1 Motion in one dimension

Proposition 1.1. *Let .... Then, it is true that*

$$\vec{F}^{ext} = m\vec{a} + (\vec{v} - \vec{u})\dot{m} \tag{1}$$

or which is equivalent,

$$\vec{F}^{ext} + \dot{m}\vec{u} = \dot{\vec{p}} \tag{2}$$

2 Oscillations

Proposition 2.1. *Let be the following differential equation*

$$\ddot{x} + \omega_0^2 x = 0, \tag{3}$$

with the initial value condition of $x(0) = x_0$ and $v(0) = v_0$. Then, the general solution is

$$x(t) = x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t, \tag{4}$$

or, which is equivalent,

$$x(t) = A\cos\left[\omega_0 t + \phi_0\right], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \phi_0 = -\arctan\frac{v_0}{\omega_0 x_0}. \tag{5}$$

Definition 2.1. Let $U(x)$ be a potential function of class $C^2(\mathbb{R})$. Then, we say x_0 *is a point of stable equilibrium* if U has a maxima in x_0.

Proposition 2.2. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{6}$$

with the initial value conditions of $x(0) = x_0$ and $v(0) = v_0$. Then, the general solution is

$$x(t) = e^{-\beta t}\left[x_0\cos\tilde{\omega}t + \frac{v_0 + \beta x_0}{\tilde{\omega}}\sin\tilde{\omega}t\right], \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \beta^2} \tag{7}$$

if $\beta < \omega_0$,

$$x(t) = e^{-\omega_0 t}\left[x_0 + (x_0\omega_0 + v_0)t\right] \tag{8}$$

if $\beta = \omega_0$, and

$$x(t) = \frac{x_0(\bar{\omega} - \beta) - v_0}{2\bar{\omega}}e^{-(\beta+\bar{\omega})t} + \frac{x_0(\bar{\omega} + \beta) + v_0}{2\bar{\omega}}e^{-(\beta-\bar{\omega})t} \qquad \bar{\omega} = \sqrt{\beta^2 - \omega_0^2} \tag{9}$$

if $\beta > \omega_0$.

Proposition 2.3. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0\cos\left[\omega t + \psi_0\right], \tag{10}$$

with the initial value conditions of $x(0) = x_0$ and $v(0) = v_0$. Then, the particular solution is

$$x_p(t) = A\cos\left[\omega t + \psi_0 - \phi_0\right], \qquad A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}} \qquad \phi_0 = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{11}$$

3 Central forces

Definition 3.1. Central force

$$\vec{F}(\vec{r}) = f(r)\vec{e}_\rho \tag{12}$$

Proposition 3.1. *All central forces are conservatives.*

Proposition 3.2. *The angular momentum with respect the origin is conserved.*

$$\dot{\vec{L}} = \vec{0} \tag{13}$$

Proposition 3.3. *The areal velocity is constant.*

$$\frac{dA}{dt} = \frac{L}{2m} = ctt \tag{14}$$

Theorem 3.4 (Bertrand's Theorem). *The only central potentials where every bounded orbit is closed are:*

$$U(r) = -\frac{k}{r}, \qquad U(r) = \frac{k}{2}r^2, \qquad k > 0 \tag{15}$$

4 Coupled oscillations 1

5 Coupled oscillations 2

6 Rotations

7 Dynamics of rigid body

Proposition 7.1. *The vector $\mathbf{\Omega}$ is independent on the origin of the system S.*

Proposition 7.2. *The energy of the rigid body is an invariant scalar under change of basis.*

8 Special relativity

Theorem 8.1. *If Maxwell's equations*

$$\langle\mathbf{\nabla}, \mathbf{E}\rangle_I = \frac{\rho}{\epsilon_0}, \tag{16}$$

$$\langle\mathbf{\nabla}, \mathbf{B}\rangle_I = 0, \tag{17}$$

$$\mathbf{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{18}$$

$$\mathbf{\nabla}\times\mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right). \tag{19}$$

are invariant under Galileo transformations, then $\mathbf{E} = \mathbf{B} = \mathbf{0}$.

Definition 8.1 (Reference system). We define a *reference system S* as a set of three axis and one origin over which we have determined an orientation. We will suppose we have selected a unit of length and that in each point a in the immobile space with respect the axis there is a clock q_a such that the clocks q_a and q_b corresponding to two different points a and b immobile with respect these axis are synchronized

Lemma 8.2. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms lines that are not contained in hyperplanes of the form $t = \mathrm{ctt}$ to lines that are not contained in hyperplanes of the form $t' = \mathrm{ctt}$.*

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Theorem 8.5. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f is an affine transformation.*

Theorem 8.6. *Let S, S' be two inertial reference systems. We can make orthogonal changes (isometries) of axis to S and S' and a change of origin of time such thate the Lorentz transformation has the form of the equation* **??**.

