

1 Arithmetic and topology

Definition 1.1. Let $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the sum “+” by components

$$(a, b) + (c, d) := (a + b, c + d). \quad (1)$$

- Product: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the product by

$$(a, b)(c, d) = (ac - bd, ad + bc). \quad (2)$$

We define the set \mathbb{C} as $(\mathbb{R}^2, +, \cdot)$.

Proposition 1.1. The set \mathbb{C} of complex numbers is an abelian field.

Proposition 1.2. Let \mathbb{C} be defined in the second way. Then,

1. \mathbb{C} is an abelian ring.
2. If we define f as

$$f : (\mathbb{C}, +, \cdot) \longrightarrow (\mathbb{R}^2, +, \cdot), \quad (x, y) \longmapsto x + yi, \quad (3)$$

then f is a morphism of rings.

3. The function f is, in fact, an isomorphism and \mathbb{C} is an abelian field.

Proposition 1.3. The subset of \mathbb{C} generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.

Theorem 1.4. \mathbb{C} is not an ordered field.

Proposition 1.5. For all $z, w \in \mathbb{C}$, we have:

1. $\bar{\bar{z}} = z$.
2. $\overline{z + w} = \bar{z} + \bar{w}$.
3. $\overline{zw} = \bar{z}\bar{w}$.
4. $z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.
5. $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.
6. The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.

Proposition 1.6. Let $z \in \mathbb{C}$. Then,

$$\operatorname{Re}\{z\} = \frac{z + \bar{z}}{2}, \quad \operatorname{Im}\{z\} = \frac{z - \bar{z}}{2i} \quad (4)$$

Proposition 1.7. Let $z, w \in \mathbb{C}$ and the following distance function.

$$\tilde{d} : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R} \quad (z, w) \longmapsto \tilde{d}(z, w) := |z - w| \quad (5)$$

Then, (\mathbb{C}, \tilde{d}) is a metric space.

Lemma 1.8. The set $B = \{B_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\}$ is a basis of the topology of \mathbb{R}^2 as a metric space. The set $D = \{D_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of \mathbb{C} as a metric space.

Proposition 1.9. The sets \mathbb{C} and \mathbb{R}^2 with the topology of metric space are homeomorphs.

Proposition 1.10. Let $z, w \in \mathbb{C}$. Then,

1. $|z| \geq 0$.
2. $|z| = 0 \Leftrightarrow z = 0$.
3. $-|z| \leq \operatorname{Re}\{z\} \leq |z|$ and $-|z| \leq \operatorname{Im}\{z\} \leq |z|$.
4. $|zw| = |z||w|$.
5. If $w \neq 0$, $|z/w| = |z|/|w|$.
6. $|z + w| \leq |z| + |w|$.
7. $|z + w| \geq ||z| - |w||$.
8. $|\operatorname{Re}\{zw\}| \leq |z||w|$ and $|\operatorname{Im}\{z\}| \leq |z||w|$.
9. $|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$.
10. $|z^n| = |z|^n$.

Proposition 1.11. Let $z \in \mathbb{C}$ and r_θ its polar form. Then,

$$z^n = (r^n)_{n\theta}. \quad (6)$$

Proposition 1.12. Let $z, w \in \mathbb{C}$. Then,

1. $\arg zw = \arg[z] + \arg[w] + 2\pi k$.
2. $\arg z^n = n \arg z + 2\pi k$.

Theorem 1.13. Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \dots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \dots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \dots, n\}$.

Proposition 1.14. Let $\{z_n\} = \{a_n + ib_n\}$ be a sequence of complex numbers. Then, it converges if and only if $\{a_n\}$ and $\{b_n\}$ converge.

Proposition 1.15. $\sum_{n=1}^{\infty} z_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

Proposition 1.16. $\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge.

