Contents

Chapter 1

Motion in one dimension

1.1 Newton 2nd Law and differential equations

Until now, we have seen how the second and third law of Newton lead to an algebraic equations with acceleration (or accelerations, in case of a system of particles), acting as an unknown (or unknowns). In general, the application of Newton's Laws give rise to a second order ordinary differential equation (or a system of ordinary differential equations) where the position of a particle is an unknown function of time, $\vec{r}(t)$, we have to determine. The most simple case, studied previously [ ], is the case of one particle with constant mass that moves in one dimension, where the equations has the form

$$\ddot{x}(t) = \frac{1}{m} F(x(t), \dot{x}(t), t), \tag{1.1}$$

where we suppose the force function F is known. We also suppose initial position and velocity, denoted by $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$ respectively, are known. With that, finding the solution to the ordinary differential equations and satisfying the given initial conditions consist the *initial value problem*. In some conditions (generally feasible for physical systems), there exist a solution to the initial value problem and is unique [ ].

1.2 Integration of the Newton 2n Law

In this section, we will consider three simple cases: forces that only depend on time, position, and velocity.

1.2.1 Time dependence

$$\ddot{x} = \frac{F(t)}{m} = \frac{d\dot{x}}{dt} \Rightarrow d\dot{x} = \frac{F(t)}{m} dt \Rightarrow \int_{\dot{x}(t_0)}^{\dot{x}(t)} d\dot{x} = \int_{t_0}^{t} \frac{F(t')}{m} dt' \Rightarrow \dot{x} = \dot{x}(t_0) + \int_{t_0}^{t} \frac{F(t')}{m} dt'$$

Integrating again

$$\frac{dx}{dt} = \dot{x(t)} \Rightarrow dx = \dot{x}(t) dt \Rightarrow \int_{x(t_0)}^{x(t)} dx = \int_{t_0}^{t} \dot{x}(t') dt' \Rightarrow x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(t') dt'$$

1.2.2 Position dependence

$$\frac{d\dot{x}}{dt} = \frac{F(x)}{m} \Rightarrow \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{F(x)}{m} = \frac{d\dot{x}}{dx}\dot{x}$$

Therefore, denoting $x(0)$ by x_0 and $x(t)$ by x, we get

$$\dot{x} d\dot{x} = \frac{F(x)}{m} dx \Rightarrow \int_{\dot{x}}^{\dot{x}} \dot{x} d\dot{x} = \int_{x_0}^{x} \frac{F(x')}{m} dx' \Rightarrow \dot{x}^2(x) = \dot{x}^2(x_0) + 2\int_{x_0}^{x} \frac{F(x')}{m} dx'$$

If we take the square root we get $\dot{x}(x)$, and if we integrate again,

$$\frac{dx}{dt} = \dot{x}(x) \Rightarrow \frac{1}{\dot{x}(x)} dx = dt \Rightarrow \int_{x_0}^{x} \frac{1}{\dot{x}(x)} dx' = \int_{0}^{t} dt' = t$$

We denote the left integral by $g(x)$, so we have now $t = g(x)$. Inverting the relation, we get

$$x(t) = g^{-1}(t), \qquad g(x) = \int_{x_0}^{x} \frac{1}{\dot{x}(x)} dx'$$

1.2.3 Velocity dependence

$$\frac{d\dot{x}}{dt} = \frac{F(\dot{x})}{m} \Rightarrow \frac{m}{F(\dot{x})}d\dot{x} = dt \Rightarrow \int_{v_0}^{v} \frac{m}{F(v)}dv' = \int_0^t dt' = t$$

Hence, denoting the left integral by $h(v)$

$$v(t) = h^{-1}(t),\ \int_{v_0}^{v} \frac{m}{F(v)}dv' \Rightarrow x(t) = x_0 + \int_0^t h^{-1}(t')dt'$$

Example 1.2.1. Let be a particle subjected to a force $F = -c\dot{x}^2$. If $x(0) = 0$ and $v(0) = v_0$, we can find its equation of motion.

$$\ddot{x} = -\frac{c}{m}\dot{x}^2 \Rightarrow \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -\frac{c}{m}\dot{x}^2 \Rightarrow -\frac{m}{c}\frac{1}{\dot{x}^2}\,\mathrm{d}\dot{x} = \mathrm{d}t \Rightarrow \frac{m}{c\dot{x}} - \frac{m}{cv_0} = t \Rightarrow \frac{m}{c\dot{x}} = t + \frac{m}{cv_0} \Rightarrow$$

$$\dot{x} = \frac{v_0}{1 + cv_0/m \cdot t} \Rightarrow x(t) = x_0 + \int\limits_0^t \dot{x}(t')\,\mathrm{d}t' = \frac{m}{c}\ln\Big(1 + \frac{cv_0}{m}t\Big).$$

1.3 Variable mass

1.3.1 Accretion mass

Proposition 1.3.1. *Let .... Then, it is true that*

$$\vec{F}^{ext} = m\vec{a} + (\vec{v} - \vec{u})\dot{m} \tag{1.2}$$

or which is equivalent,

$$\vec{F}^{ext} + \dot{m}\vec{u} = \dot{\vec{p}} \tag{1.3}$$

Proof. There are two scenarios to consider in this equation, when $m(t)$ is gaining mass and when it is losing. We will study them separately, beginning with the increasing mass situation.
Let us consider the situation of the figure, where, in an interval of time dt, an object of mass $m(t)$ with velocity $\vec{v}(t)$ gains mass as a consequence of adhering to another infinitesimal mass dm that travels at a velocity $\vec{u}(t)$. Before the fusion there are two separate masses, so the linear momentum is

$$\vec{P} = m(t)\vec{v}(t) + dm\vec{u}(t).$$

At the instant $t + dt$, however, the momentum of the system is (remembering that higher-order infinitesimals tend to zero)

$$\vec{p} + d\vec{p} = (m + dm)\,(\vec{v} + d\vec{v}) = m\vec{v} + md\vec{v} + dm\vec{v},$$

since now there is only one mass. If we take the difference of these expressions and divide it by dt, we get

$$d\vec{p} = m(t)d\vec{v} + (\vec{v} - \vec{u})\,dm \Rightarrow \vec{F}^{\text{ext}} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + (\vec{v} - \vec{u})\,\frac{dm}{dt} \Rightarrow \vec{F}^{\text{ext}} = m\vec{a} + (\vec{v} - \vec{u})\dot{m}$$

Now let us consider the second case. In this scenario, in an interval of time dt, an object of mass $m(t)$ with velocity $\vec{v}(t)$ ejects some infinitesimal amount of mass dm at a velocity $\vec{u}(t)$.

Therefore, before the a separation the system only has one mass and the linear momentum is determined by

$$\vec{P} = m\vec{v}.$$

After the separation, there two masses, one with an increase of dm and the other with $-dm$, so the linear momentum is

$$\vec{P} + d\vec{P} = (m + dm)(\vec{v} + d\vec{v}) + \vec{u}(-dm) = m\vec{v} + dm\vec{v} + \vec{v}dm - dm\vec{u}.$$

Therefore, taking the difference and dividing by d, we get

$$d\vec{P} = dm\vec{v} + (\vec{v} - \vec{u})dm \Rightarrow \vec{F}^{\text{ext}} = m\vec{a} + (\vec{v} - \vec{u})\dot{m},$$

which is the equation we got before. ■

1.4 Rocket motion

Let us suppose that a rocket burns fuel that is ejected in form of gas to propel itself. This gas is expelled at a velocity $\vec{c}$ with respect to the rocket and the mass of the rocket, including the fuel, is $m(t)$. Our proposal is to find an expression of $\vec{v}$ to describe the motion of the rocket, and for that first we need to express the magnitudes in an external frame of reference. In particular, the velocity of the fuel will be

$$\vec{u} = \vec{v} + \vec{c}.$$

Now, applying the equation of accretion mass, we obtain

$$\vec{F} = m\vec{a} + (\vec{v} - \vec{u})\dot{m} = m\vec{a} - \dot{m}\vec{c} \Rightarrow m\vec{a} = \vec{F}^{\text{ext}} + \dot{m}\vec{c}. \tag{1.4}$$

1.4.1 Rocket without gravity

Differentiable method

Since the situation we consider is a scenario without gravity, $\vec{F}^{\text{ext}} = \vec{0}$ and the problem can be solved easily. We will take the positive direction in y so $\vec{v}$ and $\vec{c}$ will be $v\vec{e}_y$ and $-c\vec{e}_y$ respectively.

$$ma = -\dot{m}c \Rightarrow m\frac{dv}{dt} = -c\frac{dm}{dt} \Rightarrow dv = -\frac{c}{m}dm \Rightarrow \int_0^v dv' = -c\int_{m_0}^m \frac{1}{m'}dm' \Rightarrow v(m) = c\ln\frac{m_0}{m}$$

This result says the final velocity is exclusively determined by the final amount of mass, independently on how the mass have been ejected, that is, without specifying $m(t)$. However, this is only valid if $m(t)$ is a differentiable function of time. Although this kind of function is which better describe the behavior of a fuel in real life, it would be interesting to see what we get with non-differentiable expressions of $m(t)$. In particular, we will discuss what happens when instantaneous transitions of mass occur, and see if in some way there is a relation between this process and the continuous one.

Discrete method

First we will consider the case where the rocket that is at rest ejects all the mass instantaneously. Considering as before $\vec{v} = v\vec{e}_y$ and $\vec{c} = -c\vec{e}_y$, the initial and final linear momentum are

$$\begin{cases} P = m_0 v_0 = 0 \\ P_f = m_f v_f + (m_0 - m_f)(v_f - c) = m_0(v_f - c) \end{cases}$$

By the principle of conservation of linear momentum, $P = P_f$ (because there is no external force) and the final velocity is

$$0 = m_0(v_f - c) \Rightarrow v_f = c. \tag{1.5}$$

This result differs from the continuous case, and it is normal since the approach is completely different. Now, to improve the method, we will consider several instantaneous ejections. In particular, the rocket will eject amounts of $\Delta m = (m_0 - m_f)/n$ and will do it n times. In the first ejection, the initial and final linear momentum are

$$\begin{cases} P_0 = m_0 v_0 = 0 \\ P_1 = (m_0 - \Delta m)v_1 + \Delta m(v_1 - c) \end{cases}$$

Applying the conservation of linear momentum, we get

$$0 = m_0 v_1 - \Delta m v_1 + \Delta m v_1 - \Delta m c = m_0 v_1 - \Delta m c \Rightarrow v_1 = \frac{\Delta m}{m_0} c = \frac{\Delta m}{m_f + \Delta m} c$$

Now we will study the second ejection, but first we need to notice an important detail. Since the fuels is ejected as a gas, after the separation of masses it is completely dispersed through the space. Hence, its linear momentum *disappears* after the ejection and it is not consider in the initial momentum of the next one. Said that, the two linear momentum of the second expulsion is

$$\begin{cases} P_1' = (m_0 - \Delta m)v_1 \\ P_2 = (m_0 - 2\Delta m)v_2 + \Delta m(v_2 - c) \end{cases} .$$

By the same procedure, v_2 is determined as follows.

$$m_0 v_1 - \Delta m v_1 = (m_0 - \Delta m)v_2 - \Delta m c \Rightarrow v_2 = v_1 + \frac{\Delta m}{m_0 - \Delta m} = \frac{\Delta m}{m_f + n\Delta m} c + \frac{\Delta m}{m_f + (n-1)\Delta m} c$$

The same reasoning can be applied in the following ejections (again considering the linear momentum of the gas does not account in the next one), and it can be proved easily (by induction for example) that the i-th velocity is

$$v_i = \sum_{k=1}^{i} \frac{\Delta m}{m_f - (n-k+1)\Delta m} c \tag{1.6}$$

Therefore, at the end of the whole process (that is, after the n ejections), the final velocity is

$$v_f = \sum_{k=1}^{n} \frac{\Delta m}{m_f + (n-k+1)\Delta m} c = \sum_{k=1}^{n} \frac{\Delta m}{m_f + k\Delta m} c,$$

where we have changed the indices to simplify the expression [2]. If we denote $m_0 - m_f$ by m_r and continue rephrasing the sum, we get

$$v_f = \sum_{k=1}^{n} \frac{\Delta m}{m_f + k\Delta m} c = \sum_{k=1}^{n} \frac{(m_0 - m_f)/n}{m_f + k(m_0 - m_f)/n} c = \sum_{k=1}^{n} \frac{m_0 - m_f}{n} \frac{1}{m_f + k(m_0 - m_f)/n} c = \frac{m_r}{n} \sum_{k=1}^{n} \frac{1}{m_f + m_r k/n} c = c\frac{m_r}{n} \sum_{k=1}^{n} f\left(\frac{m_r k}{n}\right), \; f\left(\frac{m_r k}{n}\right) = \frac{1}{m_f + m_r k/n}$$

In we make n tend to infinity, we get can apply the relation between summations and integrals [1] and finally get

$$\lim_{n\to\infty} c\frac{m_r}{n}\sum_{k=1}^{n} f\left(\frac{m_r k}{n}\right) = c\int_0^{m_r} f(m)dm = c\int_0^{m_r}\frac{1}{m_f+m}dm = c\ln\left|\frac{m_f+m_r}{m_f}\right| = c\ln\frac{m_0}{m_f},$$

which is the same equation we got in the differential form.

