Contents

Chapter 1

Harmonic oscillator

1.1 Ladder operators

Definition 1.1.1. Let $\mathcal{H}$ be a Hilbert space in a harmonic potential

$$V(\hat{x}) = \frac{\omega^2}{2}\hat{x}^2, \qquad \omega^2 = \frac{k}{m}. \tag{1.1}$$

We define the *creation* and *annihilation operators* as

$$\hat{a}^\dagger := \frac{\alpha}{\sqrt{2}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right), \qquad \hat{a} := \frac{\alpha}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \qquad \alpha := \sqrt{\frac{m\omega}{\hbar}}. \tag{1.2}$$

Proposition 1.1.1. *Let $\mathcal{H}$ be a Hilbert space in a harmonic potential. Then,*

$$\langle x| \hat{a}^\dagger = \frac{\alpha}{\sqrt{2}}\left(x - \frac{1}{\alpha^2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \qquad \langle x| \hat{a} = \frac{\alpha}{\sqrt{2}}\left(x + \frac{1}{\alpha^2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \qquad \alpha = \frac{m\omega}{\hbar}. \tag{1.3}$$

Proposition 1.1.2. *Let $\mathcal{H}$ be a Hilberet space with a harmonic potential. Then,*

$$\hat{x} = \frac{1}{\sqrt{2}\alpha}(\hat{a}^\dagger + \hat{a}), \qquad \hat{p} = i\hbar\frac{\alpha}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}). \tag{1.4}$$

Proposition 1.1.3. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{a}, \hat{a}^\dagger$ are not hermitian.*
2. *$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$.*
3. *$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$.*

Definition 1.1.2. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *number operator* as

$$\hat{N} := \hat{a}^\dagger\hat{a}. \tag{1.5}$$

Proposition 1.1.4. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{H}$ is hermitian.*
2. *$\left[\hat{N}, \hat{a}\right] = -\hat{a}$, $\left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger$,*
3. *$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\hat{I}\right)$.*

Proposition 1.1.5. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, $\hat{H}$ and $\hat{N}$ have a common basis of eigenvectors, which is countable, and*

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \qquad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{1.6}$$

$$\hat{N}\,|n\rangle = n\,|n\rangle\,, \qquad \hat{H}\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle\,, \qquad n \in \mathbb{N}. \tag{1.7}$$

Corollary 1.1.6. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n |0\rangle\,. \tag{1.8}$$

Proposition 1.1.7. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, the eigenstates form a non-dgenerate basis.*

Definition 1.1.3 (Fock states)**.** Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *Fock states* as the states that determine the basis $(|n\rangle)$ and have a well-defined number of excitations.

Definition 1.1.4. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We call the fundamental Fock state *the vaccum.*

Proposition 1.1.8. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, $\hat{a}, \hat{a}^\dagger$ and $\hat{N}$ have the following matrix representation in the basis $(|n\rangle)$.*

$$[\hat{N}]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad [\hat{a}]_B = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ 0 & 0 & \sqrt{2} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad [\hat{a}^\dagger]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.9}$$

or in coefficient representation,

$$[\hat{N}]_{ij} = (i-1)\delta_{ij}, \qquad [\hat{a}]_{ij} = \sqrt{j-1}\delta_{i,j-1}, \qquad [\hat{a}^\dagger]_{ij} = \sqrt{i-1}\delta_{i-1,j}. \tag{1.10}$$

1.2 Fock states wave functions

Proposition 1.2.1. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\varphi_0(x) = \langle x|0\rangle = \left(\frac{\beta^2}{\pi}\right)^{1/4} \exp\left(-\frac{\alpha^2 x^2}{2}\right), \tag{1.11}$$

$$\varphi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{\beta}{\sqrt{2}}x - \frac{1}{\sqrt{2}\beta}\frac{\mathrm{d}}{\mathrm{d}x}\right)\varphi_0(x) = \frac{1}{\sqrt{2^n n!}} H_n(\beta x)\varphi_0(x). \tag{1.12}$$

