

Contents

1	Vector analysis	3
1.1	Introduction	4
1.1.1	Representation of positions	4
1.1.2	properties of the euclidean space	4
1.1.3	Other details	4
1.2	Construction of general orthogonal coordinate systems	4
1.3	Properties	5
1.3.1	Products	5
2	Electrostatics	7
2.1	Coulomb's Law	8
2.2	Electric field	8
2.2.1	Field representation	8
2.3	Gauss Theorem	9
2.3.1	How to use Gauss theorem	9
2.4	Potential	9
2.4.1	Boundary conditions	10
2.4.2	Some properties	10
2.4.3	Laplace equation in one dimension	11
2.4.4	Laplace equation in two dimensions	11
2.4.5	Laplace equation in three dimensions	11
2.5	Energy	11
2.6	Conductors	12
2.7	The method of images	12
2.7.1	Separation of variables	13
3	Electrostatics in matter	15
3.1	Multipole expansion	16
4	Magnetostatics	17
4.1	Electric current	18
4.2	Continuity equation	18
4.3	Biot-Savart law	18
4.4	Force between wires	19
4.5	Ampere's Law	19
4.6	Divergence of the magnetic field	19
4.7	Vector potential	19
4.8	Scalar potential	19
5	Magnetostatics in matter	21
5.1	Multipolar expansion	22
5.2	Magnetic dipole	22
5.3	Generated field by magnetic a media	22
5.4	Magnetic intensity	23
5.5	Classification of magnetic materials	23

6	Slow variable fields	25
6.1	Faraday's Law	26
6.2	Limitations of Faraday's Law	26
6.3	Mutual inductance and autoinductance	26
6.4	Magnetic energy of coupled circuits	27
7	Maswell's equations	29
7.1	Maxwell equations	30
7.2	Scalar and vector potentials	31
8	Electromagnetic potentials and radiation field	33

Chapter 1

Vector analysis

1.1 Introduction

1.1.1 Representation of positions

We can represent locations in real world using the points in the standard euclidean affine space, that is, an affine space whose associated vector space is has as the matrix of its bilinear form the identity matrix. This representation allows describe the details of reality for two main reasons. The first one is that, since its associated vector space has as the bilinear form the standard dot product, it satisfies too the definition of metric space and then we can establish the concept of distance and all the other related topological concepts. The second one is that, although we cannot operate points of the euclidean space, we can describe changes of positions using the position vector of the associated vector space. This position vector is described as follows.

Definition 1.1.1. Let $\mathfrak{R} = (O, \mathfrak{B})$ be a reference the euclidean space \mathbb{R}^3 and P a point of the this affine space. We define the position vector \vec{r}_p of the point P as the point that satisfies

$$P = O + \vec{r}_p \quad (1.1)$$

1.1.2 properties of the euclidean space

1.1.3 Other details

We will supposed all functions are differentiable.

1.2 Construction of general orthogonal coordinate systems

We have seen that we can associated a position vector \vec{r} to some point, given a reference. With that, given a basis of the associated vector space, we relate to the point the coordinates of this vector. Usually we use the basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, that we can form from the parameters (x, y, z) that determine the position of a point. However, we can use other parametres, so now we will see how, from three parameters (that satisfy certain conditions), we can generate a basis of the vector space and hence a frame.

Suppose that q_1, q_2, q_3 are independent functions of position such that

$$q_1 = q_1(x, y, z), \quad q_2 = q_2(x, y, z), \quad q_3 = q_3(x, y, z), \quad (1.2)$$

We know that if in regions where the Jacobian $\frac{\partial(q_1, q_2, q_3)}{\partial(x, y, z)}$ is different from zero, the system of equations can be solved simultaneously for x, y , and z , giving

$$x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3) \quad (1.3)$$

and therefore,

$$\vec{r} = \vec{r}(q_1, q_2, q_3) \quad (1.4)$$

In this case every specification of numbers (x, y, z) corresponds to a specification of (q_1, q_2, q_3) . Thus, we come to regard the set of three numbers (q_1, q_2, q_3) as the *curvilinear coordinates* of a point in the euclidean space.

Let us suppose now we go from one point P with position vector \vec{r} to a point Q whose position vector is $\vec{r} + d\vec{r}$. Supposing the \vec{r} is in general differentiable [], we have the total distance traveled is

$$\overrightarrow{PQ} = d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3. \quad (1.5)$$

This is a linear combination of three vectors, but we are interested in unitary vectors, so we have to normalize them with what are called *metric factors* h_i .

$$\vec{e}_i = \frac{\partial \vec{r} / \partial q_i}{\|\partial \vec{r} / \partial q_i\|} = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial q_i}, \quad h_i := \left\| \frac{\partial \vec{r}}{\partial q_i} \right\| \quad (1.6)$$

With that, the differential of length is expressed as follows.

$$d\vec{l} := \vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3 = h_1 dq_1 \vec{e}_1 + h_2 dq_2 \vec{e}_2 + h_3 dq_3 \vec{e}_3 \quad (1.7)$$

1.3 Properties

1.3.1 Products

1. $\langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle_I = \langle \mathbf{b}, \mathbf{c} \times \mathbf{a} \rangle_I = \langle \mathbf{c}, \mathbf{a} \times \mathbf{b} \rangle_I$
2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \langle \mathbf{a}, \mathbf{c} \rangle_I - \mathbf{c} \langle \mathbf{a}, \mathbf{b} \rangle_I$
3. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b} \langle \mathbf{a}, \mathbf{c} \rangle_I - \mathbf{a} \langle \mathbf{b}, \mathbf{c} \rangle_I$
4. $\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle_I = \langle \mathbf{a}, \mathbf{c} \rangle_I \langle \mathbf{b}, \mathbf{d} \rangle_I - \langle \mathbf{a}, \mathbf{d} \rangle_I \langle \mathbf{b}, \mathbf{c} \rangle_I = \langle \mathbf{a}, \mathbf{b} \times (\mathbf{c} \times \mathbf{d}) \rangle_I$
5. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{c} \langle \mathbf{a}, \mathbf{b} \times \mathbf{d} \rangle_I - \mathbf{d} \langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle_I = \mathbf{b} \langle \mathbf{a}, \mathbf{c} \times \mathbf{d} \rangle_I - \mathbf{a} \langle \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle_I$

