1 Arithmetic and topology

Definition 1.1. Let $\mathbb{R}^2 = \{(x.y) \mid x, y \in \mathbb{R}\}$. Let us consider the following operations of addition and multiplication:

- Sum: given two $(a,b), (c,d) \in \mathbb{R}^2$ we define the sum "+" by components
$$(a,b) + (c,d) \coloneqq (a+b, c+d). \tag{1}$$
- Product: given two $(a,b), (c,d) \in \mathbb{R}^2$ we define the product by
$$(a,b)(c,d) = (ac - bd, ad + bc). \tag{2}$$

We define the set $\mathbb{C}$ as $(\mathbb{R}^2, +,)$.

Proposition 1.1. *The set $\mathbb{C}$ of complex numbers is an abelian field.*

Proposition 1.2. *Let $\mathbb{C}$ be defined in the second way. Then,*

1. *$\mathbb{C}$ is an abelian ring.*
2. *If we define f as*
$$\begin{aligned} f : (\mathbb{C}, +,) &\longrightarrow (\mathbb{R}^2, +,) \\ (x,y) &\longmapsto x + yi \end{aligned}, \tag{3}$$
then f is a morphism of rings.
3. *The function f is, in fact, an isomorphism and $\mathbb{C}$ is an abelian field.*

Proposition 1.3. *The subset of $\mathbb{C}$ generated by numbers of the form $\underline{x} = (x, 0)$ is isomorph to the set of real numbers.*

Theorem 1.4. *$\mathbb{C}$ is not an ordered field.*

Proposition 1.5. *For all $z, w \in \mathbb{C}$, we have:*

1. *$\bar{\bar{z}} = z$.*
2. *$\overline{z + w} = \bar{z} + \bar{w}$.*
3. *$\overline{zw} = \bar{z}\bar{w}$.*
4. *$z\bar{z} \in \mathbb{R}$. In particular, if $z = a + bi$, then $z\bar{z} = a^2 + b^2$.*
5. *$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$.*
6. *The inverse element of $z \in \mathbb{C}^*$ in multiplication is $z^{-1} = \bar{z}/(z\bar{z})$.*

Proposition 1.6. *Let $z \in \mathbb{C}$. Then,*
$$\operatorname{Re}\{z\} = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im}\{z\} = \frac{z - \bar{z}}{2i} \tag{4}$$

Proposition 1.7. *Let $z, w \in \mathbb{C}$ and the following distance function.*
$$\begin{aligned} \tilde{d} : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z,w) &\longmapsto \tilde{d}(z,w) \coloneqq |z - w| \end{aligned} \tag{5}$$
Then, $(\mathbb{C}, \tilde{d})$ is a metric space.

Lemma 1.8. *The set $B = \{B_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{R}^2\}$ is a basis of the topology of $\mathbb{R}^2$ as a metric space. The set $D = \{D_r(z_0) \mid r \in \mathbb{R}^+, z_0 \in \mathbb{C}\}$ is a basis of the topology of $\mathbb{C}$ as a metric space.*

Proposition 1.9. *The sets $\mathbb{C}$ and $\mathbb{R}^2$ with the topology of metric space are homeomorphs.*

Proposition 1.10. *Let $z, w \in \mathbb{C}$. Then,*

1. *$|z| \geq 0$.*
2. *$|z| = 0 \Leftrightarrow z = 0$.*
3. *$-|z| \leq \operatorname{Re}\{z\} \leq |z|$ and $-|z| \leq \operatorname{Im}\{z\} \leq |z|$.*
4. *$|zw| = |z||w|$.*
5. *If $w \neq 0$, $|z/w| = |z|/|w|$.*
6. *$|z + w| \leq |z| + |w|$.*
7. *$|z + w| \geq ||z| - |w||$.*
8. *$|\operatorname{Re}\{zw\}| \leq |z||w|$ and $|\operatorname{Im}\{z\}| \leq |z||w|$.*
9. *$|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}\{z\bar{w}\}$.*
10. *$|z^n| = |z|^n$*

