Contents

Chapter 1

Motion in one dimension

1.1   Newton 2nd Law and differential equations

Until now, we have seen how the second and third law of Newton lead to an algebraic equations with acceleration (or accelerations, in case of a system of particles), acting as an unknown (or unknowns). In general, the application of Newton's Laws give rise to a second order ordinary differential equation (or a system of ordinary differential equations) where the position of a particle is an unknown function of time, $\vec{r}(t)$, we have to determine. The most simple case, studied previously [ ], is the case of one particle with constant mass that moves in one dimension, where the equations has the form

$$\ddot{x}(t) = \frac{1}{m}F(x(t), \dot{x}(t), t), \tag{1.1}$$

where we suppose the force function F is known. We also suppose initial position and velocity, denoted by $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$ respectively, are known. With that, finding the solution to the ordinary differential equations and satisfying the given initial conditions consist the *initial value problem*. In some conditions (generally feasible for physical systems), there exist a solution to the initial value problem and is unique [ ].

1.2   Integration of the Newton 2n Law

In this section, we will consider three simple cases: forces that only depend on time, position, and velocity.

1.2.1   Time dependence

Proposition 1.2.1. *If $F = F(t)$, then*

$$\dot{x}(t) = v_0 + \int_0^t \frac{F(t')}{m}\,\mathrm{d}t', \qquad x(t) = x_0 + \int_0^t \dot{x}(t')\,\mathrm{d}t'. \tag{1.2}$$

$$\ddot{x} = \frac{F(t)}{m} = \frac{d\dot{x}}{dt} \Rightarrow d\dot{x} = \frac{F(t)}{m}dt \Rightarrow \int_{\dot{x}(t_0)}^{\dot{x}(t)} d\dot{x} = \int_{t_0}^{t} \frac{F(t')}{m}dt' \Rightarrow \dot{x} = \dot{x}(t_0) + \int_{t_0}^{t} \frac{F(t')}{m}dt'$$

Integrating again

$$\frac{dx}{dt} = \dot{x(t)} \Rightarrow dx = \dot{x}(t)dt \Rightarrow \int_{x(t_0)}^{x(t)} dx = \int_{t_0}^{t} \dot{x}(t')dt' \Rightarrow x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(t')dt'$$

1.2.2   Position dependence

Proposition 1.2.2. *If $F = F(x)$, then*

$$\dot{x}^2(t) = v_0^2 + 2\int_{x_0}^{x} \frac{F(x')}{m}\,\mathrm{d}x', \qquad g(x) = \int_{x_0}^{x} \frac{1}{\dot{x}(x')}\,\mathrm{d}x' \Rightarrow x(t) = g^{-1}(t). \tag{1.3}$$

$$\frac{d\dot{x}}{dt} = \frac{F(x)}{m} \Rightarrow \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{F(x)}{m} = \frac{d\dot{x}}{dx}\dot{x}$$

Therefore, denoting $x(0)$ by x_0 and $x(t)$ by x, we get

$$\dot{x}d\dot{x} = \frac{F(x)}{m}dx \Rightarrow \int_{\dot{x}}^{\dot{x}} \dot{x}d\dot{x} = \int_{x_0}^{x} \frac{F(x')}{m}dx' \Rightarrow \dot{x}^2(x) = \dot{x}^2(x_0) + 2\int_{x_0}^{x} \frac{F(x')}{m}dx'$$

If we take the square root we get $\dot{x}(x)$, and if we integrate again,

$$\frac{dx}{dt} = \dot{x}(x) \Rightarrow \frac{1}{\dot{x}(x)}dx = dt \Rightarrow \int_{x_0}^{x} \frac{1}{\dot{x}(x)}dx' = \int_0^t dt' = t$$

We denote the left integral by $g(x)$, so we have now $t = g(x)$. Inverting the relation, we get

$$x(t) = g^{-1}(t), \qquad g(x) = \int_{x_0}^{x} \frac{1}{\dot{x}(x)}dx'$$

1.2.3 Velocity dependence

Proposition 1.2.3. *If $F = F(v)$, then*

$$h(\dot{x}) = \int_{x_0}^{x} \frac{m}{F(\dot{x}')}\,\mathrm{d}\dot{x}', \qquad x(t) = x_0 + \int_0^t h^{-1}(t')\,\mathrm{d}t'. \tag{1.4}$$

$$\frac{d\dot{x}}{dt} = \frac{F(\dot{x})}{m} \Rightarrow \frac{m}{F(\dot{x})}d\dot{x} = dt \Rightarrow \int_{v_0}^{v} \frac{m}{F(v)}dv' = \int_0^t dt' = t$$

Hence, denoting the left integral by $h(v)$

$$v(t) = h^{-1}(t), \int_{v_0}^{v} \frac{m}{F(v)}dv' \Rightarrow x(t) = x_0 + \int_0^t h^{-1}(t')dt'$$

Example 1.2.1. Let be a particle subjected to a force $F = -c\dot{x}^2$. If $x(0) = 0$ and $v(0) = v_0$, we can find its equation of motion.

$$\ddot{x} = -\frac{c}{m}\dot{x}^2 \Rightarrow \frac{d\dot{x}}{dt} = -\frac{c}{m}\dot{x}^2 \Rightarrow -\frac{m}{c}\frac{1}{\dot{x}^2}\,\mathrm{d}\dot{x} = \mathrm{d}t \Rightarrow \frac{m}{c\dot{x}} - \frac{m}{cv_0} = t \Rightarrow \frac{m}{c\dot{x}} = t + \frac{m}{cv_0} \Rightarrow$$

$$\dot{x} = \frac{v_0}{1 + cv_0/m \cdot t} \Rightarrow x(t) = x_0 + \int_0^t \dot{x}(t')\,\mathrm{d}t' = \frac{m}{c}\ln\left(1 + \frac{cv_0}{m}t\right).$$

1.3 Variable mass

1.3.1 Accretion mass

Proposition 1.3.1. *Let .... Then, it is true that*

$$\vec{F}^{ext} = m\vec{a} + (\vec{v} - \vec{u})\dot{m} \tag{1.5}$$

or which is equivalent,

$$\vec{F}^{ext} + \dot{m}\vec{u} = \dot{\vec{p}} \tag{1.6}$$

Proof. There are two scenarios to consider in this equation, when $m(t)$ is gaining mass and when it is losing. We will study them separately, beginning with the increasing mass situation.
Let us consider the situation of the figure, where, in an interval of time dt, an object of mass $m(t)$ with velocity $\vec{v}(t)$ gains mass as a consequence of adhering to another infinitesimal mass dm that travels at a velocity $\vec{u}(t)$. Before the fusion there are two separate masses, so the linear momentum is

$$\vec{P} = m(t)\vec{v}(t) + dm\vec{u}(t).$$

At the instant $t + dt$, however, the momentum of the system is (remembering that higher-order infinitesimals tend to zero)

$$\vec{p} + d\vec{p} = (m + dm)(\vec{v} + d\vec{v}) = m\vec{v} + md\vec{v} + dm\vec{v},$$

since now there is only one mass. If we take the difference of these expressions and divide it by dt, we get

$$d\vec{p} = m(t)d\vec{v} + (\vec{v} - \vec{u})\,dm \Rightarrow \vec{F}^{\text{ext}} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + (\vec{v} - \vec{u})\frac{dm}{dt} \Rightarrow \vec{F}^{\text{ext}} = m\vec{a} + (\vec{v} - \vec{u})\dot{m}$$

Now let us consider the second case. In this scenario, in an interval of time dt, an object of mass $m(t)$ with velocity $\vec{v}(t)$ ejects some infinitesimal amount of mass dm at a velocity $\vec{u}(t)$. Therefore, before the a separation the system only has one mass and the linear momentum is determined by

$$\vec{P} = m\vec{v}.$$

After the separation, there two masses, one with an increase of dm and the other with $-dm$, so the linear momentum is

$$\vec{P} + d\vec{P} = (m + dm)(\vec{v} + d\vec{v}) + \vec{u}(-dm) = m\vec{v} + dm\vec{v} + \vec{v}dm - dm\vec{u}.$$

Therefore, taking the difference and dividing by d, we get

$$d\vec{P} = dm\vec{v} + (\vec{v} - \vec{u})dm \Rightarrow \vec{F}^{\text{ext}} = m\vec{a} + (\vec{v} - \vec{u})\dot{m},$$

which is the equation we got before.   ∎

1.4   Rocket motion

Let us suppose that a rocket burns fuel that is ejected in form of gas to propel itself. This gas is expelled at a velocity $\vec{c}$ with respect to the rocket and the mass of the rocket, including the fuel, is $m(t)$. Our proposal is to find an expression of $\vec{v}$ to describe the motion of the rocket, and for that first we need to express the magnitudes in an external frame of reference. In particular, the velocity of the fuel will be

$$\vec{u} = \vec{v} + \vec{c}.$$

Now, applying the equation of accretion mass, we obtain

$$\vec{F} = m\vec{a} + (\vec{v} - \vec{u})\dot{m} = m\vec{a} - \dot{m}\vec{c} \Rightarrow m\vec{a} = \vec{F}^{\text{ext}} + \dot{m}\vec{c}. \tag{1.7}$$

1.4.1   Rocket without gravity

Differentiable method

Proposition 1.4.1. *Let be a rocket with an initial mass m_0 that expulses fuel at velocity c with respect to the moving reference frame. If there is no any force field acting upon it, then*

$$v(m) = c\ln\frac{m_0}{m}. \tag{1.8}$$

Since the situation we consider is a scenario without gravity, $\vec{F}^{\text{ext}} = \vec{0}$ and the problem can be solved easily. We will take the positive direction in y so $\vec{v}$ and $\vec{c}$ will be $v\vec{e}_y$ and $-c\vec{e}_y$ respectively.

$$ma = -\dot{m}c \Rightarrow m\frac{dv}{dt} = -c\frac{dm}{dt} \Rightarrow dv = -\frac{c}{m}dm \Rightarrow \int_0^v dv' = -c\int_{m_0}^m \frac{1}{m'}dm' \Rightarrow v(m) = c\ln\frac{m_0}{m}$$

This result says the final velocity is exclusively determined by the final amount of mass, independently on how the mass have been ejected, that is, without specifying $m(t)$. However, this is only valid if $m(t)$ is a differentiable function of time. Although this kind of function is which better describe the behavior of a fuel in real life, it would be interesting to see what we get with non-differentiable expressions of $m(t)$. In particular, we will discuss what happens when instantaneous transitions of mass occur, and see if in some way there is a relation between this process and the continuous one.

Discrete method

First we will consider the case where the rocket that is at rest ejects all the mass instantaneously. Considering as before $\vec{v} = v\vec{e}_y$ and $\vec{c} = -c\vec{e}_y$, the initial and final linear momentum are

$$\begin{cases} P = m_0 v_0 = 0 \\ P_f = m_f v_f + (m_0 - m_f)(v_f - c) = m_0(v_f - c) \end{cases}$$

By the principle of conservation of linear momentum, $P = P_f$ (because there is no external force) and the final velocity is

$$0 = m_0(v_f - c) \Rightarrow v_f = c. \tag{1.9}$$

This result differs from the continuous case, and it is normal since the approach is completely different. Now, to improve the method, we will consider several instantaneous ejections. In particular, the rocket will eject amounts of $\Delta m = (m_0 - m_f)/n$ and will do it n times. In the first ejection, the initial and final linear momentum are

$$\begin{cases} P_0 = m_0 v_0 = 0 \\ P_1 = (m_0 - \Delta m)v_1 + \Delta m(v_1 - c) \end{cases}$$

Applying the conservation of linear momentum, we get

$$0 = m_0 v_1 - \Delta m v_1 + \Delta m v_1 - \Delta m c = m_0 v_1 - \Delta m c \Rightarrow v_1 = \frac{\Delta m}{m_0} c = \frac{\Delta m}{m_f + \Delta m} c$$

Now we will study the second ejection, but first we need to notice an important detail. Since the fuels is ejected as a gas, after the separation of masses it is completely dispersed through the space. Hence, its linear momentum *disappears* after the ejection and it is not consider in the initial momentum of the next one. Said that, the two linear momentum of the second expulsion is

$$\begin{cases} P_1' = (m_0 - \Delta m)v_1 \\ P_2 = (m_0 - 2\Delta m)v_2 + \Delta m(v_2 - c) \end{cases}.$$

By the same procedure, v_2 is determined as follows.

$$m_0 v_1 - \Delta m v_1 = (m_0 - \Delta m)v_2 - \Delta m c \Rightarrow v_2 = v_1 + \frac{\Delta m}{m_0 - \Delta m} = \frac{\Delta m}{m_f + n\Delta m}c + \frac{\Delta m}{m_f + (n-1)\Delta m}c$$

The same reasoning can be applied in the following ejections (again considering the linear momentum of the gas does not account in the next one), and it can be proved easily (by induction for example) that the i-th velocity is

$$v_i = \sum_{k=1}^{i} \frac{\Delta m}{m_f - (n-k+1)\Delta m} c \tag{1.10}$$

Therefore, at the end of the whole process (that is, after the n ejections), the final velocity is

$$v_f = \sum_{k=1}^{n} \frac{\Delta m}{m_f + (n-k+1)\Delta m} c = \sum_{k=1}^{n} \frac{\Delta m}{m_f + k\Delta m} c,$$

where we have changed the indices to simplify the expression [2]. If we denote $m_0 - m_f$ by m_r and continue rephrasing the sum, we get

$$v_f = \sum_{k=1}^{n} \frac{\Delta m}{m_f + k\Delta m} c = \sum_{k=1}^{n} \frac{(m_0 - m_f)/n}{m_f + k(m_0 - m_f)/n} c = \sum_{k=1}^{n} \frac{m_0 - m_f}{n} \frac{1}{m_f + k(m_0 - m_f)/n} c =$$

$$\frac{m_r}{n} \sum_{k=1}^{n} \frac{1}{m_f + m_r k/n} c = c\frac{m_r}{n} \sum_{k=1}^{n} f\left(\frac{m_r k}{n}\right), \ f\left(\frac{m_r k}{n}\right) = \frac{1}{m_f + m_r k/n}$$

In we make n tend to infinity, we get can apply the relation between summations and integrals [1] and finally get

$$\lim_{n \to \infty} c \frac{m_r}{n} \sum_{k=1}^{n} f\left(\frac{m_r k}{n}\right) = c \int_0^{m_r} f(m) dm = c \int_0^{m_r} \frac{1}{m_f + m} dm = c \ln \left| \frac{m_f + m_r}{m_f} \right| = c \ln \frac{m_0}{m_f},$$

which is the same equation we got in the differential form.