Nabla operator

1. $\nabla f g = f \nabla g + g \nabla f$
2. $\nabla \langle \mathbf{a}, \mathbf{b} \rangle_I = \langle \mathbf{b}, \nabla \rangle_I \mathbf{a} + \langle \mathbf{a}, \nabla \rangle_I \mathbf{b} + \mathbf{b} \times (\nabla \times \mathbf{a}) + \mathbf{a} \times (\nabla \times \mathbf{b})$
3. $\nabla \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = -\frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3}, \quad \nabla_1 \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3}$
4. $\nabla \|\mathbf{r} - \mathbf{r}'\|^n = n \|\mathbf{r} - \mathbf{r}'\|^{n-1} \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|}, \quad \langle \nabla, \|\mathbf{r} - \mathbf{r}'\|^n (\mathbf{r} - \mathbf{r}') \rangle_I = (n+3) \|\mathbf{r} - \mathbf{r}'\|^n$
5. $\nabla(f\mathbf{a}) = \langle \nabla f, \mathbf{a} \rangle_I + f \langle \nabla, \mathbf{a} \rangle_I$
6. $\langle \nabla, \mathbf{a} \times \mathbf{b} \rangle_I = \langle \mathbf{b}, \nabla \times \mathbf{a} \rangle_I - \langle \mathbf{a}, \nabla \times \mathbf{b} \rangle_I$
7. $\nabla^2 f := \langle \nabla, \nabla f \rangle_I$
8. $\langle \nabla, \nabla \times \mathbf{a} \rangle_I = 0$
9. $\langle \nabla, \mathbf{b} \rangle_I = 0 \Leftrightarrow \mathbf{b} = \nabla \times \mathbf{a}$
10. $\nabla \times (f\mathbf{a}) = (\nabla f) \times \mathbf{a} + f(\nabla \times \mathbf{a})$
11. $\nabla \times (\mathbf{a} \times \mathbf{b}) = \langle \mathbf{b}, \nabla \rangle_I \mathbf{a} - \langle \mathbf{a}, \nabla \rangle_I \mathbf{b} + \langle \nabla, \mathbf{b} \rangle_I \mathbf{a} - \langle \nabla, \mathbf{a} \rangle_I \mathbf{b}$
12. $\nabla \times \nabla f = 0$
13. $\nabla \times \mathbf{a} = \mathbf{0} \Leftrightarrow \mathbf{a} = \nabla f$
14. $\nabla^2 \mathbf{a} := \nabla \langle \nabla, \mathbf{a} \rangle_I - \nabla \times \nabla \times \mathbf{a}$
15. $\langle \nabla, \mathbf{r} \rangle_I = 3, \quad \nabla \times \mathbf{r} = \mathbf{0}$
16. $\left\langle \nabla, \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|} \right\rangle_I = \frac{2}{\|\mathbf{r} - \mathbf{r}'\|}, \quad \nabla \times \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|} = \mathbf{0}, \quad \nabla^2 \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = -4\pi \delta(\mathbf{r} - \mathbf{r}')$

Integrals

1. $\langle \nabla, \mathbf{A} \rangle_I dv = \oint_{\sigma} \langle \mathbf{A}, d\mathbf{s} \rangle_I$
2. $\int_{\Sigma} \langle \nabla \times \mathbf{A}, d\mathbf{s} \rangle_I = \oint_{\partial \Sigma} \langle \mathbf{A}, d\mathbf{r} \rangle_I$
3. $\int_V \nabla \times \mathbf{A} dv = - \oint_{\partial V} \mathbf{A} \times d\mathbf{s}$
4. $\int_V \nabla f dv = \oint_{\partial V} f d\mathbf{s}$

$$5. - \int_{\Sigma} \nabla f \times \mathbf{ds} = \oint_{\partial \Sigma} f \, \mathbf{dr}$$

$$6. \int_{\Sigma} \langle f \nabla \times \mathbf{A}, \mathbf{ds} \rangle_I = \int_{\Sigma} \langle \mathbf{A} \times \nabla f, \mathbf{ds} \rangle_I + \oint_{\partial \Sigma} \langle f \mathbf{A}, \mathbf{dr} \rangle_I$$

$$7. \int_V \langle \mathbf{B}, \nabla \times \mathbf{A} \rangle_I = \int_V \langle \mathbf{A}, \nabla \times \mathbf{B} \rangle_I \, dv + \oint_{\partial V} \langle \mathbf{A} \times \mathbf{B}, \mathbf{ds} \rangle_I$$

$$8. \text{Green's identity: } \int_v f \nabla^2 g + \langle \nabla f, \nabla g \rangle_I \, dv = \oint_{\partial V} \langle f \nabla g, \mathbf{ds} \rangle_I$$

$$9. \text{Green's second identity: } \int_v f \nabla^2 g - g \nabla^2 f \, dv = \oint_{\partial V} \langle f \nabla g - g \nabla f, \mathbf{ds} \rangle_I$$

$$10. \text{Surface vector: } \mathbf{S} := \int_{\Sigma} \mathbf{ds} = \frac{1}{2} \int_{\partial \Sigma} \mathbf{r} \times \mathbf{dr}, \quad \oint_{\Gamma} \langle \mathbf{c}, \mathbf{r} \rangle_I \, \mathbf{dr} = \mathbf{S} \times \mathbf{c}, \mathbf{c} = \text{ctt}$$

11. Helmholtz Theorem:

$$\begin{cases} \langle \nabla, \mathbf{F} \rangle_I = \rho \\ \nabla \times \mathbf{F} = \mathbf{J} \end{cases} \Rightarrow \mathbf{F} = -\nabla \phi + \nabla \times \mathbf{A}, \begin{cases} \phi = \frac{1}{4\pi} \int_V \frac{\rho(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} \, dv - \frac{1}{4\pi} \oint_{\Sigma} \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \langle \mathbf{F}(\mathbf{r}'), \mathbf{ds} \rangle_I \\ \mathbf{A} = \frac{1}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} \, dv + \frac{1}{4\pi} \oint_{\Sigma} \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \mathbf{F} \times \mathbf{ds} \end{cases}$$

Dirac delta function in one dimension

1. $\delta(x) = \delta(-x)$
2. $f(x)\delta'(x) = -f'(x)\delta(x)$
3. $\delta'(x) = -\delta'(-x)$
4. $x^n \delta(x) = 0 \forall n \in \mathbb{N}^*, x \in \mathbb{R}$
5. $\delta(ax - b) = |a|^{-1} \delta(x - (b/a)), a \neq 0.$
6. $h(x)\delta(x - a) = h(a)\delta(x - a)$
7. $h(x)\delta'(x - a) = h(a)\delta'(x - a) - h'(a)\delta(x - a)$
8. $\delta(f(x)) = \sum_n |f'(x_n)|^{-1} \delta(x - x_n), f(x_n) = 0, f'(x_n) \neq 0$
9. $\delta(x) \in C^{-2}(\mathbb{R}), \delta(x) = \theta'(x) = R''(x)$

Dirac delta function in several dimension

1. $\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$
2. $\delta(a\mathbf{r}) = |a|^{-n} \delta(\mathbf{r})$
3. $\int_{\mathbb{R}^n} f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{a}) \, dv = f(\mathbf{a})$

Bibliography

Chapter 2

Electrostatics

2.1 Coulomb's Law

Law 1 (Coulomb's Law). *Let q_1, q_2 be two charges at positions \mathbf{r}_1 and \mathbf{r}_2 respectively. Then,*

$$\mathbf{F}_{1 \rightarrow 2} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{\|\mathbf{r}_2 - \mathbf{r}_1\|^2} \mathbf{e}_{1 \rightarrow 2}. \quad (2.1)$$

The constant ϵ_0 is the *permittivity of free space*, and has the following value and units.

$$\frac{1}{4\pi\epsilon_0} \approx 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2} \quad (2.2)$$

Proposition 2.1.1. *Let \mathbf{F} be the electric force. Then, \mathbf{F} satisfies the strong form of Newton's Third Law.*

Principle 2.1.2 (Superposition principle). *The total force is the sum of forces caused by every individual charge.*

The principle of superposition is not obvious at all. If the electric force had any other form, it is perfectly possible that it would not satisfy it. For example, if the force was proportional to the square of the charges, this principle would not hold, since $(q_1 + q_2)^2 \neq q_1^2 + q_2^2$. In this case, we know it is true by empirical fact [].

