Contents

Chapter 1

Fourier series

1.1 Preliminaries

Definition 1.1.1. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We say f *has a period* T or f *is* T*-periodic* with $T > 0$ if and only if $f(x + T) = f(x)$ for all $x \in \mathbb{R}$.

Lemma 1.1.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function. Then, $f(x + T') = f(x)$ for all $x \in \mathbb{R}$ if and only if $T' = kT$ for some $k \in \mathbb{Z}$.*

Proposition 1.1.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function and $a \in \mathbb{R}$ a number. Then,*

$$\int_a^{a+T} f \,\mathrm{d}x = \int_0^T f \,\mathrm{d}x \,. \tag{1.1}$$

Lemma 1.1.3. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function continuous in $\mathbb{R}$. Then, $|f|$ is bounded.*

Proposition 1.1.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a T-periodic function. Then, there is no a power series that converges uniformly to f in $\mathbb{R}$.*

1.2 Orthogonal and orthonormal systems of functions

Definition 1.2.1. Let $f, g : [a, b] \longrightarrow \mathbb{C}$ be two integrable functions. We define the *inner product of f and g* as

$$\langle f, g\rangle_2 \coloneqq \int_a^b f(x)\overline{g(x)} \,\mathrm{d}x \,, \tag{1.2}$$

where $\overline{g}$ is the conjugate of g.

Definition 1.2.2. Let $f, g : [a, b] \longrightarrow \mathbb{C}$ two integrable functions. We define the *2-norm of f* as

$$\|f\|_2 \coloneqq \sqrt{\langle f, f\rangle_2} = \left(\int_a^b |f(x)|^2 \,\mathrm{d}x\right)^{1/2} . \tag{1.3}$$

Definition 1.2.3. Let $f, g : [a, b] \longrightarrow \mathbb{C}$ two integrable functions. We define the *distance between r and g* as

$$d(f, g) \coloneqq \|f - g\|. \tag{1.4}$$

Proposition 1.2.1. *Let $f, g : [a, b] \longrightarrow \mathbb{C}$ two integrable functions. Then,*

1. $\langle f, f\rangle_2 \geq 0$.
2. $\langle f + g, h\rangle_2 = \langle f, h\rangle_2 + \langle g, h\rangle_2$ *and* $\langle f, g + h\rangle_2 = \langle f, g\rangle_2 + \langle f, h\rangle_2$.
3. $\langle f, g\rangle_2 = \overline{\langle g, f\rangle_2}$.
4. *Fro* $\alpha \in \mathbb{C}$, $\langle \alpha f, g\rangle_2 = \alpha\langle f, g\rangle_2$ *and* $\langle f, \alpha g\rangle_2 = \overline{\alpha}\langle f, g\rangle_2$.

Note that for Riemann integrable functions it is not true that $\langle f, f\rangle_2 = 0 \Leftrightarrow f = 0$, since a function that is not zero at some point will satisfy $\langle f, f\rangle_2 = 0$. However, if we deal with the space of continuous functions in $[a, b]$ the it is true.

Theorem 1.2.2. *Let $f, g : [a, b] \longrightarrow \mathbb{C}$ two Riemann integrable functions. Then,*

$$|\langle f, g\rangle_2| \leq \|f\|_2\|g\|_2. \tag{1.5}$$

Theorem 1.2.3. *Let $f, g : [a, b] \longrightarrow \mathbb{C}$ two Riemann integrable functions. Then,*

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2. \tag{1.6}$$

Definition 1.2.4. Let $f, g : [a, b] \longrightarrow \mathbb{C}$ two Riemann integrable functions with $f \neq g$. Then,

1. we say f *and* g *are orthogonal* if and only if $\langle f, g\rangle_2 = 0$.

2. We say f *and* g *are orthonormal* if and only if $\langle f, g\rangle_2 = 0$ and $\|f\|_2 = \|g\|_2 = 1$.

3. Let $S = \{\phi_0, \phi_1, \dots\}$ be a collection of Riemann integrable functions in $[a, b]$. We say S *is an orthonormal system* if and only if $\|\phi_i\|_2 = 1\forall i$ and $\langle \phi_i, \phi_j\rangle_2 = 0\forall i \neq j$.

Definition 1.2.5. Let $\{\phi_0, \dots, \phi_n\}$ be a collection of Riemann integrable functions in $[a, b]$. We say *the collection is linearly dependent in* $[a, b]$ if and only if there exist $c_0, \dots, c_n$ with not all being zero such that

$$c_0\phi_0(x) + \cdots + c_n\phi_n(x) = 0, \forall x \in [a, b]. \tag{1.7}$$

Theorem 1.2.4. *Let $S = \{\phi_0, \phi_1, \dots\}$ be an orthonormal system in $[a, b]$ such that $\sum_{n=0}^{\infty} c_n\phi_n(x)$ converges uniformly in $[a, b]$. Let f be the function that defines the series in $[a, b]$. Thenm f is Riemann integrable in $[a, b]$ and*

$$c_n = \langle f, \phi_n\rangle_2 = \int_a^b f(x)\overline{\phi_n(x)}\,\mathrm{d}x\,, n \geq 0. \tag{1.8}$$

1.3 Fourier coefficients. Fourier series

1.3.1 Exponential form

Definition 1.3.1. Let f be a Riemann integrable function in $[0, 1]$. We define the *n-th Fourier coefficient* with $n \in \mathbb{Z}$ as

$$\hat{f}(n) = \langle f, e_n\rangle_2 = \int_0^1 f(x)\mathrm{e}^{-2\pi i n x}\,\mathrm{d}x\,. \tag{1.9}$$

The series

$$Sf(x) = \sum_{n=0}^{\infty} \hat{f}(x)\mathrm{e}^{2\pi i n x} \tag{1.10}$$

constructed by these coefficients is called the *Fourier series of* f.

