Contents

Chapter 1

Vector analysis

1.1 Introduction

1.1.1 Representation of positions

We can represent locations in real world using the points in the standard euclidean affine space, that is, an affine space whose associated vector space is has as the matrix of its bilineal form the identity matrix. This representation allows describe the details of reality for two main reasons. The first one is that, since its associated vector space has as the bilineal form the standard dot product, it satisfies too the definition of metric space and then we can stablish the concept of distance and all the other related topological concepts. The second one is that, although we cannot operate points of the euclidean space, we can describe changes of positions using the position vector of the associated vector space. This position vector is described as follows.

Definition 1.1.1. Let $\mathfrak{R} = (O, \mathfrak{B})$ be a reference the euclidean space $\mathbb{R}^3$ and P a point of the this affine space. We define the position vector $\vec{r}_p$ of the point P as the point that satisfies

$$P = O + \vec{r}_p \tag{1.1}$$

1.1.2 properties of the euclidean space

1.1.3 Other details

We will supposed all functions are differentiable.

1.2 Construction of general orthogonal coordinate systems

We have seen that we can associated a position vector $\vec{r}$ to some point, given a reference. With that, given a basis of the associated vector space, we relate to the point the coordinates of this vector. Usually we use the basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, that we can form from the parameters (x, y, z) that determine the position of a point. However, we can use other parametres, so now we will see how, from three parameters (that satisfy certain conditions), we can generate a basis of the vector space and hence a frame.
Suppose that q_1, q_2, q_3 are independent functions of position such that

$$q_1 = q_1(x, y, z), \qquad q_2 = q_2(x, y, z), \qquad q_3 = q_3(x, y, z), \tag{1.2}$$

We know that if in regions where the Jacobian $\frac{\partial(q_1,q_2,q_3)}{\partial(x,y,z)}$ is different from zero, the system of equations can be solved simultaneously for x, y, and z, giving

$$x = x(q_1, q_2, q_3), \qquad y = y(q_1, q_2, q_3), \qquad z = z(q_1, q_2, q_3) \tag{1.3}$$

and therefore,

$$\vec{r} = \vec{r}(q_1, q_2, q_3) \tag{1.4}$$

In this case every specification of numbers $(x, y.z)$ corresponds to a specification of (q_1, q_2, q_3). Thus, we come to regard the set of three numbers (q_1, q_2, q_3) as the *curvilinear coordinates* of a point in the euclidean space.
Let us suppose now we go from one point P with position vector $\vec{r}$ to a point Q whose position vector is $\vec{r} + d\vec{r}$. Supposing the $\vec{r}$ is in general differentiable [ ], we have the total distance traveled is

$$\overrightarrow{PQ} = d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3. \tag{1.5}$$

This is a linear combination of three vectors, but we are interested in unitary vectors, so we have to normalize them with what are called *metric factors* h_i.

$$\vec{e_i} = \frac{\partial \vec{r}/\partial q_i}{\|\partial \vec{r}/\partial q_i\|} = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial q_i}, \quad h_i := \left\| \frac{\partial \vec{r}}{\partial q_i} \right\| \tag{1.6}$$

With that, the differential of length is expressed as follows.

$$d\vec{l} := \vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3 = h_1 dq_1 \vec{e}_1 + h_2 dq_2 \vec{e}_2 + h_3 dq_3 \vec{e}_3 \tag{1.7}$$

These are citations of theory. [9], [13], [2], [7], [14], [5], [11], [10], [6].
These are citations of problems. [8], [3], [12], [4], [1].

Bibliography

[1] Emilio Benito Santos. Problemas de campos electromagnéticos. 1972.

[2] Lorrain Corson and P Lorrain. *Campos y ondas electromagnéticos*. 1972.

[3] Joseph A Edminister and Joseph A Edminister. *Electromagnetismo*. McGraw-Hill, 1992.

[4] A. G. Fernández. *Problemas de Campos Electromagnéticos*. McGraw-Hill, 2005.

