Contents

Chapter 1

Numeric series

1.1 Series convergence

Definition 1.1.1. Let (a_n) be a sequence of real numbers. A *numeric series* is an expression of the form

$$\sum_{n=1}^{\infty} a_n.$$

We call a_n *general term of the series* and $S_N = \sum_{n=1}^{N} a_n$, for all $N \in \mathbb{N}$, *N-th partial sum of the series*[1].

Definition 1.1.2. We say the series $\sum a_n$ is *convergent* if the sequence of partial sums is convergent, that is, if $S = \lim_{N\to\infty} S_N$ exist and it's finite. In that case, S is called the *sum of the series.* If the previous limit doesn't exists or it is infinite we say the series is *divergent*[2].

Proposition 1.1.1. *Let (a_n) be a sequence such that $\sum a_n < \infty$. Then $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ such that*

$$\left|\sum_{n=1}^{N} a_n - \sum_{n=1}^{\infty} a_n\right| < \varepsilon$$

if $N \geq n_0$.

Theorem 1.1.2 (Cauchy's test)**.** *Let (a_n) be a sequence. $\sum a_n < \infty$ if and only if $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ such that*

$$\left|\sum_{n=N}^{M} a_n\right| < \varepsilon$$

if $M \geq N \geq n_0$.

Corollary 1.1.3. *Changing a finite number of terms in a series has no effect on the convergence or divergence of the series.*

Corollary 1.1.4. *If $\sum a_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.*

Theorem 1.1.5 (Linearity)**.** *Let $\sum a_n, \sum b_n$ be two convergent series with sums A and B respectively and let λ be a real number. The series*

$$\sum_{n=1}^{\infty} (a_n + \lambda b_n)$$

is convergent and has sum $A + \lambda B$.

Theorem 1.1.6 (Associative property)**.** *Let $\sum a_n$ be a convergent series with sum A. Suppose (n_k) is a strictly increasing sequence of natural numbers. The series $\sum b_n$, with $b_k = a_{n_{k-1}+1} + \cdots + a_{n_k}$ for all $i \in \mathbb{N}$, is convergent and its sum is A.*

1.2 Non-negative terms series

Theorem 1.2.1. *Let $\sum a_n$ be a series of non-negative terms $a_n \geq 0$*[3]*. The series converges if and only if the sequence (S_N) of partial sums is bounded.*

Theorem 1.2.2 (Comparison test)**.** *Let $(a_n), (b_n) \geq 0$ be two sequences of real numbers. Suppose that exists a constant $C > 0$ and a number $n_0 \in \mathbb{N}$ such that $a_n \leq C b_n$ for all $n \geq n_0$.*

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[1] From now on we will write $\sum a_n$ to refer $\sum_{n=1}^{\infty} a_n$.

[2] We will use the notation $\sum a_n < \infty$ or $\sum a_n = +\infty$ to express that the series converges or diverges, respectively.

[3] Obviously the following results are also valid if the series is of non-positive terms or has a finite number of negative or positive terms.

1. *If* $\sum b_n < \infty \implies \sum a_n < \infty$.
2. *If* $\sum a_n = +\infty \implies \sum b_n = +\infty$.

Theorem 1.2.3 (Limit comparison test). *Let* (a_n), $(b_n) \geq 0$ *be two sequences of real numbers. Suppose that the limit* $\ell = \lim_{n\to\infty} \frac{a_n}{b_n}$ *exists.*

1. *If* $0 < \ell < \infty \implies \sum a_n < \infty \iff \sum b_n < \infty$.
2. *If* $\ell = 0$ *and* $\sum b_n < \infty \implies \sum a_n < \infty$.
3. *If* $\ell = \infty$ *and* $\sum a_n < \infty \implies \sum b_n < \infty$.

Theorem 1.2.4 (Root test). *Let* $(a_n) \geq 0$. *Suppose that the limit* $\ell = \lim_{n\to\infty} \sqrt[n]{a_n}$ *exists.*

1. *If* $\ell < 1 \implies \sum a_n < \infty$.
2. *If* $\ell > 1 \implies \sum a_n = +\infty$.

Theorem 1.2.5 (Ratio test). *Let* $(a_n) \geq 0$. *Suppose that the limit* $\ell = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ *exists.*

1. *If* $\ell < 1 \implies \sum a_n < \infty$.
2. *If* $\ell > 1 \implies \sum a_n = +\infty$.

Theorem 1.2.6 (Raabe's test). *Let* $(a_n) \geq 0$. *Suppose that the limit* $\ell = \lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right)$ *exists.*

1. *If* $\ell > 1 \implies \sum a_n < \infty$.
2. *If* $\ell < 1 \implies \sum a_n = +\infty$.

Theorem 1.2.7 (Condensation test). *Let* $(a_n) \geq 0$ *be a decreasing sequence. Then:*

$$\sum a_n < \infty \iff \sum 2^n a_{2^n} < \infty.$$

Theorem 1.2.8 (Logarithmic test). *Let* $(a_n) \geq 0$. *Suppose that the limit* $\ell = \lim_{n\to\infty} \frac{\log \frac{1}{a_n}}{\log n}$ *exists.*

1. *If* $\ell > 1 \implies \sum a_n < \infty$.
2. *If* $\ell < 1 \implies \sum a_n = +\infty$.

Theorem 1.2.9 (Integral test). *Let* $f : [1, \infty) \to (0, \infty)$ *be a decreasing function. Then:*

$$\sum f(n) < \infty \iff$$

$$\iff \exists C > 0 \text{ such that } \int_1^n f(x)dx \leq C \ \forall n.$$

4

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[4] Later we will see that this is equivalent to say that if $f : [1, \infty) \to (0, \infty)$ is a locally integrable decreasing function, then:

$$\sum f(n) < \infty \iff \int_1^\infty f(x)dx < \infty.$$

1.3 Alternating series

Definition 1.3.1. An *alternating series* is a series of the form $\sum(-1)^n a_n$, with $(a_n) \geq 0$.