Proposition 1.11. *Let $z \in \mathbb{C}$ and r_θ its polar form. Then,*
$$z^n = (r^n)_{n\theta}. \tag{6}$$

Proposition 1.12. *Let $z, w \in \mathbb{C}$. Then,*

1. *$\arg zw = \arg[z] + \arg[w] + 2\pi k$.*
2. *$\arg z^n = n \arg z + 2\pi k$.*

Theorem 1.13. *Let $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$. Then, there exist $w_1, \ldots, w_n \in \mathbb{C}$ such that $w_i^n = z$ for all $i \in \{1, \ldots, n\}$, and $w_i \neq w_j$ for all $i \neq j$. Besides, if $\omega \in \mathbb{C}$ satisfies $\omega^n = z$, then $\omega = w_k$ for some $k \in \{1, \ldots, n\}$.*

Proposition 1.14. *Let $\{z_n\} = \{a_n + ib_n\}$ be a sequence of complex numbers. Then, it converges if and only if $\{a_n\}$ and $\{b_n\}$ converge.*

Proposition 1.15. *$\sum_{n=1}^{\infty} z_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.*

Proposition 1.16. *$\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge.*

2 Sequences and limits

Definition 2.1. A *sequence of complex numbers* is an application of the form
$$\begin{aligned} \mathbb{N}_{\geq m} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned}. \tag{7}$$
We denote it by $\{z_n\}_{n=m}^{\infty}$

Theorem 2.1. *Let $z_n = z_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$ and $L = L_x + iL_y \in \mathbb{C}$. Then,*
$$\{z_n\}_{n=0}^{\infty} \to L \Leftrightarrow \{x_n\}_{n=0}^{\infty} \to L_x \wedge \{y_n\}_{n=0}^{\infty} \to L_z. \tag{8}$$

Theorem 2.2. *Let $\{z_n\}_{n=0}^{\infty}$ be a convergent sequence. Then, it is a Cauchy sequence.*

Theorem 2.3. *Let $z_n = x_n + iy_n$ be the general term of a sequence $\{z_n\}_{n=0}^{\infty}$. Then,*

$$\{z_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence } \Leftrightarrow \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \text{ are Cauchy sequences.} \tag{9}$$

Theorem 2.4. *The field $\mathbb{C}$ of complex numbers is complete.*

3 Functions

Definition 3.1. A *topology* is an ordered pair $(\mathbb{X}, \tau)$, where X is a set and τ a collection of subsets of X satisfying the following properties:

1. The empty set and X belong to τ.
2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ.
3. The intersection of any finite number of members of τ still belongs to τ.

The elements of τ are called *open sets* and the collection τ is called the *topology on X*.

Proposition 3.1. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, $f = \mathrm{Re}\{f\} + i\,\mathrm{Im}\{f\}$ is continuous at z_0 if and only if $\mathrm{Re}\{f\}$ and $\mathrm{Im}\{f\}$ are continuous at z_0.*

Proposition 3.2. *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \Omega$. Then, f is continuous at z_0 if and only if for all sequence $\{z_n\}_{n=1}^{\infty}$ of Ω convergent at z_0 it is true that the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to $f(z_0)$.*

Proposition 3.3. *Let $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. Then, λf, $f + g$, and fg are continuous at z_0. The function f/g is continuous at z_0 if $g(z_0) \neq 0$.*

Proposition 3.4. *The radius of convergence can be calculated as follows*

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}. \tag{10}$$

Theorem 3.5 (Cauchy-Hadamard Theorem)**.** *Let*

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{11}$$

be the following complex power series with $a_n, z_0 \in \mathbb{C}$ and $R \in [0, +\infty) \cup \{+\infty\}$ the radius of convergence. Then,

1. *If $|z - z_0| < R$ then S converges. In fact, for all $r < R$ we have S converges uniformly at the disc $\overline{D_r(z_0)}$.*
2. *If $|z - z_0| > R$ then S diverges.*
3. *The function $f(z) = S(z)$ is derivable at $D_R(z_0)$ and its formal derivative is*

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \tag{12}$$

with the same radius of convergence.