2.2 Electric field

2.2.1 Field representation

1. Field lines are tangent to the electric field.
2. Field lines start at positive charges and end at negative charges.
3. Field lines do not cross.
4. The number of field lines is proportional to the magnitude of the electric field.
5. At great distances of the system the electric fields are radial.
6. If the number of field lines that leave the charge is greater than the number of those that enter, the charge is positive. If they have the same number, the total charge is zero, and otherwise is negative.
7. The quotient between the values of two charges is equal to the quotient between the net numbers of field lines that leave each charge.

Streamlines equation

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \quad (2.3)$$

Proposition 2.2.1. *Let $\mathbf{r}_p \in \mathbb{R}^3$ be a vector, $B(r_p, \rho)$ an open ball of center r_p and radius δ , and $\lambda \in \mathbb{R}$ a number. Then, the integral*

$$I = \int_{B(r_p, \delta)} \frac{1}{\|\mathbf{r} - \mathbf{r}_q\|^\lambda} dv_q \quad (2.4)$$

is convergent for $\lambda < 3$ and divergent for $\lambda \geq 3$ [1].

Corollary 2.2.2. *Let $\mathbf{r}_p \in \mathbb{R}^3$ be a position vector, $K \subseteq \mathbb{R}^3$ a compact, and $\rho(\mathbf{r}_p) : K \rightarrow \mathbb{R}$ a bounded function. Then, the integral*

$$I = \int_K \rho(\mathbf{r}_p) \frac{\mathbf{r}_p - \mathbf{r}_1}{\|\mathbf{r}_p - \mathbf{r}_1\|^3} dv_q \quad (2.5)$$

is convergent [1].

Corollary 2.2.3. Let $\mathbf{r}_p \in \mathbb{R}^3$ be a position vector and $\rho(\mathbf{r}_p) : \mathbb{R}^3 \rightarrow \mathbb{R}$ a function of class $C^\infty(\mathbb{R}^3)$ and a compact support. Then, the electric field can be calculated by the integral [1]

$$\mathbf{E} = \int_{\mathbb{R}^3} \rho(\mathbf{r}_p) \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|^2} \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|} dv_q. \quad (2.6)$$

Proposition 2.2.4. Let $\mathbf{r}_p = (x^1, x^2, x^3) \in \mathbb{R}^3$ be a position vector, $\rho(\mathbf{r}_p) : \mathbb{R}^3 \rightarrow \mathbb{R}$ a function of class $C^\infty(\mathbb{R}^3)$ and a compact support, and the integral

$$\Phi(\mathbf{r}) = \int_{\mathbb{R}^3} \rho(\mathbf{r}_p) \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} dv_q. \quad (2.7)$$

Then, the integral is convergent, the function is derivable and its partial derivatives are obtained deriving under the integral sign [1]

$$\frac{\partial \Phi}{\partial x^i} = \int_{\mathbb{R}^3} \frac{\partial}{\partial x^i} \left(\frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} \right) \rho(\mathbf{r}_q) dv_q. \quad (2.8)$$

2.3 Gauss Theorem

Law 2 (Gauss's Law). Let \mathbf{E} be the electric field produced by Coulomb's Law. If it satisfies the superposition principle, then

$$\oint_{\partial V} \langle \mathbf{E}, d\mathbf{s} \rangle_I = \frac{Q_{\text{int}}}{\epsilon_0}. \quad (2.9)$$

In differential form,

$$\langle \nabla, \mathbf{E} \rangle_I = \frac{\rho}{\epsilon_0}. \quad (2.10)$$

Curl

$$\nabla \times \mathbf{E} = \mathbf{0}, \quad \oint_{\Gamma} \langle \mathbf{E}, d\mathbf{r} \rangle_I = 0. \quad (2.11)$$

2.3.1 How to use Gauss theorem

We use arguments of symmetry. See onenote exercise 3.17.

2.4 Potential

Definition 2.4.1. Potential

$$\Phi(\mathbf{r}) := \int_{\mathbf{r}_0}^{\mathbf{r}} \langle \mathbf{E}(\mathbf{r}'), d\mathbf{r}' \rangle_I, \quad [\Phi] = V := \text{C m} = \text{J C}^{-1} \quad (2.12)$$

Theorem 2.4.1.

$$\mathbf{E} = -\nabla \Phi. \quad (2.13)$$

The word *potential* could lead to some confusions since it reminds the *potential energy*. It is important to see that they are completely different terms, although they have an important relation we will see later.

Since $\mathbf{E} = -\nabla \Phi$, if we know one we can find the other. One could wonder about how the potential Φ function can carry the same information than \mathbf{E} since former is only a scalar function (one equation) while the latter is a vector function (three equations). The answer is that its components

are not completely independent. We have seen that $\nabla \times \mathbf{E} = \mathbf{0}$, and this translates to the following equations.

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}.$$

The common origin position is at infinity, but in systems with infinite size charges this convention could fail.

Proposition 2.4.2. *Let Φ be the potential generated by \mathbf{E} . Then, Φ obeys the superposition principle.*

Theorem 2.4.3. $\partial\Phi/\partial x^i$ is derivable with respect to x^i and

$$\frac{\partial^2 \Phi}{\partial (x^i)^2} = \int_{\mathbb{R}^3} \frac{\partial}{\partial x^i} \left(\frac{\rho(\mathbf{r}_p)}{\|\mathbf{r}_p - \mathbf{r}_q\|^2} \frac{x_q^i - x_p^i}{\|\mathbf{r}_p - \mathbf{r}_q\|} \right) dv_q - \rho(\mathbf{r}_p) \int_{S^2} (\omega^i)^2 d\omega, \quad (2.14)$$

which is convergent and where $S^2 = \{\mathbf{r}_q \in \mathbb{R}^3 \mid \|\mathbf{r}_q\| = 1\}$ [1].

Poisson's equation and Laplace's equation.

$$\boxed{\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}.} \quad (2.15)$$

$$\boxed{\nabla^2 \Phi = \mathbf{0}.} \quad (2.16)$$

For continuous charges (and origin at infinity)

$$\Phi(\mathbf{r}_p) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}_q)}{\|\mathbf{r}_p - \mathbf{r}_q\|} dv. \quad (2.17)$$

2.4.1 Boundary conditions

Electric field

The electric field is always discontinuous in surface densities. The tangential component (to the surface) is always continuous, but the normal has a discontinuity of the form

$$E_{\text{above}}^\perp - E_{\text{below}}^\perp = \frac{\sigma}{\epsilon_0}.$$

In general, the boundary conditions are

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \mathbf{e}_n. \quad (2.18)$$

Potential

The potential in this cases is continuous, and as boundary conditions we have

$$\frac{\partial \Phi}{\partial n} = \langle \nabla \Phi, \mathbf{e}_n \rangle_I, \quad (2.19)$$

which is the normal derivative.

2.4.2 Some properties

Theorem 2.4.4 (Earnshaw's Theorem for Electrostatics). *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone [2].*

2.4.3 Laplace equation in one dimension

$$\frac{d^2\Phi}{dx^2} = 0 \Rightarrow \Phi(x) = mx + n \quad (2.20)$$

In one dimension, Laplace equation does not tolerate maxima or minima in points different from the extreme points. Besides, it satisfies

$$\Phi(x) = \frac{\Phi(x+a) + \Phi(x-a)}{2}.$$

2.4.4 Laplace equation in two dimensions

$$\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y} = 0. \quad (2.21)$$

Again the function is the average of those values around the point, in particular the points the lie on a circle around the original point.