Theorem 1.3.1 (Leibnitz's test)**.** *Let $(a_n) \geq 0$ be a decreasing sequence such that $\lim_{n\to\infty} a_n = 0$. Then, $\sum(-1)^n a_n$ is convergent.*

Theorem 1.3.2 (Abel's summation formula)**.** *Let $(a_n), (b_n)$ be two sequences of real numbers. Then,*

$$\sum_{n=N}^{M} a_n(b_{n+1} - b_n) = a_{M+1}b_{M+1} - a_N b_N - \\ - \sum_{n=N}^{M} b_{n+1}(a_{n+1} - a_n).$$

Theorem 1.3.3 (Dirichlet's test)**.** *Let $(a_n), (b_n)$ be two sequences of real numbers such that:*

1. *$\exists C > 0$ such that $\left|\sum_{n=1}^{N} a_n\right| \leq C$ for all $N \in \mathbb{N}$.*
2. *(b_n) is monotone and $\lim_{n\to\infty} b_n = 0$.*

Then, $\sum a_n b_n$ is convergent.

Theorem 1.3.4 (Abel's test)**.** *Let $(a_n), (b_n)$ be two sequences of real numbers such that:*

1. *The series $\sum a_n$ is convergent.*
2. *(b_n) is monotone and bounded.*

Then, $\sum a_n b_n$ is convergent.

1.4 Absolute convergence and rearrangement of series

Definition 1.4.1. We say a series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent.

Theorem 1.4.1. *If a series converges absolutely, it converges.*

Definition 1.4.2. We say a sequence (b_n) is a *rearrangement of the sequence* (a_n) if exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\sigma(n)}$. A *rearrangement of the series* $\sum a_n$ is the series $\sum a_{\sigma(n)}$ for some bijection $\sigma : \mathbb{N} \to \mathbb{N}$.

Definition 1.4.3. Let $x \in \mathbb{R}$. We define the *positive part of* x as

$$x^+ = \begin{cases} x & \text{si } x \geq 0 \\ 0 & \text{si } x < 0 \end{cases}$$

Analogously, we define the *negative part of* x as

$$x^- = \begin{cases} 0 & \text{si } x \geq 0 \\ -x & \text{si } x < 0 \end{cases}$$

Note that we can write $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

Theorem 1.4.2. *A series $\sum a_n$ is absolutely convergent if and only if positive and negative terms series, $\sum a_n^+$ and $\sum a_n^-$, converge. In this case,*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-.$$

Theorem 1.4.3. *Let $\sum a_n$ be an absolutely convergent series. Then, for all bijection $\sigma : \mathbb{N} \to \mathbb{N}$, the rearranged series $\sum a_{\sigma(n)}$ is absolutely convergent and $\sum a_n = \sum a_{\sigma(n)}$.*

Theorem 1.4.4 (Riemann's theorem)**.** *Let $\sum a_n$ be a convergent series but not absolutely convergent. Then, $\forall \alpha \in \mathbb{R} \cup \{\infty\}$, there exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum a_{\sigma(n)}$ converges and $\sum a_{\sigma(n)} = \alpha$.*

Theorem 1.4.5. *A series $\sum a_n$ is absolutely convergent if and only if any rearranged series converges to the same value of $\sum a_n$.*

Chapter 2

Sequences and series of functions

2.1 Sequences of functions

Definition 2.1.1. Let $D \subseteq \mathbb{R}$. A set

$$(f_n(x)) = \{f_1(x), f_2(x), \dots, f_n(x), \dots\}$$

is a *sequence of real functions* if $f_i : D \to \mathbb{R}$ is a real-valued function. In this case we say the sequence $(f_n(x))$, or simply (f_n), is well-defined on D.

Definition 2.1.2. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say (f_n) *converges pointwise to* f *on* D if $\forall x \in D$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$

Definition 2.1.3. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say (f_n) *converges uniformly to* f *on* D if $\forall \varepsilon > 0$, $\exists n_0 : |f_n(x) - f(x)| < \varepsilon$ $\forall n \geq n_0$ and $\forall x \in D$.

Lemma 2.1.1. *Let (f_n) be an uniform convergent sequence of functions defined on $D \subseteq \mathbb{R}$ and let f be a function such that (f_n) converges pointwise to f. Then, (f_n) converges uniformly f on D.*

Lemma 2.1.2. *Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$. (f_n) converges uniformly a f en D if and only if $\lim\limits_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0$.*

Corollary 2.1.3. *A sequence of functions (f_n) converges uniformly to f on $D \subseteq \mathbb{R}$ if and only if there is a sequence (a_n), with $a_n \geq 0$ and $\lim\limits_{n\to\infty} a_n = 0$, and a number $n_0 \in \mathbb{N}$ such that $\sup\{|f_n(x) - f(x)| : x \in D\} \leq a_n$, $\forall n \geq n_0$.*

Theorem 2.1.4 (Cauchy's test)**.** *A sequence of functions (f_n) converges uniformly to f on $D \subseteq \mathbb{R}$ if and only if $\forall \varepsilon > 0$ $\exists n_0 : \sup\{|f_n(x) - f_m(x)| : x \in D\} < \varepsilon$ if $n, m \geq n_0$.*

Theorem 2.1.5. *Let (f_n) be a sequence of continuous functions defined on $D \subseteq \mathbb{R}$. If (f_n) converges uniformly to f on D, then f is continuous on D, that is, for any $x_0 \in D$, it satisfies:*

$$\lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right) = \lim_{x\to x_0} f(x).$$

Theorem 2.1.6. *Let (f_n) be a sequence of functions defined on $I = [a, b] \subseteq \mathbb{R}$. If (f_n) are Riemann-integrable on I and (f_n) converges uniformly to f on I, then f is Riemann-integrable on I and*

$$\int_a^b \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Theorem 2.1.7. *Let (f_n) be a sequence of functions defined on $I = (a, b) \subset \mathbb{R}$. If (f_n) are derivable on I, $(f_n'(x))$ converges uniformly on I and $\exists x_0 \in I : \lim\limits_{n\to\infty} f_n(x_0) \in \mathbb{R}$, then there is a function f such that (f_n) converges uniformly to f on I, f is derivable on I and $(f_n'(x))$ converges uniformly to f' on I.*

2.2 Series of functions

Definition 2.2.1. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$. The expression

$$\sum_{n=1}^{\infty} f_n(x)$$

is the *series of functions associated with* (f_n).

