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Chapter 1

Basics

1.1 Introduction

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1.2.1 Point mass model

As a first model to represent the moving objects we will use point bodies, that is, without dimensions. We will call this supposed 0-dimensional object by the name of *particle*, *point mass*, *point body*, or directly *point*. This representation of course is not perfectly accurate to express the phenomena we observe in the real world, where objects do have dimensions. However, it does not deviate so much from that neither. For instance, when a ball falls out the window, its movement will be similar to that of a really small object almost without dimensions. Even the movement of an abject of astronomic proportions such as that of a planet could also be approximated as the movement of a point because its size is a miniature compared with the distance traveled.
Moreover, the fact of using a point mass simplifies the calculations substantially. Since we don't consider the dimensions of the body, it is not necessary take into account some properties such as its shape, that could influence its movement.The model also helps in the question of finding the position of an object, because we won't need to know the position of each part but only one point.

1.2.2 Basic concepts

1.2.3 Position

We will always use orthogonal coordinate systems.
Explanation of distance and its relations with the norm...
We will present several coordinate systems. Not for every system the position needs to be a linear combination of all three basis vectors.
As a general norm, we will use a system with positive orientation. To know if a system has this property we can use the *right hand rule*. Definition (formal) of the concept of trajectory. State that we will consider always that is differentiable.

Example 1.2.1. Let us be the next equation of motion.

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto R\cos(\omega t)\vec{e}_x + R\sin(\omega t)\vec{e}_y \end{aligned}$$

One of the first things we should do always is check if the equation of motion is correctly define, if the dimensions fit... Let us remember that the trigonometric functions, logarithms, exponentials... are dimensionaless.
Now, one of the things we can see about this equation if that the distance is constant. Following the definition [ ], the distance has the following expression.

$$\|\vec{r}(t)\|^2 = R^2\left(\cos^2(\omega t) + \sin^2(\omega t)\right) = R^2$$

IF we apply now a reparametrization, we will have the same geometric property.

$$\|\vec{r}(t)\|^2 = R^2\left(\cos^2 f(t) + \sin^2 f(t)\right) = R^2$$

This shows that reparametrizations do not change the geometry of a trajectory. However, other properties do it, like the velocity or acceleration.

1.2.4 Velocity

When Aristotle studied physical phenomena, he lacked many mathematical notions and tools that would have provided his work of more precision. One of these notions was the concept of velocity, which was formulated for the first time by Galileo. When Galileo studied the motion of bodies on an inclined plane, he observed that in a certain period, the amount of distance they completed was not the same [Wikipedia]. From that, he defined a new magnitude capable of describing that phenomenon and others. We can also witness that not all bodies travel a distance at the same time. For instance, an airplane can go from one country to another in some hours, while walking would take us several weeks. This and other examples show that we need this new magnitude if we want to describe the moving objects.

Definition 1.2.1. Let $\vec{r}(t)$ be an equation of motion in some coordinate system. We define the *average velocity* $\vec{v}_m$ as the increase of the equation of motion in an given interval of time divided by this interval. Mathematically, we write it as follows.

$$\vec{v}_m = \frac{\vec{r}\,(t+\Delta t) - \vec{r}\,(t)}{\Delta t} = \frac{\Delta \vec{r}}{\Delta t} \quad [\vec{v}_m] = \mathrm{m\,s^{-1}} \tag{1.1}$$

As we did with the position's vector, we denote its norm by an independent name.

Definition 1.2.2. Let $\vec{v}_m$ be an equation of motion in some coordinate system. Let $\vec{v}_m$ its average velocity in some period. Then, we define the *average speed* as the norm of this average velocity.

$$v_m = \left\|\vec{v}_m(t)\right\| \quad [v_m] = \mathrm{m\,s^{-1}} \tag{1.2}$$

Galileo calculated the average velocity using the formula of the definition 1.2.1, with $\Delta t = 1\,\mathrm{s}$. That simplified the task of finding its value, and it worked for several cases. However, if people needed to determine the velocity in more complex movements, they would have difficulties. In that era, although mathematics was more advanced that was int Aristotle's era, it was not enough refined to these cases yet.
For a complicated trajectory, selecting an interval of one second leads to a value that is far from the real velocity. Therefore, we should choose shorter periods and measure how much the object traveled to get a more accurate value. However, if we want the most precise value, we would need each time a smaller interval, and then we could never finish our task. Fortunately, through history appeared brilliant minds like Newton and Leibniz, who worked in the infinitesimal intervals and found a method to follow each time we are in a similar situation. They formulated the concept of derivative, and this modern tool will serve for the current problem and future ones [ ].

Definition 1.2.3. Let $\vec{r}(t)$ be an equation of motion in some coordinate system. We define the *velocity* $\vec{v}(t)$ as the derivative of $\vec{r}(t)$ with respect to time.

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}\,(t+\Delta t) - \vec{r}\,(t)}{\Delta t} = \frac{d\vec{r}(t)}{dt} = \dot{\vec{r}}(t) \quad [\vec{v}] = \mathrm{m\,s^{-1}} \tag{1.3}$$

Note that since velocity is the first time derivative of the equation of motion, it will be tangent to $\vec{r}(t)$. Another important detail is that, when deriving the equation, we can not treat the basis vectors as constants. This is because, as we have seen previously, these vectors depend on the position of the point, and since it varies over time, the basis vectors too. Therefore, the extended form of velocity would look like this.

$$\vec{v}(t) = \frac{d\lambda(t)}{dt}\vec{e}_1(t) + \lambda(t)\frac{d\vec{e}_1(t)}{dt} + \frac{d\mu(t)}{dt}\vec{e}_2(t) + \mu(t)\frac{d\vec{e}_2(t)}{dt} + \frac{d\nu(t)}{dt}\vec{e}_3(t) + \nu(t)\frac{d\vec{e}_3(t)}{dt}$$

Definition 1.2.4. Let $\vec{r}(t)$ be an equation of motion in some coordinate system. We define the *speed* as norm of the velocity.

$$v(t) = \left\|\vec{v}(t)\right\| \quad [v] = \mathrm{m\,s^{-1}} \tag{1.4}$$

Sometimes instead of searching how has an object changed in position in some interval of time, we just want to know the distance traveled (the norm). As before, a point can travel a certain distance in different times, so we should also define a new magnitude to study this phenomenon.

Definition 1.2.5. Let $\vec{r}(t)$ be an equation of motion in some coordinate system. We define the *average celerity* as the distance traveled in a given interval of time divided by this interval. Mathematically, we write it as follows.

$$c_m = \frac{l}{\Delta t} = \frac{\|\Delta \vec{r}\|}{\Delta t} \quad [c_m] = \mathrm{m\,s^{-1}} \tag{1.5}$$

It is important noting that, since we always will select positive differences of time [ ], the celerity will always be positive.
And, as we did with the definitions of velocity and speed, we can talk about a celerity that works for an instant t, not only as an average for an interval of time.

Definition 1.2.6. Let $\vec{r}(t)$ be an equation of motion in some coordinate system. We define the *celerity* as the derivative of the distance traveled with respect to time.

$$c(t) = \frac{dl}{dt} \qquad [c] = \mathrm{m\,s^{-1}} \tag{1.6}$$

If we look at the definitions of speed and celerity initially different, we can see they are closely related and, in fact, are the same magnitude.

1.2.5 Acceleration

Definition 1.2.7. *Average acceleration.*

$$\vec{a}_m = \frac{\vec{v}\,(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\Delta \dot{\vec{r}}(t)}{\Delta t} \tag{1.7}$$

When we defined the average velocity we could see how to calculate it with the difference between the vector of position [ ]. It was easy because they had the same origin, but since now we have to subtract velocities, these ones do not have the same point of origin. To calculate the subtraction geometrically we should translate them first to the origin.

Definition 1.2.8. *Acceleration.*

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t) \tag{1.8}$$

In contrary to velocity, acceleration is not tangent to the trajectory.

1.3 Frenet-Serret vectors

Tangent vector.

$$\vec{e}_t(t) = \frac{\vec{v}(t)}{\left\|\vec{v}(t)\right\|} = \frac{\dot{\vec{r}}(t)}{\left\|\dot{\vec{r}}(t)\right\|} \tag{1.9}$$

We will see that the normal vector is perpendicular to the tangent one.

Proposition 1.3.1. *It is true that* $\vec{e}_t \perp \dot{\vec{e}}_n$.

Proof.

$$\langle \vec{e}_t, \vec{e}_t \rangle_I = \|\vec{e}_t\|^2 = 1 \overset{[\,]}{\Rightarrow} \frac{d\langle \vec{e}_t, \vec{e}_t \rangle_I}{dt} = \frac{d1}{dt} = 0 \Rightarrow 2\langle \vec{e}_t, \dot{\vec{e}}_t \rangle_I = 0 \Rightarrow \vec{e}_t \perp \dot{\vec{e}}_n$$

■

Then, we construct the normal vector.

$$\vec{e}_n(t) = \frac{\dot{\vec{e}}_t(t)}{\left\|\dot{\vec{e}}_t(t)\right\|} \tag{1.10}$$

Since tangent and normal have norm 1, the vector product will have also norm 1. Therefore, we define the binormal vector as follows.

$$\vec{e}_b = \vec{e}_t \times \vec{e}_n \tag{1.11}$$

With that we can be sure that the system has a positive orientation [ ]. We can express these vectors directly in terms of the equation of motion.

$$\vec{e}_t(t) = \frac{\dot{\vec{r}}(t)}{\left\|\dot{\vec{r}}(t)\right\|} \qquad \vec{e}_n(t) = \frac{\dot{\vec{r}}(t) \times \left[\ddot{\vec{r}}(t) \times \dot{\vec{r}}(t)\right]}{\left\|\dot{\vec{r}}(t)\right\| \left\|\ddot{\vec{r}}(t) \times \dot{\vec{r}}(t)\right\|} \qquad \vec{e}_b(t) = \frac{\dot{\vec{r}}(t) \times \dot{\vec{r}}(t)}{\left\|\dot{\vec{r}}(t) \times \dot{\vec{r}}(t)\right\|} \tag{1.12}$$

These expressions only work in three dimensions because the vector product is just defined in three dimensions. Clearly when we have only one dimension we can only define the tangent vector and with motion in two dimensions we can talk about tangent and normal vector, but not binormal vector. With $1D$ motion there is no problem because its unique vector can be calculated perfectly, but for the $2D$ motion we have a problem with the normal vector because it uses the vector product. To solve this, we will use another expressions presented during the obtaining of the normal vector in terms of $\vec{r}(t)$.

$$\dot{\vec{e}}_t = -\frac{\langle \dot{\vec{r}}(t), \ddot{\vec{r}}(t)\rangle_I}{\left\|\dot{\vec{r}}(t)\right\|^3}\dot{\vec{r}}(t) + \frac{1}{\left\|\dot{\vec{r}}(t)\right\|}\ddot{\vec{r}}(t)$$

$$\left\|\dot{\vec{e}}_t\right\| = \left\| -\frac{\langle \dot{\vec{r}}(t), \ddot{\vec{r}}(t)\rangle_I}{\left\|\dot{\vec{r}}(t)\right\|^3}\dot{\vec{r}}(t) + \frac{1}{\left\|\dot{\vec{r}}(t)\right\|}\ddot{\vec{r}}(t) \right\| =$$

$$\frac{\langle \dot{\vec{r}}(t), \ddot{\vec{r}}(t)\rangle_I}{\left\|\dot{\vec{r}}(t)\right\|^6}\langle \dot{\vec{r}}(t), \dot{\vec{r}}(t)\rangle_I + \frac{1}{\left\|\dot{\vec{r}}(t)\right\|^2}\langle \ddot{\vec{r}}(t), \ddot{\vec{r}}(t)\rangle_I - 2\frac{\langle \dot{\vec{r}}(t), \ddot{\vec{r}}(t)\rangle_I}{\left\|\dot{\vec{r}}(t)\right\|^3}\langle \dot{\vec{r}}(t), \ddot{\vec{r}}(t)\rangle_I =$$

$$\frac{\langle \dot{\vec{r}}(t), \ddot{\vec{r}}(t)\rangle_I}{\left\|\dot{\vec{r}}(t)\right\|^4} + \frac{\left\|\ddot{\vec{r}}(t)\right\|}{\left\|\dot{\vec{r}}(t)\right\|^2} - 2\frac{\langle \dot{\vec{r}}(t), \ddot{\vec{r}}(t)\rangle_I^2}{\left\|\dot{\vec{r}}(t)\right\|^3}$$

1.4 Advanced concepts

1.4.1 Arc length

$$L = \int_a^b dl = \int_a^b \left\|\vec{v}(t)\right\| dt \tag{1.13}$$

1.4.2 Curvature

$$\kappa_C = \frac{d\theta}{dl} \tag{1.14}$$

$$\kappa_C(t) = \frac{\left\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\right\|}{\left\|\dot{\vec{r}}(t)\right\|^3} \tag{1.15}$$

1.4.3 Osculator plane

$$\pi : P(t) + \langle \vec{v}, \vec{a}\rangle_{\mathbb{R}} \tag{1.16}$$

1.4.4 Curvature radius and curvature centre

$$\rho_C(t) = \frac{\left\|\dot{\vec{r}}(t)\right\|^3}{\left\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\right\|} \tag{1.17}$$

$$\vec{r}_C(t) = \vec{r}(t) + \rho_C(t)\vec{e}_n = \vec{r}(t) + \frac{\left\|\dot{\vec{r}}(t)\right\|^2}{\left\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\right\|}\dot{r}(t) \times \left[\ddot{\vec{r}}(t) \times \dot{\vec{r}}(t)\right] \tag{1.18}$$

1.4.5 Torsion

$$\tau(t) = \frac{\left[\dot{\vec{r}}(t), \ddot{\vec{r}}(t), \dddot{\vec{r}}(t)\right]}{\left\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\right\|^2} \tag{1.19}$$

1.5 Relations

$$\begin{bmatrix} \dot{\vec{e}}_t(t) \\ \dot{\vec{e}}_n(t) \\ \dot{\vec{e}}_b(t) \end{bmatrix} = \left\|\dot{\vec{r}}(t)\right\| \begin{bmatrix} 0 & \kappa_C(t) & 0 \\ -\kappa_C(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \vec{e}_t(t) \\ \vec{e}_n(t) \\ \vec{e}_b(t) \end{bmatrix} \tag{1.20}$$

1.6 Expressions with spacial dependence

1.7 Other expressions

[In two dimensions] We can decompose the tangent vector in the vectors $\vec{e}_x$ and $\vec{e}_y$ as

$$\vec{e}_t = \cos\theta\vec{e}_x + \sin\theta\vec{e}_y, \tag{1.21}$$

and therefore,

$$\dot{\vec{e}}_t = -\sin\theta\dot{\theta}\vec{e}_x + \cos\theta\dot{\theta}\vec{e}_y \Rightarrow \left\|\dot{\vec{e}}_t(t)\right\| = \dot{\theta}\left(\sin^2\theta + \cos^2\theta\right) = \left\|\dot{\theta}(t)\right\|. \tag{1.22}$$

With that, we can rephrase the curvature with another expression.

$$\kappa_C(t) = \left\|\frac{d\theta}{ds}\right\| = \left\|\frac{d\theta/dt}{ds/dt}\right\| = \frac{\left\|\dot{\theta}(t)\right\|}{\left\|\dot{\vec{r}}(t)\right\|} = \frac{\left\|\dot{\vec{e}}_t\right\|}{\left\|\vec{v}(t)\right\|} \tag{1.23}$$

Chapter 2

Application of the concepts in different coordinate systems

2.1 Movement in one dimension

Velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\left[x\vec{e}_x\right]}{dt} = \frac{dx}{dt}\vec{e}_x = \dot{x}\vec{e}_x \tag{2.1}$$

Acceleration

$$\vec{a} = \frac{d^2\left[x\vec{e}_x\right]}{dt^2} = \frac{d^2x}{dt^2}\vec{e}_x = \ddot{x}\vec{e}_x \tag{2.2}$$

Tangent vector

$$\vec{e}_t = \frac{\vec{v}(t)}{\left\|\vec{v}(t)\right\|} = \frac{x}{|x|}\vec{e}_x = \operatorname{sgn}(x)\vec{e}_x \tag{2.3}$$

Arc length

$$L = \int_{t_0}^{t} \left\|\vec{v}(t)\right\| dt' = \int_{t_0}^{t} \dot{x}dt' = x(t) - x(t_0) \tag{2.4}$$

2.2 Movement in two dimensions: Cartesian coordinates

2.2.1 Temporal dependence

Velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\left[x\vec{e}_x + y\vec{e}_y\right]}{dt} = \frac{dx}{dt}\vec{e}_x + \frac{dy}{dt}\vec{e}_y = \dot{x}\vec{e}_x + \dot{y}\vec{e}_y \tag{2.5}$$

Acceleration

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2\left[x\vec{e}_x + y\vec{e}_y\right]}{dt^2} = \frac{d^2x}{dt^2}\vec{e}_x + \frac{d^2y}{dt^2}\vec{e}_y = \ddot{x}\vec{e}_x + \ddot{y}\vec{e}_y \tag{2.6}$$