[5] SIR FEYNMAN. 8211; R. LEIGHTON. Física (volumen 2). Ed, 1987.

[6] David J Griffiths. Introduction to electrodynamics, 2005.

[7] John David Jackson. Classical electrodynamics, 1999.

[8] Felisa Núñez Cubero, Eloísa López Pérez, et al. *100 problemas de electromagnetismo*. 1997.

[9] J. C. Quintana. Anàlisi vectorial, 2020.

[10] Joan Costa Quintana. *Interacción electromagnética: teoría clásica*. Reverté, 2019.

[11] John R Reitz and Frederick J Milford. Fundamentos de la teoría electromagnética. Technical report, 1996.

[12] José M De Juana Sardón, José M De Juana Sardón, and Miguel Angel Herrero García. Electromagnetismo. Problemas de Exámenes resueltos. 1993.

[13] Roald K Wangsness. *Campos electromagnéticos*. 1996.

[14] Roald K Wangsness and Ronald K Wangsness. *Electromagnetic fields*, volume 2. Wiley New York, 1979.

Chapter 2

Electrostatics

2.1 Coulomb's Law

Law 1 (Coulomb's Law)**.** *Let q_1, q_2 be two charges at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then,*

$$\mathbf{F}_{1\to 2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\|\mathbf{r}_2 - \mathbf{r}_1\|^2}\mathbf{e}_{1\to 2}. \tag{2.1}$$

The constant ϵ_0 is the *permittivity of free space*, and has the following value and units.

$$\frac{1}{4\pi\epsilon_0} \approx 9\times 10^9\,\mathrm{N\,m^2\,C^{-2}} \tag{2.2}$$

Proposition 2.1.1. *Let $\mathbf{F}$ be the electric force. Then, $\mathbf{F}$ satisfies the strong form of Newton's Third Law.*

Principle 2.1.2 (Superposition principle)**.** *The total force is the sum of forces caused by every individual charge.*

The principle of superposition is not obvious at all. If the electric force had any other form, it is perfectly possible that it would not satisfy it. For example, if the force was proportional to the square of the charges, this principle would not hold, since $(q_1+q_2)^2 \neq q_1^2 + q_2$. In this case, we know it is true by empirical fact [ ].

2.2 Electric field

2.2.1 Field representation

1. Field lines are tangent to the electric field.
2. Field lines start at positive charges and end at negative charges.
3. Field lines do not cross.
4. The number of field lines is proportional to the magnitude of the electric field.
5. At great distances of the system the electric fields are radial.
6. If the number of field lines that leave the charge is greater than the number of those that enter, the charge is positive. If they have the same number, the total charge is zero, and otherwise is negative.
7. The quotient between the values of two charges is equal to the quotient between the net numbers of field lines that leave each charge.

Streamlines equation

$$\frac{\mathrm{d}x}{E_x} = \frac{\mathrm{d}y}{E_y} = \frac{\mathrm{d}y}{E_z} \tag{2.3}$$

2.3 Gauss Theorem

Law 2 (Gauss's Law)**.** *Let $\mathbf{E}$ be the electric field produced by Coulomb's Law. If it satisfies the superposition principle, then*

$$\boxed{\oint_{\partial V} \langle \mathbf{E}, \mathrm{d}\mathbf{s}\rangle_I = \frac{Q_{\text{int}}}{\epsilon_0}.} \tag{2.4}$$

In differential form,

$$\boxed{\langle \boldsymbol{\nabla}, \mathbf{E}\rangle_I = \frac{\rho}{\epsilon_0}.} \tag{2.5}$$

Curl

$$\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}, \qquad \oint_{\Gamma} \langle \mathbf{E}, \mathrm{d}\mathbf{r}\rangle_I = \mathbf{0}. \tag{2.6}$$