2 Sequences and limits

Definition 2.1. A sequence of complex numbers is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned} \quad (7)$$

We denote it by $\{z_n\}_{n=m}^{\infty}$

Theorem 2.1. Let $z_n = z_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$ and $L = L_x + iL_y \in \mathbb{C}$. Then,

$$\{z_n\}_{n=0}^{\infty} \rightarrow L \Leftrightarrow \{x_n\}_{n=0}^{\infty} \rightarrow L_x \wedge \{y_n\}_{n=0}^{\infty} \rightarrow L_y. \quad (8)$$

Theorem 2.2. Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.

Theorem 2.3. Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,

$\{z_n\}_{n=0}^{\infty}$ is a Cauchy sequence $\Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ are Cauchy sequences. (9)

Theorem 2.4. The field \mathbb{C} of complex numbers is complete.

3 Functions

Definition 3.1. A topology is an ordered pair (X, τ) , where X is a set and τ a collection of subsets of X satisfying the following properties:

1. The empty set and X belong to τ .
2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ .
3. The intersection of any finite number of members of τ still belongs to τ .

The elements of τ are called *open sets* and the collection τ is called the *topology on X* .

Proposition 3.1. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \text{Re}\{f\} + i \text{Im}\{f\}$ is continuous at z_0 if and only if $\text{Re}\{f\}$ and $\text{Im}\{f\}$ are continuous at z_0 .

Proposition 3.2. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, f is continuous at z_0 if and only if for all sequence $\{z_n\}_{n=1}^{\infty}$ of Ω convergent at z_0 it is true that the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$.

Proposition 3.3. Let $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, λf , $f + g$, and fg are continuous at z_0 . The function f/g is continuous at z_0 if $g(z_0) \neq 0$.

Proposition 3.4. The radius of convergence can be calculated as follows

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \sup \sqrt[n]{|a_n|}. \quad (10)$$

Theorem 3.5 (Cauchy-Hadamard Theorem). Let

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (11)$$

be the following complex power series with $a_n, z_0 \in \mathbb{C}$ and $R \in [0, +\infty) \cup \{+\infty\}$ the radius of convergence. Then,

1. If $|z - z_0| < R$ then S converges. In fact, for all $r < R$ we have S converges uniformly at the disc $\overline{D_r(z_0)}$.
2. If $|z - z_0| > R$ then S diverges.

3. The function $f(z) = S(z)$ is derivable at $D_R(z_0)$ and its formal derivative is

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \quad (12)$$

Theorem 3.6 (Abel's Theorem). Let be the following series

$$\sum_{n=0}^{\infty} f_n(z_0) g_n(z_0),$$

where f, g_n are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \quad (13)$$

then the original series converges uniformly in Ω .

Theorem 3.7 (Weierstrass' criterion). Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant M_n , satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at Ω for all $z_0 \in \Omega$.

Proposition 3.8. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$. Then,

1. Every connected component of Ω is a closed of Ω with a subspace topology.
2. Two connected components are the same or are disjoint.
3. Every connected of Ω is one and only one connected component.
4. Ω is the disjoint union of its connected components.

Theorem 3.9 (Analytic prolongation Principle). Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of Ω that contains z_0 (the function can be zero also out Ω , but we are not studying that).

Lemma 3.10. Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right). \quad (14)$$

Proposition 3.11. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, f is analytic at $\Omega = D_R(0)$.

Proposition 3.12. The radius of convergence of e^z is infinite.

Proposition 3.13. $e^z = e^x$ for all $x \in \mathbb{R}$.

Proposition 3.14. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

Proposition 3.15. For all $z \in \mathbb{C}$ we have $e^z \neq 0$.

Proposition 3.16. The image of e^z is \mathbb{C}^* .

Proposition 3.17. *The derivative of e^z is e^z .*

Proposition 3.18. $\overline{e^z} = e^{\bar{z}}$.

Proposition 3.19. $|e^z| = e^{\operatorname{Re}\{z\}}$.