Proposition 2.4.5. *Let $\Phi(x, y)$ be the solution of the Laplace equation in two dimensions and Γ a curve described by the condition $(x - x_0)^2 + (y - y_0)^2 = R^2$. Then,*

$$\Phi(x_0, y_0) = \frac{1}{2\pi R} \oint_{\Gamma} \Phi \, dr. \quad (2.22)$$

Besides, Φ has not local maxima or minima.

2.4.5 Laplace equation in three dimensions

In this third case the solution again is the average of the neighbor values, in particular those who form a sphere.

Proposition 2.4.6. *Let $\Phi(x, y, z)$ be the solution of the Laplace equation in three dimensions and Σ a surface described by the condition $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$. Then,*

$$\Phi(\mathbf{r}_0) = \frac{1}{4\pi R^2} \oint_{\Sigma} \Phi \, ds. \quad (2.23)$$

The solution does not admit local maxima or minima at points that do not lie in the boundary.

2.5 Energy

$$U = \frac{1}{2} \sum_{i=1}^n q_i \Phi(\mathbf{r}_i) \quad (2.24)$$

$$U = \frac{1}{2} \int_V \rho(\mathbf{r}_q) \Phi(\mathbf{r}_q) \, dv. \quad (2.25)$$

$$U = \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} \|\mathbf{E}\|^2 \, dv. \quad (2.26)$$

The integral expression that contains $\|\mathbf{E}\|^2$ it is always non-negative while the sum of potential could also be negative. Another distinction is that, if we calculate with the first mentioned expression the energy of a single particle, we will have an infinite value. This difference is due to the fact these two formulas express two slightly different quantities. The integral expression shows the energy for the creation of the whole system (creation and charges and configuration) but the sum only considers the motion of particles, supposing these are already given [2].

The step where the equations changes its expression is from 2.24 to 2.25. In the first one each potential is caused by every particle different from the i -th where it is evaluated, while the integral

takes into account the potential of dq generated by itself. In the integral there is no distinction because a point set does not modify the value of the integral, but in a discrete system there is indeed a difference.

One could wonder about where the energy is stored, if in the electric field or the charge distribution. In this section, there is no difference and we will discuss it later.

The energy does not obey the superposition principle. If we have two systems with energies U_1 and U_2 , the total energy is

$$U_t = \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} \|\mathbf{E}_1 + \mathbf{E}_2\|^2 dv = U_1 + U_2 + \epsilon_0 \int_{\mathbb{R}^3} \langle \mathbf{E}_1, \mathbf{E}_2 \rangle_I dv. \quad (2.27)$$

2.6 Conductors

Theorem 2.6.1 (Thomson's Theorem). *If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors [3].*

Proof. Jackson, exercise 1.15 ■

If we have a point charge near an uncharged conductor, the force between them is not always attractive. To see that, we will explore a particular case [4]. If we take an conductor with shape of infinitely thin inferior hemisphere and a point charge, it will experience a repulsion force for small distances from the center and attraction at long distances. At $z = 0$ and $z \rightarrow \infty$ the force is zero. This configuration can be extended to hemispheres with finite width always that satisfies $d \ll R$.

Force on surface

The electric field in a infinitesimal patch caused by the surrounding region (exterior fields and neighbor patches) is

$$\mathbf{E}_{\text{other}} = \frac{\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}}{2}. \quad (2.28)$$

The force per unit area in this patch is $\mathbf{f} = \sigma \mathbf{E}$, but only contribute the surrounding field. In the case of a conductor inside the conductor there is no field and outside $\sigma/\epsilon_0 \mathbf{e}_n$, so the force is

$$\mathbf{f} = \frac{\sigma^2}{2\epsilon_0} \mathbf{e}_n. \quad (2.29)$$

If we have two conductors with q and $-q$, there is an electric field. This field is proportional to the charge because if we double q , ρ will double and hence \mathbf{E} . The proof of why ρ doubles without changing the configuration will come later. Since the electric field is proportional, the difference of voltage between the positive and negative conductor too. We define

$$C := \frac{Q_+}{\Phi_+ - \Phi_-}, \quad [C] = \text{F} := \text{C V}^{-1}. \quad (2.30)$$

This capacitance is always positive and defined for a system of two conductors. However, in some cases one can consider the capacitance of one conductor, where the second one would be an infinite imaginary spherical shell with negative charge, whose potential won't have contribution. Then, the potential of this single conductor would have the infinite as a origin.

2.7 The method of images

With a plane there are two conditions: $\Phi = 0$ at $z = 0$ and $\Phi \rightarrow 0$ when $x^2 + y^2 + z^2 \gg d^2$. In this case, the potential and the electric field in $z \geq 0$ and the force the particle experiences are the same for a scenario with a charge q at $d\mathbf{e}_z$ and a charge $-q$ at $-d\mathbf{e}_z$. However, the energy is different. Since the electric field is the same as the alternative system but only for $z \geq 0$, the energy expressed by $\|\mathbf{E}\|^2$ has only one half of space of integration, so the total energy will be a half. In fact, this phenomenon can be generalized as follows.

Theorem 2.7.1. *The electrostatic energy of a point charge q near n perfect conductors of arbitrary shapes, each conductor being either neutral or grounded, is half the Coulombic energy between the charge q and each image charge [5].*

Theorem 2.7.2. *The electrostatic energy of a set of m point charges q_1, \dots, q_m near n perfect conductors of arbitrary shapes, each conductor being either neutral or grounded, is the Coulombic interaction energy between the real point charges plus half the sum, from $i = 1$ to $i = m$, of the Coulombic energies between charge q_i and each image charge [5].*

Sphere conductor

If now we have a conductor sphere of radius R and a point charge at $\mathbf{r} = a\mathbf{e}_y$, then we can find another image charge to solve the problem. This solution was proposed by William Thomson (also called Lord Kelvin) and published in 1848. With this method the attraction is expressed as

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q^2 Ra}{(a^2 - R^2)^2} \mathbf{e}_y. \quad (2.31)$$

2.7.1 Separation of variables

For certain configurations, if there is a side where the problem does not specifies the potential (because there is no conductor there), then we add the condition of $\Phi \rightarrow 0$ when we move away in that direction, since it is the more logical option.

Spherical coordinates with azimuthal symmetry

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) = 0. \quad (2.32)$$

For the part that depends on θ , the solution is $\Theta(\theta) = P_l(\cos \theta)$. This is restrictive for two reasons: it only works for nonnegative integer values of l , and it is only one solution (we should get two because is an equation of second order). The second problem is due the fact that a second solution does not work at $\theta = 0, \pi$, so cannot represent the physical phenomenon.

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Chapter 3

Electrostatics in matter

3.1 Multipole expansion

For a monopole, $\Phi \propto 1/r$; for a dipole, $\Phi \propto 1/r^2$; for a quadrupole, $\Phi \propto 1/r^3$; and for an octopole, $\Phi \propto 1/r^4$. Using the Legendre generating function,

$$\Phi(\mathbf{r}_p) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{\|\mathbf{r}_p\|^{n+1}} \int_V \|\mathbf{r}_q\|^n P_n(\cos \alpha) \rho(\mathbf{r}_q) dv. \quad (3.1)$$

For a *physical dipole*, the dipolar term is just an approximation. This approximation improves by reducing d . However, by making $d \rightarrow 0$, p also tend to zero and we lose the dipolar term. A *perfect or ideal dipole* would have a distance $d \rightarrow 0$ but also $q \rightarrow$ such that $qd = p$ remains constant. If we have a system of charge and move them by \mathbf{a} (or we move our origin), then the dipole term is affected.

$$\mathbf{p}' = \mathbf{p} - Q_T \mathbf{a} \quad (3.2)$$

Then, unless total charge is zero, to define the dipole moment we need to specify the origin.