Definition 2.2.2. A series of functions $\sum f_n(x)$ defined on $D \subseteq \mathbb{R}$ *converges pointwise on* D if the sequence of partials sums

$$F_N(x) = \sum_{n=1}^{N} f_n(x)$$

converges pointwise. If the pointwise limit of (F_N) is $F(x)$, we say F is the *sum of the series in a pointwise sense.*

Definition 2.2.3. A series of functions $\sum f_n(x)$ defined on $D \subseteq \mathbb{R}$ *converges uniformly on* D if the sequence of partials sums

$$F_N(x) = \sum_{n=1}^{N} f_n(x)$$

converges uniformly. If the uniform limit of (F_N) is $F(x)$, we say F is the *sum of the series in an uniform sense.*

Theorem 2.2.1 (Cauchy's test)**.** *A series of functions* $\sum f_n(x)$ *defined on* $D \subseteq \mathbb{R}$ *converges uniformly if and only if* $\forall \varepsilon > 0\ \exists n_0$ *such that*

$$\sup\left\{\left|\sum_{n=N}^{M} f_n(x)\right| : x \in D\right\} < \varepsilon$$

if $M \geq N \geq n_0$*.*

Corollary 2.2.2. *If* $\sum f_n(x)$ *is a series of continuous functions on* $D \subseteq \mathbb{R}$*, then* (f_n) *converges uniformly to zero on* D*.*

Theorem 2.2.3. *If* $\sum f_n(x)$ *is uniformly convergent series of functions on* $D \subseteq \mathbb{R}$*, then its sum function is also continuous on* D*.*

Theorem 2.2.4. *Let* (f_n) *be a sequence of functions defined on* $I = [a, b] \subseteq \mathbb{R}$*. If* (f_n) *are Riemann-integrable on* I *and* $\sum f_n(x)$ *converges uniformly on* I*, then* $\sum f_n(x)$ *is Riemann-integrable on* I *and*

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$

Theorem 2.2.5. *Let* (f_n) *be a sequence of functions defined on* $I = (a, b) \subset \mathbb{R}$*. If* (f_n) *are derivable on* I*,* $\sum f_n'(x)$ *converges uniformly on* I *and* $\exists c \in I : \sum f_n(c) < \infty$*, then* $\sum f_n(x)$ *converges uniformly on* I*,* $\sum f_n(x)$ *is derivable on* I *and*

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x).$$

Theorem 2.2.6 (Weierstraß M-test)**.** *Let* (f_n) *be a sequence of functions defined on* $D \subseteq \mathbb{R}$ *such that* $|f_n(x)| \leq M_n\ \forall x \in D$ *and suppose that* $\sum M_n$ *is a convergent numeric series. Then,* $\sum f_n(x)$ *is converges uniformly on* D*.*

Theorem 2.2.7 (Dirichlet's test)**.** *Let* $(f_n), (g_n)$ *be two sequences of functions defined on* $D \subseteq \mathbb{R}$*. Suppose:*

1. $\exists C > 0 : \sup\left\{\left|\sum_{n=1}^{N} f_n(x)\right| : x \in D\right\} \leq C,\ \forall N.$

2. $(g_n(x))$ *is a monotone sequence for all* $x \in D$ *and* $\lim_{n\to\infty} \sup\{|g_n(x)| : x \in D\} = 0$*.*

Then, $\sum f_n(x)g_n(x)$ *converges uniformly on* D*.*

Theorem 2.2.8 (Abel's test)**.** *Let* $(f_n), (g_n)$ *be two sequences of functions defined on* $D \subseteq \mathbb{R}$*. Suppose:*

1. *The series* $\sum f_n(x)$ *converges uniformly on* D*.*

2. $(g_n(x))$ *is a monotone and bounded sequence for all* $x \in D$*.*

Then, $\sum f_n(x)g_n(x)$ *converges uniformly on* D*.*

2.3 Power series

Definition 2.3.1. Let (a_n) be a sequence of real numbers and $x_0 \in \mathbb{R}$. A *power series centred on* x_0 is a series of functions of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Proposition 2.3.1. *Let $\sum a_n(x-x_0)^n$ be a power series. Suppose there exists an $x_1 \in \mathbb{R}$ such that $\sum a_n(x_1 - x_0)^n < \infty$. Then, $\sum a_n(x-x_0)^n$ converges uniformly on any closed interval $I \subset A = \{x \in \mathbb{R} : |x - x_0| < |x_1 - x_0|\}$.*

Theorem 2.3.2. *Let $\sum a_n(x-x_0)^n$ be a power series and consider*

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|} \right)^{-1} \in [0, \infty].$$

Then:

1. *If $|x - x_0| < R \implies \sum a_n(x-x_0)^n$ converges absolutely.*
2. *If $0 \le r < R \implies \sum a_n(x-x_0)^n$ converges uniformly on $[x_0 - r, x_0 + r]$.*
3. *If $|x - x_0| > R \implies \sum a_n(x-x_0)^n$ diverges.*

The number R is called radius of convergence of the power series.

