

# 1 Arithmetic and topology

**Definition 1.1.** Let  $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ . Let us consider the following operations of addition and multiplication:

- Sum: given two  $(a, b), (c, d) \in \mathbb{R}^2$  we define the sum “+” by components

$$(a, b) + (c, d) := (a + b, c + d). \quad (1)$$

- Product: given two  $(a, b), (c, d) \in \mathbb{R}^2$  we define the product by

$$(a, b)(c, d) = (ac - bd, ad + bc). \quad (2)$$

We define the set  $\mathbb{C}$  as  $(\mathbb{R}^2, +, \cdot)$ .

**Proposition 1.1.** *The set  $\mathbb{C}$  of complex numbers is an abelian field.*

**Proposition 1.2.** *Let  $\mathbb{C}$  be defined in the second way. Then,*

1.  $\mathbb{C}$  is an abelian ring.
2. If we define  $f$  as

$$f : (\mathbb{C}, +, \cdot) \longrightarrow (\mathbb{R}^2, +, \cdot), \quad (x, y) \longmapsto x + yi, \quad (3)$$

then  $f$  is a morphism of rings.

3. The function  $f$  is, in fact, an isomorphism and  $\mathbb{C}$  is an abelian field.

**Proposition 1.3.** *The subset of  $\mathbb{C}$  generated by numbers of the form  $\underline{x} = (x, 0)$  is isomorph to the set of real numbers.*

**Theorem 1.4.**  $\mathbb{C}$  is not an ordered field.

**Definition 1.2.** Let  $z = a + bi \in \mathbb{C}$ . We define the conjugate of  $z$  as

$$\bar{z} := a - bi. \quad (4)$$

**Proposition 1.5.** *For all  $z, w \in \mathbb{C}$ , we have:*

1.  $\bar{\bar{z}} = z$ .
2.  $\overline{z + w} = \bar{z} + \bar{w}$ .
3.  $\overline{zw} = \bar{z}\bar{w}$ .
4.  $z\bar{z} \in \mathbb{R}$ . In particular, if  $z = a + bi$ , then  $z\bar{z} = a^2 + b^2$ .
5.  $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$ .
6. The inverse element of  $z \in \mathbb{C}^*$  in multiplication is  $z^{-1} = \bar{z}/(z\bar{z})$ .

**Definition 1.3.** Let  $z = a + bi \in \mathbb{C}$ . We define the real part of  $z$  and imaginary part of  $z$  respectively as

$$\operatorname{Re}\{z\} := a, \quad \operatorname{Im}\{z\} := b. \quad (5)$$

**Proposition 1.6.** *Let  $z \in \mathbb{C}$ . Then,*

$$\operatorname{Re}\{z\} = \frac{z + \bar{z}}{2}, \quad \operatorname{Im}\{z\} = \frac{z - \bar{z}}{2i} \quad (6)$$

**Proposition 1.7.** *Let  $z, w \in \mathbb{C}$  and the following distance function.*

$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto \tilde{d}(z, w) := |z - w| \end{aligned} \quad (7)$$

Then,  $(\mathbb{C}, \tilde{d})$  is a metric space.

**Definition 1.4.** Let  $z = a + bi \in \mathbb{C}$ . We define the modulus of  $z$  as

$$|z| := \tilde{d}(z, 0), \quad (8)$$

which is equivalent to  $\sqrt{z\bar{z}}$ .

**Definition 1.5.** Let  $r \in \mathbb{R}^+$  and  $z_0 \in \mathbb{C}$ . We define an open disc of radius  $r$  and center  $z_0$  as follows

$$B_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}. \quad (9)$$

**Definition 1.6.** Let  $r \in \mathbb{R}^+$  and  $z_0 \in \mathbb{C}$ . We define a punctured disc of radius  $r$  and center  $z_0$  as follows

$$B_r^*(z_0) := \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}. \quad (10)$$

**Definition 1.7.** Let  $r \in \mathbb{R}^+$  and  $z_0 \in \mathbb{C}$ . We define a closed disc of radius  $r$  and center  $z_0$  as follows

$$\overline{B_r(z_0)} := \{z \in \mathbb{C} \mid |z - z_0| \leq r\}. \quad (11)$$

**Definition 1.8.** We denote by  $\mathbb{D}$  the unitary disc of center 0 and radius 1. Besides, we denote by  $\mathbb{T} \subseteq \mathbb{C}$  the unitary circumference, that is,

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}. \quad (12)$$

We also denote it by  $\mathbb{S}^1$ .

**Lemma 1.8.** *The set  $B = \{B_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\}$  is a basis of the topology of  $\mathbb{R}^2$  as a metric space. The set  $D = \{D_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$  is a basis of the topology of  $\mathbb{C}$  as a metric space.*

**Proposition 1.9.** *The sets  $\mathbb{C}$  and  $\mathbb{R}^2$  with the topology of metric space are homeomorphs.*

**Corollary 1.10.** *There is a bijection between  $B$  and  $D$ , that is, between balls of  $\mathbb{R}^2$  and discs of  $\mathbb{C}$ .*

**Proposition 1.11.** *Let  $z, w \in \mathbb{C}$ . Then,*

1.  $|z| \geq 0$ .
2.  $|z| = 0 \Leftrightarrow z = 0$ .
3.  $-|z| \leq \operatorname{Re}\{z\} \leq |z|$  and  $-|z| \leq \operatorname{Im}\{z\} \leq |z|$ .
4.  $|zw| = |z||w|$ .
5. If  $w \neq 0$ ,  $|z/w| = |z|/|w|$ .
6.  $|z + w| \leq |z| + |w|$ .
7.  $|z + w| \geq ||z| - |w||$ .
8.  $|\operatorname{Re}\{zw\}| \leq |z||w|$  and  $|\operatorname{Im}\{z\bar{w}\}| \leq |z||w|$ .
9.  $|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$ .

$$10. |z^n| = |z|^n$$

**Corollary 1.12.** Let  $z_1, \dots, z_n \in \mathbb{C}$ . Then,

$$\left| \sum_{i=1}^n z_i \right| \leq \sum_{i=1}^n |z_i|, \quad |z_1 \cdots z_n| = |z_1| \cdots |z_n|, \quad |\operatorname{Re}\{z_1 \cdots z_n\}| \leq |z_1| \cdots |z_n| \leq \operatorname{Im} \left\{ \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \right\}. \quad (13)$$

**Definition 1.9.** Let  $z \in \mathbb{C}^*$ . We define the *argument* of  $z$ , denoted by  $\arg z$ , as the real number  $\theta$  such that  $z = |z|(\cos \theta + i \sin \theta)$ . Let us observe that  $\arg z$  is not a function but a multivalued application. We define the *principal argument* of  $z$  as

$$\operatorname{Arg} z := \theta_0 \in [0, 2\pi) \mid z = |z|(\cos \theta + i \sin \theta). \quad (14)$$

In general, to make  $\theta$  to be unique, it is enough to impose it to belong to a certain semiopen interval of length  $2\pi$ . Choosing the interval  $I$  is called by *taking a determination of the argument*.

**Definition 1.10.** Given a complex number  $z$  that we can express by  $z = |z|(\cos \theta + i \sin \theta)$  for some  $\theta \in \mathbb{R}$ , we use the notation  $r = |z|$  to write

$$z = r_\theta^z = r(\cos \theta + i \sin \theta) \quad (15)$$

or simply  $r_\theta$  when it is obvious which complex number are we referring to. We call it *polar form* of  $z$ .

**Proposition 1.13.** Let  $z \in \mathbb{C}$  and  $r_\theta$  its polar form. Then,

$$z^n = (r^n)_{n\theta}. \quad (16)$$

**Corollary 1.14** (De Moivre's Formula). Let  $\theta \in \mathbb{R}$ . Then,

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta). \quad (17)$$

**Proposition 1.15.** Let  $z, w \in \mathbb{C}$ . Then,

1.  $\arg zw = \arg z + \arg w + 2\pi k$ .
2.  $\arg z^n = n \arg z + 2\pi k$ .

**Definition 1.11.** We denote the complex numbers  $z$  generated by moving the point  $z_0 = 1$  around  $\mathbb{T}$  a length  $t$  in a counter-clockwise direction by  $1_t$ . In other words,  $1_t$  are the complex numbers  $z = \cos t + i \sin t$ .

**Proposition 1.16.** Let  $f : t \rightarrow 1_t$ . Then,  $f$  is a morphism from  $(\mathbb{R}, +)$  to  $(\mathbb{T}, \cdot)$ , with  $\ker f = 2\pi\mathbb{Z}$ .

**Definition 1.12.** Let  $z \in \mathbb{C}$  and  $n \in \mathbb{N}$ . We say  $w \in \mathbb{C}$  is an  $n$ -th root of  $z$  if and only if

$$w^n = z. \quad (18)$$

**Theorem 1.17.** Let  $n \in \mathbb{N}^*$  and  $z \in \mathbb{C}$ . Then, there exist  $w_1, \dots, w_n \in \mathbb{C}$  such that  $w_i^n = z$  for all  $i \in \{1, \dots, n\}$ , and  $w_i \neq w_j$  for all  $i \neq j$ . Besides, if  $\omega \in \mathbb{C}$  satisfies  $\omega^n = z$ , then  $\omega = w_k$  for some  $k \in \{1, \dots, n\}$ .

**Theorem 1.18.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a continuous curve such that  $\gamma(t) \neq 0 \forall t \in [a, b]$ . Then, there exists a continuous determination  $\phi$  of the argument of  $\gamma$ . Then,  $\phi(t) + 2\pi k$  with  $k \in \mathbb{Z}$  is the general expression of all the argument determinations of  $\gamma$ . If  $\gamma$  is differentiable, then  $\phi$  is differentiable and  $\phi' = \operatorname{Im}\{\gamma'/\gamma\}$ .

**Definition 1.13.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a regular curve. We define the *variation of the argument along  $\gamma$*  as

$$\Delta_\gamma \arg z = \operatorname{Im} \left\{ \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \right\}. \quad (19)$$

**Definition 1.14.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve such that  $\gamma(t) \neq 0 \forall t \in [a, b]$ . Then, we define the *index of  $\gamma$  with respect to the origin* or *the number of revolutions of  $\gamma$  around the origin*

$$\operatorname{Ind}(\gamma, 0) := \frac{1}{2\pi} \Delta_\gamma \arg. \quad (20)$$

**Proposition 1.19.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a piece-wise regular curve. Then,

$$\operatorname{Ind}(\gamma, 0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \quad (21)$$

**Definition 1.15.** Let  $\gamma$  be a closed curve and  $z \notin \Gamma$ . We define the *index of  $\gamma$  with respect to  $z$*  as

$$\operatorname{Ind}(\gamma, z) := \operatorname{Ind}(\gamma - z, 0). \quad (22)$$

**Proposition 1.20.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve piece-wise of class  $C^1([a, b])$ . Then,

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} dt. \quad (23)$$

**Proposition 1.21.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a piece-wise of class  $C^1([a, b])$ . Then,  $\operatorname{Ind}(-\gamma, z) = -\operatorname{Ind}(\gamma, z)$ .

## 2 Sequences and limits

**Definition 2.1.** A *sequence of complex numbers* is an application of the form

$$\begin{aligned} \mathbb{N}_{\geq m} &\rightarrow \mathbb{C} \\ n &\mapsto z_n \end{aligned} \quad (24)$$

We denote it by  $\{z_n\}_{n=m}^\infty$

**Definition 2.2.** Let  $\{z_n\}_{n=0}^\infty$  be a sequence. We say *the sequence has limit  $L$*  or *it converges to the limit  $L$*  if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - L| < \varepsilon \forall n > n_0. \quad (25)$$

We denote it by

$$\lim_{n \rightarrow \infty} z_n = L, \quad \lim_{n \rightarrow \infty} \{z_n\}_{n=0}^\infty = L, \quad \{z_n\}_{n=0}^\infty \rightarrow L. \quad (26)$$

**Theorem 2.1.** Let  $z_n = x_n + iy_n$  be the general term of a sequence  $\{z_n\}_{n=0}^\infty$  and  $L = L_x + iL_y \in \mathbb{C}$ . Then,

$$\{z_n\}_{n=0}^\infty \rightarrow L \Leftrightarrow \{x_n\}_{n=0}^\infty \rightarrow L_x \wedge \{y_n\}_{n=0}^\infty \rightarrow L_y. \quad (27)$$

**Definition 2.3.** Let  $\{z_n\}_{n=0}^{\infty}$  be a sequence. We say it *tends to infinity* and denote it by  $\lim z_n = \infty$  if and only if

$$\forall k \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n| \geq k, \forall n > n_0. \quad (28)$$

**Definition 2.4.** Let  $\{z_n\}_{n=0}^{\infty}$  be a sequence. We say it is a *Cauchy sequence* if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |z_n - z_m| < \varepsilon, \forall n, m > n_0. \quad (29)$$

**Theorem 2.2.** Let  $\{z_n\}_{n=0}^{\infty}$  be a convergent sequence. Then, it is a Cauchy sequence.

**Theorem 2.3.** Let  $z_n = x_n + iy_n$  be the general term of a sequence  $\{z_n\}_{n=0}^{\infty}$ . Then,

$$\{z_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence} \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \text{ are Cauchy sequences.} \quad (30)$$

**Theorem 2.4.** The field  $\mathbb{C}$  of complex numbers is complete.

**Definition 2.5.** The *Riemann sphere* is a one-dimensional complex manifold which is the one-point compactification of the extended complex numbers  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , together with two charts.