1.4.2 Rocket with gravity

To study the case where we consider gravity, we only need to substitute the $\vec{F}^{\text{ext}}$ by the gravitational force in the equation 1.4. However, the resulting equation is too complex and we will consider just a particular case, where $\dot{m}$ is constant. Said that, the commented relations are expressed as follows.

$$\begin{aligned}\vec{F}^{\text{ext}} &= -(m+dm)g\vec{e}_y \approx -mg\vec{e}_y\\ \dot{m} &= -\beta, \qquad \beta > 0, \qquad [\beta] = \mathrm{kg\,s^{-1}}\end{aligned}$$

With this, we can start with the operations.

$$ma = -mg - \dot{m}c \Rightarrow \frac{dv}{dt} = -g + \frac{\beta}{m}c \Rightarrow \frac{dv}{dm}\frac{dm}{dt} = -g + \frac{\beta}{m}c = -\beta \Rightarrow dv = \left(g - \frac{\beta}{m}c\right)\frac{1}{\beta}dm$$

If we integrate this expression with $v_0 = 0$,

$$v = \int_{m_0}^{m}\left(g - \frac{\beta}{m'}c\right)\frac{1}{\beta}dm' = \frac{g}{\beta}\int_{m_0}^{m}dm' - c\int_{m_0}^{m}\frac{1}{m'}dm' = \frac{g}{\beta}(m-m_0) + c\ln\frac{m_0}{m},$$

and rearranging terms,

$$v = c\ln\frac{m_0}{m} - \frac{g}{\beta}(m_0 - m) \tag{1.7}$$

The first term is always positive, but the second one, since it has a minus sign, is negative. Therefore, in some cases the expression of v will be negative, in particular when β or c are too small. In fact we could have seen it when we isolated dv. There, if β or c are too small dv will be negative and therefore v. This means the equation is wrong in these cases, but we can solve it.

If we want it not tho happen, the expression of dv tells us that we only need the condition $m_0 g \leq \beta c$ to be satisfied, since then it would be positive from $t = 0$. However, if this is not the case, we need to modify the equation. The problem begins when we expressed $\vec{F}^{\text{ext}}$. Since we thought the rocket would have a positive direction always, it would not have contact with the floor (only during an instant), but if b or c are small this is not true. While the rocket is in the floor there is also the normal force, so the rocket won't have a negative velocity but it will remain at rest until the condition $mg \leq \beta c$ is satisfied (because then dv will be positive).

Now we have considered the details, we can find the new expressions to determine the velocity. There are two interval of time, while the rocket remains in the floor and when it starts moving. We know already the velocity of the rocket during the first interval, but we can still discover things, for example the magnitude of the normal force. Since the acceleration is zero during this interval, we have

$$m\vec{a} = \vec{F}^{\text{ext}} + \dot{m}\vec{c} \Rightarrow 0 = -mg + N - \dot{m}c \Rightarrow N = mg + \dot{m}c = mg - \beta c.$$

Before starting studying the second interval, we need first to find out at what instant t does the first interval ends. From the condition we mentioned before, the time t_b at which the rocket starts to blast-off is determined by

$$\dot{m} = -\beta \Rightarrow m = m_0 - \beta t \Rightarrow mg = \beta c = (m_0 - \beta t) \Rightarrow t_b = \frac{m_0}{\beta} - \frac{c}{g},$$

and the mass at that time,

$$m(t_b) = m_0 - \beta\left(\frac{m_0}{\beta} - \frac{c}{g}\right) = \frac{\beta c}{g} = m_b.$$

Now, we can determine the velocity as follows

$$v = \int_{m_0}^{m_b} 0 dm' + \int_{m_b}^{m}\left(g - \frac{\beta}{m'}c\right)\frac{1}{\beta}dm' = \frac{g}{\beta}\int_{m_0}^{m} dm' - c\int_{m_b}^{m}\frac{1}{m'}dm' = \frac{g}{\beta}(m - m_b) + c\ln\frac{m_b}{m} = \frac{g}{\beta}\left(m - \frac{\beta c}{g}\right) + c\ln\frac{\beta c}{mg}.$$

If we want to express the velocities of the rocket in the two cases (when the condition was satisfied from the beginning and when not), we get

$$v(t) = c\ln\frac{m_0}{m_0 - \beta t} - gt \tag{1.8}$$

$$v(t) = c\ln\frac{\beta c}{m_0 g - \beta g t} - gt + \frac{g}{\beta}\left(m_0 - \frac{\beta c}{g}\right) \tag{1.9}$$

Bibliography

[1] Relation between integral and summation, 2012.

[2] S. Widnall J. Peraire. Lecture L14 - Variable Mass Systems: The Rocket Equation, 2008.

Chapter 2

Oscillations

2.1 Simple harmonic oscillator

Proposition 2.1.1. *Let be the following differential equation*

$$\ddot{x} + \omega_0^2 x = 0, \tag{2.1}$$

with the initial value condition of $x(0) = x_0$ *and* $v(0) = v_0$*. Then, the general solution is*

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t, \tag{2.2}$$

or, which is equivalent,

$$x(t) = A \cos [\omega_0 t + \phi_0], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \phi_0 = -\arctan \frac{v_0}{\omega_0 x_0}. \tag{2.3}$$

Proof. This is a linear homogeneous differential equation of second order, and can be solve easily with the associated polynomial.

$$P(s) = s^2 + \omega_0^2 = 0 \Rightarrow s = \pm i\omega_0, \ \omega_0 > 0$$

Since the roots are complex, the solution is given by the following expression

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

and its velocity

$$v(t) = -\omega_0 c_1 \sin \omega_0 t + \omega_0 c_2 \cos \omega_0 t$$

To solve now the initial values problem, we just need to calculate $x(0)$ and $v(0)$ and make them equal to the given values.

$$\begin{cases} x(0) = c_1 = x_0 \\ v(0) = \omega_0 c_2 = v_0 \end{cases}$$

By isolating the arbitrary constants we get $c_1 = x_0$ and $c_2 = v_0/\omega_0$, so the $x(t)$ will be expressed as

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t.$$

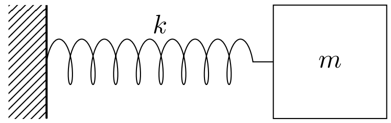
Now, to find the alternative expression, we know it can be done with the changes of variables $A = \sqrt{c_1^2 + c_2^2}$ and $\phi_0 = -\arctan (c_2/c_1)$. Since these constants are already determined, the final expression results in

$$x(t) = A \cos [\omega_0 t + \phi_0], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \phi_0 = -\arctan \frac{v_0}{\omega_0 x_0},$$

which completes the solutions we proposed. ■

The quantity $\omega_0 t + \phi_0$ has units of rad (radians) and ω_0 of $\mathrm{rad\,s^{-1}}$. Let us observe is a periodic function with

$$T = \frac{2\pi}{\omega_0} \Rightarrow f = \frac{\omega_0}{2\pi} \tag{2.4}$$



Example 2.1.1. If a mass m is subjected to a spring with an elastic constant k, the elastic force applied by the spring to the mass, by the Hooke's Law, is

$$F_{\mathrm{m}} = -kx.$$

F_{m} is the force applied to the mass when it is displaced a certain distance x of its position of equilibrium (in this case, the equilibrium occurs when the spring has its natural length).
By the Newton's second law [ ], we have

$$-kx = m\ddot{x} \Rightarrow \ddot{x} + \frac{k}{m}x = 0,$$

which is a differential equation like that of the proposition 2.1.1. Therefore, the solution is

$$x(t) = A\cos\left[\omega_0 t + \phi_0\right], \qquad \omega_0 = \sqrt{\frac{k}{m}} > 0$$

2.1.1 Phase space

The phase space of the simple harmonic oscillator is

$$\vec{u}(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} A\cos\left[\omega_0 t + \phi_0\right] \\ -\omega_0 A\sin\left[\omega_0 t + \phi_0\right] \end{pmatrix}. \tag{2.5}$$

Therefore, the system in the phase space describes an ellipse of semi-axis $a = A$ and $b = \omega_A$, and moves through it in clockwise direction.

Figure 2.1: Trajectory of the mass in the phase space with $\omega_0 > 0$.

2.1.2 Energy analysis

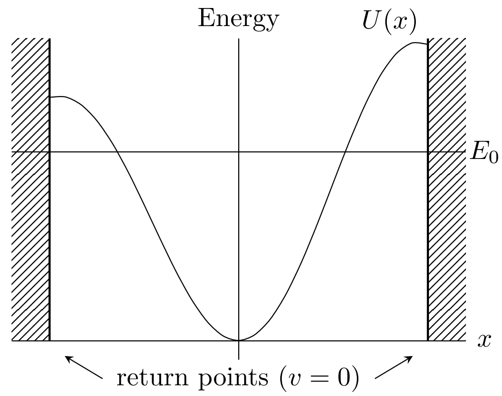
Let us start with a particular example and then make a more general discussion about the topic.

Example 2.1.2. Let m be a mass subjected to an elastic force applied by a spring as before. During the oscillation, the mass has kinetic and potential energy, so the total energy is

$$E = K + U = \frac{mv^2}{2} + \frac{kx^2}{2} = \frac{m\omega_0^2 A^2 \sin^2\left[\omega_0 t + \phi_0\right]}{2} + \frac{kA^2\cos^2\left[\omega_0 t + \phi_0\right]}{2} = \frac{kA^2 \sin^2\left[\omega_0 t + \phi_0\right]}{2} + \frac{kA^2\cos^2\left[\omega_0 t + \phi_0\right]}{2} = \frac{kA^2}{2}.$$

The expression of the energy is a constant, and since A depends on the initial conditions, E too. Note that at $x = 0$, E coincides with the kinetic energy of the mass, since $U = 0$. Therefore, it is where its speed has its maximum value. A particular case of this is if the particle starts at rest in the position $x = 0$. In this situation, $K_{\max} = 0$, and therefore the mass will remain at rest (stable equilibrium).
At $x = \pm A$, we have that $v = 0$, the motion is inverted, and E coincides with energy potential. In any other position, the total energy has a part of kinetic and a part of potential.

Now we have seen these properties, we will generalize them. For that, let us suppose a potential function $U(x)$ as the figure [ ], and let us suppose a mass m starts with an initial energy E_0.



The kinetic energy, and hence the speed, is maximum where U is minimum, that is, when $U = U_{\text{min}}$. In the figure, this occurs at $x = 0$. The extremes of $U(x)$ are points of equilibrium, since in these points

$$0 = \frac{dU}{dx}(x_{\text{extreme}}) = -F.$$

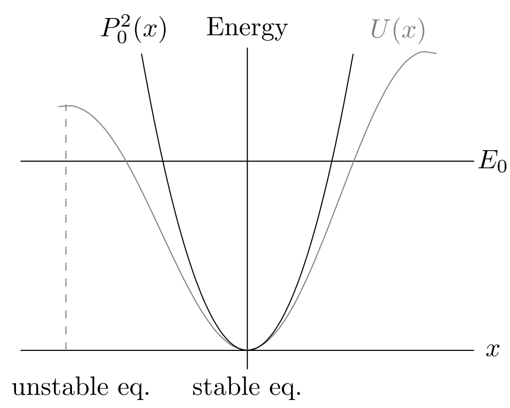
From our knowledge of calculus, we know the extremes (with $U' = 0$) can be classified thought the higher-order derivatives, in most common case the second one.

Definition 2.1.1. Let $U(x)$ be a potential function of class $C^2(\mathbb{R})$. Then, we say x_0 *is a point of stable equilibrium* if U has a maxima in x_0.

This happens if in an extreme $U'' > 0$. Around that point, small displacements make the force push the particle back to x_0.

Definition 2.1.2. Let $U(x)$ be a potential function of class $C^2(\mathbb{R})$. Then, we say x_0 *is a point of unstable equilibrium* if U has a minima in x_0.

This happens if in an extreme $U'' < 0$. Around that point, small displacements make the force push the particle away from x_0.



These derivatives serve us to find an approximate solution when $U(x)$ is too complex to study. To simplify the operations, first we will locate the coordinate origin at the minima x_0 and establishing that $U(0)=0$. Then, we can compute the Taylor polynomial to the second order and get

$$P_0^2(x)=U(0)+\frac{U'(0)}{1!}x+\frac{U''(0)}{2!}x^2=\frac{U''(0)}{2}x^2\Rightarrow U(x)\approx\frac{U''(0)}{2}x^2,$$

which is an harmonic potential. To see that, let us write the conservation of energy with this approximation

$$E=K+U\Rightarrow E=\frac{mv^2}{2}+\frac{U''(0)}{2}x^2.$$

Now, if we derive the equation with respect to x (the energy does not depend on the position so it acts as a constant), we obtain

$$ma+U''(x)x=0\Rightarrow\ddot{x}+\frac{U''(x)}{m}x=0\Rightarrow\omega_0=\sqrt{\frac{U''(x)}{m}}.$$

As we see, we obtain the same equation of a simple harmonic oscillator and therefore the particle will behave similarly at points near the minima.