Theorem 2.3.3 (Abel's theorem)**.** *Let $\sum a_n x^n$ be a power series[1] with radius of convergence R satisfying $\sum a_n R^n < \infty$. Then the series $\sum a_n x^n$ converges uniformly on $[0, R]$. In particular, if $f(x) = \sum a_n x^n$,*

$$\lim_{x\to R^-} f(x) = \sum_{n=0}^{\infty} a_n R^n.$$

Corollary 2.3.4. *Let f be the sum function of a power series $\sum a_n x^n$. Then f is continuous on the domain of convergence of the series.*

Corollary 2.3.5. *If the series $\sum a_n x^n$ has radius of convergence $R \neq 0$ and f is its sum function, then f is Riemann-integrable on any closed subinterval on the domain of convergence of the series. In particular, for $|x| < R$,*

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}\text{[2]}.$$

Corollary 2.3.6. *Let f be the sum function of the power series $\sum a_n x^n$. Then f is derivable within the domain of convergence of the series and*

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

Corollary 2.3.7. *Any function f defined as a sum of a power series $\sum a_n x^n$ is indefinitely derivable within the domain of convergence of the series and*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k},$$

for all $k \in \mathbb{N} \cup \{0\}$. In particular $f^{(k)}(0) = k! a_k$.

Definition 2.3.2. A function is *analytic* if it can be expressed locally as a power series.

[1] From now on we will suppose, for simplicity, $x_0 = 0$.

[2] The formula is also valid for $|x| = R$ if the series $\sum a_n R^n$ (or $\sum a_n(-R)^n$) is convergent.

2.4 Stone-Weierstraß approximation theorem

Definition 2.4.1. Let f be a real-valued function. We say f has *compact support*[3] if exists an $M > 0$ such that $f(x) = 0$ for all $x \in \mathbb{R} \setminus [-M, M]$.

Definition 2.4.2. Let f, g be real-valued functions with compact support. We define the convolution of f with g as

$$(f * g)(x) = \int_{\mathbb{R}} f(t)g(x-t)dt^4.$$

Definition 2.4.3. We sap a sequence of functions (ϕ_ε) with compact support is an *approximation of unity* if

1. $\phi_\varepsilon \geq 0$.
2. $\displaystyle\int_{\mathbb{R}} \phi_\varepsilon = 1$.
3. For all $\delta > 0$, $\phi_\varepsilon(t)$ converges uniformly to zero when $\varepsilon \to 0$ if $|t| > \delta$.

Lemma 2.4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. Let (ϕ_ε) be an approximation of unity. Then $(f * \phi_\varepsilon)$ converges uniformly to f on $\mathbb{R}$ when $\varepsilon \to 0$.*

Theorem 2.4.2 (Stone-Weierstraß theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists polynomials $p_n \in \mathbb{R}[x]$ such that the sequence (p_n) converge uniformly to f on $[a, b]$.*

[3] In general, the support of a function is the adherence of the set of points which are not mapped to zero.

[4] Alternatively if f, g are Riemann-integrable functions on $[a, b]$ we can define the convolution of f and g as

$$(f * g)(x) = \int_a^b f(t)g(x-t)dt.$$

Chapter 3

Improper integrals

3.1 Locally integrable functions

Definition 3.1.1. Let $f : [a, b) \to \mathbb{R}$, with $b \in \mathbb{R} \cup \{\infty\}$. We say f is *locally integrable on* $[a, b)$ if f is Riemann-integrable on $[a, x]$ for all $a \leq x < b$.

Definition 3.1.2. Let $f : [a, b) \to \mathbb{R}$ be a locally integrable function. If there exists

$$\lim_{x \to b^-} \int_a^x f$$

and it's finite, we say that the *improper integral of* f *on* $[a, b)$, $\int_a^b f$, is *convergent.*

Theorem 3.1.1 (Cauchy's test). *Let $f : [a, b) \to \mathbb{R}$ be a locally integrable function. The improper integral $\int_a^b f$ is convergent if and only if $\forall \varepsilon > 0$ $\exists b_0$, $a < b_0 < b$, such that*

$$\left| \int_{x_1}^{x_2} f \right| < \varepsilon$$

if $b_0 < x_1 < x_2 < b$.

3.2 Improper integrals of non-negative functions

Theorem 3.2.1. *Let $f : [a, b) \to \mathbb{R}$ be a locally integrable non-negative function. A necessary and sufficient condition for $\int_a^b f$ to be convergent is that the function*

$$F(x) = \int_a^x f(t)dt$$

must be bounded for all $x < b$.

Theorem 3.2.2 (Comparison test). *Let $f, g : [a, b) \to [0, +\infty)$ be two locally integrable non-negative functions. Then:*

1. *If $\exists C > 0$ such that $f(x) \leq Cg(x)$ $\forall x$ on a neighborhood of b and $\int_a^b g < \infty \implies \int_a^b f < \infty$.*

2. *Suppose the limit $\ell = \lim_{x \to b} \frac{f(x)}{g(x)}$ exists. Then,*

   (a) *If $\ell \in (0, \infty) \implies \int_a^b f < \infty \iff \int_a^b g < \infty$.*

   (b) *If $\ell = 0$ and $\int_a^b g < \infty \implies \int_a^b f < \infty$.*

   (c) *If $\ell = \infty$ and $\int_a^b f < \infty \implies \int_a^b g < \infty$.*

3.3 Absolute convergence of improper integrals

Definition 3.3.1. Let $f : [a, b) \to (0, \infty)$ be a locally integrable function. We say $\int_a^b f$ *converges absolutely* if $\int_a^b |f|$ is convergent.

Theorem 3.3.1 (Dirichlet's test)**.** *Let $f, g : [a, b) \to \mathbb{R}$ be two locally integrable functions Suppose:*

1. $\exists C > 0$ *such that* $\left|\int_a^x f(t)dt\right| \leq C$ *for all* $x \in [a, b)$.
2. g *is monotone and* $\lim_{x\to b} g(x) = 0$.