1.4.2   Rocket with gravity

Proposition 1.4.2. *Let be a rocket with an initial mass m_0 that expulses fuel at velocity c with respect to the moving reference frame. If there is a force field $\mathbf{g} = -g\mathbf{e}_y$ acting upon it and $\dot{m} = -\beta$, then*

$$v = c \ln \frac{m_0}{m} - \frac{g}{\beta}(m_0 - m) \tag{1.11}$$

if $m_0 g \leq \beta c$. If the last condition is not satisfied, then $v = 0$ for $t \leq m_0/\beta - c/g$ and after

$$v = c \ln \frac{\beta c}{mg} - \frac{g}{\beta}\left(\frac{\beta c}{g} - m\right). \tag{1.12}$$

To study the case where we consider gravity, we only need to substitute the $\vec{F}^{\text{ext}}$ by the gravitational force in the equation 1.7. However, the resulting equation is too complex and we will consider just a particular case, where $\dot{m}$ is constant. Said that, the commented relations are expressed as follows.

$$\vec{F}^{\text{ext}} = -(m + dm)g\vec{e}_y \approx -mg\vec{e}_y$$
$$\dot{m} = -\beta, \qquad \beta > 0, \qquad [\beta] = \text{kg}\,\text{s}^{-1}$$

With this, we can start with the operations.

$$ma = -mg - \dot{m}c \Rightarrow \frac{dv}{dt} = -g + \frac{\beta}{m}c \Rightarrow \frac{dv}{dm}\frac{dm}{dt} = -g + \frac{\beta}{m}c = -\beta \Rightarrow dv = \left(g - \frac{\beta}{m}c\right)\frac{1}{\beta}dm$$

If we integrate this expression with $v_0 = 0$,

$$v = \int_{m_0}^{m} \left(g - \frac{\beta}{m'}c\right)\frac{1}{\beta}dm' = \frac{g}{\beta}\int_{m_0}^{m} dm' - c\int_{m_0}^{m} \frac{1}{m'}dm' = \frac{g}{\beta}(m - m_0) + c\ln\frac{m_0}{m},$$

and rearranging terms,

$$v = c \ln \frac{m_0}{m} - \frac{g}{\beta}(m_0 - m) \tag{1.13}$$

The first term is always positive, but the second one, since it has a minus sign, is negative. Therefore, in some cases the expression of v will be negative, in particular when β or c are too small. In fact we could have seen it when we isolated dv. There, if β or c are too small dv will be negative and therefore v. This means the equation is wrong in these cases, but we can solve it.

If we want it not tho happen, the expression of dv tells us that we only need the condition $m_0 g \leq \beta c$ to be satisfied, since then it would be positive from $t = 0$. However, if this is not the case, we need to modify the equation. The problem begins when we expressed $\vec{F}^{\text{ext}}$. Since we thought the rocket would have a positive direction always, it would not have contact with the floor (only during an instant), but if b or c are small this is not true. While the rocket is in the floor there is also the normal force, so the rocket won't have a negative velocity but it will remain at rest until the condition $mg \leq \beta c$ is satisfied (because then dv will be positive).

Now we have considered the details, we can find the new expressions to determine the velocity. There are two interval of time, while the rocket remains in the floor and when it starts moving. We know already the velocity of the rocket during the first interval, but we can still discover things, for example the magnitude of the normal force. Since the acceleration is zero during this interval, we have

$$m\vec{a} = \vec{F}^{\text{ext}} + \dot{m}\vec{c} \Rightarrow 0 = -mg + N - \dot{m}c \Rightarrow N = mg + \dot{m}c = mg - \beta c.$$

Before starting studying the second interval, we need first to find out at what instant t does the first interval ends. From the condition we mentioned before, the time t_b at which the rocket starts to blast-off is determined by

$$\dot{m} = -\beta \Rightarrow m = m_0 - \beta t \Rightarrow mg = \beta c = (m_0 - \beta t) \Rightarrow t_b = \frac{m_0}{\beta} - \frac{c}{g},$$

and the mass at that time,

$$m(t_b) = m_0 - \beta \left(\frac{m_0}{\beta} - \frac{c}{g} \right) = \frac{\beta c}{g} = m_b.$$

Now, we can determine the velocity as follows

$$v = \int_{m_0}^{m_b} 0 \, dm' + \int_{m_b}^{m} \left(g - \frac{\beta}{m'} c \right) \frac{1}{\beta} dm' = \frac{g}{\beta} \int_{m_0}^{m} dm' - c \int_{m_b}^{m} \frac{1}{m'} dm' = \frac{g}{\beta}(m - m_b) + c \ln \frac{m_b}{m} =$$

$$\frac{g}{\beta} \left(m - \frac{\beta c}{g} \right) + c \ln \frac{\beta c}{mg}.$$

If we want to express the velocities of the rocket in the two cases (when the condition was satisfied from the beginning and when not), we get

$$v(t) = c \ln \frac{m_0}{m_0 - \beta t} - gt \tag{1.14}$$

$$v(t) = c \ln \frac{\beta c}{m_0 g - \beta g t} - gt + \frac{g}{\beta} \left(m_0 - \frac{\beta c}{g} \right) \tag{1.15}$$

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[2] S. Widnall J. Peraire. Lecture L14 - Variable Mass Systems: The Rocket Equation, 2008.

Chapter 2

Oscillations

2.1 Introduction

Proposition 2.1.1. *Let $P(s) = 0$ be the auxiliar equation from the differential seond order equation $P[D]x = 0$ and $\Delta = b^2 - 4ac$. Then,*

1. *If $\Delta > 0$, then there are two solutions s_1, s_2 of the auxiliar equation and the general solution of $P[D]x = 0$ is*

$$x(t) = c_1 e^{s_1 t} + x_2 e^{s_2 t}. \tag{2.1}$$

2. *If $\Delta = 0$, then there is one solution sand the general solution of $P[D]x = 0$ is*

$$x(t) = (c_1 + c_2 t)e^{st}. \tag{2.2}$$

3. *If $\Delta < 0$, then there are two solutions $s_1 = \alpha + i\beta, s_2 = \bar{s}_1$ and the general solution of $P[D]x = 0$ is*

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t), \tag{2.3}$$

or, which is equivalent,

$$x(t) = A e^{\alpha t} \cos(\beta t + \delta),$$
$$A = \sqrt{c_1^2 + c_2^2}, \qquad \tan \delta = -\frac{c_2}{c_1}.$$

Proposition 2.1.2. *Let be the differential equation $a_1 \dot{x} + a_0 x = g(t)$. Then, the particular solution is*

$$x_p(t) = \left[c + \frac{1}{a_1} \int e^{\frac{a_0}{a_1} t} g(t) \, dt \right] e^{-\frac{a_0}{a_1} t}. \tag{2.4}$$

Proposition 2.1.3. *Let $P[D]x = e^{\zeta t}$ be a differential equation of order n. If $P^{(k)}(\zeta) = 0$ for $k < n$, then the particular solution is*

$$z_p(t) = \frac{t^n e^{\zeta t}}{P^{(n)}(\zeta)}. \tag{2.5}$$

2.2 Simple harmonic oscillator

Proposition 2.2.1. *Let be the following differential equation*

$$\ddot{x} + \omega_0^2 x = 0, \tag{2.6}$$

with the initial value condition of $x(0) = x_0$ and $v(0) = v_0$. Then, the general solution is

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t, \tag{2.7}$$

or, which is equivalent,

$$x(t) = A \cos[\omega_0 t + \phi_0], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \phi_0 = -\arctan \frac{v_0}{\omega_0 x_0}. \tag{2.8}$$

Proof. This is a linear homogeneous differential equation of second order, and can be solve easily with the associated polynomial.

$$P(s) = s^2 + \omega_0^2 = 0 \Rightarrow s = \pm i\omega_0, \ \omega_0 > 0$$

Since the roots are complex, the solution is given by the following expression

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

and its velocity

$$v(t) = -\omega_0 c_1 \sin \omega_0 t + \omega_0 c_2 \cos \omega_0 t$$

To solve now the initial values problem, we just need to calculate $x(0)$ and $v(0)$ and make them equal to the given values.

$$\begin{cases} x(0) = c_1 = x_0 \\ v(0) = \omega_0 c_2 = v_0 \end{cases}$$

By isolating the arbitrary constants we get $c_1 = x_0$ and $c_2 = v_0/\omega_0$, so the $x(t)$ will be expressed as

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t.$$

Now, to find the alternative expression, we know it can be done with the changes of variables $A = \sqrt{c_1^2 + c_2^2}$ and $\phi_0 = -\arctan(c_2/c_1)$. Since these constants are already determined, the final expression results in

$$x(t) = A \cos[\omega_0 t + \phi_0], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \phi_0 = -\arctan\frac{v_0}{\omega_0 x_0},$$

which completes the solutions we proposed. ∎

The quantity $\omega_0 t + \phi_0$ has units of rad (radians) and ω_0 of $\mathrm{rad\,s}^{-1}$. Let us observe is a periodic function with

$$T = \frac{2\pi}{\omega_0} \Rightarrow f = \frac{\omega_0}{2\pi} \tag{2.9}$$

Example 2.2.1. If a mass m is subjected to a spring with an elastic constant k, the elastic force applied by the spring to the mass, by the Hooke's Law, is

$$F_{\mathrm{m}} = -kx.$$

F_{m} is the force applied to the mass when it is displaced a certain distance x of its position of equilibrium (in this case, the equilibrium occurs when the spring has its natural length).
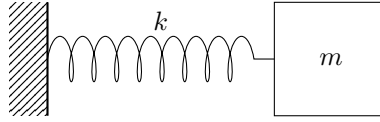
By the Newton's second law [ ], we have

$$-kx = m\ddot{x} \Rightarrow \ddot{x} + \frac{k}{m}x = 0,$$

which is a differential equation like that of the proposition 2.2.1. Therefore, the solution is

$$x(t) = A \cos[\omega_0 t + \phi_0], \qquad \omega_0 = \sqrt{\frac{k}{m}} > 0$$

2.2.1 Phase space

The phase space of the simple harmonic oscillator is

$$\vec{u}(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} A \cos[\omega_0 t + \phi_0] \\ -\omega_0 A \sin[\omega_0 t + \phi_0] \end{pmatrix}. \tag{2.10}$$

Therefore, the system in the phase space describes an ellipse of semi-axis $a = A$ and $b = \omega_A$, and moves through it in clockwise direction.

Figure 2.1: Trajectory of the mass in the phase space with $\omega_0 > 0$.

2.2.2   Energy analysis

Let us start with a particular example and then make a more general discussion about the topic.

Example 2.2.2. Let m be a mass subjected to an elastic force applied by a spring as before. During the oscillation, the mass has kinetic and potential energy, so the total energy is

$$E = K + U = \frac{mv^2}{2} + \frac{kx^2}{2} = \frac{m\omega_0^2 A^2 \sin^2\left[\omega_0 t + \phi_0\right]}{2} + \frac{kA^2 \cos^2\left[\omega_0 t + \phi_0\right]}{2} =$$
$$\frac{kA^2 \sin^2\left[\omega_0 t + \phi_0\right]}{2} + \frac{kA^2 \cos^2\left[\omega_0 t + \phi_0\right]}{2} = \frac{kA^2}{2}.$$

The expression of the energy is a constant, and since A depends on the initial conditions, E too. Note that at $x = 0$, E coincides with the kinetic energy of the mass, since $U = 0$. Therefore, it is where its speed has its maximum value. A particular case of this is if the particle starts at rest in the position $x = 0$. In this situation, $K_{\max} = 0$, and therefore the mass will remain at rest (stable equilibrium).