### 3 Functions

**Definition 3.1.** A *topology* is an ordered pair  $(X, \tau)$ , where  $X$  is a set and  $\tau$  a collection of subsets of  $X$  satisfying the following properties:

1. The empty set and  $X$  belong to  $\tau$ .
2. Any arbitrary (finite or infinite) union of members of  $\tau$  still belongs to  $\tau$ .
3. The intersection of any finite number of members of  $\tau$  still belongs to  $\tau$ .

The elements of  $\tau$  are called *open sets* and the collection  $\tau$  is called the *topology on  $X$* .

**Definition 3.2.** Let  $(X, d)$  be a metric space. A *topology on the metric space by the metric  $d$*  is the set  $\tau$  of all open sets of  $M$ .

**Definition 3.3.** Let  $A$  be a subset of a metric space  $(\mathbb{M}, d)$  and  $a$  a point in  $\mathbb{M}$ . We say that  $a$  is an *interior point of  $A$*  if there is a ball  $B_{(\mathbb{M}, d)}(a, r) \subset A$ .

**Definition 3.4.** Let  $A$  be a subset of a metric space  $(\mathbb{M}, d)$  and  $a$  a point in  $\mathbb{M}$ . We say that  $a$  is an *exterior point of  $A$*  if there is a ball such that  $B_{(\mathbb{M}, d)}(a, r) \cup A = \emptyset$ .

**Definition 3.5.** Let  $A$  be a subset of a metric space  $(\mathbb{M}, d)$  and  $a$  a point in  $\mathbb{M}$ . We say that  $a$  is a *boundary point of  $A$*  if it is not interior or exterior or, which is equivalent, if every ball  $B_{(\mathbb{M}, d)}(a, r)$  contains elements of  $A$  and  $A^c$ .

**Definition 3.6.** Let  $A$  be a subset of a metric space  $(\mathbb{M}, d)$  and  $a$  a point in  $\mathbb{M}$ . We say that  $a$  is an *accumulation point of  $A$*  if every ball with center  $a$  contains points of  $A$  different to  $a$ . In other words, every punctured ball satisfies  $B_{(\mathbb{M}, d)}^*(a, r) \cup A \neq \emptyset$ .

**Definition 3.7.** Let  $A$  be a subset of a metric space  $(\mathbb{M}, d)$ . We define the *interior of  $A$*  as the set of all interior points of  $A$ , and we denote it by  $\text{int}(A)$ .

**Definition 3.8.** Let  $A$  be a subset of a metric space  $(\mathbb{M}, d)$ . We define the *exterior of  $A$*  as the set of all exterior points of  $A$ , and we denote it by  $\text{ext}(A)$ .

**Definition 3.9.** Let  $A$  be a subset of a metric space  $(\mathbb{M}, d)$ . We define the *boundary of  $A$*  as the set of all boundary points of  $A$ , and we denote it by  $\partial A$ .

**Definition 3.10.** Let  $A$  be a subset of a metric space  $(\mathbb{M}, d)$ . We define the *closure of  $A$*  as the set of all accumulation points of  $A$ , and we denote it by  $\bar{A}$ .

**Definition 3.11.** Let  $(\mathbb{M}, d)$  be a metric space and  $A$  a subset of  $\mathbb{M}$ . We say  $A$  is an *open set* if it contains none of its boundary points, that is, if  $\partial A \cap A = \emptyset$ .

**Definition 3.12.** Let  $(\mathbb{M}, d)$  be a metric space and  $A$  a subset of  $\mathbb{M}$ . We say  $A$  is a *closed set* if it contains all its boundary points, that is, if  $\partial A \subseteq A$ .

**Definition 3.13.** Let  $(\mathbb{M}, d)$  be a metric space and  $A$  a subset of  $\mathbb{M}$ . We say  $A$  is a *bounded set* if there exist a point  $a \in \mathbb{M}$  and a positive real number  $r$  such that the ball  $B_{(\mathbb{M}, d)}(a, r)$  contains  $A$ .

**Definition 3.14.** Let  $(\mathbb{M}, d)$  be a metric space and  $A$  a subset of  $\mathbb{M}$ . We say  $A$  is a *compact set* if it is bounded and closed set.

**Proposition 3.1.** Let  $(\mathbb{M}, d)$  be a metric space and  $A$  a subset of  $\mathbb{M}$ . Then,  $A$  is open if and only if  $A^c$  is closed.

**Definition 3.15.** Let  $\Omega \subseteq \mathbb{C}$  be a set. We say  $\Omega$  is *connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets (in the topology of the subspace). More formally,  $\Omega$  is connected if there are not two open sets  $U, V \subseteq \mathbb{C}$  such that

$$U_1 = U \cap \Omega, \quad V_1 = V \cap \Omega, \quad U_1 \cap V_1 = \emptyset, \quad U_1 \cup V_1 = \Omega. \quad (31)$$

Otherwise, we say  $\Omega$  is *disconnected*.

**Definition 3.16.** Let  $\Omega \subseteq \mathbb{C}$  be a set. We say  $\Omega$  is *simply connected* if and only if every circuit is homotopic in  $\Omega$  to a point in  $\Omega$ . Equivalently,  $\Omega$  is simply connected if and only if every pair of curves with the same extremes are homotopic.

**Definition 3.17.** Let  $\Omega \subseteq \mathbb{C}$  be a set. We say  $\Omega$  is *convex* if and only if for all pair of point  $a, b \in \Omega$ , the segment defined by

$$[a, b] = \{z \mid z = (1 - t)a + tb, 0 \leq t \leq 1\} \quad (32)$$

is contained in  $\Omega$ , that is, if every pair of points can be connected by a straight line that belongs to the set.

**Definition 3.18.** Let  $\Omega \subseteq \mathbb{C}$  be a set. We say  $\Omega$  is a *star-convex set* if and only if there exists  $z_0 \in \mathbb{C}$  such that for all  $z \in \Omega$  the segment  $[z_0, z]$  is contained by  $\Omega$ .

**Definition 3.19.** Let  $(\mathbb{M}, d)$  be a metric space and  $S \subseteq \mathbb{M}$  a set. We say  $S$  is *path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

**Definition 3.20.** Let  $\Omega \subseteq \mathbb{C}$  be a set. We say  $\Omega$  is a *region or domain* if and only if it is open, non-empty, and connected.

**Definition 3.21.** Let  $\Omega \subseteq \mathbb{C}$  be a non-empty set. We say  $\Omega_1 \subseteq \Omega$  is a *connected component* of  $\Omega$  if and only if it is a maximal connected subset, that is, if  $z_0 \in \Omega_1$  and  $W$  is a connected subset of  $\mathbb{C}$  that contains  $z_0$ , then  $W \subseteq \Omega_1$ .

**Definition 3.22.** Let  $D \subseteq \mathbb{C}$  be a set. We define a *complex function*  $f$  as the application

$$f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C} \\ z \longmapsto w = f(z). \quad (33)$$

**Definition 3.23.** Let  $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$  be a function. We say it *tends to infinity at the point*  $z_0$  and denote it by  $\lim_{z \rightarrow z_0} f(z) = \infty$  if and only if

$$\forall k \in \mathbb{R}^+ \exists \delta(k) \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z)| > k. \quad (34)$$

**Definition 3.24.** Let  $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$  be a function. We write  $\lim_{z \rightarrow \infty} f(z) = L$  if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists k(\varepsilon) \in \mathbb{R}^+ \mid |z| > k \Rightarrow |f(z) - L| < \varepsilon. \quad (35)$$

**Proposition 3.2.** Let  $f_1, f_2 : \Omega \longrightarrow \mathbb{C}$  be two functions and  $z_0$  a point such that  $\lim_{z \rightarrow z_0} f_1 = w_1, \lim_{z \rightarrow z_0} f_2 = w_2$ . Then,

1.  $f_1 + f_2$  has also a limit and  $\lim_{z \rightarrow z_0} f + g = w_1 + w_2$ .
2.  $f_1 f_2$  has also a limit and  $\lim_{z \rightarrow z_0} f g = w_1 w_2$ .
3. If  $w_2 \neq 0$ , then  $f/g$  has also a limit and  $\lim_{z \rightarrow z_0} f/g = w_1/w_2$ .
4. If  $h(z)$  is a continuous function defined on a neighborhood of  $w_1$ , then  $\lim_{z \rightarrow z_0} h(f_1(z)) = h(w_1)$ .

**Definition 3.25.** Let  $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$  be a function and  $z_0 \in \Omega$ . We say  $f$  is *continuous in*  $z_0$  if and only if

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \mid |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon. \quad (36)$$

**Proposition 3.3.** Let  $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$  be a function and  $z_0 \in \Omega$ . Then,  $f = \operatorname{Re}\{f\} + i \operatorname{Im}\{f\}$  is continuous at  $z_0$  if and only if  $\operatorname{Re}\{f\}$  and  $\operatorname{Im}\{f\}$  are continuous at  $z_0$ .

**Proposition 3.4.** Let  $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$  be a function and  $z_0 \in \Omega$ . Then,  $f$  is continuous at  $z_0$  if and only if for all sequence  $\{z_n\}_{n=1}^\infty$  of  $\Omega$  convergent at  $z_0$  it is true that the sequence  $\{f(z_n)\}_{n=1}^\infty$  converges to  $f(z_0)$ .

**Proposition 3.5.** Let  $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$  be two continuous function at a point  $z_0 \in \mathbb{C}$  and  $\lambda \in \mathbb{C}$ . Then,  $\lambda f$ ,  $f + g$ , and  $fg$  are continuous at  $z_0$ . The function  $f/g$  is continuous at  $z_0$  if  $g(z_0) \neq 0$ .

**Definition 3.26.** For all  $z \in \mathbb{C}$ , we define the *complex exponential function* as the following series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \quad (37)$$

**Proposition 3.6.** The radius of convergence of  $e^z$  is infinite.

**Proposition 3.7.**  $e^z = e^x$  for all  $x \in \mathbb{R}$ .

**Proposition 3.8.**  $e^{z+w} = e^z e^w$  for all  $z, w \in \mathbb{C}$ .

**Proposition 3.9.** For all  $z \in \mathbb{C}$  we have  $e^z \neq 0$ .

**Proposition 3.10.** The image of  $e^z$  is  $\mathbb{C}^*$ .

**Proposition 3.11.** The derivative of  $e^z$  is  $e^z$ .

**Proposition 3.12.**  $\overline{e^z} = e^{\bar{z}}$ .

**Proposition 3.13.**  $|e^z| = e^{\operatorname{Re}\{z\}}$ .

**Proposition 3.14** (Euler's Formula). If  $\theta \in \mathbb{R}$ , then  $e^{xi}$  has modulus one and we have that

$$\boxed{e^{xi} = \cos x + i \sin x.} \quad (38)$$

**Corollary 3.15.** Let  $z \in \mathbb{C}^*$ . Then,

$$z = |z|e^{i\theta}, \quad (39)$$

with  $\theta \in [0, 2\pi)$ .

**Proposition 3.16.** The following function

$$\exp : (\mathbb{R}, +) \longrightarrow (\mathbb{C}^*, \cdot) \\ x \longmapsto e^{xi} \quad (40)$$

is a morphism of groups and its image is  $\mathbb{T}$ .

**Proposition 3.17.** The complex exponential function is a periodic function with period  $2\pi i$ .

**Proposition 3.18.** Let  $a \in \mathbb{C}^*$ . Then,  $e^z = a$  has infinite solutions.

**Proposition 3.19.** The equation  $e^z = 0$  does not have solutions.

**Proposition 3.20.** Let  $y_0 \in \mathbb{C}$  be a numbers,  $B := \{z \in \mathbb{C} \mid y_0 < \operatorname{Im}\{z\} < y_0 + 2\pi\}$  a set, and  $f : B \longrightarrow \mathbb{C}^*$  be the exponential function. Then,  $f$  is bijective in  $B$  ?

**Proposition 3.21.** Let  $x_0, y_0, m \in \mathbb{C}$  be two numbers with  $m \neq 0$  and  $f$  the exponential function ?. Then,

1.  $f$  transforms the line  $y = y_0$  to a line that starts at  $z = 0$  and continues with an argument  $y_0$  from the real positive axis.
2.  $f$  transforms the line  $x = x_0$  to a circle centered at the origin and radius  $r = e^{x_0}$ .

3.  $f$  transforms the line  $y = mx$  to the parametric curve  $z = e^x e^{imx}$  (a spiral).