Then, $\int_a^b fg$ *is convergent.*

Theorem 3.3.2 (Abel's test)**.** *Let $f, g : [a, b) \to \mathbb{R}$ be two locally integrable functions. Suppose:*

1. $\int_a^b f$ *is convergent.*
2. g *is monotone and bounded.*

Then, $\int_a^b fg$ *is convergent.*

3.4 Differentiation under integral sign

Theorem 3.4.1. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b] \times [c, d]$. Consider the function $F(y) = \int_a^b f(x, y)dx$ defined on $[c, d]$. Then, F is continuous, that is, if $c < y_0 < d$,*

$$\lim_{y\to y_0} F(y) = \lim_{y\to y_0} \int_a^b f(x, y)dx = \int_a^b \lim_{y\to y_0} f(x, y)dx =$$

$$= \int_a^b f(x, y_0)dx = F(y_0).$$

Theorem 3.4.2. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a Riemann-integrable function and let $F(y) = \int_a^b f(x, y)dx$. If f is differentiable with respect to y and $\partial f/\partial y$ is continuous on $[a, b] \times [c, d]$, then $F(y)$ is derivable on (c, d) and its derivative is*

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)dx,$$

for all $y \in (c, d)$.

Theorem 3.4.3. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b] \times [c, d]$. Let $a, b : [c, d] \to \mathbb{R}$ be to differentiable functions satisfying $a \le a(y) \le b(y) \le b$ for every $y \in [c, d]$. Suppose that $\partial f/\partial y$ is continuous on $\{(x, y) \in \mathbb{R}^2 : a(y) \le x \le b(y),\ c \le y \le d\}$. Then $F(y) = \int\limits_{a(y)}^{b(y)} f(x, y)dx$ is derivable on (c, d) and its derivative is*

$$F'(y) = b'(y)f(b(y), y) - a'(y)f(a(y), y)+ \\ + \int\limits_{a(y)}^{b(y)} \frac{\partial f}{\partial y}(x, y)dx,$$

for all $y \in (c, d)$.

Theorem 3.4.4. *Let $f : [a, b) \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b) \times [c, d]$. We consider $F(y) = \int\limits_a^b f(x, y)dx$. Suppose that:*

1. *$\dfrac{\partial f}{\partial y}$ is continuous on $[a, b) \times [c, d]$.*
2. *Given $y_0 \in [c, d]$, $\exists \delta > 0$ such that the integral*

$$\int\limits_a^b \sup\left\{\left|\frac{\partial f}{\partial y}(x, y)\right| : y \in (y_0 - \delta, y_0 + \delta)\right\} dx$$

exists and it's finite on $[a, b)$.

Then, $F(y)$ is derivable at y_0 and

$$F'(y_0) = \int\limits_a^b \frac{\partial f}{\partial y}(x, y_0)dx.$$

Theorem 3.4.5. *Let $f : [a, b) \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b) \times [c, d]$. Let $a, b : [c, d] \to \mathbb{R}$ be two differentiable functions satisfying $a \le a(y) \le b(y) \le b$ for every $y \in [c, d]$. We consider $F(y) = \int\limits_{a(y)}^{b(y)} f(x, y)dx$. Suppose that:*

1. *$\dfrac{\partial f}{\partial y}$ is continuous on $\{(x, y) \in \mathbb{R}^2 : a(y) \le x \le b(y),\ c \le y \le d\}$.*
2. *Given $y_0 \in [c, d]$, $\exists \delta > 0$ such that the integral*

$$\int\limits_{a(y)}^{b(y)} \sup\left\{\left|\frac{\partial f}{\partial y}(x, y)\right| : y \in (y_0 - \delta, y_0 + \delta)\right\} dx$$

exists and it's finite on $[a, b)$.

The, $F(y)$ is derivable at y_0 and

$$F'(y_0) = b'(y_0)f(b(y_0), y_0) - a'(y_0)f(a(y_0), y_0)+ \\ + \int\limits_{a(y_0)}^{b(y_0)} \frac{\partial f}{\partial y}(x, y_0)dx.$$

3.5 Gamma function

Definition 3.5.1. For $x > 0$, *Gamma function* is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Theorem 3.5.1. *Gamma function is a generalization of the factorial. In fact, for $x > 0$ we have*

$$\Gamma(x+1) = x\Gamma(x).$$

In particular, $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

Theorem 3.5.2. *Gamma function satisfies:*

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x \sqrt{2\pi x}} = 1.$$

Corollary 3.5.3 (Stirling's formula)**.**

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.$$

Chapter 4

Fourier series

4.1 Preliminaries

Definition 4.1.1. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We say f *has a period* T or f *is* T*-periodic* with $T > 0$ if and only if $f(x+T) = f(x)$ for all $x \in \mathbb{R}$.

Lemma 4.1.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function. Then, $f(x+T') = f(x)$ for all $x \in \mathbb{R}$ if and only if $T' = kT$ for some $k \in \mathbb{Z}$.*

Proposition 4.1.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function and $a \in \mathbb{R}$ a number. Then,*

$$\int_{a}^{a+T} f \,\mathrm{d}x = \int_{0}^{T} f \,\mathrm{d}x \,. \tag{4.1}$$

Lemma 4.1.3. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function continuous in $\mathbb{R}$. Then, $|f|$ is bounded.*

Proposition 4.1.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function. Then, there is no a power series that converges uniformly to f in $\mathbb{R}$.*

4.2 Orthogonal and orthonormal systems of functions

Definition 4.2.1. Let $f, g : [a,b] \longrightarrow \mathbb{C}$ be two integrable functions. We define the *inner product of f and g* as

$$\langle f, g\rangle_2 \coloneqq \int_{a}^{b} f(x)\overline{g(x)} \,\mathrm{d}x \,, \tag{4.2}$$

where $\overline{g}$ is the conjugate of g.

