1 Motion in one dimension

Proposition 1.1. *Let .... Then, it is true that*

$$\vec{F}^{ext} = m\vec{a} + (\vec{v} - \vec{u})\dot{m} \tag{1}$$

or which is equivalent,

$$\vec{F}^{ext} + \dot{m}\vec{u} = \dot{\vec{p}} \tag{2}$$

2 Oscillations

Proposition 2.1. *Let be the following differential equation*

$$\ddot{x} + \omega_0^2 x = 0, \tag{3}$$

with the initial value condition of $x(0) = x_0$ and $v(0) = v_0$. Then, the general solution is

$$x(t) = x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t, \tag{4}$$

or, which is equivalent,

$$x(t) = A\cos\left[\omega_0 t + \phi_0\right], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \phi_0 = -\arctan\frac{v_0}{\omega_0 x_0}. \tag{5}$$

Definition 2.1. Let $U(x)$ be a potential function of class $C^2(\mathbb{R})$. Then, we say x_0 *is a point of stable equilibrium* if U has a maxima in x_0.

Proposition 2.2. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{6}$$

with the initial value conditions of $x(0) = x_0$ and $v(0) = v_0$. Then, the general solution is

$$x(t) = e^{-\beta t}\left[x_0\cos\tilde{\omega}t + \frac{v_0 + \beta x_0}{\tilde{\omega}}\sin\tilde{\omega}t\right], \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \beta^2} \tag{7}$$

if $\beta < \omega_0$,

$$x(t) = e^{-\omega_0 t}\left[x_0 + (x_0\omega_0 + v_0)t\right] \tag{8}$$

if $\beta = \omega_0$, and

$$x(t) = \frac{x_0(\bar{\omega} - \beta) - v_0}{2\bar{\omega}}e^{-(\beta+\bar{\omega})t} + \frac{x_0(\bar{\omega} + \beta) + v_0}{2\bar{\omega}}e^{-(\beta-\bar{\omega})t}, \qquad \bar{\omega} = \sqrt{\beta^2 - \omega_0^2} \tag{9}$$

if $\beta > \omega_0$.

Proposition 2.3. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0\cos\left[\omega t + \psi_0\right], \tag{10}$$

with the initial value conditions of $x(0) = x_0$ and $v(0) = v_0$. Then, the particular solution is

$$x_p(t) = A\cos\left[\omega t + \psi_0 - \phi_0\right], \qquad A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}, \qquad \phi_0 = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{11}$$

3 Central forces

Definition 3.1. Central force

$$\vec{F}(\vec{r}) = f(r)\vec{e}_\rho \tag{12}$$

Proposition 3.1. *All central forces are conservatives.*

Proposition 3.2. *The angular momentum with respect the origin is conserved.*

$$\dot{\vec{L}} = \vec{0} \tag{13}$$

Proposition 3.3. *The areal velocity is constant.*

$$\frac{dA}{dt} = \frac{L}{2m} = ctt \tag{14}$$

Theorem 3.4 (Bertrand's Theorem)**.** *The only central potentials where every bounded orbit is closed are:*

$$U(r) = -\frac{k}{r}, \qquad U(r) = \frac{k}{2}r^2, \qquad k > 0 \tag{15}$$

4 Coupled oscillations 1

5 Coupled oscillations 2

6 Rotations

7 Dynamics of rigid body

Proposition 7.1. *The vector $\mathbf{\Omega}$ is independent on the origin of the system S.*

Proposition 7.2. *The energy of the rigid body is an invariant scalar under change of basis.*

8 Special relativity

Theorem 8.1. *If Maxwell's equations*

$$\langle\mathbf{\nabla}, \mathbf{E}\rangle_I = \frac{\rho}{\epsilon_0}, \tag{16}$$

$$\langle\mathbf{\nabla}, \mathbf{B}\rangle_I = 0, \tag{17}$$

$$\mathbf{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{18}$$

$$\mathbf{\nabla}\times\mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right). \tag{19}$$

are invariant under Galileo transformations, then $\mathbf{E} = \mathbf{B} = \mathbf{0}$.