Example 2.1.3. Let us suppose there is a spring subjected to the roof and with a mass m with a natural length l_0 (figure a). Then, we hang a mass m in the extreme so the spring stretches until reach a new position of equilibrium (figure b). Finally, we agitate the system and make the mass oscillate.

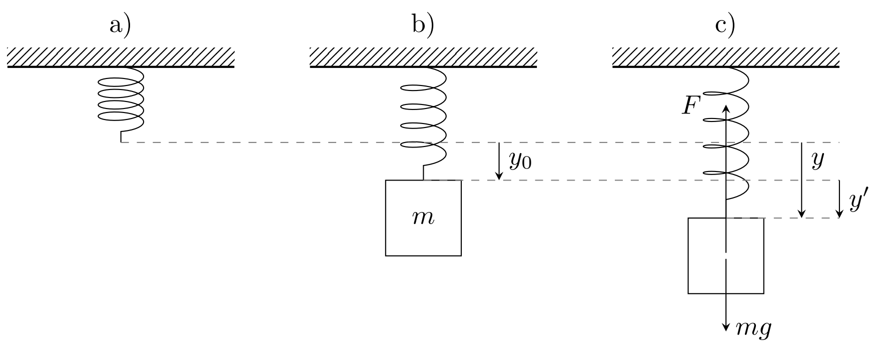


Figure 2.2: Springs.

We will solve the problem of finding th equation of motion in two ways, with Newton's second Law, and with conservation of energy. We will take the positive direction in $-y$, so the forces for the laws of motion are

$$F=mg-F_{\mathrm{m}}=mg-ky.$$

Therefore, the second law of motion let us write

$$mg-ky=m\ddot{y}\Rightarrow\ddot{y}+\frac{k}{m}y-g=0.$$

This equation does not have the form 2.1, so we could think the behavior of the spring is different from the horizontal case. Nevertheless, there is a detail we have not considered while obtaining the equation. We usually take the origin of coordinates at the equilibrium point, but this time there are two, the natural length and the equilibrium with the mass. Since now we are in a scenario with mass, we need to put the origin in y_0. To see how does change the expression, let us make the change of variable $y' = y - y_0$. Before substituting it in the equation, we need to calculate y_0, ans since this is the position where the net force is zero, it is determined as follows.

$$0 = mg - ky_0 \Rightarrow y_0 = \frac{mg}{k}$$

Now, introducing the expression in the differential equation, we get

$$\ddot{y}' + \frac{k}{m}y' + \frac{k}{m}y_0 - g = 0 \Rightarrow \ddot{y}' + \frac{k}{m}y = 0 \Rightarrow y' = A\cos\left[\omega_0 t + \phi_0\right] = y - y_0 \Rightarrow$$
$$y(t) = \frac{mg}{k} + A\cos\left[\omega_0 t + \phi_0\right], \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

We see then, the mass also behaves as a simple harmonic oscillator but with a shift in the position of equilibrium.
SOLUTION BY ENERGIES

2.1.3 Examples

Simple pendulum

Let us suppose there is a point mass m subjected by an inextensible mass-less rope of natural length l. The other side of the rope is fixed in the roof. The forces that act over the mass at an angle θ from its position of equilibrium are shown in the figure.

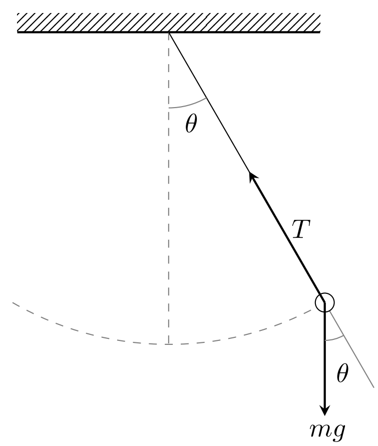


Figure 2.3: Simple pendulum at an angle θ.

The letter T denotes the tension (we will use τ for period), and it is the responsible for the mass moving over a circumference of radius l (the dashed line). If we study the tangential direction of the forces, by the Newton's Second Law we get

$$-mg\sin\theta = ma_t = ml\alpha = ml\ddot{\theta} \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0,$$

where we have written the wight in negative because it acts opposite to the increase of mass. This equation is similar to that of the proposition 2.1.1, but it has a $\sin x$ instead of x. This makes the problem more complex and the system becomes a non simple harmonic oscillator. Therefore, we will simplify the problem and let the exact solution for a later section. In order to get a a simpler problem, we will take the Taylor approximation of $\sin\theta$ to the first order, that is, with $\sin\theta \approx \theta$ (only for small angles). With that, we get the equation of a SHO and therefore we can obtain the equation of motion.

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \Rightarrow \theta(t) = A\cos\left[\omega_0 t + \phi_0\right], \qquad \omega_0 = \sqrt{\frac{g}{l}}$$

Note that ω_0 is not the derivative of θ and ϕ_0 is not θ_0. Rather, magnitudes refer to the phase and frequency of the oscillation of the angle.
ENERGIES AND OTHERS——————-

$$U(\theta) = mgh = mg(l - l\cos\theta) = mgl(1 - \cos\theta)$$

Taylor second order

$$U'(\theta) = mgl\sin\theta \Rightarrow U'(0) = 0, \qquad U''(\theta) = mgl\cos\theta \Rightarrow U''(0) = mgl, \qquad U(\theta) = \frac{1}{2!}mgl\theta^2 \Rightarrow ml\alpha$$

Then,

$$ma + \frac{ds}{dU} = 0 \Rightarrow ml\alpha + \frac{1}{l}\frac{dU}{d\theta} = 0 \Rightarrow ml\ddot{\theta} + \frac{mgl}{l}\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

where we have used that $ds = ld\theta$.

Physical pendulum

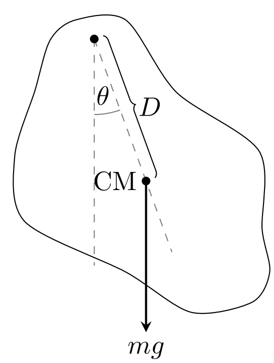


Figure 2.4: Physical pendulum at an angle θ.

We know

$$\tau_e = I_e\ddot{\theta}$$

In this case,

$$\tau = -mgD\sin\theta$$

Then,

$$\ddot{\theta} + \frac{mgD}{I_e}\theta = 0 \tag{2.6}$$

The solution is

$$\theta(t) = A\cos\left[\omega_0 t + \phi_0\right], \qquad \omega_0 = \sqrt{\frac{mgD}{I_e}} \tag{2.7}$$

By energies

$$U \approx \frac{1}{2}mgD\theta^2$$

LC circuit

Let be a circuit

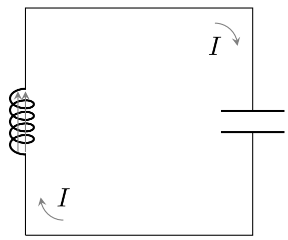


Figure 2.5: LC circuit.

By Faraday Law

$$\epsilon_{\text{ind}} = \oint \langle \vec{E}, d\vec{r} \rangle_I = -\frac{d}{dt} \int_S \langle \vec{B}, d\vec{s} \rangle_I$$

In inductors,

$$\Phi = LI \Rightarrow \frac{d\Phi}{dt} = L\frac{dI}{dt} \Rightarrow \oint \langle \vec{E}, d\vec{r} \rangle_I = -L\dot{I}$$

Since the capacitor is charging,

$$I = \dot{q}$$

Besides,

$$-L\ddot{q} = -L\dot{I} = \oint \langle \vec{E}, d\vec{r} \rangle_I = \oint_{\text{capacitor}} \langle \vec{E}, d\vec{r} \rangle_I = V = \frac{q}{C} \tag{2.8}$$

Finally,

$$L\ddot{q} + \frac{q}{C} = 0 \tag{2.9}$$

If we derive it

$$L\ddot{I} + \frac{I}{C} = 0$$

The solution for the first one is

$$q(t) = A\cos\left[\omega_0 t + \phi_0\right], \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{2.10}$$

Derivating,

$$I = -\omega_0 A \sin\left[\omega_0 t + \phi_0\right] = A' \cos\left[\omega_0 t + \phi_0'\right], \qquad A' = \omega_0 A, \phi_0' = \phi_0 + \frac{\pi}{2}$$

2.2 Damped harmonic oscillator

2.2.1 Evolution equation and solution

Proposition 2.2.1. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{2.11}$$

with the initial value conditions of $x(0) = x_0$ *and* $v(0) = v_0$*. Then, the general solution is*

$$x(t) = e^{-\beta t}\left[x_0 \cos\tilde{\omega}t + \frac{v_0 + \beta x_0}{\tilde{\omega}} \sin\tilde{\omega}t\right], \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \beta^2} \tag{2.12}$$

if $\beta < \omega_0$,

$$x(t) = e^{-\omega_0 t}\left[x_0 + (x_0\omega_0 + v_0)t\right] \tag{2.13}$$

if $\beta = \omega_0$, *and*

$$x(t) = \frac{x_0(\bar{\omega} - \beta) - v_0}{2\bar{\omega}} e^{-(\beta+\bar{\omega})t} + \frac{x_0(\bar{\omega} + \beta) + v_0}{2\bar{\omega}} e^{-(\beta-\bar{\omega})t}, \qquad \bar{\omega} = \sqrt{\beta^2 - \omega_0^2} \tag{2.14}$$

if $\beta > \omega_0$.

Example 2.2.1. Mass m submerged in a fluid and subjected to a string of elastic constant k

$$F = -kx$$

$$F_f = -b\dot{x}$$

By Newtons' law

$$-kx - b\dot{x} = m\ddot{x} \Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}, \beta = \frac{b}{2m}$$

2.2.2 Underdamped harmonic oscillator

The general solution is

$$x(t) = e^{-\beta t}\left[x_0 \cos\tilde{\omega}t + \frac{v_0 + \beta x_0}{\tilde{\omega}} \sin\tilde{\omega}t\right], \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \beta^2}$$

or, which is equivalent,

$$x(t) = Ae^{-\beta t}\cos\left[\tilde{\omega}t + \phi_0\right], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0 + \beta x_0}{\tilde{\omega}}\right)^2}, \qquad \phi_0 = -\arctan\frac{v_0 + \beta x_0}{x_0\tilde{\omega}}$$

Definition 2.2.1. Amplitude as a function of time.

$$A(t) = Ae^{-\beta t} \tag{2.15}$$

Period of the amplitude

$$\tau_{e^{-1}} = \frac{1}{\beta}$$

Period between zeros

$$\tilde{T} = \frac{2\pi}{\tilde{\omega}} > T_0$$

Quality factor

Definition 2.2.2. The quality factor Q is defined as

$$Q \coloneqq \frac{\omega_0}{2\beta} \tag{2.16}$$

From that,

$$\tilde{\omega} = \omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

$$N_{e^{-1}} \approx \frac{Q}{\pi} \tag{2.17}$$

Quality factor and energy

$$E = \frac{\mu}{2}\left(v^2 + \omega_0^2 x^2\right) \tag{2.18}$$

Example 2.2.2. The example [ ] with a mass and a string without friction results in

$$\frac{m}{2}\left(v^2 + \omega_0 x^2\right)$$

Example 2.2.3. The simple pendulum.

$$K = \frac{mv^2}{2} = \frac{ml^2\dot{\theta}^2}{2}, \qquad U = mgl(1-\cos\theta) \approx \frac{mgl\theta^2}{2} \tag{2.19}$$

Then,

$$E = U + K = \frac{mgl\theta^2}{2} + \frac{ml^2\dot{\theta}^2}{2} = \frac{ml^2}{2}\left(\dot{\theta}^2 + \frac{g}{l}\theta^2\right) = \frac{\mu}{2}(\dot{\theta}^2 + \omega_0^2\theta^2), \qquad \mu = ml^2, \omega_0^2 = \frac{g}{l}.$$

For general underdamped oscillators.

$$E = \frac{\mu A^2}{2}\left(\tilde{\omega}^2 \sin^2\left[\tilde{\omega}t + \phi_0\right] + \omega_0^2 \sin^2\left[\tilde{\omega}t + \phi_0\right]\right) e^{-2\beta t} \approx$$
$$\frac{\mu A^2}{2}\left(\omega_0^2 \sin^2\left[\tilde{\omega}t + \phi_0\right] + \omega_0^2 \sin^2\left[\tilde{\omega}t + \phi_0\right]\right) e^{-2\beta t} = \frac{\mu\omega_0^2 A^2}{2} e^{-2\beta t}$$

so

$$E(t) = E_0 e^{-2\beta t} \tag{2.20}$$

The rhythm at which the energy is dissipated is

$$\left|\frac{dE}{dt}\right| = 2\beta E_0 e^{-2\beta t} = 2\beta E \Rightarrow \frac{E}{|dE/dt|} = \frac{1}{2\beta}$$