Tangent vector

$$\vec{e}_t = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\dot{x}\vec{e}_x + \dot{y}\vec{e}_y}{\sqrt{\dot{x}^2 + \dot{y}^2}} \tag{2.7}$$

Normal vector

$$\ddot{\vec{r}} \times \dot{\vec{r}} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \ddot{x} & \ddot{x} & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \begin{vmatrix} \ddot{x} & \ddot{y} \\ \dot{x} & \dot{y} \end{vmatrix} \vec{e}_z = \left(\ddot{x}\dot{y} - \dot{x}\ddot{y}\right)\vec{e}_z$$

$$\dot{\vec{r}} \times \left[\ddot{\vec{r}} \times \dot{\vec{r}}\right] = \left[\dot{x}\vec{e}_x + \dot{y}\vec{e}_y\right] \times \left(\ddot{x}\dot{y} - \dot{x}\ddot{y}\right)\vec{e}_z = \dot{x}\left(\ddot{x}\dot{y} - \dot{x}\ddot{y}\right)\vec{e}_x \times \vec{e}_z + \dot{y}\left(\ddot{x}\dot{y} - \dot{x}\ddot{y}\right)\vec{e}_y \times \vec{e}_z =$$
$$\dot{x}\left(\dot{x}\ddot{y} - \ddot{x}\dot{y}\right)\vec{e}_y + \dot{y}\left(\ddot{x}\dot{y} - \dot{x}\ddot{y}\right)\vec{e}_x$$

$$\vec{e}_n = \frac{\dot{\vec{r}} \times \left[\ddot{\vec{r}} \times \dot{\vec{r}}\right]}{\left\|\dot{\vec{r}}(t)\right\| \left\|\ddot{\vec{r}}(t) \times \dot{\vec{r}}(t)\right\|} = \frac{\dot{x}\left(\dot{x}\ddot{y} - \ddot{x}\dot{y}\right)\vec{e}_y + \dot{y}\left(\ddot{x}\dot{y} - \dot{x}\ddot{y}\right)\vec{e}_x}{\sqrt{\dot{x}^2 + \dot{y}^2}\left(\ddot{x}\dot{y} - \dot{x}\ddot{y}\right)} = \frac{\dot{y}\vec{e}_x - \dot{x}\vec{e}_y}{\sqrt{\dot{x}^2 + \dot{y}^2}} \tag{2.8}$$

Arc length

$$L = \int_{t_0}^{t} \|\vec{v}\| dt' = \int_{t_0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2}dt' \tag{2.9}$$

Curvature

$$\kappa_C = \frac{\left\|\dot{\vec{r}} \times \ddot{\vec{r}}\right\|}{\left\|\dot{\vec{r}}\right\|^3} = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

Curvature radius
Curvature center

2.2.2 Dependence of x

Tangent vector

$$\vec{e}_t = \frac{dx/dt\vec{e}_x + dy/dt\vec{e}_y}{\sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2}} = \frac{dx\vec{e}_x + dy\vec{e}_y}{\sqrt{dx^2 + dy^2}} = \frac{\vec{e}_x + dy/dx\vec{e}_y}{\sqrt{1 + (dy/dx)^2}} = \frac{\vec{e}_x + f'(x)\vec{e}_y}{\sqrt{1 + f'(x)^2}} \tag{2.10}$$

Normal vector

$$\vec{e}_n = \frac{\dot{y}\vec{e}_x - \dot{x}\vec{e}_y}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \cdots = \frac{f'(x)\vec{e}_x - \vec{e}_y}{\sqrt{f'(x)^2 + 1}} \tag{2.11}$$

Arc length

$$L = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + f'(x)^2} dx \tag{2.12}$$

Curvature

$$\tan\theta = \frac{dy}{x} \Rightarrow \theta = \arctan\frac{dy}{dx} \Rightarrow \frac{d\theta}{dx} = \frac{1}{1 + (dy/dx)^2}\frac{d^2y}{dx^2} \Rightarrow d\theta = \frac{1}{1 + (dy/dx)^2}\frac{d^2y}{dx^2}dx$$

$$\kappa_C = \frac{d\theta}{ds} = \frac{1/\left(1 + (dy/dx)^2\right)\left(d^2y/dx^2\right)dx}{\sqrt{1 + \left(dy/dx\right)^2}dx} = \frac{dy^2/dx^2}{\left(1 + \left(dy/dx\right)^2\right)^{3/2}} = \frac{f''(x)}{\left(1 + f'(x)^2\right)^{3/2}} \tag{2.13}$$

Curvature radius
Curvature center

2.3 Movement in two dimensions: polar coordinates

Unit vectors [ ].

$$\begin{cases} \vec{e}_\rho = \cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y \\ \vec{e}_\varphi = -\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y \end{cases} \tag{2.14}$$

Partial derivatives

$$\frac{\partial\vec{e}_\rho}{\partial\rho} = 0 \qquad \frac{\partial\vec{e}_\rho}{\partial\varphi} = -\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y = \vec{e}_\varphi$$

$$\frac{\partial\vec{e}_\varphi}{\partial\rho} = 0 \qquad \frac{\partial\vec{e}_\varphi}{\partial\varphi} = \cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y = -\vec{e}_\rho$$

Vector derivatives [ ].

$$\dot{\vec{e}}_\rho = \frac{d\vec{e}_\rho}{dt} = \frac{\partial\vec{e}_\rho}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\vec{e}_\rho}{\partial\varphi}\frac{d\varphi}{dt} = \dot{\varphi}\vec{e}_\varphi \tag{2.15}$$

$$\dot{\vec{e}}_\varphi = \frac{d\vec{e}_\varphi}{dt} = \frac{\partial\vec{e}_\varphi}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\vec{e}_\varphi}{\partial\varphi}\frac{d\varphi}{dt} = -\dot{\varphi}\vec{e}_\rho \tag{2.16}$$

Velocity

$$\vec{v} = \frac{d\left[\rho\vec{e}_\rho\right]}{dt} = \frac{d\rho}{dt}\vec{e}_\rho + \rho\frac{d\vec{e}_\rho}{dt} = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi \tag{2.17}$$

Acceleration

$$\vec{a} = \frac{d\left[\dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi\right]}{dt} = \frac{d\dot{\rho}}{dt}\vec{e}_\rho + \dot{\rho}\frac{d\vec{e}_\rho}{dt} + \frac{d\left[\rho\dot{\varphi}\right]}{dt}\vec{e}_\varphi + \rho\dot{\varphi}\frac{d\vec{e}_\varphi}{dt} \tag{2.18}$$

$$\ddot{\rho}\vec{e}_\rho + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \left(\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\vec{e}_\varphi - \rho\dot{\varphi}\dot{\varphi}\vec{e}_\rho = \left(\ddot{\rho} - \rho\dot{\varphi}^2\right)\vec{e}_\rho + \left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\vec{e}_\varphi \tag{2.19}$$

Tangent vector

$$\vec{e}_t = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi}{\sqrt{\dot{\rho}^2 + \rho^2\dot{\varphi}^2}} \tag{2.20}$$

Normal vectorn

$$\ddot{\vec{r}} \times \dot{\vec{r}} = \begin{vmatrix} \vec{e}_\rho & \vec{e}_\varphi & \vec{e}_z \\ \ddot{\rho} - \rho\ddot{\varphi}^2 & 2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi} & 0 \\ \dot{\rho} & \rho\dot{\varphi} & 0 \end{vmatrix} = \begin{vmatrix} \ddot{\rho} - \rho\ddot{\varphi}^2 & 2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi} \\ \dot{\rho} & \rho\dot{\varphi} \end{vmatrix}\vec{e}_z = \left(\rho\dot{\varphi}\left(\ddot{\rho} - \rho\ddot{\varphi}^2\right) - \dot{\rho}\left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\right)\vec{e}_z$$

$$\dot{\vec{r}} \times \left[\ddot{\vec{r}} \times \dot{\vec{r}}\right] = \left[\dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi\right] \times \left(\rho\dot{\varphi}\left(\ddot{\rho} - \rho\ddot{\varphi}^2\right) - \dot{\rho}\left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\right)\vec{e}_z =$$

$$\dot{\rho}\left(\rho\dot{\varphi}\left(\ddot{\rho} - \rho\ddot{\varphi}^2\right) - \dot{\rho}\left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\right)\vec{e}_\rho \times \vec{e}_z + \rho\dot{\varphi}\left(\rho\dot{\varphi}\left(\ddot{\rho} - \rho\ddot{\varphi}^2\right) - \dot{\rho}\left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\right)\vec{e}_\varphi \times \vec{e}_z =$$

$$\dot{\rho}\left(\dot{\rho}\left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right) - \rho\dot{\varphi}\left(\ddot{\rho} - \rho\ddot{\varphi}^2\right)\right)\vec{e}_\varphi + \rho\dot{\varphi}\left(\rho\dot{\varphi}\left(\ddot{\rho} - \rho\ddot{\varphi}^2\right) - \dot{\rho}\left(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}\right)\right)\vec{e}_\rho$$

2.4 Movement in three dimensions: Frenet-Serret coordinates

$$\vec{v} = \|\vec{v}\|\,\vec{e}_t \Rightarrow \vec{a} = \frac{d\,\|\vec{v}\|}{dt}\vec{e}_t + \|\vec{v}\|\,\frac{d\vec{e}_t}{dt} = \dot{v}\vec{e}_t + v\dot{\vec{e}}_n \tag{2.21}$$

We could think that to obtain the tangential component of the acceleration vector would be projecting it with the dot product, that is, $a_t = \langle\vec{a}, \vec{e}_t\rangle_I$. In fact, these two forms are equivalent, as we will see now.

$$a_t = \frac{d\,\|\vec{v}\|}{dt} = \frac{d\sqrt{\langle\vec{v},\vec{v}\rangle_I}}{dt} = \frac{1}{2\sqrt{\langle\vec{v},\vec{v}\rangle_I}}\frac{d\langle\vec{v},\vec{v}\rangle_I}{dt} = \frac{1}{2\sqrt{\langle\vec{v},\vec{v}\rangle_I}}2\langle\dot{\vec{v}},\vec{v}\rangle_I = \frac{1}{\|\vec{v}\|}\langle\vec{a},\vec{v}\rangle_I = \left\langle\vec{a}, \frac{\vec{v}}{\|\vec{v}\|}\right\rangle_I = \langle\vec{a},\vec{e}_t\rangle_I$$

With respect to the other component, we could expect that it will have the form $\lambda\vec{e}_2$. To see that, we can do some substitutions.

$$\vec{a}_n = v\dot{\vec{e}}_t = v\left\|\dot{\vec{e}}_t\right\|\vec{e}_n = v^2\frac{\left\|\dot{\vec{e}}_t\right\|}{\|\vec{v}\|}\vec{e}_n = v^2\kappa_C\vec{e}_n = \frac{v^2}{\rho_C}\vec{e}_n \tag{2.22}$$

In the first substitution we already obtained the form we wanted, but the last one is also interesting and we will discuss it later. As in the case of tangencial component, a way to obtain the normal component would be with the relation $a_n = \langle\vec{a}, \vec{e}_n\rangle_I$. We will see now that this is equivalent to the previous expression.

$$a_n = \langle \vec{a}, \vec{e}_n \rangle_I = \left\langle \vec{a}, \frac{\dot{\vec{e}}_t}{\left\|\dot{\vec{e}}_t\right\|} \right\rangle_I = \frac{1}{\left\|\dot{\vec{e}}_t\right\|} \langle \vec{a}, \dot{\vec{e}}_t \rangle_I = \frac{1}{\left\|\dot{\vec{e}}_t\right\|} \left\langle \vec{a}, \frac{\vec{a}v - \vec{v}\dot{v}}{v^2} \right\rangle_I = \frac{1}{\left\|\dot{\vec{e}}_t\right\|} \left\langle \vec{a}, \frac{\vec{a}v - \vec{e}_t a_t}{v} \right\rangle_I =$$

$$\frac{1}{\left\|\dot{\vec{e}}_t\right\| v} \left(a^2 - a_t^2\right) = \frac{a_n^2}{\left\|\dot{\vec{e}}_t\right\| v} \Rightarrow a_n \left\|\dot{\vec{e}}_t\right\| v = a_n^2 \Rightarrow a_n = \left\|\dot{\vec{e}}_t\right\| v$$

The other two expressions of the components of acceleration, $\dot{v}$ and v^2/ρ_C are important because they reflect the geometric properties of these acceleration.
Change of coordinates of velocity.
If i want to decompose the velocity to some component $\vec{e}_i$, it is a good idea to use the frenet serret formulas, since they take directly the speed. It can be done as follows.

$$\vec{v}_i = \langle \vec{v}, \vec{e}_i \rangle_I = v \langle \vec{e}_t, \vec{e}_i \rangle_I \tag{2.23}$$

Chapter 3

Some examples

3.1 Motion in one dimension

3.1.1 Constant velocity

$$v(t) = v = \frac{dx}{dt} \Rightarrow dx = vdt \Rightarrow \int_{x_0}^{x} dx' = v \int_{t_0}^{t} dt' \Rightarrow x - x_0 = v(t - t_0) \Rightarrow$$
$$x(t) = x_0 + v(t - t_0)$$

3.1.2 Constant acceleration

Equation of velocity as a function of time.

$$a(t) = a = \frac{dv}{dt} \Rightarrow dv = adt \Rightarrow \int_{v_0}^{v} dv' = a \int_{t_0}^{t} dt' \Rightarrow v - v_0 = a(t - t_0) \Rightarrow$$
$$v(t) = v_0 + a(t - t_0)$$

Equation of function as a function of position.

$$v(t) = v_0 + a(t - t_0) = \frac{dx}{dt} \Rightarrow dx = \big(v_0 + a(t - t_0)\big)\, dt \Rightarrow \int_{x_0}^{x} dx' = \int_{t_0}^{t} v_0 + a(t' - t_0) dt' \Rightarrow$$
$$x - x_0 = v_0(t - t_0) + \frac{a}{2}(t - t_0)^2 \Rightarrow x(t) = x_0 + v_0(t - t_0) + \frac{a}{2}(t - t_0)^2$$

Equation of velocity as a function of position.

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx} v \Rightarrow adx = vdv \Rightarrow a \int_{x_0}^{x} dx' = \int_{v_0}^{v} dv' \Rightarrow a(x - x_0) = \frac{v^2}{2} - \frac{v_0^2}{2} \Rightarrow$$
$$v^2 = v_0^2 + 2a(x - x_0)$$

3.2 Motion in two dimensions

3.2.1 MCU

It is determined by the fact that ρ is constant.

$$\begin{cases} \rho(t) = \rho \\ \varphi(t) = \varphi(t) \end{cases} \tag{3.1}$$

We know that $\vec{r}(t) = \rho(t)\vec{e}_\rho$. Now, the velocity is

$$\vec{v} = \frac{d\left[\rho\vec{e}_\rho\right]}{dt} = \rho\frac{d\vec{e}_\rho}{dt} = \rho\dot{\varphi}\vec{e}_\varphi := \rho\omega\vec{e}_\varphi. \tag{3.2}$$

And acceleration,

$$\vec{a} = \frac{d\left[\rho\omega\vec{e}_\varphi\right]}{dt} = \rho\frac{d\omega}{dt}\vec{e}_\varphi + \rho\omega\frac{d\vec{e}_\varphi}{dt} = \rho\dot{\omega}\vec{e}_\varphi - \rho\omega\dot{\varphi}\vec{e}_\rho := \rho\alpha\vec{e}_\varphi - \rho\omega^2\vec{e}_\rho \tag{3.3}$$

In this case, where $\rho = \rho_C$, we have that

$$v = \left\|\dot{l}\right\| = \rho|\omega| \tag{3.4}$$

$$a_n = \frac{v^2}{\rho_C} = \frac{v^2}{\rho} = \rho\omega^2 \tag{3.5}$$

Frenet serret equations

$$\vec{e}_t = \frac{\rho\omega\vec{e}_\varphi}{\rho|\omega|} = \pm\vec{e}_\varphi \tag{3.6}$$

$$\vec{e}_n = \mp\frac{\omega\vec{e}r}{|\omega|} = -\vec{e}_r \tag{3.7}$$

Radius of curvature

$$\rho_C = \frac{\|\rho\omega\vec{e}_t\|}{\|\mp\omega\vec{e}_\rho\|} = \rho \tag{3.8}$$

3.2.2 Parabola

Equation of motion

$$\begin{cases} x(t) = x_0 + v_{0x}(t-t_0) \\ y(t) = y_0 + v_{0y}(t-t_0) - \frac{g}{2}(t-t_9)^2 \end{cases} \tag{3.9}$$

Simplified equation

$$\begin{cases} x(t) = v_0\cos\theta t \\ y(t) = v_0\sin\theta t - \frac{g}{2}t^2 \end{cases} \tag{3.10}$$

From these equations we can eliminate t and write the trajectory as

$$y = x\tan\theta - \frac{g\sec^2\theta}{2v_0^2}x^2 = x\tan\theta - \frac{\sec^2\theta}{2L_0}x^2,$$

where we have used $L_0/v_0^2/g)$. By completing squares, we see this last equations is equivalent to

$$y - \frac{L_0\sin^2\theta}{2} = -\frac{\sec^2\theta}{2L_0}\left[x - \frac{L_0\sin{(2\theta)}}{2}\right]^2.$$

Then, in terms of X (the part inside the claudators) and Y (the left side), the trajectory is expressed as

$$Y = -\frac{\sec^2\theta}{2L_0}X^2 \tag{3.11}$$

We know the arc length during the whole trajectory is

$$L = \int_{-L_0\sin{(2\theta)}/2}^{L_0\sin 2\theta/2}\sqrt{1+Y'(X)^2}dX = 2\int_0^{L_0\sin{(2\theta)}/2\sqrt{1+\left(\frac{\sec^2\theta}{L_0}X\right)^2}dX}.$$

By doing the change of variable $X \to u$, such that

$$X = uL_0\cos^2\theta,$$

we get

$$L = 2L_0\cos^2\theta\int_0^{\tan\theta}\sqrt{1+u^2}du,$$

which is an elemental integral (the indefinite integral is $(1/2)\ln{(u+\sqrt{1+u^2})} + (u/2)\sqrt{1+u^2}$). Substituting we get

$$L = L_0\left[\cos^2\theta + \ln{(\tan\theta+\sec\theta)} + \sin\theta\right],$$

that can also be written as

$$L = L_0\left(\frac{\cos^2\theta}{2}\ln\frac{1+\sin\theta}{1-\sin\theta} + \sin\theta\right) \tag{3.12}$$

The graphic of L/L_0 as a function of θ is shown in the following figure. Note that at $\theta = \pi/2$ we have $L = L_0$, as it must be.