Definition 4.2.2. Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two integrable functions. We define the *2-norm of f* as

$$\|f\|_2 \coloneqq \sqrt{\langle f, f\rangle_2} = \left(\int_{a}^{b} |f(x)|^2 \,\mathrm{d}x\right)^{1/2} . \tag{4.3}$$

Definition 4.2.3. Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two integrable functions. We define the *distance between r and g* as

$$d(f,g) \coloneqq \|f - g\|. \tag{4.4}$$

Proposition 4.2.1. *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two integrable functions. Then,*

1. $\langle f, f\rangle_2 \geq 0$.
2. $\langle f+g, h\rangle_2 = \langle f, h\rangle_2 + \langle g, h\rangle_2$ *and* $\langle f, g+h\rangle_2 = \langle f, g\rangle_2 + \langle f, h\rangle_2$.
3. $\langle f, g\rangle_2 = \overline{\langle g, f\rangle_2}$.
4. *Fro* $\alpha \in \mathbb{C}$, $\langle \alpha f, g\rangle_2 = \alpha\langle f, g\rangle_2$ *and* $\langle f, \alpha g\rangle_2 = \overline{\alpha}\langle f, g\rangle_2$.

Note that for Riemann integrable functions it is not true that $\langle f, f\rangle_2 = 0 \Leftrightarrow f = 0$, since a function that is not zero at some point will satisfy $\langle f, f\rangle_2 = 0$. However, if we deal with the space of continuous functions in $[a,b]$ the it is true.

Theorem 4.2.2. *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two Riemann integrable functions. Then,*

$$|\langle f, g\rangle_2| \leq \|f\|_2 \|g\|_2. \tag{4.5}$$

Theorem 4.2.3. *Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two Riemann integrable functions. Then,*

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2. \tag{4.6}$$

Definition 4.2.4. Let $f, g : [a,b] \longrightarrow \mathbb{C}$ two Riemann integrable functions with $f \neq g$. Then,

1. we say *f and g are orthogonal* if and only if $\langle f, g\rangle_2 = 0$.

2. We say *f and g are orthonormal* if and only if $\langle f, g\rangle_2 = 0$ and $\|f\|_2 = \|g\|_2 = 1$.

3. Let $S = \{\phi_0, \phi_1, \dots\}$ be a collection of Riemann integrable functions in $[a, b]$. We say *S is an orthonormal system* if and only if $\|\phi_i\|_2 = 1 \forall i$ and $\langle \phi_i, \phi_j\rangle_2 = 0 \forall i \neq j$.

Definition 4.2.5. Let $\{\phi_0, \dots, \phi_n\}$ be a collection of Riemann integrable functions in $[a, b]$. We say *the collection is linearly dependent in* $[a, b]$ if and only if there exist $c_0, \dots, c_n$ with not all being zero such that

$$c_0\phi_0(x) + \cdots + c_n\phi_n(x) = 0, \forall x \in [a, b]. \tag{4.7}$$

Theorem 4.2.4. *Let $S = \{\phi_0, \phi_1, \dots\}$ be an orthonormal system in $[a, b]$ such that $\sum_{n=0}^{\infty} c_n\phi_n(x)$ converges uniformly in $[a, b]$. Let f be the function that defines the series in $[a, b]$. Thenm f is Riemann integrable in $[a, b]$ and*

$$c_n = \langle f, \phi_n\rangle_2 = \int_a^b f(x)\overline{\phi_n(x)}\,\mathrm{d}x\,, n \geq 0. \tag{4.8}$$

4.3 Fourier coefficients. Fourier series

4.3.1 Exponential form

Definition 4.3.1. Let f be a Riemann integrable function in $[0, 1]$. We define the *n-th Fourier coefficient* with $n \in \mathbb{Z}$ as

$$\hat{f}(n) = \langle f, e_n\rangle_2 = \int_0^1 f(x)\mathrm{e}^{-2\pi i n x}\,\mathrm{d}x\,. \tag{4.9}$$

The series

$$Sf(x) = \sum_{n=0}^{\infty} \hat{f}(x)\mathrm{e}^{2\pi i n x} \tag{4.10}$$

constructed by these coefficients is called the *Fourier series of f*.

Proposition 4.3.1. *Let f be a Riemann integrable function in $[0, 1]$ and $\lambda, \mu \in \mathbb{C}$ two numbers. In relation to Fourier coefficients, the following statements are true.*

1. $\lambda\widehat{f + \mu}g(n) = \lambda\hat{f}(b) + \mu\hat{g}(n)$.

2. *If $\tau \in (0, 1)$ and $f_\tau(x) \coloneqq f(x - \tau)$, then $\hat{f}_\tau(n) = \mathrm{e}^{-2\pi i n\tau}\hat{f}(n)$.*

3. *If f is even, then $\hat{f}(n) = \hat{f}(-n)\forall n$ and if f is odd, $\hat{f}(n) = -\hat{f}(-n)\forall n$.*

4. *If f' exists and it is continuous, then $\hat{f}'(n) = 2\pi i n\hat{f}(n)$.*

Definition 4.3.2. Let f, g be two Riemann integrable functions in $[0, 1]$. We define the *convolution of f and g* as

$$(f * g)(x) = \int_0^1 f(t)g(x - t)\,\mathrm{d}t\,, \tag{4.11}$$

defined for $x \in \mathbb{R}$.

Proposition 4.3.2. *Let f, g be two Riemann integrable functions in $[0, 1]$. Then, $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.*

4.3.2 Fourier series in terms of sine and cosine

We can write the Fourier series as

$$Sf(x) = A_0 + 2\sum_{n=0}^{\infty} A_n \cos(2\pi nx) + B_n \sin(2\pi nx), \tag{4.12}$$

$$A_0 = \int_0^1 f(x)\,\mathrm{d}x\,, \qquad A_n = \int_0^1 f(x)\cos(2\pi nx)\,\mathrm{d}x\,, \qquad B_n = \int_0^1 f(x)\sin(2\pi nx)\,\mathrm{d}x\,. \tag{4.13}$$

Proposition 4.3.3. *Let f be a Riemann integrable function in $[0,1]$. If f is even, then its Fourier series is written as*

$$Sf(x) = A_0 + 2\sum_{n=0}^{\infty} A_n \cos(2\pi nx). \tag{4.14}$$

If f is odd, then its Fourier series is written as

$$Sf(x) = 2\sum_{n=0}^{\infty} B_n \sin(2\pi nx), \tag{4.15}$$

4.3.3 Fourier series in terms of sine or cosine

Definition 4.3.3. LEt f be a Riemann integrable function in $[0,1/2]$. We define the *even extension* and *odd extension*, respectively, as