Definition 8.1 (Reference system)**.** We define a *reference system S* as a set of three axis and one origin over which we have determined an orientation. We will suppose we have selected a unit of length and that in each point a in the immobile space with respect the axis there is a clock q_a such that the clocks q_a and q_b corresponding to two different points a and b immobile with respect these axis are synchronized

Lemma 8.2. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms lines that are not contained in hyperplanes of the form $t =$ ctt to lines that are not contained in hyperplanes of the form $t' =$ ctt.*

Lemma 8.3. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms planes that are not contained in hyperplanes of the form $t = \mathrm{ctt}$ to planes that are not contained in hyperplanes of the form $t' = \mathrm{ctt}$.*

Lemma 8.4. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms hyperplanes that are not of the form $t = \mathrm{ctt}$ to hyperplanes that are not of the form $t' = \mathrm{ctt}$.*

Theorem 8.5. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f is an affine transformation.*

Theorem 8.6. *Let S, S' be two inertial reference systems. We can make orthogonal changes (isometries) of axis to S and S' and a change of origin of time such thate the Lorentz transformation has the form of the equation* **??**.

Lemma 8.7. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation and $r \subseteq \mathbb{R}^4$ a line with a timelike direction vector. Then, f transforms r to a line with a timelike direction vector.*

Lemma 8.8. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation and V and admissible plane (hyperplane). Then, $f(V)$ is an admissible plane (hyperplane).*

Theorem 8.9. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f is an affine transformation.*

Theorem 8.10. *(Lorenz transformation) Let S, S' be two reference systems with the same origin such that S' moves with a constant velocity $\mathbf{v} = v\mathbf{e}_x$. Then,*

$$P_{\mathrm{s}'} = \Lambda P_{\mathrm{s}} \Leftrightarrow P^{\nu}_{\mathrm{s}'} = \Lambda^{\nu}_{\mu} P^{\mu}_{\mathrm{s}}, \qquad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

Proposition 8.11. *Let $\mathbf{R} \in L$ be a vector and $a \in \mathbb{R}$ a scalar. Then, $\|a\mathbf{R}\|_m = |a|\|\mathbf{R}\|_m$.*

Proposition 8.12. *Every subspace W of L is either timelike, spacelike, or lightlike. Besides,*

1. *S is timelike $\Leftrightarrow W^{\perp}$ is spacelike.*
2. *S is spacelike $\Leftrightarrow W^{\perp}$ is timelike.*
3. *W is lightlike $\Leftrightarrow W^{\perp}$ is lightlike.*

Proposition 8.13. *Two orthogonal vectors different from zero and non spacelike are necessarily lightlike and collinear. In particular, there is not a subspace of dimension 2 where $\langle , \rangle$ is null.*

Proposition 8.14. *Let $\mathbf{R}_1, \mathbf{R}_2 \in T$ be two timelike vectors. Then, the following statements are true.*

1. *$|\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m| \geq \|\mathbf{R}_1\|_m \|\mathbf{R}_2\|_m$, and the equality is equivalent to both vectors being collinear.*
2. *$\mathbf{R}_1, \mathbf{R}_2$ are in the same time cone (C_+ or C_-) if and only if $\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m < 0$. In this case,*

   (a) There is a unique $\varphi \in \mathbb{R}$ such that

   $$\cosh \varphi = -\frac{\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m}{\|\mathbf{R}_1\|_m \|\mathbf{R}_2\|_m}. \tag{21}$$

   We call this φ the hyperbolic angle.

   (b) $\|\mathbf{R}_1\|_m + \|\mathbf{R}_2\|_m \leq \|\mathbf{R}_1 + \mathbf{R}_2\|_m$.

