Contents

Chapter 1

Fourier series

1.1 Preliminaries

Definition 1.1.1. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We say *f has a period T* or *f is T-periodic* with $T > 0$ if and only if $f(x+T) = f(x)$ for all $x \in \mathbb{R}$.

Lemma 1.1.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function. Then, $f(x+T') = f(x)$ for all $x \in \mathbb{R}$ if and only if $T' = kT$ for some $k \in \mathbb{Z}$.*

Proposition 1.1.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function and $a \in \mathbb{R}$ a number. Then,*

$$\int_a^{a+T} f \, \mathrm{d}x = \int_0^T f \, \mathrm{d}x \, . \tag{1.1}$$

Lemma 1.1.3. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function continuous in $\mathbb{R}$. Then, $|f|$ is bounded.*

Proposition 1.1.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function. Then, there is no a power series that converges uniformly to f in $\mathbb{R}$.*

1.2 Orthogonal and orthonormal systems of functions

Definition 1.2.1. Let $f, g : [a,b] \longrightarrow \mathbb{C}$ be two integrable functions. We define the *inner product of f and g* as

$$\langle f, g \rangle_2 \coloneqq \int_a^b f(x)\overline{g(x)} \, \mathrm{d}x \, , \tag{1.2}$$

where $\overline{g}$ is the conjugate of g.

Definition 1.2.2. Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two integrable functions. We define the *2-norm of f* as

$$\|f\|_2 \coloneqq \sqrt{\langle f, f \rangle_2} = \left(\int_a^b |f(x)|^2 \, \mathrm{d}x \right)^{1/2} . \tag{1.3}$$

Definition 1.2.3. Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two integrable functions. We define the *distance between r and g* as

$$d(f,g) \coloneqq \|f - g\|. \tag{1.4}$$

Proposition 1.2.1. *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two integrable functions. Then,*

1. *$\langle f, f \rangle_2 \geq 0$.*
2. *$\langle f+g, h \rangle_2 = \langle f, h \rangle_2 + \langle g, h \rangle_2$ and $\langle f, g+h \rangle_2 = \langle f, g \rangle_2 + \langle f, h \rangle_2$.*
3. *$\langle f, g \rangle_2 = \overline{\langle g, f \rangle_2}$.*
4. *Fro $\alpha \in \mathbb{C}$, $\langle \alpha f, g \rangle_2 = \alpha \langle f, g \rangle_2$ and $\langle f, \alpha g \rangle_2 = \overline{\alpha} \langle f, g \rangle_2$.*

Note that for Riemann integrable functions it is not true that $\langle f, f \rangle_2 = 0 \Leftrightarrow f = 0$, since a function that is not zero at some point will satisfy $\langle f, f \rangle_2 = 0$. However, if we deal with the space of continuous functions in $[a,b]$ the it is true.

Theorem 1.2.2. *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two Riemann integrable functions. Then,*

$$|\langle f, g \rangle_2| \leq \|f\|_2 \|g\|_2. \tag{1.5}$$

Theorem 1.2.3. *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two Riemann integrable functions. Then,*

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2. \tag{1.6}$$

Definition 1.2.4. Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two Riemann integrable functions with $f \neq g$. Then,

1. we say *f and g are orthogonal* if and only if $\langle f, g \rangle_2 = 0$.

2. We say f *and* g *are orthonormal* if and only if $\langle f, g \rangle_2 = 0$ and $\|f\|_2 = \|g\|_2 = 1$.

3. Let $S = \{\phi_0, \phi_1, \dots\}$ be a collection of Riemann integrable functions in $[a, b]$. We say S *is an orthonormal system* if and only if $\|\phi_i\|_2 = 1 \forall i$ and $\langle \phi_i, \phi_j \rangle_2 = 0 \forall i \neq j$.

Definition 1.2.5. Let $\{\phi_0, \dots, \phi_n\}$ be a collection of Riemann integrable functions in $[a, b]$. We say *the collection is linearly dependent in* $[a, b]$ if and only if there exist $c_0, \dots, c_n$ with not all being zero such that

$$c_0\phi_0(x) + \cdots + c_n\phi_n(x) = 0, \forall x \in [a, b]. \tag{1.7}$$

Theorem 1.2.4. *Let $S = \{\phi_0, \phi_1, \dots\}$ be an orthonormal system in $[a, b]$ such that $\sum_{n=0}^{\infty} c_n\phi_n(x)$ converges uniformly in $[a, b]$. Let f be the function that defines the series in $[a, b]$. Thenm f is Riemann integrable in $[a, b]$ and*

$$c_n = \langle f, \phi_n \rangle_2 = \int_a^b f(x)\overline{\phi_n(x)} \, \mathrm{d}x \, , n \geq 0. \tag{1.8}$$

1.3 Fourier coefficients. Fourier series

1.3.1 Exponential form

Definition 1.3.1. Let f be a Riemann integrable function in $[0, 1]$. We define the *n-th Fourier coefficient* with $n \in \mathbb{Z}$ as

$$\hat{f}(n) = \langle f, e_n \rangle_2 = \int_0^1 f(x)\mathrm{e}^{-2\pi i n x} \, \mathrm{d}x \, . \tag{1.9}$$

The series

$$Sf(x) = \sum_{n=0}^{\infty} \hat{f}(x)\mathrm{e}^{2\pi i n x} \tag{1.10}$$

constructed by these coefficients is called the *Fourier series of f*.

Proposition 1.3.1. *In relation to Fourier coefficients, the following statements are true.*

1. $\widehat{\lambda f + \mu g}(n) = \lambda\hat{f}(b) + \mu\hat{g}(n)$.

2. *If $\tau \in (0, 1)$ and $f_\tau(x) \coloneqq f(x - \tau)$, then $\hat{f}_\tau(n) = \mathrm{e}^{-2\pi i n \tau}\hat{f}(n)$.*

3. *If f is even, then $\hat{f}(n) = \hat{f}(-n) \forall n$ and if f is odd, $\hat{f}(n) = -\hat{f}(-n) \forall n$.*

4. *If f' exists and it is continuous, then $\hat{f}'(n) = 2\pi i n\hat{f}(n)$.*