Frenet-Serret vectors
Let us parameterize the trajectory as in the equation 3.11 writing $Y = -aX^2$, where $a = \sec^2\theta/2L_0 > 0$ and $X = v_0\cos\theta t - L_0\sin(2\theta)/2$. Therefore,

$$\vec{r} = X\vec{e}_x - aX^2\vec{e}_y, \qquad \vec{v} = \dot{X}\vec{e}_x - 2aX\dot{X}\vec{e}_y, \qquad \vec{a} = -2a\dot{X}^2\vec{e}_y, \tag{3.13}$$

where we have used that $\ddot{X} = 0$. For the tangent vector we have

$$\vec{e}_t = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{e}_x - 2aX\vec{e}_y}{\sqrt{1+4a^2X^2}},$$

that, as we observe, is independent of $\dot{X}$, as it must be, since $\vec{e}_t$ only depends on the trajectory, not in how do we move through it. Differentiating this last expression we get

$$\dot{\vec{e}}_t = -\frac{4a^2X\vec{e}_x + 2a\vec{e}_y}{(1+4a^2X^2)^{3/2}}\dot{X}.$$

Normalizing this vector we get the normal vector

$$\vec{e}_n = \frac{\dot{\vec{e}}_t}{\left\|\dot{\vec{e}}_t\right\|} = -\frac{2aX\vec{e}_x + \vec{e}_y}{\sqrt{1+4a^2X^2}},$$

which does not depend on $\dot{x}$ either for the same reason.
Normal and tangent acceleration and radius of curvature.
The normal acceleration is obtained projecting the acceleration onto the vector $\vec{e}_n$. Hence,

$$a_n = \langle\vec{a}, \vec{e}_n\rangle_I = \frac{2a\dot{X}^2}{\sqrt{1+4a^2X^2}},$$

where we have considered that $\langle\vec{v}, \vec{e}_n\rangle_I = 0$. The normal acceleration can also be written as

$$a_n = \frac{2a}{(1+4a^aX^2)^{3/2}}(1+4a^2X^2)\dot{X}^2 = \frac{2a}{(1+4a^aX^2)^{3/2}}\|\vec{v}\|^2.$$

By comparing with the general expression $a_n = v^2/\rho_C$ we get the radius of curvature of the trajectory, which is

$$\rho_C = \frac{(1+4a^2X^2)^{3/2}}{2a}.$$

The tangential acceleration is

$$a_t = \langle\vec{a}, \vec{e}_t\rangle_I = \frac{4a^2X\dot{X}^2}{\sqrt{1+4a^2X^2}}$$

Center of curvature. Let us remember that the center of curvature of a point $\vec{r}$ of the trajectory has to be placed at a distance ρ_C of $\vec{r}$ in the direction of $\vec{e}_n$, that is, $\vec{r}_C = \vec{r} + \rho_C\vec{e}_n$. Therefore,

$$\vec{r}_C = X\vec{e}_x - aX^2\vec{e}_y - \frac{(1+4a^2X^2)^{3/2}}{2a}\frac{2aX\vec{e}_x + \vec{e}_y}{\sqrt{1+4a^2X^2}} = -4a^2X^3\vec{e}_x - \frac{1+6a^2X^2}{2a}\vec{e}_y$$

The following figure shows the (parabolic) trajectory of a bullet (continuous line) for $v_0 = 100\,\mathrm{m\,s^{-1}}$ and $\theta = \pi/5\,\mathrm{rad}$, and three osculatric circumferences (discontinuous lines) and their corresponding centers. The smallest is tangent to the vertex of the parabola ($t_0 = v_0\sin\theta/g \approx 9.7\,\mathrm{s}$). The other ones are tangent to the trajectory at $t = 7.7\,\mathrm{s}$ and $t = 5.7\,\mathrm{s}$. The normal associated vectors to these points (scaled by a factor 80 to be visible) are also shown in the figure.

Chapter 4

Newton's laws

4.1 Introduction

4.1.1 Historic context

4.2 First Law

Law 1. *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon [1].*

This principle was already formulated by Galileo, and it is called also the *law of inertia.* It is important that this law only works for inertial reference systems [ ].

Definition 4.2.1. Let a be a particle. Then, we say *a is a free particle* if and only if it is not subjected to no net force.

4.3 Second law

Law 2. *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of right line in which that force is impressed.*

Mathematically, we express this law as follows.

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \left[\vec{F}\right] = \mathrm{N} = \mathrm{kg\,m\,s^{-2}} \tag{4.1}$$

Note that we are deriving the momentum, which is a product, so the total derivative would look like this.

$$\vec{F} = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt} \tag{4.2}$$

Generally the mass $m(t)$ will be in fact a constant, but in some cases it will change with time, so we should not forget the first part of the derivative. In the particular case of the constant mass, the second law will look like the equality $\vec{F} = m\vec{a}$.
In this form, the second law seems more a definition than a principle. There is an alternative manner to express the equality that could look more like a principle, which is the following one:

$$\frac{\vec{F}\left(\vec{r}(t), \dot{\vec{r}}(t), t\right)}{m} = \ddot{\vec{r}}(t) \tag{4.3}$$

What we are saying now not only is the relation we expressed before, but also saying that, for a given force $\vec{F}$, there is a equation of motion that satisfies this equality. This is a system of three differential equations of second order, and now it shows the relation between all the magnitudes. It has 2×3 arbitrary constants: $\vec{r}_0 = \vec{r}(0), \dot{\vec{r}}_0 = \dot{\vec{r}}(0)$.

4.3.1 Some details

We denote the force that exercises a point masses 1 to a body 2 by $\vec{F}_{1\to 2}$. Note that, since the force goes from $\vec{r}_1$ to $\vec{r}_2$, the force is proportional to the vector that goes from $\vec{r}_1$ to $\vec{r}_2$, in other words, $\vec{F}_{1\to 2} = \alpha(\vec{r}_2 - \vec{r}_1)$. We do not know the value of α, but if we normalize the vector $\vec{r}_{1\to 2}$, we know that the multiplicative constant will be the norm of the force [ ]. Therefore, we get

$$\vec{F}_{1\to 2} = F\vec{e}_{1\to 2} \tag{4.4}$$

Definition 4.3.1. Let $\vec{r}_1$ and $\vec{r}_2$ the equations of motion and let $\vec{F}_{1\to 2}$. We define the *line of action of the force $\vec{F}_{1\to 2}$* as the line [ ] that passes through $\vec{r}_1$ and has $\vec{F}_{1\to 2}$ as the director subspace [ ].

Let us note an important detail. Since $\vec{e}_2 = \vec{e}_1 + \vec{r}_{1\to 2} = \vec{e}_1 + \alpha\vec{F}_{1\to 2}$ [ ], we have that $\vec{r}_2$ also belongs to the line of action of $\vec{F}_{1\to 2}$.

4.3.2 Determinism and numeric solutions

The second law of motion inspires the ideology of determinism. If we think about each instant as a frame, the equation of force shows us that each frame influences (as the equation states) to the next one, so all the path would be determined by just the first conditions. To see that, let us denote each instant by t_i and the intervals of time $\epsilon = t_{i+1} - t_i$. Thus, we have that position, velocity, acceleration and force at each instant are $\vec{r}_i = \vec{r}(t_i)$, $\vec{v}_i = \vec{v}(t_i)$, $\vec{a}_i = \vec{a}(t_i)$, and $\vec{F}_i = \vec{F}(\vec{r}_i, \vec{v}_i, t_i)$, respectively.
If we approximate velocity and acceleration as being changed regularly between frames, we will obtain the following relations.

$$\vec{v}_i = \frac{\vec{r}_{i+1} - \vec{r}_i}{\epsilon} \Rightarrow \vec{r}_{i+1} = \vec{r}_i + \epsilon\vec{v}_i \tag{4.5}$$

$$\vec{a}_i = \frac{\vec{v}_{i+1} - \vec{v}_i}{\epsilon} \Rightarrow \vec{v}_{i+1} = \vec{v}_i + \epsilon\vec{a}_i = \vec{v}_i + \epsilon\frac{\vec{F}_i}{m} \tag{4.6}$$

This shows how a frame influences the next one and, generally, the whole path. Apart from that, this method of approximation, called the Euler's method, can also be useful to solve some special case where we need numerical answers. We will discuss it later [ ].

4.4 Third law

Law 3. *To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

One could wonder if this law is really necessary as the previous ones. At first, it seems a principle as the other two, but let as see a situation that could make us doubt about this.
Let be two m_1 and m_2 together to which we apply some force $\vec{F}$. If we study the two objects as a unique system, the acceleration will result as follows.

$$\vec{F} = \frac{d\left[m_1 + m_2\right]}{dt}\vec{v} + (m_1 + m_2)\frac{d\vec{v}}{dt} = (m_1 + m_2)\vec{a} \Rightarrow \vec{a} = \frac{\vec{F}}{m_1 + m_2}$$

Let us now study the second mass. We know that it will move, so there should be a force pushing it. We will denote this force that goes from the first object to the second by $\vec{F}_{1\to2}$. We also know that since the two bodies are together they will have the same acceleration (and the same as that of the global system). Therefore, if we apply the second law to m_1 we get that

$$\vec{F}_{1\to2} = \frac{d\vec{p}_2}{dt} = m_2\vec{a} = m_2\frac{\vec{F}}{m_1 + m_2}$$

That remains the first body to study. For that, we will suppose an imaginary force denoted by $\vec{F}_{2\to1}$ that will push it (in case our supposition was not correct, after some calculations the value of this force will be 0). Applying now the second law, we obtain the following result.

$$\vec{F} + \vec{F}_{2\to1} = \frac{d\vec{p}_1}{dt} = m_1\vec{a} = m_1\frac{\vec{F}}{m_1 + m_2} \Rightarrow \vec{F}_{2\to1} = m_1\frac{\vec{F}}{m_1 + m_2} - \vec{F} =$$
$$m_1\frac{\vec{F}}{m_1 + m_2} - (m_1 + m_2)\frac{\vec{F}}{m_1 + m_2} = -m_2\frac{\vec{F}}{m_1 + m_2} \Rightarrow \vec{F}_{2\to1} = -\vec{F}_{1\to2}$$

That means that there exists an equal force but opposed to $\vec{F}_{1\to2}$, which is what the third law states. Then, it is the action-reaction law unnecessary? The answer is that the third law is in fact necessary, and is not something we can prove from the other ones. With the previous example it seems that we have performed a sort of proof, but actually we have made a mistake. The error is that we have supposed the previous laws to work for systems with more than one object. This is not stated, so we can not apply the equations to two masses. The reality is that, having the third law of motion is what allows us to use the other laws to more objects. In other words, the capability to study several bodies is a consequence of the third law.

4.4.1 Some details

We have seen that in a system with two point masses and a force $\vec{F}_{1\to 2}$, the vector position of both objects belong to the line of action of that force [ ]. We just saw also that, for the third law, there is another force of the $\vec{F}_{2\to 1}$, and since $\vec{F}_{2\to 1} = -\vec{F}_{1\to 2}$ (are proportional), this new force is also a director subspace of the line of action. Besides, since $\vec{r}_2$ belongs to the line [], we see that it is possible determine the line of action by the pair by the position of the first object and its force or the position of the second object and its force.
That has an immediate consequence that will be useful in the future [ ]. Let us suppose again a system two particles an a force as before. Let us suppose too that there are no more forces (apart from that emerged from the third law) between this particles or any other external force.

Limitations

The third law is not absolutely general, mathematically speaking (physically speaking, we should know every possible force in the universe and see if each one satisfies the third law). It is true for central forces, since

$$\vec{F}_{1\to 2} = f(r)\vec{e}_{1\to 2} = f(\|\vec{r}_2 - \vec{r}_1\|)\vec{e}_{1\to 2} = f(\|\vec{r}_1 - \vec{r}_2\|)(-1)\vec{e}_{2\to 1} = -\vec{F}_{2\to 1},$$

but it is not necessarily true for other kind of forces, like velocity dependent interactions.

4.5 Interpretations

Marion (p.50): the first and second laws are more as definitions for force. The third law, however, is indeed a law, since it concerns the real relations between physical phenomena.
Lindsay and Margenau(Li36): the first and second laws are in fact laws, and the third law is a consequence of these two.

4.6 Some forces

4.6.1 Gravitational force

We know the mathematical expression for gravity for the Newton's Law of Universal Gravitation. In case of the Earth, it can be approximated as the second expression.

$$\vec{F}_{1\to 2} = -G\frac{m_1 m_2}{\|\vec{r}_2 - \vec{r}_1\|^2}\frac{\vec{r}_2 - \vec{r}_1}{\|\vec{r}_2 - \vec{r}_1\|} \qquad \vec{F}_p \approx -mg\vec{e}_z \tag{4.7}$$

Let us suppose a system of two objects in the ground such that m_1 is above m_2. Let us study each object separately.
The object of above, m_1, is pushed to the ground because of the gravity caused by m_2 and by the mass of the Earth, However, it does not move, which means there is another extra force that pulls it upwards and compensates the gravity. We call it the *normal force*, and it is perpendicular to the surface of contact [ ]. We should not think the normal force is the *reaction* of the gravity. As we have seen, if there is a force that exercises an object a to an object b, the reaction goes from b to a. Normal force and gravitational force act to the same object, so they cannot be on reaction to the other. We will see the reactions of these forces in the next objects.
The second object is pushed by the gravity of m_1 and the Earth. The first one is the reaction from the force of gravity caused by m_2 that acted to m_1. Since it does not moves either, there is a force compensating the forces of gravity (which are clearly not balanced). This is a normal force that goes from m_1 to m_2 (reaction from $\vec{F}_{N,2\to 1}$) and one that goes from the Earth to m_2.
Finally, the Earth is pushed by m_1 and m_2 and balanced by a normal force $\vec{F}_{N,2\to E}$. These two gravitational force are the reactions of $\vec{F}_{g,E\to 1}$ and $\vec{F}_{g,E\to 2}$ respectively.
If we take off the mass m_2, m_1 will be in the air. In that moment there is no a normal force because it is not in contact with anything, but the force of gravity of the Earth continues pushing

it. Therefore the forces are not balanced and, by the Newton's second Law, it will accelerate. In particular,

$$\vec{a} = \frac{F}{m_1} = -\frac{1}{m_1} G \frac{M m_1}{r^2} \vec{e}_z = -G \frac{M}{r^2} \vec{e}_Z \tag{4.8}$$

Which shows the acceleration caused by gravity does not depends on the mass of the object. This field of acceleration is called the *gravitational field*.

Artificial gravity

The *weight* we are used to think about is not actually the force that gravity exerts on us. In fact, our concept of weight consists on the normal force we feel when we are subjected to an acceleration (generally produced by gravity). The greater normal force we feel, the greater the normal force the weighing machine feels (since it is its reaction), and therefore, the larger the number shown.
As we have seen, an object that moves in a circular trajectory presents a normal acceleration to the center. Let us take some large object with the shape of a ring and a person in the edge. The object is initially at rest, and when the object starts rotating, he/she will feel an acceleration in the perpendicular direction of the surface. This is equivalent to the surface being at rest and the person having the same acceleration but in the opposite direction. Hence, he/she will experience this situation as having a gravity, concretely $ma_n = m\omega^2 R$. With that, we conclude that with some specific values for ω and R, we could create an "artificial gravity" that would seem identical to the real gravity.
Note that, if we have a construction like this we could simulate gravity with a construction we have just observed, it would only work at a certain distance of the center. If we approach or go away from the center, we will change R and therefore the gravity will feel. In particular, at the center $R = 0$ so we will experience weightlessness.