Proposition 8.15. *The Lorentz-Minkowski metric (using the proper orthonormal basis) can be expressed by the bilinear form η*

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

Proposition 8.16. *The transformation in $\mathbb{M} = \mathbb{R}^4$ (using the proper orthonormal basis) can be expressed by the matrix Λ*

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

Proposition 8.17. *The Lorenz-Minkowski metric is invariant under Lorentz transformations.*

Proposition 8.18. *Let S, S' be two inertial reference systems such that the velocity of S' is $\mathbf{w} = w\mathbf{e}_x$ with respect to S. Then,*

$$v^1_{\mathrm{s}'} = \frac{v^1_{\mathrm{s}} - w}{1 - \beta_v \beta_w}, \qquad v^2_{\mathrm{s}'} = \frac{1}{\gamma_w} \frac{v^2_{\mathrm{s}}}{1 - \beta_v \beta_w}, \qquad v^3_{\mathrm{s}'} = \frac{1}{\gamma_w} \frac{v^3_{\mathrm{s}}}{1 - \beta_v \beta_w}, \tag{24}$$

Proposition 8.19. *If the system has a general velocity $\mathbf{w}$, then*

$$\mathbf{v}' = \frac{1}{1 - \langle \boldsymbol{\beta}_v, \boldsymbol{\beta}_2 \rangle_I} \left[\frac{\mathbf{v}}{\gamma_w} - \mathbf{w} + \frac{1}{c^2} \frac{\gamma_w}{1 + \gamma_w} \langle \mathbf{v}, \mathbf{w} \rangle_I \mathbf{w} \right], \tag{25}$$

$$\mathbf{v} = \frac{1}{1 + \langle \boldsymbol{\beta}_{v'}, \boldsymbol{\beta}_w \rangle_I} \left[\frac{\mathbf{v}'}{\gamma_w} + \mathbf{w} + \frac{1}{c^2} \frac{\gamma_w}{1 + \gamma_w} \langle \mathbf{w}', \mathbf{w} \rangle_I \mathbf{v} \right]. \tag{26}$$

Proposition 8.20. *Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a 4-velocity $\mathbf{U}$. Then,*

$$\mathbf{U} = (\gamma c, \gamma \mathbf{v}), \qquad -\langle \mathbf{U}, \mathbf{U} \rangle_m = c^2. \tag{27}$$

Proposition 8.21. *Let p be a particle of velocity $\mathbf{U}$ and acceleration $\mathbf{A}$. Then, $\langle \mathbf{A}, \mathbf{U} \rangle_m = 0$.*

Proposition 8.22. *Let p be a particle of 4-momentum $\mathbf{P}$. Then,*

$$\mathbf{P} = (E/c, \mathbf{p}) = (\gamma mc, \gamma m \mathbf{v}). \tag{28}$$

Theorem 8.23.

$$E^2 = p^2 c^2 + m^2 c^4. \tag{29}$$

Proposition 8.24. *There are three possible cases: stationary particle with mass, moving particle with mass, particle with no mass.*

$$E = mc^2, \qquad E^2 = m^2 c^4 + \|p\|^2 c^2, \qquad E = pc \tag{30}$$

Theorem 8.25. *(Work-Energy theorem)*

$$W = \Delta E. \tag{31}$$

Theorem 8.26. *Let p be a particle of velocity $\mathbf{v}$. Then, the kinetic energy is obtained by the expression*

$$T = \int_\Gamma \langle \mathbf{F}, \mathrm{d}\mathbf{r} \rangle_I = (\gamma(\dot{\mathbf{r}}) - 1)mc^2 \tag{32}$$

Theorem 8.27 (Compton scattering)**.**

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta). \tag{33}$$

Theorem 8.28 (Center of momentum)**.** *Let be a system of particles $p_1, \dots, p_n$ with energies $E_1, \dots, E_n$ and momentum $\mathbf{p}_1, \dots, \mathbf{p}_n$. Then, the center of momentum system has a velocity determined by the expression*

$$\mathbf{v}_{\mathrm{cp}} = \frac{1}{E_t} \sum_{i=1}^n \|\mathbf{p}_i\|^2 c^2. \tag{34}$$