At $x = \pm A$, we have that $v = 0$, the motion is inverted, and E coincides with energy potential. In any other position, the total energy has a part of kinetic and a part of potential.

Now we have seen these properties, we will generalize them. For that, let us suppose a potential function $U(x)$ as the figure [ ], and let us suppose a mass m starts with an initial energy E_0.
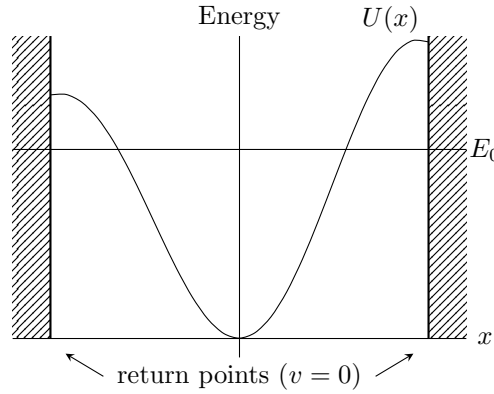
The kinetic energy, and hence the speed, is maximum where U is minimum, that is, when $U = U_{\min}$. In the figure, this occurs at $x = 0$. The extremes of $U(x)$ are points of equilibrium, since in these points

$$0 = \frac{dU}{dx}(x_{\text{extreme}}) = -F.$$

From our knowledge of calculus, we know the extremes (with $U' = 0$) can be classified thought the higher-order derivatives, in most common case the second one.

Definition 2.2.1. Let $U(x)$ be a potential function of class $C^2(\mathbb{R})$. Then, we say x_0 *is a point of stable equilibrium* if U has a maxima in x_0.

This happens if in an extreme $U'' > 0$. Around that point, small displacements make the force push the particle back to x_0.

Definition 2.2.2. Let $U(x)$ be a potential function of class $C^2(\mathbb{R})$. Then, we say x_0 *is a point of unstable equilibrium* if U has a minima in x_0.

This happens if in an extreme $U'' < 0$. Around that point, small displacements make the force push the particle away from x_0.

These derivatives serve us to find an approximate solution when $U(x)$ is too complex to study. To simplify the operations, first we will locate the coordinate origin at the minima x_0 and establishing that $U(0) = 0$. Then, we can compute the Taylor polynomial to the second order and get

$$P_0^2(x) = U(0) + \frac{U'(0)}{1!}x + \frac{U''(0)}{2!}x^2 = \frac{U''(0)}{2}x^2 \Rightarrow U(x) \approx \frac{U''(0)}{2}x^2,$$
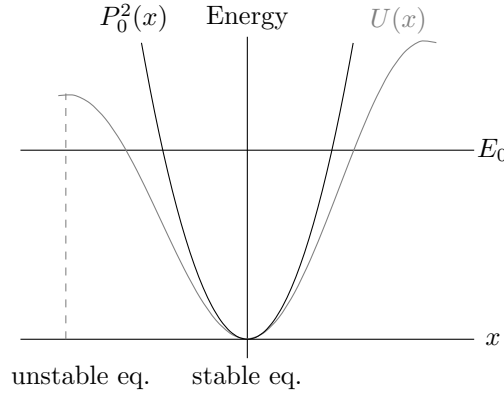
which is an harmonic potential. To see that, let us write the conservation of energy with this approximation

$$E = K + U \Rightarrow E = \frac{mv^2}{2} + \frac{U''(0)}{2}x^2.$$

Now, if we derive the equation with respect to x (the energy does not depend on the position so it acts as a constant), we obtain

$$ma + U''(x)x = 0 \Rightarrow \ddot{x} + \frac{U''(x)}{m}x = 0 \Rightarrow \omega_0 = \sqrt{\frac{U''(x)}{m}}.$$

As we see, we obtain the same equation of a simple harmonic oscillator and therefore the particle will behave similarly at points near the minima.

Proposition 2.2.2. *Let p be a particles on which a potential field U acts. If x_0 is a local minima of U and we make the change $x \to x - x_0$ and $U \to U - U(x_0)$, then for small deviations from the equilibrium point*

$$U \approx \frac{U''(0)}{2}x^2, \qquad \ddot{x} + \frac{U''(x)}{m}x = 0, \qquad \omega_0 = \sqrt{\frac{U''(x)}{m}}. \tag{2.11}$$

Example 2.2.3. Let us suppose there is a spring subjected to the roof and with a mass m with a natural length l_0 (figure a). Then, we hang a mass m in the extreme so the spring stretches until reach a new position of equilibrium (figure b). Finally, we agitate the system and make the mass oscillate.
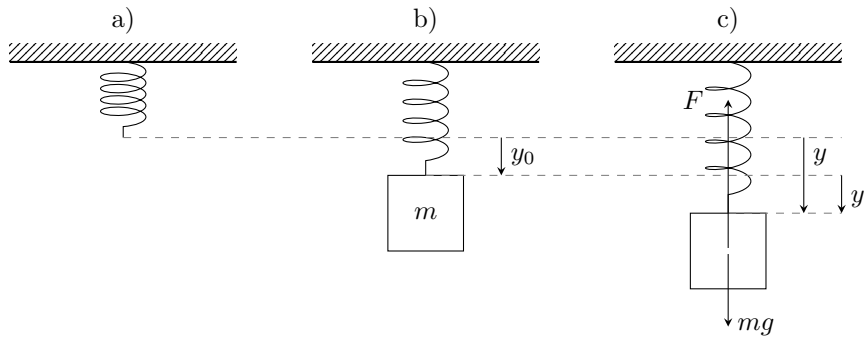
Figure 2.2: Springs.

We will solve the problem of finding th equation of motion in two ways, with Newton's second Law, and with conservation of energy. We will take the positive direction in $-y$, so the forces for the laws of motion are

$$F = mg - F_{\mathrm{m}} = mg - ky.$$

Therefore, the second law of motion let us write

$$mg - ky = m\ddot{y} \Rightarrow \ddot{y} + \frac{k}{m}y - g = 0.$$

This equation does not have the form 2.6, so we could think the behavior of the spring is different from the horizontal case. Nevertheless, there is a detail we have not considered while obtaining the equation. We usually take the origin of coordinates at the equilibrium point, but this time there are two, the natural length and the equilibrium with the mass. Since now we are in a scenario with mass, we need to put the origin in y_0. To see how does change the expression, let us make the change of variable $y' = y - y_0$. Before substituting it in the equation, we need to calculate y_0, ans since this is the position where the net force is zero, it is determined as follows.

$$0 = mg - ky_0 \Rightarrow y_0 = \frac{mg}{k}$$

Now, introducing the expression in the differential equation, we get

$$\ddot{y}' + \frac{k}{m}y' + \frac{k}{m}y_0 - g = 0 \Rightarrow \ddot{y}' + \frac{k}{m}y = 0 \Rightarrow y' = A\cos\left[\omega_0 t + \phi_0\right] = y - y_0 \Rightarrow$$

$$y(t) = \frac{mg}{k} + A\cos\left[\omega_0 t + \phi_0\right], \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

We see then, the mass also behaves as a simple harmonic oscillator but with a shift in the position of equilibrium.
SOLUTION BY ENERGIES

2.2.3   Examples

Simple pendulum

Proposition 2.2.3. *Let be a pendulum formed by an ideal bar of length l and with mass in the extrem m. Let θ be the angle that describes the bar with the vertical axis. If over the system acts a constant force $\mathbf{F} = -mg\mathbf{e}_y$, then the motion of θ is described by the following differential equation*

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0, \tag{2.12}$$

and for small angles,

$$\ddot{\theta} + \frac{g}{l}\theta = 0. \tag{2.13}$$

Let us suppose there is a point mass m subjected by an inextensible mass-less rope of natural length l. The other side of the rope is fixed in the roof. The forces that act over the mass at an angle θ from its position of equilibrium are shown in the figure.
The letter T denotes the tension (we will use τ for period), and it is the responsible for the mass moving over a circumference of radius l (the dashed line). If we study the tangential direction of the forces, by the Newton's Second Law we get

$$-mg\sin\theta = ma_t = ml\alpha = ml\ddot{\theta} \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0,$$

where we have written the wight in negative because it acts opposite to the increase of mass. This equation is similar to that of the proposition 2.2.1, but it has a $\sin x$ instead of x. This makes the problem more complex and the system becomes a non simple harmonic oscillator. Therefore, we will simplify the problem and let the exact solution for a later section. In order to get a a simpler problem, we will take the Taylor approximation of $\sin\theta$ to the first order, that is, with $\sin\theta \approx \theta$ (only for small angles). With that, we get the equation of a SHO and therefore we can obtain the equation of motion.

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \Rightarrow \theta(t) = A\cos\left[\omega_0 t + \phi_0\right], \qquad \omega_0 = \sqrt{\frac{g}{l}}$$
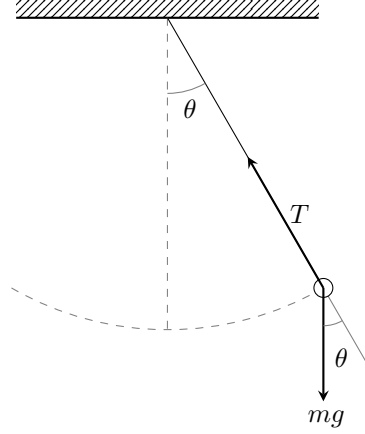
Figure 2.3: Simple pendulum at an angle θ.

Note that ω_0 is not the derivative of θ and ϕ_0 is not θ_0. Rather, magnitudes refer to the phase and frequency of the oscillation of the angle.

ENERGIES AND OTHERS—————————-

$$U(\theta) = mgh = mg(l - l\cos\theta) = mgl(1 - \cos\theta)$$

Taylor second order

$$U'(\theta) = mgl\sin\theta \Rightarrow U'(0) = 0, \qquad U''(\theta) = mgl\cos\theta \Rightarrow U''(0) = mgl, \qquad U(\theta) = \frac{1}{2!}mgl\theta^2 \Rightarrow ml\alpha$$

Then,

$$ma + \frac{ds}{dU} = 0 \Rightarrow ml\alpha + \frac{1}{l}\frac{dU}{d\theta} = 0 \Rightarrow ml\ddot{\theta} + \frac{mgl}{l}\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

where we have used that $ds = ld\theta$.

Physical pendulum

Proposition 2.2.4. *Let be an object of mass m that can rotate with respect a fixed point O and over which acts a constant force $\mathbf{F} = -mg\mathbf{e}_y$. Let D be the distance beteween its center of mass and O, and θ the angle that describes the center of mass with respect the vertical axis. Then, the motion of θ is described by the following differential equation.*

$$\ddot{\theta} + \frac{mgD}{I_e}\sin\theta = 0, \tag{2.14}$$

and for small angles,

$$\ddot{\theta} + \frac{mgD}{l}\theta = 0. \tag{2.15}$$

We know

$$\tau_e = I_e\ddot{\theta}$$

In this case,

$$\tau = -mgD\sin\theta$$

Then,

$$\ddot{\theta} + \frac{mgD}{I_e}\theta = 0 \tag{2.16}$$

The solution is

$$\theta(t) = A\cos\left[\omega_0 t + \phi_0\right], \qquad \omega_0 = \sqrt{\frac{mgD}{I_e}} \tag{2.17}$$

By energies

$$U \approx \frac{1}{2}mgD\theta^2$$
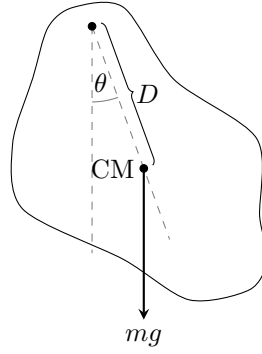
Figure 2.4: Physical pendulum at an angle θ.

LC circuit

Proposition 2.2.5. *Let be the circuit described by the figure 2.2.3. Then, the behavior of q and I is described by the followind differential equations.*

$$\ddot{q} + \frac{1}{LC}q = 0, \qquad \ddot{I} + \frac{1}{LC}I = 0. \tag{2.18}$$

Let be a circuit
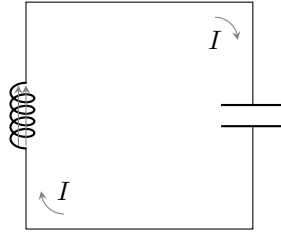
Figure 2.5: LC circuit.

