

1 Arithmetic and topology

Definition 1.1. Let $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the sum “+” by components

$$(a, b) + (c, d) := (a + b, c + d). \quad (1)$$

- Product: given two $(a, b), (c, d) \in \mathbb{R}^2$ we define the product by

$$(a, b)(c, d) = (ac - bd, ad + bc). \quad (2)$$

We define the set \mathbb{C} as $(\mathbb{R}^2, +, \cdot)$.

Proposition 1.1. *The set \mathbb{C} of complex numbers is an abelian field.*

Proposition 1.2. *Let \mathbb{C} be defined in the second way. Then,*

1. \mathbb{C} is an abelian ring.
2. If we define f as

$$f : (\mathbb{C}, +, \cdot) \longrightarrow (\mathbb{R}^2, +, \cdot), \quad (x, y) \longmapsto x + yi, \quad (3)$$

then f is a morphism of rings.

3. The function f is, in fact, an isomorphism and \mathbb{C} is an abelian field.

Proposition 1.3. *The subset of \mathbb{C} generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.*

Theorem 1.4. \mathbb{C} is not an ordered field.

Definition 1.2. Let $z = a + bi \in \mathbb{C}$. We define the conjugate of z as

$$\bar{z} := a - bi. \quad (4)$$

Proposition 1.5. *For all $z, w \in \mathbb{C}$, we have:*

1. $\bar{\bar{z}} = z$.
2. $\overline{z + w} = \bar{z} + \bar{w}$.
3. $\overline{zw} = \bar{z}\bar{w}$.
4. $z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.
5. $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.
6. The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.

Definition 1.3. Let $z = a + bi \in \mathbb{C}$. We define the real part of z and imaginary part of z respectively as

$$\operatorname{Re}\{z\} := a, \quad \operatorname{Im}\{z\} := b. \quad (5)$$

Proposition 1.6. *Let $z \in \mathbb{C}$. Then,*

$$\operatorname{Re}\{z\} = \frac{z + \bar{z}}{2}, \quad \operatorname{Im}\{z\} = \frac{z - \bar{z}}{2i} \quad (6)$$

Proposition 1.7. *Let $z, w \in \mathbb{C}$ and the following distance function.*

$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto \tilde{d}(z, w) := |z - w| \end{aligned} \quad (7)$$

Then, (\mathbb{C}, \tilde{d}) is a metric space.

Definition 1.4. Let $z = a + bi \in \mathbb{C}$. We define the modulus of z as

$$|z| := \tilde{d}(z, 0), \quad (8)$$

which is equivalent to $\sqrt{z\bar{z}}$.

Definition 1.5. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define an open disc of radius r and center z_0 as follows

$$B_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}. \quad (9)$$

Definition 1.6. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a punctured disc of radius r and center z_0 as follows

$$B_r^*(z_0) := \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}. \quad (10)$$

Definition 1.7. Let $r \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. We define a closed disc of radius r and center z_0 as follows

$$\overline{B_r(z_0)} := \{z \in \mathbb{C} \mid |z - z_0| \leq r\}. \quad (11)$$

Definition 1.8. We denote by \mathbb{D} the unitary disc of center 0 and radius 1. Besides, we denote by $\mathbb{T} \subseteq \mathbb{C}$ the unitary circumference, that is,

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}. \quad (12)$$

We also denote it by \mathbb{S}^1 .

Lemma 1.8. *The set $B = \{B_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\}$ is a basis of the topology of \mathbb{R}^2 as a metric space. The set $D = \{D_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of \mathbb{C} as a metric space.*

Proposition 1.9. *The sets \mathbb{C} and \mathbb{R}^2 with the topology of metric space are homeomorphs.*

Corollary 1.10. *There is a bijection between B and D , that is, between balls of \mathbb{R}^2 and discs of \mathbb{C} .*

Proposition 1.11. *Let $z, w \in \mathbb{C}$. Then,*

1. $|z| \geq 0$.
2. $|z| = 0 \Leftrightarrow z = 0$.
3. $-|z| \leq \operatorname{Re}\{z\} \leq |z|$ and $-|z| \leq \operatorname{Im}\{z\} \leq |z|$.
4. $|zw| = |z||w|$.
5. If $w \neq 0$, $|z/w| = |z|/|w|$.
6. $|z + w| \leq |z| + |w|$.
7. $|z + w| \geq ||z| - |w||$.
8. $|\operatorname{Re}\{zw\}| \leq |z||w|$ and $|\operatorname{Im}\{z\bar{w}\}| \leq |z||w|$.
9. $|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$.

$$10. |z^n| = |z|^n$$

Corollary 1.12. Let $z_1, \dots, z_n \in \mathbb{C}$. Then,

$$\left| \sum_{i=1}^n z_i \right| \leq \sum_{i=1}^n |z_i|, \quad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \quad \left| \operatorname{Re} \left\{ \sum_{i=1}^n z_i \right\} \right| \leq \sum_{i=1}^n |z_i|. \quad (13)$$

Definition 1.9. Let $z \in \mathbb{C}^*$. We define the *argument* of z , denoted by $\arg z$, as the real number θ such that $z = |z|(\cos \theta + i \sin \theta)$. Let us observe that $\arg z$ is not a function but a multivalued application. We define the *principal argument* of z as

$$\operatorname{Arg} z := \theta_0 \in [0, 2\pi) \mid z = |z|(\cos \theta + i \sin \theta). \quad (14)$$

In general, to make θ to be unique, it is enough to impose it to belong to a certain semiopen interval of length 2π . Choosing the interval I is called by *taking a determination of the argument*.

Definition 1.10. Given a complex number z that we can express by $z = |z|(\cos \theta + i \sin \theta)$ for some $\theta \in \mathbb{R}$, we use the notation $r = |z|$ to write

$$z = r_\theta z = r(\cos \theta + i \sin \theta) \quad (15)$$

or simply r_θ when it is obvious which complex number are we referring to. We call it *polar form* of z .

Proposition 1.13. Let $z \in \mathbb{C}$ and r_θ its polar form. Then,

$$z^n = (r^n)_{n\theta}. \quad (16)$$

Corollary 1.14 (De Moivre's Formula). Let $\theta \in \mathbb{R}$. Then,

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta). \quad (17)$$

Proposition 1.15. Let $z, w \in \mathbb{C}$. Then,

1. $\arg zw = \arg z + \arg w + 2\pi k$.
2. $\arg z^n = n \arg z + 2\pi k$.