Definition 3.1.1. Dipolar electric moment

$$\mathbf{p} := \int_{\Omega} \mathbf{r} \rho(\mathbf{r}) dv, \quad [p] = \text{C m}. \quad (3.3)$$

Theorem 3.1.1 (Green's Reciprocity Theorem). *Let $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ two charge distributions producing two potentials $\Phi_1(\mathbf{r})$, $\Phi_2(\mathbf{r})$ respectively. Then,*

$$\int_{\mathbb{R}^3} \rho_1 \Phi_2 dv = \int_{\mathbb{R}^3} \rho_2 \Phi_1 dv. \quad (3.4)$$

Some relations for l.i.h. materials

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E} \Rightarrow (\epsilon - \epsilon_0) \mathbf{E} = \mathbf{P} \quad (3.5)$$

$$\text{div} \mathbf{D} = \epsilon \text{div} \mathbf{E} \Rightarrow \rho_f = \epsilon_r (\rho_f + \rho_b) \Rightarrow \rho_f = \frac{\epsilon_r}{1 - \epsilon_r} \rho_b \quad (3.6)$$

Definition 3.1.2. Electric polarization vector,

$$\mathbf{P} := \frac{1}{\Delta V} \sum_{i=1}^n \mathbf{p}_i, \quad [P] = \text{C m}^{-2} \quad (3.7)$$

Definition 3.1.3. Bounded charge or polarization charge

$$\rho_p := -\text{div} \mathbf{P}, \quad [\rho_p] = \text{C m}^{-3}, \quad (3.8)$$

$$\sigma_p := \langle \mathbf{p}, \mathbf{n} \rangle_I, \quad [\sigma_p] = \text{C m}^{-2}. \quad (3.9)$$

Proposition 3.1.2. *In the interior and exterior of a dielectric,*

$$\mathbf{E}(\mathbf{r}_p) = \frac{1}{4\pi\epsilon_0} \oint_{\partial\Omega} \langle \sigma_p(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3}, d\mathbf{s} \rangle_I + \frac{1}{4\pi\epsilon_0} \int_{\Omega} \rho_p(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} dv. \quad (3.10)$$

Parameters of a dielectric (p.8)

- Linear: is not a function of \mathbf{E} ,
- Constant/homogenous: does not depend on the position
- Depends on the direction: matrix. If isotropic, the matrix is λI .

Chapter 4

Magnetostatics

4.1 Electric current

Definition 4.1.1. Density current

$$\mathbf{J} := \rho \mathbf{v}, \quad [J] = \text{A m}^{-2}; \mathbf{K} := \sigma \mathbf{v}, \quad [K] = \text{A m}^{-1}. \quad (4.1)$$

Definition 4.1.2. Electric current

$$I := \int_{\sigma} \langle \mathbf{J}, d\mathbf{s} \rangle_I, \quad \int_{\gamma} \langle \mathbf{K}, d\mathbf{r} \rangle_I, \quad [I] = \text{C m}^{-1} = \text{A}. \quad (4.2)$$

Axiom 1 (Ohm's law).

$$\mathbf{J} = g\mathbf{E}, \quad [g] = \Omega^{-1} \text{ m}^{-1} = \text{S m}^{-1}, \quad (4.3)$$

where g is called the conductivity.

Proposition 4.1.1 (Ohm's macroscopic law).

$$\Delta V = IR, \quad R = \frac{\rho l}{s}, \quad [\rho] = \Omega \text{ m}, \quad (4.4)$$

where ρ is called the resistivity.

Proposition 4.1.2 (Joule's law).

$$P = IV. \quad (4.5)$$

4.2 Continuity equation

Theorem 4.2.1 (Continuity equation).

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad (4.6)$$

Theorem 4.2.2 (Kirchhoff's circuit laws). 1.

2. For intensity,

$$\sum_{i=1}^n I_i = 0. \quad (4.7)$$

3. For voltage,

$$\sum_{i=1}^n V_i = 0. \quad (4.8)$$

4.3 Biot-Savart law

Axiom 2.

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{r}_q) \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} dv_q, \quad [B] = \text{Wb m}^{-2} = \text{T} = \text{N s m}^{-1} \text{C}^{-1}. \quad (4.9)$$

Definition 4.3.1. Magnetic permeability

$$\mu_0 := \frac{1}{\epsilon_0 c^2}, \quad [\mu_0] = \text{N A}^{-2}. \quad (4.10)$$

Axiom 3.

$$\mathbf{F}(\mathbf{r}_p) = \int_V \mathbf{J}(\mathbf{r}_p) \times \mathbf{B}(\mathbf{r}_p) dv_p \quad (4.11)$$

Axiom 4 (Lorentz Force).

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}, \quad \mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} q\mathbf{v} \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3}. \quad (4.12)$$

4.4 Force between wires

Theorem 4.4.1.

$$\mathbf{F}_{1 \rightarrow 2} = \frac{\mu_0}{4\pi} I_1 I_2 \int_{\gamma_1} \int_{\gamma_2} d\mathbf{r}_2 \times \left(d\mathbf{r}_1 \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \right) = -\frac{\mu_0}{4\pi} I_1 I_2 \int_{\gamma_1} \int_{\gamma_2} \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \langle d\mathbf{r}_2, d\mathbf{r}_1 \rangle_I. \quad (4.13)$$

Corollary 4.4.2. $\mathbf{F}_{1 \rightarrow 2} = -\mathbf{F}_{2 \rightarrow 1}$.

4.5 Ampere's Law

Theorem 4.5.1.

$$\mathbf{B} = \nabla \times \left(\frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}_q)}{\|\mathbf{r}_p - \mathbf{r}_q\|} dv_q \right). \quad (4.14)$$

Theorem 4.5.2.

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.} \quad (4.15)$$

Corollary 4.5.3.

$$\oint_{\partial \Sigma} \langle \mathbf{B}, d\mathbf{r} \rangle_I = \mu_0 \int_{\Sigma} \langle \mathbf{J}, d\mathbf{s} \rangle_I \quad (4.16)$$

4.6 Divergence of the magnetic field

Theorem 4.6.1.

$$\operatorname{div} \mathbf{B} = 0. \quad (4.17)$$

4.7 Vector potential

Definition 4.7.1.

$$\mathbf{A}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{r}_q) \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} dv_q, \quad [A] = \text{T m}. \quad (4.18)$$

Proposition 4.7.1. *The vector potential is not unique, \mathbf{A} and $\mathbf{A}' = \mathbf{A} + \nabla \xi$ are equivalent.*

Proposition 4.7.2.

$$\phi_m = \int_{\partial \Sigma} \langle \mathbf{A}, d\mathbf{r} \rangle_I. \quad (4.19)$$