Theorem 3.6 (Abel's Theorem)**.** *Let be the following series*

$$\sum_{n=0}^{\infty} f_n(z_0) g_n(z_0),$$

where f, g_n are complex functions. If $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly converging at $\Omega \subseteq \mathbb{C}$ and exists $M \geq 0$ such that for $z_0 \in \Omega$,

$$|g_1(z_0)| + \sum_{n=1}^{\infty} |g_n(z_0) - g_{n+1}(z_0)| \leq M, \tag{13}$$

then the original series converges uniformly in Ω.

Theorem 3.7 (Weierstrass' criterion)**.** *Let $f_n, n \in \mathbb{N}$ be a sequence of functions defined at $\Omega \subseteq \mathbb{C}$ such that $|f_n(z)| < M_n$ for all $z \in \Omega, n \geq 1$, and certain constant M_n, satisfying $\sum_{n=0}^{\infty} M_n < \infty$. Then, $\sum_{n=0}^{\infty} f_n(z_0)$ is uniformly convergent at Ω for all $z_0 \in \Omega$.*

Proposition 3.8. *Let $\varnothing \neq \Omega \subseteq \mathbb{C}$. Then,*

1. *Every connected component of Ω is a closed of Ω with a subspace topology.*
2. *Two connected components are the same or are disjoint.*
3. *Every connected of Ω is one and only one connected component.*
4. *Ω is the disjoint union of its connected components.*

Theorem 3.9 (Analytic prolongation Principle)**.** *Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ an analytic function and $z_0 \in \Omega$ such that $f^{(n)}(z_0 = 0)$ for all $n \in \mathbb{N}$. Then, $f(z) = 0(z)$ at the connected component of Ω that contains z_0 (the function can be zero also out Ω, but we are not studying that).*

Lemma 3.10. *Given two sequences $\{a_n\}, \{b_n\} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, with at least one of them absolutely convergent, then*

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right). \tag{14}$$

Proposition 3.11. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined at $D_R(0)$ for certain $R \in \mathbb{R}^+$. Then, f is analytic at $\Omega = D_R(0)$.*

Proposition 3.12. *The radius of convergence of e^z is infinite.*

Proposition 3.13. *$\mathrm{e}^z = \mathrm{e}^x$ for all $x \in \mathbb{R}$.*

Proposition 3.14. *$\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$ for all $z, w \in \mathbb{C}$.*

Proposition 3.15. *For all $z \in \mathbb{C}$ we have $\mathrm{e}^z \neq 0$.*

Proposition 3.16. *The image of e^z is $\mathbb{C}^*$.*

Proposition 3.17. *The derivative of* e^z *is* e^z.

Proposition 3.18. $\overline{e^z} = e^{\bar{z}}$.

Proposition 3.19. $|e^z| = e^{\text{Re}\{z\}}$.

Proposition 3.20 (Euler's Formula)**.** *If* $\theta \in \mathbb{R}$*, then* e^{xi} *has modulus one and we have that*

$$\boxed{e^{xi} = \cos x + i \sin x.} \tag{15}$$

Proposition 3.21. *The following function*

$$\begin{aligned} \exp : (\mathbb{R}, +) &\longrightarrow (\mathbb{C}^*, \cdot) \\ x &\longmapsto e^{xi} \end{aligned} \tag{16}$$

is a morphism of groups and its image is $\mathbb{T}$.

Proposition 3.22. *The complex exponential function is a periodic function with period* $2\pi i$.