Definition 1.11. Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $w \in \mathbb{C}$ is an n -th root of z if and only if

$$w^n = z. \quad (18)$$

Theorem 1.16. Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \dots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \dots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \dots, n\}$.

Definition 1.12. Let $z_n \in \mathbb{C}$ and $n \in \mathbb{N}$. We say $\lim_{n \rightarrow \infty} z_n = l$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0(\varepsilon) \in \mathbb{N}^* \mid |z_n - l| < \varepsilon, n \geq n_0. \quad (19)$$

Proposition 1.17. Let $\{z_n\} = \{a_n + ib_n\}$ be a sequence of complex numbers. Then, it converges if and only if $\{a_n\}$ and $\{b_n\}$ converge.

Definition 1.13. We say $\sum_{n=1}^\infty z_n$ converges if and only if $S_n := \sum_{n=1}^N z_n$ has limit at $n \rightarrow \infty$.

Proposition 1.18. $\sum_{n=1}^\infty z_n$ converges if and only if $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ converge.

Definition 1.14. We say $\sum_{n=1}^\infty z_n$ converges absolutely if and only if $\sum_{n=1}^\infty |z_n|$ converges.

Proposition 1.19. $\sum_{n=1}^\infty |z_n|$ converges if and only if $\sum_{n=1}^\infty |a_n|$ and $\sum_{n=1}^\infty |b_n|$ converge.

2 Sequences and limits

Definition 2.1. A sequence of complex numbers is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned} \quad (20)$$

We denote it by $\{z_n\}_{n=m}^\infty$

Definition 2.2. Let $\{z_n\}_{n=0}^\infty$ be a sequence. We say the sequence has limit L or it converges to the limit L if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \quad (21)$$

We denote it by

$$\lim_{n \rightarrow \infty} z_n = L, \quad \lim_{n \rightarrow \infty} \{z_n\}_{n=0}^\infty = L, \quad \{z_n\}_{n=0}^\infty \rightarrow L. \quad (22)$$

Theorem 2.1. Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^\infty$ and $L = L_x + iL_y \in \mathbb{C}$. Then,

$$\{z_n\}_{n=0}^\infty \rightarrow L \Leftrightarrow \{x_n\}_{n=0}^\infty \rightarrow L_x \wedge \{y_n\}_{n=0}^\infty \rightarrow L_y. \quad (23)$$

Definition 2.3. Let $\{z_n\}_{n=0}^\infty$ be a sequence. We say it tends to infinity and denote it by $\lim z_n = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \quad (24)$$

Definition 2.4. Let $\{z_n\}_{n=0}^\infty$ be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \quad (25)$$

Theorem 2.2. Let $\{z_n\}_{n=0}^\infty$ be a convergent sequence. Then, it is a Cauchy sequence.

Theorem 2.3. Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^\infty$. Then,

$$\{z_n\}_{n=0}^\infty \text{ is a Cauchy sequence} \Leftrightarrow \{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty \text{ are Cauchy sequences} \quad (26)$$

Theorem 2.4. The field \mathbb{C} of complex numbers is complete.

Definition 2.5. The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, together with two charts.

3 Functions

Definition 3.1. A *topology* is an ordered pair (\mathbb{X}, τ) , where X is a set and τ a collection of subsets of X satisfying the following properties:

1. The empty set and X belong to τ .
2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ .
3. The intersection of any finite number of members of τ still belongs to τ .

The elements of τ are called *open sets* and the collection τ is called the *topology on X* .

Definition 3.2. Let (\mathbb{X}, d) be a metric space. A *topology on the metric space by the metric d* is the set τ of all open sets of M .

Definition 3.3. Let A be a subset of a metric space (\mathbb{M}, d) and a a point in \mathbb{M} . We say that a is an *interior point of A* if there is a ball $B_{(\mathbb{M}, d)}(a, r) \subset A$.

Definition 3.4. Let A be a subset of a metric space (\mathbb{M}, d) and a a point in \mathbb{M} . We say that a is an *exterior point of A* if there is a ball such that $B_{(\mathbb{M}, d)}(a, r) \cup A = \emptyset$.

Definition 3.5. Let A be a subset of a metric space (\mathbb{M}, d) and a a point in \mathbb{M} . We say that a is a *boundary point of A* if it is not interior or exterior or, which is equivalent, if every ball $B_{(\mathbb{M}, d)}(a, r)$ contains elements of A and A^c .

Definition 3.6. Let A be a subset of a metric space (\mathbb{M}, d) and a a point in \mathbb{M} . We say that a is an *accumulation point of A* if every ball with center a contains points of A different to a . In other words, every punctured ball satisfies $B_{(\mathbb{M}, d)}^*(a, r) \cup A \neq \emptyset$.

Definition 3.7. Let A be a subset of a metric space (\mathbb{M}, d) . We define the *interior of A* as the set of all interior points of A , and we denote it by $\text{int}(A)$.

Definition 3.8. Let A be a subset of a metric space (\mathbb{M}, d) . We define the *exterior of A* as the set of all exterior points of A , and we denote it by $\text{ext}(A)$.

Definition 3.9. Let A be a subset of a metric space (\mathbb{M}, d) . We define the *boundary of A* as the set of all boundary points of A , and we denote it by ∂A .

Definition 3.10. Let A be a subset of a metric space (\mathbb{M}, d) . We define the *closure of A* as the set of all accumulation points of A , and we denote it by \bar{A} .

Definition 3.11. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . We say A is an *open set* if it contains none of its boundary points, that is, if $\partial A \cap A = \emptyset$.

Definition 3.12. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . We say A is a *closed set* if it contains all its boundary points, that is, if $\partial A \subseteq A$.

Definition 3.13. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . We say A is a *bounded set* if there exist a point $a \in \mathbb{M}$ and a positive real number r such that the ball $B_{(\mathbb{M}, d)}(a, r)$ contains A .

Definition 3.14. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . We say A is a *compact set* if it is a bounded and closed set.

Proposition 3.1. Let (\mathbb{M}, d) be a metric space and A a subset of \mathbb{M} . Then, A is open if and only if A^c is closed.

Definition 3.15. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is *connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally, Ω is connected if there are not two open sets $U, V \subseteq \mathbb{C}$ such that

$$U_1 = U \cap \Omega, \quad V_1 = V \cap \Omega, \quad U_1 \cap V_1 = \emptyset, \quad U_1 \cup V_1 = \Omega. \quad (27)$$

Otherwise, we say Ω is *disconnected*.