By Faraday Law

$$\epsilon_{\text{ind}} = \oint \langle \vec{E}, d\vec{r} \rangle_I = -\frac{d}{dt} \int_S \langle \vec{B}, d\vec{s} \rangle_I$$

In inductors,

$$\Phi = LI \Rightarrow \frac{d\Phi}{dt} = L\frac{dI}{dt} \Rightarrow \oint \langle \vec{E}, d\vec{r} \rangle_I = -L\dot{I}$$

Since the capacitor is charging,

$$I = \dot{q}$$

Besides,

$$-L\ddot{q} = -L\dot{I} = \oint \langle \vec{E}, d\vec{r} \rangle_I = \oint_{\text{capacitor}} \langle \vec{E}, d\vec{r} \rangle_I = V = \frac{q}{C} \tag{2.19}$$

Finally,

$$L\ddot{q} + \frac{q}{C} = 0 \tag{2.20}$$

If we derive it

$$L\ddot{I} + \frac{I}{C} = 0$$

The solution for the first one is

$$q(t) = A\cos[\omega_0 t + \phi_0], \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{2.21}$$

Derivating,

$$I = -\omega_0 A \sin [\omega_0 t + \phi_0] = A' \cos [\omega_0 t + \phi_0'], \qquad A' = \omega_0 A, \phi_0' = \phi_0 + \frac{\pi}{2}$$

2.3   Damped harmonic oscillator

2.3.1   Evolution equation and solution

Proposition 2.3.1. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{2.22}$$

with the initial value conditions of $x(0) = x_0$ and $v(0) = v_0$. Then

1. If $\beta < \omega_0$, we denote $\tilde{\omega} = \sqrt{\omega_0^2 - \beta^2}$ and the solution is

$$x(t) = e^{-\beta t} \left[x_0 \cos \tilde{\omega} t + \frac{v_0 + \beta x_0}{\tilde{\omega}} \sin \tilde{\omega} t \right]. \tag{2.23}$$

2. If $\beta = \omega_0$, the solution is

$$x(t) = e^{-\omega_0 t} \left[x_0 + (x_0 \omega_0 + v_0) t \right]. \tag{2.24}$$

3. If $\beta > \omega_0$, we denote $\bar{\omega} = \sqrt{\beta^2 - \omega_0^2}$ and the solution is

$$x(t) = \frac{x_0(\bar{\omega} - \beta) - v_0}{2\bar{\omega}} e^{-(\beta + \bar{\omega})t} + \frac{x_0(\bar{\omega} + \beta) + v_0}{2\bar{\omega}} e^{-(\beta - \bar{\omega})t}. \tag{2.25}$$

Example 2.3.1. Mass m submerged in a fluid and subjected to a string of elastic constant k

$$F = -kx$$

$$F_f = -b\dot{x}$$

By Newtons' law

$$-kx - b\dot{x} = m\ddot{x} \Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}, \beta = \frac{b}{2m}$$

2.3.2   Underdamped harmonic oscillator

The general solution is

$$x(t) = e^{-\beta t} \left[x_0 \cos \tilde{\omega} t + \frac{v_0 + \beta x_0}{\tilde{\omega}} \sin \tilde{\omega} t \right], \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \beta^2}$$

or, which is equivalent,

$$x(t) = Ae^{-\beta t} \cos [\tilde{\omega} t + \phi_0], \qquad A = \sqrt{x_0^2 + \left(\frac{v_0 + \beta x_0}{\tilde{\omega}}\right)^2}, \qquad \phi_0 = -\arctan \frac{v_0 + \beta x_0}{x_0 \tilde{\omega}}$$

Definition 2.3.1. Amplitude as a function of time.

$$A(t) = Ae^{-\beta t} \tag{2.26}$$

Definition 2.3.2. Period of the amplitude

$$\tau_{e^{-1}} = \frac{1}{\beta}$$

Period between zeros

$$\tilde{T} = \frac{2\pi}{\tilde{\omega}} > T_0$$

Quality factor

Definition 2.3.3. The quality factor Q is defined as

$$Q := \frac{\omega_0}{2\beta} \tag{2.27}$$

From that,

Proposition 2.3.2.

$$\tilde{\omega} = \omega_0\sqrt{1 - \frac{1}{4Q^2}}, \qquad N_{e^{-1}} \approx \frac{Q}{\pi} \tag{2.28}$$

$$\tilde{\omega} = \omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

$$N_{e^{-1}} \approx \frac{Q}{\pi} \tag{2.29}$$

Quality factor and energy

$$E = \frac{\mu}{2}\left(v^2 + \omega_0^2 x^2\right) \tag{2.30}$$

Example 2.3.2. The example [ ] with a mass and a string without friction results in

$$\frac{m}{2}\left(v^2 + \omega_0 x^2\right)$$

Example 2.3.3. The simple pendulum.

$$K = \frac{mv^2}{2} = \frac{ml^2\dot{\theta}^2}{2}, \qquad U = mgl(1 - \cos\theta) \approx \frac{mgl\theta^2}{2} \tag{2.31}$$

Then,

$$E = U + K = \frac{mgl\theta^2}{2} + \frac{ml^2\dot{\theta}^2}{2} = \frac{ml^2}{2}\left(\dot{\theta}^2 + \frac{g}{l}\theta^2\right) = \frac{\mu}{2}(\dot{\theta}^2 + \omega_0^2\theta^2), \qquad \mu = ml^2, \omega_0^2 = \frac{g}{l}.$$

For general underdamped oscillators.

$$E = \frac{\mu A^2}{2}\left(\tilde{\omega}^2 \sin^2\left[\tilde{\omega}t + \phi_0\right] + \omega_0^2 \sin^2\left[\tilde{\omega}t + \phi_0\right]\right)e^{-2\beta t} \approx$$

$$\frac{\mu A^2}{2}\left(\omega_0^2 \sin^2\left[\tilde{\omega}t + \phi_0\right] + \omega_0^2 \sin^2\left[\tilde{\omega}t + \phi_0\right]\right)e^{-2\beta t} = \frac{\mu\omega_0^2 A^2}{2}e^{-2\beta t}$$

so

$$E(t) = E_0 e^{-2\beta t} \tag{2.32}$$

The rhythm at which the energy is dissipated is

$$\left|\frac{dE}{dt}\right| = 2\beta E_0 e^{-2\beta t} = 2\beta E \Rightarrow \frac{E}{|dE/dt|} = \frac{1}{2\beta}$$

Then,

$$\tau_E = \frac{1}{2\beta} \tag{2.33}$$

This does not differ much from $\Delta E/\tilde{T}$. This is because if $\beta \ll \omega_0$, then $|dE/dt|$ does not change considerably in an interval of time $\tilde{T}$.
We have then,

$$\left|\frac{dE}{dt}\right| \approx \frac{\Delta E}{2\pi/\omega_0} \Rightarrow 2\pi\frac{E}{\Delta E} \approx \omega_0\frac{E}{|dE/dt|} \approx \frac{\omega_0}{2\beta} = Q$$

Then,

$$\frac{Q}{2\pi} = \frac{E}{\Delta E} \tag{2.34}$$

2.3.3 Critically damped harmonic oscillator

2.3.4 Overdamped harmonic oscillator

2.3.5 Examples

Example 2.3.4. We know

$$V = IR$$

Then,

$$-L\ddot{q} = -L\dot{I} = \oint \langle \vec{E}, d\vec{r} \rangle_I = \int_{\text{capacitor}} \langle \vec{E}, d\vec{r} \rangle_I + \int_{\text{resistance}} \langle \vec{E}, d\vec{r} \rangle_I = \frac{q}{C} + IR = \frac{q}{C} + R\dot{q}$$

so

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0 \Rightarrow \ddot{I} + \frac{1}{RL}\dot{I} + \frac{1}{LC}I = 0 \Rightarrow \beta = \frac{R}{2L}, \omega_0 = \frac{1}{\sqrt{LC}} \tag{2.35}$$

The system is underdamped if

$$\frac{R}{2}\sqrt{\frac{C}{L}} < 1$$

The charge is given by

$$q(t) = Ae^{-\beta t} \cos\left[\tilde{\omega}t + \phi_0\right] \tag{2.36}$$

And the current

$$I(t) = A'e^{-\beta t} \cos\left[\tilde{\omega}t + \phi_0'\right], \qquad A' =, \phi_0' = \tag{2.37}$$

2.4 Forced harmonic oscillator

2.4.1 Evolution equation and solution

Proposition 2.4.1. *Let be the following differential equation*

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0 \cos\left[\omega t + \psi_0\right], \tag{2.38}$$

with the initial value conditions of $x(0) = x_0$ and $v(0) = v_0$. Then, the particular solution is

$$x_p(t) = A\cos\left[\omega t + \psi_0 - \phi_0\right], \qquad A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}, \qquad \phi_0 = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}. \tag{2.39}$$

Proof. We know

$$f_0 \cos\left[\omega_0 t + \psi_0\right] = \text{Re}\left\{f_0 e^{i\psi_0} e^{i\omega t}\right\} \tag{2.40}$$

Then

$$P[D]x(t) = (D^2 + 2\beta D + \omega_0)z(t) = \tilde{f}_0 e^{i\omega t}, \qquad \tilde{f}_0 := f_0 e^{i\psi_0}$$

Then

$$x_p(t) = \text{Re}\{z_p(t)\}$$

Then

$$z_p(t) = \frac{\tilde{f}_0 e^{i\omega t}}{P(i\omega)} = \frac{\tilde{f}_0 e^{i\omega t}}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

Then

$$z_p = \frac{\tilde{f}_0 e^{-i\arctan\left(2\beta\omega/(\omega_0^2 - \omega^2)\right)}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}} := Ae^{-i(\phi_0 - \psi_0)}$$

where

$$A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}, \qquad \phi_0 = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{2.41}$$

Taking the real part, we get

$$A\cos\left[\omega t + \psi_0 - \phi_0\right]. \tag{2.42}$$

■

2.4.2 Underdamped forced harmonic oscillator

$$c_1 = A_0 \cos \left[\psi_0 - \psi_0 \right]$$
$$c_2 = \frac{A v_0 + \omega A \sin \left[\phi_0 - \psi_0 \right] + \beta (x_0 - A \cos \left[\psi_0 - \phi_0 \right])}{\tilde{\omega}}$$

2.4.3 Critically damped forced harmonic oscillator

$$c_1 = x_0 - A \cos \left[\psi_0 - \phi_0 \right]$$
$$c_2 = v_0 + \omega_0 x_0 + A \left[\omega \sin \left[\psi_0 - \phi_0 \right] - \omega_0 \cos \left[\psi_0 - \phi_0 \right] \right]$$

2.4.4 Overdamped forced harmonic oscillator

$$c_1 = \frac{A(\beta - \bar{\omega}) \cos \left[\psi_0 - \phi_0 \right] - A\omega \sin \left[\psi_0 - \phi_0 \right] - (\bar{\omega} + \beta)x_0 - v_0}{2\bar{\omega}}$$
$$c_2 = \frac{A\omega \sin \left[\psi_0 - \phi_0 \right] - A(\beta + \bar{\omega}) \cos \left[\psi_0 - \phi_0 \right] + (\bar{\omega} + \beta)x_0 + v_0}{2\bar{\omega}}$$

2.4.5 Solution analysis

Forçat sense amortiment (el numerador de l'amplitud té valor absolut però el desfasament li dona el signe)

$$\omega \neq \omega_0 : x(t) = \left[x_0 - \frac{f_0}{\omega_0^2 - \omega^2} \right] \cos \omega_0 t` + \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t + \frac{f_0}{\omega_0^2 - \omega^2} \cos \omega t$$
$$\omega = \omega_0 : x(t) = x_0 \cos \omega_0 t + \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t + \frac{f_0 t}{2\omega_0} \sin \omega_0 t$$

2.4.6 Resonance in amplitude

Resonance

$$\omega_r = \sqrt{\omega_0^2 - 2\beta^2}, \qquad A_r = f_0/2\beta \sqrt{\omega_0^2 - \beta^2}, \qquad \text{a prop de } \omega_0 : A\omega_0^2/f_0 \approx QA$$

2.4.7 Quality factor, resonance in energy and bandwidth

Ressonància i energia (a prop de ω_0 i amb $f = f_0 \cos \omega t$)

$$E = \frac{\mu \omega_0^2 A^2}{2}, A = \frac{f_0}{2\omega_0 \sqrt{(\omega - \omega_0)^2 + \beta^2}}, \omega \approx \omega_0, \beta \ll \omega_0, E_{\max} = \frac{\mu f_0^2}{8\beta^2}, Q = \frac{\omega_0}{\Delta \omega}, \omega_{1,2} = \omega_0 \pm \beta$$

2.4.8 Examples. RLC circuit with generator in series and parallel

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{V_{\text{in}}}{L} \Rightarrow q = \frac{\epsilon_0/\omega}{\sqrt{(L\omega - 1/C\omega)^2 + R^2}} \cos (\omega t - \delta), \delta = -\arctan \frac{R}{L\omega - 1/C\omega}$$

$$X = X_L - X_C = L\omega - 1/C\omega, \ Z = \sqrt{X^2 + R^2}, \delta' = \delta - \pi/2 = \arctan X/R$$

$$r = \frac{V_{\text{out}}^2}{V_{\text{in}}^2} = \frac{R^2}{Z^2}, r_{\max} = 1 \Rightarrow \omega_{1,2}(r_{\max}/2) = \sqrt{\frac{1}{Lc} - \left(\frac{R}{2L} \right)} \pm \frac{R}{2L}, Q = \frac{\omega_0}{\Delta \omega} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