**Definition 3.27.** Let  $z \in \mathbb{C}$  be a number. We define the complex trigonometric functions as

$$\cos z := \frac{e^{zi} + e^{-zi}}{2}, \quad (41)$$

$$\sin z := \frac{e^{zi} - e^{-zi}}{2i}, \quad (42)$$

$$\tan z := \frac{e^{zi} - e^{-zi}}{e^{zi} + e^{-zi}}. \quad (43)$$

**Proposition 3.22.** For all  $z \in \mathbb{C}$ ,

$$\sin^2 z + \cos^2 z = 1. \quad (44)$$

**Proposition 3.23.** For all  $z \in \mathbb{C}$ ,

$$\cos(-z) = \cos(z), \quad \sin(-z) = -\sin(z). \quad (45)$$

**Proposition 3.24.** For all  $z, w \in \mathbb{C}$ ,

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \quad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \quad (46)$$

**Proposition 3.25.** The functions  $\cos z, \sin z$  have period of  $2\pi$ .

**Proposition 3.26.** Let  $z_0 \in \mathbb{C}$ . Then,  $z_0$  is root of  $\sin z$  ( $\cos z$ ) if and only if it is a root of  $\sin x$  ( $\cos x$ ).

**Definition 3.28.** Let  $z \in \mathbb{C}$  be a number. We define the complex hyperbolic functions as

$$\cosh z := \frac{e^z + e^{-z}}{2}, \quad (47)$$

$$\sinh z := \frac{e^z - e^{-z}}{2}, \quad (48)$$

$$\tanh z := \frac{e^z - e^{-z}}{e^z + e^{-z}}. \quad (49)$$

**Proposition 3.27.** For all  $z \in \mathbb{C}$ ,

$$\cosh^2 z - \sinh^2 z = 1. \quad (50)$$

**Proposition 3.28.** For all  $z \in \mathbb{C}$ ,

$$\cosh(-z) = \cosh(z), \quad \sinh(-z) = -\sinh(z). \quad (51)$$

**Proposition 3.29.** For all  $z, w \in \mathbb{C}$ ,

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \quad (52)$$

$$\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \quad (53)$$

**Proposition 3.30.** For all  $z \in \mathbb{C}$ ,

$$\cosh z = \cos(iz), \quad \cos z = \cosh(iz) \quad (54)$$

$$\sinh z = -i \sin(iz), \quad \sin z = -i \sinh(iz) \quad (55)$$

**Proposition 3.31.** For all  $z = x + iy \in \mathbb{C}$ ,

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \quad (56)$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \quad (57)$$

$$\tan(x + iy) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i \frac{\sinh y}{\cos(2x) + \cosh(2y)}. \quad (58)$$

**Proposition 3.32.** For all  $z = x + iy \in \mathbb{C}$ ,

$$\tanh(x + iy) = \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)} + i \frac{\sin(2y)}{\cosh(2x) + \cos(2y)}. \quad (59)$$

**Proposition 3.33.** For all  $z = x + iy$ ,

$$|\cos z| = \sqrt{\cosh^2 y - \sin^2 x} = \sqrt{\cos^2 x + \sinh^2 y}, \quad (60)$$

$$|\sin z| = \sqrt{\sinh^2 x + \sin^2 y} = \sqrt{\cosh^2 x - \cos^2 y}. \quad (61)$$

**Corollary 3.34.** For all  $z = x + iy$ ,

$$|\sinh y| \leq |\cos z| \leq \cosh y, \quad |\sinh y| \leq |\sin z| \leq \cosh y. \quad (62)$$

**Proposition 3.35.** The roots of the function  $\sinh z$  are of the form  $z_n = n\pi$  and those for the function  $\cosh z$  are of the form  $w_n = (2n + 1)\pi/2i$ .

**Definition 3.29.** Let  $D \subseteq \mathbb{C}$  be a set. We define a *multivalued function* from  $D$  to  $\mathbb{C}$  as a subset of  $D \times \mathbb{C}$  such that for every  $z \in D$  there exists a number  $y \in \mathbb{C}$  such that  $(z, y) \in f$ .

**Definition 3.30.** For  $z \in \mathbb{C}^*$ , we call the *natural logarithm* of  $z$  every number  $w$  such that  $e^w = z$ , that is,

$$\ln z := \{w \in \mathbb{C} \mid e^w = z\}. \quad (63)$$

**Proposition 3.36.** Given  $z \in \mathbb{C}$  we can define  $\ln z$  from the natural logarithm of a real number as

$$\ln z = \ln |z| + i \arg z = \ln |z| + i \operatorname{Arg} z + 2\pi ki. \quad (64)$$

**Definition 3.31.** We define the *principal natural logarithm* of  $z$  as the value defined by the principal argument of  $z$ , that is,

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z. \quad (65)$$

**Definition 3.32.** We define the *determination  $I$*  (with  $I$  being a semiopen interval) of the logarithm as

$$\log_I z := \ln |z| + i \arg_I z. \quad (66)$$

**Definition 3.33.** Let  $E \subseteq \mathbb{C}^*$  be a connected set. We define the *continuous determination of the logarithm in  $E$*  as the continuous function  $g : E \rightarrow \mathbb{C}$  such that  $e^{g(z)} = z$ . More generally, if  $f : E \rightarrow \mathbb{C}$  is a function such that  $f(z) \neq 0$  for all  $z \in E$ , then we define the *continuous determination of  $\ln f$*  as a function  $g : E \rightarrow \mathbb{C}$  such that  $e^{g(z)} = f(z)$ .

**Proposition 3.37.** Let  $z, w \in \mathbb{C}$  two numbers. Then,

$$1. \ln(zw) = \ln z + \ln w + 2\pi ki, \quad k \in \mathbb{Z}.$$

2. If we want to stay in the principal argument,

$$\ln(zw) = \begin{cases} \ln z + \ln w, & \text{if } \operatorname{Arg} z + \operatorname{Arg} w < 2\pi \\ \ln z + \ln w - 2\pi i, & \text{if } \operatorname{Arg} z + \operatorname{Arg} w \geq 2\pi \end{cases}. \quad (67)$$

3. SEARCH MORE PROPERTIES

**Definition 3.34.** Let  $z \in \mathbb{C}$  be a number. We define the *complex trigonometric inverse functions* as

$$\arcsin z := -i \ln \left( iz + \sqrt{1 - z^2} \right), \quad (68)$$

$$\arccos z := -i \ln \left( z + \sqrt{z^2 - 1} \right), \quad (69)$$

$$\arctan z := -\frac{i}{2} \ln \frac{1 + iz}{1 - iz}. \quad (70)$$

**Definition 3.35.** Let  $z \in \mathbb{C}$  be a number. We define the *complex hyperbolic inverse functions* as

$$\operatorname{arcsinh} z := \ln \left( z + \sqrt{1 + z^2} \right), \quad (71)$$

$$\operatorname{arccosh} z := \ln \left( z + \sqrt{z^2 - 1} \right), \quad (72)$$

$$\operatorname{artanh} z := \frac{1}{2} \ln \frac{1 + z}{1 - z}. \quad (73)$$

**Definition 3.36.** Let  $z, a \in \mathbb{C}$  with  $z \neq 0$ . Then, we define the *complex power function* as

$$z^a := e^{a \ln z}. \quad (74)$$

If  $E \subseteq \mathbb{C}^*$  is a connected set and  $f : E \rightarrow \mathbb{C}$  a function such that  $f(z) \neq 0$  for all  $z \in E$ , and  $w \in \mathbb{C}$  a number, we define a *continuous determination of  $f^w$*  as a continuous function  $g : E \rightarrow \mathbb{C}$  such that  $g(z) \in [f(z)]^w$ .

**Proposition 3.38.** If  $a = \alpha + \beta i$  and  $z = re^{\theta i}$ , then

$$z^a = e^{\alpha \ln r - \beta(\theta + 2\pi k)} e^{(\beta \ln r + \alpha(\theta + 2\pi k))i}, \quad (75)$$

$$|z^a| = e^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \quad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k). \quad (76)$$

**Proposition 3.39.** Let  $a, z \in \mathbb{C}$  be two numbers. Then,

1. If  $a = n \in \mathbb{Z}$ , the complex power is a function and

$$z^n = r^n e^{n\theta i}. \quad (77)$$

2. If  $a = n/m \in \mathbb{Q}$ , there are  $n$  values and

$$z^a = \sqrt[m]{r^n} e^{(\theta + 2\pi k)n/mi}. \quad (78)$$

3. If  $a$  is irrational, the norm is uniquely determined but the argument has infinite values.
4. If  $a \in \mathbb{C} \setminus \mathbb{R}$ , the argument is uniquely determined and the norm has infinite values.

**Proposition 3.40.** Let  $z, w \in \mathbb{C}$ . Then,

1.  $(e^b)^a = e^{a(b + 2\pi ki)}$

**Definition 3.37.** A *Riemann surface*  $X$  is a connected complex 1-manifold.

**Definition 3.38.** We define a *sheet* as each of the complex planes of the Riemann surface.

**Definition 3.39.** We define a *cut* as the line (not necessarily straight) of union between sheets.

**Definition 3.40.** We define a *branch point* as a point where start or finish a cut.

## 4 Derivatives

**Definition 4.1.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function and  $z_0 \in \Omega$  an interior point. We define the *derivative of  $f$  at  $z_0$*  as

$$f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (79)$$

in case the limit exists. If  $f$  has derivative, we say  $f$  is *derivable at  $z_0$* .

**Definition 4.2.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function and  $z_0 \in \mathbb{C}$  a point. We say  $f$  is *holomorphic at  $\Omega$*  if and only if it is  $\mathbb{C}$ -derivable at every point of  $\Omega$ . In that case, it is defined the function  $f' : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  that associates each point  $z$  of  $\Omega$  with  $f'(z)$ .

**Definition 4.3.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function. We define the *domain of holomorphism* as the region where  $f$  is derivable. We say  $f$  is *entire* if and only if the domain of holomorphism is  $\mathbb{C}$ .

**Definition 4.4.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function and  $z_0 \in \mathbb{C}$  a point. We say  $f$  is *holomorphic at  $z_0$*  if and only if it is holomorphic at some neighborhood of  $z_0$ .

**Proposition 4.1.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function and  $z_0 \in \mathbb{C}$  a point. If  $f$  is derivable at  $z_0$ , then it is continuous at  $z_0$ .

**Theorem 4.2.** Let  $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be two functions and  $z_0 \in \Omega$  a point. Then, the following statements are true.

1. If  $f$  is constant at  $\Omega$ , then  $f$  is derivable at  $z_0$  and  $f'(z_0) = 0$ .
2. If  $f(z) = z$  in every point of  $\Omega$ , then  $f$  is derivable at  $z_0$  and  $f'(z_0) = 1$ .
3. If  $f, g$  are derivable at  $z_0$  and  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha f + \beta g$  are derivable at  $z_0$  and  $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$ .
4. If  $f, g$  are derivable at  $z_0$ , then  $fg$  is derivable at  $z_0$  and

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \quad (80)$$

5. If  $f, g$  are derivable at  $z_0$  and  $g(z_0) \neq 0$ , then  $f/g$  is derivable at  $z_0$  and

$$\left( \frac{f}{g} \right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \quad (81)$$

**Theorem 4.3.** Let  $f : \Omega_1 \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a derivable function at a point  $z_0 \in \mathbb{C}$  and  $g : \Omega_2 \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be another derivable function at a point  $f(z_0) \in \Omega_2$ . Then,  $g \circ f$  is derivable at  $z_0$  and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0). \quad (82)$$

**Theorem 4.4.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a function of class  $C^1(\Omega)$  with  $\Omega$  an open set, injective, and derivable at every point of  $\Omega$  with non-zero derivative. Then,

1.  $\Omega' = f(\Omega)$  is an open subset of  $\mathbb{C}$ .
2. The inverse function  $f^{-1}$  exist, it is well defined and it is derivable at  $\Omega'$ .
3. If  $z \in \Omega$  and  $z' = f(z)$ , then

$$(f^{-1})'(z') = \frac{1}{f'(z)}. \quad (83)$$

**Proposition 4.5.** A determination of  $\ln z$  with  $z \in \mathbb{C}$  is continuous except in a semiline.

**Theorem 4.6.** Let  $\Omega \subseteq \mathbb{C}$  be an open set and  $\phi \in C(\Omega)$ , such that  $e^{\phi(z)} = z$  for all  $z \in \Omega$ . Then, we have  $\phi \in H(\Omega)$  and

$$\phi'(z) = \frac{1}{z}, \forall z \in \Omega. \quad (84)$$

**Proposition 4.7.** A determination of  $\ln z$  with  $z \in \mathbb{C}$  is holomorphic except in a semiline.

**Proposition 4.8.** Let  $I = [\theta, \theta + 2\pi)$  a determination of the logarithm,  $\ln_I z$ . Then,  $\ln_I z$  is holomorphic except in the semiline  $L_\theta = \{re^{i\theta} \in \mathbb{C} \mid r \geq 0\}$ .