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [0,1/2] \\ f(-x), & \text{if } x \in [-1/2,0] \end{cases}\,, \qquad \tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [0,1/2] \\ -f(-x), & \text{if } x \in [-1/2,0] \end{cases}\,. \tag{4.16}$$

Proposition 4.3.4. *Let $f : [0,1/2] \longrightarrow \mathbb{C}$ be a Riemann integrable function. If we make the even extension of f, then*

$$Sf(x) = 2A_0 + 4\sum_{n=0}^{\infty} A_n \cos(2\pi nx), \qquad A_0 = \int_0^{1/2} f(x)\,\mathrm{d}x\,, \qquad A_n = \int_0^{1/2} f(x)\cos(2\pi nx)\,\mathrm{d}x\,. \tag{4.17}$$

If we make the odd extension of f, then

$$Sf(x) = 4\sum_{n=0}^{\infty} B_n \sin(2\pi nx), \qquad B_n = \int_0^{1/2} f(x)\sin(2\pi nx)\,\mathrm{d}x\,. \tag{4.18}$$

4.3.4 Change of period

For a function $f : \mathbb{R} \longrightarrow \mathbb{C}$ T-periodic,

$$Sf(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\mathrm{e}^{2\pi nxi/T}, \qquad \hat{f}(n) = \int_{-T/2}^{T/2} f(x)\mathrm{e}^{-2\pi inx/T}\,\mathrm{d}x\,. \tag{4.19}$$

or

$$Sf(x) = A_0 + 2\sum_{n=1}^{\infty} A_n \cos\left[\frac{2\pi nx}{T}\right] + B_n \sin\left[\frac{2\pi nx}{T}\right], \tag{4.20}$$

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,\mathrm{d}x\,, A_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\cos\left[\frac{2\pi nx}{T}\right]\mathrm{d}x\,, B_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\sin\left[\frac{2\pi nx}{T}\right]\mathrm{d}x\,. \tag{4.21}$$

4.4 Punctual convergence of Fourier series

We denote the N-th partial sum as

$$S_n f := \sum_{n=-N}^{N} \hat{f}(n) \mathrm{e}^{2\pi i n x}. \tag{4.22}$$

4.4.1 Dirichlet nucleus

$$S_N f(x) = \int_0^1 f(t) \sum_{n=-N}^{N} \mathrm{e}^{e\pi i n(x-t)} \, \mathrm{d}t \tag{4.23}$$

Definition 4.4.1. We define the Dirichlet nucles of N-th order as

$$D_N(t) = \sum_{n=-N}^{N} \mathrm{e}^{2\pi i n t}. \tag{4.24}$$

This way, we have

$$S_N f(x) = (f * D_N)(x). \tag{4.25}$$

$$D_n(t) = \frac{\sin[(2N+1)\pi t]}{\sin \pi t}. \tag{4.26}$$

Proposition 4.4.1. *The Dirichlet nucleus has the following elemental properties.*

1. *D_N is a periodic function of period 1.*
2. *D_N is an even function.*
3. $\displaystyle\int_0^1 D_N(t) dt = 1.$

Definition 4.4.2 (Dirichlet kernel)**.** We define the *Dirichlet kernel of order N* as

$$D_N(t) = \frac{1}{T} \sum_{n=-N}^{N} e^{\frac{2\pi i n t}{T}} = \frac{1}{T} \frac{\sin\left(\frac{(2N+1)\pi t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}.$$

Proposition 4.4.2. *The Dirichlet kernel has the following properties:*

1. *D_N is a T-periodic and even function.*
2. $\displaystyle\int_0^T D_N(t) dt = 1$ *for all N.*

Proposition 4.4.3. *Sigui $f \in L^1([-T/2, T/2])$. Then*

$$S_N f(x) = (f * D_N)(x) = \int_{-T/2}^{T/2} f(x-t) D_N(t) dt =$$

$$= \int_0^{T/2} [f(x+t) + f(x-t)] D_N(t) dt.$$

Lemma 4.4.4 (Riemann-Lebesgue lemma)**.** *Let* $f \in L^1([-T/2, T/2])$ *and* $\lambda \in \mathbb{R}$*. Then:*

$$\lim_{\lambda\to\infty} \int_{-T/2}^{T/2} f(t)\sin(\lambda t)dt = \lim_{\lambda\to\infty} \int_{-T/2}^{T/2} f(t)\cos(\lambda t)dt = 0.$$

In particular, $\lim_{|n|\to\infty} \widehat{f}(n) = 0$.

Theorem 4.4.5. *Let* $f \in L^1([-T/2, T/2])$ *be a function having left and right-hand side derivatives at* x_0*, that is, there exists the following limits*

$$f'(x_0^+) = \lim_{t\to 0^+} \frac{f(x_0+t) - f(x_0^+)}{t},$$
$$f'(x_0^-) = \lim_{t\to 0^-} \frac{f(x_0+t) - f(x_0^-)}{t},$$

(supposing left and right-hand side exists). Then,

$$\lim_{N\to\infty} S_N f(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

Theorem 4.4.6 (Dini's theorem)**.** *Let* $f \in L^1([-T/2, T/2])$*,* $x_0 \in (-T/2, T/2)$ *and* $\ell \in \mathbb{R}$ *such that*

$$\int_0^{\delta} \frac{|f(x_0+t) + f(x_0-t) - 2\ell|}{t} dt < \infty$$

for some $\delta > 0$*. Then* $\lim_{N\to\infty} S_N f(x_0) = \ell$*.*

Theorem 4.4.7 (Lipschitz's theorem)**.** *Let* $f \in L^1([-T/2, T/2])$ *such that at a point* $x_0 \in (-T/2, T/2)$ *it satisfies* $|f(x_0+t) - f(x_0)| \le L|t|$ *for some constant* L *and for* $|t| < \delta$*. Then* $\lim_{N\to\infty} S_N f(x_0) = f(x_0)$*.*