Proposition 1.2.2. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $\hat{\sigma}$ a sequence formed by k $\hat{a}$ and l $\hat{a}^\dagger$. Then,*

$$\langle n|\,\hat{\sigma}\,|n\rangle \leftrightarrow k = l. \tag{1.13}$$

Proposition 1.2.3. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\langle \hat{x}\rangle_n = 0, \qquad \left\langle \hat{x}^2\right\rangle = \frac{\hbar}{2m\omega}(2n+1), \qquad \langle \hat{p}\rangle_n = 0, \qquad \left\langle \hat{p}^2\right\rangle = \frac{\hbar m\omega}{2}(2n+1), \tag{1.14}$$

$$\Delta x \Delta p = \frac{\hbar}{2}(2n+1). \tag{1.15}$$

Proposition 1.2.4. *Let $\mathcal{H}$ a Hilbert space with a harmonic potential. Then,*

$$\langle T\rangle = \langle V\rangle\,. \tag{1.16}$$

1.3 Coherent states

Definition 1.3.1. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define a *coherent state* as a state $|\alpha\rangle \in \mathcal{H}$ such that

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{1.17}$$

Definition 1.3.2. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *displaced state* as the state $|\psi_\alpha\rangle \in \mathcal{H}$ determined by

$$\psi_\alpha(x) = \psi_0(x - x_0). \tag{1.18}$$

Proposition 1.3.1. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and a force $F = f$. Then, the fundamental state is a displaced state with $x_0 = f/m\omega^2$.*

Proposition 1.3.2. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\psi_\alpha\rangle \in \mathcal{H}$ a displaced state with displacement x_0. Then, $|\psi_\alpha\rangle$ is a coherent state with eigenvalue*

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}}x_0. \tag{1.19}$$

Proposition 1.3.3. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha \hat{a}^\dagger} |0\rangle . \tag{1.20}$$

Proposition 1.3.4. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle$ a coherent state. Then,*

$$\left\langle \hat{N} \right\rangle_\alpha = |\alpha|^2, \qquad p_{|\alpha\rangle}(n) = \mathrm{e}^{-\langle \hat{N} \rangle} \frac{\left\langle \hat{N} \right\rangle^n}{n!}. \tag{1.21}$$

Theorem 1.3.5 (Baker-Campbell-Hausdorff formula)**.** *Let $\mathcal{H}$ be a Hilbert space and $\hat{A}, \hat{B} : \mathcal{H} \longrightarrow \mathcal{H}$ two operators such that $\left[\left[\hat{A}, \hat{B}\right], \hat{A}\right], \left[\left[\hat{A}, \hat{B}\right], \hat{B}\right] = 0$. Then,*

$$\exp\left(\hat{A} + \hat{B}\right) = \exp\left(-\frac{1}{2}\left[\hat{A}, \hat{B}\right]\right) \exp\left(\hat{A}\right) \exp\left(\hat{B}\right). \tag{1.22}$$

Proposition 1.3.6. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\left[\bar{\alpha}\hat{a}, \alpha\hat{a}^\dagger\right] = |\alpha|^2 \hat{I}, \qquad |\alpha\rangle = \exp\left(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}\right) |0\rangle \coloneqq \hat{\mathcal{D}}(\alpha) |0\rangle . \tag{1.23}$$

Definition 1.3.3. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *displacement operator* as

$$\hat{\mathcal{D}}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}\right). \tag{1.24}$$

Proposition 1.3.7. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{\lceil}(\alpha)$ is unitary.*
2. *$\hat{\mathcal{D}}^\dagger(\alpha) = \hat{\mathcal{D}}(-\alpha)$.*
3. *$\hat{\mathcal{D}}(\alpha)\hat{\mathcal{D}}^\dagger(\alpha) = \hat{I}$.*

Proposition 1.3.8. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\hat{\mathcal{D}}(\alpha) = \exp\left(-i\frac{x_0\hat{p} - p_0\hat{x}}{\hbar}\right) = \exp\left(-\frac{i}{2}\frac{x_0 p_0}{\hbar}\right) \exp\left(i\frac{p_0\hat{x}}{\hbar}\right) \exp\left(-i\frac{x_0\hat{p}}{\hbar}\right), \tag{1.25}$$

$$x_0 = \sqrt{2} l \operatorname{Re}\{\alpha\}, \qquad p_0 = \sqrt{2}\frac{l}{\hbar} \operatorname{Im}\{\alpha\}, \qquad l = \sqrt{\frac{\hbar}{m\omega}}. \tag{1.26}$$