**Definition 4.5.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function. We say  $f$  is of class  $C^1(\Omega)$  or simply  $f \in C^1(\Omega)$  if and only if, using  $f = u + iv$  with  $u = \operatorname{Re}\{f\}$ ,  $v = \operatorname{Im}\{f\}$ , the partial derivatives of  $u$  and  $v$  as a two variable real functions exist and are continuous. In other words,  $f \in C^1(\Omega)$  if and only if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \quad (85)$$

exist and are continuous.

**Theorem 4.9** (Cauchy-Riemann conditions). Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function and  $z_0 \in \Omega$  an interior point. Then,  $f$  is derivable at  $z_0$  if and only if is differentiable at  $z_0$  and  $df(z_0)$  is  $\mathbb{C}$ -linear, that is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad (86)$$

which are known as Cauchy-Riemann conditions.

**Theorem 4.10.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function and  $z_0 \in \Omega$  an interior point. If  $u, v$  satisfy the Cauchy-Riemann equation and their partial derivatives are continuous, then  $f$  is derivable.

**Definition 4.6.** We define the operators

$$\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad (87)$$

that act over the functions such that the real and imaginary part  $u, v$  have partial derivatives.

**Proposition 4.11.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a function of class  $C^1(\Omega)$ . Then, for all  $z_0 \in \Omega$

$$f(z_0 + h) = f(z_0) + \left( \frac{\partial f}{\partial z} \right)_{z_0} h + \left( \frac{\partial f}{\partial \bar{z}} \right)_{z_0} \bar{h} + o(|h|^2). \quad (88)$$

**Theorem 4.12.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function and  $z_0$  an interior point. Then, at  $z_0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}. \quad (89)$$

## 5 Series

**Definition 5.1.** We say  $\sum_{n=1}^{\infty} z_n$  converges if and only

if  $S_n := \sum_{n=1}^N z_n$  has limit at  $n \rightarrow \infty$ .

**Proposition 5.1.**  $\sum_{n=1}^{\infty} z_n$  converges if and only if

$\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  converge.

**Definition 5.2.** We say  $\sum_{n=1}^{\infty} z_n$  converges absolutely if

and only if  $\sum_{n=1}^{\infty} |z_n|$  converges.

**Proposition 5.2.**  $\sum_{n=1}^{\infty} |z_n|$  converges if and only if

$\sum_{n=1}^{\infty} |a_n|$  and  $\sum_{n=1}^{\infty} |b_n|$  converge.

**Proposition 5.3.** 1. A series converges absolutely with sum  $S$  if and only if every rearrangement is convergent with the same sum  $S$ .

2. An absolutely convergent series can be summed by blocks in an arbitrary way.

**Proposition 5.4.** Let  $\sum_n a_n, \sum_n b$  be two absolutely convergent series with sums  $A$  and  $B$  respectively.

Then, the series  $\sum_k c_k$  with  $c_k = \sum_{n=0}^k a_n b_{k-n}$  is absolutely convergent with sum  $AB$ .

**Theorem 5.5** (Weierstrass M-test). If  $|f_n(p)| < M_n$  for all  $p \in X, n \geq 1$  and  $\sum_{n=0}^{\infty} M_n < \infty$ , then the series

$\sum_{n=0}^{\infty} f_n(p)$  is uniformly convergent on  $X$ .

**Lemma 5.6** (Abel's summation formula). Let  $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$  be two sequences of complex numbers and  $A_n = a_1 + \dots + a_n$ . Then,

$$\sum_{k=1}^n a_k b_k = A_n b_{n+1} - \sum_{k=1}^n A_k (b_{k+1} - b_k). \quad (90)$$

**Theorem 5.7** (Dirichlet's criteria). Let  $\sum_{n=1}^{\infty} f_n(p)g_n(p)$  be a series where  $f_n(p)$  are complex and  $g_n(p)$  are real for all  $p \in X, n \geq 1$ . If we denote  $F_n(p) = f_1(p) + \dots + f_n(p)$ , there exists a constant  $M$  such that  $|F_n(p)| \leq M$  for all  $n \geq 1, p \in X$ ,  $g_n(p)$  is monotonous decreasing and converges uniformly to zero on  $X$ , then the series  $\sum_{n=1}^{\infty} f_n(p)g_n(p)$  is uniformly convergent on  $X$ .

**Theorem 5.8** (Abel's criteria). Let  $\sum_{n=1}^{\infty} f_n(p)g_n(p)$  be a series where  $f_n(p), g_n(p)$  are complex. If  $\sum_{n=1}^{\infty} f_n(p)$  is uniformly convergent on  $X$  and there exists a number  $M \in \mathbb{R}^+$  such that for all  $p \in X$

$$|g_1(p)| + \sum_{n=1}^{\infty} |g_n(p) - g_{n+1}(p)| \leq M, \quad (91)$$

then the series  $\sum_{n=1}^{\infty} f_n(p)g_n(p)$  is uniformly convergent on  $X$ .

**Definition 5.3.** We define a *complex power series* as a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad a_n, z, z_0 \in \mathbb{C}. \quad (92)$$

We call the term  $a_n$  the  $n$ -th coefficient of the series. In case  $a_n = 0 \forall n \leq m$ , we will start the counting directly from  $m$ .

**Definition 5.4.** Let  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  be a power series. We define its *domain of convergence* as

$$E := \left\{ z \in \mathbb{C} \mid \sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ converges} \right\}. \quad (93)$$

**Theorem 5.9.** Let  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  be a power series and  $R = 1/\rho$ , where  $\rho = \limsup_n |a_n|^{1/n}$ . Then, the series converges uniformly on the compacts of the open disc  $D(z_0, R)$ , converges absolutely at every point  $z \in D$  and diverges outside  $\bar{D}$ . Hence, the set of converges  $E$  satisfies  $D \subseteq E \subseteq \bar{D}$  and  $D = \text{int} E$ .

**Definition 5.5.** Radius of convergence.

**Proposition 5.10.** Let  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  be a power series and  $R = \lim_{n \rightarrow \infty} |a_n|/|a_{n+1}|$ . If the limit exists, then  $R$  is the radius of convergence.

**Theorem 5.11** (Cauchy-Hadamard Theorem). Let

$$S = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad (94)$$

be the following complex power series with  $a_n, z_0 \in \mathbb{C}$  and  $R \in [0, +\infty) \cup \{+\infty\}$  the radius of convergence. Then,

1. If  $|z - z_0| < R$  then  $S$  converges. In fact, for all  $r < R$  we have  $S$  converges uniformly at the disc  $\overline{D_r(z_0)}$ .
2. If  $|z - z_0| > R$  then  $S$  diverges.
3. The function  $f(z) = S(z)$  is derivable at  $D_R(z_0)$  and its formal derivative is

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \quad (95)$$

with the same radius of convergence.

**Definition 5.6.** Let  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  be a series,  $S = E \cap C(z_0, R)$  non empty, and  $m > 1$  a real number. We define

$$S_m := \{z \in \mathbb{C} \mid |z - z_0| < R, d(z, S) \leq m(R - |z - a|)\}. \quad (96)$$

**Definition 5.7** (Stolz angle). Let  $S$  be formed by one point  $w$ . We define the *Stolz angle* as the angle generated by the  $S_m$ .

**Theorem 5.12** (Abel's theorem). Let  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  be a series with  $S$  non empty and such that the series converges uniformly on it. Then, the series converges uniformly on  $S_m$  for all  $m > 1$ . In particular, the sum function is continuous on  $S_m$  and one has

$$\lim_{z \rightarrow w, z \in S_m} \sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} a_n(w - z_0)^n, \quad w \in S. \quad (97)$$

**Theorem 5.13.** Let  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  be a series with radius of convergence  $R$ . Then,  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  is holomorphic on  $D(a, R)$  and it has a derivative

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \quad \forall z \in D. \quad (98)$$

**Proposition 5.14.** Let  $f : D(a, R) \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function. If there exists a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ , convergent on  $D$  such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad |z - z_0| < R, \quad (99)$$

then the series is unique. In fact,  $f$  is infinitely holomorphic and the coefficients  $a_n$  are determined by  $f$  with the relation

$$a_n = \frac{f^n(z_0)}{n!}, \quad n \in \mathbb{N}. \quad (100)$$

**Definition 5.8.** Let  $f : D(a, R) \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function. We say  $f$  admits a *series expansion* if and only if there exists a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ , convergent on  $D$  such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad |z - z_0| < R. \quad (101)$$



**Definition 5.9.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function with  $\Omega$  an open set. We say  $f$  is *analytic on*  $\Omega$  if and only if it admits locally a series expansion, that is, if for every point  $z_0 \in \Omega$  there exists a disc  $D(z_0, \delta)$  and a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \forall z \in D.$$

**Theorem 5.15.** Let  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  on  $D(z_0, R)$  and  $w_0 \in D(z_0, R_0)$ . Then, the series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!} (z - z_1)^n$  has a radius of convergence  $R_1 \geq R_0 - |z_0 - z_1|$  and it satisfies

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!} (z - z_1)^n, \quad \text{if } |z - z_1| < R - |z_0 - z_1|. \quad (102)$$

**Corollary 5.16.** Let  $R$  be the radius of convergence of the function

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Then  $f$  has as Taylor polynomial of degree  $m$  around  $z_0$  the following one

$$P_{m,f,z_0}(z) = \sum_{n=0}^m a_n(z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}. \quad (103)$$

**Proposition 5.17.** Let  $\emptyset \neq \Omega \subseteq \mathbb{C}$ . Then,

1. Every connected component of  $\Omega$  is a closed of  $\Omega$  with a subspace topology.
2. Two connected components are the same or are disjoint.
3. Every connected of  $\Omega$  is one and only one connected component.
4.  $\Omega$  is the disjoint union of its connected components.

## 6 Holomorphic functions

**Definition 6.1.** Let  $I = [a, b] \subseteq \mathbb{R}$  be a closed interval. We define a *curve* as an application of the form

$$\gamma : I \rightarrow \mathbb{C} \\ t \mapsto \gamma_1(t) + i\gamma_2(t). \quad (104)$$

**Definition 6.2.** Let  $I = [a, b] \subseteq \mathbb{R}$  be a closed interval and  $D \subseteq \mathbb{C}$  a domain. We define an *arc* as a continuous application of the form

$$\gamma : I \rightarrow D \\ t \mapsto \gamma_1(t) + i\gamma_2(t). \quad (105)$$

Equivalently, we can say an arc is a curve restricted to some interval.

**Definition 6.3.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We call  $\gamma(a)$  and  $\gamma(b)$  the *extremes of*  $\gamma$ . In particular, we call  $\gamma(a)$  the *initial point* and  $\gamma(b)$  the *final point*.

**Definition 6.4.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We define the *route or graph of*  $\gamma$  as

$$\gamma^* := \{z \in D \mid z = \gamma(t), t \in I\}. \quad (106)$$

**Definition 6.5.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We say  $\gamma$  is *closed* if and only if  $\gamma(a) = \gamma(b)$ .

**Definition 6.6.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We say  $\gamma$  is *simple* if and only if there is no two numbers  $t_1, t_2 \in (a, b)$  such that  $\gamma(t_1) = \gamma(t_2)$ . We also call it a *Jordan curve*, and if it is closed, a *circuit*.

**Definition 6.7.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We say  $\gamma$  is *differentiable* if for al value  $t_0 \in [a, b]$  there exists the limit

$$\gamma'(t_0) = \lim_{t \rightarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}. \quad (107)$$

For  $t_0 = a$  or  $t_0 = b$  we consider the laterals limits from the right and from the left respectively.

**Definition 6.8.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We say  $\gamma$  is of *class*  $C^1$  if and only if  $\gamma'$  exists and is continuous at  $[a, b]$ .

**Definition 6.9.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We say  $\gamma$  is *regular or smooth* if and only if it is differentiable and  $\gamma'$  never vanishes.

**Definition 6.10.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We say  $\gamma$  is *piece-wise of class*  $C^1$  if and only if  $\gamma'$  exists and is continuous in  $I$  except in a finite number of points where  $\gamma$  has lateral derivatives.

**Definition 6.11.** Let  $\gamma : [a, b] \rightarrow D$  be an arc. We define the *opposite arc* as

$$-\gamma : [-b, -a] \rightarrow \mathbb{C} \\ t \mapsto \gamma(-t). \quad (108)$$

**Definition 6.12.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be an arc. We say  $\Gamma(s), s \in [c, d] \subseteq \mathbb{R}$  has been obtained from  $\gamma(t), t \in [a, b]$  by a *change of parametrization* if and only if the new parameter  $s$  and the original parameter  $t$  are related by a relation  $t = \phi(s)$ , where  $\phi : [c, d] \rightarrow [a, b]$  is an homeomorphism that satisfies  $\Gamma(s) = \gamma(\phi(s)) = (\gamma \circ \phi)(s)$ . We call  $\Gamma$  the *reparametrization of*  $\gamma$ .