Proposition 8.29. *Let p be a particle of mass m moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Then,*

$$\langle \mathbf{f}, \mathbf{v} \rangle_I = \frac{\mathrm{d}E}{\mathrm{d}t} = m\gamma^3 \langle \mathbf{v}, \mathbf{a} \rangle_I, \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\gamma\mathbf{a} + m\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v}, \tag{35}$$

$$\mathbf{F} = \left(\frac{\gamma}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, \gamma\mathbf{f} \right) = \left(\frac{m\gamma^4}{c}\langle \mathbf{v}, \mathbf{a} \rangle_I, m\gamma^2\mathbf{a} + m\frac{\gamma^4}{c^2}\langle \mathbf{v}, \mathbf{a} \rangle_I \mathbf{v} \right) \tag{36}$$

Proposition 8.30. *Let p be a particle of mass m moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Then,*

1. *If $\mathbf{f} \parallel \mathbf{v}$, then*

$$f = m\gamma^3 a, \qquad m\gamma^3 := \textit{longitudinal mass}. \tag{37}$$

2. *If $\mathbf{f} \perp \mathbf{v}$, then*

$$f = m\gamma a, \qquad m\gamma := \textit{transverse mass}. \tag{38}$$

Theorem 8.31. *Let p be a particle of mass m with $v_0 = x_0 = t_0 = 0$ on which a constant force F acts. If we denote $a_0 = \gamma^3 a = F/m$ (which is constant), then*

$$x(t) = \frac{c^2}{a_0}\left[\sqrt{1 + \frac{a_0^2 t^2}{c^2}} - 1 \right], \qquad v(t) = \frac{a_0 t}{\sqrt{1 + a_0^2 t^2 / c^2}}, \tag{39}$$

and in the limit cases,

$$t \to \infty : v(t) \approx c, x(t) \approx ct - \frac{c^2}{a_0}, \tag{40}$$

$$a_0 t \ll c : v(t) \approx a_0 t, x(t) \approx \frac{a_0}{2}t^2. \tag{41}$$

Theorem 8.32. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$ and $x_0 = -k/mc^2, t_0 = 0$, then*

$$E = \gamma mc^2 + \frac{k}{x} = 0, \qquad \omega^2 x^2 + v^2 = c^2, \tag{42}$$

$$\omega = \frac{c}{x_0}, \qquad T = \frac{\pi}{2}\frac{x_0}{c}, \tag{43}$$

$$x(t) = x_0 \cos\omega t, \qquad v(t) = -c\sin\omega t, \tag{44}$$

while n classical formalism the period would be $\frac{\pi}{2\sqrt{2}}\frac{x_0}{c} < T_{\mathrm{rel}}$, which is normal since in the classical framework the mass can have an infinite velocity while in the relativistic one is bounded by c.

Theorem 8.33. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$ and $x_0 = -k/mc^2, t_0 = 0$, then with respect to the particle reference,*

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\gamma} = \cos\omega t = \frac{x}{x_0}, \qquad \tau = \int \gamma \,\mathrm{d}t = \frac{1}{\omega}\sin\omega t, \tag{45}$$

$$x^2 = x_0^2 - c^2\tau^2, \qquad T' = \frac{x_0}{c}, \qquad \frac{L'}{x_0} = \frac{T}{T'}. \tag{46}$$

Theorem 8.34. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$, then*

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\gamma^3}{c^2}\langle \mathbf{v}, \mathbf{a} \rangle_I \mathbf{v}, \qquad \gamma m\beta^2 c^2 = -\frac{k}{r},$$

$$-1 < \frac{T}{U} = -\frac{\gamma}{\gamma + 1} < -\frac{1}{2}, \qquad E = \frac{mc^2}{\gamma}.$$