Equivalence principle

Let us suppose there is a mass m in an elevator that is broken. As we have seen, all objects will fall at the same acceleration (g), so we could wonder what number would show a weighing machine if the mass is above it in the elevator. We know that this number is in fact the norm of the normal force [ ], so we just need to find out which is its value.

$$\sum F = ma \Rightarrow N - mg = ma = mg \Rightarrow N = 0$$

Therefore, we would not notice any weight, and it is the same as if we were in a system without gravity. This phenomenon is called the *equivalence principle*

Inertial mass and gravitational mass

We have seen previously the definition of inertial mass []. Now, we have just seen the gravitational force acts proportionally to some variable m, which is called the gravitational mass. It has been determined experimentally that these two things are equal []. This equality leads to the Equivalence Principle.
The first experiments realized to prove that are known from Galileo and Newton. Galileo threw two objects from the Pisa tower to see they fall at same acceleration and Newton measured the periods of different pendulum with objects of different masses but same form. More precise experiments were done by Eötvös, and Dicke (using concepts explained in future sections).

Superposition principle

To deal with a system of masses (discrete or continuous), we must **assume** the superposition principle.

4.6.2 Spring force

$$\vec{F}_s = -kx\vec{e}_x \Rightarrow x(t) = A\cos(\omega t + \phi_0), \omega = \sqrt{\frac{k}{m}} \tag{4.9}$$

Pendulum of length l.

$$\omega^2 = \frac{g}{l} \tag{4.10}$$
$$s = \theta l, x = l\sin\theta \tag{4.11}$$
$$A_0 = l\theta_0 \approx l\sin\theta_0 \tag{4.12}$$

4.6.3 Friction

As we have seen, in a static body there is the gravitational force and the normal force. Let us suppose we are pushing a big object with a small force. This object does not move, so there is an extra force that compensates us. We call this force *friction*.
The force of friction acts in the opposite direction as the force is acting, and if we increase a bit our force, the object continue without moving. This means that the force of friction is also increasing and that is proportional to the applied force.
However, if we continue increasing our force, at some point we achieve to move the object. Now, friction is constant. Besides, by experimental results we can see that this force does not depends on the surface of contact, only on the mass of the object. In summary, the force of friction acts as follows.

$$F_f = \begin{cases} F, F \leq \mu_s F_N \\ \mu_k F_N, F > \mu_s F_N \end{cases} \qquad \mu_k < \mu_s \tag{4.13}$$

Example of horizontal plane (when acting the kinetic coefficient).
Example of inclined plane (also with the kinetic coefficient).

Recent observations

Friction is still a field of research. Recent discoveries have shown the force of friction is not directly proportional to the load but to the microscopic area of contact between two objects (not the apparent contact area). Then, μN is in fact an approximation. We also believe that the static friction is larger than kinetic friction because the bonding of atoms between the two objects does not have as much time to develop in kinetic motion.

4.6.4 Fluid friction

The dependence can vary in different orders of the velocity.

Example 4.6.1. Let us take the first order, that is, $F_f = -kv$. If we take the initial conditions $x_0 = y_0 = 0$, the equations of motion result in

$$\begin{cases} x(t) = \dfrac{v_0\cos\theta}{k}\left(1 - e^{-kt}\right) \\ y(t) = -\dfrac{gt}{k} + \dfrac{kv_0\sin\theta + g}{k^2}\left(1 - e^{-kt}\right) \end{cases}.$$

Let us see one representation of parabolic (not actual parabola) for different values of k.
To see the range of the motion, we get a transcendental equation, and we can solve it by perturbation method (gives a linear approximation) and numerical method.

4.6.5 Tension

In physics we usually consider systems with rows without mass. We will see now what does it imply. As every other object, it satisfies the Laws of Motion, and will experience a force in one extreme, called *tension* and denoted by T, and another in the other extreme, denoted by T'. If we apply the second law and consider the row does not have mass, we get

$$T - T' = ma = 0 \cdot a = a \Rightarrow T = T'. \tag{4.14}$$

As we see, T and T' have the same magnitude only when the row does not have mass. Otherwise, they would be different.

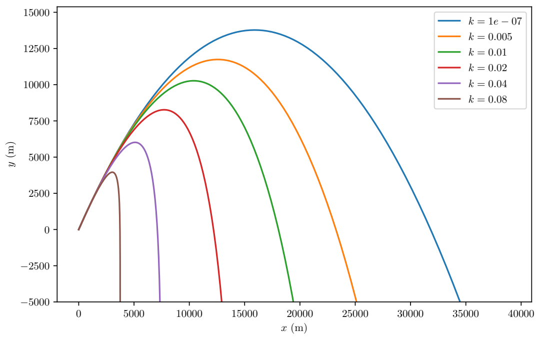

Figure 4.1: Trajectory of the particle for different values of k.

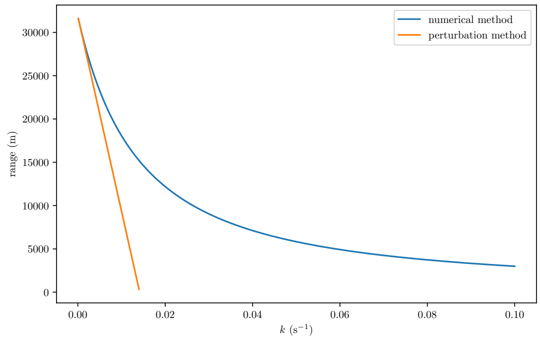

Figure 4.2: Range as a function of k calculated by numerical method and perturbation method.

4.6.6 Pulleys

Let us now consider now a pulley without mass and frictionless. Apply the second law of motion, we get

$$T_r \cos\theta - T_l \cos\theta = ma = 0 \cdot a = 0 \Rightarrow T_r = T_l \tag{4.15}$$

Note this is independent on if the row has mass or not. If there was friction or mass, they would be different.

4.6.7 Rotation

The static friction causes the wheel to rotate.
We do not consider where the force is applied.
Let us suppose there is a circle rotating without sliding at an angular velocity ω. Since it is not sliding, the linear velocity of the surface and of the object have the same value and it is $v = \omega R$. The point of contact between the circle and the floor has no velocity, since the velocity of the object in one direction is countered by the linear velocity in the opposite direction. By contrast, in the highest point has a velocity of $2v$ because now the velocity of translation and rotation have the same direction. The responsible of the rotation without sliding is the static friction, since if there wasn't friction the object would not rotate, and if the friction was dynamic the velocity would not be constant.

Example 4.6.2. ABS brakes
Let us suppose a will that is rotating without sliding. In the ABS brakes system there are two forces with the same magnitude acting in opposite directions. By the second law of motion, this forces will balance each other and therefore the wheel would not brake*. Therefore, the responsible to brake the wheel is the force of friction (static) with the floor. If we exceed the value of $\mu_s g$ (maximum acceleration), the point of contact will have a nonzero velocity and the wheel will start sliding. Then, the wheel will be subjected to the forces $F - F_f = ma$, so the new (dynamic) acceleration will have a smaller value. The ABS system puts brakes to not exceed the value of $\mu_s g$.
*Note: Note this is not correct because we are not considering the rotation, that we will study in the future.

Example 4.6.3. Suppose a bucket is tied to a rope and we make it rotate. What would be the minimum velocity at the highest point? Since the motion is circular, there will be a normal acceleration, but since it is minimum, the normal force between the bucket and the water is 0. With that, we get

$$R\omega^2 \geq \frac{F_n}{m} = \frac{mg + N}{m} = \frac{mg + 0}{m} = g \Rightarrow R\omega^2 \geq g$$

4.7 Systems of reference

We will considered only changes of references with translations (no rotations, for example).
We will follow the notation of $\vec{r}$ for the stationary observer, $\vec{r}'$ for the observer in movement, and $\vec{R}$ the difference of position between these observers.
We will accept the following axioms.

Axiom 1. *Time is absolute, that is, it does not depend on the movement of the observer. Mathematically, we can express this axiom as follows.*

$$t - t' = \alpha \tag{4.16}$$

Being α a constant.

Axiom 2. *Space is absolute.*

The relation of positions with respect to each observer is the next one.

$$\vec{r}(t)' = \vec{r}(t) - \vec{R}(t) \tag{4.17}$$

Let us suppose that $\vec{R}(t)$ is a function at least 2 times differentiable. Then, we have that

$$\dot{\vec{r}}'(t) = \dot{\vec{r}}(t) - \dot{\vec{R}}(t) \Rightarrow \ddot{\vec{r}}'(t) = \ddot{\vec{r}}(t) - \ddot{\vec{R}}(t) \Rightarrow m\ddot{\vec{r}}'(t) = m\ddot{\vec{r}}(t) - m\ddot{\vec{R}}(t) \Rightarrow m\ddot{\vec{r}}'(t) = \vec{F}(t) - m\ddot{\vec{R}}(t)$$

Therefore, if change the system of reference, we do not have the expression $m\ddot{\vec{r}}'(t) = \vec{F}(t)$ that states the 2nd Law [ ], but a variation.
We call this extra term the *inertial form*, and we denote it by $\vec{F}_{iner}$ but with the sign changed, that is

$$\vec{F}_{iner}(t) := -m\ddot{\vec{R}}(t) \tag{4.18}$$

We can see that the second law is satisfied if $\vec{R}(t)$ is less than 2 times differentiable. In this case we say that it is an *inertial system of reference.*
One example of changing of system of reference with acceleration is when we are in a car that is rotating (has a normal acceleration). In this case, we notice a force that pushes us to the most external (with respect to the circular trajectory) car wall.

4.7.1 Galileo transformations

We say the system has a standard configuration if $O' = O$ and $t' = t = 0$.
Can relate this to a matrix transformation.
We can see that some concepts are preserved: force, mass, time, distance

$$\begin{cases} x' = x + vt \\ y' = y \\ z' = z \end{cases}, \quad \begin{cases} v'_x = v_x - V \\ v'_y = v_y \\ v'_z = v_z \end{cases}, \quad \begin{cases} \vec{a}' = \vec{a} \\ \vec{F}' = \vec{v} \end{cases} \tag{4.19}$$

With these transformations we can see that physical laws do not change (covariance) [ ].
*Ways to construct more inertial systems of reference: translation from the origin, reorientation, change of time, $v(t) = v$, reflection.

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Chapter 5

Dynamics for system of particles I

5.1 Revision of Newton's Third Law

From the section 4.4, we have seen the statement of Newton's Third Law which have allowed us to derive a great amount of important consequences. This statement was expressed as follows.

Law 3 (Weak form). *To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

However, as it is written in parenthesis, this is in fact the *weak form* of the third law, although it was the original expression. We call it this way because to find the properties derived in the following sections, we need a stronger form of the third law. Since it has more restriction, the number of cases where it is applicable is smaller, but still enough to study most of classical interactions.

Law 3 (Strong form). *The forces exerted by two particles i, j on each other, in addition to being equal and opposite, must lie on the straight line joining the two particles.*

5.2 Linear momentum of a system

We have seen previously the definition of linear momentum [ ], and that if there is no force, then linear momentum is preserved. We could ask about what would happen if we have a system of several particles. First, we have to define some concepts.

Definition 5.2.1. Let $\mathbb{S}$ be a system of n particles. We define the *linear momentum of the system* as the sum of all linear momentums of each particle of the $\mathbb{S}$.

$$\vec{p}_{\mathbb{S}} = \sum_{i=1}^{n} \vec{p}_i \tag{5.1}$$

Note that, if there is another reference system S' that is inertial, then $\vec{p}_{\mathbb{S}} = \vec{p}'_{\mathbb{S}}$ [ ].
In this system, the total force applied to a particle i is

$$\vec{F}_{\to i} = \vec{F}^{\text{ext}}_{\to i} + \sum_{j=1}^{n} \vec{F}_{j\to i}, \tag{5.2}$$

and considering that $\vec{F}_{i\to i} = 0$.

Definition 5.2.2. Let $\mathbb{S}$ be a system of n particles. We define the *external force applied to the system* as the sum of all external forces that act in each particle of the $\mathbb{S}$.

$$\vec{F}^{\text{ext}}_{\mathbb{S}} = \sum_{i=1}^{n} \vec{F}^{\text{ext}}_{i} \tag{5.3}$$

Again, if there is another reference system S' that is inertial, then $\vec{F}_{\mathbb{S}} = \vec{F}'_{\mathbb{S}}$ [ ].
From these definitions, we haven an important consequence.

Proposition 5.2.1. *Let $\mathbb{S}$ be a system of n particles whose interactions obey the weak form of Law 3. Then,*

$$\frac{d\vec{p}_{\mathbb{S}}}{dt} = \vec{F}^{ext}_{\mathbb{S}}. \tag{5.4}$$

Proof.

$$\dot{\vec{p}}_{\mathbb{S}} = \sum_{i=1}^{n} \dot{\vec{p}}_i = \sum_{i=1}^{n} \vec{F}_{\to i} = \sum_{i=1}^{n} \left(\vec{F}^{\text{ext}}_{\to i} + \sum_{j=1, j\neq i}^{n} \vec{F}_{j\to i} \right) = \sum_{i=1}^{n} \vec{F}^{\text{ext}}_{\to i} + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \vec{F}_{j\to i}$$

Let us see the second summation has a 0 value. Since these forces obey the weak form of the Netown's Third Law,

$$\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \vec{F}_{j\to i} = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} -\vec{F}_{i\to j} = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} -\vec{F}_{j\to i} = -\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \vec{F}_{j\to i}$$

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \vec{F}_{j\to i} = 0.$$

Therefore,

$$\dot{\vec{p}}_{\mathbb{S}} = \sum_{i=1}^{n} \vec{F}^{\text{ext}}_{\to i} + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \vec{F}_{j\to i} = \sum_{i=1}^{n} \vec{F}^{\text{ext}}_{\to i} = \vec{F}^{\text{ext}}_{\mathbb{S}}$$

■

Corollary 5.2.2. *If a system $\mathbb{S}$ is isolated (there are no external forces), then the momentum of the system is conserved.*

Corollary 5.2.3. *If a system $\mathbb{S}$ has a net force $\vec{0}$ in one component, then the momentum of the system in that component is conserved.*

Mass of a system $\mathbb{S}$.

$$m_{\mathbb{S}} = \sum_{i=1}^{n} m_i \tag{5.5}$$

Definition 5.2.3. The center of mass.

$$\vec{r}_{\text{CM}} := \frac{1}{m_{\mathbb{S}}} \sum_{i=1}^{n} m_i \vec{r}_i \tag{5.6}$$

The center of mass is the weighted average (in masses) of the positions that occupy the n particles of the system $\mathbb{S}$.
The center of mass does not have to coincide in position with any particle of the system.
If we differentiate it, we get

$$\dot{\vec{r}}_{CM} = \frac{1}{m_{\mathbb{S}}} \sum_{i=1}^{n} m_i \dot{\vec{r}}_i = \frac{1}{m_{\mathbb{S}}} \sum_{i=1}^{n} \vec{p}_i = \frac{\vec{p}_{\mathbb{S}}}{m_{\mathbb{S}}} \Rightarrow \vec{p}_{\mathbb{S}} = m_{\mathbb{S}} \dot{\vec{r}}_{CM}, \tag{5.7}$$

that is, the linear momentum of a system is the product of its mass and the velocity of its center of mass.
Since the absence of external forces implied the conservation of momentum of the system, by the above equality, the velocity of the center of mass too. In case there was some external force, we would have

$$\vec{F}^{\text{ext}}_{\mathbb{S}} = \dot{\vec{p}}_{\mathbb{S}} = m_{\mathbb{S}} \ddot{\vec{r}}_{\mathbb{S}} \tag{5.8}$$

which has the form of the Newton's second Law.

Corollary 5.2.4. *For an isollated system, the velocity of the center of mass is constant.*

Continuous case of center of mass.

$$\vec{r}_{CM} = \frac{1}{m_{\mathbb{S}}} \int_V \vec{r} dm, \qquad m_{\mathbb{S}} = \int dm = \int_V \rho(\vec{r}) dV \tag{5.9}$$

5.3 Angular momentum and torque

5.3.1 One particle

Angular momentum (with respect to some origin O)

$$\vec{L}_O = \vec{r} \times \vec{p} \tag{5.10}$$

Torque (with respect to some origin O)

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{5.11}$$

From the definitions, we have

$$\dot{\vec{L}}_O = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} = \vec{\tau}_O \tag{5.12}$$

This means that there is no torque, the angular momentum is conserved.