Proposition 3.20 (Euler's Formula). *If $\theta \in \mathbb{R}$, then e^{xi} has modulus one and we have that*

$$\boxed{e^{xi} = \cos x + i \sin x.} \quad (15)$$

Proposition 3.21. *The following function*

$$\begin{aligned} \exp : (\mathbb{R}, +) &\longrightarrow (\mathbb{C}^*, \cdot) \\ x &\longmapsto e^{xi} \end{aligned} \quad (16)$$

is a morphism of groups and its image is \mathbb{T} .

Proposition 3.22. *The complex exponential function is a periodic function with period $2\pi i$.*

Proposition 3.23. *Let $a \in \mathbb{C}^*$. Then, $e^z = a$ has infinite solutions.*

Proposition 3.24. *For all $z \in \mathbb{C}$,*

$$\sin^2 z + \cos^2 z = 1. \quad (17)$$

Proposition 3.25. *For all $z \in CC$,*

$$\cos(-z) = \cos(z), \quad \sin(-z) = -\sin(z). \quad (18)$$

Proposition 3.26. *For all $z, w \in \mathbb{C}$,*

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \quad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \quad (19)$$

Proposition 3.27. *The functions $\cos z, \sin z$ have period of 2π .*

Proposition 3.28. *Let $z_0 \in \mathbb{C}$. Then, z_0 is root of $\sin z$ ($\cos z$) if and only if it is a root of $\sin x$ ($\cos x$).*

Proposition 3.29. *For all $z \in \mathbb{C}$,*

$$\sinh^2 z - \cosh^2 z = 1. \quad (20)$$

Proposition 3.30. *For all $z \in CC$,*

$$\cosh(-z) = \cosh(z), \quad \sinh(-z) = -\sinh(z). \quad (21)$$

Proposition 3.31. *For all $z, w \in \mathbb{C}$,*

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \quad \sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \quad (22)$$

Proposition 3.32. *For all $z \in \mathbb{C}$,*

$$\cosh z = \cos(iz), \quad \cos z = \cosh(iz) \quad \sinh z = -i \sin(iz), \quad \sin z = i \sinh(iz) \quad (23)$$

Proposition 3.33. *The roots of the function $\sinh z$ are of the form $z_n = n\pi$ and those for the function $\cosh z$ are of the form $w_n = (2n+1)\pi/2i$.*

Proposition 3.34. *Given $z \in \mathbb{C}$ we can define $\ln z$ from the natural logarithm of a real number as*

$$\ln z = \ln |z| + i \arg z = \ln |z| + i \operatorname{Arg} z + 2\pi ki. \quad (24)$$

Proposition 3.35. *Let $z, w \in \mathbb{C}$ two numbers. Then,*

$$1. \quad \ln(zw) = \ln z + \ln w + 2\pi ki, k \in \mathbb{Z}.$$

2. *If we want to stay in the principal argument,*

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \text{if } \operatorname{Arg} z + \operatorname{Arg} w < 2\pi \\ \ln z + \ln w - 2\pi i, & \text{if } \operatorname{Arg} z + \operatorname{Arg} w \geq 2\pi \end{cases}. \quad (25)$$

3. *SEARCH MORE PROPERTIES*

Proposition 3.36. *If $a = \alpha + \beta i$ and $z = re^{\theta i}$, then*

$$z^a = e^{\alpha \ln r - \beta(\theta + 2\pi k)} e^{\beta \ln r + \alpha(\theta + 2\pi k)}, \quad (26)$$

$$|z^a| = e^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \quad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k) \quad (27)$$

Proposition 3.37. *Let $z, w \in \mathbb{C}$. Then,*

$$1. \quad (e^b)^a = e^{a(b + 2\pi ki)}$$

4 Derivatives

Definition 4.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We define the *derivative of f at z_0* as

$$f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (28)$$

in case the limit exists. If f has derivative, we say f is \mathbb{C} -derivable at z_0 .