**Definition 6.13.** Let  $\gamma_1 : I_1 \rightarrow \mathbb{C}$  and  $\gamma_2 : I_2 \rightarrow \mathbb{C}$  be two arcs. We say they are *equivalent* if and only if there exists a bijective, monotone, and continuous function  $\rho : I_2 \rightarrow I_1$  such that  $\gamma_2 = \gamma_1 \circ \rho$ . If  $\rho$  is an increasing function we say  $\gamma_1$  and  $\gamma_2$  have the *same orientation*; otherwise, we say  $\gamma_1$  and  $\gamma_2$  have *opposite orientations*.

**Definition 6.14.** Let  $\gamma_1 : [a, b] \rightarrow \mathbb{C}$  and  $\gamma_2 : [c, d] \rightarrow \mathbb{C}$  be two arcs such that  $[a, b] \cap [c, d] = \emptyset$ . We define the application  $\gamma_1 \cup \gamma_2$  (sometimes denoted by  $\gamma_1 + \gamma_2$ ) as

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t), & \text{if } a \leq t \leq b \\ \gamma_2(t - b + c), & \text{if } b \leq t \leq b + d - c \end{cases} \quad (109)$$

We say  $\gamma_1, \gamma_2$  can be joined/added or that there exists its union/sum if and only if  $\gamma_1(b) = \gamma_2(c)$ . In this case  $\gamma_1 + \gamma_2$  is an arc, and we call it *the sum arc of  $\gamma_1$  plus  $\gamma_2$* .

**Definition 6.15.** We define the *segment of extremes*  $z_1, z_2 \in \mathbb{C}$  as the arc defined by the expression

$$\begin{aligned} [z_1, z_2] : [0, 1] &\rightarrow \mathbb{C} \\ t &\mapsto (1 - t)z_1 + tz_2. \end{aligned} \quad (110)$$

**Definition 6.16.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function. We say  $f$  is *polygonal* if and only if it can be expressed as a finite union of segments, that is, if there exist a natural number  $n$  and points  $\{z_0, \dots, z_n\}$  such that

$$p = [z_0, z_1] \cup \dots \cup [z_{n-1}, z_n]. \quad (111)$$

**Definition 6.17.** Let  $\gamma : [a, b] \rightarrow D$  be an arc with  $a, b$  finite. We say  $\gamma$  is a *basic curve* if and only if  $\gamma \in C^1((a, b)) \cap C([a, b])$  and there exist  $\lim_{t \rightarrow a^+} \gamma'(t)$ ,  $\lim_{t \rightarrow b^-} \gamma'(t)$ .

**Definition 6.18.** A *path* is a function  $\gamma : [a, b] \rightarrow \mathbb{C}$  such that there exist basic curves  $\gamma_j : [a_j, b_j] \rightarrow \mathbb{C}$ ,  $j \in \{1, \dots, k\}$  such that  $\gamma = \gamma_1 + \dots + \gamma_k$  and therefore  $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$  and  $a = a_1, b = a_k$ .

**Definition 6.19.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a continuous curve and  $a_1, \dots, a_l \in \mathbb{R}$  such that  $a = a_0 \leq \dots \leq a_l \leq b = a_{l+1}$ . We say  $\gamma$  is *piece-wise differentiable* if and only if

$$\gamma \in C^1 \left( \bigcup_{j=0}^l (\alpha_j, \alpha_{j+1}) \right),$$

$$\forall j \in \{0, \dots, l+1\} \exists \lim_{t \rightarrow a_j^+} \gamma'(t) \text{ (except if } j = l+1), \lim_{t \rightarrow a_j^-} \gamma'(t) \text{ (except if } j = 0).$$

Equivalently, we can think about a piece-wise differentiable curve as a differentiable path.

**Theorem 6.1.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function of class  $C^1(\Omega)$  with  $\Omega$  an open set and  $\phi : I \rightarrow \Omega$  a basic curve. Then,  $\psi = f \circ \phi$  is a basic curve (hence a derivable curve) and its real derivative is

$$\psi'(t) = f'(\phi(t))\phi'(t). \quad (112)$$

**Definition 6.20.** Let  $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbb{C}$  be two curves. We say  $\gamma_1, \gamma_2$  are *homotopic* if and only if there exists a continuous function  $h(t, s) : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$  such that

1.  $h(t, 0) = \gamma_1(t), t \in [0, 1]$ .
2.  $h(t, 1) = \gamma_2(t), t \in [0, 1]$ .

$$3. h(0, s) = \gamma_1(0) = \gamma_2(0), s \in [0, 1].$$

$$4. h(1, s) = \gamma_1(1) = \gamma_2(1), s \in [0, 1].$$

**Definition 6.21.** Let  $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbb{C}$  be two circuits. We say  $\gamma_1, \gamma_2$  are *homotopic* if and only if there exists a continuous function  $h(t, s) : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$  such that

$$1. h(t, 0) = \gamma_1(t), t \in [0, 1].$$

$$2. h(t, 1) = \gamma_2(t), t \in [0, 1].$$

$$3. h(0, s) = h(1, s), s \in [0, 1].$$

**Definition 6.22.** Let  $f : [a, b] \rightarrow \mathbb{C}$  be a function with the notation  $f = u + iv$ . We define the integral of  $f$  as

$$\int_a^b f(t) dt := \int_a^b u(t) dt + i \int_a^b v(t) dt. \quad (113)$$

**Proposition 6.2.** Let  $f, g : [a, b] \rightarrow \mathbb{C}$  be two integrable functions and  $\lambda, \mu \in \mathbb{C}$  two numbers. Then,

$$\int_a^b \lambda f + \mu g dt = \lambda \int_a^b f dt + \mu \int_a^b g dt. \quad (114)$$

**Proposition 6.3.** Let  $f : [a, b] \rightarrow \mathbb{C}$  be an integrable function. Then,

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt. \quad (115)$$

**Definition 6.23.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve of class  $C^1([a, b])$  and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a continuous function in  $\Gamma \subseteq \Omega$ . Then, we define the *line integral of  $f$  over  $\gamma$*  as

$$\int_\gamma f(z) dz := \int_a^b f(\gamma(t))\gamma'(t) dt. \quad (116)$$

**Proposition 6.4.** The previous definition is well defined.

**Proposition 6.5.** If we use the notation  $f = u + iv$  and  $\gamma = x + iy$ , then the integral has the form

$$\int_\gamma f = \int_a^b u \frac{dx}{dt} + v \frac{dy}{dt} dt + i \int_a^b v \frac{dx}{dt} + u \frac{dy}{dt} dt. \quad (117)$$

**Definition 6.24.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve of class  $C^1([a, b])$  and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a continuous function in  $\Gamma \subseteq \Omega$ . Then, we define the *line integral of  $f$  over  $\gamma$  with respect the differential of length* as

$$\int_\gamma f(z) ds := \int_\gamma f(z) |dz| = \int_a^b f(\gamma(t)) |\gamma'(t)| dt. \quad (118)$$

**Theorem 6.6.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve of class  $C^1([a, b])$ ,  $f, g : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  two functions, and  $\lambda, \mu \in \mathbb{C}$  two numbers. Then,

$$\int_{\gamma} \lambda f + \mu g \, dz = \lambda \int_{\gamma} f \, dz + \mu \int_{\gamma} g \, dz. \quad (119)$$

**Theorem 6.7.** Let  $\gamma_1, \gamma_2$  be two equivalent curves of the same orientation and of class  $C^1$  on their respective domains and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a continuous function in  $\Gamma_1, \Gamma_2 \subseteq \Omega$ . Then,

$$\int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz. \quad (120)$$

**Proposition 6.8.** Let  $\gamma_1, \dots, \gamma_n$  be  $n$  curves of class  $C^1$  on their respective domains and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a continuous function in  $\Gamma_1, \dots, \Gamma_n \subseteq \Omega$ . If we define  $\gamma = \gamma_1 + \dots + \gamma_n$ , then

$$\int_{\gamma} f(z) \, dz = \sum_{i=1}^n \int_{\gamma_i} f(z) \, dz. \quad (121)$$

**Proposition 6.9.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve of class  $C^1([a, b])$  and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a continuous function in  $\Gamma \subseteq \Omega$ . Then,

$$\left| \int_{\gamma} f(z) \, dz \right| \leq \int_{\gamma} |f| \, ds. \quad (122)$$

**Corollary 6.10.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve of class  $C^1([a, b])$  and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a continuous function in  $\Gamma \subseteq \Omega$ . If  $|f(z)| \leq M$  for all  $z \in \Gamma$ , then,

$$\left| \int_{\gamma} f(z) \, dz \right| \leq ML(\gamma). \quad (123)$$

**Proposition 6.11.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve of class  $C^1([a, b])$  and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a continuous function in  $\Gamma \subseteq \Omega$ . Then,

$$\text{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} \, dw. \quad (124)$$

**Proposition 6.12.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve of class  $C^1([a, b])$  and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  a continuous function in  $\Gamma \subseteq \Omega$ . Then,

$$|\text{Ind}(\gamma, z)| \leq \frac{1}{2\pi} \frac{L(\gamma)}{|z - \Gamma|}. \quad (125)$$

**Proposition 6.13.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a curve of class  $C^1([a, b])$  and  $\{f_n\}_{n=0}^{\infty}$  a sequence of continuous functions on  $\Gamma$  such that  $\sum_{n=0}^{\infty} f_n$  converges uniformly on  $\Gamma$ . Then,  $\sum_{n=0}^{\infty} \int_{\gamma} f_n \, dz$  converges and

$$\int_{\gamma} \sum_{n=0}^{\infty} f_n(z) \, dz = \sum_{n=0}^{\infty} \int_{\gamma} f_n(z) \, dz. \quad (126)$$

**Definition 6.25.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function. We say  $f$  has a *primitive* on  $\Omega$  if and only if there exists a function  $F : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  such that  $F' = f \forall z \in \Omega$ .

**Definition 6.26.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function. We say  $f$  has a *local primitive* on  $D$  if and only if for all  $z$  there exists a neighborhood where  $f$  has a primitive.

**Theorem 6.14** (Fundamental theorem of complex calculus). Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function with  $\Omega$  a domain. Then, the line integral of  $f$  is independent on the path on  $\Omega$  if and only if  $f$  has an holomorphic primitive  $F$  such that  $F' = f$  on  $\Omega$ . In that case,

$$\int_{\gamma} f(z) \, dz = F(\gamma(b)) - F(\gamma(a)). \quad (127)$$

**Theorem 6.15.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a continuous function on a star domain  $S \subseteq \Omega$ . Then,  $f$  has an holomorphic primitive  $F$  on  $S$  if and only if

$$\int_{\partial \Delta} f(z) \, dz = 0 \quad (128)$$

for all triangle  $\Delta \subseteq \Omega$ .

**Proposition 6.16.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function with no roots on a domain  $D \subseteq \Omega$ . Then, there is a determination of the logarithm of  $f$  on  $D$  if and only if  $f'/f$  has an holomorphic primitive on  $D$ .

**Proposition 6.17.** Let  $K \subseteq \mathbb{C}$  be a compact set. Then,

1. If  $\alpha \in V_{\infty}$ , then the non-bounded component of  $\mathbb{C} \setminus K$ , then there exists a determination of  $\log(z - \alpha)$  in a neighborhood of  $K$ .
2. If  $\alpha, \beta$  belong to the same bounded component of  $\mathbb{C} \setminus K$ , then there exists a determination of  $\log\left(\frac{z - \alpha}{z - \beta}\right)$  in a neighborhood of  $K$ .

**Theorem 6.18** (Green's theorem). Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain with piece-wise regular and positively oriented boundary. Let  $\mathbf{F} = (P, Q)$  be a vector field with  $P, Q$  being differentiable functions on a neighborhood of  $\bar{\Omega}$  such that  $\partial_x P - \partial_y Q$  is continuous on  $\bar{\Omega}$ . Then,

$$\int_{\partial \Omega} \langle \mathbf{F}, ds \rangle_I = \int_{\partial \Omega} P \, dx + Q \, dy = \iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx \, dy. \quad (129)$$

**Theorem 6.19** (Cauchy's integral theorem). Let  $\Omega$  be a bounded domain with piece-wise regular and positively oriented boundary and  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  an holomorphic function in a neighborhood of  $\bar{\Omega}$ . Then,

$$\int_{\partial \Omega} f(z) \, dz = 0. \quad (130)$$

**Corollary 6.20.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function in a domain  $D \subseteq \Omega$ . Then,  $f$  has local primitive on  $D$ . If  $D$  is a star domain,  $f$  has a global holomorphic primitive.

**Corollary 6.21.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function with no roots in a domain  $D \subseteq \Omega$ . Then,  $f$  has a local determination of the logarithm on  $D$ . If  $D$  is a star domain,  $f$  has a global determination of the logarithm.

