $\Omega$. In that case, $F = u + i\tilde{u}$.*

**Proposition 6.28.** *Let $u$ be an harmonic function on a domain $\Omega$. Then, it has an harmonic conjugate if an only if the closed form $\mathrm{d}^* u$ is exact on $\Omega$, that is, if $\int_\gamma \mathrm{d}^* u = 0$ for all closed curve $\gamma$ such that $\Gamma \subseteq \Omega$, condition that is is always locally completed. If $\Omega$ is a star domain, every harmonic function on $\Omega$ has a harmonic conjugate function on $\Omega$.*

# 7 Local properties of holomorphic functions

**Lemma 7.1.** *Let $a \in \mathbb{C}$ be a number and $f = 1/|z - a|$. Then, $f$ is Lebesgue-integrable on every subset of $\mathbb{C}$ of finite measure.*

**Theorem 7.2** (Cauchy-Green formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and $f$ a differentiable function on a neighborhood of $\bar{\Omega}$ such that $\overline{\partial} f$ is continuous on $\bar{\Omega}$. Then, for all $z_0 \in \Omega$,*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0}\, \mathrm{d}z - \frac{1}{\pi} \int_{\Omega} \frac{\overline{\partial} f(z)}{z - z_0}\, \mathrm{d}m(z)\,. \tag{142}$$

**Corollary 7.3** (Cauchy's integral formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piece-wise regular and positively oriented boundary, and $f$ an holomorphic function on a neighborhood of $\bar{\Omega}$. Then,*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0}\, \mathrm{d}z\,. \tag{143}$$

**Corollary 7.4.** *Let $f$ be a differentiable function on $\mathbb{C}$ with compact support and $\overline{\partial} f$ continuous on $\mathbb{C}$. Then,*

$$f(z_0) = -\frac{1}{\pi} \int_{\Omega} \frac{\overline{\partial} f(z)}{z - z_0}\, \mathrm{d}m(z)\,. \tag{144}$$

**Proposition 7.5.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and $\gamma$ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z_0) f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0}\, \mathrm{d}z\,. \tag{145}$$

**Corollary 7.6.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and $\gamma_1, \gamma_2$ two homotopic, piece-wise regular, and positively oriented curves such that $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then,*

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z - z_0}\, \mathrm{d}z - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z - z_0}\, \mathrm{d}z\,. \tag{146}$$

**Theorem 7.7.** *Let $f$ be an holomorphic function on a disc $D(z_0, R)$. Then, there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence greater or equal to $R$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall z \in D(z_0, R). \tag{147}$$

**Theorem 7.8.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function with $\Omega$ an open set. Then, $f$ is holomorphic on $\Omega$ if and only if $f$ is analytic on $\Omega$. More precisely, every holomorphic function $f$ on $\Omega$ is indefinitely holomorphic on $\Omega$, and for all $z_0 \in \Omega$ the Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \tag{148}$$

*is valid on the greatest disc centered on $z_0$ and contain on $\Omega$, which is $D(z_0, \delta(z_0))$, where $\delta(z_0) = \inf\{|z_0 - w|, w \notin \Omega\}$.*

**Proposition 7.9.** *Let $f$ be a function and $\Omega$ a bounded domain with piece-wise regular and positively oriented boundary. If $f$ is holomorphic on a neighborhood of $\bar{\Omega}$, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - z_0)^{n+1}}\, \mathrm{d}z\,. \tag{149}$$

**Proposition 7.10.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function and $\gamma$ a piece-wise regular and positively oriented curve such that $\Gamma \subseteq \Omega$. Then,*

$$\operatorname{Ind}(\gamma, z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}}\, \mathrm{d}z\,. \tag{150}$$

**Theorem 7.11** (Maximum modulus principle). *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with $\Omega$ a domain. If $f$ is not constant, then $|f|$ does not have any local maxima on $\Omega$.*

**Corollary 7.12.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $f$ an holomorphic function on a neighborhood of $\bar{\Omega}$ or, more generally, $f \in C(\bar{\Omega}) \cap H(\Omega)$. Let $M$ be the maxima of $|f|$ on $\partial\Omega$. Then, one has*

$$|f(z)| \leq M, \qquad \text{for all } z \in \Omega. \tag{151}$$

*In other words,* $\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|$.