Then,

$$\tau_E = \frac{1}{2\beta} \tag{2.21}$$

This does not differ much from $\Delta E/\tilde{T}$. This is because if $\beta \ll \omega_0$, then $|dE/dt|$ does not change considerably in an interval of time $\tilde{T}$.
We have then,

$$\left|\frac{dE}{dt}\right| \approx \frac{\Delta E}{2\pi/\omega_0} \Rightarrow 2\pi\frac{E}{\Delta E} \approx \omega_0 \frac{E}{|dE/dt|} \approx \frac{\omega_0}{2\beta} = Q$$

Then,

$$\frac{Q}{2\pi} = \frac{E}{\Delta E} \tag{2.22}$$

2.2.3 Critically damped harmonic oscillator

2.2.4 Overdamped harmonic oscillator

2.2.5 Examples

Example 2.2.4. We know

$$V = IR$$

Then,

$$-L\ddot{q} = -L\dot{I} = \oint \langle \vec{E}, d\vec{r} \rangle_I = \int_{\text{capacitor}} \langle \vec{E}, d\vec{r} \rangle_I + \int_{\text{resistance}} \langle \vec{E}, d\vec{r} \rangle_I = \frac{q}{C} + IR = \frac{q}{C} + R\dot{q}$$

so

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0 \Rightarrow \ddot{I} + \frac{1}{RL}\dot{I} + \frac{1}{LC}I = 0 \Rightarrow \beta = \frac{R}{2L}, \omega_0 = \frac{1}{\sqrt{LC}} \tag{2.23}$$

The system is underdamped if

$$\frac{R}{2}\sqrt{\frac{C}{L}} < 1$$

The charge is given by

$$q(t) = Ae^{-\beta t} \cos\left[\tilde{\omega} t + \phi_0\right] \tag{2.24}$$

And the current

$$I(t) = A' e^{-\beta t} \cos\left[\tilde{\omega} t + \phi_0'\right], \qquad A' = , \phi_0' = \tag{2.25}$$

2.3 Forced harmonic oscillator

2.3.1 Evolution equation and solution

Proposition 2.3.1. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0 \cos\left[\omega t + \psi_0\right], \tag{2.26}$$

with the initial value conditions of $x(0) = x_0$ and $v(0) = v_0$. Then, the particular solution is

$$x_p(t) = A\cos\left[\omega t + \psi_0 - \phi_0\right], \qquad A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}, \qquad \phi_0 = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}. \tag{2.27}$$

Proof. We know

$$f_0 \cos\left[\omega_0 t + \psi_0\right] = \operatorname{Re}\left(f_0 e^{i\psi_0} e^{i\omega t}\right) \tag{2.28}$$

Then

$$P[D]x(t) = (D^2 + 2\beta D + \omega_0)z(t) = \tilde{f}_0 e^{i\omega t}, \qquad \tilde{f}_0 := f_0 e^{i\psi_0}$$

Then

$$x_p(t) = \operatorname{Re}\left(z_p(t)\right)$$

Then

$$z_p(t) = \frac{\tilde{f}_0 e^{i\omega t}}{P(i\omega)} = \frac{\tilde{f}_0 e^{i\omega t}}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

Then

$$z_p = \frac{\tilde{f}_0 e^{-i\arctan\left(2\beta\omega/(\omega_0^2 - \omega^2)\right)}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}} := Ae^{-i(\phi_0 - \psi_0)}$$

where

$$A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}, \qquad \phi_0 = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{2.29}$$

Taking the real part, we get

$$A\cos\left[\omega t + \psi_0 - \phi_0\right]. \tag{2.30}$$

∎

2.3.2 Underdamped forced harmonic oscillator

$$c_1 = A_0 \cos[\psi_0 - \psi_0]$$
$$c_2 = \frac{Av_0 + \omega A \sin[\phi_0 - \psi_0] + \beta(x_0 - A\cos[\psi_0 - \phi_0])}{\tilde{\omega}}$$

2.3.3 Critically damped forced harmonic oscillator

$$c_1 = x_0 - A\cos[\psi_0 - \phi_0]$$
$$c_2 = v_0 + \omega_0 x_0 + A[\omega \sin[\psi_0 - \phi_0] - \omega_0 \cos[\psi_0 - \phi_0]]$$

2.3.4 Overdamped forced harmonic oscillator

$$c_1 = \frac{A(\beta - \bar{\omega})\cos[\psi_0 - \phi_0] - A\omega \sin[\psi_0 - \phi_0] - (\bar{\omega} + \beta)x_0 - v_0}{2\bar{\omega}}$$
$$c_2 = \frac{A\omega \sin[\psi_0 - \phi_0] - A(\beta + \bar{\omega})\cos[\psi_0 - \phi_0] + (\bar{\omega} + \beta)x_0 + v_0}{2\bar{\omega}}$$

2.3.5 Solution analysis

Forçat sense amortiment (el numerador de l'amplitud té valor absolut però el desfasament li dona el signe)

$$\omega \neq \omega_0 : x(t) = \left[x_0 - \frac{f_0}{\omega_0^2 - \omega^2}\right]\cos\omega_0 t` + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t + \frac{f_0}{\omega_0^2 - \omega^2}\cos\omega t$$
$$\omega = \omega_0 : x(t) = x_0 \cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t + \frac{f_0 t}{2\omega_0}\sin\omega_0 t$$

2.3.6 Resonance in amplitude

Resonance

$$\omega_r = \sqrt{\omega_0^2 - 2\beta^2}, \qquad A_r = f_0/2\beta\sqrt{\omega_0^2 - \beta^2}, \qquad \text{a prop de } \omega_0 : A\omega_0^2/f_0 \approx QA$$

2.3.7 Quality factor, resonance in energy and bandwidth

Ressonància i energia (a prop de ω_0 i amb $f = f_0 \cos\omega t$)

$$E = \frac{\mu\omega_0^2 A^2}{2}, A = \frac{f_0}{2\omega_0\sqrt{(\omega - \omega_0)^2 + \beta^2}}, \omega \approx \omega_0, \beta \ll \omega_0, E_{\max} = \frac{\mu f_0^2}{8\beta^2}, Q = \frac{\omega_0}{\Delta\omega}, \omega_{1,2} = \omega_0 \pm \beta$$

2.3.8 Examples. RLC circuit with generator in series and parallel

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{V_{\text{in}}}{L} \Rightarrow q = \frac{\epsilon_0/\omega}{\sqrt{(L\omega - 1/C\omega)^2 + R^2}}\cos(\omega t - \delta), \delta = -\arctan\frac{R}{L\omega - 1/C\omega}$$
$$X = X_L - X_C = L\omega - 1/C\omega,\ Z = \sqrt{X^2 + R^2}, \delta' = \delta - \pi/2 = \arctan X/R$$
$$r = \frac{V_{\text{out}}^2}{V_{\text{in}}^2} = \frac{R^2}{Z^2}, r_{\max} = 1 \Rightarrow \omega_{1,2}(r_{\max}/2) = \sqrt{\frac{1}{Lc} - \left(\frac{R}{2L}\right)} \pm \frac{R}{2L}, Q = \frac{\omega_0}{\Delta\omega} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

2.4 Fourier series

2.4.1 Fourier analysis of the forced harmonic oscillator

$$f(t+T) = f(t), \qquad \omega \coloneqq \frac{2\pi}{T}$$

We have

$$f(t) = a_0 + \sum_{l=1}^{\infty} a_l \cos\omega_l t + \sum_{l=1}^{\infty} b_l \sin\omega_l t = A_0 + \sum_{l=1}^{\infty} A_l \cos\left[\omega_l t + \psi_l\right], \qquad \omega_l = l\omega$$

Answer of the system

$$x(t) = \frac{A_0}{\omega_0^2} + \sum_{l=1}^{\infty} \frac{A}{\sqrt{\left(\omega_0^2 - \omega_l^2\right)^2 + 4\beta^2\omega_l^2}} \cos\left[\omega_l + \psi_l - \phi_l\right], \qquad \phi_l = \arctan\frac{2\beta\omega_l}{\omega_0^2 - \omega_l} \tag{2.31}$$

2.4.2 Obtention of Fourier coefficients

$$a_0 = \frac{1}{T}\int_n^{n+T} f(t)dt, a_l = \frac{2}{T}\int_n^{n+T} f(t)\cos\omega_l t dt, b_l = \frac{2}{T}\int_n^{n+T} f(t)\sin\omega_l t dt \tag{2.32}$$

2.4.3 Examples

Example 2.4.1. The force is

$$f(t) = \Big\{ f_0/2, \quad 0 \leq t \leq T/2 - f_0/2, \quad T/2 \leq t \leq T$$

The answer is

$$x(t) = \frac{2f_0}{\pi}\sum_{l=0}^{\infty}\frac{\sin\left[(2l+1)\omega t\right]}{2l+1}$$

Example 2.4.2. The force is

$$f(t) = \Big\{ f_0/2, \quad 0 \leq t \leq T/4 - f_0/2, \text{otherwise}$$

The answer is

$$x(t) = -\frac{2f_0}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\cos\left[(2l+1)\omega t\right]$$

2.5 Impulsive forces

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \delta(t-t_0) \Rightarrow x(t) = G(t-t_0) \Leftrightarrow P[D]G(t-t_0) = \delta(t-t_0)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \Rightarrow x(t) = \int_{-\infty}^{\infty} f(t')G(t-t')dt'$$

Some properties

$$\theta(t-t_0) + \theta(t_0-t) = 1, \dot{\theta}(t) = \delta(t), t > T \Rightarrow |x(t)| \leq Ke^{-\beta t}$$

2.6 Non-linear oscillator

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0, \theta \in [-\pi, \pi] \Rightarrow T = 4\sqrt{\frac{l}{g}}K(k) = 2\pi\sqrt{\frac{l}{g}}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}\cdot(n!)^2}\right)^2$$

Brachristocrone (in a pendulum)

$$x = a\phi - a\sin\phi, \qquad y = a\cos\phi - a \tag{2.33}$$

General periods

$$T = \int_0^{x_0} \frac{1}{\sqrt{2/m(U_0 - U)}}dx \tag{2.34}$$

2.7 Extra

2.7.1 Some integrals

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x \tag{2.35}$$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \operatorname{arcsinh} x \tag{2.36}$$

$$\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x} \tag{2.37}$$

$$\operatorname{arccoth} x = \frac{1}{2}\ln\frac{x+1}{x-1} \tag{2.38}$$

$$\tan\frac{\theta}{2} = u \tag{2.39}$$

$$\int \frac{1}{a^2x^2 - b^2}dx = -\frac{1}{ab}\operatorname{arccoth}\frac{ax}{b} \tag{2.40}$$

$$\int \frac{1}{\sqrt{1-z^4}} = 1,311\ldots \tag{2.41}$$

$$\int_0^1 \frac{z^{2p}}{\sqrt{1-z^2}} = \frac{\pi(2p)!}{2^{2p+1}(p!)^2} \tag{2.42}$$

$$\int \tau^p e^{-\beta\tau}\sin[\omega\tau]d\tau = (-1)^p\frac{d^p}{x\beta^p}\operatorname{Im}\left(\frac{e^{-\beta+i\omega}\tau}{-\beta+i\omega}\right) \tag{2.43}$$

Chapter 3

Central forces

3.1 Definition of central force and properties

Definition 3.1.1. Central force

$$\vec{F}(\vec{r}) = f(r)\vec{e}_\rho \tag{3.1}$$

Definition 3.1.2. The origin $\vec{r} = \vec{0}$ is the *center of forces.*

Proposition 3.1.1. *All central forces are conservatives.*

3.2 Conservation of angular momentum and areolar velocity

Proposition 3.2.1. *The angular momentum with respect the origin is conserved.*

$$\dot{\vec{L}} = \vec{0} \tag{3.2}$$

Proposition 3.2.2. *The areal velocity is constant.*

$$\frac{dA}{dt} = \frac{L}{2m} = ctt \tag{3.3}$$

This is known as Kepler's Second Law [], published by Johannes Kepler (1571-1630) in 1609 after an exhaustive study of the compilations made by Tycho Brahe (1546-1601) of the position of the planet Mars.