Definition 3.16. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is *simply connected* if and only if every circuit is homotopic in Ω to a point in Ω . Equivalently, Ω is simply connected if and only if every pair of curves with the same extremes are homotopic.

Definition 3.17. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is *convex* if and only if for all pair of point $a, b \in \Omega$, the segment defined by

$$[a, b] = \{z \mid z = (1-t)a + tb, 0 \leq t \leq 1\} \quad (28)$$

is contained in Ω , that is, if every pair of points can be connected by a straight line that belongs to the set.

Definition 3.18. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is a *star-convex set* if and only if there exists $z_0 \in \mathbb{C}$ such that for all $z \in \Omega$ the segment $[z_0, z]$ is contained by Ω .

Definition 3.19. Let (\mathbb{M}, d) be a metric space and $S \subseteq \mathbb{M}$ a set. We say S is *path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

Definition 3.20. Let $\Omega \subseteq \mathbb{C}$ be a set. We say Ω is a *region or domain* if and only if it is open, non-empty, and connected.

Definition 3.21. Let $\Omega \subseteq \mathbb{C}$ be a non-empty set. We say $\Omega_1 \subseteq \Omega$ is a *connected component of Ω* if and only if it is a maximal connected subset, that is, if $z_0 \in \Omega_1$ and W is a connected subset of \mathbb{C} that contains z_0 , then $W \subseteq \Omega_1$.

Definition 3.22. Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function f* as the application

$$f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C} \\ z \longmapsto w = f(z). \quad (29)$$

Definition 3.23. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say it *tends to infinity at the point z_0* and denote it by $\lim_{z \rightarrow z_0} f(z) = \infty$ if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \quad (30)$$

Definition 3.24. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. We write $\lim_{z \rightarrow \infty} f(z) = L$ if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \quad (31)$$

Definition 3.25. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. We say f is *continuous in z_0* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon. \quad (32)$$

Proposition 3.2. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$ is continuous at z_0 if and only if $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are continuous at z_0 .

Proposition 3.3. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, f is continuous at z_0 if and only if for all sequence $\{z_n\}_{n=1}^\infty$ of Ω convergent at z_0 it is true that the sequence $\{f(z_n)\}_{n=1}^\infty$ converges to $f(z_0)$.

Proposition 3.4. Let $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, λf , $f + g$, and fg are continuous at z_0 . The function f/g is continuous at z_0 if $g(z_0) \neq 0$.

Definition 3.26. We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n, z, z_0 \in \mathbb{C}. \quad (33)$$

We call the term a_n the *n -th coefficient of the series*. In case $a_n = 0 \forall n \leq m$, we will start the counting directly from m .

Definition 3.27. Radius of convergence.

Proposition 3.5. The radius of convergence can be calculated as follows

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \sup \sqrt[n]{|a_n|}. \quad (34)$$

Theorem 3.6 (Cauchy-Hadamard Theorem). Let

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (35)$$

be the following complex power series with $a_n, z_0 \in \mathbb{C}$ and $R \in [0, +\infty) \cup \{+\infty\}$ the radius of convergence. Then,

1. If $|z - z_0| < R$ then S converges. In fact, for all $r < R$ we have S converges uniformly at the disc $\overline{D_r(z_0)}$.
2. If $|z - z_0| > R$ then S diverges.
3. The function $f(z) = S(z)$ is derivable at $D_R(z_0)$ and its formal derivative is

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \quad (36)$$

with the same radius of convergence.

Definition 3.28. Let $f(z) = \sum a_n (z - z_0)^n$ be a series with radius of convergence R . Then, its formal derivative is

$$f'(z) = \frac{df}{dz}. \quad (37)$$

Corollary 3.7. Let $f(z) = \sum a_n (z - z_0)^n$ be a series with radius of convergence R . Then, f is infinitely derivable at $D_R(z_0)$.

Corollary 3.8. Let R be the radius of convergence of the function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then f has as Taylor polynomial of degree m around z_0 the following one

$$P_{m,f,z_0}(z) = \sum_{n=0}^m a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}. \quad (38)$$

Theorem 3.9 (Abel's Theorem). Let be the following series

$$\sum_{n=0}^{\infty} f_n(z_0) g_n(z_0),$$

where f, g_n are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \quad (39)$$

then the original series converges uniformly in Ω .

Theorem 3.10 (Weierstrass' criterion). Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant M_n , satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at Ω for all $z_0 \in \Omega$.

Definition 3.29. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a complex function with Ω an open set. We say f is *complex analytic* if and only if for all $z_0 \in \Omega$ exists a real number $R(z_0)$ and a sequence $\{a_n\} \subseteq \mathbb{C}$ (that can also depend on z_0) such that is $z \in D_R(z_0)$, then formally it is true that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \quad (40)$$

We denote the set of complex analytic functions with domain Ω by $C^\omega(\Omega)$.

Corollary 3.11. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. If $f \in C^\omega(\Omega)$, then $f \in C^\infty(\Omega)$.

Corollary 3.12. Let z_0 . Then, the coefficients a_n of the local expression of f given by the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ are determined by

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \quad (41)$$

Proposition 3.13. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$. Then,

1. Every connected component of Ω is a closed of Ω with a subspace topology.
2. Two connected components are the same or are disjoint.
3. Every connected of Ω is one and only one connected component.
4. Ω is the disjoint union of its connected components.

Theorem 3.14 (Analytic prolongation Principle). *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of Ω that contains z_0 (the function can be zero also out Ω , but we are not studying that).*

Corollary 3.15. *Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a function with Ω a region. Then, the following statements are equivalent:*

1. $f(z) = 0$ for all $z \in \Omega$.
2. There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.

Corollary 3.16 (Analytic Prolongation Principle). *Let $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ two analytic functions with Ω a region. Then, the following statements are equivalent:*

1. $f(z) = g(z)$ for all $z \in \Omega$.
2. There exists a $z_0 \in \Omega$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}$.

Lemma 3.17. *Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then*

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right). \quad (42)$$

Corollary 3.18. *Let $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ two analytic functions. Then, fg is analytic.*

Proposition 3.19. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, f is analytic at $\Omega = D_R(0)$.*

Definition 3.30. For all $z \in \mathbb{C}$, we define the complex exponential function as the following series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \quad (43)$$

Proposition 3.20. *The radius of convergence of e^z is infinite.*

Proposition 3.21. $e^z = e^x$ for all $x \in \mathbb{R}$.

Proposition 3.22. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

Proposition 3.23. For all $z \in \mathbb{C}$ we have $e^z \neq 0$.