2.4 Potential

Definition 2.4.1. Potential

$$\Phi(\mathbf{r}) \coloneqq \int_{\mathbf{r}_0}^{\mathbf{r}} \langle \mathbf{E}(\mathbf{r}'), \mathrm{d}\mathbf{r}' \rangle_I, \qquad [\Phi] = \mathrm{V} \coloneqq \mathrm{C\,m} = \mathrm{J\,C}^{-1} \tag{2.7}$$

We can see that

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi. \tag{2.8}$$

The word *potential* could lead to some confusions since it reminds the *potential energy*. It is important to see that they are completely different terms, although they have an important relation we will see later.
Since $\mathbf{E} = -\boldsymbol{\nabla}\Phi$, if we know one we can find the other. One could wonder about how the potential Φ function can carry the same information than $\mathbf{E}$ since former is only a scalar function (one equation) while the latter is a vector function (three equations). The answer is that its components are not completely independent. We have seen that $\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}$, and this translates to the following equations.

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}.$$

The common origin position is at infinity, but in systems with infinite size charges this convention could fail.

Proposition 2.4.1. *Let Φ be the potential generated by $\mathbf{E}$. Then, Φ obeys the superposition principle.*

Poisson's equation and Laplace's equation.

$$\boxed{\nabla^2\Phi = -\frac{\rho}{\epsilon_0}.} \tag{2.9}$$

$$\boxed{\nabla^2\Phi = \mathbf{0}.} \tag{2.10}$$

For continuous charges (and origin at infinity)

$$\Phi(\mathbf{r}_p) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}_q)}{\|\mathbf{r}_p - \mathbf{r}_q\|}\, \mathrm{d}v\,. \tag{2.11}$$

2.4.1 Boundary conditions

Electric field

The electric field is always discontinuous in surface densities. The tangential component (to the surface) is always continuous, but the normal has a discontinuity of the form

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}.$$

In general, the boundary conditions are

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\mathbf{e}_n. \tag{2.12}$$

Potential

The potential in this cases is continuous, and as boundary conditions we have

$$\frac{\partial\Phi}{\partial n} = \langle \boldsymbol{\nabla}\Phi, \mathbf{e}_n \rangle_I, \tag{2.13}$$

which is the normal derivative.

2.4.2 Some properties

Theorem 2.4.2 (Earnshaw's Theorem for Electrostatics)**.** *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone [1].*

2.4.3 Laplace equation in one dimension

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = 0 \Rightarrow \Phi(x) = mx + n \tag{2.14}$$

In one dimension, Laplace equation does not tolerate maxima or minima in points different from the extreme points. Besides, it satisfies

$$\Phi(x) = \frac{\Phi(x+a) + \Phi(x-a)}{2}.$$

2.4.4 Laplace equation in two dimensions

$$\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y} = 0. \tag{2.15}$$

Again the function is the average of those values around the point, in particular the points the lie on a circle around the original point.

Proposition 2.4.3. *Let $\Phi(x, y)$ be the solution of the Laplace equation in two dimensions and Γ a curve described by the condition $(x - x_0)^2 + (y - y_0)^2 = R^2$. Then,*

$$\Phi(x_0, y_0) = \frac{1}{2\pi R} \oint_\Gamma \Phi \,\mathrm{d}r\,. \tag{2.16}$$

Besides, Φ has not local maxima or minima.

2.4.5 Laplace equation in three dimensions

In this third case the solution again is the average of the neighbor values, in particular those who form a sphere.

Proposition 2.4.4. *Let $\Phi(x, y, z)$ be the solution of the Laplace equation in three dimensions and Σ a surface described by the condition $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$. Then,*

$$\Phi(\mathbf{r}_0) = \frac{1}{4\pi R^2} \oint_\Sigma \Phi \,\mathrm{d}s\,. \tag{2.17}$$

The solution does not admit local maxima or minima at points that do not lie in the boundary.