Proposition 1.3.1. *Let f be a Riemann integrable function in $[0, 1]$ and $\lambda, \mu \in \mathbb{C}$ two numbers. In relation to Fourier coefficients, the following statements are true.*

1. $\lambda\widehat{f + \mu}g(n) = \lambda\hat{f}(b) + \mu\hat{g}(n)$.

2. *If $\tau \in (0, 1)$ and $f_\tau(x) \coloneqq f(x - \tau)$, then $\hat{f}_\tau(n) = \mathrm{e}^{-2\pi i n\tau}\hat{f}(n)$.*

3. *If f is even, then $\hat{f}(n) = \hat{f}(-n)\forall n$ and if f is odd, $\hat{f}(n) = -\hat{f}(-n)\forall n$.*

4. *If f' exists and it is continuous, then $\hat{f}'(n) = 2\pi i n\hat{f}(n)$.*

Definition 1.3.2. Let f, g be two Riemann integrable functions in $[0, 1]$. We define the *convolution of f and g* as

$$(f * g)(x) = \int_0^1 f(t)g(x - t)\,\mathrm{d}t\,, \tag{1.11}$$

defined for $x \in \mathbb{R}$.

Proposition 1.3.2. *Let f, g be two Riemann integrable functions in $[0, 1]$. Then, $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.*

1.3.2 Fourier series in terms of sine and cosine

We can write the Fourier series as

$$Sf(x) = A_0 + 2\sum_{n=0}^{\infty} A_n \cos(2\pi nx) + B_n \sin(2\pi nx), \tag{1.12}$$

$$A_0 = \int_0^1 f(x)\,\mathrm{d}x\,, \qquad A_n = \int_0^1 f(x)\cos(2\pi nx)\,\mathrm{d}x\,, \qquad B_n = \int_0^1 f(x)\sin(2\pi nx)\,\mathrm{d}x\,. \tag{1.13}$$

Proposition 1.3.3. *Let f be a Riemann integrable function in $[0,1]$. If f is even, then its Fourier series is written as*

$$Sf(x) = A_0 + 2\sum_{n=0}^{\infty} A_n \cos(2\pi nx). \tag{1.14}$$

If f is odd, then its Fourier series is written as

$$Sf(x) = 2\sum_{n=0}^{\infty} B_n \sin(2\pi nx), \tag{1.15}$$

1.3.3 Fourier series in terms of sine or cosine

Definition 1.3.3. LEt f be a Riemann integrable function in $[0,1/2]$. We define the *even extension* and *odd extension*, respectively, as

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [0,1/2] \\ f(-x), & \text{if } x \in [-1/2,0] \end{cases}, \qquad \tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [0,1/2] \\ -f(-x), & \text{if } x \in [-1/2,0] \end{cases}. \tag{1.16}$$

Proposition 1.3.4. *Let $f : [0,1/2] \longrightarrow \mathbb{C}$ be a Riemann integrable function. If we make the even extension of f, then*

$$Sf(x) = 2A_0 + 4\sum_{n=0}^{\infty} A_n \cos(2\pi nx), \qquad A_0 = \int_0^{1/2} f(x)\,\mathrm{d}x\,, \qquad A_n = \int_0^{1/2} f(x)\cos(2\pi nx)\,\mathrm{d}x\,. \tag{1.17}$$

If we make the odd extension of f, then

$$Sf(x) = 4\sum_{n=0}^{\infty} B_n \sin(2\pi nx), \qquad B_n = \int_0^{1/2} f(x)\sin(2\pi nx)\,\mathrm{d}x\,. \tag{1.18}$$

1.3.4 Change of period

For a function $f : \mathbb{R} \longrightarrow \mathbb{C}$ T-periodic,

$$Sf(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\mathrm{e}^{2\pi nxi/T}, \qquad \hat{f}(n) = \int_{-T/2}^{T/2} f(x)\mathrm{e}^{-2\pi inx/T}\,\mathrm{d}x\,. \tag{1.19}$$

or

$$Sf(x) = A_0 + 2\sum_{n=1}^{\infty} A_n \cos\left[\frac{2\pi nx}{T}\right] + B_n \sin\left[\frac{2\pi nx}{T}\right], \tag{1.20}$$

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,\mathrm{d}x\,, A_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\cos\left[\frac{2\pi nx}{T}\right]\mathrm{d}x\,, B_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\sin\left[\frac{2\pi nx}{T}\right]\mathrm{d}x\,. \tag{1.21}$$

1.4 Punctual convergence of Fourier series

We denote the N-th partial sum as

$$S_n f := \sum_{n=-N}^{N} \hat{f}(n) \mathrm{e}^{2\pi i n x}. \tag{1.22}$$

1.4.1 Dirichlet nucleus

$$S_N f(x) = \int_0^1 f(t) \sum_{n=-N}^{N} \mathrm{e}^{e\pi i n(x-t)} \,\mathrm{d}t \tag{1.23}$$

Definition 1.4.1. We define the Dirichlet nucles of N-th order as

$$D_N(t) = \sum_{n=-N}^{N} \mathrm{e}^{2\pi i n t}. \tag{1.24}$$

This way, we have

$$S_N f(x) = (f * D_N)(x). \tag{1.25}$$

$$D_n(t) = \frac{\sin[(2N+1)\pi t]}{\sin \pi t}. \tag{1.26}$$

Proposition 1.4.1. *The Dirichlet nucleus has the following elemental properties.*

1. *D_N is a periodic function of period* 1.
2. *D_N is an even function.*
3. $\int_0^1 D_N(t) dt = 1$.