Proposition 3.24. *The image of e^z is \mathbb{C}^* .*

Proposition 3.25. *The derivative of e^z is e^z .*

Proposition 3.26. $\overline{e^z} = e^{\bar{z}}$.

Proposition 3.27. $|e^z| = e^{\operatorname{Re}\{z\}}$.

Proposition 3.28 (Euler's Formula). *If $\theta \in \mathbb{R}$, then $e^{x i}$ has modulus one and we have that*

$$\boxed{e^{x i} = \cos x + i \sin x.} \quad (44)$$

Corollary 3.29. *Let $z \in \mathbb{C}^*$. Then,*

$$z = |z| e^{i\theta}, \quad (45)$$

with $\theta \in [0, 2\pi)$.

Proposition 3.30. *The following function*

$$\begin{aligned} \exp : (\mathbb{R}, +) &\longrightarrow (\mathbb{C}^*, \cdot) \\ x &\longmapsto e^{x i} \end{aligned} \quad (46)$$

is a morphism of groups and its image is \mathbb{T} .

Proposition 3.31. *The complex exponential function is a periodic function with period $2\pi i$.*

Proposition 3.32. *Let $a \in \mathbb{C}^*$. Then, $e^z = a$ has infinite solutions.*

Definition 3.31. Let $z \in \mathbb{C}$ be a number. We define the complex trigonometric functions as

$$\cos z := \frac{e^{z i} + e^{-z i}}{2}, \quad (47)$$

$$\sin \theta := \frac{e^{z i} - e^{-z i}}{2}, \quad (48)$$

$$\tan z := \frac{e^{z i} - e^{-z i}}{e^{z i} + e^{-z i}}. \quad (49)$$

Proposition 3.33. For all $z \in \mathbb{C}$,

$$\sin^2 z + \cos^2 z = 1. \quad (50)$$

Proposition 3.34. For all $z \in \mathbb{C}$,

$$\cos(-z) = \cos(z), \quad \sin(-z) = -\sin(z). \quad (51)$$

Proposition 3.35. For all $z, w \in \mathbb{C}$,

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \quad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \quad (52)$$

Proposition 3.36. *The functions $\cos z, \sin z$ have period of 2π .*

Proposition 3.37. *Let $z_0 \in \mathbb{C}$. Then, z_0 is root of $\sin z$ ($\cos z$) if and only if it is a root of $\sin x$ ($\cos x$).*

Definition 3.32. Let $z \in \mathbb{C}$ be a number. We define the complex hyperbolic functions as

$$\cosh z := \frac{e^z + e^{-z}}{2}, \quad (53)$$

$$\sinh \theta := \frac{e^z - e^{-z}}{2}, \quad (54)$$

$$\tanh z := \frac{e^z - e^{-z}}{e^z + e^{-z}}. \quad (55)$$

Proposition 3.38. For all $z \in \mathbb{C}$,

$$\sinh^2 z - \cosh^2 z = 1. \quad (56)$$

Proposition 3.39. For all $z \in \mathbb{C}$,

$$\cosh(-z) = \cosh(z), \quad \sinh(-z) = -\sinh(z). \quad (57)$$

Proposition 3.40. For all $z, w \in \mathbb{C}$,

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \quad \sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \quad (58)$$

Proposition 3.41. For all $z \in \mathbb{C}$,

$$\cosh z = \cos(iz), \quad \cos z = \cosh(iz), \quad \sinh z = -i \sin(iz), \quad \sin z = -i \sinh(iz) \quad (59)$$

Proposition 3.42. The roots of the function $\sinh z$ are of the form $z_n = n\pi$ and those for the function $\cosh z$ are of the form $w_n = (2n+1)\pi/2i$.

Definition 3.33. Let $D \subseteq \mathbb{C}$ be a set. We define a multivalued function from D to \mathbb{C} as a subset of $D \times \mathbb{C}$ such that for every $z \in D$ there exists a number $y \in \mathbb{C}$ such that $(z, y) \in f$.

Definition 3.34. For $z \in \mathbb{C}^*$, we call the natural logarithm of z every number w such that $e^w = z$, that is,

$$\ln z := \{w \in \mathbb{C} \mid e^w = z\}. \quad (60)$$

Proposition 3.43. Given $z \in \mathbb{C}$ we can define $\ln z$ from the natural logarithm of a real number as

$$\ln z = \ln |z| + i \arg z = \ln |z| + i \operatorname{Arg} z + 2\pi ki. \quad (61)$$

Definition 3.35. We define the principal natural logarithm of z as the value defined by the principal argument of z , that is,

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z. \quad (62)$$

Definition 3.36. We define the determination I (with I being a semiopen interval) of the logarithm as

$$\log_I z := \ln |z| + i \arg_I z. \quad (63)$$

Proposition 3.44. Let $z, w \in \mathbb{C}$ two numbers. Then,

1. $\ln(zw) = \ln z + \ln w + 2\pi ki, k \in \mathbb{Z}$.
2. If we want to stay in the principal argument,

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \text{if } \operatorname{Arg} z + \operatorname{Arg} w < 2\pi \\ \ln z + \ln w - 2\pi i, & \text{if } \operatorname{Arg} z + \operatorname{Arg} w \geq 2\pi \end{cases} \quad (64)$$

3. SEARCH MORE PROPERTIES

Definition 3.37. Let $z \in \mathbb{C}$ be a number. We define the complex trigonometric inverse functions as

$$\arcsin z := -i \ln \left(iz + \sqrt{1 - z^2} \right), \quad (65)$$

$$\arccos z := -i \ln \left(z + \sqrt{z^2 - 1} \right), \quad (66)$$

$$\arctan z := -\frac{i}{2} \ln \frac{1 + iz}{1 - iz}. \quad (67)$$

Definition 3.38. Let $z \in \mathbb{C}$ be a number. We define the complex hyperbolic inverse functions as

$$\operatorname{arcsinh} z := \ln \left(z + \sqrt{1 + z^2} \right), \quad (68)$$

$$\operatorname{arccosh} z := \ln \left(z + \sqrt{z^2 - 1} \right), \quad (69)$$

$$\operatorname{arctanh} z := \frac{1}{2} \ln \frac{1 + z}{1 - z}. \quad (70)$$

Definition 3.39. Let $z, a \in \mathbb{C}$. Then, we define the complex power function as

$$z^a := e^{a \ln z} \quad (71)$$

Proposition 3.45. If $a = \alpha + \beta i$ and $z = re^{\theta i}$, then

$$z^a = e^{\alpha \ln r - \beta(\theta + 2\pi k)} e^{\beta \ln r + \alpha(\theta + 2\pi k)}, \quad (72)$$

$$|z^a| = e^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \quad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k) \quad (73)$$

Proposition 3.46. Let $z, w \in \mathbb{C}$. Then,

$$1. (e^b)^a = e^{a(b + 2\pi ki)}$$

Definition 3.40. A Riemann surface X is a connected complex 1-manifold.