5.3.2 System of particles

Angular momentum (with respect to some origin O)

$$\vec{L}_{\mathbb{S},O} = \sum_{i=1}^{n} \vec{L}_O \tag{5.13}$$

Torque (with respect to some origin O)

Definition 5.3.1. Let $\mathbb{S}$ be a system of n particles. We define *the external torque applied to the system* as the sum of all external torques that act in each particle of the $\mathbb{S}$.

$$\vec{\tau}_{\mathbb{S},O}^{\text{ext}} = \sum_{i=1}^{n} \vec{\tau}_O^{\text{ext}} \tag{5.14}$$

Proposition 5.3.1. *Let $\mathbb{S}$ be a system of n particles where the internal forces obey the strong form of Law 3. Then,*

$$\vec{\tau}_{\mathbb{S},O}^{ext} = \dot{\vec{L}}_{\mathbb{S},O}, \tag{5.15}$$

Proof.

$$\vec{L}_{\mathbb{S},O} = \sum_{i=1}^{n} \vec{L}_i \Rightarrow \dot{\vec{L}}_{\mathbb{S},O} = \sum_{i=1}^{n} \frac{d}{dt}\left(\vec{r}_i \times \vec{p}_i\right) = \sum_{i=1}^{n} \vec{v}_i \times \vec{p}_i + \vec{r}_i \times \vec{F}_{\to i} = \sum_{i=1}^{n} \vec{v}_i \times \vec{p}_i + \vec{r}_i \times \vec{F}_{\to i} =$$

$$\sum_{i=1}^{n} \vec{r}_i \times \left(\vec{F}_{\to i}^{\text{ext}} + \sum_{j=1}^{n} \vec{F}_{j\to i}\right) = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{\to i}^{\text{ext}} + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \vec{r}_i \times \vec{F}_{j\to i}$$

Let us see the second summation is 0. Since $\vec{F}_{j\to i} = -\vec{F}_{i\to j}$,

$$\sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \vec{r}_i \times \vec{F}_{j\to i} = \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \vec{r}_i \times \left(-\vec{F}_{i\to j}\right) = -\sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \vec{r}_i \times \vec{F}_{i\to j} =$$

$$-\sum_{j=1}^{n} \sum_{i=1,i\neq j}^{n} \vec{r}_j \times \vec{F}_{j\to i} \Rightarrow \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \vec{r}_i \times \vec{F}_{j\to i} =$$

$$\frac{1}{2}\left(\sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \vec{r}_i \times \vec{F}_{j\to i} + \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \vec{r}_i \times \vec{F}_{j\to i}\right) =$$

$$\frac{1}{2}\left(\sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \vec{r}_i \times \vec{F}_{j\to i} - \sum_{j=1}^{n} \sum_{i=1,i\neq j}^{n} \vec{r}_j \times \vec{F}_{j\to i}\right) = \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \left(\vec{r}_i - \vec{r}_j\right) \times \vec{F}_{j\to i}.$$

The proposition requires the strong form of the Newton's Third Law, so we always have $\left(\vec{r}_i - \vec{r}_j\right) \times \vec{F}_{j\to i} = \vec{0}$. Finally,

$$\vec{L}_{\mathbb{S},O} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{\to i}^{\text{ext}} + \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \vec{r}_i \times \vec{F}_{j\to i} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{\to i}^{\text{ext}} = \vec{\tau}_{\mathbb{S},O}^{\text{ext}}.$$

■

Therefore, if there is no external torque, then angular momentum is conserved.
Equilibrium conditions.

$$\vec{F}_{\mathbb{S}}^{\text{ext}} = \vec{\tau}_{\mathbb{S},O}^{\text{ext}} = 0 \tag{5.16}$$

Proposition 5.3.2. *If a system $\mathbb{S}$ is in equilibrium conditions with respect to an origin point O, then it is in equilibrium condition with respect to any other point.*

Changes of origin points (but static ones).

$$\vec{L}_{\mathbb{S},O'} = \vec{L}_{\mathbb{S},O} + \vec{r}(O)_{O'} \times \vec{p}_{\mathbb{S}} \tag{5.17}$$

$$\vec{\tau}^{\text{ext}}_{\mathbb{S},O'} = \vec{\tau}^{\text{ext}}_{\mathbb{S},O} + \vec{r}(O)_{O'} \times \vec{F}^{\text{ext}}_{\mathbb{S}} \tag{5.18}$$

Gravity (supposing constant field).

$$d\vec{F}_g = gdm(-\vec{e}_z) \Rightarrow d\vec{\tau}_g = gdm\vec{r} \times (-\vec{e}_z) \tag{5.19}$$

center of gravity.

$$\vec{r}_{CG} = \frac{1}{m_{\mathbb{S}}} \int_V gdm\vec{r} \times (-\vec{e}_z) \tag{5.20}$$

$$\vec{F}_{g,\mathbb{S}} = -m_{\mathbb{S}} g\vec{e}_z \tag{5.21}$$

$$\vec{\tau}_g = \vec{r}_{CM} \times \vec{F}_{g,\mathbb{S}} \tag{5.22}$$

Chapter 6

Work and energy

6.1 One particle

Work

$$dW = \langle \vec{F}, d\vec{r} \rangle_I \qquad W_{\mathscr{C}} = \int_{\mathscr{C}} \langle \vec{F}, d\vec{r} \rangle_I \tag{6.1}$$

If there is no ambiguity about the followed curve, we will just write W.

Definition 6.1.1. Power

$$dP = \frac{dW}{dt} = \langle \vec{F}, \vec{v} \rangle_I \qquad P_{\mathscr{C}} = \int_{\mathscr{C}} \langle \vec{F}, \vec{v} \rangle_I \tag{6.2}$$

Kinetic energy

$$K = \frac{mv^2}{2} \tag{6.3}$$

Proposition 6.1.1. *Let $\vec{F}$ be the net force over a particle. Then, if m is constant, the work W of this forces satisfies the following relation.*

$$W = \Delta K \tag{6.4}$$

Proof.

$$W_{\mathscr{C}} = \int_{\mathscr{C}} \left\langle \vec{F}, d\vec{r} \right\rangle_I = \int_{\mathscr{C}} \langle \frac{d\vec{p}}{dt}, d\vec{r} \rangle_I = m \int_{\mathscr{C}} \left\langle \frac{d\vec{v}}{dt}, d\vec{r} \right\rangle_I = m \int_{\mathscr{C}} \left\langle d\vec{v}, \frac{d\vec{r}}{dt} \right\rangle_I m \int_{\mathscr{C}} \langle d\vec{v}, \vec{v} \rangle_I =$$
$$m \int_{\mathscr{C}} \langle \vec{v}, d\vec{v} \rangle_I = m \int_{\mathscr{C}} \frac{1}{2} d\langle \vec{v}, \vec{v} \rangle_I = \frac{m}{2} \int_{\mathscr{C}} dv^2 = \frac{m}{2} \Delta v^2 = \Delta K$$

■

Definition 6.1.2. We say a force $\vec{F}$ is conservative if for all path $\mathscr{C}$ of fixed extremes A and B the work depends only of A and B.

$$W_{\mathscr{C}} = \int_{\mathscr{C}} \langle \vec{F}, d\vec{r} \rangle_I = W_{AB} \tag{6.5}$$

This means we can associate an potential energy function to this force such that $\vec{F} = -\vec{\nabla} U$, or what is the same,

$$W_{\mathscr{C}} = \int_{\mathscr{C}} \langle \vec{F}, d\vec{r} \rangle_I = -\int_{\mathscr{C}} \langle \vec{\nabla} U, d\vec{r} \rangle_I = -\int_{\mathscr{C}} dU = U_A - U_B \tag{6.6}$$

We define this potential energy from a position of reference.

$$U = -\int_{\vec{r}_0}^{\vec{r}} \langle \vec{F}, d\vec{r} \rangle_I \tag{6.7}$$

We could think about $U(\vec{r})$ as the work done by an external force $\vec{F}^{\text{ext}}$ we have to do contrary to the $\vec{F}$ that pushes a particle from $\vec{r}_0$ such that its kinetic energy is conserved. The selection of $\vec{r}_0$ is completely arbitrary, and we can choose it depending on the simplicity of calculations.

$$W_{field} = -\Delta U \qquad W^{\text{ext}} = \Delta U \tag{6.8}$$

Definition 6.1.3. Mechanic energy.The mechanic energy of a particle (with a kinetic energy K) subjected to a conservative force $\vec{F}$ of potential energy U is

$$E = K + U \tag{6.9}$$

Proposition 6.1.2. *If a particle is subjected to a conservative force, then $\Delta E = 0$*

Proof.

$$\Delta K = W = \int_{\vec{r}_A}^{\vec{r}_B} \langle \vec{F}, d\vec{r} \rangle_I = \int_{\vec{r}_A}^{\vec{r}_0} \langle \vec{F}, d\vec{r} \rangle_I + \int_{\vec{r}_0}^{\vec{r}_B} \langle \vec{F}, d\vec{r} \rangle_I = U_A - U_B = -\Delta U \Rightarrow$$
$$\Delta K + \Delta U = 0 = \Delta E$$

■

If there is a non-conservative force, its work to the system produces a change in the mechanic energy.

$$W_{nc} = \Delta E \tag{6.10}$$

Non conservative forces: friction.
Conservative forces: spring, gravity

6.2 More general potential functions

6.2.1 Time dependent potential function

The expressions $W = \Delta K$, $\vec{F} = -\vec{\nabla} U$, and $E = K + U$ are direct consequences from the Newton Laws. However, the other ones are only true by the assumption that the potential is a time independent function. Let us study a potential function of the form

$$U = U(\vec{r}, t). \tag{6.11}$$

In this case, the work has the following form

$$W = \int_a^b \vec{F}(\gamma(t), t)\gamma'(t)dt \tag{6.12}$$

We will see that it is not more true that $W = -\Delta U$,

$$W = \int_\Gamma \langle \vec{F}, d\vec{r} \rangle_I = -\int_\Gamma \langle \vec{\nabla} U, d\vec{r} \rangle_I = -\int_\Omega dU + \int_a^b \frac{\partial U}{\partial t} dt = -\Delta U + \int_a^b \frac{\partial U}{\partial t} dt.$$

Then, the energy is not conserved

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{mv^2}{2} + U\right] m\langle \vec{v}, \vec{a} \rangle_I + \langle \vec{\nabla} U, \vec{v} \rangle_I + \frac{\partial U}{\partial t} = \langle m\vec{a} + \vec{\nabla} U, \vec{v} \rangle_I + \frac{\partial U}{\partial t} = \frac{\partial U}{\partial t}. \tag{6.13}$$

6.2.2 Velocity dependent potential function

Now let us consider velocity dependent potential, that is, a potential function of the form

$$U = U(\vec{r}, \vec{v}). \tag{6.14}$$

Now, the works has the following form

$$W = \int_a^b \vec{F}(\gamma(t), \gamma'(t))\gamma'(t)dt. \tag{6.15}$$

In terms of potential,

$$W = \int_\Gamma \langle \vec{F}, d\vec{r} \rangle_I = -\int_\Gamma \langle \vec{\nabla} U, d\vec{r} \rangle_I = -\int_\Omega dU + \int_a^b \frac{\partial U}{\partial \dot{x}} + \frac{\partial U}{\partial \dot{y}} + \frac{\partial U}{\partial \dot{z}} dt =$$
$$-\Delta U + \int_a^b \frac{\partial U}{\partial \dot{x}} + \frac{\partial U}{\partial \dot{y}} + \frac{\partial U}{\partial \dot{z}} dt.$$

Similarly,

$$\frac{dE}{dt} = \frac{\partial U}{\partial \dot{x}} \ddot{x} + \frac{\partial U}{\partial \dot{y}} \ddot{y} + \frac{\partial U}{\partial \dot{z}} \ddot{z}. \tag{6.16}$$

Example 6.2.1. Liénard-Wiechert potential.

$$U = k\frac{1}{1-\langle\vec{e}_{12},\vec{\beta}\rangle_I}\frac{1}{\|\vec{r}_2-\vec{r}_1\|} \tag{6.17}$$

6.3 Several particles

Proposition 6.3.1. *If two particles exert a mutual conservative force $\vec{F}_{1\to 2}$ and $\vec{F}_{2\to 1}$ which is independent of any other degree of freedom of any bigger system they are apart of, and obeys the Newton's Third Law as $\vec{F}_{1\to 2} = -\vec{F}_{2\to 1}$, with the forces collinear to particles' relative orientation, then this mutual force can be written in the form*

$$\vec{F}_{i\to j} = -\vec{\nabla}_j V(\|\vec{r}_i-\vec{r}_j\|) \tag{6.18}$$

for some appropriate potential V.

Proof. Under the assumption of independence on other degree of freedom, the force has an associated potential function of the form

$$\vec{F}_{i\to j} = \vec{\nabla}_j V(\vec{r}_i,\vec{r}_j).$$

The potential function can be transformed by a change of variable, always that this change establishes a bijective relation. In particular, a possible approach is take another pair of linearly independent vectors formed as linear combinations of $\vec{r}_i$ and $\vec{r}_j$. We will take $\vec{r} = \vec{r}_i - \vec{r}_j$ and $\vec{R} = \frac{1}{2}\vec{r}_1 + \frac{1}{2}\vec{r}_2$. This way, the gradient operators become

$$\begin{cases} \vec{\nabla}_{\vec{r}_i} = \vec{\nabla}_{\vec{r}} + \dfrac{1}{2}\vec{\nabla}_{\vec{R}}, \\ \vec{\nabla}_{\vec{r}_j} = -\vec{\nabla}_{\vec{r}} + \dfrac{1}{2}\vec{\nabla}_{\vec{R}}, \end{cases}$$

and applying the weak form of Newton's Third Law,

$$\vec{0} = \vec{F}_{i\to j} + \vec{F}_{j\to i} = -\vec{\nabla}_{\vec{r}_j}V(\vec{r},\vec{R}) - \vec{\nabla}_{\vec{r}_i}V(\vec{r},\vec{R}) = -\vec{\nabla}_{\vec{R}}V(\vec{r},\vec{R}) \Rightarrow \vec{0} = \vec{\nabla}_{\vec{R}}V(\vec{r},\vec{R}),$$

which shows there is no dependence on the positions but only the difference $\vec{r}$, that is, $V = V(\vec{r}_i - \vec{r}_j)$. To see now the unique dependence is on its norm, we will set a coordinate system with origin at $\vec{r}_i$ and compute the gradient of V_{ij} at $\vec{r}_i$ using spherical coordinates.

$$\vec{\nabla}V_{ij} = \frac{\partial V_{ij}}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial V_{ij}}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V_{ij}}{\partial\varphi}\vec{e}_\varphi$$

In this coordinate system, the radial vector corresponds to the vector $\vec{r}_i - \vec{r}_j$ normalized. We know this pair of forces obey the strong form of Newton's Third Law, so the gradient can only have the radial component (otherwise the forces won't lie on he joining line). Therefore, V has no dependence on θ or φ but only the actual norm of the difference vector, showing that V has the form $V(\|\vec{r}_i - \vec{r}_j\|)$. ■

For simplicity, we will suppose that all intern forces are conservatives and that are central forces, that is, the forces $\vec{F}_{i\to j}$ only depend exclusively on the distance between particles i and j. In this case, it is true that

$$W_{\mathbb{S}} = -\Delta U_{\mathbb{S}}^{\text{ext}} - \Delta U_{\mathbb{S}}^{\text{int}}, \tag{6.19}$$

where

$$U_{\mathbb{S}}^{\text{ext}} = \sum_{i=1}^{n} U_i^{\text{ext}}, \qquad U_{\mathbb{S}}^{\text{int}} := \sum_{i,j=1 j<i}^{n} U_{j\to i}, \tag{6.20}$$

Note that the second expression (the internal potential energy of the system) has as many terms as different ways to couple the particles of the system.
In this expression, U_i^{ext} is the potential energy of the external force $\vec{F}_{\to i}^{\text{ext}}$ that acts over i and $U_{j\to i}$ is the potential energy of the intern force $\vec{F}_{j\to i}$. This last one ia function of $r_{j\to i} = \left\|\vec{r}_{j\to i}\right\|$. We can do it for the previous proposition.