Proposition 3.23. *Let* $a \in \mathbb{C}^*$*. Then,* $e^z = a$ *has infinite solutions.*

Proposition 3.24. *For all* $z \in \mathbb{C}$,

$$\sin^2 z + \cos^2 z = 1. \tag{17}$$

Proposition 3.25. *For all* $z \in CC$,

$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z). \tag{18}$$

Proposition 3.26. *For all* $z, w \in \mathbb{C}$,

$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w, \qquad \sin(z \pm w) = \sin z \cos w \pm \cos z \sin w. \tag{19}$$

Proposition 3.27. *The functions* $\cos z, \sin z$ *have period of* 2π.

Proposition 3.28. *Let* $z_0 \in \mathbb{C}$*. Then,* z_0 *is root of* $\sin z$ *(*$\cos z$*) if and only if it is a root of* $\sin x$ *(*$\cos x$*).*

Proposition 3.29. *For all* $z \in \mathbb{C}$,

$$\sinh^2 z - \cosh^2 z = 1. \tag{20}$$

Proposition 3.30. *For all* $z \in CC$,

$$\cosh(-z) = \cosh(z), \qquad \sinh(-z) = -\sinh(z). \tag{21}$$

Proposition 3.31. *For all* $z, w \in \mathbb{C}$,

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w, \qquad \sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w. \tag{22}$$

Proposition 3.32. *For all* $z \in \mathbb{C}$,

$$\cosh z = \cos(iz), \cos z = \cosh(iz) \qquad \sinh z = -i \sin(iz), \sin z = -i \sinh(iz) \tag{23}$$

Proposition 3.33. *The roots of the function* $\sinh z$ *are of the form* $z_n = n\pi$ *and those for the function* $\cosh z$ *are of the form* $w_n = (2n+1)\pi/2i$.

Proposition 3.34. *Given* $z \in \mathbb{C}$ *we can define* $\ln z$ *from the natural logarithm of a real number as*

$$\ln z = \ln |z| + i \arg z = \ln |z| + i\text{Arg}z + 2\pi k i. \tag{24}$$

Proposition 3.35. *Let* $z, w \in \mathbb{C}$ *two numbers. Then,*

1. $\ln(zw) = \ln z + \ln w + 2\pi k i, k \in \mathbb{Z}$.
2. *If we want to stay in the principal argument,*
$$\ln(zw) = \begin{cases} \ln z + \ln w, & \text{if } \text{Arg}z + \text{Arg}w < 2\pi \\ \ln z + \ln w - 2\pi i, & \text{if } \text{Arg}z + \text{Arg}w \geq 2\pi \end{cases}. \tag{25}$$
3. *SEARCH MORE PROPERTIES*

Proposition 3.36. *If* $a = \alpha + \beta i$ *and* $z = re^{\theta i}$*, then*

$$z^a = e^{\alpha \ln r - \beta(\theta + 2\pi k)} e^{\beta \ln r + \alpha(\theta + 2\pi k)}, \tag{26}$$

$$|z^a| = e^{\alpha \ln |z| - \beta(\arg z + 2\pi k)}, \qquad \arg(z^a) = \beta \ln |z| + \alpha(\arg z + 2\pi k) \tag{27}$$

Proposition 3.37. *Let* $z, w \in \mathbb{C}$*. Then,*

1. $(e^b)^a = e^{a(b + 2\pi k i)}$

4 Derivatives

Definition 4.1. Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$ a point. We define the *derivative of f at z_0* as

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{28}$$

in case the limit exists. If f has derivative, we say f *is* $\mathbb{C}$*-derivable at* z_0.

Proposition 4.1. *Let* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be a function and* $z_0 \in CC$ *a point. If* f *is derivable at* z_0*, then it is continuous at* z_0.