Definition 3.41. We define a sheet as each of the complex planes of the Riemann surface.

Definition 3.42. We define a cut as the line (not necessarily straight) of union between sheets.

Definition 3.43. We define a branch point as a point where start or finish a cut.

4 Derivatives

Definition 4.1. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We define the derivative of f at z_0 as

$$f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (74)$$

in case the limit exists. If f has derivative, we say f is \mathbb{C} -derivable at z_0 .

Definition 4.2. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say f is holomorphic at Ω if and only if it is \mathbb{C} -derivable at every point of Ω . In that case, it is defined the function $f' : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ that associates each point z of Ω with $f'(z)$.

Definition 4.3. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. We define the domain of holomorphism as the region where f is derivable. We say f is entire if and only if the domain of holomorphism is \mathbb{C} .

Definition 4.4. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We say f is holomorphic at z_0 if and only if it is holomorphic at some neighborhood of z_0 .

Proposition 4.1. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$ a point. If f is derivable at z_0 , then it is continuous at z_0 .

Theorem 4.2. Let $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be two functions and $z_0 \in \Omega$ a point. Then, the following statements are true.

1. If f is constant at Ω , then f is derivable at z_0 and $f'(z_0) = 0$.
2. If $f(z) = z$ in every point of Ω , then f is derivable at z_0 and $f'(z_0) = 1$.
3. If f, g are derivable at z_0 and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ are derivable at z_0 and $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.
4. If f, g are derivable at z_0 , then fg is derivable at z_0 and

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \quad (75)$$

5. If f, g are derivable at z_0 and $g(z_0) \neq 0$, then f/g is derivable at z_0 and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \quad (76)$$

Theorem 4.3. Let $f : \Omega_1 \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a derivable function at a point $z_0 \in \Omega_1$ and $g : \Omega_2 \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be another derivable function at a point $f(z_0) \in \Omega_2$. Then, $g \circ f$ is derivable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \quad (77)$$

Definition 4.5. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. We say f is of class $C^1(\Omega)$ or simply $f \in C^1(\Omega)$ if and only if, using $f = u + iv$ with $u = \operatorname{Re}\{f\}, v = \operatorname{Im}\{f\}$, the partial derivatives of u and v as a two variable real functions exist and are continuous. In other words, $f \in C^1(\Omega)$ if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \quad (78)$$

exist and are continuous.

Theorem 4.4. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a function of class $C^1(\Omega)$ with Ω an open set, injective, and derivable at every point of Ω with non-zero derivative. Then,

1. $\Omega' = f(\Omega)$ is an open subset of \mathbb{C} .
2. The inverse function f^{-1} exist, it is well defined and it is derivable at Ω' .
3. If $z \in \Omega$ and $z' = f(z)$, then

$$(f^{-1})'(z') = \frac{1}{f'(z)}. \quad (79)$$

Proposition 4.5. A determination of $\ln z$ with $z \in \mathbb{C}$ is continuous except in a semiline.

Theorem 4.6. Let $\Omega \subseteq \mathbb{C}$ be an open set and $\phi \in C(\Omega)$, such that $e^{\phi(z)} = z$ for all $z \in \Omega$. Then, we have $\phi \in H(\Omega)$ and

$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \quad (80)$$

Proposition 4.7. A determination of $\ln z$ with $z \in \mathbb{C}$ is holomorphic except in a semiline.

Proposition 4.8. Let $I = [\theta, \theta + 2\pi)$ a determination of the logarithm, $\ln_I z$. Then, $\ln_I z$ is holomorphic except in the semiline $L_\theta = \{re^{i\theta} \in \mathbb{C} \mid r \geq 0\}$.

Theorem 4.9. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function in Ω . Then, f is holomorphic in Ω . In particular, given a point $z_0 \in \mathbb{C}$, $f'(z_0) = a_1$ where a_1 is the first coefficient of the power series that represents f in a neighborhood of z_0 .

Definition 4.6. We define the operators

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad (81)$$

that act over the functions such that the real and imaginary part u, v have partial derivatives.

Proposition 4.10. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ a function of class $C^1(\Omega)$. Then, for all $z_0 \in \Omega$

$$f(z_0 + h) = f(z_0) + \left(\frac{\partial f}{\partial z} \right)_{z_0} h + \left(\frac{\partial f}{\partial \bar{z}} \right)_{z_0} \bar{h} + o(|h|^2). \quad (82)$$

Corollary 4.11. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$. Then, f is holomorphic in Ω if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ at } \Omega. \quad (83)$$

Definition 4.7. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function of class $C^1(\Omega)$ such that $f = u + iv$ with $u = \operatorname{Re}\{f\}, v = \operatorname{Im}\{f\}$ and $z_0 \in \mathbb{C}$ a point. Then, we call $(\partial_{\bar{z}} f)_{z_0} = 0$ the *Cauchy-Riemann condition*, which is equivalent to

$$\left(\frac{\partial u}{\partial x} \right)_{z_0} = \left(\frac{\partial v}{\partial y} \right)_{z_0}, \quad \left(\frac{\partial v}{\partial x} \right)_{z_0} = - \left(\frac{\partial u}{\partial y} \right)_{z_0}, \quad (84)$$

which are called the *Cauchy-Riemann equations*.

5 Holomorphic functions

Definition 5.1. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\gamma : I \rightarrow \mathbb{C} \\ t \mapsto \gamma_1(t) + i\gamma_2(t). \quad (85)$$

Definition 5.2. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval and $D \subseteq \mathbb{C}$ a domain. We define an *arc* as a continuous application of the form

$$\gamma : I \rightarrow D \\ t \mapsto \gamma_1(t) + i\gamma_2(t). \quad (86)$$

Equivalently, we can say an arc is a curve restricted to some interval.

Definition 5.3. Let $\gamma : [a, b] \rightarrow D$ be an arc. We call $\gamma(a)$ and $\gamma(b)$ *the extremes of γ* . In particular, we call $\gamma(a)$ the *initial point* and $\gamma(b)$ the *final point*.