2.5   Fourier series

2.5.1   Fourier analysis of the forced harmonic oscillator

$$f(t + T) = f(t), \qquad \omega := \frac{2\pi}{T}$$

We have

$$f(t) = a_0 + \sum_{l=1}^{\infty} a_l \cos \omega_l t + \sum_{l=1}^{\infty} b_l \sin \omega_l t = A_0 + \sum_{l=1}^{\infty} A_l \cos [\omega_l t + \psi_l], \qquad \omega_l = l\omega$$

Answer of the system

$$x(t) = \frac{A_0}{\omega_0^2} + \sum_{l=1}^{\infty} \frac{A}{\sqrt{(\omega_0^2 - \omega_l^2)^2 + 4\beta^2 \omega_l^2}} \cos [\omega_l + \psi_l - \phi_l], \qquad \phi_l = \arctan \frac{2\beta\omega_l}{\omega_0^2 - \omega_l} \qquad (2.43)$$

2.5.2   Obtention of Fourier coefficients

$$a_0 = \frac{1}{T} \int_n^{n+T} f(t) dt, \, a_l = \frac{2}{T} \int_n^{n+T} f(t) \cos \omega_l t dt, \, b_l = \frac{2}{T} \int_n^{n+T} f(t) \sin \omega_l t dt \qquad (2.44)$$

2.5.3   Examples

Example 2.5.1. The force is

$$f(t) = \begin{cases} f_0/2, & 0 \le t \le T/2 - f_0/2, & T/2 \le t \le T \end{cases}$$

The answer is

$$x(t) = \frac{2f_0}{\pi} \sum_{l=0}^{\infty} \frac{\sin [(2l + 1)\omega t]}{2l + 1}$$

Example 2.5.2. The force is

$$f(t) = \begin{cases} f_0/2, & 0 \le t \le T/4 - f_0/2, \text{otherwise} \end{cases}$$

The answer is

$$x(t) = -\frac{2f_0}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l + 1} \cos [(2l + 1)\omega t]$$

2.6   Impulsive forces

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \delta(t - t_0) \Rightarrow x(t) = G(t - t_0) \Leftrightarrow P[D]G(t - t_0) = \delta(t - t_0)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \Rightarrow x(t) = \int_{-\infty}^{\infty} f(t')G(t - t')dt'$$

Some properties

$$\theta(t - t_0) + \theta(t_0 - t) = 1, \dot{\theta}(t) = \delta(t), t > T \Rightarrow |x(t)| \le Ke^{-\beta t}$$

2.7   Non-linear oscillator

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0, \theta \in [-\pi, \pi] \Rightarrow T = 4\sqrt{\frac{l}{g}}K(k) = 2\pi\sqrt{\frac{l}{g}}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}\cdot(n!)^2}\right)^2$$

Brachristocrone (in a pendulum)

$$x = a\phi - a\sin\phi, \qquad y = a\cos\phi - a \tag{2.45}$$

General periods

$$T = \int\limits_0^{x_0} \frac{1}{\sqrt{2/m(U_0 - U)}}dx \tag{2.46}$$

2.8   Extra

2.8.1   Some integrals

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x \tag{2.47}$$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \operatorname{arcsinh} x \tag{2.48}$$

$$\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x} \tag{2.49}$$

$$\operatorname{arccoth} x = \frac{1}{2}\ln\frac{x+1}{x-1} \tag{2.50}$$

$$\tan\frac{\theta}{2} = u \tag{2.51}$$

$$\int \frac{1}{a^2x^2 - b^2}dx = -\frac{1}{ab}\operatorname{arccoth}\frac{ax}{b} \tag{2.52}$$

$$\int \frac{1}{\sqrt{1-z^4}} = 1,311\ldots \tag{2.53}$$

$$\int\limits_0^1 \frac{z^{2p}}{\sqrt{1-z^2}} = \frac{\pi(2p)!}{2^{2p+1}(p!)^2} \tag{2.54}$$

$$\int \tau^p e^{-\beta\tau}\sin[\omega\tau]d\tau = (-1)^p\frac{d^p}{x\beta^p}\operatorname{Im}\left\{\frac{e^{-\beta+i\omega}\tau}{-\beta+i\omega}\right\} \tag{2.55}$$

Chapter 3

Central forces

3.1   Definition of central force and properties

Definition 3.1.1. Central force

$$\vec{F}(\vec{r}) = f(r)\vec{e}_\rho \tag{3.1}$$

Definition 3.1.2. The origin $\vec{r} = \vec{0}$ is the *center of forces*.

Proposition 3.1.1. *All central forces are conservatives.*

3.2   Conservation of angular momentum and areolar velocity

Proposition 3.2.1. *The angular momentum with respect the origin is conserved.*

$$\dot{\vec{L}} = \vec{0} \tag{3.2}$$

Proposition 3.2.2. *The areal velocity is constant.*

$$\frac{dA}{dt} = \frac{L}{2m} = ctt \tag{3.3}$$

This is known as Kepler's Second Law [], published by Johannes Kepler (1571-1630) in 1609 after an exhaustive study of the compilations made by Tycho Brahe (1546-1601) of the position of the planet Mars.

3.3   Trajectory equation

Forces

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \end{cases} \tag{3.4}$$

We see

$$r^2\dot{\theta} = \text{ctt}, \qquad \dot{\theta}\frac{L}{mr^2} = \frac{l}{r^2} \Rightarrow \ddot{r} - \frac{l^2}{r^3} = \frac{f(r)}{m} \tag{3.5}$$

Finally, we get the *trajectory equation*

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2}\frac{1}{u^2}f\left(\frac{1}{u}\right) := \psi(u) \tag{3.6}$$

Then we get $u(\theta)$ and $r(\theta)$. Now,

$$t = \frac{1}{l}\int r^2(\theta)d\theta := g(\theta) \Rightarrow \theta(t) = g^{-1}(t) \Rightarrow r(t) = r(g^{-1}(t)) \tag{3.7}$$

Besides, we can know the force from the motion

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^2}{ml^2}f(r) \tag{3.8}$$

3.4   Mechanic energy conservation and orbits

Energy

$$E = \frac{m\dot{r}^2}{2} + U_{\text{ef}}(r) = \frac{m\dot{r}^2}{2} + \frac{ml^2}{2r^2} + U(r) \tag{3.9}$$

The term of angular momentum is equivalent to say $f_{\text{centr}} = mr\dot{\theta}^2\vec{e}_\rho$.
Inertial forces for rotating coordinates (the right component)

$$\vec{F}' = \vec{F} + mr\dot{\theta}^2\vec{e}_\rho - 2m\dot{r}\dot{\theta}\vec{e}_\theta - mr\ddot{\theta}\vec{e}_\theta \tag{3.10}$$

3.4.1 Orbits characterization

Circular orbits are at minima of $U_{\text{eff}}(r)$. In the extremes (maxima/minima) it is satisfied that

$$\dot{r} = \pm\sqrt{\frac{2}{m}(E - U_{(}eff)(r))} = 0 \Rightarrow E - U(r) - \frac{ml^2}{2r^2} = 0 \tag{3.11}$$

Then,

$$\Delta\theta = 2\int_{r_{\min}}^{r_{\max}} \frac{l}{r^2}\sqrt{2/m(E - U_{\text{eff}}(r))}dr \tag{3.12}$$

The orbits are closed if and only if

$$\Delta\theta = 2\pi\frac{p}{q}, \qquad p, q \in \mathbb{N} \tag{3.13}$$

More in general,

$$\theta(r) = \int \frac{L}{r^2}\frac{1}{\sqrt{2m\left(E - U(r) - \frac{L^2}{2mr^2}\right)}}dr \tag{3.14}$$

Notice that, since l is constant and r^2 and the square root always are positive, $\theta(r)$ will increase monotonically with time.

Theorem 3.4.1 (Bertrand's Theorem). *The only central potentials where every bounded orbit is closed are:*

$$U(r) = -\frac{k}{r}, \qquad U(r) = \frac{k}{2}r^2, \qquad k > 0 \tag{3.15}$$

These two potential have $(p, q) = (1, 1)$ for $-k/r$ and $(p, q) = (1, 2)$ for $kr^2/2$.

3.5 Potential $-k/r$

Proposition 3.5.1. *If $U = -k/r$, then*

$$r(\theta) = \frac{\alpha}{\epsilon\cos\theta + \text{sgn}\,k}, \qquad \alpha = \frac{L^2}{m|k|}, \qquad \epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}. \tag{3.16}$$

Which is a conic section with one focus at the origin. Johann Bernoulli (1667-1748) appears to have been the first to prove that all possible orbits of a body moving in a potential proportional to $1/r$ are conic sections (1710) [ ].

Definition 3.5.1. The pericenter is the minimum value of $r(\theta)$.

Definition 3.5.2. The apocenter is the maximum value of $r(\theta)$.

Definition 3.5.3. Apsides are the turning points.

3.5.1 Conic sections

Definition 3.5.4. Conic section

$$\epsilon = \frac{d(P, F)}{d(P, D)} = \frac{\sqrt{x^2 + y^2}}{|x - d|} = \frac{r}{|r\cos\theta - d|}. \tag{3.17}$$

Definition 3.5.5. Latus rectum

$$\alpha := \frac{\bar{L}L'}{2} = r(\pi/2) = \epsilon d \tag{3.18}$$

Ellipse $(0 < \varepsilon < 1)$

Theorem 3.5.2. *If* $0 < \epsilon < 1$, *then*

$$r(\theta) = \frac{\alpha}{\varepsilon \cos \theta + 1}, \tag{3.19}$$

$$r_{\min} = \frac{\alpha}{1 + \varepsilon}, \qquad r_{\max} = \frac{\alpha}{1 - \varepsilon}, \qquad a^2 = b^2 + c^2, \tag{3.20}$$

$$a = \frac{r_{min} + r_{max}}{2} = \frac{\alpha}{1 - \varepsilon^2}, \qquad b = \frac{\alpha}{\sqrt{1 - \epsilon^2}}, \qquad c = \frac{\alpha\epsilon}{1 - \epsilon^2}. \tag{3.21}$$

Hyperbola $(1 < \varepsilon)$

Proposition 3.5.3. *If* $1 < \epsilon$, *then*

$$r(\theta) = \frac{\alpha}{\varepsilon \cos \theta \pm 1}, \tag{3.22}$$

$$\tan \theta = \pm\sqrt{\varepsilon^2 - 1}, \qquad r_{min} = r(0) = \frac{\alpha}{\varepsilon + 1}, \qquad c^2 = a^2 + b^2, \tag{3.23}$$

$$c = \frac{\alpha\epsilon}{\epsilon^2 - 1}, \qquad a = c - r_{\min} = \frac{\alpha}{\epsilon^2 - 1}, \qquad b = \frac{\alpha}{\sqrt{\epsilon^2 - 1}}. \tag{3.24}$$

Parabola $(\varepsilon = 1)$

Proposition 3.5.4. *If* $\epsilon = 1$, *then*

$$r(\theta) = \frac{\alpha}{\cos \theta + 1}, \qquad x = -\frac{y^2 - \alpha^2}{2\alpha} \tag{3.25}$$

Circunference of radius R $(d = R/\epsilon, \varepsilon \to 0)$

Theorem 3.5.5. *If* $d = R/\epsilon$ *and* $\epsilon \to 0$, *then*

$$r(\theta) = R, \qquad \alpha = R. \tag{3.26}$$

General formulae

Theorem 3.5.6. *In general,*

$$a = \frac{\alpha}{|1 - \varepsilon^2|}, \qquad b = \frac{\alpha}{\sqrt{|1 - \varepsilon^2|}}, \qquad c = \frac{\alpha\varepsilon}{|1 - \varepsilon^2|}. \tag{3.27}$$

Theorem 3.5.7. *Let* $U = -k/r$. *Then,*

$$a = \frac{|k|}{2|E|}, \qquad b = \frac{L}{\sqrt{2m|E|}}. \tag{3.28}$$

1. *If* $E < 0$, *then* $0 < \epsilon < 1$ *(elipse) and* $k > 0$.