Theorem 4.2. *Let* $f, g : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *be two functions and* $z_0 \in \Omega$ *a point. Then, the following statements are true.*

1. *If* f *is constant at* Ω*, then* f *is derivable at* z_0 *and* $f'(z_0) = 0$.
2. *If* $f(z) = z$ *in every point of* Ω*, then* f *is derivable at* z_0 *and* $f'(z_0) = 1$.
3. *If* f, g *are derivable at* z_0 *and* $\alpha, \beta \in \mathbb{C}$*, then* $\alpha f + \beta g$ *are derivable at* z_0 *and* $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.
4. *If* f, g *are derivable at* z_0*, then* fg *is derivable at* z_0 *and*
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{29}$$
5. *If* f, g *are derivable at* z_0 *and* $g(z_0) \neq 0$*, then* f/g *is derivable at* z_0 *and*
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{30}$$

Theorem 4.3. *Let $f:\Omega_1\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be a derivable function at a point $z_0\in\mathbb{C}$ and $g:\Omega_2\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be another derivable function at a point $f(z_0)\in\Omega_2$. Then, $g\circ f$ is derivable at z_0 and*

$$(g\circ f)'(z_0)=g'(f(z_0))f'(z_0). \tag{31}$$

Theorem 4.4. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ a function of class $C^1(\Omega)$ with Ω an open set, injective, and derivable at every point of Ω with non-zero derivative. Then,*

1. *$\Omega'=f(\Omega)$ is an open subset of $\mathbb{C}$.*
2. *The inverse function f^{-1} exist, it is well defined and it is derivable at Ω'.*
3. *If $z\in\Omega$ and $z'=f(z)$, then*

$$(f^{-1})'(z')=\frac{1}{f'(z)}. \tag{32}$$

Proposition 4.5. *A determination of $\ln z$ with $z\in\mathbb{C}$ is continuous except in a semiline.*

Theorem 4.6. *Let $\Omega\subseteq\mathbb{C}$ be an open set and $\phi\in C(\Omega)$, such that $\mathrm{e}^{\phi(z)}=z$ for all $z\in\Omega$. Then, we have $\phi\in H(\Omega)$ and*

$$\phi'(z)=\frac{1}{z},\forall z\in\Omega. \tag{33}$$

Proposition 4.7. *A determination of $\ln z$ with $z\in\mathbb{C}$ is holomorphic except in a semiline.*

Proposition 4.8. *Let $I=[\theta,\theta+2\pi)$ a determination of the logarithm, $\ln_I z$. Then, $\ln_I z$ is holomorphic except in the semiline $L_\theta=\left\{r\mathrm{e}^{\theta i}\in\mathbb{C}\,\middle|\,r\geq 0\right\}$.*

Theorem 4.9. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be an analytic function in Ω. Then, f is holomorphic in Ω. In particular, given a point $z_0\in\mathbb{C}$, $f'(z_0)=a_1$ where a_1 is the first coefficient of the power series that represents f in a neighborhood of z_0.*

Proposition 4.10. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ a function of class $C^1(\Omega)$. Then, for all $z_0\in\Omega$*

$$f(z_0+h)=f(z_0)+\left(\frac{\partial f}{\partial z}\right)_{z_0}h+\left(\frac{\partial f}{\partial\bar z}\right)_{z_0}\bar h+o(|h|^2). \tag{34}$$

5 Holomorphic functions

Definition 5.1. Let $I=[a,b]\subseteq\mathbb{R}$ be a closed interval. We define a *curve* as an application of the form

$$\begin{aligned}\gamma: I&\longrightarrow\mathbb{C}\\ t&\longmapsto\gamma_1(t)+i\gamma_2(t)\end{aligned}. \tag{35}$$

Theorem 5.1. *Let $f:\Omega\subseteq\mathbb{C}\longrightarrow\mathbb{C}$ be an holomorphic function of class $C^1(\Omega)$ with Ω an open set and $\phi: I\longrightarrow\Omega$ a basic curve. Then, $\psi=f\circ\phi$ is a basic curev (hence a derivable curve) and its real derivative is*

$$\psi'(t)=f'(\phi(t))\phi'(t). \tag{36}$$

6 Local properties of holomorphic functions

7 Isolated singularities of holomorphic functions

8 Homology

9 Harmonic functions

10 Conforming representation

11 Riemann theorem

12 Runge theorem

13 Zeros of holomorphic functions

14 Fourier transform

Definition 14.1. Let $f\in L^1(\mathbb{R})$ be a function and $\xi\in\mathbb{R}$ a number. We define the *Fourier transform of f at the point ξ* as

$$\boxed{\hat f(\xi)\coloneqq\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x\,.} \tag{37}$$

Proposition 14.1. *Let $f\in L^1(\mathbb{R})$ be a function. Then, the application*

$$\begin{aligned}\mathscr{F}\{f\}:\mathbb{R}&\longrightarrow\mathbb{C}\\ \xi&\longmapsto\hat f(\xi)\end{aligned} \tag{38}$$

is a well defined application.