Proof. Let us calculate the work of all the forces that act in the system. Let $\mathscr{C}_i$ be the trajectory that follows a particle i. Let us denote by $\vec{r}_i^A$ and $\vec{r}_i^B$ the extremes of $\mathscr{C}_i$. Let us denote $\vec{r}_{j\to i} := \vec{r}_i - \vec{r}_j$.
The total work is

$$w = \sum_{i=1}^{n} W_{\mathscr{C}_i} = \sum_{i=1}^{n} \int_{\mathscr{C}_i} \langle \vec{F}_{\to i}, d\vec{r}_i \rangle_I = \sum_{i=1}^{n} \int_{\mathscr{C}} \left\langle \vec{F}_{\to i}^{\text{ext}} + \sum_{j=1}^{n} \vec{F}_{j\to i}, d\vec{r}_i \right\rangle_I =$$

$$\sum_{i=1}^{n} \int_{\mathscr{C}} \left\langle \vec{F}_{\to i}^{\text{ext}}, d\vec{r}_i \right\rangle_I + \sum_{i=1}^{n} \int_{\mathscr{C}} \left\langle \sum_{j=1}^{n} \vec{F}_{j\to i}, d\vec{r}_i \right\rangle_I = \sum_{i=1}^{n} \int_{\mathscr{C}} \left\langle \vec{F}_{\to i}^{\text{ext}}, d\vec{r}_i \right\rangle_I + \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathscr{C}_i} \left\langle \vec{F}_{j\to i}, d\vec{r}_i \right\rangle_I$$

$$:= W^{\text{ext}} + W^{\text{int}}$$

Therefore, the work done by external forces is $W^{\text{ext}} = -\Delta U^{\text{ext}}$, where

$$W_i^{\text{ext}} = \int_{\mathscr{C}_i} \left\langle \vec{F}_{\to i}^{\text{ext}}, d\vec{r}_i \right\rangle_I = U_i^{\text{ext}}(\vec{r}_i^A) - U_i^{\text{ext}}(\vec{r}_i^B) = -\Delta U_i^{\text{ext}}.$$

The work done by internal forces is

$$W^{\text{int}} = \sum_{i=j, i<j}^{n} \int_{\mathscr{C}_i} \left\langle \vec{F}_{j\to i}, d\vec{r}_i \right\rangle_I + \sum_{i=j, i>j}^{n} \int_{\mathscr{C}_i} \left\langle \vec{F}_{j\to i}, d\vec{r}_i \right\rangle_I =$$

$$\sum_{i=j, i<j}^{n} \int_{\mathscr{C}_i} \left\langle \vec{F}_{j\to i}, d\vec{r}_i \right\rangle_I + \sum_{j=i, j>i}^{n} \int_{\mathscr{C}_j} \left\langle \vec{F}_{i\to j}, d\vec{r}_j \right\rangle_I =$$

$$\sum_{i=j, i<j}^{n} \int_{\mathscr{C}_i} \left\langle \vec{F}_{j\to i}, d\vec{r}_i \right\rangle_I - \sum_{j=i, i<j}^{n} \int_{\mathscr{C}_j} \left\langle \vec{F}_{j\to i}, d\vec{r}_j \right\rangle_I =$$

$$\sum_{i=j, i<j}^{n} \left(\int_{\mathscr{C}_i} \left\langle \vec{F}_{j\to i}, d\vec{r}_i \right\rangle_I - \int_{\mathscr{C}_j} \left\langle \vec{F}_{j\to i}, d\vec{r}_j \right\rangle_I \right)$$

We will denote each term of the sum by $W_i j$.
Let us suppose that at the instant t the position of particles i an j are $\vec{r}_i$ and $\vec{r}_j$, respectively, and that at an instant $t + dt$ they are in $\vec{r}_i + d\vec{r}_i$ and $\vec{r}_i + d\vec{r}_j$.
Since the intern forces are central, $\vec{F}_{j\to i}$ only depends on the norm $r_{j\to i}$, that is, the force is expressed as $\vec{F}_{j\to i}(r_{j\to i})$. The infinitesimal work done between t and $t + dt$ is

$$dW_{j\to i} = \left\langle \vec{F}_{j\to i}(r_{j\to i}), d\vec{r}_i \right\rangle_I - \left\langle \vec{F}_{j\to i}(r_{j\to i}), d\vec{r}_j \right\rangle_I =$$

$$\left\langle \vec{F}_{j\to i}(r_{j\to i}), \frac{d\vec{r}_i}{dt} \right\rangle_I dt - \left\langle \vec{F}_{j\to i}(r_{j\to i}), \frac{d\vec{r}_j}{dt} \right\rangle_I dt =$$

$$\left\langle \vec{F}_{j\to i}(r_{j\to i}), \vec{v}_i \right\rangle_I dt - \left\langle \vec{F}_{j\to i}(r_{j\to i}), \vec{v}_j \right\rangle_I dt =$$

$$\left\langle \vec{F}_{j\to i}(r_{j\to i}), (\vec{v}_i - \vec{v}_j) \right\rangle_I dt = \left\langle \vec{F}_{j\to i}(r_{j\to i}), \vec{v}_{j\to i} \right\rangle_I dt =$$

$$\left\langle \vec{F}_{j\to i}(r_{j\to i}), d\vec{r}_{j\to i} \right\rangle_I$$

Hence, (let us remember that the work cannot depend on the trajectories $\mathscr{C}_i$ and $\mathscr{C}_j$)

$$W_{ij} = \in_{\vec{r}_{j\to i}^A}^{\vec{r}_{j\to i}^B} \left\langle \vec{F}_{j\to i, d\vec{r}_{j\to i}} \right\rangle_I = U_{j\to i}\left(r_{j\to i}^A\right) - U_{j\to i}\left(r_{j\to i}^B\right) = -\Delta U_{j\to i.}$$

And, finally,

$$W^{\text{int}} = \sum_{i,j=1, j<i}^{n} W_i j = -\Delta \left(\sum_{i,j=1, j<i}^{n} U_{j\to i} \right) := -\Delta U_{\text{int}} \Rightarrow W^{\text{int}} = -\Delta U^{\text{int}}$$

■

Kinetic energy of a system of particles.

$$K_{\mathbb{S}} = \sum_{i=1}^{n} K_i = \sum_{i=1}^{n} \frac{m_i v_i^2}{2} \tag{6.21}$$

If we apply the work-kinetic energy theorem to every particle, we get

$$\Delta K_{\mathbb{S}} = W = -\Delta U_{\mathbb{S}}^{\text{ext}} + \Delta U_{\mathbb{S}}^{\text{int}} \tag{6.22}$$

And hence,

$$\Delta K_{\mathbb{S}} + \Delta U_{\mathbb{S}}^{\text{ext}} + \Delta U_{\mathbb{S}}^{\text{int}} = W_{nc,\mathscr{C}}^{\text{ext}} \tag{6.23}$$

If we define the mechanic energy of a system as the sum of other energies as before, we have that in absence of non conservatives forces

$$0 = \Delta E \coloneqq \Delta \left(\Delta K_{\mathbb{S}} + \Delta U_{\mathbb{S}}^{\text{ext}} + \Delta U_{\mathbb{S}}^{\text{int}} \right) \tag{6.24}$$

And, since internal potential energy is a function of the distances ($U_i^{\text{int}}(\|\vec{r}_{i\to j}\|)$), if we have a system where distances between particles are conserved (rigid solid, we will study them later), the final internal potential energy is the same as the initial and we could express the conservation of energy as

$$0 = \Delta K_{\mathbb{S}} + \Delta U_{\mathbb{S}}^{\text{ext}} \tag{6.25}$$

Exercise. We can chose $U_{i\to j}$ and $U_{j\to i}$ such that $U_{i\to j} = U_{j\to i}$.

$$U_{i\to j}(\mathbf{r}_{i\to j}^A) = -\int_{\mathbf{r}_0}^{\mathbf{r}_A} \langle \mathbf{F}_{i\to j}, \mathrm{d}\mathbf{r}_{i\to j}\rangle_I = -\int_{\mathbf{r}_0}^{\mathbf{r}_A} \langle -\mathbf{F}_{j\to i}, -\,\mathrm{d}\mathbf{r}_{j\to i}\rangle_I = -\int_{\mathbf{r}_0}^{\mathbf{r}_A} \langle \mathbf{F}_{j\to i}, \mathrm{d}\mathbf{r}_{j\to i}\rangle_I = U_{j\to i}(\mathbf{r}_{j\to i}^A).$$

Proposition 6.3.2. *LetU^{int} be the internal energy of the system. Then,*

$$U^{\text{int}} = \frac{1}{2}\sum_{i,j=1}^{n} U_{i\to j}. \tag{6.26}$$

Proof.

$$U^{\text{int}} = \sum_{i<j}^{n} U_{i\to j} = \frac{1}{2}\left(\sum_{i<j}^{n} U_{i\to j} + \sum_{i<j}^{n} U_{i\to j}\right) = \frac{1}{2}\left(\sum_{i<j}^{n} U_{i\to j} + \sum_{i<j}^{n} U_{j\to i}\right) =$$
$$\frac{1}{2}\left(\sum_{i<j}^{n} U_{i\to j} + \sum_{j<i}^{n} U_{i\to j}\right) = \frac{1}{2}\sum_{i,j=1}^{n} U_{i\to j}.$$

■

Proposition 6.3.3. *The kinetic energy of a system can be written as*

$$\boxed{K_{\mathbb{S}} = K_{\mathbb{S},\text{CM}} + \frac{m_{\mathbb{S}} v_{\text{CM}}^2}{2}.} \tag{6.27}$$

Proof. Let us remember that $\vec{v}_i = \vec{v}_{i,CM} + \vec{v}_{CM}$. Then,

$$v_i^2 = v_{i,CM}^2 + v_{CM}^2 + 2\langle \vec{v}_{i,CM}, \vec{v}_{CM}\rangle_I.$$

The kinetic energy of the system can be written as

$$K_{\mathbb{S}} = \sum_{i=1}^{n} \frac{1}{2} m_i v_{i,CM}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i v_{CM}^2 + \sum_{i=1}^{n} m_i \langle \vec{v}_{i,CM}, \vec{v}_{CM}\rangle_I + =$$
$$K_{\mathbb{S},CM} + \frac{1}{2}\left(\sum_{i=!}^{n} m_i\right) v_{CM}^2 + m_{\mathbb{S}}\left\langle \sum_{i=1}^{n} \frac{m_i \vec{v}_{i,CM}}{m_{\mathbb{S}}}, \vec{v}_{CM}\right\rangle_I = K_{\mathbb{S},CM} + \frac{1}{2} m_{\mathbb{S}} v_{CM}^2$$

■

We call the first term kinetic energy of translation and the second one kinetic energy of rotation

Chapter 7

Gravitation

7.1 Gravitational potential

Example 7.1.1. Observations have shown the orbital speed of masses in many spiral galaxies rotating about their center behaves approximately constant as a function of the radial distance. In particular, the first measurement of this in good agreement with modern data was published in 1957 by Henk van de Hulst and collaborators [1].
We know the orbital speed is determined by the expression $v = \sqrt{GM/R}$. Then, a constant graph is inconsistent with the observations if we suppose most of the mass is at the center. From the shape of the graph, the only possible way for this to occur is if the mass increases in a proportion similar to R. However, the density shows that it is not the case, so we conclude there is an extra quantity of mass that we cannot, the dark matter.

Example 7.1.2. Let us consider a ring of radius a and mass M with uniform density λ. We will find the position of equilibrium and see if it is a stable or unstable equilibrium. The potential of a differential of mass is given by

$$dV = -G\frac{1}{b}dM = -G\frac{a\rho}{b}d\varphi.$$

If we express now b in terms of known variables,

$$b = \|\vec{r}_p - \vec{r}_m\| = \|a\cos\varphi\vec{e}_x + a\sin\varphi Y - r'\vec{e}_x\| = a\sqrt{1 + \left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\varphi}. \tag{7.1}$$

With that, the integral becomes

$$V = -G\int_M \frac{1}{b}dM = -\rho a G\int_0^{2\pi}\frac{1}{b}d\varphi = -\rho G\int_0^{2}\pi\left[1 + \left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\varphi\right]^{-1/2}d\varphi,$$

which is a difficult integral to evaluate. Instead, we will consider postions near the equilibrium $r' = 0$. If $r' \ll a$, we can expand the denominator and get

$$V = -\rho G\int_0^{2\pi} 1 + \frac{r'}{a}\cos\varphi + \frac{1}{2}\left(\frac{r'}{a}\right)^2(3\cos^2\varphi - 1)d\varphi = -\frac{MG}{a}\left[1 + \frac{1}{4}\left(\frac{r'}{a}\right)^2\right].$$

If we derivate the expression $U = mV$, we see the exact equilibrium is at $r' = 0$, and if we take the second derivative, we have

$$\frac{d^2U}{dr'^2}(0) = -G\frac{mM}{2a^3} < 0,$$

which shows it is an unstable equilibrium.

7.1.1 Poisson equation

We see that

$$\boxed{\vec{\nabla}^2\Phi = 4\pi G\rho.} \tag{7.2}$$

In case that $\rho = 0$, we get the Laplace equation.

7.1.2 Ocean tides

The acceleration of m in the inertial reference system is

$$m\ddot{\vec{r}}_m = -G\frac{mM_E}{r^2}\vec{e}_r - G\frac{mM_M}{R^2}\vec{e}_R,$$

and the force the center of mass of the earth experiences from the Moon is

$$M_E\ddot{\vec{r}}_E = -G\frac{M_EM_M}{D^2}\vec{e}_D.$$

If we study the movement of the mass m in the Earth system, we get

$$m\ddot{\vec{r}} = -G\frac{mM_E}{r^2}\vec{e}_r - GmM_m\left(\frac{\vec{e}_R}{R^2} - \frac{\vec{e}_D}{D^2}\right) = \vec{F}_E + \vec{F}_T$$

The first part is the force generated by the Earth and the second the *tidal force*.

$$\vec{F}_T = -GmM_m\left(\frac{\vec{e}_R}{R^2} - \frac{\vec{e}_D}{D^2}\right) \tag{7.3}$$

If we study the tidal force in the nearest and furthest positions, the unitary vectors have the same direction and the force has only one component (horizontal). In the furthest point $R > D$ and the expression inside parenthesis becomes negative, resulting in a positive force. In the nearest point occurs the opposite and the force is negative.
Let us now find a more simplified expression for these cases. In fact, since $D \gg r$, this situation is equivalent to the scenario where the mass is the most near to the moon. The force in these situations is

$$F_{Tx} = -GmM_M\left(\frac{1}{R^2} - \frac{1}{D^2}\right) = -GmM_M\left(\frac{1}{(D+r)^2} - \frac{1}{D^2}\right) =$$
$$-G\frac{mM_M}{D^2}\left(\frac{1}{(1+r/D)^2} - \frac{1}{D^2}\right) \approx 2G\frac{mM_M r}{D^3}.$$

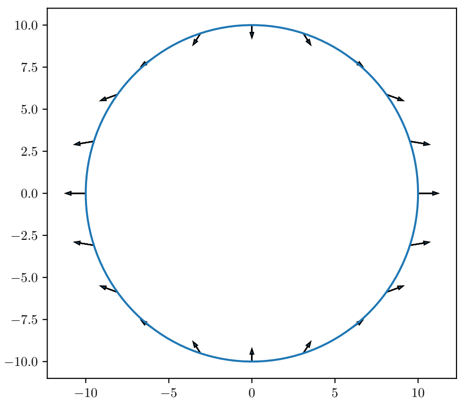
Now we will consider the top and bottom position. In both cases $R \approx B$ and the horizontal component is canceled. The only vector that has a vertical component is $\vec{e}_D$, which we can approximate by $r/D\vec{e}_y$. In this case, the force is

$$F_{Ty} = -G\frac{mM_M}{D^2}\frac{r}{D} = -G\frac{mM_M r}{D^3}.$$

For the other extreme, the expression results equal but with opposite sign. Now, for a general point, we can use the same arguments and see the total force will be a combinations from these two. The unique difference is that r in the expressions will now be x for the horizontal and y for the vertical components.

$$\vec{F}_{Tx} = 2G\frac{mM_M r}{D^3}\cos\theta\vec{e}_x - G\frac{mM_M r}{D^3}\sin\theta\vec{e}_y \tag{7.4}$$

A representation of this force is in the following figure, which represent coherently how the ocean tides behave.



Height changes

Now we know the responsible forces for the ocean tides behavior (in an approximated way), we can calculate the maximum height change in the ocean tides. If we have a height change h, the energy change is mgh. Then, we can isolate h if we know the work done. To find the work expression, we will need to select two points and a certain path (the result is independent but allows us simplify the calculations). The initial point will be at the positive y extreme, the final point at the positive x extreme, and the path will be $\vec{r}_0 \to \vec{0} \to \vec{r}_f$. In a real case the top position would not be at $R\vec{e}_y$ because it is a slightly different height and the same for the right one, but these values are so small that we can neglect them. Hen, the work done is

$$W = \int_\Gamma \langle \vec{F}, d\vec{r} \rangle_I = \int_{p_0}^{0} \langle \vec{F}, d\vec{r} \rangle_I + \int_0^{p_f} \langle \vec{F}, d\vec{f} \rangle_I = \int_{r+\delta_1}^{0} F_{Ty} dy + \int_0^{r+\delta_2} F_{Tx} dx \approx$$

$$\int_r^0 F_{Ty} dy + \int_0^r F_{Tx} dx = G\frac{mM_M}{D^3} \int_r^0 -y dy + G\frac{mM_M}{D^3} \int_0^r dx dx = G\frac{mM_M}{D^3} \left(\frac{r^2}{2} + r^2 \right) =$$

$$G\frac{3mM_M r^2}{2D^3}.$$

Then, the change in height is

$$mgh = G\frac{3mM_M r^2}{2D^3} \Rightarrow h = G\frac{3M_M r^2}{2gD^3} = 0.54\,\text{m}. \tag{7.5}$$

Bibliography

[1] H. C. van de Hulst, E. Raimond, and H. van Woerden. Rotation and density distribution of the Andromeda nebula derived from observations of the 21-cm line. *Bulletin Astronomical Institute of the Netherlands*, 14:1, nov 1957.