Definition 5.4. Let $\gamma : [a, b] \rightarrow D$ be an arc. We define the *route or graph of γ* as

$$\gamma^* := \{z \in D \mid z = \gamma(t), t \in I\}. \quad (87)$$

Definition 5.5. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *closed* if and only if $\gamma(a) = \gamma(b)$.

Definition 5.6. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *simple* if and only if there is no two numbers $t_1, t_2 \in (a, b)$ such that $\gamma(t_1) = \gamma(t_2)$. We also call it a *Jordan curve*, and if it is closed, a *circuit*.

Definition 5.7. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *differentiable* if for all value $t_0 \in [a, b]$ there exists the limit

$$\gamma'(t_0) = \lim_{t \rightarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \quad (88)$$

For $t_0 = a$ or $t_0 = b$ we consider the laterals limits from the right and from the left respectively.

Definition 5.8. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is of *class C^1* if and only if γ' exists and is continuous at $[a, b]$.

Definition 5.9. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *regular or smooth* if and only if it is differentiable and γ' never vanishes.

Definition 5.10. Let $\gamma : [a, b] \rightarrow D$ be an arc. We say γ is *piece-wise of class C^1* if and only if γ' exists and is continuous in I except in a finite number of points where γ has lateral derivatives.

Definition 5.11. Let $\gamma : [a, b] \rightarrow D$ be an arc. We define the *opposite arc* as

$$\begin{aligned} -\gamma : [-b, -a] &\rightarrow \mathbb{C} \\ t &\mapsto \gamma(-t) \end{aligned} \quad (89)$$

Definition 5.12. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be an arc. We say $\Gamma(s), s \in [c, d] \subseteq \mathbb{R}$ *has been obtained from $\gamma(t), t \in [a, b]$ by a change of parametrization* if and only if the new parameter s and the original parameter t are related by a relation $t = \phi(s)$, where $\phi : [c, d] \rightarrow [a, b]$ is an homeomorphism that satisfies $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$. We call Γ the *reparametrization of γ* .

Definition 5.13. Let $\gamma_1 : I_1 \rightarrow \mathbb{C}$ and $\gamma_2 : I_2 \rightarrow \mathbb{C}$ be two arcs. We say they are *equivalent* if and only if there exists a bijective, monotone, and continuous function $\rho : I_2 \rightarrow I_1$ such that $\gamma_2 = \gamma_1 \circ \rho$. If ρ is an increasing function we say γ_1 and γ_2 *have the same orientation*; otherwise, we say γ_1 and γ_2 *have opposite orientations*.

Definition 5.14. Let $\gamma_1 : [a, b] \rightarrow \mathbb{C}$ and $\gamma_2 : [c, d] \rightarrow \mathbb{C}$ be two arcs such that $[a, b] \cap [c, d] = \emptyset$. We define

the application $\gamma_1 \cup \gamma_2$ (sometimes denoted by $\gamma_1 + \gamma_2$) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \leq t \leq b \\ \gamma_2(t - b + c), & \text{if } b \leq t \leq b + d - c \end{cases} \quad (90)$$

We say γ_1, γ_2 can be joined/added or that there exists its union/sum if and only if $\gamma_1(b) = \gamma_2(c)$. In this case $\gamma_1 + \gamma_2$ is an arc, and we call it *the sum arc of γ_1 plus γ_2* .

Definition 5.15. We define the *segment of extremes* $z_1, z_2 \in \mathbb{C}$ as the arc defined by the expression

$$\begin{aligned} [z_1, z_2] : [0, 1] &\rightarrow \mathbb{C} \\ t &\mapsto (1 - t)z_1 + tz_2 \end{aligned} \quad (91)$$

Definition 5.16. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a function. We say f is *polygonal* if and only if can be expressed as a finite union of segments, that is, if there exist a natural number n and points $\{z_0, \dots, z_n\}$ such that

$$p = [z_0, z_1] \cup \dots \cup [z_{n-1}, z_n]. \quad (92)$$

Definition 5.17. Let $\gamma : [a, b] \rightarrow D$ be an arc with a, b finite. We say γ is a *basic curve* if and only if $\gamma \in C^1((a, b)) \cap C([a, b])$ and there exist $\lim_{t \rightarrow a^+} \gamma'(t), \lim_{t \rightarrow b^-} \gamma'(t)$.

Definition 5.18. A *path* is a function $\gamma : [a, b] \rightarrow \mathbb{C}$ such that there exist basic curves $\gamma_j : [a_j, b_j] \rightarrow \mathbb{C}, j \in \{1, \dots, k\}$ such that $\gamma = \gamma_1 + \dots + \gamma_k$ and therefore $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ and $a = a_1, b = a_k$.

Definition 5.19. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a continuous curve and $a_1, \dots, a_l \in \mathbb{R}$ such that $a = a_0 \leq \dots \leq a_l \leq b = a_{l+1}$. We say γ is *piece-wise differentiable* if and only if

$$\gamma \in C^1 \left(\bigcup_{j=0}^l (\alpha_j, \alpha_{j+1}) \right),$$

$$\forall j \in \{0, \dots, l+1\} \exists \lim_{t \rightarrow a_j^+} \gamma'(t) \text{ (except if } j = l+1), \lim_{t \rightarrow a_j^-} \gamma'(t) \text{ (except if } j = 0)$$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

Theorem 5.1. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with Ω an open set and $\phi : I \rightarrow \Omega$ a basic curve. Then, $\psi = f \circ \phi$ is a basic curve (hence a derivable curve) and its real derivative is

$$\psi'(t) = f'(\phi(t))\phi'(t). \quad (93)$$

Definition 5.20. Let $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbb{C}$ be two curves. We say γ_1, γ_2 are *homotopic* if and only if there exists a continuous function $h(t, s) : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$ such that

1. $h(t, 0) = \gamma_1(t), t \in [0, 1]$.
2. $h(t, 1) = \gamma_2(t), t \in [0, 1]$.
3. $h(0, s) = \gamma_1(0) = \gamma_2(0), s \in [0, 1]$.

4. $h(1, s) = \gamma_1(1) = \gamma_2(1), s \in [0, 1]$.

Definition 5.21. Let $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbb{C}$ be two circuits. We say γ_1, γ_2 are *homotopic* if and only if there exists a continuous function $h(t, s) : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$ such that

1. $h(t, 0) = \gamma_1(t), t \in [0, 1]$.
2. $h(t, 1) = \gamma_2(t), t \in [0, 1]$.
3. $h(0, s) = h(1, s), s \in [0, 1]$.

