

1 Harmonic oscillator

Definition 1.1. Let \mathcal{H} be a Hilbert space in a harmonic potential

$$V(\hat{x}) = \frac{\omega^2}{2} \hat{x}^2, \quad \omega^2 = \frac{k}{m}. \quad (1)$$

We define the *creation* and *annihilation operators* as

$$\hat{a}^\dagger := \frac{\alpha}{\sqrt{2}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right), \quad (2)$$

$$\hat{a} := \frac{\alpha}{\sqrt{2}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right), \quad (3)$$

$$\alpha := \sqrt{\frac{m\omega}{\hbar}}. \quad (4)$$

Proposition 1.1. Let \mathcal{H} be a Hilbert space in a harmonic potential. Then,

$$\langle x | \hat{a}^\dagger = \frac{\alpha}{\sqrt{2}} \left(x - \frac{1}{\alpha^2} \frac{d}{dx} \right), \quad (5)$$

$$\langle x | \hat{a} = \frac{\alpha}{\sqrt{2}} \left(x + \frac{1}{\alpha^2} \frac{d}{dx} \right), \quad (6)$$

$$\alpha = \frac{m\omega}{\hbar}. \quad (7)$$

Proposition 1.2. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then,

$$\hat{x} = \frac{1}{\sqrt{2}\alpha} (\hat{a}^\dagger + \hat{a}), \quad \hat{p} = i\hbar \frac{\alpha}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}). \quad (8)$$

Proposition 1.3. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then,

1. \hat{a}, \hat{a}^\dagger are not hermitian.

2. $[\hat{a}, \hat{a}^\dagger] = \hat{I}$.

3. $\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$.

Definition 1.2. Let \mathcal{H} be a Hilbert space with a harmonic potential. We define the *number operator* as

$$\hat{N} := \hat{a}^\dagger \hat{a}. \quad (9)$$

Proposition 1.4. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then,

1. \hat{H} is hermitian.

2. $[\hat{N}, \hat{a}] = -\hat{a}$, $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$,

3. $\hat{H} = \hbar\omega \left(\hat{N} + \frac{1}{2} \hat{I} \right)$.

Proposition 1.5. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then, \hat{H} and \hat{N} have a common basis of eigenvectors, which is countable, and

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle, \quad (10)$$

$$\hat{N} |n\rangle = n |n\rangle, \quad \hat{H} |n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle, \quad (11)$$

$$n \in \mathbb{N}. \quad (12)$$

Corollary 1.6. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then,

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle. \quad (13)$$

Proposition 1.7. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then, the eigenstates form a non-degenerate basis.

Definition 1.3 (Fock states). Let \mathcal{H} be a Hilbert space with a harmonic potential. We define the *Fock states* as the states that determine the basis $(|n\rangle)$ and have a well-defined number of excitations.

Definition 1.4. Let \mathcal{H} be a Hilbert space with a harmonic potential. We call the fundamental Fock state the *vacuum*.

Proposition 1.8. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then, \hat{a}, \hat{a}^\dagger and \hat{N} have the following matrix representation in the basis $(|n\rangle)$.

$$[\hat{N}]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (14)$$

$$[\hat{a}]_B = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ 0 & 0 & \sqrt{2} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (15)$$

$$[\hat{a}^\dagger]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (16)$$

or in coefficient representation,

$$[\hat{N}]_{ij} = (i-1)\delta_{ij}, \quad (17)$$

$$[\hat{a}]_{ij} = \sqrt{j-1}\delta_{i,j-1}, \quad (18)$$

$$[\hat{a}^\dagger]_{ij} = \sqrt{i-1}\delta_{i-1,j}. \quad (19)$$

Proposition 1.9. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then,

$$\varphi_0(x) = \langle x|0\rangle = \left(\frac{\beta^2}{\pi} \right)^{1/4} \exp\left(-\frac{\alpha^2 x^2}{2} \right), \quad (20)$$

$$\varphi_n(x) = \frac{1}{\sqrt{n!}} \left(\frac{\beta}{\sqrt{2}} x - \frac{1}{\sqrt{2}\beta} \frac{d}{dx} \right) \varphi_0(x) = \quad (21)$$

$$\frac{1}{\sqrt{2^n n!}} H_n(\beta x) \varphi_0(x). \quad (22)$$

Proposition 1.10. Let \mathcal{H} be a Hilbert space with a harmonic potential and $\hat{\sigma}$ a sequence formed by k \hat{a} and l \hat{a}^\dagger . Then,

$$\langle n | \hat{\sigma} | n \rangle \leftrightarrow k = l. \quad (23)$$

Proposition 1.11. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then,

$$\langle \hat{x} \rangle_n = 0, \quad \langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega}(2n+1), \quad (24)$$

$$\langle \hat{p} \rangle_n = 0, \quad \langle \hat{p}^2 \rangle = \frac{\hbar m\omega}{2}(2n+1), \quad (25)$$

$$\Delta x \Delta p = \frac{\hbar}{2}(2n+1). \quad (26)$$

Proposition 1.12. Let \mathcal{H} a Hilbert space with a harmonic potential. Then,

$$\langle T \rangle = \langle V \rangle. \quad (27)$$

Definition 1.5. Let \mathcal{H} be a Hilbert space with a harmonic potential. We define a *coherent state* as a state $|\alpha\rangle \in \mathcal{H}$ such that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \quad (28)$$

Definition 1.6. Let \mathcal{H} be a Hilbert space with a harmonic potential. We define the *displaced state* as the state $|\psi_\alpha\rangle \in \mathcal{H}$ determined by

$$\psi_\alpha(x) = \psi_0(x - x_0). \quad (29)$$

Proposition 1.13. Let \mathcal{H} be a Hilbert space with a harmonic potential and a force $F = f$. Then, the fundamental state is a displaced state with $x_0 = f/m\omega^2$.