**Theorem 7.13** (Cauchy's inequality). *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on a neighborhood of the disc $\bar{D}(z_0, R)$ and $|f(z)| \leq M$ for $z \in C(z_0, R)$. Then,*

$$\left|f^{(n)}(z_0)\right| \leq M \frac{n!}{R^n}. \tag{152}$$

**Corollary 7.14.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function with $\Omega$ a domain such that $|f(z)| \leq M, z \in \Omega$. Then,*

$$\left|f^{(n)}(z)\right| \leq M \frac{n!}{d(z, U^c)^n}, \qquad z \in U, n \in \mathbb{N}. \tag{153}$$

**Theorem 7.15** (Liouville's theorem). *Let $f$ be a bounded entire function. Then, $f$ is constant. Also, a function $u$ harmonic and bounded on $\mathbb{C}$ is constant.*

**Theorem 7.16** (Fundamental theorem of algebra). *Let $P(<) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree $n$ of complex coefficients and $n \geq 1$. Then, $P$ has exactly $n$ roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ (some of which can be counted with their multiplicity) and*

$$P(z) = a_n \prod_{i=1}^{n} (z - \alpha_i). \tag{154}$$

# 8 Isolated singularities of holomorphic functions

**Theorem 8.1.** *Every holomorphic function on an annulus admits a Laurent expansion.*

**Proposition 8.2.** *Let $f$ be an holomorphic function on an annulus $C(z_0, R_2, R_1)$. If $f$ has an isolated singularity at $z_0$, then its Laurent expansion is uniquely determined by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \tag{155}$$

*where $a_n$ is independent of $r$, $r \in (R_2, R_1)$.*

# 9 Homology

# 10 Harmonic functions

**Theorem 10.1.** *Let $f \in H(\Omega), C^1(\Omega)$ be a function. If $f = u + iv$, then $u, v$ are harmonic functions on $\Omega$.*

# 11 Conforming representation

# 12 Riemann theorem

# 13 Runge theorem

# 14 Zeros of holomorphic functions

**Definition 14.1.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0$ a number. We say $z_0$ *is a zero of order $n$ of $f$* if and only if $f^{(k)}(z_0) = 0$ for all $0 \leq k \leq n$.

**Proposition 14.1.** *The zeros of finite order of an holomorphic function are isolated points.*

**Proposition 14.2.** *All the zeros of an non null analytic function are isolated points and of finite order.*

**Theorem 14.3** (Weierstrass Factorization Theorem). *content...*

# 15 Fourier transform

**Definition 15.1.** Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform of $f$ at the point $\xi$* as

$$\boxed{\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \mathrm{e}^{-i\xi x} \, \mathrm{d}x \,.} \tag{156}$$

**Proposition 15.1.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, the application*

$$\begin{aligned} \mathscr{F}\{f\} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \hat{f}(\xi) \end{aligned} \tag{157}$$

*is a well defined application.*

**Definition 15.2.** Let $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. We say *the functions $f_n$ converge to $f$ with a norm $\|\cdot\|_p$* or *converge in $L^p(\mathbb{R})$* if and only if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0. \tag{158}$$

**Theorem 15.2.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.*

1. *The Fourier transform of $f$ satisfies*

$$\mathscr{F}\{f\} \in L^\infty(\mathbb{R}), \qquad \|\mathscr{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|f\|_1. \tag{159}$$

2. *$\mathscr{F}\{f\}$ is $\mathbb{C}$ linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,*

$$\mathscr{F}\{\alpha f + \beta g\} = \alpha \mathscr{F}\{f\} + \beta \mathscr{F}\{g\}. \tag{160}$$

3. *If $g(x) = \bar{f}(x)$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \tag{161}$$

4. *If $g(x) = g(\lambda x)$ and $\lambda \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \frac{1}{|\lambda|}\hat{f}\left(\frac{\xi}{\lambda}\right). \tag{162}$$

5. *If $g(x) = f(x-a)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \mathrm{e}^{-ia\xi}\hat{f}(\xi). \tag{163}$$

6. *If $g(x) = \mathrm{e}^{iax}f(x)$ with $\alpha \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \hat{f}(\xi - a) \tag{164}$$

7. *If $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$ when $n \to \infty$, then $\mathscr{F}\{f_n\} \to \mathscr{F}\{f\}$ uniformly in $\mathbb{R}$.*