6 Local properties of holomorphic functions

7 Isolated singularities of holomorphic functions

8 Homology

9 Harmonic functions

10 Conforming representation

11 Riemann theorem

12 Runge theorem

13 Zeros of holomorphic functions

14 Fourier transform

Definition 14.1. Let $f \in L^1(\mathbb{R})$ be a function and $\xi \in \mathbb{R}$ a number. We define the *Fourier transform* of f at the point ξ as

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx. \quad (94)$$

Proposition 14.1. Let $f \in L^1(\mathbb{R})$ be a function. Then, the application

$$\begin{aligned} \mathcal{F}\{f\} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \hat{f}(\xi) \end{aligned} \quad (95)$$

is a well defined application.

Definition 14.2. Let $\{f_n\}_{n \in \mathbb{N}} \subseteq L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. We say the functions f_n converge to f with a norm $\|\cdot\|_p$ or converge in $L^p(\mathbb{R})$ if and only if

$$\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0. \quad (96)$$

Theorem 14.2. Let $f \in L^1(\mathbb{R})$ be a function. Then, the following statements are true.

1. The Fourier transform of f satisfies

$$\mathcal{F}\{f\} \in L^\infty(\mathbb{R}), \quad \|\mathcal{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|f\|_1. \quad (97)$$

2. $\mathcal{F}\{f\}$ is \mathbb{C} linear, that is, for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$,

$$\mathcal{F}\{\alpha f + \beta g\} = \alpha \mathcal{F}\{f\} + \beta \mathcal{F}\{g\}. \quad (98)$$

3. If $g(x) = \bar{f}(x)$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \quad (99)$$

4. If $g(x) = g(\lambda x)$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \frac{1}{\lambda} \hat{f}\left(\frac{\xi}{\lambda}\right). \quad (100)$$

5. If $g(x) = f(x - a)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = e^{-ia\xi} \hat{f}(\xi). \quad (101)$$

6. If $g(x) = e^{iax} f(x)$ with $\alpha \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \hat{f}(\xi - a) \quad (102)$$

7. If $\{f_n\}_{n \in \mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \rightarrow f$ in $L^1(\mathbb{R})$ when $n \rightarrow \infty$, then $\mathcal{F}\{f_n\} \rightarrow \mathcal{F}\{f\}$ uniformly in \mathbb{R} .

8. The Fourier transform $\mathcal{F}\{f\}$ is a continuous function in \mathbb{R} , $\mathcal{F}\{f\} \in C(\mathbb{R})$.

Proposition 14.3. Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and $\lim_{|x| \rightarrow \infty} |f(x)| = 0$. Then,

$$\hat{f}'(\xi) = i\xi \hat{f}(\xi). \quad (103)$$

Corollary 14.4. Let $f \in L^1(\mathbb{R})$ be a function such that there exists its n -th derivative $f^{(n)} \in L^1(\mathbb{R})$ and $\lim_{|x| \rightarrow \infty} |f(x)| = 0$. Then,

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi). \quad (104)$$

Definition 14.3. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{C}$ be a function. We define the support of f as

$$\text{supp } f := \overline{\{x \in I \mid f(x) \neq 0\}}. \quad (105)$$

Definition 14.4. We define the set $\mathcal{D}(\mathbb{R})$ as

$$\mathcal{D}(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}) \mid \text{supp } \varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \quad (106)$$

Theorem 14.5. Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_n \in \mathcal{D}(\mathbb{R})$ such that

$$\lim_{h \rightarrow \infty} \int_{\mathbb{R}} |f - \phi_n| dx = 0, \quad (107)$$

that is, we have convergence of ϕ_n to f with norm $\|\cdot\|_1$.

Proposition 14.6. Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.

Proposition 14.7. Let $f \in L^1(\mathbb{R})$ be a function. Then, $|\hat{f}(\xi)| \leq \|f\|_1$.

Theorem 14.8. Let $f \in L^1(\mathbb{R})$ be a function. Then,

$$\lim_{|x| \rightarrow \infty} \hat{f}(x) = 0. \quad (108)$$

Theorem 14.9. The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathcal{F}\{f\} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$.

Definition 14.5. We define the Schwartz space as

$$S(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{C} \mid f \in C^\infty(\mathbb{R}) \wedge \forall n, m \in \mathbb{N} \exists c_{n,m} < \infty \text{ such that } (1 + |x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}.$$

Proposition 14.10. Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \rightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,

1. $f + g \in S(\mathbb{R})$.
2. $\lambda f \in S(\mathbb{R})$.
3. $f g \in S(\mathbb{R})$.
4. $P f \in S(\mathbb{R})$.

Theorem 14.11. Let $I, J \subseteq \mathbb{R}$ be two intervals with I compact and J open. Let $f : I \times J \rightarrow \mathbb{R}$ be a function such that

1. $f(\cdot, \lambda)$ is Riemann-integrable in I for all $\lambda \in J$,
2. $f(x, \cdot)$ is derivable in J for all $x \in I$.

If $\partial_\lambda f$ is continuous in $I \times J$, then

1. $\partial_\lambda f(\cdot, \lambda)$ is Riemann-integrable for all $\lambda \in J$.
2. $F(\lambda) = \int_I f(x, \lambda) dx$ is derivable with continuous derivative in J for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.

$$F'(\lambda) = \frac{d}{d\lambda} \int_I f(x, \lambda_0) dx = \int_I \frac{\partial f}{\partial \lambda}(x, \lambda_0) dx, \forall \lambda_0 \in J. \quad (109)$$

Proposition 14.12. Let $f \in S(\mathbb{R})$. Then,

1. $S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.
2. $\widehat{x f}(\xi) = (i D_\xi \hat{f})(\xi)$ for all $\xi \in \mathbb{R}$.

Corollary 14.13. Let $f \in s(\mathbb{R})$. Then,

$$\widehat{x^n f}(\xi) = (i^n D^n \hat{f})(\xi), \forall n \in \mathbb{N}. \quad (110)$$

Proposition 14.14. The Fourier transform \mathcal{F} restricted to $S(\mathbb{R})$ is an automorphism, that is, if $f \in S(\mathbb{R})$ then $\mathcal{F}\{f\} = \hat{f} \in S(\mathbb{R})$.

Lemma 14.15. If $G(x) = e^{-x^2/2}$, then $\hat{G}(\xi) = e^{-\xi^2/2}$. We observe hence that G is a fixed point of \mathcal{F} .