Theorem 8.7. *(Lorenz transformation) Let S, S' be two reference systems with the same origin such that S' moves with a constant velocity $\mathbf{v} = v\mathbf{e}_x$. Then,*

$$P_{\mathrm{s}'} = \Lambda P_{\mathrm{s}} \Leftrightarrow P^{\nu}_{\mathrm{s}'} = \Lambda^{\nu}_{\mu} P^{\mu}_{\mathrm{s}}, \qquad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

.

Proposition 8.8. *Let $\mathbf{R} \in L$ be a vector and $a \in \mathbb{R}$ a scalar. Then, $\|\mathbf{R}\|_m = |a|\|\mathbf{R}\|_m$.*

Proposition 8.9. *Every subspace W of L is either timelike, spacelike, or lightlike. Besides,*

1. *S is timelike $\Leftrightarrow W^{\perp}$ is spacelike.*
2. *S is spacelike $\Leftrightarrow W^{\perp}$ is timelike.*
3. *W is lightlike $\Leftrightarrow W^{\perp}$ is lightlike.*

Proposition 8.10. *Two orthogonal vectors different from zero and non spacelike are necessarily lightlike and collinear. In particular, there is not a subspace of dimension 2 where $\langle , \rangle$ is null.*

Proposition 8.11. *Let $\mathbf{R}_1, \mathbf{R}_2 \in T$ be two timelike vectors. Then, the following statements are true.*

1. *$|\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m| \geq \|\mathbf{R}_1\|_m \|\mathbf{R}_2\|_m$, and the equality is equivalent to both vectors being collinear.*
2. *$\mathbf{R}_1, \mathbf{R}_2$ are in the same time cone (C_+ or C_-) if and only if $\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m < 0$. In this case,*
   (a) *There is a unique $\varphi \in \mathbb{R}$ such that*
   $$\cosh\varphi = -\frac{\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m}{\|\mathbf{R}_1\|_m \|\mathbf{R}_2\|_m}. \tag{21}$$
   We call this φ the hyperbolic angle.
   (b) *$\|\mathbf{R}_1\|_m + \|\mathbf{R}_2\|_m \leq \|\mathbf{R}_1 + \mathbf{R}_2\|_m$.*

Proposition 8.12. *The Lorenz-Minkowski metric is invariant under Lorentz transformations.*

Proposition 8.13. *Let S, S' be two inertial reference systems such that the velocity of S' is $\mathbf{w} = w\mathbf{e}_x$ with respect to S. Then,*

$$v^1_{\mathrm{s}'} = \frac{v^1_{\mathrm{s}} - w}{1 - \beta_v \beta_w}, \qquad v^2_{\mathrm{s}'} = \frac{1}{\gamma_w}\frac{v^2_{\mathrm{s}}}{1 - \beta_v \beta_w}, \qquad v^3_{\mathrm{s}'} = \frac{1}{\gamma_w}\frac{v^3_{\mathrm{s}}}{1 - \beta_v \beta_w}, \tag{22}$$

Proposition 8.14. *If the system has a general velocity $\mathbf{w}$, then*

$$\mathbf{v}' = \frac{1}{1 - \langle \boldsymbol{\beta}_v, \boldsymbol{\beta}_2 \rangle_I}\left[\frac{\mathbf{v}}{\gamma_w} - \mathbf{w} + \frac{1}{c^2}\frac{\gamma_w}{1 + \gamma_w}\langle \mathbf{v}, \mathbf{w} \rangle_I \mathbf{w}\right], \tag{23}$$

$$\mathbf{v} = \frac{1}{1 + \langle \boldsymbol{\beta}_{v'}, \boldsymbol{\beta}_w \rangle_I}\left[\frac{\mathbf{v}'}{\gamma_w} + \mathbf{w} + \frac{1}{c^2}\frac{\gamma_w}{1 + \gamma_w}\langle \mathbf{w}', \mathbf{w} \rangle_I \mathbf{v}\right]. \tag{24}$$

Proposition 8.15. *Let S, S' two inertial reference systems. Then,*

$$\mathbf{U}_{\mathrm{s}'} = \boldsymbol{\Lambda}\mathbf{U}_{\mathrm{s}} \Leftrightarrow U^{\nu}_{\mathrm{s}'} = \Lambda^{\nu}_{\mu} U^{\mu}_{\mathrm{s}}. \tag{25}$$

Theorem 8.16.