**Theorem 6.22** (Cauchy's integral theorem for homotopic curves). Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function with  $\Omega$  a domain and  $\gamma_1, \gamma_2$  two homotopic curves such that  $\Gamma_1, \Gamma_2 \subseteq \Omega$ . Then,

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz. \quad (131)$$

**Theorem 6.23** (Cauchy's general integral theorem). Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a regular function on  $\Omega$  except a finite numbers of points where  $f$  is continuous. If  $\gamma$  is a constant curve, then

$$\oint_{\gamma} f(z) dz = 0. \quad (132)$$

**Theorem 6.24** (Morera's theorem). Let  $f$  be a continuous function in a region  $\Omega$ . If

$$\oint_{\gamma} f(z) dz = 0 \quad (133)$$

for all simple and closed curve  $\gamma$  such that  $\Gamma \subseteq \Omega$ , then  $f$  is analytic on  $\Omega$ .

**Theorem 6.25.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a differentiable function on a domain  $D$ . Then,  $f = u + iv$  is holomorphic if and only if the field  $\bar{f} = (u, -v)$  is locally conservative and locally solenoidal.

**Definition 6.27.** Let  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a differentiable vector field on a domain  $D \subseteq \mathbb{R}^n$ . We say the field is *holomorphic* if and only if is locally conservative and locally solenoidal, that is, it satisfies

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}, \quad \forall i, j; \quad \text{div } \mathbf{F} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} = 0, \quad \text{on } D. \quad (134)$$

**Definition 6.28.** Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$  be a scalar field two times differentiable on an open set  $\Omega \subseteq \mathbb{R}^n$ . We say the field is *harmonic* if and only if  $\nabla^2 \Phi = 0$  on  $\Omega$ .

**Theorem 6.26.** Holomorphic vector fields are the fields that are locally the gradient of an harmonic function. Holomorphic functions are the functions  $f$  that, locally, satisfy  $f = \Phi_x + i\Phi_y$  with  $\Phi$  harmonic.

**Definition 6.29.** Let  $u$  be an harmonic real function on a domain  $\Omega \subseteq \mathbb{C}$ . We say a differentiable function  $\tilde{u}$  on  $\Omega$  is the *harmonic conjugate* of  $u$  if and only if  $d\tilde{u} = d^*u$ , that is, if the function  $f = u + i\tilde{u}$  is holomorphic on  $\Omega$ .

**Theorem 6.27.** Let  $u$  be an harmonic real function on a domain  $\Omega \subseteq \mathbb{C}$  and  $f = \bar{\nabla}u$ . Then,  $u$  has an harmonic conjugate on  $\Omega$ ,  $\tilde{u}$ , if and only if  $f$  has an holomorphic primitive  $F$  on  $\Omega$ . In that case,  $F = u + i\tilde{u}$ .

**Proposition 6.28.** Let  $u$  be an harmonic function on a domain  $\Omega$ . Then, it has an harmonic conjugate if and only if the closed form  $d^*u$  is exact on  $\Omega$ , that is, if  $\int_{\gamma} d^*u = 0$  for all closed curve  $\gamma$  such that  $\Gamma \subseteq \Omega$ , condition that is always locally completed. If  $\Omega$  is a star domain, every harmonic function on  $\Omega$  has a harmonic conjugate function on  $\Omega$ .

## 7 Local properties of holomorphic functions

**Lemma 7.1.** Let  $a \in \mathbb{C}$  be a number and  $f = 1/|z - a|$ . Then,  $f$  is Lebesgue-integrable on every subset of  $\mathbb{C}$  of finite measure.

**Theorem 7.2** (Cauchy-Green formula). Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain with piece-wise regular and positively oriented boundary, and  $f$  a differentiable function on a neighborhood of  $\bar{\Omega}$  such that  $\bar{\partial}f$  is continuous on  $\bar{\Omega}$ . Then, for all  $z_0 \in \Omega$ ,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0} dz - \frac{1}{\pi} \int_{\Omega} \frac{\bar{\partial}f(z)}{z - z_0} dm(z). \quad (135)$$

**Corollary 7.3** (Cauchy's integral formula). Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain with piece-wise regular and positively oriented boundary, and  $f$  an holomorphic function on a neighborhood of  $\bar{\Omega}$ . Then,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0} dz. \quad (136)$$

**Corollary 7.4.** Let  $f$  be a differentiable function on  $\mathbb{C}$  with compact support and  $\bar{\partial}f$  continuous on  $\mathbb{C}$ . Then,

$$f(z_0) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial}f(z)}{z - z_0} dm(z). \quad (137)$$

**Proposition 7.5.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function and  $\gamma$  a piece-wise regular and positively oriented curve such that  $\Gamma \subseteq \Omega$ . Then,

$$\text{Ind}(\gamma, z_0)f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz. \quad (138)$$

**Corollary 7.6.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function and  $\gamma_1, \gamma_2$  two homotopic, piece-wise regular, and positively oriented curves such that  $\Gamma_1, \Gamma_2 \subseteq \Omega$ . Then,

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z - z_0} dz. \quad (139)$$

**Theorem 7.7.** Let  $f$  be an holomorphic function on a disc  $D(z_0, R)$ . Then, there exists a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  with radius of convergence greater or equal to  $R$  such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad \forall z \in D(z_0, R). \quad (140)$$

**Theorem 7.8.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function with  $\Omega$  an open set. Then,  $f$  is holomorphic on  $\Omega$  if and only if  $f$  is analytic on  $\Omega$ . More precisely, every holomorphic function  $f$  on  $\Omega$  is indefinitely holomorphic on  $\Omega$ , and for all  $z_0 \in \Omega$  the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad (141)$$

is valid on the greatest disc centered at  $z_0$  and contained on  $\Omega$ , which is  $D(z_0, \delta(z_0))$ , where  $\delta(z_0) = \inf\{|z_0 - w|, w \notin \Omega\}$ .

**Theorem 7.9.** The assignation  $f \rightarrow \left(\frac{f^{(n)}(0)}{n!}\right)_{n=0}^{\infty}$  is a bijection between the space of entire functions and the space formed by the sequences  $\{a_n\}_{n=0}^{\infty}$  such that the series  $\sum_{n=0}^{\infty} a_n z^n$  has an infinite radius of convergence, that is,  $\lim_{n \rightarrow \infty} |a_n|^{1/n} = 0$ .

**Theorem 7.10** (Morera's theorem). Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function of class  $C(\Omega)$  with  $\Omega$  an open set. Then,  $f$  is holomorphic on  $\Omega$  if and only if

$$\int_{\partial \Delta} f(z) dz = 0 \quad (142)$$

for all triangle  $\Delta \subseteq \Omega$ .

**Theorem 7.11.** Let  $f$  be a function continuous on an open set  $\Omega$  and holomorphic on  $\Omega \setminus E$ , where  $E$  is a finite collection of points and segments. Then,  $f$  is holomorphic on  $\Omega$ .

**Proposition 7.12.** Let  $f$  be a function and  $\Omega$  a bounded domain with piece-wise regular and positively oriented boundary. If  $f$  is holomorphic on a neighborhood of  $\bar{\Omega}$ , then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (143)$$

**Proposition 7.13.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function and  $\gamma$  a piece-wise regular and positively oriented curve such that  $\Gamma \subseteq \Omega$ . Then,

$$\text{Ind}(\gamma, z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (144)$$

**Lemma 7.14.** let  $\Omega \subseteq \mathbb{C}$  be a domain,  $f \in H(\Omega)$  a function, and  $z_0 \in \Omega$  a number. Then, the following statements are equivalent.

1.  $f^{(n)}(z_0) = 0$  for all  $n \in \mathbb{N}$ .
2.  $f(z) = 0$  for all  $z$  in a neighborhood of  $z_0$ .
3.  $f$  is identically null on  $\Omega$ .

**Definition 7.1.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function and  $z_0$  a number. We say  $z_0$  is a zero of order  $n$  of  $f$  if and only if  $f^{(k)}(z_0) = 0$  for all  $0 \leq k \leq n$ . We call  $k$  the order of  $z_0$  as a zero of  $f$ .

**Proposition 7.15.** The zeros of finite order of an holomorphic function are isolated points.

**Proposition 7.16.** All the zeros of a non null analytic function are isolated points and of finite order.

**Definition 7.2.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function with  $\Omega$  a domain. Then, we denote the set of zeros of  $f$  as

$$Z(f) := \{w \in \Omega \mid f(w) = 0\}. \quad (145)$$

**Theorem 7.17.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function with  $\Omega$  a domain such that  $f \not\equiv 0$ . Then,  $Z(f) \subseteq \Omega$  is a closed set without accumulation points. In particular,  $Z(f)$  is a finite or countable set and on every compact of  $\Omega$  there is a finite number of zeros of  $f$ .

**Theorem 7.18** (Principle of analytic continuation). Let  $f, g$  be two holomorphic functions on a domain  $\Omega \subseteq \mathbb{C}$ . Then,  $f(z) = g(z)$  for all  $z \in \Omega$  if and only if they satisfy one of the following conditions.

1. There exists a point  $w \in \Omega$  such that  $f^{(n)}(w) = g^{(n)}(w)$  for all  $n \in \mathbb{N}$ , that is,  $|f(z) - g(z)| = o(|z - w|^n)$ , if  $z \rightarrow w$ , for all  $n \in \mathbb{N}$ .
2. There exists a set  $\Psi \subseteq \Omega$  that contains an accumulation point on  $\Omega$  and  $f(z) = g(z)$  for all  $z \in \Psi$ .
3. There exists an open set  $\Psi \subseteq \Omega$  such that  $f(z) = g(z)$  for all  $z \in \Psi$ .

**Theorem 7.19** (Schwarz reflection principle). Let  $\Omega \subseteq \mathbb{C}$  be a symmetric domain and  $f \in H(\Omega)$  such that  $f(x) \in \mathbb{R}$  for all  $x \in \Omega \cap \mathbb{R}$ . Then,  $f(\bar{z}) = \overline{f(z)}$  for all  $z \in \Omega$ .

**Theorem 7.20.** Every analytic function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is the restriction on  $\mathbb{R}$  of an holomorphic function  $F : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  defined in a symmetric domain  $\Omega$ , that is,  $\mathbb{R} \subseteq \Omega$  and  $f = F|_{\mathbb{R}}$ .

**Theorem 7.21.** Let  $f, g$  be two analytic functions on a domain  $\Omega \subseteq \mathbb{R}^2$ . Then,  $f(x, y) = g(x, y)$  for all  $(x, y) \in \Omega$  if and only if they satisfy one of the following conditions.

1. There exists a point  $(x_0, y_0) \in \Omega$  such that

$$\frac{\partial^{n+m} f}{\partial x^n \partial y^m}(x_0, y_0) = \frac{\partial^{n+m} g}{\partial x^n \partial y^m}(x_0, y_0) \quad (146)$$

for all  $n, m \in \mathbb{N}$ , that is,  $|f(x, y) - g(x, y)| = o\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right)$ , if  $(x, y) \rightarrow (x_0, y_0)$  for all  $n \in \mathbb{N}$ .

2. There exists an open set  $\Psi$  such that  $f(x, y) = g(x, y)$  for all  $(x, y) \in \Psi$ .

**Theorem 7.22** (Maximum modulus principle). Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function with  $\Omega$  a domain. If  $f$  is not constant, then  $|f|$  does not have any local maxima on  $\Omega$ .

**Corollary 7.23.** Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain and  $f$  an holomorphic function on a neighborhood of  $\bar{\Omega}$  or, more generally,  $f \in C(\bar{\Omega}) \cap H(\Omega)$ . Let  $M$  be the maxima of  $|f|$  on  $\partial\Omega$ . Then, one has

$$|f(z)| \leq M, \quad \text{for all } z \in \Omega. \quad (147)$$

In other words,  $\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|$ .

**Theorem 7.24** (Cauchy's inequality). Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function on a neighborhood of the disc  $\bar{D}(z_0, R)$  and  $|f(z)| \leq M$  for  $z \in C(z_0, R)$ . Then,

$$|f^{(n)}(z_0)| \leq M \frac{n!}{R^n}. \quad (148)$$

**Corollary 7.25.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function with  $\Omega$  a domain such that  $|f(z)| \leq M, z \in \Omega$ . Then,

$$|f^{(n)}(z)| \leq M \frac{n!}{d(z, U^c)^n}, \quad z \in U, n \in \mathbb{N}. \quad (149)$$

**Theorem 7.26** (Liouville's theorem). Let  $f$  be a bounded entire function. Then,  $f$  is constant. Also, a function  $u$  harmonic and bounded on  $\mathbb{C}$  is constant.

**Theorem 7.27** (Fundamental theorem of algebra). Let  $P(<) = a_0 + a_1 z + \dots + a_n z^n$  be a polynomial of degree  $n$  of complex coefficients and  $n \geq 1$ . Then,  $P$  has exactly  $n$  roots  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$  (some of which can be counted with their multiplicity) and

$$P(z) = a_n \prod_{i=1}^n (z - \alpha_i). \quad (150)$$

## 8 Isolated singularities of holomorphic functions

**Definition 8.1.** We say  $f$  has an isolated singularity at  $z_0$  if and only if  $f$  is holomorphic on  $D_r^*(z_0)$  for some  $r \in \mathbb{R}^+$ . We say the singularity is removable if and only if  $f$  can be extended to an holomorphic function on  $D_r(z_0)$ .