2.5 Energy

$$U = \frac{1}{2}\sum_{i=1}^{n} q_i \Phi(\mathbf{r}_i) \tag{2.18}$$

$$U = \frac{1}{2}\int_V \rho(\mathbf{r}_q)\Phi(\mathbf{r}_q)\,\mathrm{d}v\,. \tag{2.19}$$

$$U = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} \|\mathbf{E}\|^2\,\mathrm{d}v\,. \tag{2.20}$$

The integral expression that contains $\|\mathbf{E}\|^2$ it is always non-negative while the sum of potential could also be negative. Another distinction is that, if we calculate with the first mentioned expression the energy of a single particle, we will have an infinite value. This difference is due to the

fact these two formulas express two slightly different quantities. The integral expression shows the energy for the creation of the whole system (creation and charges and configuration) but the sum only considers the motion of particles, supposing these are already given [1].
The step where the equations changes its expression is from 2.18 to 2.19. In the first one each potential is caused by every particle different from the i-th where it is evaluated, while the integral takes into account the potential of dq generated by itself. In the integral there is no distinction because a point set does not modify the value of the integral, but in a discrete system there is indeed a difference.
One could wonder about where the energy is stored, if in the electric field or the charge distribution. In this section, there is no difference and we will discuss it later.
The energy does not obey the superposition principle. If we have two systems with energies U_1 and U_2, the total energy is

$$U_t = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} \|\mathbf{E}_1 + \mathbf{E}_2\|^2 \,\mathrm{d}v = U_1 + U_2 + \epsilon_0 \int_{\mathbb{R}^3} \langle \mathbf{E}_1, \mathbf{E}_2 \rangle_I \,\mathrm{d}v\,. \tag{2.21}$$

2.6 Conductors

Theorem 2.6.1 (Thomson's Theorem)**.** *If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors [2].*

Proof. Jackson, exercise 1.15 ■

If we have a point charge near an uncharged conductor, the force between them is not always attractive. To see that, we will explore a particular case [3]. If we take an conductor with shape of infinitely thin inferior hemisphere and a point charge, it will experience a repulsion force for small distances from the center and attraction at long distances. At $z = 0$ and $z \to \infty$ the force is zero. This configuration can be extended to hemispheres with finite width always that satisfies $d \ll R$.

Force on surface

The electric field in a infinitesimal patch caused by the surrounding region (exterior fields and neighbor patches) is

$$\mathbf{E}_{\text{other}} = \frac{\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}}{2}. \tag{2.22}$$

The force per unit area in this patch is $\mathbf{f} = \sigma\mathbf{E}$, but only contribute the surrounding field. In the case of a conductor inside the conductor there is no field and outside $\sigma/\epsilon_0 \mathbf{e}_n$, so the force is

$$\mathbf{f} = \frac{\sigma^2}{2\epsilon_0}\mathbf{e}_n. \tag{2.23}$$

If we have two conductors with q and $-q$, there is an electric field. This field is proportional to the charge because if we double q, ρ will double and hence $\mathbf{E}$. The proof of why ρ doubles without changing the configuration will come later. Since the electric field is proportional, the difference of voltage between the positive and negative conductor too. We define

$$C \coloneqq \frac{Q_+}{\Phi_+ - \Phi_-}, \qquad [C] = \mathrm{F} \coloneqq \mathrm{C\,V^{-1}}. \tag{2.24}$$

This capacitance is always positive and defined for a system of two conductors. However, in some cases one can consider the capacitance of one conductor, where the second one would be an infinite imaginary spherical shell with negative charge, whose potential won't have contribution. Then, the potential of this single conductor would have the infinite as a origin.

2.7 The method of images

With a plane there are two conditions: $\Phi = 0$ at $z = 0$ and $\Phi \to 0$ when $x^2 + y^2 + z^2 \gg d^2$. In this case, the potential and the electric field in $z \geq 0$ and the force the particle experiences are the same for a scenario with a charge q at $d\mathbf{e}_z$ and a charge $-q$ at $-d\mathbf{e}_z$. However, the energy is different. Since the electric field is the same as the alternative system but only for $z \geq 0$, the energy expressed by $\|\mathbf{E}\|^2$ has only one half of space of integration, so the total energy will be a half. In fact, this phenomenon can be generalized as follows.