Theorem 8.35. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k > 0$, then it is not possible falling to the origin, and if $k < 0$, then it is possible if $Lc \leq k$ (in this case $p_r \to \infty$).*

Theorem 8.36. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k > 0$, then it is always possible to escape to the infinity is always possible, and if $k < 0$, it is possible if $E > mc^2$.*

Theorem 8.37. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. Then,*

$$L = \gamma mr^2\dot{\theta} = \mathrm{ctt}, \qquad E = \gamma mc^2 + \frac{k}{r} = \mathrm{ctt}, \qquad \mathbf{p} = \gamma m(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e} \tag{47}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\dot{r}) - \frac{L^2}{\gamma mr^3} = \frac{k}{r^2}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\gamma mk}{L^2}, \tag{48}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + (1 - \alpha^2)\frac{1}{r} = -\frac{kE}{L^2c^2}, \ \alpha^2 = \frac{k^2}{L^2c^2}. \tag{49}$$

Proposition 8.38. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If the variation of r is negligible, then $\alpha^2 < 1$.*

Proposition 8.39. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If $\alpha^2 <$, $E < mc^2$, and $E > 0$, then the trajectory of p is bounded.*

Theorem 8.40. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If $\alpha^2 < 1$, $E < mc^2$, and $E > mc^2\sqrt{1-\alpha^2}$, then the trajectory of p is determined by the expression*

$$r = \frac{a(1-e^2)}{1+e\cos\left(\sqrt{1-\alpha^2}\theta\right)}, \tag{50}$$

$$\frac{1}{a} = \frac{E}{k}\left[1 - \frac{m^2c^4}{E^2}\right],\ e = \frac{1}{\alpha}\sqrt{1+(\alpha^2-1)\frac{m^2c^4}{E^2}}, \tag{51}$$

which is an ellipse with a precession $2\pi(1/\sqrt{1-\alpha^2}-1)$ per revolution (and $\pi\alpha^2$ if $\alpha^2 \ll 1$).

Theorem 8.41. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If p has a closed bounded trajectory, then the average of $\frac{\mathrm{d}\langle \mathbf{r},\mathbf{p}\rangle_I}{\mathrm{d}t} = 0$ on an interval of nT.*

Proposition 8.42. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If p has a closed bounded trajectory, then*

$$E = \left\langle \frac{1}{\gamma} \right\rangle mc^2. \tag{52}$$

9 Generalized coordinates

Definition 9.1. Let S be a system of particles $p_1,\ldots,p_n$ with masses $m_1,\cdots,m_n$. Then, we say the system has *non stationary holonomic constraints* or *rheonomic constraints* if and only if there is a function $\mathbf{f}: \mathbb{R}^{3n}\times\mathbb{R} \longrightarrow \mathbb{R}^k$ such that

$$\mathbf{f}(\mathbf{r}_1,\ldots,\mathbf{r}_n,t) = \mathbf{0}. \tag{53}$$

Proposition 9.1. *Let S be a system of n particles with a constraint $\mathbf{f}: \mathbb{R}^{3n}\times\mathbb{R} \longrightarrow \mathbb{R}^k$ and V the set of possible velocities at an instant t. If $\dot{\mathbf{x}} = (\mathbf{v}_1,\ldots,\mathbf{v}_n)$, then*

$$\sum_{i=1}^{n} \langle \boldsymbol{\nabla}_i f_j, \mathbf{v}_i\rangle_I + \frac{\partial f_j}{\partial t} = 0,\ j = 1,\ldots,k. \tag{54}$$

Theorem 9.2 (D'Alembert's principle)**.** *Let S be a system of particles. Then,*

$$\boxed{\sum_{i=1}^{n} \langle \mathbf{F}_i - m\mathbf{a}_i, \delta\mathbf{r}_i\rangle_I = 0,\ \forall\,\delta\mathbf{r}_i\,.} \tag{55}$$