Proposition 4.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in CC$ a point. If f is derivable at z_0 , then it is continuous at z_0 .*

Theorem 4.2. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two functions and $z_0 \in \Omega$ a point. Then, the following statements are true.*

1. *If f is constant at Ω , then f is derivable at z_0 and $f'(z_0) = 0$.*

2. *If $f(z) = z$ in every point of Ω , then f is derivable at z_0 and $f'(z_0) = 1$.*

3. *If f, g are derivable at z_0 and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ are derivable at z_0 and $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.*

4. *If f, g are derivable at z_0 , then fg is derivable at z_0 and*

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \quad (29)$$

5. *If f, g are derivable at z_0 and $g(z_0) \neq 0$, then f/g is derivable at z_0 and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \quad (30)$$

Theorem 4.3. Let $f : \Omega_1 \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a derivable function at a point $z_0 \in \mathbb{C}$ and $g : \Omega_2 \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be another derivable function at a point $f(z_0) \in \Omega_2$. Then, $g \circ f$ is derivable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \quad (31)$$

Theorem 4.4. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a function of class $C^1(\Omega)$ with Ω an open set, injective, and derivable at every point of Ω with non-zero derivative. Then,

1. $\Omega' = f(\Omega)$ is an open subset of \mathbb{C} .
2. The inverse function f^{-1} exist, it is well defined and it is derivable at Ω' .
3. If $z \in \Omega$ and $z' = f(z)$, then

$$(f^{-1})'(z') = \frac{1}{f'(z)}. \quad (32)$$

Proposition 4.5. A determination of $\ln z$ with $z \in \mathbb{C}$ is continuous except in a semiline.

Theorem 4.6. Let $\Omega \subseteq \mathbb{C}$ be an open set and $\phi \in C(\Omega)$, such that $e^{\phi(z)} = z$ for all $z \in \Omega$. Then, we have $\phi \in H(\Omega)$ and

$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \quad (33)$$

Proposition 4.7. A determination of $\ln z$ with $z \in \mathbb{C}$ is holomorphic except in a semiline.

Proposition 4.8. Let $I = [\theta, \theta + 2\pi)$ a determination of the logarithm, $\ln_I z$. Then, $\ln_I z$ is holomorphic except in the semiline $L_\theta = \{re^{i\theta} \in \mathbb{C} \mid r \geq 0\}$.

Theorem 4.9. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function in Ω . Then, f is holomorphic in Ω . In particular, given a point $z_0 \in \mathbb{C}$, $f'(z_0) = a_1$ where a_1 is the first coefficient of the power series that represents f in a neighborhood of z_0 .

Proposition 4.10. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a function of class $C^1(\Omega)$. Then, for all $z_0 \in \Omega$

$$f(z_0 + h) = f(z_0) + \left(\frac{\partial f}{\partial z}\right)_{z_0} h + \left(\frac{\partial f}{\partial \bar{z}}\right)_{z_0} \bar{h} + o(|h|^2). \quad (34)$$

5 Holomorphic functions

Definition 5.1. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{aligned} \gamma : I &\rightarrow \mathbb{C} \\ t &\mapsto \gamma_1(t) + i\gamma_2(t). \end{aligned} \quad (35)$$

Theorem 5.1. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with Ω an open set and $\phi : I \rightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curve (hence a derivable curve) and its real derivative is

$$\psi'(t) = f'(\phi(t))\phi'(t). \quad (36)$$

6 Local properties of holomorphic functions

7 Isolated singularities of holomorphic functions

8 Homology

9 Harmonic functions

10 Conforming representation

11 Riemann theorem

12 Runge theorem

13 Zeros of holomorphic functions

14 Fourier transform

Definition 14.1. Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform* of f at the point ξ as

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx. \quad (37)$$

Proposition 14.1. Let $f \in L^1(\mathbb{R})$ be a function. Then, the application

$$\begin{aligned} \mathcal{F}\{f\} : \mathbb{R} &\rightarrow \mathbb{C} \\ \xi &\mapsto \hat{f}(\xi) \end{aligned} \quad (38)$$

is a well defined application.