**Definition 8.2.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function on a disc  $D_r^*(z_0)$ . We say  $f$  has a pole of order  $k$  at  $z_0$  if and only if there exist  $\alpha \in \mathbb{C}, k \in \mathbb{N}_{\geq 1}$  such that  $f(z) \propto \alpha(z - z_0)^k$  when  $z \rightarrow z_0$ . We call  $k$  the multiplicity of the pole or order of the pole.

**Definition 8.3.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function with  $\Omega$  a domain. We say  $f$  is meromorphic on  $\Omega$  if and only if there exists a set  $A \subseteq \Omega$ , discrete and closed on  $\Omega$ , such that  $f$  is defined and holomorphic on  $\Omega \setminus A$  and has a pole on every point  $z \in A$ .

**Proposition 8.1.**  $f$  has a pole of order  $k$  at  $z_0$  if and only if there exists an holomorphic function  $g(z)$  in a neighborhood of  $z_0$  such that  $g(z_0) \neq 0$  and

$$f(z) = \frac{g(z)}{(z - z_0)^k}. \quad (151)$$

**Theorem 8.2.** Every holomorphic function on an annulus admits a Laurent expansion.

**Proposition 8.3.** Let  $f$  be an holomorphic function on an annulus  $C(z_0, R_2, R_1)$ . If  $f$  has an isolated singularity at  $z_0$ , then its Laurent expansion is uniquely determined by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad (152)$$

where  $a_n$  is independent of  $r, r \in (R_2, R_1)$ .

**Definition 8.4.** Let  $f \in H(D_\epsilon^*(z_0))$  be an holomorphic function with a Laurent expansion  $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$  around  $z_0$ . We define the residue of  $f$  at  $z_0$  as

$$\text{Res}(f, z_0) := a_{-1} = \frac{1}{2\pi i} \int_{C(z_0, r)} f(z) dz. \quad (153)$$

**Theorem 8.4.** Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain with piece-wise regular and positively oriented boundary. Let  $\Psi$  be an open set such that  $\bar{\Omega} \subseteq \Psi, X \subseteq \Psi$  a closed set formed by isolated points (the accumulation points of  $X$ , if there are, must be in  $\partial\Psi$ ) such that  $X \cap \partial\Omega = \emptyset$ , and  $f$  an holomorphic function on the open set  $\Psi \setminus X$ . Then,

$$\frac{1}{2\pi i} \int_{\partial\Omega} f(z) dz = \sum_{w \in X \cap \Omega} \text{Res}(f, w). \quad (154)$$

**Theorem 8.5.** For a general curve,

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{i=1}^n \text{Ind}(\gamma, z_i) \text{Res}(f, z_i). \quad (155)$$

**Proposition 8.6.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function on a neighborhood of  $z_0$  with  $z_0$  a pole. Then,

1. If  $z_0$  is a removable singularity,  $\text{Res}(f, z_0) = 0$ .

2. If  $z_0$  is a simple singularity,

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z). \quad (156)$$

3. If  $z_0$  is a singularity of order  $k$ ,

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} [(z - z_0) f(z)]. \quad (157)$$

4. If  $z_0$  is an essential singularity, the residue  $a_{-1}$  must be obtained directly from the Laurent series.

**Proposition 8.7.** If  $f = g/h$ , with  $f, g$  holomorphic in a neighborhood of  $z_0, g(z_0) \neq 0, h(z_0) = 0, h'(z_0) \neq 0$ , then

$$\text{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}. \quad (158)$$

**Proposition 8.8.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a meromorphic function in a neighborhood of  $z_0 \in \mathbb{C}$ . If we denote  $f(z) = (z - z_0)^m g(z)$  with  $w \in \mathbb{Z}^*$  (depending on the sign  $z_0$  can be a zero or a pole), then  $z_0$  is a single singularity of  $f'/f$  and  $\text{Res}(f'/f, z_0) = m$ .

**Proposition 8.9.** Let  $f(z) = g(\frac{1}{z-z_0})$  be a function with  $g(w)$  an entire function that admits an expansion  $g(w) = \sum_{n=0}^{\infty} b_n w^n$ . If  $g$  is not a polynomial, then  $f$  has an essential singularity at  $z_0$  and  $\text{Res}(f, z_0) = g'(0) = b_1$ .

**Proposition 8.10.** Let  $f$  be a function with a simple pole at  $z_0$  and  $g$  an holomorphic function in a neighborhood of  $z_0$ . Then,  $fg$  has a simple singularity at  $z_0$  and  $\text{Res}(fg, z_0) = g(z_0) \text{Res}(f, z_0)$ .

**Definition 8.5.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function. We say  $f$  is holomorphic at infinity if and only if  $g(w) = f(1/w)$  is holomorphic at the origin.

**Proposition 8.11.** Let  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be an holomorphic function on  $D_\epsilon^*(\infty)$ . Then,

$$\text{Res}(f, \infty) = -a_{-1} = -\frac{1}{2\pi i} \int_{C(z_0, r)} f(z) dz. \quad (159)$$

**Proposition 8.12.** Let  $f$  be a meromorphic function on the Riemann sphere. Then,  $f$  is a rational function. Besides, if  $X$  is the set formed by the poles of  $F$  and the infinite point, then  $X$  is finite and

$$\sum_{w \in X} \text{Res}(f, w) = 0 \Leftrightarrow \text{Res}(f, \infty) = - \sum_{w \in X \setminus \{\infty\}} \text{Res}(f, w). \quad (160)$$

**Theorem 8.13.** Let  $\Omega$  be a bounded domain with a piece-wise regular and positively oriented boundary. Let  $\Psi$  be an open set such that  $\bar{\Omega} \subseteq \Psi$ ,  $f$  a meromorphic function on  $\Psi$  and  $h$  an holomorphic function on  $\Psi$ . Let  $\{a_j\}$  be the zeros of  $f$  on  $\Psi$  and  $n_j$  the multiplicities of  $a_j$ , and let  $\{b_j\}$  be the poles of  $f$  on  $\Psi$  and  $m_j$  the multiplicities of  $b_j$ . If there is no zeros or poles on  $\partial\Omega$ , then

$$\frac{1}{2\pi i} \int_{\partial\Omega} h(z) \frac{f'(z)}{f(z)} dz = \sum_{a_j \in \Omega} h(a_j) n_j - \sum_{b_j \in \Omega} h(b_j) m_j. \quad (161)$$

**Corollary 8.14.** Let  $\Omega$  be a bounded domain with a piece-wise regular and positively oriented boundary and  $f$  a meromorphic function on a neighborhood of  $\bar{\Omega}$  that does not have zeros or poles on  $\partial\Omega$ . Let  $N$  the total number of zeros of  $f$  on  $\Omega$  and  $P$  the total number of poles on  $\Omega$  (counting multiplicities). If we denote  $\Gamma = f(\partial\Omega)$ , then

$$\text{Ind}(\Gamma, 0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = N - P. \quad (162)$$

## 9 Homology

## 10 Harmonic functions

**Theorem 10.1.** Let  $f \in H(\Omega), C^1(\Omega)$  be a function. If  $f = u + iv$ , then  $u, v$  are harmonic functions on  $\Omega$ .

## 11 Conforming representation

## 12 Riemann theorem

## 13 Runge theorem

## 14 Zeros of holomorphic functions

**Theorem 14.1** (Weierstrass Factorization Theorem). content...

## 15 Fourier transform

**Definition 15.1.** Let  $f \in L^1(\mathbb{R})$  be a function and  $\xi \in \mathbb{R}$  a number. We define the Fourier transform of  $f$  at the point  $\xi$  as

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx. \quad (163)$$

**Proposition 15.1.** Let  $f \in L^1(\mathbb{R})$  be a function. Then, the application

$$\begin{aligned} \mathcal{F}\{f\} : \mathbb{R} &\rightarrow \mathbb{C} \\ \xi &\mapsto \hat{f}(\xi) \end{aligned} \quad (164)$$

is a well defined application.

**Definition 15.2.** Let  $\{f_n\}_{n \in \mathbb{N}} \subseteq L^p(\mathbb{R})$  and  $f \in L^p(\mathbb{R})$  with  $1 \leq p \leq \infty$ . We say the functions  $f_n$  converge to  $f$  with a norm  $\|\cdot\|_p$  or converge in  $L^p(\mathbb{R})$  if and only if

$$\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0. \quad (165)$$

**Theorem 15.2.** Let  $f \in L^1(\mathbb{R})$  be a function. Then, the following statements are true.

1. The Fourier transform of  $f$  satisfies

$$\mathcal{F}\{f\} \in L^\infty(\mathbb{R}), \quad \|\mathcal{F}\{f\}\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|f\|_1. \quad (166)$$

2.  $\mathcal{F}\{f\}$  is  $\mathbb{C}$  linear, that is, for all  $\alpha, \beta \in \mathbb{C}$  and  $f, g \in L^1(\mathbb{R})$ ,

$$\mathcal{F}\{\alpha f + \beta g\} = \alpha \mathcal{F}\{f\} + \beta \mathcal{F}\{g\}. \quad (167)$$

3. If  $g(x) = \bar{f}(x)$ , then for all  $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \overline{\hat{f}(-\xi)}. \quad (168)$$

4. If  $g(x) = g(\lambda x)$  and  $\lambda \in \mathbb{R}$ , then for all  $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \frac{1}{|\lambda|} \hat{f}\left(\frac{\xi}{\lambda}\right). \quad (169)$$

5. If  $g(x) = f(x - a)$  with  $a \in \mathbb{R}$ , then for all  $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = e^{-ia\xi} \hat{f}(\xi). \quad (170)$$

6. If  $g(x) = e^{iax} f(x)$  with  $\alpha \in \mathbb{R}$ , then for all  $\xi \in \mathbb{R}$

$$\hat{g}(\xi) = \hat{f}(\xi - a) \quad (171)$$

7. If  $\{f_n\}_{n \in \mathbb{N}} \subseteq L^1(\mathbb{R})$ ,  $f \in L^1(\mathbb{R})$  and  $f_n \rightarrow f$  in  $L^1(\mathbb{R})$  when  $n \rightarrow \infty$ , then  $\mathcal{F}\{f_n\} \rightarrow \mathcal{F}\{f\}$  uniformly in  $\mathbb{R}$ .

8. The Fourier transform  $\mathcal{F}\{f\}$  is a continuous function in  $\mathbb{R}$ ,  $\mathcal{F}\{f\} \in C(\mathbb{R})$ .

**Proposition 15.3.** Let  $f \in L^1(\mathbb{R})$  be a function such that there exists its derivative  $f' \in L^1(\mathbb{R})$  and  $\lim_{|x| \rightarrow \infty} |f(x)| = 0$ . Then,

$$\hat{f}'(\xi) = i\xi \hat{f}(\xi). \quad (172)$$

**Corollary 15.4.** Let  $f \in L^1(\mathbb{R})$  be a function such that there exists its  $n$ -th derivative  $f^{(n)} \in L^1(\mathbb{R})$  and  $\lim_{|x| \rightarrow \infty} |f(x)| = 0$ . Then,

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi). \quad (173)$$

**Definition 15.3.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{C}$  be a function. We define the support of  $f$  as

$$\text{supp } f := \overline{\{x \in I \mid f(x) \neq 0\}}. \quad (174)$$

**Definition 15.4.** We define the set  $\mathcal{D}(\mathbb{R})$  as

$$\mathcal{D}(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}) \mid \text{supp } \varphi \text{ compact}\} \subseteq L^1(\mathbb{R}). \quad (175)$$

**Theorem 15.5.** Let  $f \in L^1(\mathbb{R})$  be a function. Then, there exists a sequence of functions  $\phi_n \in \mathcal{D}(\mathbb{R})$  such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f - \phi_n| dx = 0, \quad (176)$$

that is, we have convergence of  $\phi_n$  to  $f$  with norm  $\|\cdot\|_1$ .

**Proposition 15.6.** Let  $f \in L^1(\mathbb{R})$  be a function. Then,  $\hat{f} \in C(\mathbb{R})$ .

**Proposition 15.7.** Let  $f \in L^1(\mathbb{R})$  be a function. Then,  $|\hat{f}(\xi)| \leq \|f\|_1$ .

**Theorem 15.8.** Let  $f \in L^1(\mathbb{R})$  be a function. Then,

$$\lim_{|x| \rightarrow \infty} \hat{f}(x) = 0. \quad (177)$$

**Theorem 15.9.** The application Fourier transform goes from  $L^1(\mathbb{R})$  to  $C_0(\mathbb{R})$ , that is,  $\mathcal{F}\{f\} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ .

**Definition 15.5.** We define the Schwartz space as

$$S(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{C} \mid f \in C^\infty(\mathbb{R}) \wedge \forall n, m \in \mathbb{N} \exists c_{n,m} < \infty \text{ such that } (1 + |x|)^m \cdot |D^n f(x)| \leq c_{n,m}, \forall x \in \mathbb{R}\}.$$

**Proposition 15.10.** Let  $f, g \in S(\mathbb{R})$  be two functions,  $\lambda \in \mathbb{C}$  a number, and  $P : \mathbb{R} \rightarrow \mathbb{C}$  a polynomial of complex coefficients. Then,

1.  $f + g \in S(\mathbb{R})$ .
2.  $\lambda f \in S(\mathbb{R})$ .
3.  $fg \in S(\mathbb{R})$ .
4.  $Pf \in S(\mathbb{R})$ .

**Theorem 15.11.** Let  $I, J \subseteq \mathbb{R}$  be two intervals with  $I$  compact and  $J$  open. Let  $f : I \times J \rightarrow \mathbb{R}$  be a function such that

1.  $f(\cdot, \lambda)$  is Riemann-integrable in  $I$  for all  $\lambda \in J$ ,
2.  $f(x, \cdot)$  is derivable in  $J$  for all  $x \in I$ .

If  $\partial_\lambda f$  is continuous in  $I \times J$ , then

1.  $\partial_\lambda f(\cdot, \lambda)$  is Riemann-integrable for all  $\lambda \in J$ .
2.  $F(\lambda) = \int_I f(x, \lambda) dx$  is derivable with continuous derivative in  $J$  for all  $\lambda \in J$  and it satisfies the rule of derivation over the integral sign.

$$F'(\lambda) = \frac{d}{d\lambda} \int_I f(x, \lambda_0) dx = \int_I \frac{\partial f}{\partial \lambda}(x, \lambda_0) dx, \forall \lambda_0 \in J \quad (178)$$

**Proposition 15.12.** Let  $f \in S(\mathbb{R})$ . Then,

1.  $S(\mathbb{R}) \subseteq L^1(\mathbb{R})$ .
2.  $\widehat{xf}(\xi) = (iD_\xi \hat{f})(\xi)$  for all  $\xi \in \mathbb{R}$ .

**Corollary 15.13.** Let  $f \in s(\mathbb{R})$ . Then,

$$\widehat{x^n f}(\xi) = (i^n D^n \hat{f})(\xi), \forall n \in \mathbb{N}. \quad (179)$$

**Proposition 15.14.** The Fourier transform  $\mathcal{F}$  restricted to  $S(\mathbb{R})$  is an automorphism, that is, if  $f \in S(\mathbb{R})$  then  $\mathcal{F}\{f\} = \hat{f} \in S(\mathbb{R})$ .

**Lemma 15.15.** If  $G(x) = e^{-x^2/2}$ , then  $\hat{G}(\xi) = e^{-\xi^2/2}$ . We observe hence that  $G$  is a fixed point of  $\mathcal{F}$ .

**Lemma 15.16.** If  $f, g \in S(\mathbb{R})$ , then

$$\int_{\mathbb{R}} f(\xi) \hat{g}(\xi) d\xi = \int_{\mathbb{R}} \hat{f}(\tau) g(\tau) d\tau. \quad (180)$$

**Lemma 15.17.** Let  $f, g \in S(\mathbb{R})$  and  $\lambda \in \mathbb{R}$ . Then,

1.  $g(\lambda x) \hat{f}(x)$  converges to  $g(0) \hat{f}(x)$  uniformly in  $\mathbb{R}$  when  $\lambda \rightarrow \infty$ .
2.  $f(\lambda x) \hat{g}(x)$  converges to  $f(0) \hat{g}(x)$  uniformly in  $\mathbb{R}$  when  $\lambda \rightarrow \infty$ .

**Lemma 15.18.** Let  $f, g \in s(\mathbb{R})$ . Then,

$$f(0) \int_{\mathbb{R}} \hat{g}(\xi) d\xi = g(0) \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad (181)$$



**Lemma 15.19.** Let  $f \in s(\mathbb{R})$  be a function. Then,

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) d\xi. \quad (182)$$

**Corollary 15.20** (Inversion formula). Let  $f \in S(\mathbb{R})$ . Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi, \forall x \in \mathbb{R}. \quad (183)$$

**Theorem 15.21** (Inversion of  $\mathcal{F}$  in  $S(\mathbb{R})$ ). Let  $\mathcal{F} : S(\mathbb{R}) \rightarrow S(\mathbb{R})$ , defined by  $\mathcal{F}\{f\} = \hat{f}$  with  $\hat{f} \in s(\mathbb{R})$ . Then,  $\mathcal{F}$  is an linear isomorphism in the vector space  $S(\mathbb{R})$  and  $\mathcal{F}^4 = Id$ . In particular,  $\mathcal{F}^{-1} = \mathcal{F}^3$  and if  $f \in S(\mathbb{R})$ , then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}\{f\}(\xi) e^{ix\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} e^{ix\xi} d\xi. \quad (184)$$

In fact,  $\mathcal{F}$  is an homomorphism (its inverse is continuous) if we consider  $S(\mathbb{R})$  as the metric space  $(S(\mathbb{R}), \|\cdot\|_{n,m})$ .

**Theorem 15.22** (Inversion of  $\mathcal{F}$  for discontinuities). Let  $f$  be a absolutely Riemann-integrable function in  $\mathbb{R}$  with  $f$  and  $f'$  piece-wise continuous. Then,

$$\frac{f(x^-) + f(x^+)}{2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f} e^{ix\xi} d\xi. \quad (185)$$

**Definition 15.6.** Let  $f$  be a Riemann-integrable function in  $\mathbb{R}$ . We define the *Fourier transform of cosine kind* as

$$\hat{f}_c(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\xi x) f(x) dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(\xi x) f_e(x) dx, \quad (186)$$

and the *Fourier transform of sine kind* as

$$\hat{f}_s(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\xi x) f(x) dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(\xi x) f_o(x) dx. \quad (187)$$

**Proposition 15.23.** Let  $\hat{f}_c, \hat{f}_s$  be the Fourier transform of cosine and sine kinds of  $f$ . Then,  $\hat{f}_c(\xi)$  is even,  $\hat{f}_s(\xi)$  is odd, and  $\hat{f}(\xi) = \hat{f}_c(\xi) - i\hat{f}_s(\xi)$ .

**Theorem 15.24.** Let  $f$  be a absolutely Riemann-integrable function in  $\mathbb{R}$  with  $f$  and  $f'$  piece-wise continuous. Then,

$$\frac{f_e(x^-) + f_e(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c \cos(\xi x) d\xi, \quad (188)$$

$$\frac{f_o(x^-) + f_o(x^+)}{2} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s \sin(\xi x) d\xi. \quad (189)$$

**Theorem 15.25** (Tonelli's Theorem). Let  $f : I \times J \rightarrow \mathbb{R}^2$  two functions with  $I, J \subseteq \mathbb{R}$  such that  $f(x, y) \geq 0$  for all  $(x, y) \in I \times J$ . Then,

$$\int_{I \times J} f dx dy = \int_I \int_J f(x, y) dy dx = \int_J \int_I f(x, y) dx dy. \quad (190)$$

Besides, if these integrals are finite, then  $f \in L^1(\mathbb{R})$ .

**Corollary 15.26.** Let  $f, g \in L^1(\mathbb{R})$ . Then,  $F(x, t) = f(t)g(x-t) \in L^1(\mathbb{R}^2)$ .

**Definition 15.7.** Let  $f, g \in L^1(\mathbb{R})$  two function. We define the *convolution of  $f$  and  $g$*  as

$$(f * g) : \mathbb{R} \rightarrow \mathbb{C} \\ x \mapsto \int_{\mathbb{R}} f(t)g(x-t) dt, \quad (191)$$

which is from  $L^1(\mathbb{R})$ .

**Proposition 15.27.** Let  $f, g \in L^1(\mathbb{R})$  be two functions. Then  $\widehat{f * g} = \sqrt{2\pi} \hat{f} \hat{g}$ .

**Proposition 15.28.** Let  $f \in L^1(\mathbb{R})$  be a function and  $g = f^2$ . Then,

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \hat{f} * \hat{f} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) \hat{f}(\xi - t) dt. \quad (192)$$

**Theorem 15.29.** Let  $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$  and  $\phi \in S(\mathbb{R})$ . Then,  $f * \phi \in C^\infty(\mathbb{R})$ .

**Theorem 15.30.** Let  $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$  with  $\text{supp } f$  compact and  $\phi \in D(\mathbb{R})$ . Then,  $f * \phi \in D(\mathbb{R})$  and  $\text{supp } \{f * \phi\} \subseteq \text{supp } f + \text{supp } \phi$ .

**Definition 15.8.** We say the functions  $\phi_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$  continuous in a compact support are an *approximation of the unity* if and only if

1.  $\phi_\epsilon \geq 0$  for all  $\epsilon$ .

2.  $\int_{\mathbb{R}} \phi_\epsilon(x) dx = 1$ .

3. For all  $\delta > 0$  it is satisfied that

$$\lim_{\epsilon \rightarrow 0} \left\{ \sup_{|t| > \delta} \phi_\epsilon(t) \right\} = 0. \quad (193)$$

**Theorem 15.31.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function with compact support  $\{\phi_\epsilon\}$  approximation of the unity. Then, when  $\epsilon \rightarrow 0$   $f * \phi_\epsilon$  converges uniformly in  $\mathbb{R}$  to  $f$ .

**Corollary 15.32.** Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a continuous function with compact support  $\{\phi_\epsilon\}$  approximation of the unity. Then, when  $\epsilon \rightarrow 0$   $f * \phi_\epsilon$  converges uniformly in  $\mathbb{R}$  to  $f$ .

**Theorem 15.33** (Weierstrass polynomial approximation). Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then, there exist polynomials  $P_n$  with  $n \in \mathbb{N}$  such that  $P_n$  converge uniformly to  $f$  in  $[a, b]$ .

**Theorem 15.34.** Let  $f \in L^p(\mathbb{R})$  be a function. Then, there exists a sequence of function  $f_n \in D(\mathbb{R})$  of the form  $f_n \rightarrow f$  with norm  $\|\cdot\|_p$  (that is, convergence in  $L^p$ ), and if  $f \in C^k(\mathbb{R})$  with  $k \geq 0$ , then

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \quad (194)$$

with  $\|f\|_{C^k(\mathbb{R})} = \max_{0 \leq l \leq k} (\sup_{x \in \mathbb{R}} |D^l f(x)|)$  being a norm.

**Lemma 15.35.** Let  $f \in L^1(\mathbb{R})$  be a function such that for all  $\phi \in S(\mathbb{R})$  it is satisfied that  $\int_{\mathbb{R}} f(x)\phi(x) dx = 0$ .

Then,  $f \equiv 0$ .

**Corollary 15.36.** The Fourier transform  $\mathcal{F}$  is injective since  $\mathcal{F}\{f\} = \hat{f} = 0 \Leftrightarrow f = 0$  in  $L^1(\mathbb{R})$  (the zero function class) and  $\mathcal{F}$  is a linear application.

**Theorem 15.37** (Inversion theorem in  $L^1(\mathbb{R})$ ). Let  $f \in L^1(\mathbb{R})$  be a function such that  $\hat{f} \in L^1(\mathbb{R})$ . Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi, \quad \forall x \in \mathbb{R}. \quad (195)$$

## 16 Fourier transform 2

**Theorem 16.1** (Parseval formula). Let  $f, g \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$  be two functions. Then,

$$\int_{\mathbb{R}} f(x) \overline{g(x)} dx = \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi. \quad (196)$$

**Theorem 16.2** (Plancherel Theorem). Let  $f \in S(\mathbb{R}) \subseteq L^2(\mathbb{R})$  be a function. Then,

$$\boxed{\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi,} \quad (197)$$

that is,  $\|f\|_2 = \|\hat{f}\|_2$  and  $\mathcal{F}$  is an isometry between vector spaces.

**Definition 16.1.** Let  $f \in S(\mathbb{R})$  be a function. We define the following quantities

$$E(f) := \int_{\mathbb{R}} |f(x)|^2 dx, \quad (198)$$

$$\sigma(f)^2 := \int_{\mathbb{R}} |xf(x)|^2 dx. \quad (199)$$

**Theorem 16.3.** Let  $f \in S(\mathbb{R})$  be a function. Then,

$$\sigma(f)\sigma(\hat{f}) \geq \frac{E(f)}{2}. \quad (200)$$

## 17 Multidimensional fourier transform

**Theorem 17.1.** For several variables

$$\mathcal{F}\{f(x_1, \dots, x_n)\} = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) e^{-i(x_1\xi_1 + \dots + x_n\xi_n)} d\mathbf{x} \quad (201)$$

or simpler,

$$\boxed{\mathcal{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\langle \mathbf{x}, \boldsymbol{\xi} \rangle_I} d\hat{\mathbf{x}}.} \quad (202)$$