4.8 Scalar potential

Theorem 4.8.1. *If $\mathbf{J} = \mathbf{0}$, then*

$$\nabla^2 \Phi_m = 0 \quad (4.20)$$

Chapter 5

Magnetostatics in matter

5.1 Multipolar expansion

5.2 Magnetic dipole

Definition 5.2.1. *Dipolar magnetic moment or magnetic moment.*

$$\mathbf{m} := \frac{1}{2} \int \mathbf{r}_q \times \mathbf{J}(\mathbf{r}_q) dv_q, \quad [m] = \text{A m}^2 = \text{J T}^{-1}. \quad (5.1)$$

Proposition 5.2.1. *For a wire,*

$$\mathbf{m} = \frac{1}{2} I \oint_{\gamma} \mathbf{r} \times d\mathbf{r}. \quad (5.2)$$

Proposition 5.2.2. *Let γ be a curve such that Γ is contained in a plane. Then, $\mathbf{m} = I \mathbf{S}\mathbf{n}$.*

Theorem 5.2.3. *If ρ_m, ρ_q are constants, $\mathbf{m} = \frac{q}{2m} \mathbf{L}$.*

Proposition 5.2.4. *If $\|\mathbf{r}\| \gg 1$, then*

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{\|\mathbf{r}\|^3}, \quad (5.3)$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[\frac{3\langle \mathbf{m}, \mathbf{r} \rangle_I \mathbf{r}}{\|\mathbf{r}\|^5} - \frac{m}{\|\mathbf{r}\|^3} \right] = -\mu_0 \nabla \Phi_m, \quad \Phi_m = \frac{1}{4\pi} \frac{\langle \mathbf{m}, \mathbf{r} \rangle_I}{\|\mathbf{r}\|^3}. \quad (5.4)$$

Definition 5.2.2. The north pole of a magnet is where the magnetic field lines leave the magnet, which is equivalent to the extreme that points to the geographic north pole.

Theorem 5.2.5. *Magnetic force and torque on a dipole*

$$\mathbf{F} = \nabla \langle \mathbf{m}, \mathbf{B} \rangle_I, \quad \boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}. \quad (5.5)$$

5.3 Generated field by magnetic a media

Definition 5.3.1. *Magnetization*

$$\mathbf{M} := \frac{1}{\Delta V} \sum_{i=1}^n \mathbf{m}_i, \quad \frac{1}{\Delta V} \frac{1}{2} \int \mathbf{r} \times \mathbf{J} dv, \quad [M] = \text{A m}^{-1}. \quad (5.6)$$

Theorem 5.3.1. *At the exterior (there is no \mathbf{J}).*

$$\mathbf{A}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_M(\mathbf{r}_p)}{\|\mathbf{r}_p - \mathbf{r}_q\|} dv_q + \frac{\mu_0}{4\pi} \oint_{\partial V} \frac{\mathbf{K}_M(\mathbf{r}_p)}{\|\mathbf{r}_p - \mathbf{r}_q\|} ds_q, \quad (5.7)$$

where $\mathbf{J}_m := \text{curl } \mathbf{M}$, $\mathbf{K}_m := \mathbf{M} \times \mathbf{n}$.

Theorem 5.3.2. *At the exterior,*

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int_V [\mathbf{J}_f(\mathbf{r}_q) + \mathbf{J}_m(\mathbf{r}_q)] \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} dv_q, \quad (5.8)$$

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int_{\text{int } V} \rho_m(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} dv_q + \frac{\mu_0}{4\pi} \oint_{\partial V} \sigma_m(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} ds_q. \quad (5.9)$$

where $\rho_m := -\text{div } \mathbf{M}$, $\sigma_m := \langle \mathbf{M}, \mathbf{n} \rangle_I$.

Theorem 5.3.3. *In general,*

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int_{\text{int } V} [\mathbf{J}_f(\mathbf{r}_q) + \mathbf{J}_m(\mathbf{r}_q)] \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} dv_q \quad (5.10)$$

$$+ \frac{\mu_0}{4\pi} \oint_{\partial V} [\mathbf{K}_f(\mathbf{r}_q) + \mathbf{K}_m(\mathbf{r}_q)] \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} ds_q. \quad (5.11)$$

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int_{\text{int } V} \rho_m(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} dv_q + \frac{\mu_0}{4\pi} \oint_{\partial V} \sigma_m(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} ds_q \quad (5.12)$$

$$+ \frac{\mu_0}{4\pi} \int_{\text{int } V} \mathbf{J}_f(\mathbf{r}_q) \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} dv_q + \frac{\mu_0}{4\pi} \oint_{\partial V} \mathbf{K}_f(\mathbf{r}_q) \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} ds_q + \mu_0 \mathbf{M}(\mathbf{r}_p). \quad (5.13)$$

5.4 Magnetic intensity

Definition 5.4.1. Magnetic intensity

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \quad [\mathbf{H}] = \text{A m}^{-1}. \quad (5.14)$$

Theorem 5.4.1. *Ampere's theorem for magnetic intensity*

$$\text{curl } \mathbf{H} = \mathbf{J}_f \Rightarrow \oint_{\partial \Sigma} \langle \mathbf{H}, d\mathbf{r} \rangle_I = \int_{\Sigma} \langle \mathbf{J}_f, d\mathbf{s} \rangle_I \quad (5.15)$$

Definition 5.4.2. *Magnetic poles density,*

$$\rho_m := \text{div } \mathbf{H}, \quad \sigma_m := \langle \mathbf{M}, \mathbf{n} \rangle_I. \quad (5.16)$$

5.5 Classification of magnetic materials

Definition 5.5.1. *Magnetic susceptibility,*

$$\mathbf{M} = \chi_m \mathbf{H}, \quad [\chi_m] = 1. \quad (5.17)$$

- Diamagnetic: the imantation is oposed to \mathbf{H} . The magnetic susceptibility is negative and with a small value.
- Paramagnetic: the imantation has de direction of \mathbf{H} . The magnetic susceptibility has a small value.
- Ferromagnetic: non zero imantation when $\mathbf{H} = \mathbf{0}$. The magnetic susceptibility is positive and with a high value.
- Antiferromagnetic: the total imantation is null.
- Ferrimagnetic: the toatl imantion is not null.

Chapter 6

Slow variable fields

6.1 Faraday's Law

Axiom 5 (Lorentz force for non-stationary conditions).

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (6.1)$$

Definition 6.1.1. Emf,

$$\mathcal{E} := \oint_{\gamma} \langle \mathbf{E}_{\text{ef}}, d\mathbf{r} \rangle_I. \quad (6.2)$$

Proposition 6.1.1. For open circuits, $\mathcal{E} = \Delta\Psi$.

Axiom 6 (Faraday's Law).

$$\mathcal{E} = \frac{d\phi}{dt} \Leftrightarrow \oint_{\partial\Sigma} \langle \mathbf{E}, d\mathbf{r} \rangle_I = -\frac{d}{dt} \int_{\Sigma} \langle \mathbf{B}, d\mathbf{s} \rangle_I \quad (6.3)$$

Theorem 6.1.2. Maxwell-Faraday law for stationary systems (general, by contrast to the original Faraday's law)

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (6.4)$$

Theorem 6.1.3. Faraday's law for moving systems.