Theorem 14.2. *Let $f\in L^1(\mathbb{R})$ be a function. Then, the following statements are true.*

1. *The Fourier transform of f satisfies*

$$\mathscr{F}\{f\}\in L^\infty(\mathbb{R}),\qquad\|\mathscr{F}\{f\}\|_\infty\leq\frac{1}{\sqrt{2\pi}}\|f\|_1. \tag{39}$$

2. *$\mathscr{F}\{f\}$ is $\mathbb{C}$ linear, that is, for all $\alpha,\beta\in\mathbb{C}$ and $f,g\in L^1(\mathbb{R})$,*

$$\mathscr{F}\{\alpha f+\beta g\}=\alpha\mathscr{F}\{f\}+\beta\mathscr{F}\{g\}. \tag{40}$$

3. *If $g(x)=\bar f(x)$, then for all $\xi\in\mathbb{R}$*

$$\hat g(\xi)=\overline{\hat f(-\xi)}. \tag{41}$$

4. *If $g(x)=g(\lambda x)$, then for all $\xi\in\mathbb{R}$*

$$\hat g(\xi)=\frac{1}{\lambda}\hat f\left(\frac{\xi}{\lambda}\right). \tag{42}$$

5. *If $g(x) = f(x-a)$ with $a \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \mathrm{e}^{-ia\xi}\hat{f}(\xi). \tag{43}$$

6. *If $g(x) = \mathrm{e}^{iax} f(x)$ with $\alpha \in \mathbb{R}$, then for all $\xi \in \mathbb{R}$*

$$\hat{g}(\xi) = \hat{f}(\xi - a) \tag{44}$$

7. *If $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$ when $n \to \infty$, then $\mathscr{F}\{f_n\} \to \mathscr{F}\{f\}$ uniformly in $\mathbb{R}$.*

8. *The Fourier transform $\mathscr{F}\{f\}$ is a continuous function in $\mathbb{R}$, $\mathscr{F}\{f\} \in C(\mathbb{R})$.*

Proposition 14.3. *Let $f \in L^1(\mathbb{R})$ be a function such that there exists its derivative $f' \in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty} |f(x)| = 0$. Then,*

$$\hat{f}'(\xi) = i\xi\hat{f}(\xi). \tag{45}$$

Theorem 14.4. *Let $f \in L^1(\mathbb{R})$ be a function. Then, there exists a sequence of functions $\phi_\in \mathscr{D}(\mathbb{R})$ such that*

$$\lim_{h\to\infty} \int\limits_{\mathbb{R}} |f - \phi_n| \,\mathrm{d}x = 0, \tag{46}$$

that is, we have convergence of ϕ_n to f with norm $\|\cdot\|_1$.

Proposition 14.5. *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\hat{f} \in C(\mathbb{R})$.*

Proposition 14.6. *Let $f \in L^1(\mathbb{R})$ be a function. Then, $\left|\hat{f}(\xi)\right| \le \|f\|_1$.*

Theorem 14.7. *Let $f \in L^1(\mathbb{R})$ be a function. Then,*

$$\lim_{|x|\to\infty} \hat{f}(x) = 0. \tag{47}$$

Theorem 14.8. *The application Fourier transform goes from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$, that is, $\mathscr{F}\{f\} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.*

Proposition 14.9. *Let $f, g \in S(\mathbb{R})$ be two functions, $\lambda \in \mathbb{C}$ a number, and $P : \mathbb{R} \longrightarrow \mathbb{C}$ a polynomial of complex coefficients. Then,*