Proposition 1.14. Let \mathcal{H} be a Hilbert space with a harmonic potential and $|\psi_\alpha\rangle \in \mathcal{H}$ a displaced state with displacement x_0 . Then, $|\psi_\alpha\rangle$ is a coherent state with eigenvalue

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}}x_0. \quad (30)$$

Proposition 1.15. Let \mathcal{H} be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle. \quad (31)$$

Proposition 1.16. Let \mathcal{H} be a Hilbert space with a harmonic potential and $|\alpha\rangle$ a coherent state. Then,

$$\langle \hat{N} \rangle_\alpha = |\alpha|^2, \quad p_{|\alpha\rangle}(n) = e^{-\langle \hat{N} \rangle} \frac{\langle \hat{N} \rangle^n}{n!}. \quad (32)$$

Theorem 1.17 (Baker-Campbell-Hausdorff formula). Let \mathcal{H} be a Hilbert space and $\hat{A}, \hat{B} : \mathcal{H} \rightarrow \mathcal{H}$ two operators such that $[[\hat{A}, \hat{B}], \hat{A}] = [[\hat{A}, \hat{B}], \hat{B}] = 0$. Then,

$$\exp(\hat{A} + \hat{B}) = \exp\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right) \exp(\hat{A}) \exp(\hat{B}). \quad (33)$$

Proposition 1.18. Let \mathcal{H} be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,

$$[\bar{\alpha}\hat{a}, \alpha\hat{a}^\dagger] = |\alpha|^2 \hat{I}, \quad (34)$$

$$|\alpha\rangle = \exp(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}) |0\rangle := \hat{D}(\alpha) |0\rangle. \quad (35)$$

Definition 1.7. Let \mathcal{H} be a Hilbert space with a harmonic potential. We define the *displacement operator* as

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}). \quad (36)$$

Proposition 1.19. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then,

1. $\hat{I}(\alpha)$ is unitary.

2. $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha)$.

3. $\hat{D}(\alpha)\hat{D}^\dagger(\alpha) = \hat{I}$.

Proposition 1.20. Let \mathcal{H} be a Hilbert space with a harmonic potential. Then,

$$\hat{D}(\alpha) = \exp\left(-i\frac{x_0\hat{p} - p_0\hat{x}}{\hbar}\right) = \quad (37)$$

$$\exp\left(-\frac{i}{2}\frac{x_0 p_0}{\hbar}\right) \exp\left(i\frac{p_0\hat{x}}{\hbar}\right) \exp\left(-i\frac{x_0\hat{p}}{\hbar}\right), \quad (38)$$

$$x_0 = \sqrt{2l} \operatorname{Re}\{\alpha\}, \quad p_0 = \sqrt{2}\frac{l}{\hbar} \operatorname{Im}\{\alpha\}, \quad (39)$$

$$l = \sqrt{\frac{\hbar}{m\omega}}. \quad (40)$$

Proposition 1.21. Let \mathcal{H} be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,

$$\langle x|\alpha\rangle = \psi_\alpha(x) = \quad (41)$$

$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(\frac{ip_0}{2\hbar}(2x - x_0)\right) \exp\left(-\frac{(x - x_0)^2}{4\sigma_x^2}\right), \quad (42)$$

$$\frac{1}{4\sigma_x^2} = \frac{1}{2}\frac{m\omega}{\hbar} \quad (43)$$

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re}\{\alpha\}, \quad p_0 = \sqrt{2\hbar m\omega} \operatorname{Im}\{\alpha\} \quad (44)$$

Proposition 1.22. Let \mathcal{H} be a Hilbert space with a harmonic potential and $\{|\alpha\rangle\}$ the set of coherent states. Then, they form a overcomplete basis, that is, not for all pair of states $|\alpha\rangle, |\alpha'\rangle$ it is satisfied $\langle \alpha'|\alpha\rangle = 0$. Hence,

$$\hat{I} = \frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha, \quad |\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}. \quad (45)$$

Besides, $\langle\alpha|\beta\rangle \rightarrow 0$ if and only if $|\alpha - \beta| \gg 1$.

Proposition 1.23. Let \mathcal{H} be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,

$$|\alpha\rangle(t) = e^{i\omega t/2} |\alpha(t)\rangle = e^{i\omega t/2} |\alpha_0 e^{i\omega t}\rangle. \quad (46)$$

Proposition 1.24. Let \mathcal{H} be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,

$$\langle \hat{x} \rangle = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t), \quad (47)$$

$$\langle \hat{p} \rangle = p_0 \cos(\omega t) - m\omega x_0 \sin(\omega t). \quad (48)$$

Definition 1.8. Let \mathcal{H} be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ a state. We say $|\psi\rangle$ is a *minimum uncertainty state* if and only if

$$\Delta x \Delta p = \frac{\hbar}{2}. \quad (49)$$