3.3 Trajectory equation

Forces

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \end{cases} \tag{3.4}$$

We see

$$r^2\dot{\theta} = \text{ctt}, \qquad \dot{\theta}\frac{L}{mr^2} = \frac{l}{r^2} \Rightarrow \ddot{r} - \frac{l^2}{r^3} = \frac{f(r)}{m} \tag{3.5}$$

Finally, we get the *trajectory equation*

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2}\frac{1}{u^2}f\left(\frac{1}{u}\right) \coloneqq \psi(u) \tag{3.6}$$

Then we get $u(\theta)$ and $r(\theta)$. Now,

$$t = \frac{1}{l}\int r^2(\theta)d\theta \coloneqq g(\theta) \Rightarrow \theta(t) = g^{-1}(t) \Rightarrow r(t) = r(g^{-1}(t)) \tag{3.7}$$

Besides, we can know the force from the motion

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^2}{ml^2}f(r) \tag{3.8}$$

3.4 Mechanic energy conservation and orbits

Energy

$$E = \frac{m\dot{r}^2}{2} + U_{\text{ef}}(r) = \frac{m\dot{r}^2}{2} + \frac{ml^2}{2r^2} + U(r) \tag{3.9}$$

The term of angular momentum is equivalent to say $f_{\text{centr}} = mr\dot{\theta}^2\vec{e}_\rho$.
Inertial forces for rotating coordinates (the right component)

$$\vec{F}' = \vec{F} + mr\dot{\theta}^2\vec{e}_\rho - 2m\dot{r}\dot{\theta}\vec{e}_\theta - mr\ddot{\theta}\vec{e}_\theta \tag{3.10}$$

3.4.1 Orbits characterization

Circular orbits are at minima of $U_{\text{eff}}(r)$. In the extremes (maxima/minima) it is satisfied that

$$\dot{r} = \pm\sqrt{\frac{2}{m}(E - U_(eff)(r))} = 0 \Rightarrow E - U(r) - \frac{ml^2}{2r^2} = 0 \tag{3.11}$$

Then,

$$\Delta\theta = 2\int_{r_{\min}}^{r_{\max}} \frac{l}{r^2}\sqrt{2/m(E - U_{\text{eff}}(r))}dr \tag{3.12}$$

The orbits are closed if and only if

$$\Delta\theta = 2\pi\frac{p}{q}, \qquad p, q \in \mathbb{N} \tag{3.13}$$

More in general,

$$\theta(r) = \int \frac{L}{r^2}\frac{1}{\sqrt{2m\left(E - U(r) - \frac{L^2}{2mr^2}\right)}}dr \tag{3.14}$$

Notice that, since l is constant and r^2 and the square root always are positive, $\theta(r)$ will increase monotonically with time.

Theorem 3.4.1 (Bertrand's Theorem)**.** *The only central potentials where every bounded orbit is closed are:*

$$U(r) = -\frac{k}{r}, \qquad U(r) = \frac{k}{2}r^2, \qquad k > 0 \tag{3.15}$$

These two potential have $(p, q) = (1, 1)$ for $-k/r$ and $(p, q) = (1, 2)$ for $kr^2/2$.

3.5 Potential $-k/r$

In this case, we have that

$$r(\theta) = \frac{\alpha}{\epsilon\cos\theta - \operatorname{sgn} k}, \qquad \alpha = \frac{L^2}{m|k|}, \qquad \epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$$

Which is a conic section with one focus at the origin. Johann Bernoulli (1667-1748) appears to have been the first to prove that all possible orbits of a body moving in a potential proportional to $1/r$ are conic sections (1710) [ ].

Definition 3.5.1. The pericenter is the minimum value of $r(\theta)$.

Definition 3.5.2. The apocenter is the maximum value of $r(\theta)$.

Definition 3.5.3. Apsides are the turning points.

3.5.1 Conic sections

$$\epsilon = \frac{d(P,F)}{d(P,D)} = \frac{\sqrt{x^2+y^2}}{|x-d|} = \frac{r}{|r\cos\theta - d|} \tag{3.16}$$

Definition 3.5.4. Latus rectum

$$\alpha := \frac{L\bar{L}'}{2} = r(\pi/2) = \epsilon d \tag{3.17}$$

Ellipse $(0 << \varepsilon < 1)$

$$r(\theta) = \frac{\alpha}{\varepsilon\cos\theta + 1} \tag{3.18}$$

$$r_{\min} = \frac{\alpha}{1+\varepsilon}, \qquad r_{\max} = \frac{\alpha}{1-\varepsilon}, \qquad a = \frac{r_{\min} + r_{\max}}{2} = \frac{\alpha}{1-\varepsilon^2} \tag{3.19}$$

Hyperbola $(1 < \varepsilon)$

$$r(\theta) = \frac{\alpha}{\varepsilon\cos\theta \pm 1} \tag{3.20}$$

$$\tan\theta = \pm\sqrt{\varepsilon^2 - 1}, \qquad r_{\min} = r(0) = \frac{\alpha}{\varepsilon + 1} \tag{3.21}$$

Parabola $(\varepsilon = 1)$

$$r(\theta) = \frac{\alpha}{\cos\theta + 1} \tag{3.22}$$

$$x = -\frac{y^2 - \alpha^2}{2\alpha} \tag{3.23}$$

Circunference of radius R $(d = R/\epsilon, \varepsilon \to 0)$

$$r(\theta) = R \tag{3.24}$$

General formulae

$$a = \frac{\alpha}{|1-\varepsilon^2|}, \qquad b = \frac{\alpha}{\sqrt{|1-\varepsilon^2|}}, \qquad c = \frac{\alpha\varepsilon}{|1-\varepsilon^2|} \tag{3.25}$$

$$T^2 = \frac{4\pi^2 m a^3}{k} \tag{3.26}$$

This is known as Kepler's Third Law, published by Kepler in 1619. Kepler's work preceded by almost 80 years Newton's enunciation of his general laws of motion. Indeed, Newton's conclusions were based to a great extent on Kepler's pioneering studies (and on those of Galileo and Huygens).
However, this Law is just an approximation. If we apply the reduced mass to the formula (which is the real situation), we will have different values of period for each planet (since the reduced mass is different), although Kepler said the period does not depend on the small mass. In particular, since the gravitational force is given by

$$\vec{F}(r) = -G\frac{Mm}{r^2},$$

the expression for the square of the period becomes

$$T^2 = \frac{4\pi^2 a^3}{G(M+m)} \approx \frac{4\pi^2 a^3}{GM}, \qquad m \ll M \tag{3.27}$$

Kepler's Laws can now be summarized:

Law 1. *Planets move in elliptical orbits about the Sun with the Sun at one focus.*

Law 2. *The area per unit time swept out by a radius vector from the Sun to a planet is constant.*

Law 3. *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.*

3.6 More general potentials

3.6.1 Almost circular orbits. Stability

3.6.2 Perturbation theory (Poincaré-Lindstedt method). Proof of Bertrand's theorem

3.7 Two body problem

3.8 Particle dispersion

We consider non-bounded orbits and potential functions that diminish faster than $1/r$. We select potential functions such that $U(\infty) = 0$.
If $U(r)$ is not bounded, there are not non-bounded orbits.
Conditions at infinity

$$E(\infty) = K_0 + U(\infty) = \frac{mv_0^2}{2}, \qquad l = bv_0 = b\sqrt{\frac{2K_0}{m}} \tag{3.28}$$

Angle from x axis to the asymptote

$$\Theta = \int_{r_{\min}}^{\infty} \frac{l}{r^2} \frac{1}{\sqrt{2/m(E - U_{\text{eff}}(r))}} dr = \int_{r_{\min}}^{\infty} \frac{b}{r^2} \frac{1}{\sqrt{1 - U/K_0 - b^2/r^2}} \tag{3.29}$$

Dispersion angle

$$\vartheta = |\pi - 2\Theta| \tag{3.30}$$

Kappa

$$\kappa := \frac{|k|}{2K_0} \tag{3.31}$$

Potential $U = -k/r$

$$\cos\Theta = -\frac{\operatorname{sgn} k}{\sqrt{1 + (b/\kappa)}^2} \Rightarrow \cot\frac{\vartheta}{2} = \frac{b}{\kappa} \tag{3.32}$$

Potential of impenetrable sphere

$$U(r) = \begin{cases} 0 & \text{if } r > R, \\ \infty & \text{if } r \le R \end{cases} \Rightarrow \Theta = \begin{cases} \dfrac{\pi}{2} & \text{if } b \ge R, \\ \arcsin\dfrac{b}{R} & \text{if } b < R \end{cases} \Rightarrow \cos\frac{\vartheta}{2} = \begin{cases} 1, & \text{if } b \ge R, \\ \dfrac{b}{R} & \text{if } b < R \end{cases} \tag{3.33}$$

3.9 Cross section

3.9.1 Cross section of a center of forces. Rutherford's problem

$$I = \#\text{ projectiles/A/t that cross the surface } \perp \text{ at their velocities} \tag{3.34}$$

$$dN = \#\text{ projectiles deflected at } d\Omega \text{ at } \vartheta \text{ / t/phyto} \tag{3.35}$$

Differential cross section

$$d\sigma(\vartheta) = \frac{dN}{I}, \qquad [d\sigma] = \text{ m}^2\,\text{sr}^{-1}, \qquad \text{bar}: \text{ b} = 10^{-28}\,\text{m}^2 = 100\,\text{fm}^2 \tag{3.36}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{I}\frac{dN}{d\Omega}, \qquad d\Omega = 2\pi\sin\vartheta d\vartheta \Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\vartheta}\left|\frac{db}{d\vartheta}\right| \tag{3.37}$$

Potential $-k/r$

$$b = \kappa\cot\frac{\vartheta}{2} \Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{\kappa}{2}\right)^2 \csc^4\frac{\vartheta}{2} \tag{3.38}$$

Potential of impenetrable sphere

$$b = R\cos\frac{\vartheta}{2} \Rightarrow \frac{d\sigma}{d\Omega} = \frac{R^2}{4} \Rightarrow \sigma = \int\frac{d\sigma}{d\Omega}d\Omega = \pi R^2 \tag{3.39}$$

Discrete case

$$\frac{d\sigma}{d\Omega} \approx \frac{N_{\text{pr}}^{(j)}/T/N_{\text{ph}}}{N_{\text{pr,T}}/A/T\cdot 2\pi\theta^{(j)}\sin\theta^{(j)}}, \qquad N_{\text{pr, T}} = \sum N_{\text{pr}}^{(j)} \tag{3.40}$$

3.9.2 Cross section in CM system and LAB system

$$m \to \mu = \frac{m_1 m_2}{M}, \qquad \text{position vector of projectile} \to \text{vector } \vec{r} = \vec{r}_1 - \vec{r}_2 \tag{3.41}$$

LAB system

$$d\sigma(\psi) = \frac{\#\text{ deflected projectiles at } d\Omega \text{ at } \psi \text{ /t/phyto}}{\text{flux of projectiles (relative to the phyto)}} \tag{3.42}$$

Relation

$$d\sigma(\psi) = d\sigma'(\vartheta), \qquad \tan\psi = \frac{\sin\vartheta}{m_1/m_2 + \cos\vartheta}, \qquad \frac{d\Omega'}{d\Omega} = \frac{\sin\vartheta}{\sin\psi}\frac{d\vartheta}{d\psi} \Rightarrow \tag{3.43}$$

$$\frac{d\sigma}{2\Omega}(\psi) = \frac{d\sigma'}{d\Omega'}(\vartheta)\frac{\sin\vartheta}{\sin\psi}\frac{d\vartheta}{d\psi} = \frac{d\sigma'}{d\Omega'}(\vartheta)\frac{\left(\sqrt{1 - x^2\sin^2\psi} + x\cos\psi\right)^2}{\sqrt{1 - x^2\sin^2\psi}}, \qquad x = \frac{m_1}{m_2}(\leq 1) \tag{3.44}$$

Potential $-k/r$

$$\frac{d\sigma'}{d\Omega'}(\vartheta) = \left(\frac{k}{4K_0'}\right)\csc^4\frac{\vartheta}{2}, \qquad x = 1 \Rightarrow \vartheta = 2\psi, K_0' = \frac{K_0}{2} \Rightarrow \frac{d\sigma}{d\Omega}(\psi) = \left(\frac{k}{K_0}\right)^2\frac{\cos\psi}{\sin^4\psi} \tag{3.45}$$

Relation between energies

$$\frac{K_0'}{K_0} = \frac{m_2}{m_1 + m_2}, \quad \frac{K_1'}{K_0} = \frac{m_2^2}{(m_1 + m_2)^2}, \quad \frac{K_2'}{K_0} = \frac{m_1 m_2}{(m_1 + m_2)^2}, \tag{3.46}$$

$$\frac{K_1}{K_0} = 1 - 2\frac{m_1 m_2}{m_1 + m_2}(1 - \cos\vartheta), \quad \frac{K_2}{K_0} = 2\frac{m_1 m_2}{(m_1 + m_2)^2}(1 - \cos\vartheta) \tag{3.47}$$

Chapter 4

Coupled oscillations 1

4.1 Problem approach

Situation: two blocks, three springs (k, k_{12}, k) with fixed extremes

$$\begin{aligned}
m\ddot{x}_1 &= -kx_1 + k_{12}(x_2 - x_1) \\
m\ddot{x}_2 &= -kx_2 - k_{12}(x_2 - x_1)
\end{aligned}$$

$$\begin{aligned}
m\ddot{x}_1 + (k + k_{12})x_1 - k_{12}x_2 = 0 \\
m\ddot{x}_2 + (k + k_{12})x_2 - k_{12}x_1 = 0
\end{aligned}$$

4.2 Normal coordinates and normal modes

Example 4.2.1.

$$q \coloneqq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \frac{1}{m}\begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix}$$

$$\ddot{q} + \Omega q = 0 \tag{4.1}$$

4.3 Weak coupling

$$\epsilon \coloneqq \frac{k_{12}}{2k} \ll 1 \Rightarrow \omega_1 \approx \omega_0(1 - \epsilon), \quad \omega_2 \approx \omega_0(1 + \epsilon)$$

With $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ and $x_1 = d$:

$$\begin{aligned}
x_1(t) = \cos[\epsilon\omega_0 t] \cos[\omega_0 t] \\
x_2(t) = \sin[\epsilon\omega_0 t] \sin[\omega_0 t]
\end{aligned}$$

Chapter 5

Coupled oscillations 2

5.1 General theory of near equilibrium oscillations

5.1.1 General coordinates (introduction)

The positions of n particles (with three components) with respect to a IRS are $\{\vec{r}_a\}_{a=1}^{N} = \{x_\alpha\}_{\alpha=1}^{3N}$. If we have k non redundant independent variables, then

$$x_\alpha = x_\alpha(\vec{q}, t), \qquad \vec{q} = \begin{pmatrix} q_1 & \dots & q_k \end{pmatrix}^t \tag{5.1}$$

We usually take $\vec{q}$ dimensionally homogeneous and the dependence of time through $\vec{q}$, that is, $x_\alpha(t) = x_\alpha(\vec{q}(t))$.
We suppose there are not non-conservative forces.