1. *$f + g \in S(\mathbb{R})$.*
2. *$\lambda f \in S(\mathbb{R})$.*
3. *$fg \in S(\mathbb{R})$.*
4. *$Pf \in S(\mathbb{R})$.*

Theorem 14.10. *Let $I, J \subseteq \mathbb{R}$ be two intervals with I compact and J open. Let $f : I \times J \longrightarrow \mathbb{R}$ be a function such that*

1. *$f(\cdot, \lambda)$ is Riemann-integrable in I for all $\lambda \in J$,*
2. *$f(x, \cdot)$ is derivable in J for all $x \in I$.*

If $\partial_\lambda f$ is continuous in $I \times J$, then

1. *$\partial_\lambda f(\cdot, \lambda)$ is Riemann-integrable for all $\lambda \in J$.*
2. *$F(\lambda) = \int\limits_I f(x, \lambda)\,\mathrm{d}x$ is derivable with continuous derivative in J for all $\lambda \in J$ and it satisfies the rule of derivation over the integral sign.*

$$F'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int\limits_I f(x, \lambda_0)\,\mathrm{d}x = \int\limits_I \frac{\partial f}{\partial \lambda}(x, \lambda_0)\,\mathrm{d}x \,, \forall \lambda_0 \in J \tag{48}$$

Proposition 14.11. *Let $f \in S(\mathbb{R})$. Then,*

1. *$S(\mathbb{R}) \subseteq L^1(\mathbb{R})$.*
2. *$\widehat{xf}(\xi) = (iD_\xi \hat{f})(\xi)$ for all $\xi \in \mathbb{R}$.*

Proposition 14.12. *The Fourier transform $\mathscr{F}$ restricted to $S(\mathbb{R})$ is an automorphism, that is, if $f \in S(\mathbb{R})$ then $\mathscr{F}\{f\} = \hat{f} \in S(\mathbb{R})$.*

Lemma 14.13. *If $G(x) = \mathrm{e}^{-x^2/2}$, then $\hat{G}(\xi) = e^{-\xi^2/2}$. We observe hence that G is a fixed point of $\mathscr{F}$.*

Lemma 14.14. *If $f, g \in S(\mathbb{R})$, then*

$$\int\limits_{\mathbb{R}} f(\xi)\hat{g}(\xi)\,\mathrm{d}\xi = \int\limits_{\mathbb{R}} \hat{f}(\tau)g(\tau)\,\mathrm{d}\tau \,. \tag{49}$$

Lemma 14.15. *Let $f, g \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,*

1. *$g(\lambda x)\hat{f}(x)$ converges to $g(0)\hat{f}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*
2. *$f(\lambda x)\hat{g}(x)$ converges to $f(0)\hat{g}(x)$ uniformly in $\mathbb{R}$ when $\lambda \to \infty$.*

Lemma 14.16. *Let $f, g \in s(\mathbb{R})$. Then,*

$$f(0)\int\limits_{\mathbb{R}} \hat{g}(\xi)\,\mathrm{d}\xi = g(0)\int\limits_{\mathbb{R}} \hat{f}(\xi)\,\mathrm{d}\xi \,. \tag{50}$$

Lemma 14.17. *Let $f \in s(\mathbb{R})$ be a function. Then,*

$$f(0) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \hat{f}(\xi)\,\mathrm{d}\xi \,. \tag{51}$$

Theorem 14.18 (Inversion of $\mathscr{F}$ in $S(\mathbb{R})$)**.** *Let $\mathscr{F} : S(\mathbb{R}) \longrightarrow S(\mathbb{R})$, defined by $\mathscr{F}\{f\} = \hat{f}$ with $\hat{f} \in s(\mathbb{R})$. Then, $\mathscr{F}$ is an linear isomorphism in the vector space $S(\mathbb{R})$ and $\mathscr{F}^4 = Id$. In particular, $\mathscr{F}^{-1} = \mathscr{F}^3$ and if $f \in S(\mathbb{R})$, then*

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \mathscr{F}\{f\}(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \hat{f}\mathrm{e}^{ix\xi}\,\mathrm{d}\xi \,.} \tag{52}$$

In fact, $\mathscr{F}$ is an homemorphism (its inverse is continuous) if we consider $S(\mathbb{R})$ as the metric space $(S(\mathbb{R}), \|\cdot\|_{n,m})$.