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \quad (6.5)$$

Applications: electric generators, electric motors, Foucault/eddy currents, circuits, Hall effect, and obtaining B from

$$\int \mathcal{E} dt = -\phi_b = -SB. \quad (6.6)$$

6.2 Limitations of Faraday's Law

6.3 Mutual inductance and autoinductance

Definition 6.3.1. Mutual inductance or induction coefficient.

$$M_{2 \leftarrow 1} := \frac{\mu_0}{4\pi} \oint_{\gamma_2} \oint_{\gamma_1} \frac{1}{\|\mathbf{r}_2 - \mathbf{r}_1\|} \langle d\mathbf{r}_2, d\mathbf{r}_1 \rangle_I. \quad (6.7)$$

Corollary 6.3.1. $M_{2 \leftarrow 1} = M_{1 \leftarrow 2}$.

Proposition 6.3.2. $\phi_{2 \leftarrow 1} = M_{2 \leftarrow 1} I_1$.

Corollary 6.3.3. $\mathcal{E}_2 = -\dot{\phi}_m = -M_{2 \leftarrow 1} \dot{I}_1$.

Definition 6.3.2.

$$L := \frac{\mu_0}{4\pi} \oint_{\gamma} \oint_{\gamma} \frac{1}{\|\mathbf{r}_2 - \mathbf{r}_1\|} \langle d\mathbf{r}_2, d\mathbf{r}_1 \rangle_I. \quad (6.8)$$

Proposition 6.3.4. Equations for a transformer (the last one is the coupling coefficient).

$$\frac{n_1}{n_2} = \frac{\phi_1}{\phi_{2 \leftarrow 1}} = \frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{L_1}{M_{2 \leftarrow 1}} = \frac{M_{1 \leftarrow 2}}{L_2} = \sqrt{\frac{L_1}{L_2}}, \quad (6.9)$$

$$L_1 L_2 = M_{2 \rightarrow 1}^2, \quad k := \frac{M_{2 \leftarrow 1}}{\sqrt{L_1 L_2}}. \quad (6.10)$$

6.4 Magnetic energy of coupled circuits

Theorem 6.4.1. *For rigid and stationary circuits without hysteresis, $W_m = I\phi_m$.*

Theorem 6.4.2. *Stationary circuits and without hysteresis,*

$$W_m = \frac{1}{2} \sum_{i=1}^n I_i \phi_i = \frac{1}{2} \sum_{i=1}^n I_i \oint_{\gamma_i} \langle \mathbf{A}_i, d\mathbf{r} \rangle_I = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n M_{ij} I_i I_j. \quad (6.11)$$

Corollary 6.4.3. *Particular systems.*

1. *Coil/bobbin:* $W_m = \frac{1}{2} L I^2$
2. *Two coupled circuits:* $\frac{1}{2} L_1 I_1^2 + M I_1 I_2 + \frac{1}{2} L_2 I_2^2$

Theorem 6.4.4. *For static circuits*

$$W_m = \frac{1}{2} \int_{\mathbb{R}^3} \langle \mathbf{H}, \mathbf{B} \rangle_I dv = W_b. \quad (6.12)$$

Theorem 6.4.5. *With and without battery respectively*

$$\mathbf{F} = \nabla W_m|_I, \quad \mathbf{F} = \nabla W_m|_\Phi. \quad (6.13)$$

Others

Definition 6.4.1. *Magnetic reluctance*

$$\mathcal{R} := \oint_{\gamma} \frac{1}{\mu S} d\mathbf{r}. \quad (6.14)$$

Proposition 6.4.6. $nI = \phi_m \mathcal{R}$.

Proposition 6.4.7 (Langevin diamagnetism). *Let S be a system of N atoms per unit volume separated by a mean square distance $\langle r^2 \rangle_I$. If the atoms have Z orbiting electrons of mass m_e and charge q , then*

$$\chi_e = -\frac{q^2 Z}{6m_e} \mu_0 N \langle r^2 \rangle_I. \quad (6.15)$$

Proposition 6.4.8. *Formulas for cylindrical coil*

$$L = \frac{\mu_0 N^2 \pi r^2}{l} \quad (6.16)$$

Proposition 6.4.9. *Formulas for toroidal coil*

$$L = \mu_0 N^2 \left(b - \sqrt{b^2 - a^2} \right), \quad W_m = \frac{\mu_0 N^2 I^2}{2} \left(b - \sqrt{b^2 - a^2} \right). \quad (6.17)$$

Chapter 7

Maswell's equations

7.1 Maxwell equations

Definition 7.1.1. Displacement current

$$\mathbf{J}_d := \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (7.1)$$

Proposition 7.1.1. Lorentz force

$$\mathbf{F} = \int_{\Omega} \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \, dv. \quad (7.2)$$

Theorem 7.1.2 (Maxwell's equations in vacuum).

$$\operatorname{div} \mathbf{E} = \frac{\rho_t}{\epsilon_0} \quad (7.3)$$

$$\operatorname{div} \mathbf{B} = 0 \quad (7.4)$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.5)$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J}_t + \mu_0 \mathbf{J}_d, \quad (7.6)$$

in integral form,

$$\oint_{\partial\Omega} \langle \mathbf{E}, d\mathbf{s} \rangle_I = \frac{1}{\epsilon_0} \int_{\Omega} \rho_t \, dv \quad (7.7)$$

$$\oint_{\partial\Omega} \langle \mathbf{B}, d\mathbf{s} \rangle_I = 0 \quad (7.8)$$

$$\oint_{\partial\Sigma} \langle \mathbf{E}, d\mathbf{r} \rangle_I = -\frac{d}{dt} \int_{\Sigma} \langle \mathbf{B}, d\mathbf{s} \rangle_I \quad (7.9)$$

$$\oint_{\partial\Sigma} \langle \mathbf{B}, d\mathbf{r} \rangle_I = \mu_0 \int_{\Sigma} \langle \mathbf{J}_t, d\mathbf{s} \rangle_I + \mu_0 \epsilon_0 \frac{d}{dt} \int_{\Sigma} \langle \mathbf{E}, d\mathbf{s} \rangle_I. \quad (7.10)$$

Theorem 7.1.3 (Maxwell's equations in media).

$$\operatorname{div} \mathbf{D} = \rho_f \quad (7.11)$$

$$\operatorname{div} \mathbf{B} = 0 \quad (7.12)$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.13)$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \quad (7.14)$$

in integral form,

$$\oint_{\partial\Omega} \langle \mathbf{D}, d\mathbf{s} \rangle_I = \int_{\Omega} \rho_f \, dv \quad (7.15)$$

$$\oint_{\partial\Omega} \langle \mathbf{B}, d\mathbf{s} \rangle_I = 0 \quad (7.16)$$

$$\oint_{\partial\Sigma} \langle \mathbf{E}, d\mathbf{r} \rangle_I = -\frac{d}{dt} \int_{\Sigma} \langle \mathbf{B}, d\mathbf{s} \rangle_I \quad (7.17)$$

$$\oint_{\partial\Sigma} \langle \mathbf{H}, d\mathbf{r} \rangle_I = \int_{\Sigma} \langle \mathbf{J}_f, d\mathbf{s} \rangle_I + \frac{d}{dt} \int_{\Sigma} \langle \mathbf{D}, d\mathbf{s} \rangle_I \quad (7.18)$$

Definition 7.1.2.