8. *The Fourier transform $\mathscr{F}\{f\}$ is a continuous function in $\mathbb{R}$, $\mathscr{F}\{f\} \in C(\mathbb{R})$.*

**Proposition 15.3.** *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty}|f(x)| = 0$. Then,*

$$\hat{f}'(\xi) = i\xi\hat{f}(\xi). \tag{165}$$

**Corollary 15.4.** *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its $n$-th derivative $f^{(n)} \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty}|f(x)| = 0$. Then,*

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n\hat{f}(\xi). \tag{166}$$

**Definition 15.3.** Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We define the support of $f$ as

$$\operatorname{supp} f := \overline{\{x \in I \mid f(x) \neq 0\}}. \tag{167}$$

**Definition 15.4.** We define the set $\mathscr{D}(\mathbb{R})$ as

$$\mathscr{D}(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}) \mid \operatorname{supp}\varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \tag{168}$$

**Theorem 15.5.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_\in \mathscr{D}(\mathbb{R})$ such that*

$$\lim_{h\to\infty}\int_{\mathbb{R}}|f - \phi_n|\,\mathrm{d}x = 0, \tag{169}$$

*that is, we have convergence of $\phi_n$ to $f$ with norm $\|\cdot\|_1$.*

**Proposition 15.6.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.*

**Proposition 15.7.** *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\left|\hat{f}(\xi)\right| \leq \|f\|_1$.*

**Theorem 15.8.** *Let $f \in L^1(\mathbb{R})$ be a function. Then,*

$$\lim_{|x|\to\infty}\hat{f}(x) = 0. \tag{170}$$

**Theorem 15.9.** *The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.*

**Definition 15.5.** We define the *Schwartz space* as

$$S(\mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{C} \mid f \in C^\infty(\mathbb{R}) \wedge \forall n, m \in \mathbb{N}\,\exists c_{n,m} < \infty \\ \text{such that } (1+|x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}.$$

**Proposition 15.10.** *Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \longrightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,*

1. *$f + g \in S(\mathbb{R})$.*
2. *$\lambda f \in S(\mathbb{R})$.*
3. *$fg \in S(\mathbb{R})$.*
4. *$Pf \in S(\mathbb{R})$.*

**Theorem 15.11.** *Let $I, J \subseteq \mathbb{R}$ be two intervals with $I$ compact and $J$ open. Let $f : I \times J \longrightarrow \mathbb{R}$ be a function such that*

1. *$f(\cdot, \lambda)$ is Riemann-integrable in $I$ for all $\lambda \in J$,*
2. *$f(x, \cdot)$ is derivable in $J$ for all $x \in I$.*

*If $\partial_\lambda f$ is continuous in $I \times J$, then*

1. *$\partial_\lambda f(\cdot, \lambda)$ is Riemann-integrable for all $\lambda \in J$.*
2. *$F(\lambda) = \int_I f(x, \lambda)\,\mathrm{d}x$ is derivable with continuous derivative in $J$ for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.*

$$F'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_I f(x, \lambda_0)\,\mathrm{d}x = \int_I \frac{\partial f}{\partial\lambda}(x, \lambda_0)\,\mathrm{d}x\,, \forall\lambda_0 \in J \tag{171}$$

**Proposition 15.12.** *Let $f \in S(\mathbb{R})$. Then,*

1. *$S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.*
2. *$\widehat{xf}(\xi) = (iD_\xi\hat{f})(\xi)$ for all $\xi \in \mathbb{R}$.*

**Corollary 15.13.** *Let $f \in s(\mathbb{R})$. Then,*

$$\widehat{x^n f}(\xi) = (i^n D^n\hat{f})(\xi), \forall n \in \mathbb{N}. \tag{172}$$

**Proposition 15.14.** *The Fourier transform $\mathscr{F}$ restricted to $S(\mathbb{R})$ is an automorphism, that is, if $f \in S(\mathbb{R})$ then $\mathscr{F}\{f\} = \hat{f} \in S(\mathbb{R})$.*

**Lemma 15.15.** *If $G(x) = \mathrm{e}^{-x^2/2}$, then $\hat{G}(\xi) = e^{-\xi^2/2}$. We observe hence that $G$ is a fixed point of $\mathscr{F}$.*

**Lemma 15.16.** *If $f, g \in S(\mathbb{R})$, then*

$$\int_{\mathbb{R}} f(\xi)\hat{g}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}}\hat{f}(\tau)g(\tau)\,\mathrm{d}\tau\,. \tag{173}$$

**Lemma 15.17.** *Let $f, g \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,*

1. *$g(\lambda x)\hat{f}(x)$ converges to $g(0)\hat{f}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*

2. $f(\lambda x)\hat{g}(x)$ converges to $f(0)\hat{g}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.