Theorem 14.2. Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.

1. The Fourier transform of f satisfies

$$\mathcal{F}\{f\} \in L^\infty(\mathbb{R}), \quad \|\mathcal{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|f\|_1. \quad (39)$$

2. $\mathcal{F}\{f\}$ is \mathbb{C} linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,

$$\mathcal{F}\{\alpha f + \beta g\} = \alpha \mathcal{F}\{f\} + \beta \mathcal{F}\{g\}. \quad (40)$$

3. If $g(x) = \bar{f}(x)$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \quad (41)$$

4. If $g(x) = g(\lambda x)$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \frac{1}{\lambda} \hat{f}\left(\frac{\xi}{\lambda}\right). \quad (42)$$

5. If $g(x) = f(x - a)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = e^{-ia\xi} \hat{f}(\xi). \quad (43)$$

6. If $g(x) = e^{iax} f(x)$ with $\alpha \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \hat{f}(\xi - a) \quad (44)$$

7. If $\{f_n\}_{n \in \mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \rightarrow f$ in $L^1(\mathbb{R})$ when $n \rightarrow \infty$, then $\mathcal{F}\{f_n\} \rightarrow \mathcal{F}\{f\}$ uniformly in \mathbb{R} .

8. The Fourier transform $\mathcal{F}\{f\}$ is a continuous function in \mathbb{R} , $\mathcal{F}\{f\} \in C(\mathbb{R})$.

Proposition 14.3. Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and $\lim_{|x| \rightarrow \infty} |f(x)| = 0$. Then,

$$\hat{f}'(\xi) = i\xi \hat{f}(\xi). \quad (45)$$

Theorem 14.4. Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_n \in \mathcal{D}(\mathbb{R})$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f - \phi_n| dx = 0, \quad (46)$$

that is, we have convergence of ϕ_n to f with norm $\|\cdot\|_1$.

Proposition 14.5. Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.

Proposition 14.6. Let $f \in L^1(\mathbb{R})$ be a function. Then, $|\hat{f}(\xi)| \leq \|f\|_1$.

Theorem 14.7. Let $f \in L^1(\mathbb{R})$ be a function. Then,

$$\lim_{|x| \rightarrow \infty} \hat{f}(x) = 0. \quad (47)$$

Theorem 14.8. The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathcal{F}\{f\} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$.

Proposition 14.9. Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \rightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,

1. $f + g \in S(\mathbb{R})$.
2. $\lambda f \in S(\mathbb{R})$.
3. $fg \in S(\mathbb{R})$.
4. $Pf \in S(\mathbb{R})$.

Theorem 14.10. Let $I, J \subseteq \mathbb{R}$ be two intervals with I compact and J open. Let $f : I \times J \rightarrow \mathbb{R}$ be a function such that

1. $f(\cdot, \lambda)$ is Riemann-integrable in I for all $\lambda \in J$,
2. $f(x, \cdot)$ is derivable in J for all $x \in I$.

If $\partial_\lambda f$ is continuous in $I \times J$, then

1. $\partial_\lambda f(\cdot, \lambda)$ is Riemann-integrable for all $\lambda \in J$.

2. $F(\lambda) = \int_I f(x, \lambda) dx$ is derivable with continuous derivative in J for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.

$$F'(\lambda) = \frac{d}{d\lambda} \int_I f(x, \lambda_0) dx = \int_I \frac{\partial f}{\partial \lambda}(x, \lambda_0) dx, \forall \lambda_0 \in J \quad (48)$$

Proposition 14.11. Let $f \in S(\mathbb{R})$. Then,

1. $S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.
2. $\widehat{x f}(\xi) = (i D_\xi \hat{f})(\xi)$ for all $\xi \in \mathbb{R}$.

Proposition 14.12. The Fourier transform \mathcal{F} restricted to $S(\mathbb{R})$ is an automorphism, that is, if $f \in S(\mathbb{R})$ then $\mathcal{F}\{f\} = \hat{f} \in S(\mathbb{R})$.