Chapter 8

Dynamics for system of particles II

8.1 Angular momentum and torque II

Let $\vec{r}_i$ be the position of a particle i from the origin, $\vec{r}_i'$ the position from the center of mass, and $\vec{r}_{CM}$ the position of the center of mass with respect to the origin. Then,

$$\vec{r}_i = \vec{r}_i' + \vec{r}_{CM} \Rightarrow \vec{v}_i = \vec{v}_i' + \vec{v}_{CM} \Rightarrow m_i\vec{v}_i = m_i\vec{v}_i' + m_i\vec{v}_{CM} \Rightarrow \vec{p}_i = \vec{p}_i' + m_i\vec{v}_{CM} \tag{8.1}$$

The angular momentum of the system $\mathbb{S}$ with respect to the center of mass is

$$\vec{L}_{\mathbb{S},CM} = \sum_{i=1}^{n} \vec{L}_{i,CM} = \sum_{i=1}^{n} \vec{r}_i' \times \vec{p}_i' = \sum_{i=1}^{n} m_i\vec{r}_i' \times \dot{\vec{r}}_i \tag{8.2}$$

From this, we obtain the following relation.

$$\vec{L}_{\mathbb{S}} = \vec{L}_{\mathbb{S},CM} + \vec{r}_{CM} \times \vec{p}_{CM} \tag{8.3}$$

Proof.

$$\begin{gathered}
\vec{L}_{\mathbb{S},0} = \sum_{i=1}^{n} \vec{L}_{i,0} = \sum_{i?1}^{n} \left(\vec{r}_i' + \vec{r}_{CM}\right) \times \left(\vec{p}_i' + m_i\vec{v}_{CM}\right) = \\
\sum_{i=1}^{n} \vec{r}_i' \times \vec{p}_i' + \sum_{i=1}^{n} \vec{r}_i' \times m_i\vec{v}_{CM} + \sum_{i=1}^{n} \vec{r}_{CM} \times \vec{p}_i' + \sum_{i=1}^{n} \vec{r}_{CM} \times m_i\vec{v}_{CM} = \\
\vec{L}_{\mathbb{S},CM} + \vec{v}_{CM} \times \sum_{i=1}^{n} m_i\vec{r}_i' + \vec{r}_{CM} \times \sum_{i=1}^{n} \vec{p}_i' + \vec{r}_{CM} \times \vec{v}_{CM} \sum_{i=1}^{n} m_i = \\
\vec{L}_{\mathbb{S},CM} + \vec{v}_{CM} m_{\mathbb{S}} \vec{r}_{CM}' + \vec{r}_{CM} \times \vec{0} + \vec{r}_{CM} \times \vec{v}_{CM} m_{\mathbb{S}} = \vec{L}_{\mathbb{S},CM} + \vec{v}_{CM} m_{\mathbb{S}} \vec{0} + \vec{r}_{CM} \times \vec{p}_{CM} = \\
\vec{L}_{\mathbb{S},CM} + \vec{r}_{CM} \times \vec{p}_{CM}
\end{gathered}$$

■

We call the first term spin angular momentum and the second one orbital angular momentum. This relations leads to another important one.
The orbital angular momentum is calculated as if the whole system was reduced to a single particle of mass $m_{\mathbb{S}}$ located in the center of mass.

$$\dot{\vec{L}}_{\mathbb{S},CM} = \vec{M}_{\mathbb{S},CM}^{\text{ext}} \tag{8.4}$$

Proof.

$$\begin{gathered}
\vec{L}_{\mathbb{S}} = \vec{L}_{\mathbb{S},CM} + \vec{r}_{\mathbb{S}} \times \vec{p}_{\mathbb{S}} \Rightarrow \dot{\vec{L}}_{\mathbb{S}} = \dot{\vec{L}}_{\mathbb{S},CM} + \vec{v}_{CM} \times \vec{p}_{CM} + \vec{r}_{CM} \times \vec{F}_{\mathbb{S}} = \dot{\vec{L}}_{\mathbb{S},CM} + \vec{r}_{CM} \times \vec{F}_{\mathbb{S}}^{\text{ext}} \Rightarrow \\
\dot{\vec{L}}_{\mathbb{S}} - \vec{r}_{CM} \times \vec{F}_{\mathbb{S}}^{\text{ext}} = \dot{\vec{L}}_{\mathbb{S},CM} \Rightarrow \vec{\tau}_{\mathbb{S}}^{\text{ext}} - \vec{r}_{CM} \times \vec{F}_{\mathbb{S}}^{\text{ext}} = \dot{\vec{L}}_{\mathbb{S},CM}
\end{gathered}$$

Let us see that the first part of the equality is $\vec{\tau}_{\mathbb{S},CM}^{\text{ext}}$.

$$\begin{gathered}
\vec{\tau}_{\mathbb{S},CM}^{\text{ext}} = \sum_{i=1}^{n} \vec{\tau}_{i,CM} = \sum_{i=1}^{n} \vec{r}_{i,CM} \times \vec{F}_{\to i}^{\text{ext}} = \sum_{i=1}^{n} \left(\vec{r}_i - \vec{r}_{CM}\right) \times \vec{F}_{\to i}^{\text{ext}} = \\
\sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{\to i}^{\text{ext}} - \vec{r}_{CM} \times \sum_{i=1}^{n} \vec{F}_{\to i}^{\text{ext}} = \vec{\tau}_{\mathbb{S}}^{\text{ext}} - \vec{r}_{CM} \times \vec{F}_{\mathbb{S}}^{\text{ext}}
\end{gathered}$$

With that, we can conclude the equality we wanted to prove. ■

And it works even if the center of mass is accelerating.

8.2 Rigid solid

The rigid solid is a system of particles whose relative distances are constant (due to intern forces).
*Internal potential energy is conserved (see previous chapter).

Theorem 8.2.1. *Steiner's Theorem.*

$$\boxed{I = I_{\mathrm{CM}} + md^2} \tag{8.5}$$

Proof. Presentation of first year. ■

The forces that maintain the axis of rotation fixed have no contribution at $\tau^{\mathrm{ext}}_{\mathrm{CM},z'}$: see README.

Chapter 9

Application to elastic collisions

9.1 Introduction

We will now study the elastic collision between two particles. This scenario is in fact a general case of interaction between two particles, since the unique restriction we will apply is the conservation of linear momentum and the conservation of energy (there is no change in the intern energy because the collision i elastic nor external potential energy because we do not consider other external force). Therefore, we could apply the following reasoning to a wide range of cases, independent on the forces that act during the interaction. A simple example of this situation is the actual contact between two billiard balls, but we could also use it in the gravitation interaction of an object around another one or even in several quantum mechanical systems (these conservation laws are also satisfied).
This particular situation of elastic collision is called *scattering*, and it consists in the interaction of a moving particle with a another one at rest. After the forces have acted, the initial moving particle scatters and gets a final velocity at a certain angle from the original and the second particle obtains a velocity at another angle (we will specify the notation later). In spite the known laws of conservation, we can not determine these angles (for that we need to known the force that acts during the collision), but we will still be able to obtain information about the velocities of the final state. In an actual collision the final state is clear, but in another kind of force (such as a field force), the notions of initial and final state should be specified (usually we take the initial and final state when the forces are not enough intense to be considered).

9.2 Notation

The system is based on an particle of mass m_1 with an initial velocity $\vec{u}_1$ and a second particle of mass m_2 at rest ($\vec{u}_2 = 0$). We will denote initial velocities by u (as we have just done) and final velocities by v. Since velocities can be different depending on the system of reference, we will denote the system where $\vec{u}_2 = 0$ by the LAB system. A part from this, we will also use the CM system, since we have seen it has some convenient properties. We will add an apostrophe to velocity vectors to denote they are with respect to the CM system, and use $\vec{V}$ to denote the velocity of the center of mass with respect to the LAB system. With all that, the notation of velocities would be expressed as follows. We need to define now the angles, but before, we will see

	LAB system		CM system	
	Initial velocity	Final velocity	Initial velocity	Final velocity
m_1	$\vec{u}_1$	$\vec{v}_1$	$\vec{u}'_1$	$\vec{v}'_1$
m_2	$\vec{u}_2$	$\vec{v}_2$	$\vec{u}'_2$	$\vec{v}'_2$

how many angles we really need to define. For this, we will apply the following proposition.

Proposition 9.2.1. *The set of all initial velocities and final velocities are coplanar (lie in the same plane).*

Proof. We know the initial momentum is $\vec{P}_0 = m_1\vec{u}_1$ and the final momentum is $\vec{P}_f = m_1\vec{v}_1 + m_2\vec{v}_2$. By the conservation of linear momentum, the initial and final momentum are the same, so we have

$$\vec{0} = \vec{P}_0 - \vec{P}_f \Rightarrow \vec{0} = m_1\vec{u}_1 - m_1\vec{v}_1 - m_2\vec{v}_2,$$

with $m_1, m_2 \neq 0$. Therefore, the vectors $\vec{u}_1$, $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent and generate a vector space of dimension one (lie in the same line) or, in the most general case, dimension two (lie in the same plane). ∎

Since we are studying the most general case we have the situation lies in a plane, so we will select a coordinate system that contains this plain and with the direction of $\vec{u}_1$ as our x axis. With that, we only need to denote two angles for the LAB system. We will use the letter ψ to denote the angle of scattering of m_1 and the letter ζ to denote the angle of $\vec{v}_2$. Now, we could think we need two other angles for the CM system, bu in fact we need only one. Since in the CM system $\vec{P}' = 0$

always, after the collision we will have $m_1\vec{v}_1' + m_2\vec{v}_2' = 0$, so they lie in the same line and therefore their angles are directly related. We will denote the angle of scattering of $\vec{v}_1'$ by ϑ, and the other one will be $\pi - \vartheta$. To illustrate the angles ψ, ζ, and ϑ we have the following diagrams.

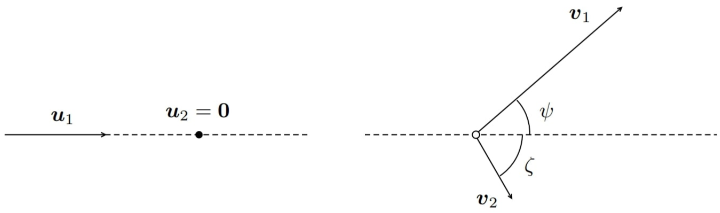


Figure 9.1: Velocity vectors before and after the collision in the LAB system.

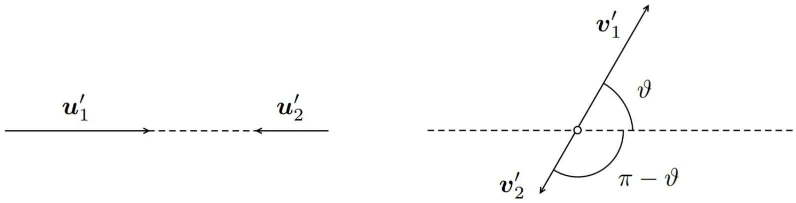


Figure 9.2: Velocity vectors before and after the collision in the CM system.

Regarding the velocity $\vec{v}_1'$ and its angle of scattering ϑ, there is an important detail we will study later but that is convenient to observe now. If we again represent the velocities of velocities in a circle (we will justify why a circle later []), we can see a distinction depending on the values of $\vec{v}_1'$ (figure of below). If $\vec{V} < \vec{v}_1'$, all velocities lie inside the circle, and it seems that for any angle θ, there will be a corresponding angle ψ associated. However, if $\vec{V} > \vec{v}_1'$, the velocity vector of the center of mass is located out of the circle. If this happens, there will be two possible values of ϑ that will result in the same ψ. Therefore, one of our tasks apart from determining the final velocities, is find the condition that will differentiate when occurs the first case or the second one.

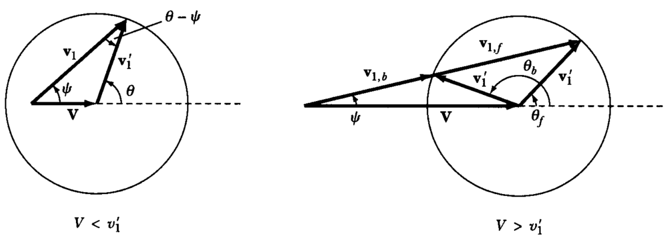


Figure 9.3: Possible values of ϑ to the same angle ψ depending on the value of V/v_1'.

9.3 Conservation laws

Since we are dealing with the CM system, the first we need to know is its velocity. Knowing $\vec{u}_2 = \vec{0}$, we get

$$\vec{V} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}\vec{u}_1. \tag{9.1}$$

With that, the expression of $\vec{u}_2'$ can be immediately obtained.

$$\vec{u}_2 = \vec{u}_2' + \vec{V} = \vec{0} \Rightarrow \vec{u}_2' = -\vec{V} \Rightarrow \vec{u}_2' = -\frac{m_1}{m_1 + m_2}\vec{u}_1 \tag{9.2}$$

Besides, since the linear momentum of the system with respect to the CM is always 0, we can determine the relation between the velocities in that reference system. In the initial sate we have

$$m_1\vec{u}_1' + m_2\vec{u}_2' = \vec{0} \Rightarrow \vec{u}_1' = -\frac{m_2}{m_1}\vec{u}_2' = \frac{m_2}{m_1+m_2}\vec{u}_1, \tag{9.3}$$

and in the final state,

$$m_1\vec{v}_1' + m_2\vec{v}_2' = \vec{0} \Rightarrow \vec{v}_1' = -\frac{m_2}{m_1}\vec{v}_2'. \tag{9.4}$$

Now we can use the conservation of energy law to derive more relations between velocities. However, since the expressions contains the norms, we will just be able to relate these scalar quantities, without knowing the directions.

$$\frac{m_1u_1'^2}{2} + \frac{m_2u_2'^2}{2} = \frac{m_1v_1'^2}{2} + \frac{m_2v_2'^2}{2} \overset{\text{Eq. 9.3}}{\Rightarrow} \frac{m_1}{2}\frac{m_2^2}{m_1^2}u_2'^2 + \frac{m_2u_2'^2}{2} = \frac{m_1v_1'^2}{2} + \frac{m_2v_2'^2}{2} \overset{\text{Eq. 9.4}}{\Rightarrow}$$

$$\left(\frac{m_2^2}{2m_1} + \frac{m_2}{2}\right)u_2'^2 = \frac{m_1}{2}\frac{m_2^2}{m_1^2}v_2'^2 + \frac{m_2}{2}v_2'^2 = \left(\frac{m_2^2}{2m_1} + \frac{m_2}{2}\right)v_2'^2 \Rightarrow u_2' = v_2' \tag{9.5}$$

If we use this new relation again in the original equation, we get

$$\frac{m_1u_1'^2}{2} + \frac{m_2u_2'^2}{2} = \frac{m_1v_1'^2}{2} + \frac{m_2v_2'^2}{2} \Rightarrow \frac{m_1u_1'^2}{2} + \frac{m_2v_2'^2}{2} = \frac{m_1v_1'^2}{2} + \frac{m_2v_2'^2}{2} \Rightarrow u_1' = v_1', \tag{9.6}$$

and finally, knowing the relations of the first equations, all velocity norms are related as follows.

$$v_1' = u_1' = \frac{m_2}{m_1+m_2}u_1 \qquad v_2' = u_2' = V = \frac{m_1}{m_1+m_2}u_1 \tag{9.7}$$

9.4 Study of x_1

To determine the velocities of the final state we can establish the x component and y component of each vector or its norm and angle. We have obtained several relations between about speeds, so our procedure will consist on determine the angles too to express the velocities. Besides, for every obtained magnitude, we will discuss the different elements of the expression and see how it behaves for different values of its parameters. We will start now by studying the velocity of m_1, so these magnitudes are v_1 and ψ.

Velocity

From the figure 9.3, we can get the following relations.

$$\begin{cases} v_1\cos\psi = V + v_1'\cos\vartheta \\ v_1\sin\psi = v_1'\sin\vartheta \end{cases} \tag{9.8}$$

If we square the equations and add them, we obtain the following result

$$\begin{cases} v_1^2\cos^2\psi = v_1'^2\cos^2\vartheta + V^2 + 2v_1'V\cos\vartheta \\ v_1^2\sin^2\psi = v_1'^2\sin^2\psi \end{cases} \Rightarrow v_1^2 = v_1'^2 + V^2 + 2Vv_1'\cos\vartheta.$$

And using the final relations of the equation 9.7, get

$$v_1^2 = \left[1 - 2\frac{m_1m_2}{(m_1+m_2)^2}(1-\cos\vartheta)\right]u_1^2. \tag{9.9}$$

Since we are studying the speed of m_1 and not its velocity, some properties will be equivalent if we take v_1 or v_1^2, so for now we will continue with this squared expression. Here, the role that play m_1 and m_2 are the same, and hence there won't be any difference between cases like $m_1 < m_2$ and

$m_2 > m_1$. This lets us study the speed more briefly, beginning by determining the maxima and minima.

$$\frac{dv_1^2}{d\vartheta} = -2\frac{m_1 m_2}{m_1+m_2}u_1^2 \sin\vartheta = 0 \Rightarrow \sin\vartheta = 0$$

We see that the maxima and minima do not depend on any variable, and in fact, this makes sense with the previous diagrams. From these (take for example the figure 9.3), we can see that the maximum speed v_1 will occur at $\vartheta = 0$ and minimum at $\vartheta = \pi$. If we compute its second derivative at 0 and π, we can see that they respectively are maxima and minima. In this cases, the values of v_1 are

$$v_{1,\max}^2 = u_1^2 \Rightarrow v_{1,\max} = u_1, \tag{9.10}$$

$$v_{1,\min}^2 = \left[1 - 4\frac{m_1 m_2}{(m_1+m_2)^2}\right]u_1^2 = \frac{(m_1-m_2)^2}{(m_1+m_2)^2}u_1^2 \Rightarrow v_{1,\min} = \frac{|m_1-m_2|}{m_1+m_2}u_1, \tag{9.11}$$

where we take the absolute value of the difference because we are studying speeds, which are always positive. We obtain that the maximum value of v_1 always is the same, but the minimum not. Therefore, in a graph we should see that, for different values of m_1/m_2, the lines come from the same point but diverge at $\vartheta = \pi$. This is represented in the graph 9.4.