Lemma 14.16. If $f, g \in S(\mathbb{R})$, then

$$\int_{\mathbb{R}} f(\xi) \hat{g}(\xi) d\xi = \int_{\mathbb{R}} \hat{f}(\tau) g(\tau) d\tau. \quad (111)$$

Lemma 14.17. Let $f, g \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,

1. $g(\lambda x) \hat{f}(x)$ converges to $g(0) \hat{f}(x)$ uniformly in \mathbb{R} when $\lambda \rightarrow \infty$.
2. $f(\lambda x) \hat{g}(x)$ converges to $f(0) \hat{g}(x)$ uniformly in \mathbb{R} when $\lambda \rightarrow \infty$.

Lemma 14.18. Let $f, g \in s(\mathbb{R})$. Then,

$$f(0) \int_{\mathbb{R}} \hat{g}(\xi) d\xi = g(0) \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad (112)$$

Lemma 14.19. Let $f \in s(\mathbb{R})$ be a function. Then,

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad (113)$$

Corollary 14.20 (Inversion formula). Let $f \in S(\mathbb{R})$. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi, \forall x \in \mathbb{R}. \quad (114)$$

Theorem 14.21 (Inversion of \mathcal{F} in $S(\mathbb{R})$). Let $\mathcal{F} : S(\mathbb{R}) \rightarrow S(\mathbb{R})$, defined by $\mathcal{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, \mathcal{F} is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathcal{F}^4 = Id$. In particular, $\mathcal{F}^{-1} = \mathcal{F}^3$ and if $f \in S(\mathbb{R})$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}\{f\}(\xi) e^{ix\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} e^{ix\xi} d\xi. \quad (115)$$

In fact, \mathcal{F} is an homomorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(S(\mathbb{R}), \|\cdot\|_{n,m})$.

Theorem 14.22 (Inversion of \mathcal{F} for discontinuities). Let f be a absolutely Riemann-integrable function in \mathbb{R} with f and f' piece-wise continuous. Then,

$$\frac{f(x^-) + f(x^+)}{2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} e^{ix\xi} d\xi. \quad (116)$$

Definition 14.6. Let f be a Riemann-integrable function in \mathbb{R} . We define the Fourier transform of cosine kind as

$$\hat{f}_c(\xi) := \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(\xi x) f(x) dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(\xi x) f_e(x) dx, \quad (117)$$

and the Fourier transform of sine kind as

$$\hat{f}_s(\xi) := \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\xi x) f(x) dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\xi x) f_o(x) dx. \quad (118)$$

Proposition 14.23. Let \hat{f}_c, \hat{f}_s be the Fourier transform of cosine and sine kinds of f . Then, $\hat{f}_c(\xi)$ is even, $\hat{f}_s(\xi)$ is odd, and $\hat{f}(\xi) = \hat{f}_c(\xi) - i\hat{f}_s(\xi)$.

Theorem 14.24. Let f be a absolutely Riemann-integrable function in \mathbb{R} with f and f' piece-wise continuous. Then,

$$\frac{f_e(x^-) + f_e(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c \cos(\xi x) d\xi, \quad (119)$$

$$\frac{f_o(x^-) + f_o(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s \sin(\xi x) d\xi. \quad (120)$$

Theorem 14.25 (Tonelli's Theorem). Let $f : I \times J \rightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x, y) \geq 0$ for all $(x, y) \in I \times J$. Then,

$$\int_{I \times J} f dx dy = \int_I \int_J f(x, y) dy dx = \int_J \int_I f(x, y) dx dy. \quad (121)$$

Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.

Corollary 14.26. Let $f, g \in L^1(\mathbb{R})$. Then, $F(x, t) = f(t)g(x-t) \in L^1(\mathbb{R}^2)$.

Definition 14.7. Let $f, g \in L^1(\mathbb{R})$ two function. We define the convolution of f and g as

$$(f * g) : \mathbb{R} \rightarrow \mathbb{C} \\ x \mapsto \int_{\mathbb{R}} f(t)g(x-t) dt, \quad (122)$$

which is from $L^1(\mathbb{R})$.

Proposition 14.27. Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi} \hat{f} \hat{g}$.

Theorem 14.28. Let $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^\infty(\mathbb{R})$.

Theorem 14.29. Let $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$ with $\text{supp } f$ compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and $\text{supp } \{f * \phi\} \subseteq \text{supp } f + \text{supp } \phi$.

Definition 14.8. We say the functions $\phi_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ continuous in a compact support are an approximation of the unity if and only if

1. $\phi_\epsilon \geq 0$ for all ϵ .
2. $\int_{\mathbb{R}} \phi_\epsilon(x) dx = 1$.
3. For all $\delta > 0$ it is satisfied that

$$\lim_{\epsilon \rightarrow 0} \left\{ \sup_{|t| > \delta} \phi_\epsilon(t) \right\} = 0. \quad (123)$$

Theorem 14.30. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \rightarrow 0$ $f * \phi_\epsilon$ converges uniformly in \mathbb{R} to f .

Corollary 14.31. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \rightarrow 0$ $f * \phi_\epsilon$ converges uniformly in \mathbb{R} to f .

Theorem 14.32 (Weierstrass polynomial approximation). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials P_n with $n \in \mathbb{N}$ such that P_n converge uniformly to f in $[a, b]$.

Theorem 14.33. Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \rightarrow f$ with norm $\|\cdot\|_p$ (that is, convergence in L^p), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \quad (124)$$

with $\|f\|_{C^k(\mathbb{R})} = \max_{0 \leq l \leq k} (\sup_{x \in \mathbb{R}} |D^l f(x)|)$ being a norm.

Lemma 14.34. Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\int_{\mathbb{R}} f(x)\phi(x) dx = 0$.

Then, $f \equiv 0$.

Corollary 14.35. The Fourier transform \mathcal{F} is injective since $\mathcal{F}\{f\} = \hat{f} = 0 \Leftrightarrow f = 0$ in $L^1(\mathbb{R})$ (the zero function class) and \mathcal{F} is a linear application.

Theorem 14.36 (Inversion theorem in $L^1(\mathbb{R})$). Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi, \forall x \in \mathbb{R}. \quad (125)$$

15 Multidimensional fourier transform

Theorem 15.1. For several variables

$$\mathcal{F}\{f(x_1, \dots, x_n)\} = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) e^{-i(x_1\xi_1 + \dots + x_n\xi_n)} d\mathbf{x} \quad (126)$$

or simpler,

$$\mathcal{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} d\mathbf{x}. \quad (127)$$