1 Harmonic oscillator

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space in a harmonic potential

$$V(\hat{x}) = \frac{\omega^2}{2}\hat{x}^2, \qquad \omega^2 = \frac{k}{m}. \tag{1}$$

We define the *creation* and *annihilation operators* as

$$\hat{a}^\dagger \coloneqq \frac{\alpha}{\sqrt{2}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right), \tag{2}$$

$$\hat{a} \coloneqq \frac{\alpha}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \tag{3}$$

$$\alpha \coloneqq \sqrt{\frac{m\omega}{\hbar}}. \tag{4}$$

Proposition 1.1. *Let $\mathcal{H}$ be a Hilbert space in a harmonic potential. Then,*

$$\langle x|\, \hat{a}^\dagger = \frac{\alpha}{\sqrt{2}}\left(x - \frac{1}{\alpha^2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{5}$$

$$\langle x|\, \hat{a} = \frac{\alpha}{\sqrt{2}}\left(x + \frac{1}{\alpha^2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{6}$$

$$\alpha = \frac{m\omega}{\hbar}. \tag{7}$$

Proposition 1.2. *Let $\mathcal{H}$ be a Hilberet space with a harmonic potential. Then,*

$$\hat{x} = \frac{1}{\sqrt{2}\alpha}(\hat{a}^\dagger + \hat{a}), \qquad \hat{p} = i\hbar\frac{\alpha}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}). \tag{8}$$

Proposition 1.3. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{a}, \hat{a}^\dagger$ are not hermitian.*
2. *$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$.*
3. *$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$.*

Definition 1.2. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *number operator* as

$$\hat{N} \coloneqq \hat{a}^\dagger\hat{a}. \tag{9}$$

Proposition 1.4. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{H}$ is hermitian.*
2. *$\left[\hat{N}, \hat{a}\right] = -\hat{a}$, $\left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger$,*
3. *$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\hat{I}\right)$.*

Proposition 1.5. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, $\hat{H}$ and $\hat{N}$ have a common basis of eigenvectors, which is countable, and*

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \qquad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{10}$$

$$\hat{N}\,|n\rangle = n\,|n\rangle\,, \qquad \hat{H}\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle\,, \tag{11}$$

$$n \in \mathbb{N}. \tag{12}$$

Corollary 1.6. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle\,. \tag{13}$$

Proposition 1.7. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, the eigenstates form a nondgenerate basis.*

Definition 1.3 (Fock states)**.** Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *Fock states* as the states that determine the basis $(|n\rangle)$ and have a well-defined number of excitations.

Definition 1.4. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We call the fundamental Fock state *the vaccum.*

Proposition 1.8. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, $\hat{a}, \hat{a}^\dagger$ and $\hat{N}$ have the following matrix representation in the basis $(|n\rangle)$.*

$$[\hat{N}]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{14}$$

$$[\hat{a}]_B = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ 0 & 0 & \sqrt{2} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{15}$$

$$[\hat{a}^\dagger]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{16}$$

or in coefficient representation,

$$[\hat{N}]_{ij} = (i-1)\delta_{ij}, \tag{17}$$

$$[\hat{a}]_{ij} = \sqrt{j-1}\delta_{i,j-1}, \tag{18}$$

$$[\hat{a}^\dagger]_{ij} = \sqrt{i-1}\delta_{i-1,j}. \tag{19}$$