4.5 Uniform convergence

Definition 4.5.1. Let $\sum a_n$ be a series with partial sums S_k. The series $\sum a_n$ is called *Cesàro summable* with sum S if

$$\lim_{N\to\infty} \frac{S_1 + \cdots + S_N}{N} = S.$$

Definition 4.5.2 (Fejer kernel)**.** We define the *Fejer kernel of order* N as

$$K_N(t) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(t) = \frac{1}{T(N+1)} \frac{\sin^2\left(\frac{(N+1)\pi t}{T}\right)}{\sin^2\left(\frac{\pi t}{T}\right)},$$

being $D_k(t)$ the Dirichlet kernel of order k, $0 \le k \le N$.

Proposition 4.5.1. *The Fejer kernel has the following properties:*

1. K_N *is a* T*-periodic, even and non-negative function.*
2. $\displaystyle\int_{-T/2}^{T/2} K_N(t)dt = 1$ *for all* N*.*
3. *For all* $\delta > 0$*,* $\lim_{N\to\infty} \sup\{|K_N(t)| : \delta \le |t| \le T/2\} = 0$*.*

Definition 4.5.3. Let $f \in L^1([-T/2, T/2])$. We define the *Fejér means* $\sigma_N f$, for all $N \in \mathbb{N}$, as

$$\sigma_N f(x) = \frac{S_0 f(x) + \cdots + S_N f(x)}{N+1}.$$

Proposition 4.5.2. *Let $f \in L^1([-T/2, T/2])$. Then*

$$\sigma_N f(x) = (f * K_N)(x) = \int_{-T/2}^{T/2} f(x-t) K_N(t) dt =$$

$$= \int_0^{T/2} [f(x+t) + f(x-t)] K_N(t) dt.$$

Theorem 4.5.3 (Fejér's theorem). *Let $f \in L^1([-T/2, T/2])$ be a function having left and right-hand side limits at point x_0. Then,*

$$\lim_{N\to\infty} \sigma_N f(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

In particular, if f is continuous at x_0, $\lim_{N\to\infty} \sigma_N f(x_0) = f(x_0)$.

Theorem 4.5.4 (Fejér's theorem). *Let f be continuous on $[-T/2, T/2]$. Then $\sigma_N f$ converges uniformly to f on $[-T/2, T/2]$.*

Corollary 4.5.5. *Let f be continuous on $[-T/2, T/2]$. Then there exists a sequence of trigonometric polynomials that converge uniformly to f on $[-T/2, T/2]$. In fact,*

$$\sigma_N f(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \widehat{f}(k) e^{2\pi i k x}.$$

Corollary 4.5.6. *Let f and g be continuous functions on $[-T/2, T/2]$ such that $Sf(x) = Sg(x)$. Then $f = g$.*

4.6 Convergence in norm

Definition 4.6.1. We say a sequence (f_N) *converge to f in norm* $L^2([-T/2, T/2])$ if $\lim_{N\to\infty} \|f_N - f\| = 0$.

Theorem 4.6.1. *Let $f \in L^2([-T/2, T/2])$. Then, $\lim_{N\to\infty} \|\sigma_N f - f\| = 0$.*

Corollary 4.6.2. *Let $f \in L^1([-T/2, T/2])$. Then $\lim_{N\to\infty} \|\sigma_N f - f\|_1 = 0$.*

Corollary 4.6.3. *Let $f, g \in L^1([-T/2, T/2])$ be functions such that $Sf(x) = Sg(x)$. Then $\lim_{N\to\infty} \|g - f\|_1 = 0$.*

Theorem 4.6.4 (Bessel's inequality). *Let $f \in L^2(I)$, where I is any interval on the real line. Then:*

$$T \sum_{n=-N}^{N} |\widehat{f}(n)|^2 \le \|f\|^2,$$

$$\frac{T}{2} \left(\frac{|a_0|^2}{2} + \sum_{n=1}^{N} |a_n|^2 + |b_n|^2 \right) \le \|f\|^2,$$

for all $N \in \mathbb{N}$.

Theorem 4.6.5. *$S_N f$ is the trigonometric polynomial of degree N that best approximates f in norm L^2.*

Corollary 4.6.6. *Let $f \in L^2([-T/2, T/2])$. Then, $\lim_{N\to\infty} \|S_N f - f\| = 0$.*

Theorem 4.6.7 (Parseval's identity)**.** *Let $f, g \in L^2([-T/2, T/2])$ be bounded functions. Then*

$$\langle f, g\rangle = T \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}.$$

In particular, if $f = g$, then:

$$\|f\|^2 = T \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2,$$

$$\|f\|^2 = \frac{T}{2}\left(\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2\right).$$

4.7 Applications of Fourier series

Theorem 4.7.1 (Wirtinger's inequality)**.** *Let f be a function such that $f(0) = f(T)$, $f' \in L^2([0, T])$ and $\int_a^b f(t)dt = 0$. Then,*

$$\int_0^T |f(x)|^2 dx \le \frac{T^2}{4\pi^2} \int_0^T |f'(x)|^2 dx,$$

with equality if and only if $f(x) = A\cos\left(\frac{2\pi x}{T}\right) + B\sin\left(\frac{2\pi x}{T}\right)$.

Theorem 4.7.2 (Wirtinger's inequality)**.** *Let $f \in \mathcal{C}^1([a, b])$ with $f(a) = f(b) = 0$. Then,*

$$\int_a^b |f(x)|^2 dx \le \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 dx.$$

Theorem 4.7.3 (Isoperimetric inequality)**.** *Let c be a simple and closed curve of class $\mathcal{C}^1$ whose length is L. If A_c is the area enclosed by c, then*

$$A_c \le \frac{L^2}{4\pi},$$

with equality if and only if c is a circle.