Theorem 14.19 (Tonelli's Theorem)**.** *Let $f : I \times J \longrightarrow \mathbb{R}^2$ two functions with $I, J \subseteq \mathbb{R}$ such that $f(x,y) \ge 0$ for all $(x,y) \in I \times J$. Then,*

$$\int\limits_{I\times J} f\,\mathrm{d}x\,\mathrm{d}y = \int\limits_I \int\limits_J f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int\limits_J \int\limits_I f(x,y)\,\mathrm{d}x\,\mathrm{d}y \,. \tag{53}$$

Besides, if these integrals are finite, then $f \in L^1(\mathbb{R})$.

Proposition 14.20. *Let $f, g \in L^1(\mathbb{R})$ be two functions. Then $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$.*

Theorem 14.21. *Let $f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty$ and $\phi \in S(\mathbb{R})$. Then, $f * \phi \in C^\infty(\mathbb{R})$.*

Theorem 14.22. *Let $f \in L^p(\mathbb{R}), 1 \leq p < +\infty$ with $\operatorname{supp} f$ compact and $\phi \in D(\mathbb{R})$. Then, $f * \phi \in D(\mathbb{R})$ and $\operatorname{supp}\{f * \phi\} \subseteq \operatorname{supp} f + \operatorname{supp} \phi$.*

Theorem 14.23. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with compact support $\{\phi_\epsilon\}$ approximation of the unity. Then, when $\epsilon \to 0$ $f * \phi_\epsilon$ converges uniformly in $\mathbb{R}$ to f.*

Theorem 14.24 (Weierstrass polynomial approximation)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then, there exist polynomials P_n with $n \in \mathbb{N}$ such that P_n converge uniformly to f in $[a, b]$.*

Theorem 14.25. *Let $f \in L^p(\mathbb{R})$ be a function. Then, there exists a sequence of function $f_n \in D(\mathbb{R})$ of the form $f_n \to f$ with norm $\|\cdot\|_p$ (that is, convergence in L^p), and if $f \in C^k(\mathbb{R})$ with $k \geq 0$, then*

$$\lim_{n\to\infty} \|f_n - f\|_{C^k(\mathbb{R})} = 0, \tag{54}$$

with $\|f\|_{C^k(\mathbb{R})} = \max_{0\leq l\leq k} \left(\sup_{x\in\mathbb{R}} \left|D^l f(x)\right|\right)$ being a norm.

Lemma 14.26. *Let $f \in L^1(\mathbb{R})$ be a function such that for all $\phi \in S(\mathbb{R})$ it is satisfied that $\int\limits_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x = 0$. Then, $f \equiv 0$.*

Theorem 14.27 (Inversion theorem in $L^1(\mathbb{R})$)**.** *Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \hat{f}(\xi) \mathrm{e}^{ix\xi}\,\mathrm{d}\xi\,, \forall x \in \mathbb{R}. \tag{55}$$

15 Multidimensional fourier transform

Theorem 15.1. *For several variables*

$$\mathscr{F}\{f(x_1, \ldots, x_n)\} = \frac{1}{(2\pi)^{n/2}} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} f(x_1, \ldots, x_n) \mathrm{e}^{-i(x_1\xi_1+\cdots+x_n\xi \tag{56}$$

or simpler,

$$\boxed{\mathscr{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{e}^{-i\langle \mathbf{x}, \boldsymbol{\xi}\rangle_I}\,\mathrm{d}\hat{x}\,.} \tag{57}$$