5.1.2 Near equilibrium potential energy

Since $U(\vec{q}) = V(x_\alpha(\vec{q}))$ and $x_\alpha^{(0)} = x_\alpha(q^{(0)})$ (equilibrium position,)

$$\left.\frac{\partial U}{\partial q_i}\right|_{\vec{q}^{(0)}} = \sum_\alpha \left.\frac{\partial V}{\partial x_\alpha}\right|_{x_\alpha^{(0)}} \left.\frac{\partial x_\alpha}{\partial q_i}\right|_{\vec{q}^{(0)}} = \sum_\alpha -F_\alpha \left.\frac{\partial x_\alpha}{\partial q_i}\right|_{\vec{q}^{(0)}} = 0 \tag{5.2}$$

And if we choose a general coordinates such that $\vec{q}^{(0)} = \vec{0}$ (for instance with $\vec{q} \mapsto \vec{q} - \vec{q}^{(0)}$), we get the Hessian matrix (non negatively defined)

$$U(\vec{q}) \approx \frac{1}{2} \sum_{ij} \frac{\partial^2 U}{\partial q_i \partial q_j} q_i q_j = \frac{1}{2} \sum_{ij} A_{ij} q_i q_j = \frac{1}{2} \vec{q}^t A_{ij} \vec{q} \tag{5.3}$$

5.1.3 Near equilibrium kinetic energy

$$\dot{x}_\alpha = \sum_i \frac{\partial x_\alpha}{\partial q_i} \dot{q}_i, \Rightarrow K = \sum_\alpha \frac{1}{2} m_\alpha \dot{x}_\alpha^2 = \frac{1}{2} \sum_{ij} \sum_\alpha m_\alpha \frac{\partial x_\alpha}{\partial q_i} \frac{\partial x_\alpha}{\partial q_j} \dot{q}_i \dot{q}_j \coloneqq \frac{1}{2} \sum_{ij} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{\vec{q}}^t m \dot{\vec{q}} \tag{5.4}$$

The matrix $m = [m_{ij}]$ is always positively defined.

5.1.4 Near equilibrium mechanical energy

$$E = K + U = \frac{1}{2} \dot{\vec{q}}^t m \dot{\vec{q}} + \frac{1}{2} \vec{q}^t A \vec{q} = \frac{\mu}{2} (\dot{\vec{q}} M \dot{\vec{q}} + \vec{q}^t \Omega \vec{q}) \tag{5.5}$$

5.1.5 Diagonalization of K and U. Normal modes

We define $y = M^{1/2} q, q = M^{-1/2} y$. We know $O^t M O = M_d$, so $M^{1/2} = O M_d^{1/2} O^t$

$$E = \frac{\mu}{2} \left[\dot{\vec{y}}^t \dot{\vec{y}} + \vec{y}^t (M^{1/2} \Omega M^{-1/2}) \vec{y} \right] \coloneqq \frac{\mu}{2} \left[\dot{\vec{y}}^t \dot{\vec{y}} + \vec{y}^t \tilde{\Omega} \vec{y} \right], \qquad \tilde{\Omega} \text{ non-neg. defined and sym.} \tag{5.6}$$

We diagonalize $\tilde{\Omega}$ and obtain $\tilde{\lambda}_j = \omega_j^2$ and $\tilde{u}_j$. With $y = \sum_k \eta_k \tilde{u}_k$, we get $\tilde{u}_j^t \tilde{u}_k = \delta_{jk}$ and

$$E = \mu \sum_j \frac{1}{2} \dot{\eta}_j^2 + \frac{1}{2} \omega_j^2 \eta_j^2 \tag{5.7}$$

Making now $u_j = M^{-1/2}\tilde{u}_j$, we get

$$\Omega u_j - \omega_j^2 M u_j = 0, \qquad u_j^t M u_k = \delta_{jk}, \qquad q = \sum_k \eta_k u_k, \qquad u_j^t M q = \eta_j \tag{5.8}$$

Initial conditions

$$\eta_j(0) = v_j^t M q(0), \qquad \dot{\eta}_j(0) = v_j^t M \dot{q}(t) \tag{5.9}$$

$$\eta_j(t) = \eta_j(0)\cos\omega_j t + \frac{\dot{\eta}_j(0)}{\omega_j}\sin\omega_j t, \qquad q(t) = \sum_k \eta_k(t) u_k, \qquad E = \frac{\mu}{2}\sum_j \omega_j^2 A_j^2 \tag{5.10}$$

5.2 Many oscillators

5.2.1 Approach and evolution equations

$$m\ddot{q}_r = T_r \sin\theta_r - T_{r-1}\sin\theta_{r-1}, \tan\theta_r = \frac{q_{r+1}-q_r}{d} \Rightarrow \ddot{q}_r + \frac{T}{md}(2q_r - q_{r-1} - q_{r+1}) = 0 \tag{5.11}$$

5.2.2 Normal modes

Now we search the eigenfrequencies

$$2a_r - a_{r+1} - a_{r-1} = \frac{md}{T}\omega^2 a_r, a_r = a\sin\mu r \Rightarrow \omega = 2\sqrt{\frac{T}{md}}\sin\frac{\mu}{2} \tag{5.12}$$

We impose that $q_0 = q_{n+1} = 0$.

$$\mu_j = \frac{j\pi}{n+1} \Rightarrow \omega_j = 2\sqrt{\frac{T}{md}}\sin\frac{j\pi}{2(n+1)} \tag{5.13}$$

We normalize and

$$\left\|v^{(j)}\right\|^2 = \sqrt{\frac{n+1}{2}} \Rightarrow v_r^{(j)} = \sqrt{\frac{2}{n+1}}\sin\frac{rj\pi}{n+1}, \qquad v^{(j)t}v^k = \delta_{jk} \tag{5.14}$$

Relations

$$q_r(t) = \sum_{j=1}^n \eta_j(t) v_r^{(j)} = \sum_{j=1}^n \eta_j(t)\sqrt{\frac{2}{n+1}}\sin\frac{rj\pi}{n+1} \tag{5.15}$$

$$\eta_j = \sum_{r=1}^n q_r(t) v_r^{(j)} = \sum_{r=1}^n q_r(t)\sqrt{\frac{2}{n+1}}\sin\frac{rj\pi}{n+1} \tag{5.16}$$

If $\dot{q}_r(0) = 0, \forall r$, then

$$q_r(t) = \frac{2}{n+1}\sum_{j=1}^n \left\{\sum_{s=1}^n q_s(0)\sin\frac{sj\pi}{n+1}\right\}\sin\frac{rj\pi}{n+1}\cos\omega_j t \tag{5.17}$$

5.2.3 Energy and normal modes

$$E = \sum_{r=1} \frac{1}{2}m\dot{q}_r^2 + \frac{T}{2d}(q_{r+1}-q_r)^2 \tag{5.18}$$

Diagonalization

$$E = \frac{1}{2}m(\dot{q}^t\dot{q} + q^t\Omega q) = \frac{1}{2}\sum_j m\omega_j^2 A_j^2 \tag{5.19}$$

5.2.4 Continuum limit

Initial substitutions

$$(n+1)d = L, \qquad \rho = \frac{M}{L} \to \frac{m}{d}, \qquad x = rd, u = \frac{v}{\sqrt{d}}, \qquad \tilde{\eta}_j = \sqrt{d}\eta_j \tag{5.20}$$

$$\int_0^L u^{(j)}(x)u^{(k)}(x)dx = \delta_{jk}, \quad q(x) = \sum_{j=1}^{\infty} \tilde{\eta}_j u^{(j)}(x), \quad \tilde{\eta}_j = \int_0^L u^{(j)}(x)q(x)dx, \quad \omega_j = \sqrt{\frac{T}{\rho}}\frac{j\pi}{L} \tag{5.21}$$

5.2.5 Wave equation derivation from the limit of discrete evolution equations

$$\frac{\partial^2 q}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = 0, c = \sqrt{\frac{T}{\rho}}, \qquad \text{b. conditions } q(0,t) = q(L,t) = 0 \tag{5.22}$$

Energy

$$E = \int_0^L \frac{\rho}{2}\left(\frac{\partial q}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial q}{\partial x}\right)^2 dx \tag{5.23}$$

5.3 Wave equation derivation from the continuum

$$\frac{\partial^2 q}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = 0, c = \sqrt{\frac{T}{\rho}}, \qquad \text{b. conditions } q(0,t) = q(L,t) = 0 \tag{5.24}$$

5.4 Oscillation modes for the rope. Boundary conditions and initial conditions

General solution (to obtain the normal modes)

$$q(x,t) = u(x)\left(c_1\cos\omega t + c_2\sin\omega t\right) \Rightarrow u''(x) + \frac{\omega^2}{c^2}u(x) = 0 \Rightarrow u(x) = a\cos\left[\frac{\omega}{c}x\right] + b\sin\left[\frac{\omega}{c}x\right] \tag{5.25}$$

Relations

$$f\lambda = c, \qquad \int_0^L u^{(j)}(x)u^{(k)}(x)dx = \delta_{jk} \tag{5.26}$$

For the fixed string ($u(0) = u(L) = 0$)

$$\omega_j = \frac{j\pi c}{L}, \qquad u_j(x) = \sqrt{\frac{2}{L}}\sin\frac{j\pi x}{L}, \tag{5.27}$$

Relations

$$q(x,t) = \sum_{j=1}^{\infty}\eta_j(t)u_j(x), \qquad q(x,0) = g(x), \dot{q}(x,0) = h(x) \tag{5.28}$$

$$\eta_k(0) = \int_0^L u_k(x)g(x)dx, \qquad \eta_k(0) = \int_0^L u_k(x)h(x)dx \tag{5.29}$$

5.5 General solution of the wave equation. Uniqueness theorem

General (and unique) solution and d'Alembert's formula

$$q(x,t) = \psi(x-ct) + \phi(x+ct) = \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')dx' \tag{5.30}$$

Decomposition

$$\psi(x-ct) = \frac{g(x-ct)}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} h(x')dx' + \frac{\psi(x_0)-\phi(x_0)}{2}, \tag{5.31}$$

$$\phi(x+ct) = \frac{g(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} h(x')dx' - \frac{\psi(x_0)-\phi(x_0)}{2} \tag{5.32}$$

"Energy"

$$\varepsilon = \int_0^L \left(\frac{\partial q}{\partial t}\right)^2 + c^2\left(\frac{\partial q}{\partial x}\right)^2 dx, \qquad \frac{d\varepsilon}{dt} = 0 \tag{5.33}$$

Chapter 6

Rotations

6.1 Mathematical background

6.1.1 Vectors and matrices

$$[u \times v]_k = \sum_{i,j} \epsilon_{ijk} u_i v_j = \sum_{i,j} \epsilon_{kij} u_i v_j$$
$$\langle \vec{u}, \vec{v} \times \vec{w} \rangle_I = \sum_{i,j,k} \epsilon_{ijk} u_i v_j w_k = \det\{[\mathsf{u}, \mathsf{v}, \mathsf{w}]\}$$
$$\sum_{i,j,j} \epsilon_{ijk} A_{ii'} A_{jj'} A_{kk'} = \epsilon_{i'j'k'} \det\{A\}$$
$$\sum_{ij} \epsilon_{ijl} A_{ii'} A_{jj'} = \det\{A\} \sum_{k'} \epsilon_{i'j'k'} A^{-1}_{k'l}$$
$$\epsilon_{ijk}\epsilon_{i'j'k'} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} & \delta_{ik'} \\ \delta_{ji'} & \delta_{jj'} & \delta_{jk'} \\ \delta_{ki'} & \delta_{kj'} & \delta_{kk'} \end{vmatrix}$$
$$\sum_{k} \epsilon_{ijk}\epsilon_{i'j'k} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}$$
$$\sum_{j,k} \epsilon_{ijk}\epsilon_{i'jk} = 2!\delta_{ii'}$$
$$\sum_{i,j,k} \epsilon_{ijk}\epsilon_{ijk} = 3!$$
$$\det\{A\} = \frac{1}{3!} \sum_{ijk,i'j'k'} \epsilon_{ijk}\epsilon_{i'j'k'} A_{ii'} A_{jj'} A_{kk'}$$
$$A^{-1}_{k'k} = \frac{1}{2\det\{A\}} \sum_{ij,i'j'} \epsilon_{ijk}\epsilon_{i'j'k'} A_{ii'} A_{jj'}, \quad \det\{A\} \neq 0$$
$$\vec{u} \times (\vec{b} \times \vec{w}) = \langle \vec{u}, \vec{w} \rangle_I \vec{v} - \langle \vec{u}, \vec{v} \rangle_I \vec{w}$$
$$\langle \vec{u} \times \vec{v}, \vec{u}' \times \vec{v}' \rangle_I = \langle \vec{u}, \vec{u}' \rangle_I \langle \vec{v}, \vec{v}' \rangle_I - \langle \vec{u}, \vec{v}' \rangle_I \langle \vec{v}, \vec{u}' \rangle_I$$

Linear transformation

$$f_A(u) = \mathsf{Au} \tag{6.1}$$

6.1.2 Basis

Let $\mathscr{B}$ be a basis. We denote the matrix of $\mathbf{v}$ of components in $\mathscr{B}$ as

$$C_{\mathscr{B}}(\mathbf{v}) = \mathsf{v}_B = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \tag{6.2}$$

If we use the canonical basis we won't specify the basis. We will write directly u.