Theorem 2.7.1. *The electrostatic energy of a point charge q near n perfect conductors of arbitrary shapes, each conductor being either neutral or grounded, is half the Coulombic energy between the charge q and each image charge [4].*

Theorem 2.7.2. *The electrostatic energy of a set of m point charges $q_1, \ldots, q_m$ near n perfect conductors of arbitrary shapes, each conductor being either neutral or grounded, is the Coulombic interaction energy between the real point charges plus half the sum, from $i = 1$ to $i = m$, of the Coulombic energies between charge q_i and each image charge [4].*

Sphere conductor

If now we have a conductor sphere of radius R and a point charge at $\mathbf{r} = a\mathbf{e}_y$, then we can find another image charge to solve the problem. This solution was proposed by William Thomson (also called Lord Kelvin) and published in 1848. With this method the attraction is expressed as

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q^2 Ra}{\left(a^2 - R^2\right)^2} \mathbf{e}_y. \tag{2.25}$$

2.7.1 Separation of variables

For certain configurations, if there is a side where the problem does not specifies the potential (because there is no conductor there), then we add the condition of $\Phi \to 0$ when we move away in that direction, since it is the more logical option.

Spherical coordinates with azimutal symmetry

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) = 0. \tag{2.26}$$

For the part that depends on θ, the solution is $\Theta(\theta) = P_l(\cos\theta)$. This is restrictive for two reasons: it only works for nonnegative integer values of l, and it is only one solution (we should get two because is an equation of second order). The second problem is due the fact that a second solution does not work at $\theta = 0, \pi$, so cannot represent the physical phenomenon.

Bibliography

[1] D.J. Griffiths. *Introduction to Electrodynamics*. Always learning. Pearson, 2013.

[2] J.D. Jackson. *Classical Electrodynamics*. Wiley, 2012.

[3] Michael Levin and Steven G. Johnson. Is the electrostatic force between a point charge and a neutral metallic object always attractive? *American Journal of Physics*, 79(8):843–849, Aug 2011.

[4] M M Taddei, T N C Mendes, and C Farina. Subtleties in energy calculations in the image method. *European Journal of Physics*, 30(5):965–972, Jul 2009.

Chapter 3

Electrostatics in matter

3.1 Multipole expansion

For a monopole, $\Phi \propto 1/r$; for a dipole, $\Phi \propto 1/r^2$; for a quadrupole, $\Phi \propto 1/r^3$; and for an octopole, $\Phi \propto 1/r^4$. Using the Legendre generating function,

$$\boxed{\Phi(\mathbf{r_p}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{\|\mathbf{r}_p\|^{n+1}} \int\limits_V \|\mathbf{r}_q\|^n \mathrm{P}_n(\cos\alpha)\rho(\mathbf{r}_q)\,\mathrm{d}v\,.} \tag{3.1}$$

For a *physical dipole*, the dipolar term is just an approximation. This approoximation improves by reducing d. However, by making $d \to 0$, p also tend to zero and we lose the dipolar term. A *perfect or ideal dipole* would have a distance $d \to 0$ but also $q \to$ such that $qd = p$ remains constant.
If we have a system of charge and move them by $\mathbf{a}$ (or we move our origin), then the dipole term is affected.

$$\mathbf{p}' = \mathbf{p} - Q_T\mathbf{a} \tag{3.2}$$

Then, unless total charge is zero, to define the dipole moment we need to specify the origin.

Theorem 3.1.1 (Green's Reciprocity Theorem)**.** *Let $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ two charge distributions producing two potentials $\Phi_1(\mathbf{r}), \Phi_2(\mathbf{r})$ respectively. Then,*

$$\boxed{\int\limits_{\mathbb{R}^3} \rho_1\Phi_2\,\mathrm{d}v = \int\limits_{\mathbb{R}^3} \rho_2\Phi_1\,\mathrm{d}v\,.} \tag{3.3}$$