2. *If* $E > 0$, *then* $\epsilon > 1$ *(hyperbola).*

3. *If* $E = 0$, *then* $\epsilon = 1$ *(parabola).*

Proposition 3.5.8. *If* $U = -k/r$ *and* $0 < \epsilon < 1$, *then*

$$T^2 = \frac{4\pi^2 m a^3}{k}. \tag{3.29}$$

This is known as Kepler's Third Law, published by Kepler in 1619. Kepler's work preceded by almost 80 years Newton's enunciation of his general laws of motion. Indeed, Newton's conclusions were based to a great extent on Kepler's pioneering studies (and on those of Galileo and Huygens). However, this Law is just an approximation. If we apply the reduced mass to the formula (which is the real situation), we will have different values of period for each planet (since the reduced mass

is different), although Kepler said the period does not depend on the small mass. In particular, since the gravitational force is given by

$$\vec{F}(r) = -G\frac{Mm}{r^2},$$

the expression for the square of the period becomes

$$T^2 = \frac{4\pi^2 a^3}{G(M+m)} \approx \frac{4\pi^2 a^3}{GM}, \qquad m \ll M \tag{3.30}$$

Kepler's Laws can now be summarized:

Law 1. *Planets move in elliptical orbits about the Sun with the Sun at one focus.*

Law 2. *The area per unit time swept out by a radius vector from the Sun to a planet is constant.*

Law 3. *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.*

3.6 More general potentials

Proposition 3.6.1. *For any attractive force $f(r)$ there exist circular orbits (not necessarily stable) of arbitrary radius.*

3.6.1 Almost circular orbits. Stability

Proposition 3.6.2. *If $u = u_0 + h$ with $|h| \ll u_0$, then at first order*

$$\frac{\mathrm{d}^2 h}{\mathrm{d}\theta^2} + (1 - \psi_0')h = 0. \tag{3.31}$$

1. *If $1 - \psi_0' < 0$, the orbit is unstable.*

2. *If $1 - \psi_0' > 0$, the orbit is stable with $h = A\cos(\alpha\theta)$, $\alpha = \sqrt{1 - \psi_0'}$ (if $r_{\min} = r(\theta = 0)$). Besides, $r_0^2 A$ is approximately the amplitude of the oscillation of r.*

Proposition 3.6.3 (Stability condition). *The solution of $h'' + (1 - \psi_0')h = 0$ is stable if and only if*

$$\alpha^2 = 3 + \frac{r_0 f'(r_0)}{f(r_0)} > 0. \tag{3.32}$$

Proposition 3.6.4. *If the orbit is closed, then $\alpha = p/q \in \mathbb{Q}$.*

Theorem 3.6.5. *The only potential functions such that every bounded orbit is closed are those of the form*

$$f(r) = -\frac{k}{r^{r-\alpha^2}}, \qquad \alpha = \frac{p}{q} \in \mathbb{Q},\ k > 0. \tag{3.33}$$

3.6.2 Perturbation theory (Poincaré-Lindstedt method). Proof of Bertrand's theorem

$$h'' + \alpha^2 h = \frac{\lambda}{2}\psi_0'' h^2 + \frac{\lambda^2}{6}\psi_0^{(3)} h^3 + \cdots \tag{3.34}$$

We make the expansion

$$h = h_0 + \lambda h_1 + \lambda^2 h_2 + \cdots, \qquad \beta^2 = \alpha^2 + \beta_1^2 \lambda + \beta_2^2 \lambda^2 + \cdots, \qquad v = v_0 + v_1 \lambda + v_2 \lambda^2 + \cdots \tag{3.35}$$

and denote $h(\theta) = v(\phi), \phi = \beta\theta$. The equation is $\beta^2 v'' + \alpha^2 v = \frac{\lambda}{2}\psi_0'' h^2 + \frac{\lambda^2}{6}\psi_0^{(3)} h^3 + \cdots$.

Proposition 3.6.6. *The first terms are*

$$\beta_0^2 = \alpha^2, \qquad \beta_1^2 = 0, \beta_2^2 = -\frac{5(\psi_0'')^2 + 3\alpha^2\psi_0^{(r)}}{24\alpha^2}A^2, \tag{3.36}$$

$$v_0 = A\cos\phi \tag{3.37}$$

$$v_1 = \frac{\psi_0''}{12\alpha^2}A^2(3 - \cos 2\phi - 2\cos\phi) \tag{3.38}$$

$$v_2 = \frac{1}{576}\left[29\left(\frac{\psi_0''}{\alpha^2}\right)^2\right]A^3\cos\phi - \frac{1}{12}\left(\frac{\psi_0''}{\alpha^2}\right)^2 A^3 + \tag{3.39}$$

$$\frac{1}{36}\left(\frac{\psi_0''}{\alpha^2}\right)^2 A^3\cos 2\phi + \frac{1}{192}\left[\left(\frac{\psi_0''}{\alpha^2}\right)^2 - \frac{\psi_0^{(3)}}{\alpha^2}\right]A^3\cos 3\phi. \tag{3.40}$$

Theorem 3.6.7 (Bertrand's theorem)**.** *The only potential functions such that every bounded orbit is closed are*

$$U_1 = \frac{kr^2}{2}, \qquad U_2 = -\frac{k}{r}. \tag{3.41}$$

3.7   Two body problem

3.8   Particle dispersion

We consider non-bounded orbits and potential functions that diminish faster than $1/r$. We select potential functions such that $U(\infty) = 0$.
If $U(r)$ is not bounded, there are not non-bounded orbits.
Conditions at infinity

$$E(\infty) = K_0 + U(\infty) = \frac{mv_0^2}{2}, \qquad l = bv_0 = b\sqrt{\frac{2K_0}{m}} \tag{3.42}$$

Angle from x axis to the asymptote

$$\Theta = \int_{r_{\min}}^{\infty} \frac{l}{r^2}\frac{1}{\sqrt{2/m(E - U_{\text{eff}}(r))}}dr = \int_{r_{\min}}^{\infty} \frac{b}{r^2}\frac{1}{\sqrt{1 - U/K_0 - b^2/r^2}} \tag{3.43}$$

Dispersion angle

$$\vartheta = |\pi - 2\Theta| \tag{3.44}$$

Kappa

$$\kappa := \frac{|k|}{2K_0} \tag{3.45}$$

Potential $U = -k/r$

$$\cos\Theta = -\frac{\operatorname{sgn} k}{\sqrt{1 + (b/\kappa)^2}} \Rightarrow \cot\frac{\vartheta}{2} = \frac{b}{\kappa} \tag{3.46}$$

Potential of impenetrable sphere

$$U(r) = \begin{cases} 0 & \text{if } r > R, \\ \infty & \text{if } r \le R \end{cases} \Rightarrow \Theta = \begin{cases} \dfrac{\pi}{2} & \text{if } b \ge R, \\ \arcsin\dfrac{b}{R} & \text{if } b < R \end{cases} \Rightarrow \cos\frac{\vartheta}{2} = \begin{cases} 1, & \text{if } b \ge R, \\ \dfrac{b}{R} & \text{if } b < R \end{cases} \tag{3.47}$$

3.9 Cross section

3.9.1 Cross section of a center of forces. Rutherford's problem

$$I = \# \text{ projectiles/A/t that cross the surface} \perp \text{ at their velocities} \tag{3.48}$$

$$dN = \# \text{ projectiles deflected at } d\Omega \text{ at } \vartheta \text{ / t/phyto} \tag{3.49}$$

Differential cross section

$$d\sigma(\vartheta) = \frac{dN}{I}, \qquad [d\sigma] = \text{m}^2\,\text{sr}^{-1}, \qquad \text{bar}: \text{b} = 10^{-28}\,\text{m}^2 = 100\,\text{fm}^2 \tag{3.50}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{I}\frac{dN}{d\Omega}, \qquad d\Omega = 2\pi\sin\vartheta d\vartheta \Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\vartheta}\left|\frac{db}{d\vartheta}\right| \tag{3.51}$$

Potential $-k/r$

$$b = \kappa\cot\frac{\vartheta}{2} \Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{\kappa}{2}\right)^2\csc^4\frac{\vartheta}{2} \tag{3.52}$$

Potential of impenetrable sphere

$$b = R\cos\frac{\vartheta}{2} \Rightarrow \frac{d\sigma}{d\Omega} = \frac{R^2}{4} \Rightarrow \sigma = \int\frac{d\sigma}{d\Omega}d\Omega = \pi R^2 \tag{3.53}$$

Discrete case

$$\frac{d\sigma}{d\Omega} \approx \frac{N_{\text{pr}}^{(j)}/T/N_{\text{ph}}}{N_{\text{pr,T}}/A/T \cdot 2\pi\theta^{(j)}\sin\theta^{(j)}}, \qquad N_{\text{pr, T}} = \sum N_{\text{pr}}^{(j)} \tag{3.54}$$

3.9.2 Cross section in CM system and LAB system

$$m \to \mu = \frac{m_1 m_2}{M}, \qquad \text{position vector of projectile} \to \text{vector } \vec{r} = \vec{r_1} - \vec{r_2} \tag{3.55}$$

LAB system

$$d\sigma(\psi) = \frac{\# \text{ deflected projectiles at } d\Omega \text{ at } \psi \text{ /t/phyto}}{\text{flux of projectiles (relative to the phyto)}} \tag{3.56}$$

Relation

$$d\sigma(\psi) = d\sigma'(\vartheta), \qquad \tan\psi = \frac{\sin\vartheta}{m_1/m_2 + \cos\vartheta}, \qquad \frac{d\Omega'}{d\Omega} = \frac{\sin\vartheta}{\sin\psi}\frac{d\vartheta}{d\psi} \Rightarrow \tag{3.57}$$

$$\frac{d\sigma}{2\Omega}(\psi) = \frac{d\sigma'}{d\Omega'}(\vartheta)\frac{\sin\vartheta}{\sin\psi}\frac{d\vartheta}{d\psi} = \frac{d\sigma'}{d\Omega'}(\vartheta)\frac{\left(\sqrt{1 - x^2\sin^2\psi} + x\cos\psi\right)^2}{\sqrt{1 - x^2\sin^2\psi}}, \qquad x = \frac{m_1}{m_2}(\leq 1) \tag{3.58}$$

Potential $-k/r$

$$\frac{d\sigma'}{d\Omega'}(\vartheta) = \left(\frac{k}{4K_0'}\right)\csc^4\frac{\vartheta}{2}, \qquad x = 1 \Rightarrow \vartheta = 2\psi, K_0' = \frac{K_0}{2} \Rightarrow \frac{d\sigma}{d\Omega}(\psi) = \left(\frac{k}{K_0}\right)^2\frac{\cos\psi}{\sin^4\psi} \tag{3.59}$$

Relation between energies

$$\frac{K_0'}{K_0} = \frac{m_2}{m_1 + m_2}, \quad \frac{K_1'}{K_0} = \frac{m_2^2}{(m_1 + m_2)^2}, \quad \frac{K_2'}{K_0} = \frac{m_1 m_2}{(m_1 + m_2)^2}, \tag{3.60}$$

$$\frac{K_1}{K_0} = 1 - 2\frac{m_1 m_2}{m_1 + m_2}(1 - \cos\vartheta), \quad \frac{K_2}{K_0} = 2\frac{m_1 m_2}{(m_1 + m_2)^2}(1 - \cos\vartheta) \tag{3.61}$$

Chapter 4

Coupled oscillations 1

4.1   Problem approach

Situation: two blocks, three springs (k, k_{12}, k) with fixed extremes

$$m\ddot{x}_1 = -kx_1 + k_{12}(x_2 - x_1)$$
$$m\ddot{x}_2 = -kx_2 - k_{12}(x_2 - x_1)$$

$$m\ddot{x}_1 + (k + k_{12})x_1 - k_{12}x_2 = 0$$
$$m\ddot{x}_2 + (k + k_{12})x_2 - k_{12}x_1 = 0$$

4.2   Normal coordinates and normal modes

Example 4.2.1.

$$q := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \frac{1}{m}\begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix}$$

$$\ddot{q} + \Omega q = 0 \tag{4.1}$$

4.3   Weak coupling

$$\epsilon := \frac{k_{12}}{2k} \ll 1 \Rightarrow \omega_1 \approx \omega_0(1 - \epsilon), \quad \omega_2 \approx \omega_0(1 + \epsilon)$$

With $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ and $x_1 = d$:

$$x_1(t) = \cos[\epsilon\omega_0 t]\cos[\omega_0 t]$$
$$x_2(t) = \sin[\epsilon\omega_0 t]\sin[\omega_0 t]$$

Chapter 5

Coupled oscillations 2

5.1 General theory of near equilibrium oscillations

5.1.1 General coordinates (introduction)

The positions of n particles (with three components) with respect to a IRS are $\{\vec{r}_a\}_{a=1}^N = \{x_\alpha\}_{\alpha=1}^{3N}$. If we have k non redundant independent variables, then

$$x_\alpha = x_\alpha(\vec{q}, t), \qquad \vec{q} = \begin{pmatrix} q_1 & \dots & q_k \end{pmatrix}^t \tag{5.1}$$

We usually take $\vec{q}$ dimensionally homogeneous and the dependence of time through $\vec{q}$, that is, $x_\alpha(t) = x_\alpha(\vec{q}(t))$.
We suppose there are not non-conservative forces.