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \quad (7.19)$$

Proposition 7.1.4.

$$\operatorname{div} \mathbf{J}_p + \frac{\partial \rho_p}{\partial t} = 0. \quad (7.20)$$

Theorem 7.1.5. *Electromagnetic energy*

$$W_{em} = \frac{1}{2} \int_{\Omega} \langle \mathbf{H}, \mathbf{B} \rangle_I + \langle \mathbf{D}, \mathbf{E} \rangle_I \, dv. \quad (7.21)$$

Definition 7.1.3. *Poynting vector, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.*

Theorem 7.1.6.

$$\frac{dW_k}{dt} + \frac{dW_{em}}{dt} = - \int_{\Sigma} \langle \mathbf{S}, d\mathbf{s} \rangle_I, \quad (7.22)$$

$$\int_{\Omega} \langle \mathbf{J}, \mathbf{E} \rangle_I \, dv + \frac{\partial}{\partial t} \int_{\Omega} \frac{\langle \mathbf{H}, \mathbf{B} \rangle_I + \langle \mathbf{D}, \mathbf{E} \rangle_I}{2} \, dv = - \oint_{\partial\Omega} \langle \mathbf{E} \times \mathbf{H}, d\mathbf{s} \rangle_I. \quad (7.23)$$

Corollary 7.1.7.

$$\eta = \frac{dW}{dV} = \frac{\langle \mathbf{H}, \mathbf{B} \rangle_I + \langle \mathbf{D}, \mathbf{E} \rangle_I}{2}. \quad (7.24)$$

Definition 7.1.4. Maxwell stress tensor

$$\sigma_{ij} := \epsilon \left(E_i E_j - \frac{\delta_{ij}}{2} E^2 \right) + \frac{1}{\mu} \left(B_i B_j - \frac{\delta_{ij}}{2} B^2 \right), \quad (7.25)$$

$$\boldsymbol{\sigma} = \epsilon \left(\mathbf{E} \otimes \mathbf{E} - \frac{1}{2} \mathbb{I} \right) + \frac{1}{\mu} \left(\mathbf{B} \otimes \mathbf{B} - \frac{1}{2} \mathbb{I} \right). \quad (7.26)$$

Theorem 7.1.8. *Electromagnetic force per unit volume*

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{f} + \epsilon \mu \frac{\partial \mathbf{S}}{\partial t}. \quad (7.27)$$

Definition 7.1.5. Electromagnetic impulse

$$\mathbf{P}_{em} := \frac{1}{u^2} \int_{\Omega} \mathbf{S} \, dv. \quad (7.28)$$

Theorem 7.1.9. *Boundary conditions*

$$\langle \mathbf{n}_{1 \rightarrow 2}, \mathbf{B}_2 - \mathbf{B}_1 \rangle_I = 0 \quad (7.29)$$

$$\mathbf{n}_{1 \rightarrow 2} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mathbf{K}_f + \mathbf{K}_m \quad (7.30)$$

$$\langle \mathbf{n}_{1 \rightarrow 2}, \mathbf{H}_2 - \mathbf{H}_1 \rangle_I = \sigma_m \quad (7.31)$$

$$\mathbf{n}_{1 \rightarrow 2} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f \quad (7.32)$$

$$\langle \mathbf{n}_{1 \rightarrow 2}, \mathbf{D}_2 - \mathbf{D}_1 \rangle_I = \sigma_f \quad (7.33)$$

$$\mathbf{n}_{1 \rightarrow 2} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0. \quad (7.34)$$

7.2 Scalar and vector potentials

Proposition 7.2.1. *There exists functions Φ, \mathbf{A} such that*

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \mathbf{E} = -\operatorname{grad} \Phi - \frac{\partial \mathbf{A}}{\partial t}. \quad (7.35)$$

Definition 7.2.1. Gauge transformations

$$\Phi' = \Phi - \frac{\partial \xi}{\partial t}, \quad \mathbf{A}' = \mathbf{A} + \operatorname{grad} \xi. \quad (7.36)$$

Definition 7.2.2. Coulomb and Lorenz conditions

$$\nabla^2 \xi = -\operatorname{div} \mathbf{A} \Rightarrow \operatorname{div} \mathbf{A}' = 0, \quad \nabla^2 \xi - \epsilon \mu \frac{\partial \xi}{\partial t} = 0 \Rightarrow \operatorname{div} \mathbf{A}' + \epsilon \mu \frac{\partial \Phi'}{\partial t} = 0. \quad (7.37)$$

Proposition 7.2.2. In general,

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\operatorname{div} \mathbf{A}) = -\frac{\rho_t}{\epsilon}, \quad \nabla^2 \mathbf{A} - \epsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left[\operatorname{div} \mathbf{A} + \epsilon \mu \frac{\partial \Phi}{\partial t} \right] = -\mu \mathbf{J}_t. \quad (7.38)$$

Theorem 7.2.3. If Coulomb conditions,

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon}, \quad \nabla^2 \mathbf{A} - \epsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2} - \epsilon \mu \frac{\partial}{\partial t}(\nabla \Phi) = -\mu \mathbf{J}_t \quad (7.39)$$

and search solutions.

Theorem 7.2.4. If Lorenz condition,

$$\nabla^2 \Phi - \epsilon \mu \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho_t}{\epsilon}, \quad \nabla^2 \mathbf{A} - \epsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_t, \quad (7.40)$$

which have the solutions

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon} \int_{\Omega} \rho(\mathbf{r}_q, t_0) \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} dv_q + \Phi_0, \quad (7.41)$$

$$\mathbf{A}(\mathbf{r}_p, t) = \frac{\mu}{4\pi} \int_{\Omega} \mathbf{J}(\mathbf{r}_q, t_0) \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} dv_q + \mathbf{A}_0, \quad (7.42)$$

where

$$t_0 := t - \frac{\|\mathbf{r}_p - \mathbf{r}_q\|}{u}, \quad u := \frac{1}{\sqrt{\epsilon\mu}}, \quad c := \frac{1}{\sqrt{\epsilon_0\mu_0}}. \quad (7.43)$$

Corollary 7.2.5. Single particle, Lienard-Wiechert potentials ($\mathbf{n} = (\mathbf{r}_p - \mathbf{r}_q)/\|\mathbf{r}_p - \mathbf{r}_q\|$)

$$\Phi(\mathbf{r}_p, t) = \frac{1}{4\pi\epsilon} \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} \frac{1}{1 - \langle \mathbf{v}, \mathbf{n} \rangle_I / u} \Big|_{t_0}, \quad \mathbf{A}(\mathbf{r}_p, t) = \frac{\mu}{4\pi} \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} \frac{q\mathbf{v}}{1 - \langle \mathbf{v}, \mathbf{n} \rangle_I / u} \Big|_{t_0}, \quad (7.44)$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|^2} \frac{(1 - \beta^2)(\mathbf{n} - \boldsymbol{\beta})}{(1 - \langle \boldsymbol{\beta}, \mathbf{n} \rangle_I)^3} + \frac{q}{4\pi\epsilon_0} \frac{1}{c\|\mathbf{r}_p - \mathbf{r}_q\|} \frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \langle \boldsymbol{\beta}, \mathbf{n} \rangle_I)^3}, \quad (7.45)$$

$$\mathbf{B} = \frac{1}{c} \mathbf{n} \times \mathbf{E}. \quad (7.46)$$

Chapter 8

Electromagnetic potentials and radiation field

Theorem 8.0.1. *Wave equations*

$$\nabla^2 \mathbf{E} - \mu g \frac{\partial \mathbf{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad (8.1)$$

$$\nabla^2 \mathbf{H} - \mu g \frac{\partial \mathbf{H}}{\partial t} - \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \quad (8.2)$$