1 Mathematical formulism of quantum mechanics

Definition 1.1. We define a *Hilbert space* $\mathcal{H}$as a vector space over the field $\mathbb{C}$ where there is an inner product $\langle , \rangle$ that has the properties of

1. Linearity: $\langle\phi| \left(a\left|\psi_1\right\rangle + b\left|\psi_2\right\rangle\right) = a\left\langle\phi|\psi_1\right\rangle + b\left\langle\phi|\psi_2\right\rangle$,
2. Positivity: $\langle\psi|\psi\rangle > 0, \forall\psi \neq 0$,
3. Hermiticity: $\langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle}$,

and such that is complete in the norm $\||\psi\rangle\| \coloneqq \sqrt{\langle\psi|\psi\rangle}$.

Definition 1.2. Let $\mathcal{H}$be a Hilbert space. We define an *orthonormal basis* $\mathcal{B} = (|e_i\rangle)$ as a collection of vectors that satisfy the following conditions

1. $\forall\left|\psi\right\rangle \in \mathcal{H}\ \exists!(\alpha_1, \dots, \alpha_n)$ such tat $|\psi\rangle = \sum_{i=1}^n \alpha_i |e_i\rangle$,
2. $\forall i, j \leq n\ \langle e_i|e_j\rangle = \delta_{ij}$.

Proposition 1.1. *Let $\mathcal{H}$be a Hilbert space, $|\psi\rangle \in \mathcal{H}$ a vector and $\mathcal{B} = (|e_i\rangle)$ a basis. Then,*

$$|\psi\rangle = \sum_{i=1}^n \alpha_i |e_i\rangle, \text{ with } \alpha_i = \langle e_i|\psi\rangle. \tag{1}$$

Definition 1.3. Let $\mathcal{H}$be a Hilbert space, $|\psi\rangle \in \mathcal{H}$ a vector and $\mathcal{B} = (|e_i\rangle)$ a basis. We define the *representation of $|\psi\rangle$ in the basis $\mathcal{B}$* as

$$|\psi\rangle_{\mathcal{B}} \coloneqq \begin{pmatrix} \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}_{(|e_i\rangle)}. \tag{2}$$

Definition 1.4. Let $\mathcal{H}$be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ a vector. We define a *bra* as

$$\langle\psi| \coloneqq |\psi\rangle^\dagger \in \mathcal{H}. \tag{3}$$

Proposition 1.2. *Let $\mathcal{H}$be a Hilbert space, $|\psi\rangle \in \mathcal{H}$ a vector and $\mathcal{B} = (|e_i\rangle)$ a basis. If $|\psi\rangle = \sum \alpha_i i |e_i\rangle$, then*

$$\langle\psi| = \sum_{i=1}^n \overline{\alpha_i} \langle e_i|. \tag{4}$$

Definition 1.5. Let $\mathcal{H}$be a Hilbert space. we define the *inner product* as the function $f_{in} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ that acts as follows

$$f_{in}\left(\langle\psi|, \langle\phi|\right) \coloneqq \langle\psi|\phi\rangle = |\psi\rangle^\dagger |\phi\rangle = \tag{5}$$

$$\begin{pmatrix} \overline{\alpha_1} & \dots & \overline{\alpha_n} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \sum_{i=1}^n \overline{\alpha_i}\beta_i. \tag{6}$$

Proposition 1.3. *The inner product satisfies the conditions of hermitic product for a Hilbert space.*

Definition 1.6. Let $\mathcal{H}$be a Hilbert space and $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ two vectors. We say they are *orthogonal* if and only if $\langle\psi|\phi\rangle = 0$.

Definition 1.7. Let $\mathcal{H}$be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ a vector. We say it is normalized if and only if $\langle\psi|\psi\rangle = 1$.

Definition 1.8. Let $\mathcal{H}$ be a Hilbert space. We dfien the *exterior product* as the function $f_{ext} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}_{2\times 2}$ that acts as follows

$$f_{ext}\left(|\psi\rangle, |\phi\rangle\right) \coloneqq |\psi\rangle\langle\phi| = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \begin{pmatrix} \overline{\beta_1} & \dots & \overline{\beta_n} \end{pmatrix} = \tag{7}$$

$$\begin{pmatrix} \alpha_1\overline{\beta_1} & \dots & \alpha_1\overline{\beta_n} \\ \vdots & \ddots & \vdots \\ \alpha_n\overline{\beta_1} & \dots & \alpha_n\overline{\beta_n} \end{pmatrix}. \tag{8}$$

Definition 1.9. Let $\mathcal{H}$ be a Hilbert space. we define a *linear operator* as an application $A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ such that if $a, b \in \mathbb{C}$ and $\psi_1, \psi_2 \in \mathcal{H}$, then

$$A\left(a\left|\psi_1\right\rangle + b\left|b_2\right\rangle\right) = aA\left(|\psi_1\rangle\right) + bA\left(|\psi_2\rangle\right). \tag{9}$$

We denote the matrix form of A as $\hat{A}$.

Proposition 1.4. *Let $\mathcal{H}$ be a Hilbert space, A an operator and $B = (|e_i\rangle)$ a basis. If $|\psi\rangle = \sum \alpha_i |e_i\rangle$, then $\hat{A}|\psi\rangle$ is uniquely determined by the elements $|f_i\rangle = |f(e_i)\rangle$ as follows*

$$A\left(|\psi\rangle\right) = \sum_{i=1}^n \alpha_i |f_i\rangle. \tag{10}$$

Proposition 1.5. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator and $\mathcal{B} = (|e_i\rangle)$ a basis. Then,*

$$\hat{A} = \sum_{i=1}^n |f_i\rangle\langle e_i|. \tag{11}$$

Proposition 1.6. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator and $\mathcal{B} = (|e_i\rangle)$ a basis. Then,*

$$\hat{A} = \sum_{i=1}^n A_{ij} |e_i\rangle\langle e_j|, \text{ with } A_{ij} = f_i^{(j)} = \langle e_i| \hat{A} |e_j\rangle. \tag{12}$$

Proposition 1.7. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator and $\mathcal{B} = (|e_i\rangle)$ a basis. Then,*

$$\hat{A}|e_k\rangle = \sum_{l=1}^n \hat{a}_{lk} |e_l\rangle. \tag{13}$$

Proposition 1.8. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator, $\mathcal{B} = (|e_i\rangle)$ a basis and $|\psi\rangle \in \mathcal{H}$ a vector. If $|\psi\rangle = \sum \alpha_i |e_i\rangle$ and $|\phi\rangle = \hat{A}|\psi\rangle = \sum \beta_i |e_i\rangle$, then*

$$\beta_n = \langle e_n|\phi\rangle = \sum_{k=1}^n \hat{A}_{nk}\alpha_k. \tag{14}$$