5.1.2 Near equilibrium potential energy

Since $U(\vec{q}) = V(x_\alpha(\vec{q}))$ and $x_\alpha^{(0)} = x_\alpha(q^{(0)})$ (equilibrium position,)

$$\left.\frac{\partial U}{\partial q_i}\right|_{\vec{q}^{(0)}} = \sum_\alpha \left.\frac{\partial V}{\partial x_\alpha}\right|_{x_\alpha^{(0)}} \left.\frac{\partial x_\alpha}{\partial q_i}\right|_{\vec{q}^{(0)}} = \sum_\alpha -F_\alpha \left.\frac{\partial x_\alpha}{\partial q_i}\right|_{\vec{q}^{(0)}} = 0 \tag{5.2}$$

And if we choose a general coordinates such that $\vec{q}^{(0)} = \vec{0}$ (for instance with $\vec{q} \mapsto \vec{q} - \vec{q}^{(0)}$), we get the Hessian matrix (non negatively defined)

$$U(\vec{q}) \approx \frac{1}{2} \sum_{ij} \frac{\partial^2 U}{\partial q_i \partial q_j} q_i q_j = \frac{1}{2} \sum_{ij} A_{ij} q_i q_j = \frac{1}{2} \vec{q}^t A_{ij} \vec{q} \tag{5.3}$$

5.1.3 Near equilibrium kinetic energy

$$\dot{x}_\alpha = \sum_i \frac{\partial x_\alpha}{\partial q_i} \dot{q}_i, \Rightarrow K = \sum_\alpha \frac{1}{2} m_\alpha \dot{x}_\alpha^2 = \frac{1}{2} \sum_{ij} \sum_\alpha m_\alpha \frac{\partial x_\alpha}{\partial q_i} \frac{\partial x_\alpha}{\partial q_j} \dot{q}_i \dot{q}_j := \frac{1}{2} \sum_{ij} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{\vec{q}}^t m \dot{\vec{q}}$$
$$\tag{5.4}$$

The matrix $m = [m_{ij}]$ is always positively defined.

5.1.4 Near equilibrium mechanical energy

$$E = K + U = \frac{1}{2} \dot{\vec{q}}^t m \dot{\vec{q}} + \frac{1}{2} \vec{q}^t A \vec{q} = \frac{\mu}{2} (\dot{\vec{q}} M \dot{\vec{q}} + \vec{q}^t \Omega \vec{q}) \tag{5.5}$$

5.1.5 Diagonalization of K and U. Normal modes

We define $y = M^{1/2} q, q = M^{-1/2} y$. We know $O^t M O = M_d$, so $M^{1/2} = O M_d^{1/2} O^t$

$$E = \frac{\mu}{2} \left[\dot{\vec{y}}^t \dot{\vec{y}} + \vec{y}^t (M^{1/2} \Omega M^{-1/2}) \vec{y} \right] := \frac{\mu}{2} \left[\dot{\vec{y}}^t \dot{\vec{y}} + \vec{y}^t \tilde{\Omega} \vec{y} \right], \qquad \tilde{\Omega} \text{ non-neg. defined and sym.} \tag{5.6}$$

We diagonalize $\tilde{\Omega}$ and obtain $\tilde{\lambda}_j = \omega_j^2$ and $\tilde{u}_j$. With $y = \sum_k \eta_k \tilde{u}_k$, we get $\tilde{u}_j^t \tilde{u}_k = \delta_{jk}$ and

$$E = \mu \sum_j \frac{1}{2} \dot{\eta}_j^2 + \frac{1}{2} \omega_j^2 \eta_j^2 \tag{5.7}$$

Making now $u_j = M^{-1/2} \tilde{u}_j$, we get

$$\Omega u_j - \omega_j^2 M u_j = 0, \qquad u_j^t M u_k = \delta_{jk}, \qquad q = \sum_k \eta_k u_k, \qquad u_j^t M q = \eta_j \tag{5.8}$$

Initial conditions

$$\eta_j(0) = v_j^t M q(0), \qquad \dot{\eta}_j(0) = v_j^t M \dot{q}(t) \tag{5.9}$$

$$\eta_j(t) = \eta_j(0) \cos \omega_j t + \frac{\dot{\eta}_j(0)}{\omega_j} \sin \omega_j t, \qquad q(t) = \sum_k \eta_k(t) u_k, \qquad E = \frac{\mu}{2} \sum_j \omega_j^2 A_j^2 \tag{5.10}$$

5.2 Many oscillators

5.2.1 Approach and evolution equations

$$m\ddot{q}_r = T_r \sin\theta_r - T_{r-1}\sin\theta_{r-1}, \tan\theta_r = \frac{q_{r+1}-q_r}{d} \Rightarrow \ddot{q}_r + \frac{T}{md}(2q_r - q_{r-1} - q_{r+1}) = 0 \quad (5.11)$$

5.2.2 Normal modes

Now we search the eigenfrequencies

$$2a_r - a_{r+1} - a_{r-1} = \frac{md}{T}\omega^2 a_r, a_r = a\sin\mu r \Rightarrow \omega = 2\sqrt{\frac{T}{md}}\sin\frac{\mu}{2} \quad (5.12)$$

We impose that $q_0 = q_{n+1} = 0$.

$$\mu_j = \frac{j\pi}{n+1} \Rightarrow \omega_j = 2\sqrt{\frac{T}{md}}\sin\frac{j\pi}{2(n+1)} \quad (5.13)$$

We normalize and

$$\left\|v^{(j)}\right\|^2 = \sqrt{\frac{n+1}{2}} \Rightarrow v_r^{(j)} = \sqrt{\frac{2}{n+1}}\sin\frac{rj\pi}{n+1}, \qquad v^{(j)t}v^k = \delta_{jk} \quad (5.14)$$

Relations

$$q_r(t) = \sum_{j=1}^{n}\eta_j(t)v_r^{(j)} = \sum_{j=1}^{n}\eta_j(t)\sqrt{\frac{2}{n+1}}\sin\frac{rj\pi}{n+1} \quad (5.15)$$

$$\eta_j = \sum_{r=1}^{n}q_r(t)v_r^{(j)} = \sum_{r=1}^{n}q_r(t)\sqrt{\frac{2}{n+1}}\sin\frac{rj\pi}{n+1} \quad (5.16)$$

If $\dot{q}_r(0) = 0, \forall r$, then

$$q_r(t) = \frac{2}{n+1}\sum_{j=1}^{n}\left\{\sum_{s=1}^{n}q_s(0)\sin\frac{sj\pi}{n+1}\right\}\sin\frac{rj\pi}{n+1}\cos\omega_j t \quad (5.17)$$

5.2.3 Energy and normal modes

$$E = \sum_{r=1}\frac{1}{2}m\dot{q}_r^2 + \frac{T}{2d}(q_{r+1}-q_r)^2 \quad (5.18)$$

Diagonalization

$$E = \frac{1}{2}m(\dot{q}^t\dot{q} + q^t\Omega q) = \frac{1}{2}\sum_j m\omega_j^2 A_j^2 \quad (5.19)$$

5.2.4 Continuum limit

Initial substitutions

$$(n+1)d = L, \qquad \rho = \frac{M}{L} \to \frac{m}{d}, \qquad x = rd, u = \frac{v}{\sqrt{d}}, \qquad \tilde{\eta}_j = \sqrt{d}\eta_j \quad (5.20)$$

$$\int_0^L u^{(j)}(x)u^{(k)}(x)dx = \delta_{jk}, \quad q(x) = \sum_{j=1}^{\infty}\tilde{\eta}_j u^{(j)}(x), \quad \tilde{\eta}_j = \int_0^L u^{(j)}(x)q(x)dx, \quad \omega_j = \sqrt{\frac{T}{\rho}}\frac{j\pi}{L} \quad (5.21)$$

5.2.5   Wave equation derivation from the limit of discrete evolution equations

$$\frac{\partial^2 q}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = 0, c = \sqrt{\frac{T}{\rho}}, \qquad \text{b. conditions } q(0,t) = q(L,t) = 0 \tag{5.22}$$

Energy

$$E = \int_0^L \frac{\rho}{2}\left(\frac{\partial q}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial q}{\partial x}\right)^2 dx \tag{5.23}$$

5.3   Wave equation derivation from the continuum

$$\frac{\partial^2 q}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = 0, c = \sqrt{\frac{T}{\rho}}, \qquad \text{b. conditions } q(0,t) = q(L,t) = 0 \tag{5.24}$$

5.4   Oscillation modes for the rope. Boundary conditions and initial conditions

General solution (to obtain the normal modes)

$$q(x,t) = u(x)\left(c_1 \cos\omega t + c_2 \sin\omega t\right) \Rightarrow u''(x) + \frac{\omega^2}{c^2}u(x) = 0 \Rightarrow u(x) = a\cos\left[\frac{\omega}{c}x\right] + b\sin\left[\frac{\omega}{c}x\right] \tag{5.25}$$

Relations

$$f\lambda = c, \qquad \int_0^L u^{(j)}(x)u^{(k)}(x)dx = \delta_{jk} \tag{5.26}$$

For the fixed string $(u(0) = u(L) = 0)$

$$\omega_j = \frac{j\pi c}{L}, \qquad u_j(x) = \sqrt{\frac{2}{L}}\sin\frac{j\pi x}{L}, \tag{5.27}$$

Relations

$$q(x,t) = \sum_{j=1}^{\infty} \eta_j(t)u_j(x), \qquad q(x,0) = g(x), \dot{q}(x,0) = h(x) \tag{5.28}$$

$$\eta_k(0) = \int_0^L u_k(x)g(x)dx, \qquad \eta_k(0) = \int_0^L u_k(x)h(x)dx \tag{5.29}$$

5.5   General solution of the wave equation. Uniqueness theorem

General (and unique) solution and d'Alembert's formula

$$q(x,t) = \psi(x-ct) + \phi(x+ct) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')dx' \tag{5.30}$$

Decomposition

$$\psi(x - ct) = \frac{g(x - ct)}{2} - \frac{1}{2c} \int_{x_0}^{x-ct} h(x')dx' + \frac{\psi(x_0) - \phi(x_0)}{2}, \tag{5.31}$$

$$\phi(x + ct) = \frac{g(x + ct)}{2} + \frac{1}{2c} \int_{x_0}^{x+ct} h(x')dx' - \frac{\psi(x_0) - \phi(x_0)}{2} \tag{5.32}$$

"Energy"

$$\varepsilon = \int_0^L \left(\frac{\partial q}{\partial t}\right)^2 + c^2 \left(\frac{\partial q}{\partial x}\right)^2 dx, \qquad \frac{d\varepsilon}{dt} = 0 \tag{5.33}$$

Chapter 6

Rotations

6.1 Mathematical background

6.1.1 Vectors and matrices

$$[u \times v]_k = \sum_{i,j} \epsilon_{ijk} u_i v_j = \sum_{i,j} \epsilon_{kij} u_i v_j$$

$$\langle \vec{u}, \vec{v} \times \vec{w} \rangle_I = \sum_{i,j,k} \epsilon_{ijk} u_i v_j w_k = \det\{[\mathsf{u}, \mathsf{v}, \mathsf{w}]\}$$

$$\sum_{i,j,j} \epsilon_{ijk} A_{ii'} A_{jj'} A_{kk'} = \epsilon_{i'j'k'} \det\{A\}$$

$$\sum_{ij} \epsilon_{ijl} A_{ii'} A_{jj'} = \det\{A\} \sum_{k'} \epsilon_{i'j'k'} A_{k'l}^{-1}$$

$$\epsilon_{ijk} \epsilon_{i'j'k'} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} & \delta_{ik'} \\ \delta_{ji'} & \delta_{jj'} & \delta_{jk'} \\ \delta_{ki'} & \delta_{kj'} & \delta_{kk'} \end{vmatrix}$$

$$\sum_k \epsilon_{ijk} \epsilon_{i'j'k} = \delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}$$

$$\sum_{j,k} \epsilon_{ijk} \epsilon_{i'jk} = 2! \delta_{ii'}$$

$$\sum_{i,j,k} \epsilon_{ijk} \epsilon_{ijk} = 3!$$

$$\det\{A\} = \frac{1}{3!} \sum_{ijk,i'j'k'} \epsilon_{ijk} \epsilon_{i'j'k'} A_{ii'} A_{jj'} A_{kk'}$$

$$A_{k'k}^{-1} = \frac{1}{2 \det\{A\}} \sum_{ij,i'j'} \epsilon_{ijk} \epsilon_{i'j'k'} A_{ii'} A_{jj'}, \quad \det\{A\} \neq 0$$

$$\vec{u} \times (\vec{b} \times \vec{w}) = \langle \vec{u}, \vec{w} \rangle_I \vec{v} - \langle \vec{u}, \vec{v} \rangle_I \vec{w}$$

$$\langle \vec{u} \times \vec{v}, \vec{u}' \times \vec{v}' \rangle_I = \langle \vec{u}, \vec{u}' \rangle_I \langle \vec{v}, \vec{v}' \rangle_I - \langle \vec{u}, \vec{v}' \rangle_I \langle \vec{v}, \vec{u}' \rangle_I$$

Linear transformation

$$f_A(u) = \mathsf{A}u \tag{6.1}$$

6.1.2 Basis

Let $\mathscr{B}$ be a basis. We denote the matrix of $\mathbf{v}$ of components in $\mathscr{B}$ as

$$C_{\mathscr{B}}(\mathbf{v}) = \mathsf{v}_B = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \tag{6.2}$$

If we use the canonical basis we won't specify the basis. We will write directly u.