**Lemma 15.18.** *Let $f, g \in s(\mathbb{R})$. Then,*

$$f(0)\int_{\mathbb{R}} \hat{g}(\xi)\,\mathrm{d}\xi = g(0)\int_{\mathbb{R}} \hat{f}(\xi)\,\mathrm{d}\xi\,. \tag{174}$$

**Lemma 15.19.** *Let $f \in s(\mathbb{R})$ be a function. Then,*

$$f(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(\xi)\,\mathrm{d}\xi\,. \tag{175}$$

**Corollary 15.20** (Inversion formula)**.** *Let $f \in S(\mathbb{R})$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,, \forall x \in \mathbb{R}. \tag{176}$$

**Theorem 15.21** (Inversion of $\mathscr{F}$ in $S(\mathbb{R})$)**.** *Let $\mathscr{F} : S(\mathbb{R}) \longrightarrow S(\mathbb{R})$, defined by $\mathscr{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, $\mathscr{F}$ is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathscr{F}^4 = Id$. In particular, $\mathscr{F}^{-1} = \mathscr{F}^3$ and if $f \in S(\mathbb{R})$, then*

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathscr{F}\{f\}(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}\mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,.} \tag{177}$$

*In fact, $\mathscr{F}$ is an homemorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(S(\mathbb{R}), \|\cdot\|_{n,m})$.*

**Theorem 15.22** (Inversion of $\mathscr{F}$ for discontinuities)**.** *Let $f$ be a absolutely Riemann-integrable function in $\mathbb{R}$ with $f$ and $f'$ piece-wise continuous. Then,*

$$\frac{f(x^-) + f(x^+)}{2} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}\mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,. \tag{178}$$

**Definition 15.6.** Let $f$ be a Riemann-integrable function in $\mathbb{R}$. We define the *Fourier transform of cosine kind* as

$$\hat{f}_c(\xi) \coloneqq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(\xi x)f(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \cos(\xi x)f_e(x)\,\mathrm{d}x\,, \tag{179}$$

and the *Fourier transform of sine kind* as

$$\hat{f}_s(\xi) \coloneqq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin(\xi x)f(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \sin(\xi x)f_o(x)\,\mathrm{d}x\,. \tag{180}$$

**Proposition 15.23.** *Let $\hat{f}_c, \hat{f}_s$ be the Fourier transform of cosine and sine kinds of $f$. Then, $\hat{f}_c(\xi)$ is even, $\hat{f}_s(\xi)$ is odd, and $\hat{f}(\xi) = \hat{f}_c(\xi) - i\hat{f}_s(\xi)$.*

**Theorem 15.24.** *Let $f$ be a absolutely Riemann-integrable function in $\mathbb{R}$ with $f$ and $f'$ piece-wise continuous. Then,*

$$\frac{f_e(x^-) + f_e(x^+)}{2} = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \hat{f}_c\cos(\xi x)\,\mathrm{d}\xi\,, \tag{181}$$

$$\frac{f_o(x^-) + f_o(x^+)}{2} = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \hat{f}_s\sin(\xi x)\,\mathrm{d}\xi\,. \tag{182}$$

**Theorem 15.25** (Tonelli's Theorem)**.** *Let $f : I \times J \longrightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x, y) \geq 0$ for all $(x, y) \in I \times J$. Then,*

$$\int_{I\times J} f\,\mathrm{d}x\,\mathrm{d}y = \int_I\int_J f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_J\int_I f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{183}$$

*Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.*

**Corollary 15.26.** *Let $f, g \in L^1(\mathbb{R})$. Then, $F(x, t) = f(t)g(x - t) \in L^1(\mathbb{R}^2)$.*

**Definition 15.7.** Let $f, g \in L^1(\mathbb{R})$ two function. We define the *convolution of $f$ and $g$* as

$$\begin{aligned}(f * g) : \mathbb{R} &\longrightarrow \mathbb{C}\\ x &\longmapsto \int_{\mathbb{R}} f(t)g(x-t)\,\mathrm{d}t\text{'}\end{aligned} \tag{184}$$

which is from $L^1(\mathbb{R})$.