Theorem 9.3. *Let S be a system of n particles with generalized coordinates $q^1,\ldots,q^r$. Then,*

$$\boxed{\sum_{i=1}^{n}\left\langle \mathbf{F}_i, \frac{\partial \mathbf{p}_i}{\partial q^j}\right\rangle_I = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^j} - \frac{\partial T}{\partial q^j},\ j=1,\ldots,r.} \tag{56}$$

And if $\mathbf{F}$ is derived from a potential $\Phi(\mathbf{r})$, then

$$\boxed{\frac{\partial L}{\partial q^j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^j} = 0,\ j=1,\ldots,r.} \tag{57}$$

Theorem 9.4. *Let $J : C^2[x_0,x_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$J(y) = \int_{x_0}^{x_1} f(x,y,y')\,\mathrm{d}x\,,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0,x_1] \mid y(x_0) = y_1 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are given real numbers. If $y \in S$ is an extremal for J, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{58}$$

for all $x \in [a_0,x_1]$.

Theorem 9.5 (Lagrange multipliers method for non-holonomic constraints)**.** *If we want to find an extrema having a set of m non-holonomic constraints*

$$\begin{aligned} \overline{\delta f_1} &= A_{11}\delta u_1 + \cdots + A_{1n}\delta u_n &&= 0,\\ \vdots &= \ddots && = \vdots\\ \overline{\delta f_m} &= A_{m1}\delta u_1 + \cdots + A_{mn}\delta u_n &&= 0, \end{aligned} \tag{59}$$

Then we can proceed as the original multipliers method by considering every variation as independent and solving the following equation.

$$\delta F + \lambda_1\overline{\delta f_1} + \cdots + \lambda_m\overline{\delta f_m} = 0 \tag{60}$$

Theorem 9.6. *Let f be a continuous functions with a variation $\delta f = \epsilon\phi$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta y = \delta\frac{\mathrm{d}}{\mathrm{d}x}y, \qquad \delta\int_a^b f(x)\,\mathrm{d}x = \int_a^b \delta f(x)\,\mathrm{d}x\,. \tag{61}$$

Theorem 9.7. *Let $J : \mathbf{C}^2[t_0,t_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$J(\mathbf{q}) = \int_{t_0}^{t_1} L(\mathbf{q},\dot{\mathbf{q}},t)\,\mathrm{d}t\,, \tag{62}$$

where $\mathbf{q} = (q_1,\ldots,q_n)$, and L has continuous second-order partial derivatives with respect to t, q_k, and $\dot{q}_k$, $k = 1,\ldots,n$. Let

$$S = \left\{\mathbf{q} \in \mathbf{C}^2[t_0,t_1] \,\middle|\, \mathbf{q}(t_0) = \mathbf{q}_0, \mathbf{q}(t_1) = \mathbf{q}_1\right\}, \tag{63}$$

where $\mathbf{q}_0, \mathbf{q}_1 \in \mathbb{R}^n$ are given vectors. If $\mathbf{q}$ is an extremal for J in S then for $k = 1,\ldots,n$

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} = 0.} \tag{64}$$

Theorem 9.8. *If we have a set of holonomic constraints*

$$\begin{aligned} f_1(q_1,\ldots,q_n,t) &= 0,\\ \ldots &= \vdots,\\ f_m(q_1,\ldots,q_n,t) &= 0, \end{aligned} \tag{65}$$

then we can treat each variable as independent and search the stationary value of

$$J' = \int_{t_1}^{t_2} L + \sum_{k=1}^{m} \lambda_m f_m \,\mathrm{d}t\,, \tag{66}$$

which leads to the equation

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} + \lambda_1 \frac{\partial f_1}{\partial q_k} + \cdots + \lambda_m \frac{\partial f_m}{\partial q_k} = 0.} \tag{67}$$