6.1.3 Change of basis

Let $\mathscr{B}_1, \mathscr{B}_2$ be two basis. Then, the matrix to change the basis $\mathsf{M}(\mathscr{B}_2 \leftarrow \mathscr{B}_1)$ has the columns formed by the vectors of $\mathscr{B}_1$ written in $\mathscr{B}_2$. This relation can be expressed as follows

$$C_{\mathscr{B}_2}(\mathbf{v}) = \begin{pmatrix} C_{\mathscr{B}_2}(\mathbf{u_1}) & \dots & C_{\mathscr{B}_2}(\mathbf{u}_n) \end{pmatrix} C_{\mathscr{B}_1}(\mathbf{v}) \Leftrightarrow \mathsf{v}_{B_2} = \begin{pmatrix} \mathsf{u}_{1,B} & \dots & \mathsf{u}_{n,B} \end{pmatrix} \mathsf{v}_{B_1}. \tag{6.3}$$

In particular, taking the canonical basis $\mathscr{C}$ as $\mathscr{B}_1$, $\mathsf{M}(\mathscr{C} \leftarrow \mathscr{B}_1)$ has the vectors of $\mathscr{B}_1$ as the columns. We denote $\mathsf{C} := \mathsf{M}(\mathscr{C} \leftarrow \mathscr{B})$.
For a linear transformation f_A with a matrix $A = M(f;\mathscr{C} \leftarrow \mathscr{C})$, we have the matrix of f_A associated to a basis $\mathscr{B}$ is

$$\mathsf{A}_{\mathscr{B}} = \mathsf{M}(f;\mathscr{B} \leftarrow \mathscr{B}) = \mathsf{M}(\mathscr{B} \leftarrow \mathscr{C})\mathsf{M}(f;\mathscr{C} \leftarrow \mathscr{C})\mathsf{M}(\mathscr{C} \leftarrow \mathscr{B}) = \mathsf{C}^{-1}\mathsf{A}_{\mathscr{C}}\mathsf{C}. \tag{6.4}$$

$$\mathbf{e}'_i = f_A(\mathbf{e}_i) \Leftrightarrow \mathsf{e}'_i = \mathsf{A}\mathsf{e}_i, \mathbf{v} = \sum_{i=1}^{n} \alpha'_i \mathbf{e}'_i = \sum_{i=1}^{n} \alpha'_i f_A(\mathbf{e}_i) = f_A\left(\sum_{i=1}^{n} \alpha'_i \mathbf{e}_i\right) \tag{6.5}$$

6.1.4 Orthogonal transformations

Properties

- $\mathsf{O}^{-1} = \mathsf{O}^t$
- $\det\{\mathsf{O}\} = \pm 1$
- They preserve the angles, the lengths, and the dot product, that is, $\langle \vec{u}, \vec{v} \rangle_I = \langle \vec{u}_{\mathscr{B}}, \vec{v}_{\mathscr{B}} \rangle_I$
- They form a group

2D: In 2D, all have the form have two possible forms (the transformations associated to the rotation matrix is R_θ and and to the reflection matrix by S). The matrix R_θ has eigenvalues $\lambda = e^{\pm\theta i}$.

$$\mathsf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \mathsf{R}_\theta\mathsf{S} = \mathsf{R}_\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.6}$$

3D: The eigenvalues are $\lambda_0 = \pm 1, \lambda_+ = e^{\theta i}, \lambda_- = e^{-\theta i}$ with eigenvectors $\vec{n}_0, \vec{n}_+, \vec{n}_-$. Taking a basis $(\vec{n}_1, \vec{n}_2, \vec{n}_0)$, a general transformation matrix is

$$\mathsf{M}_{\mathscr{B}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \tag{6.7}$$

To make sure the convention of the sign of $\vec{n}_0$ is correct, it must be true that $\mathsf{C}\mathsf{M}_{\mathscr{B}}\mathsf{C} = \mathsf{M}$. If not, change the sign of $\vec{n}_0$ and other to keep the orientation. If we take some vector $\vec{u}$, take $\vec{u}_\perp = \vec{u} - \langle \vec{n}, \vec{u} \rangle_I \vec{n}$ (Gram-Schmidt), $\vec{n}_1 = \vec{u}_\perp / \|\vec{u}_\perp\|$, and $\vec{n}_2 = \vec{n} \times \vec{n}_1$ to make $\mathscr{B} = (\vec{n}_1, \vec{n}_2, \vec{n})$
Axis rotations

$$\mathsf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad \mathsf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \mathsf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.8}$$

Rotation of a vector $\vec{u}$

$$R_{\vec{n}}(\psi)\vec{u} = \vec{u}\cos\psi + (\vec{n} \times \vec{u})\sin\psi + \vec{n}\langle \vec{n}, \vec{u} \rangle_I (1 - \cos\psi), \tag{6.9}$$

$$[R_{\vec{n}}(\psi)]_{ij} = \delta_{ij}\cos\psi - \sum_k \epsilon_{ijk} n_k \sin\psi + n_i n_j (1 - \cos\psi) \tag{6.10}$$

6.2 Infinitesimal rotations. Angular velocity and acceleration

$$R_n(d\psi)\vec{u} = \vec{u} + (\vec{n} \times \vec{u})d\psi \Rightarrow d\vec{u} = (\vec{n} \times \vec{u})d\psi \Rightarrow \dot{\vec{u}} = \dot{\psi}\vec{n} \times \vec{u} \tag{6.11}$$

With $\vec{u} = \vec{r}$ in an instantaneous rotation over the center of curvature (without variation of length),

$$\vec{v} = \dot{\psi}\vec{n} \times \vec{r} = \vec{\omega} \times \vec{r} \Rightarrow \vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.12}$$

Infinitesimal rotations (only) commute

$$R_{\vec{n}'}(d\psi')R_{\vec{n}}(d\psi)\vec{u} = \vec{u} + (\vec{n}d\psi + \vec{n}'d\psi') \times \vec{u} \tag{6.13}$$

6.3 Rotating reference systems

6.3.1 Rotating and fixed system

We denote $\mathscr{B}$ the fixed basis and $\mathscr{B}'$ the rotating basis. We denote $\vec{u}'$ the vector with respect to u_i' the i-th component of $\vec{u}$ with respect to $\mathscr{B}'$ and $\vec{e}_i'$ the i-th vector of the basis $\mathscr{B}'$. If the system rotates at ω,

$$\dot{\vec{e}}_i' = \vec{\omega} \times \vec{e}_i', \qquad \frac{d\vec{u}}{dt} = \left(\frac{d\vec{u}}{dt}\right)' + \vec{\omega} \times \vec{u} \tag{6.14}$$

If $\vec{u} = \vec{r}$,

$$\vec{v} = (\vec{v})' + \vec{\omega} \times \vec{r}, \qquad \vec{a} = (\vec{a})' + \vec{\alpha} \times \vec{r} + 2\vec{\omega} \times (\vec{v})' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.15}$$

If the system moves,

$$\vec{a} = (\vec{a})' + \ddot{\vec{R}} + \vec{\alpha} \times \vec{r} + 2\vec{\omega} \times (\vec{v})' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.16}$$

6.3.2 Inertial forces

$$m(\vec{a})' = (\vec{F})' = \vec{F} - m\ddot{\vec{R}} - m\vec{\alpha} \times \vec{r} - 2m\vec{\omega} \times (\vec{v})' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.17}$$

And

$$\vec{F}_{\text{cor}} = -2m\vec{\omega} \times (\vec{v})', \qquad \vec{F}_{\text{cf}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.18}$$

Observations

- We use ALWAYS x, y, z for the vector in rotating system and x_f, y_f, z_f for the fixed system.

6.4 Solid rigid kinematics

Chapter 7

Dynamics of rigid body

7.1 Part 1

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_O \tag{7.1}$$
$$\mathbf{a} = \mathbf{A} + \boldsymbol{\alpha} \times \mathbf{r}_O + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{7.2}$$

7.2 Part 2

Now, $\mathbf{r}$ is the position from the center of mass

$$T = \frac{1}{2}\int_V \|\mathbf{v}\|^2 \rho(\mathbf{r})\,\mathrm{d}v \tag{7.3}$$
$$E = \frac{1}{2}M\|\mathbf{V}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^t\mathbf{I}\boldsymbol{\Omega} \tag{7.4}$$

Expressions and properties for inertia tensor

$$I_{ij} = \int_V \rho(\mathbf{r})\left[\|\mathbf{r}\|\delta_{ij} - x_i x_j\right]\mathrm{d}v\,, \qquad \mathbf{I} = \int_V \rho(\mathbf{r})\left[\|\mathbf{r}\|\mathbf{E}_r - \mathbf{r}\otimes\mathbf{r}\right]\mathrm{d}v\,, \tag{7.5}$$
$$\mathbf{I} = \int \begin{pmatrix} \|\mathbf{r}\|^2 - x^2 & -xy & -xz \\ -yx & \|\mathbf{r}\|^2 - y^2 & -yz \\ -zx & -zy & \|\mathbf{r}\|^2 - z^2 \end{pmatrix} \mathrm{d}m \tag{7.6}$$
$$\mathbf{J} = \mathbf{I} + m\left[\langle\mathbf{R},\mathbf{R}\rangle_I \mathbf{E}_3 - \mathbf{R}\otimes\mathbf{R}\right], \qquad J_{ij} = I_{ij} + m(\|\mathbf{R}\|^2\delta_{ij} - a_i a_j) \tag{7.7}$$
$$\mathbf{I}_{B_2} = \mathbf{R}\mathbf{I}_{B_1}\mathbf{R}^t \tag{7.8}$$

1. It is a symmetric tensor, so it is diagonalizable and $\forall i, \lambda_i \in \mathbb{R}^+$.
2. We can always select a basis (that can be moving basis) such that inertia tensor is diagonal. In that case,
$$T_{\text{rot}} = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) \tag{7.9}$$

Angular momentum (with origin at CM)

$$\mathbf{L} = \mathbf{I}\cdot\boldsymbol{\Omega}, \tag{7.10}$$
$$T_{\text{rot}} = \frac{1}{2}\langle\boldsymbol{\Omega},\mathbf{L}\rangle_I \tag{7.11}$$
$$\mathbf{L}_D = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3) \tag{7.12}$$

Spherical tops

1. The inertia tensor is diagonal independently on the orientation of the basis.
2. Choosing $\mathbf{e}_3 \parallel \mathbf{L}$, we get $\Omega_3 = L/I_1, \Omega_1 = \Omega_2 = 0$.