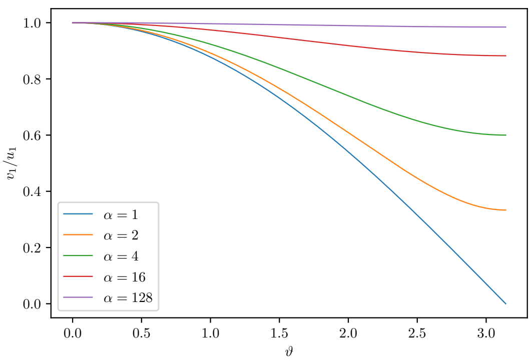


Figure 9.4: Function of $v_1(\vartheta)$ for different values of m_1/m_2.

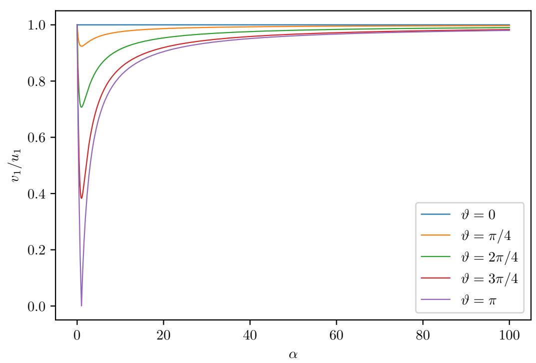


Figure 9.5: Function of $v_1(\alpha)$ for different values of ϑ.

As we can see, the behavior of $v_{1,\max}$ and $v_{1,\min}$ are the same we obtained by the expressions. In particular, when $m_1 = m_2$, the minimum speed has $|m_1 - m_1| = 0$ multiplying, which leads to a value of zero. Besides, note that we have put ratios like $m_2/m_1 = 4$ but not $m_1/m_2 = 4$, which is because their roles are the same, as we commented before (although there is a subtraction it has an absolute value so the order does not matter).
However, the ratio m_2/m_1 (or its inverse) does indeed play an important role in v_1. We have just seen that $v_{1,\min}$ depends on it, but in fact the speed depends always in this ration except at $\vartheta = 0$. Therefore, we can study too for what values of m_2/m_1 will the velocity has a minima or what happens at the limits. For that, we will express first the equation in terms of $\alpha = m_2/m_1$.

$$v_1^2 = \left[1 - 2\frac{\alpha}{(1+\alpha)^2}(1-\cos\vartheta)\right]u_1^2$$

If we compute its derivative and make equal to zero, we get

$$\frac{dv_1^2}{d\alpha} = 2\frac{\alpha-1}{(1+\alpha)^3}u_1^2(1-\cos\vartheta) = 0 \Rightarrow \alpha = 1,$$

which shows us that the minimum velocity always occurs at $m_1 = m_2$. We can see this in the graph 9.5, and as we commented before, it works for every value of ϑ except zero. With that only remains to discuss what happens at limits. As said before (and as we can see in the graph), the situation of $m_2 \gg m_1$ is equivalent to $m_2 \ll m_1$, and applying the limit, we have that $v_1 = u_1$ for every angle. With all this, we have determined the behavior of v_1 depending on ϑ and the ratio m_2/m_1, so we can proceed to study the angle ψ.

Angle

In the relations 9.8, we squared each equation, but another option is to divide the second equation by the first one. In that case, we get

$$\tan\psi = \frac{v_1'\sin\vartheta}{V + v_1'\cos\vartheta} \overset{\text{Eq. 9.7}}{=} \frac{m_2u_1\sin\vartheta/(m_1+m_2)}{m_1u_1/(m_1+m_2) + m_2u_2\cos\vartheta/(m_1+m_2)} \Rightarrow$$

$$\tan\psi = \frac{\sin\vartheta}{m_1/m_2 + \cos\vartheta} \tag{9.12}$$

A part from this relation between ψ and ϑ there is an interesting detail that occurred while simplifying the expression, and is that $V/v_1' = m_1/m_2$. We discussed the importance of this quotient when we talked about the possible number of values of ϑ that could correspond to the same value of ψ (see figure 9.3). Now we see the situation is in fact governed by m_1/m_2, which corresponds only to previously known variable. If m_1/m_2 is greater than 1, there will be two possible values, otherwise there will be a one to one correspondence.
The diagram 9.3 helps us to see this intuitively, but we still can prove it mathematically. To do that, we will study the function $\psi(\vartheta)$ and its derivative. If there are two possible values (since ψ starts increasing), at some point will decrease to reach the same value, and therefore there will be a maximum ($\psi' = 0$). Said that, let us start by calculating its derivative.

$$\psi(\vartheta) = \arctan\left(\frac{\sin\vartheta}{m_1/m_2 + \cos\vartheta}\right) \Rightarrow \frac{d\psi}{d\vartheta} = \frac{m_2^2 + m_1m_2\cos\vartheta}{m_2^2 + m_1^2 + 2m_1m_2\cos\vartheta} \tag{9.13}$$

If we make it equal to 0, we obtain a relation we can study in three cases.

$$\frac{d\psi}{d\vartheta} = 0 \Rightarrow m_2^2 + m_1m_2\cos\vartheta = 0 \Rightarrow \cos\vartheta = -\frac{m_2}{m_1} \tag{9.14}$$

If $m_2 > m_1$, there is no ϑ that can satisfy the equation. Besides, the expression of $d\psi/d\vartheta$ will be always positive and therefore the function is an strictly increasing function and therefore is bijective, that is, there will be a unique ϑ for a unique ψ. The maximum possible value of ψ will be at the end of the interval and will be $\psi(\pi) = \pi$
If $m_2 = m_1$, there is only one possible value of ϑ in the interval $[0, \pi]$ that can satisfy this, and is $\vartheta = \pi$. Since the derivative is null only at this point and positive at the rest of the interval, the function is bijective with a one to one correspondence. This maximum value of ψ at the end has the value of $\psi(\pi) = \pi/2$.
If $m_2 < m_1$, there is a possible one possible value that satisfies the condition and that will generate a maximum inside the interval. Therefore, after the maximum the function will decrease and cover some values that did before, that is, two possible values of ϑ can correspond to the same value of ψ. The maximum value of ψ will be determined by

$$\tan\psi = \frac{\sqrt{1-\cos^2\vartheta}}{m_1/m_2 + \cos\vartheta} \Rightarrow \tan\psi = \frac{m_2}{\sqrt{m_1^2 - m_2^2}} \Rightarrow \sin\psi = \frac{m_2}{m_1}. \tag{9.15}$$

We could in fact have derived the expression of maximum angle geometrically, as we have done with other relations. In the figure of below, we can see the maximum value can only be done when $\vec{v}_1$ and $\vec{v}_1'$ are perpendicular. At this point, $\sin\psi = V/v_1$, and by the previous discussions, this equals to m_2/m_1.
Note that, since m_2/m_1 is always less than one (because we are in this case) and the function goes from 0 to ψ_{max}, it never reaches the second, third or fourth quadrant. Therefore, in this case, respecting to LAB system, there is no scattering backwards. All these related dependencies on the angle ϑ and the different cases for m_2/m_1 are represented by the graph 9.7.
As we see, the function has a one to one correspondence when $m_2 \geq m_1$. Besides, the maximum angle is reached at $\vartheta = \pi$, and is $\psi_{\max} = \pi$ when m_2 is greater than m_1 and $\psi_{\max} = \pi/2$ when is equal, again as we calculated. For the cases where $m_1 < m_2$ there is a maxima which depends on the ratio $\alpha = m_2/m_1$, and the angle reached at the end is $\psi = 0$. All this representations and formulae confirms the important paper of the relation between masses in the final angle ψ, so as we

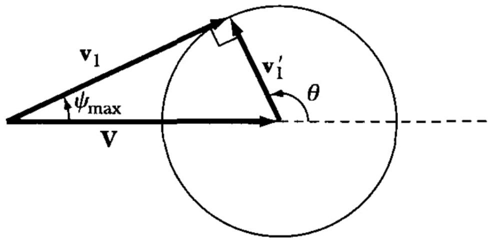


Figure 9.6: Relation between final velocities when the angle ψ is maximum.

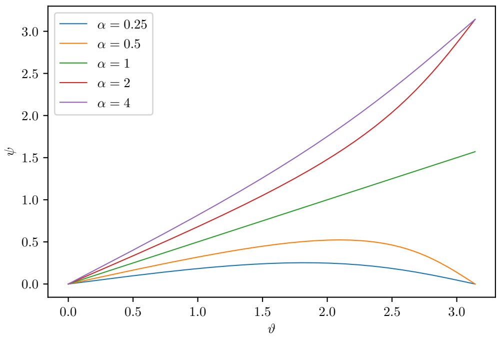


Figure 9.7: Function of $\psi(\vartheta)$ for different values of m_2/m_1.

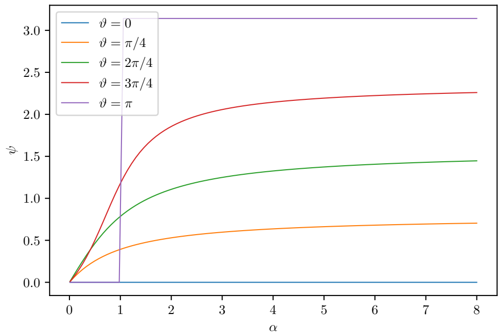


Figure 9.8: Function of $\psi(\alpha)$ for different values of ϑ.

did before, it would be convenient too to study the angle as a function of α, having the following expression.

$$\psi(\vartheta) = \arctan\left(\frac{\sin\vartheta}{1/\alpha + \cos\vartheta}\right) \tag{9.16}$$

If we compute its derivative, we get

$$\frac{d\psi}{d\alpha} = \frac{\sin\vartheta}{\alpha^2 + 2\alpha\cos\vartheta + 1}.$$

Since we are studying with positive values of ϑ, $\sin\vartheta$ is positive and therefore $d\psi/d\alpha$ too. This means the $\psi(\alpha)$ is an increasing function, so the maximum value will be reached when $m_2 \gg m_1$. The graph 9.8 shows this always increasing behavior of ψ, with two particular cases. In the case where $\vartheta = 0$, independently on the ratio, $\psi = 0$, and this is exactly what we got in the graph 9.7. The second particular case is when $\vartheta = \pi$, because it depends strongly on the cases $m_2 < m_1$, $m_2 = m_1$, and $m_2 > m_1$. In the first case we see $\psi = 0$, in the second one $\pi/2$, and in the third one π, without considering the actual value of α. These are the results we got while studying when $\psi(\vartheta)$ was bijective.

Now we have studied the possible values of maximum angles, let us now discuss our expression at limits: when $m_2 \ll m_1$, when $m_2 = m_1$, and when $m_2 \gg m_1$. If $m_2 \ll m_1$, the expression 9.12 tends to 0 and, independently on the value of ϑ, we will have $\psi = 0$. If $m_2 \gg m_1$, the expression 9.12 becomes $\tan\vartheta$ and therefore we conclude that $\psi = \vartheta$. Note that this result is coherent with the fact that, when we studied the case $m_2 > m_1$, the function $\psi(\vartheta)$ was bijective and $\psi(\pi) = \pi$. Finally, if $m_2 = m_1$, the function can be expressed by some trigonometric properties as

$$\tan\psi = \frac{\sin\vartheta}{1 + \cos\vartheta} = \tan\frac{\vartheta}{2},$$

from which we conclude that $\psi = \vartheta/2$. Again, this matches with our previous discussion, where we obtained that $\psi_{\max} = \pi/2$. With all this we conclude our discussion on the angle of the first particle.

Velocity and angle

We have observed in detail how the angle and magnitude behave depending on the angle of scattering and the ratio between masses. However, we still have to see the behavior of the velocity as a whole, which still has new things to observe. Using again the equations 9.7 on the relations 9.8, we obtain

$$\vec{v}_1 = \frac{u_1}{m_1 + m_2} \left[(m_1 + m_2 \cos\vartheta)\vec{e}_x + m_2 \sin\vartheta \vec{e}_y \right] \tag{9.17}$$

9.5 Study of m_2

Velocity

$$v_2^2 = \left[2\frac{m_1^2}{(m_1+m_2)^2}(1-\cos\vartheta) \right] u_1^2 \tag{9.18}$$

Angle

$$2\zeta = \pi - \vartheta \tag{9.19}$$

9.6 Relation between energies

	LAB system	CM system
Initial kinetic energy	$K_0 = \frac{1}{2}m_1u_1^2$	$K_0' = \frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2$
Final kinetic energy	$K_1 = \frac{1}{2}m_1v_1^2$	$K_1' = \frac{1}{2}m_1v_1'^2$
	$K_2 = \frac{1}{2}m_2v_2$	$K_2' = \frac{1}{2}m_2v_2'^2$

From these, we obtain

$$\frac{K_0'}{K_0} = \frac{m_1u_1'^2/2 + m_2u_2'^2/2}{m_1u_1^2/2} \overset{\text{Eq 9.7}}{=} \frac{m_1m_2^2u_2^2/2(m_1+m_2)^2 + m_2m_1u_1^2/2(m_1+m_2)^2}{m_1u_1^2/2} = \frac{m_2}{m_1+m_2}$$

$$\frac{K_1'}{K_0} = \frac{m_1v_1'^2/2}{m_1u_1^2/2} \overset{\text{Eq 9.7}}{=} \frac{m_1m_2^2u_1^2/2(m_1+m_2)^2}{m_1u_1^2/2} = \frac{m_2^2}{(m_1+m_2)^2}$$

$$\frac{K_2'}{K_0} = \frac{m_2v_2'^2/2}{m_1u_1^2/2} \overset{\text{Eq 9.7}}{=} \frac{m_2m_1^2u_1^2/2(m_1+m_2)^2}{m_1u_1^2/2} = \frac{m_1m_2}{(m_1+m_2)^2}$$

$$\frac{K_1}{K_0} = \frac{m_1v_1^2/2}{m_1u_1^2/2} \overset{\text{Eq 9.9}}{=} \frac{m_1u_1^2/2}{m_1u_1^2/2}\left[1 - 2\frac{m_1m_2}{m_1+m_2}(1-\cos\vartheta)\right] = 1 - 2\frac{m_1m_2}{m_1+m_2}(1-\cos\vartheta)$$

$$\frac{K_2}{K_0} = \frac{m_2v_2^2/2}{m_1u_1^2/2} \overset{\text{Eq 9.18}}{=} \frac{m_2u_1^2/2}{m_1u_1^2/2}2\frac{m_1^2}{(m_1+m_1)^2}(1-\cos\vartheta) = 2\frac{m_1m_2}{(m_1+m_2)^2}(1-\cos\vartheta)$$

9.7 Observations from graphics and applets

9.7.1 When $x < 1$

If $x < 1$, in the applet *GeoDispEB-cdf* we can see the angle $\psi(\vartheta)$ has the range of values $[0, \pi]$. We can see this too in the graph, but with a difference that could lead us to errors. Taking the value of $x = 0,5$, in the graph $\psi(\vartheta)$ increases until it reaches the value of $\pi/2$ but then continues in a negative range. This error has an explanation, and is for the relation $\tan\psi = \sin\vartheta/(x + \cos\vartheta)$.

When $\cos\vartheta$ becomes less than $-x$ the value of the right side becomes negative. The applet represent correctly this having a tangent negative in the second quadrant. However, since the graph makes $\psi = \arctan f(\vartheta)$, it confuses the value of tangent of the second quadrant and the fourth. Therefore, it thinks it has a negative angle. If we correct this adding *pi* to the negative region of ψ we would see a continuous and bijective function going from 0 to π, the range of values we see in the applet. In the applet there another interesting detail. While $\psi(\theta)$ is changing, the velocity v_1 not only changes in angle but also in norm. It does it exactly such that it draws a perfect circle with center in the extreme of the vector $\vec{V}$. This means that we could integrate these velocities (which are constant), and instead of having a circle of velocities we would have a circle of distance (an actual circle) with center $\vec{V}t$.

9.7.2 When $x = 1$

If $x = 1$, the applet shows us that the range of ψ is $[0, \pi/2]$. Besides, we see that the circle we commented before still exists but now the point of impact is in the edge of the circle. Finally, since velocities from the CM form a right line that passes through the center of the circle, it draws the diameter. And since the velocities from LAB reference are segments that go from this diameter segment to the point of impact, which is in the edge, for Tale's theorem, they draw a right angle. In other words, if $m_1 = m_2$, the angle after the impact is of 90 degrees.
In relation to the graph, since now there is no point where the expression becomes negative, the confusion of before does not happen. Therefore we can see a graph that goes from 0 to $\pi/2$ continues. In fact, it behaves almost like a line.

9.7.3 When $x > 1$

If $x > 1$, the circle drawn only is formed by the extreme of the vector v_1. The other velocity and the point of impact are out the circle. From this final fact we can see that now there is now a one-to-one correspondence between these angles. Two angles ϑ can correspond to the same ψ (but not in the inverse relation). The fact that it forms an angle ψ two times shows us that at some point it starts decreasing of angle, so there is a maximum angle $\psi_{max} < \pi/2$.
We can see this facts in the graph too. For example, for $x = 1,5$ we see the function $\psi(\vartheta)$ is not bijective and that there exists a maximum value for ψ.