**Proposition 15.27.** *Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$.*

**Proposition 15.28.** *Let $f \in L^1(\mathbb{R})$ be a function and $g = f^2$. Then,*

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\hat{f} * \hat{f} = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)\hat{f}(\xi - t)\,\mathrm{d}t\,. \tag{185}$$

**Theorem 15.29.** *Let $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^\infty(\mathbb{R})$.*

**Theorem 15.30.** *Let $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$ with* $\operatorname{supp} f$ *compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and* $\operatorname{supp}\{f * \phi\} \subseteq \operatorname{supp} f + \operatorname{supp}\phi$.

**Definition 15.8.** We say the functions $\phi_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ continuous in a compact support are *an approximation of the unity* if and only if

1. $\phi_\epsilon \geq 0$ for all $\epsilon$.
2. $\displaystyle\int_{\mathbb{R}} \phi_\epsilon(x)\,\mathrm{d}x = 1.$
3. For all $\delta > 0$ it is satisfied that

$$\lim_{\epsilon\to 0}\left\{\sup_{|t|>\delta}\phi_\epsilon(t)\right\} = 0. \tag{186}$$

**Theorem 15.31.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to $f$.*

**Corollary 15.32.** *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to $f$.*

**Theorem 15.33** (Weierstrass polynomial approximation)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials $P_n$ with $n \in \mathbb{N}$ such that $P_n$ converge uniformly to $f$ in $[a,b]$.*

**Theorem 15.34.** *Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \to f$ with norm $\|\cdot\|_p$ (that is, convergence in $L^p$), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then*

$$\lim_{n\to\infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \tag{187}$$

*with $\|f\|_{C^k(\mathbb{R})} = \max_{0\leq l\leq k}\left(\sup_{x\in\mathbb{R}}\left|D^l f(x)\right|\right)$ being a norm.*

**Lemma 15.35.** *Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\int\limits_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x = 0$. Then, $f \equiv 0$.*

**Corollary 15.36.** *The Fourier transform $\mathscr{F}$ is injective since $\mathscr{F}\{f\} = \hat{f} = 0 \Leftrightarrow f = 0$ in $L^1(\mathbb{R})$ (the zero function class) and $\mathscr{F}$ is a linear application.*

**Theorem 15.37** (Inversion theorem in $L^1(\mathbb{R})$)**.** *Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \hat{f}(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,, \forall x \in \mathbb{R}. \tag{188}$$

# 16 Fourier transform 2

**Theorem 16.1** (Parseval formula)**.** *Let $f, g \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be two functions. Then,*

$$\int\limits_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x = \int\limits_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi\,. \tag{189}$$

**Theorem 16.2** (Plancherel Theorem)**.** *Let $f \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ be a function. Then,*

$$\boxed{\int\limits_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x = \int\limits_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2\mathrm{d}\xi\,,} \tag{190}$$

*that is, $\|f\|_2 = \left\|\hat{f}_2\right\|$ and $\mathscr{F}$ is an isometry between vector spaces.*

**Definition 16.1.** Let $f \in S(\mathbb{R})$ be a function. We define the following quantities

$$E(f) \coloneqq \int\limits_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x\,, \tag{191}$$

$$\sigma(f)^2 \coloneqq \int\limits_{\mathbb{R}} |xf(x)|^2\,\mathrm{d}x\,. \tag{192}$$

**Theorem 16.3.** *Let $f \in S(\mathbb{R})$ be a function. Then,*

$$\sigma(f)\sigma(\hat{f}) \geq \frac{E(f)}{2}. \tag{193}$$

# 17 Multidimensional fourier transform

**Theorem 17.1.** *For several variables*

$$\mathscr{F}\{f(x_1,\ldots,x_n)\} = \frac{1}{(2\pi)^{n/2}} \int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} f(x_1,\ldots,x_n)\mathrm{e}^{-i(x_1\xi_1+\cdots+x_n\xi \tag{194}$$

*or simpler,*

$$\boxed{\mathscr{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} f(\mathbf{x})\mathrm{e}^{-i\langle\mathbf{x},\boldsymbol{\xi}\rangle_I}\,\mathrm{d}\hat{x}\,.} \tag{195}$$