Eulerian angles

$$\begin{aligned} \theta &= \arg(\mathbf{e}_z, \mathbf{e}_3'), \theta \in [0,\pi] \\ \phi &= \arg(\mathbf{e}_x, \mathbf{e}_n), \phi \in [0,2\pi], \\ \psi &= \arg(\mathbf{e}_n, \mathbf{e}_1'), \psi \in [0,2\pi] \end{aligned} \qquad (\mathbf{e}_\phi)_P = \begin{pmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{pmatrix},\ (\mathbf{e}_\theta)_P = \begin{pmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{pmatrix},\ (\mathbf{e}_\psi)_P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.13}$$

$$\mathbf{\Omega} = \dot{\phi}\mathbf{e}_\phi + \dot{\theta}\mathbf{e}_\theta + \dot{\psi}\mathbf{e}_\psi \Rightarrow (\mathbf{\Omega})_B = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix} \tag{7.14}$$

1. $\Omega_1^2 + \Omega_2^2 = \dot{\psi}^2\sin^2\theta + \dot{\theta}^2$

$$\mathbf{R}(\phi,\theta,\psi) = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi\cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{7.15}$$

Euler equations for rigid body

$$\dot{\mathbf{L}} + \mathbf{\Omega}\times\mathbf{L} = \boldsymbol{\tau}, \qquad \mathbf{I}\dot{\mathbf{\Omega}} + \mathbf{\Omega}\times(\mathbf{I}\mathbf{\Omega}) = \boldsymbol{\tau} \tag{7.16}$$

In basis with principal axis:

$$\begin{cases} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = \tau_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \tau_3 \end{cases} \tag{7.17}$$

Symmetric top

$$\dot{\theta} = 0 \Rightarrow \dot{\phi} = \Omega_{\text{prec}}, \dot{\psi} = \omega \tag{7.18}$$

Misael's equations

$$\sin\theta = \frac{\Omega_1}{\sqrt{\Omega_1^2 + (\Omega_3 - \omega)^2}}, \qquad \cos\theta = \frac{\Omega_3 - \omega}{\sqrt{\Omega_1^2 + (\Omega_3 - \omega)^2}}, \qquad \tan\theta = \frac{\Omega_1}{\Omega_3 - \omega} \tag{7.19}$$

$$\Omega_1^2 + (\Omega_3 - \omega)^2 = \Omega_{\text{prec}}^2, \qquad \Omega_1 = \sqrt{\Omega_{\text{prec}}^2 - \frac{\omega^2}{(I_1/I_3 - 1)^2}} \tag{7.20}$$

$$\sin\theta = \frac{\Omega_1}{\Omega_{\text{prec}}}, \qquad \cos\theta = \frac{\omega}{\Omega_{\text{prec}}}\frac{1}{I_1/I_3 - 1}, \qquad \tan\theta = \sqrt{\left(\frac{\Omega_p}{\omega}\right)^2\left(\frac{I_1}{I_3} - 1\right)^2 - 1}, \tag{7.21}$$

$$\frac{\Omega_1}{\Omega_3} = \sqrt{\left(\frac{\Omega_{\text{prec}}}{\omega}\right)^2\left(1 - \frac{I_3}{I_1}\right)^2 - \left(\frac{I_3}{I_1}\right)^2} \tag{7.22}$$

$$\tan\alpha = \frac{(\Omega_{\text{prec}}/\omega)\sin\theta}{1 + (\Omega_{\text{prec}}/\omega)\cos\theta} \tag{7.23}$$

Asymmetric top

1. Three possible movements: precession ($\dot{\phi}$), intrinsic rotation ($\dot{\psi}$), and nutation ($\dot{\theta}$).

7.2.1 Rotation of symmetric top under the action of gravity

Notation: now $\mathbf{r}$ is the vector of position from the fixed point and $\mathbf{R}$ is the vector of CM from the fixed point.

$$T_{\mathrm{rot}} = \frac{1}{2}\mathbf{\Omega}^t\mathbf{I}\mathbf{\Omega} = \frac{1}{2}\langle \mathbf{\Omega}, \mathbf{L}\rangle_I, \qquad I_{ij} = \int \|\mathbf{r}\|^2\delta_{ij} - x_i x_j \,\mathrm{d}m \tag{7.24}$$

$$T_{\mathrm{rot}} = \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1^2 a^2}{2I_3} \tag{7.25}$$

$$I_1\ddot{\theta} + I_1\frac{a\sin^3\theta(b - a\cos\theta) - (b - a\cos\theta)^2\sin\theta\cos\theta}{\sin^4\theta} - MgR\sin\theta = 0 \tag{7.26}$$

$$\dot{\phi}\sin^2\theta = b - a\cos\theta, \qquad \dot{\psi} = \frac{I_1 a}{I_3} - \dot{\phi}\cos\theta \tag{7.27}$$

Possible movements with constant angle

$$\Omega_{\mathrm{p,fast}} = \frac{L_3}{I_1\cos\theta}, \qquad \Omega_{\mathrm{p,slow}} = \frac{MgR}{L_3} \tag{7.28}$$

Determination of movement of the angle through effective potential

Chapter 8

Special relativity

Linear momentum

$$\mathbf{p} = \frac{1}{\sqrt{1-\|\boldsymbol{\beta}\|^2}} m\mathbf{v} = \gamma m\mathbf{v} \tag{8.1}$$

Addition of velocities

$$\mathbf{u}' = \frac{1}{1-\langle \boldsymbol{\beta}_u, \boldsymbol{\beta}_v \rangle_I} \left[\frac{\mathbf{u}}{\gamma_v} - \mathbf{v} + \frac{1}{c^2} \frac{\gamma_v}{1+\gamma_v} \langle \mathbf{u}, \mathbf{v} \rangle_I \mathbf{v} \right], \tag{8.2}$$

$$\mathbf{u} = \frac{1}{1+\langle \boldsymbol{\beta}_{u'}, \boldsymbol{\beta}_v \rangle_I} \left[\frac{\mathbf{u}'}{\gamma_v} + \mathbf{v} + \frac{1}{c^2} \frac{\gamma_v}{1+\gamma_v} \langle \mathbf{u}', \mathbf{v} \rangle_I \mathbf{v} \right] \tag{8.3}$$

4-vector

$$\mathbf{R} = (ct, x, y, z) \tag{8.4}$$

Lorentz transformation

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad \mathbf{R}' = \boldsymbol{\Lambda}\mathbf{R} \tag{8.5}$$

Minkowski bilinear form

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.6}$$

Invariant space-time interval

$$\Delta s \coloneqq \Delta\mathbf{R}^t \boldsymbol{\eta} \Delta\mathbf{R} = \langle \Delta\mathbf{R}, \Delta\mathbf{R} \rangle_m = \langle \Delta\mathbf{R}', \Delta\mathbf{R}' \rangle_m \tag{8.7}$$

Proper space-time

$$c\,\mathrm{d}\tau = \mathrm{d}s \tag{8.8}$$

Some relations

$$\mathrm{d}t = \gamma\,\mathrm{d}\tau \tag{8.9}$$

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \langle \mathrm{d}\mathbf{r}\,, \mathrm{d}\mathbf{r} \rangle_I \tag{8.10}$$

$$\mathbf{V} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}\tau} = (\gamma c, \gamma\mathbf{v}) \tag{8.11}$$

$$\mathbf{P} = (\gamma mc, \mathbf{p}), \qquad \mathbf{p} = m\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau} \tag{8.12}$$

Chapter 9

Generalized coordinates

We can characterize the system by n variables $q_1, \dots, q_n$, which we can see as the n components of a n-dimensional point in a bigger space. This is the *configuration space*, and the movement of this general point represents the behavior of the whole system. This simplifies the problem extremely for some particular cases. For instance, a rigid body needs only six degrees of freedom, so once we determine them as a function of time, a six dimensional point can represent the whole body. We call this point *C-point* and its path on the space *C-curve*.
While changing one set of coordinates to another, the geometry is distorted, but other important properties remain. In particular, properties of space like area, distances, and angles are modified, bu other topological properties are the same: a point remains a point, a curve remains a curve, adjacent curves remain adjacent curves, a neighborhood remains a neighborhood, continuous differentiable curves remain continuous differentiable curves.
This can be seen too as a transformation, where each new coordinate q_i' is a function of all the others from the first set $f_i(x_1, \dots, x_n)$. These functions must be finite, single valued and with non zero jacobian. This mapping has interesting properties in infinitesimal regions: a straight line remains a straight line, parallel lines remain parallel lines, a parallelepiped remains a parallelepiped. Geometrically, the determinant (the jacobian) represents the ratio between the new differential of volume with respect the original one.
–
This is strongly related to Riemannian geometry, which is based on the concept of the line element $\mathrm{d}\bar{s}$, defined as distance between two neighboring points, expressed in terms of coordinate systems. For a general set of coordinates x_1, x_2, x_3 it satisfies

$$\mathrm{d}\bar{s}^2 = \begin{pmatrix} \mathrm{d}x_1 & \mathrm{d}x_2 & \mathrm{d}x_3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{22} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{pmatrix}, \tag{9.1}$$

and it is independent on the coordinates, that is, if we select x_1', x_2', x_3', the value will be the same (g_{ij} are not constant, change in space). This forms a tensor of second order. In particular is an invariant tensor and, if we add that $g_{ij} = g_{ji}$, then we have a symmetric invariant tensor of second order.

9.0.1 Kinetic energy and two approaches

We know

$$T = \sum_{i=1}^{N} \frac{m_i v_i^2}{2}$$

with $\mathbf{v} = \mathrm{d}\bar{s}/dt$. If we consider now a single particle of mass $m = 1$ with the same kinetic energy,

$$T = \frac{1}{2}\left(\frac{\mathrm{d}\bar{s}}{\mathrm{d}t}\right)^2 \Rightarrow \mathrm{d}\bar{s}^2 = 2T dt = \sum_{i=1}^{N} m_i v_i^2 dt^2 = \sum_{i=1}^{N} m_i (\mathrm{d}x_i{}^2 + \mathrm{d}y_i{}^2 + \mathrm{d}z_i{}^2) =$$
$$\sum_{i=1}^{N} \left(\sqrt{m_i}\,\mathrm{d}x_i\right)^2 + \left(\sqrt{m_i}\,\mathrm{d}y_i\right)^2 + \left(\sqrt{m_i}\,\mathrm{d}z_i\right)^2.$$

If we take each term as a coordinate, we have that $\mathrm{d}\bar{s}$ has 3N components, so the total energy can be seen as the energy of a single particle moving in 3N dimensions. Although the expression of $\mathrm{d}\bar{s}$ is in a Riemannian general form, the geometry is euclidean (even changing the coordinates).

If there are now some constraints of the form

$$\begin{aligned} f_1(x_1,\dots,z_N) &= 0 \\ \vdots &= 0 \\ f_m(x_1,\dots,z_N) &= 0. \end{aligned}$$

Then, there are $n = 3N - m$ independent variables. These form subspaces where the C-point is constrained, and these hypersurfaces are not necessarily euclidean. If we started by expressing the system directly from the independent variables $q_1, \dots, q_n$, since these make the C-point line in a non necessarily euclidean space, the differential of length now is expressed in a general geometry (before was euclidean) determined by

$$\mathrm{d}\bar{s}^2 = \sum_{i,j=1}^{n} a_{ij}\, \mathrm{d}q_i\, \mathrm{d}q_j. \tag{9.2}$$

With that, we can transform the problem of a mechanical system to a riemannian geometry problem.

9.0.2 Holonomic and non holonomic constraints

Theorem 9.0.1. *Let $J : C^2[x_0, x_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')\,\mathrm{d}x\,,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_1 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are given real numbers. If $y \in S$ is an extremal for J, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{9.3}$$

for all $x \in [a_0, x_1]$.

Theorem 9.0.2 (Lagrange multipliers method for non-holonomic constraints)**.** *If we want to find an extrema having a set of m non-holonomic constraints*

$$\begin{aligned} \overline{\delta f_1} &= A_{11}\delta u_1 + \dots + A_{1n}\delta u_n &&= 0, \\ \vdots &= \ddots &&= \vdots \\ \overline{\delta f_m} &= A_{m1}\delta u_1 + \dots + A_{mn}\delta u_n &&= 0, \end{aligned} \tag{9.4}$$

Then we can proceed as the original multipliers method by considering every variation as independent and solving the following equation.

$$\delta F + \lambda_1 \overline{\delta f_1} + \dots + \lambda_m \overline{\delta f_m} = 0 \tag{9.5}$$

Theorem 9.0.3. *Let f be a continuous functions with a variation $\delta f = \epsilon\phi$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta y = \delta \frac{\mathrm{d}}{\mathrm{d}x} y, \qquad \delta \int_a^b f(x)\,\mathrm{d}x = \int_a^b \delta f(x)\,\mathrm{d}x\,. \tag{9.6}$$

Theorem 9.0.4. *Let* $J : \mathbf{C}^2[t_0, t_1] \longrightarrow \mathbb{R}$ *be a functional of the form*

$$J(\mathbf{q}) = \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}, t)\, \mathrm{d}t\,, \tag{9.7}$$

where $\mathbf{q} = (q_1, \ldots, q_n)$*, and* L *has continuous second-order partial derivatives with respect to* t, q_k*, and* $\dot{q}_k$*,* $k = 1, \ldots, n$*. Let*

$$S = \left\{ \mathbf{q} \in \mathbf{C}^2[t_0, t_1] \,\middle|\, \mathbf{q}(t_0) = \mathbf{q}_0, \mathbf{q}(t_1) = \mathbf{q}_1 \right\}, \tag{9.8}$$

where $\mathbf{q}_0, \mathbf{q}_1 \in \mathbb{R}^n$ *are given vectors. If* $\mathbf{q}$ *is an extremal for* J *in* S *then for* $k = 1, \ldots, n$

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} = 0.} \tag{9.9}$$

Theorem 9.0.5. *If we have a set of holonomic constraints*

$$\begin{aligned} f_1(q_1, \ldots, q_n, t) &= 0, \\ \ldots &= \vdots, \\ f_m(q_1, \ldots, q_n, t) &= 0, \end{aligned} \tag{9.10}$$

then we can treat each variable as independent and search the stationary value of

$$J' = \int_{t_1}^{t_2} L + \sum_{k=1}^{m} \lambda_m f_m \, \mathrm{d}t\,, \tag{9.11}$$

which leads to the equation

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} + \lambda_1 \frac{\partial f_1}{\partial q_k} + \cdots + \lambda_m \frac{\partial f_m}{\partial q_k} = 0.} \tag{9.12}$$