Proposition 1.3.9. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\langle x|\alpha\rangle = \psi_\alpha(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(\frac{ip_0}{2\hbar}(2x - x_0)\right) \exp\left(-\frac{(x - x_0)^2}{4\sigma_x^2}\right), \qquad \frac{1}{4\sigma_x^2} = \frac{1}{2}\frac{m\omega}{\hbar} \tag{1.27}$$

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re}\{\alpha\}, \qquad p_0 = \sqrt{2\hbar m\omega} \operatorname{Im}\{\alpha\} \tag{1.28}$$

Proposition 1.3.10. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $\{|\alpha\rangle\}$ the set of coherent states. Then, they form a* overcomplete basis*, that is, not for all pair of states $|\alpha\rangle, |\alpha'\rangle$ it is satisfied $\langle\alpha'|\alpha\rangle = 0$. Hence,*

$$\hat{I} = \frac{1}{\pi} \int |\alpha\rangle\langle\alpha| \, \mathrm{d}^2\alpha , \qquad |\langle\alpha|\beta\rangle|^2 = \mathrm{e}^{-|\alpha - \beta|^2}. \tag{1.29}$$

Besides,$\langle\alpha|\beta\rangle \to 0$ if and only if $|\alpha - \beta| \gg 1$.

1.3.1 Coherent states dynamics

Proposition 1.3.11. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$|\alpha\rangle(t) = \mathrm{e}^{i\omega t/2}|\alpha(t)\rangle = \mathrm{e}^{i\omega t/2}\left|\alpha_0\mathrm{e}^{i\omega t}\right\rangle. \tag{1.30}$$

Proposition 1.3.12. *Let $\mathcal{H}$ be a Hilbert space witha harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\langle\hat{x}\rangle = x_0\cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t), \qquad \langle\hat{p}\rangle = p_0\cos(\omega t) - m\omega x_0\sin(\omega t). \tag{1.31}$$

1.4 Minimum uncertainty states

Definition 1.4.1. Let $\mathcal{H}$ be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ a state. We say $|\psi\rangle$ is a *minimum uncertainty state* if and only if

$$\Delta x\Delta p = \frac{\hbar}{2}. \tag{1.32}$$

Proposition 1.4.1. *Let $\mathcal{H}$ be a Hilbert state, $|\in\rangle\,\mathcal{H}$ a state and $|\psi_x\rangle = \hat{\delta x}|\psi\rangle, |\psi_p\rangle = \hat{\delta p}|\psi\rangle$. Then,*

$$\langle\psi_x|\psi_x\rangle\langle\psi_p|\psi_p\rangle \geq |\langle\psi_x|\psi_p\rangle|^2. \tag{1.33}$$

and the equality only occures when there exists a $\lambda \in \mathbb{C}$ such that $|\psi_p\rangle = \lambda|\psi_x\rangle$.

Proposition 1.4.2. *Let $\mathcal{H}$ be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ be a state. Then,*

$$\Big|\langle\psi|\hat{\delta x}\hat{\delta p}|\psi\rangle\Big|^2 \geq \frac{1}{4}\Big|\langle\psi|\left[\hat{\delta x}, \hat{\delta p}\right]|\psi\rangle\Big|^2, \tag{1.34}$$

and the equality only occures when $\{\hat{\delta x}, \hat{\delta p}\} = 0$.

Proposition 1.4.3. *Let $\mathcal{H}$ be a Hilbert space and $|\in\rangle\,\mathcal{H}$ a minimum uncertainty state. Then,*

$$\langle x|\psi\rangle = \psi(x) = C\exp\left[-\frac{|\lambda|}{2}(x - \langle x\rangle)^2\right]\exp\left[\frac{ix\langle p\rangle}{\hbar}\right], \tag{1.35}$$

for some $\lambda \in \mathbb{C}$ and with variance $\Delta x^2 = \hbar/2|\lambda|$.