Lemma 14.13. If $G(x) = e^{-x^2/2}$, then $\hat{G}(\xi) = e^{-\xi^2/2}$. We observe hence that G is a fixed point of \mathcal{F} .

Lemma 14.14. If $f, g \in S(\mathbb{R})$, then

$$\int_{\mathbb{R}} f(\xi) \hat{g}(\xi) d\xi = \int_{\mathbb{R}} \hat{f}(\tau) g(\tau) d\tau. \quad (49)$$

Lemma 14.15. Let $f, g \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,

1. $g(\lambda x) \hat{f}(x)$ converges to $g(0) \hat{f}(x)$ uniformly in \mathbb{R} when $\lambda \rightarrow \infty$.
2. $f(\lambda x) \hat{g}(x)$ converges to $f(0) \hat{g}(x)$ uniformly in \mathbb{R} when $\lambda \rightarrow \infty$.

Lemma 14.16. Let $f, g \in s(\mathbb{R})$. Then,

$$f(0) \int_{\mathbb{R}} \hat{g}(\xi) d\xi = g(0) \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad (50)$$

Lemma 14.17. Let $f \in s(\mathbb{R})$ be a function. Then,

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad (51)$$

Theorem 14.18 (Inversion of \mathcal{F} in $S(\mathbb{R})$). Let $\mathcal{F} : S(\mathbb{R}) \rightarrow S(\mathbb{R})$, defined by $\mathcal{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, \mathcal{F} is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathcal{F}^4 = \text{Id}$. In particular, $\mathcal{F}^{-1} = \mathcal{F}^3$ and if $f \in S(\mathbb{R})$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}\{f\}(\xi) e^{ix\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} e^{ix\xi} d\xi. \quad (52)$$

In fact, \mathcal{F} is an homomorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(S(\mathbb{R}), \|\cdot\|_{n,m})$.

Theorem 14.19 (Tonelli's Theorem). Let $f : I \times J \rightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x, y) \geq 0$ for all $(x, y) \in I \times J$. Then,

$$\int_{I \times J} f dx dy = \int_I \int_J f(x, y) dy dx = \int_J \int_I f(x, y) dx dy. \quad (53)$$

Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.

Proposition 14.20. Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi} \hat{f} \hat{g}$.

Theorem 14.21. Let $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^\infty(\mathbb{R})$.

Theorem 14.22. Let $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$ with $\text{supp } f$ compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and $\text{supp } \{f * \phi\} \subseteq \text{supp } f + \text{supp } \phi$.

Theorem 14.23. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \rightarrow 0$ $f * \phi_\epsilon$ converges uniformly in \mathbb{R} to f .

Theorem 14.24 (Weierstrass polynomial approximation). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials P_n with $n \in \mathbb{N}$ such that P_n converge uniformly to f in $[a, b]$.

Theorem 14.25. Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \rightarrow f$ with norm $\|\cdot\|_p$ (that is, convergence in L^p), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \quad (54)$$

with $\|f\|_{C^k(\mathbb{R})} = \max_{0 \leq l \leq k} (\sup_{x \in \mathbb{R}} |D^l f(x)|)$ being a norm.

Lemma 14.26. Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\int_{\mathbb{R}} f(x) \phi(x) dx = 0$. Then, $f \equiv 0$.

Theorem 14.27 (Inversion theorem in $L^1(\mathbb{R})$). Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi, \forall x \in \mathbb{R}. \quad (55)$$

15 Multidimensional fourier transform

Theorem 15.1. For several variables

$$\mathcal{F}\{f(x_1, \dots, x_n)\} = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) e^{-i(x_1\xi_1 + \cdots + x_n\xi_n)} d\xi_1 \cdots d\xi_n \quad (56)$$

or simpler,

$$\boxed{\mathcal{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\langle \mathbf{x}, \boldsymbol{\xi} \rangle_I} d\hat{\mathbf{x}}.} \quad (57)$$