$$E^1 = p^2c^2 + m^2c^4. \tag{26}$$

Theorem 8.17. *(Work-Energy theorem)*

$$W = \Delta E. \tag{27}$$

9 Generalized coordinates

Definition 9.1. Let S be a system of particles $p_1, \dots, p_n$ with masses $m_1, \cdots, m_n$. Then, we say the system has *non stationary holonomic constraints* or *rheonomic constraints* if and only if there is a function $\mathbf{f} : \mathbb{R}^{3n} \times \mathbb{R} \longrightarrow \mathbb{R}^k$ such that

$$\mathbf{f}(\mathbf{r}_1, \dots, \mathbf{r}_n, t) = \mathbf{0}. \tag{28}$$

Proposition 9.1. *Let S be a system of n particles with a constraint $\mathbf{f} : \mathbb{R}^{3n} \times \mathbb{R} \longrightarrow \mathbb{R}^k$ and V the set of possible velocities at an instant t. If $\dot{\mathbf{x}} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$, then*

$$\sum_{i=1}^{n} \langle \boldsymbol{\nabla}_i f_j, \mathbf{v}_i \rangle_I + \frac{\partial f_j}{\partial t} = 0, \ j = 1, \dots, k. \tag{29}$$

Theorem 9.2 (D'Alembert's principle)**.** *Let S be a system of particles. Then,*

$$\boxed{\sum_{i=1}^{n} \langle \mathbf{F}_i - m\mathbf{a}_i, \delta\mathbf{r}_i \rangle_I = 0, \ \forall\, \delta\mathbf{r}_i\,.} \tag{30}$$

Theorem 9.3. *Let S be a system of n particles with generalized coordinates $q^1, \dots, q^r$. Then,*

$$\boxed{\sum_{\mu=1}^{n} \left\langle \mathbf{F}_\mu, \frac{\partial \mathbf{p}_\mu}{\partial q^\nu} \right\rangle_I = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\nu} - \frac{\partial T}{\partial q^\nu}, \ \nu = 1, \dots, r.} \tag{31}$$

And if $\mathbf{F}$ is derived from a potential $\Phi(\mathbf{r})$, then

$$\boxed{\frac{\partial L}{\partial q^\nu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\nu} = 0, \ \nu = 1, \dots, r.} \tag{32}$$

Theorem 9.4. *Let $J : C^2[x_0, x_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')\,\mathrm{d}x\,,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_1 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are given real numbers. If $y \in S$ is an extremal for J, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{33}$$

for all $x \in [a_0, x_1]$.

Theorem 9.5 (Lagrange multipliers method for non-holonomic constraints)**.** *If we want to find an extrema having a set of m non-holonomic constraints*

$$\begin{aligned} \overline{\delta f_1} &= A_{11}\delta u_1 + \cdots + A_{1n}\delta u_n &= 0, \\ \vdots &= \ddots &= \vdots \\ \overline{\delta f_m} &= A_{m1}\delta u_1 + \cdots + A_{mn}\delta u_n &= 0, \end{aligned} \tag{34}$$

Then we can proceed as the original multipliers method by considering every variation as independent and solving the following equation.

$$\delta F + \lambda_1 \overline{\delta f_1} + \cdots + \lambda_m \overline{\delta f_m} = 0 \tag{35}$$

Theorem 9.6. *Let f be a continuous functions with a variation $\delta f = \epsilon\phi$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta y = \delta\frac{\mathrm{d}}{\mathrm{d}x}y, \qquad \delta\int_a^b f(x)\,\mathrm{d}x = \int_a^b \delta f(x)\,\mathrm{d}x\,. \tag{36}$$

Theorem 9.7. *Let $J : \mathbf{C}^2[t_0, t_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$J(\mathbf{q}) = \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}, t)\,\mathrm{d}t\,, \tag{37}$$

where $\mathbf{q} = (q_1, \ldots, q_n)$, and L has continuous second-order partial derivatives with respect to t, q_k, and $\dot{q}_k$, $k = 1, \ldots, n$. Let

$$S = \left\{\mathbf{q} \in \mathbf{C}^2[t_0, t_1] \,\middle|\, \mathbf{q}(t_0) = \mathbf{q}_0, \mathbf{q}(t_1) = \mathbf{q}_1\right\}, \tag{38}$$

where $\mathbf{q}_0, \mathbf{q}_1 \in \mathbb{R}^n$ are given vectors. If $\mathbf{q}$ is an extremal for J in S then for $k = 1, \ldots, n$

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} = 0.} \tag{39}$$

Theorem 9.8. *If we have a set of holonomic constraints*

$$\begin{aligned} f_1(q_1, \ldots, q_n, t) &= 0, \\ \ldots &= \vdots, \\ f_m(q_1, \ldots, q_n, t) &= 0, \end{aligned} \tag{40}$$

then we can treat each variable as independent and search the stationary value of

$$J' = \int_{t_1}^{t_2} L + \sum_{k=1}^{m} \lambda_m f_m\,\mathrm{d}t\,, \tag{41}$$

which leads to the equation

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} + \lambda_1\frac{\partial f_1}{\partial q_k} + \cdots + \lambda_m\frac{\partial f_m}{\partial q_k} = 0.} \tag{42}$$