6.1.3 Change of basis

Let $\mathscr{B}_1, \mathscr{B}_2$ be two basis. Then, the matrix to change the basis $\mathsf{M}(\mathscr{B}_2 \leftarrow \mathscr{B}_1)$ has the columns formed by the vectors of $\mathscr{B}_1$ written in $\mathscr{B}_2$. This relation can be expressed as follows

$$C_{\mathscr{B}_2}(\mathbf{v}) = \begin{pmatrix} C_{\mathscr{B}_2}(\mathbf{u_1}) & \dots & C_{\mathscr{B}_2}(\mathbf{u_n}) \end{pmatrix} C_{\mathscr{B}_1}(\mathbf{v}) \Leftrightarrow \mathsf{v}_{B_2} = \begin{pmatrix} \mathsf{u}_{1,B} & \dots & \mathsf{u}_{n,B} \end{pmatrix} \mathsf{v}_{B_1}. \tag{6.3}$$

In particular, taking the canonical basis $\mathscr{C}$ as $\mathscr{B}_1$, $\mathsf{M}(\mathscr{C} \leftarrow \mathscr{B}_1)$ has the vectors of $\mathscr{B}_1$ as the columns. We denote $\mathsf{C} := \mathsf{M}(\mathscr{C} \leftarrow \mathscr{B})$.

For a linear transformation f_A with a matrix $A = M(f; \mathscr{C} \leftarrow \mathscr{C})$, we have the matrix of f_A associated to a basis $\mathscr{B}$ is

$$\mathsf{A}_{\mathscr{B}} = \mathsf{M}(f; \mathscr{B} \leftarrow \mathscr{B}) = \mathsf{M}(\mathscr{B} \leftarrow \mathscr{C}) \mathsf{M}(f; \mathscr{C} \leftarrow \mathscr{C}) \mathsf{M}(\mathscr{C} \leftarrow \mathscr{B}) = \mathsf{C}^{-1} \mathsf{A}_{\mathscr{C}} \mathsf{C}. \tag{6.4}$$

$$\mathbf{e}'_i = f_A(\mathbf{e}_i) \Leftrightarrow \mathbf{e}'_i = \mathsf{A}\mathbf{e}_i, \mathbf{v} = \sum_{i=1}^n \alpha'_i \mathbf{e}'_i = \sum_{i=1}^n \alpha'_i f_A(\mathbf{e}_i) = f_A\left(\sum_{i=1}^n \alpha'_i \mathbf{e}_i\right) \tag{6.5}$$

6.1.4   Orthogonal transformations

Properties

- $\mathsf{O}^{-1} = \mathsf{O}^t$

- $\det\{\mathsf{O}\} = \pm 1$

- They preserve the angles, the lengths, and the dot product, that is, $\langle \vec{u}, \vec{v} \rangle_I = \langle \vec{u}_{\mathscr{B}}, \vec{v}_{\mathscr{B}} \rangle_I$

- They form a group

2D: In 2D, all have the form have two possible forms (the transformations associated to the rotation matrix is R_θ and and to the reflection matrix by S). The matrix R_θ has eigenvalues $\lambda = e^{\pm \theta i}$.

$$\mathsf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \mathsf{R}_\theta \mathsf{S} = \mathsf{R}_\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.6}$$

3D: The eigenvalues are $\lambda_0 = \pm 1, \lambda_+ = e^{\theta i}, \lambda_- = e^{-\theta i}$ with eigenvectors $\vec{n}_0, \vec{n}_+, \vec{n}_-$. Taking a basis $(\vec{n}_1, \vec{n}_2, \vec{n}_0)$, a general transformation matrix is

$$\mathsf{M}_{\mathscr{B}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \tag{6.7}$$

To make sure the convention of the sign of $\vec{n}_0$ is correct, it must be true that $\mathsf{C}\mathsf{M}_{\mathscr{B}}\mathsf{C} = \mathsf{M}$. If not, change the sign of $\vec{n}_0$ and other to keep the orientation. If we take some vector $\vec{u}$, take $\vec{u}_\perp = \vec{u} - \langle \vec{n}, \vec{u} \rangle_I \vec{n}$ (Gram-Schmidt), $\vec{n}_1 = \vec{u}_\perp / \|\vec{u}_\perp\|$, and $\vec{n}_2 = \vec{n} \times \vec{n}_1$ to make $\mathscr{B} = (\vec{n}_1, \vec{n}_2, \vec{n})$
Axis rotations

$$\mathsf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad \mathsf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \mathsf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.8}$$

Rotation of a vector $\vec{u}$

$$R_{\vec{n}}(\psi)\vec{u} = \vec{u}\cos\psi + (\vec{n} \times \vec{u})\sin\psi + \vec{n}\langle \vec{n}, \vec{u} \rangle_I (1 - \cos\psi), \tag{6.9}$$

$$[R_{\vec{n}}(\psi)]_{ij} = \delta_{ij}\cos\psi - \sum_k \epsilon_{ijk} n_k \sin\psi + n_i n_j (1 - \cos\psi) \tag{6.10}$$

6.2   Infinitesimal rotations. Angular velocity and acceleration

$$R_n(d\psi)\vec{u} = \vec{u} + (\vec{n} \times \vec{u})d\psi \Rightarrow d\vec{u} = (\vec{n} \times \vec{u})d\psi \Rightarrow \dot{\vec{u}} = \dot{\psi}\vec{n} \times \vec{u} \tag{6.11}$$

With $\vec{u} = \vec{r}$ in an instantaneous rotation over the center of curvature (without variation of length),

$$\vec{v} = \dot{\psi}\vec{n} \times \vec{r} = \vec{\omega} \times \vec{r} \Rightarrow \vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.12}$$

Infinitesimal rotations (only) commute

$$R_{\vec{n}'}(d\psi')R_{\vec{n}}(d\psi)\vec{u} = \vec{u} + (\vec{n}d\psi + \vec{n}'d\psi') \times \vec{u} \tag{6.13}$$

6.3   Rotating reference systems

6.3.1   Rotating and fixed system

We denote $\mathscr{B}$ the fixed basis and $\mathscr{B}'$ the rotating basis. We denote $\vec{u}'$ the vector with respect to u_i' the i-th component of $\vec{u}$ with respect to $\mathscr{B}'$ and $\vec{e}_i'$ the i-th vector of the basis $\mathscr{B}'$. If the system rotates at ω,

$$\dot{\vec{e}}_i' = \vec{\omega} \times \vec{e}_i', \qquad \frac{d\vec{u}}{dt} = \left(\frac{d\vec{u}}{dt}\right)' + \vec{\omega} \times \vec{u} \tag{6.14}$$

If $\vec{u} = \vec{r}$,

$$\vec{v} = (\vec{v})' + \vec{\omega} \times \vec{r}, \qquad \vec{a} = (\vec{a})' + \vec{\alpha} \times \vec{r} + 2\vec{\omega} \times (\vec{v})' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.15}$$

If the system moves,

$$\vec{a} = (\vec{a})' + \ddot{\vec{R}} + \vec{\alpha} \times \vec{r} + 2\vec{\omega} \times (\vec{v})' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.16}$$

6.3.2   Inertial forces

$$m(\vec{a})' = (\vec{F})' = \vec{F} - m\ddot{\vec{R}} - m\vec{\alpha} \times \vec{r} - 2m\vec{\omega} \times (\vec{v})' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.17}$$

And

$$\vec{F}_{\text{cor}} = -2m\vec{\omega} \times (\vec{v})', \qquad \vec{F}_{\text{cf}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{6.18}$$

Observations

- We use ALWAYS x, y, z for the vector in rotating system and x_f, y_f, z_f for the fixed system.

6.4   Solid rigid kinematics

Chapter 7

Dynamics of rigid body

7.1   Part 1

Now, $\mathbf{r}$ is the position in a reference system S that moves such that $\mathbf{r}$ is constant. The vector $\mathbf{r}_f$ represents a reference system that has origin and basis fixed.

$$\mathbf{v}_f = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}_O \tag{7.1}$$

$$\mathbf{a}_f = \mathbf{A} + \boldsymbol{\alpha} \times \mathbf{r}_O + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{7.2}$$

Proposition 7.1.1. *The vector $\boldsymbol{\Omega}$ is independent on the origin of the system S.*

7.2   Part 2

7.2.1   Kinetic energy and inertia tensor

Now, $\mathbf{r}$ is the position from the center of mass

$$T = \frac{1}{2} \int\limits_V \|\mathbf{v}_f\|^2 \rho(\mathbf{r}) \, \mathrm{d}v \tag{7.3}$$

$$E = \frac{1}{2} M \|\mathbf{V}\|^2 + \frac{1}{2} \boldsymbol{\Omega}^t \mathbf{I} \boldsymbol{\Omega} \tag{7.4}$$

Proposition 7.2.1. *The energy of the rigid body is an invariant scalar under change of basis.*

Expressions and properties for inertia tensor

$$I_{ij} = \int\limits_V \rho(\mathbf{r}) \left[\|\mathbf{r}\| \delta_{ij} - x_i x_j\right] \mathrm{d}v, \qquad \mathbf{I} = \int\limits_V \rho(\mathbf{r}) \left[\|\mathbf{r}\| \mathbf{E}_r - \mathbf{r} \otimes \mathbf{r}\right] \mathrm{d}v, \tag{7.5}$$

$$\mathbf{I} = \int \begin{pmatrix} \|\mathbf{r}\|^2 - x^2 & -xy & -xz \\ -yx & \|\mathbf{r}\|^2 - y^2 & -yz \\ -zx & -zy & \|\mathbf{r}\|^2 - z^2 \end{pmatrix} \mathrm{d}m \tag{7.6}$$

$$\mathbf{J} = \mathbf{I} + m\left[\langle \mathbf{R}, \mathbf{R}\rangle_I \mathbf{E}_3 - \mathbf{R} \otimes \mathbf{R}\right], \qquad J_{ij} = I_{ij}^{\mathrm{cm}} + m(\|\mathbf{R}\|^2 \delta_{ij} - a_i a_j), \qquad \mathbf{a} = \mathbf{r}_{o \to o_{\mathrm{cm}}} \tag{7.7}$$

7.2.2   Diagonalization of inertia tensor

1. Change of basis of inertia tensor.

$$\mathbf{I}_{B_2} = \mathbf{T} \mathbf{I}_{B_1} \mathbf{T}^t, \qquad \mathbf{T} = \mathbf{M}(B_2 \leftarrow B_1) \tag{7.8}$$

2. It is a symmetric tensor, so it is diagonalizable and $\forall i, \lambda_i \in \mathbb{R}^+$.

3. We can always select a basis (that can be moving basis) such that inertia tensor is diagonal. In that case,

$$T_{\mathrm{rot}} = \frac{1}{2}(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2) \tag{7.9}$$

7.2.3   Angular momentum (with origin at CM)

$$\mathbf{L} := \mathbf{I} \cdot \boldsymbol{\Omega}, \tag{7.10}$$

$$T_{\mathrm{rot}} = \frac{1}{2} \langle \boldsymbol{\Omega}, \mathbf{L} \rangle_I \tag{7.11}$$

$$\mathbf{L}_D = (I_1 \Omega_1, I_2 \Omega_2, I_3 \Omega_3) \tag{7.12}$$

7.2.4   Spherical tops

1. The inertia tensor is diagonal independently on the orientation of the basis.

2. Since the eigenvalue has as eigenspace $\mathbb{R}^3$, we can select arbitrarily the basis.

3. Choosing $\mathbf{e}_3 \parallel \mathbf{L}$, we get $\Omega_3 = L/I_1, \Omega_1 = \Omega_2 = 0$.

7.2.5   Symmetric top

1. We can choose $\mathbf{e}_1$ and $\mathbf{e}_2$ but not $\mathbf{e}_3$. We denote $\theta = \arg(\mathbf{e}_3, \mathbf{L})$.

2. We select $\mathbf{e}_2 \in \pi(\mathbf{0}; \mathbf{e}_x, \mathbf{e}_y)$. We call $\mathbf{e}_3$ the *axis of the top* and $\mathbf{v}_a$ the velocity of a point of the axis.

$$\mathbf{e}_2, \mathbf{v}_a \perp \pi(\mathbf{r}_{\rm cm}; \mathbf{e}_3, \mathbf{L}), \qquad L_2 = \Omega_2 = 0, \qquad \mathbf{\Omega} \in \pi(\mathbf{r}_{\rm cm}; \mathbf{e}_3, \mathbf{L}) \qquad (7.13)$$

3. The top rotates along $\mathbf{L}$ at a constant angle θ of and with constant velocity angular velocity $\Omega_{\rm prec}$.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \langle \mathbf{L}, \mathbf{v}_a \rangle_I = 0 \Rightarrow \langle \mathbf{L}, \mathbf{r}_a \rangle_I = \text{ctt} \qquad (7.14)$$

Some relations

$$\mathbf{\Omega} = \Omega_{\rm prec}\mathbf{e}_L + \omega_{\rm rot}\mathbf{e}_3 = \Omega_1\mathbf{e}_1 + \Omega_3\mathbf{e}_3, \qquad \Omega_{\rm prec}\sin\theta = \Omega_1, \qquad \Omega_{\rm prec}\cos\theta + \omega_{\rm rot} = \Omega_3 \quad (7.15)$$

$$L\sin\theta = L_1, \qquad L\cos\theta = L_3, \qquad \Omega_{\rm prec} = \frac{L}{I_1}, \qquad \omega_{\rm rot} = \left(1 - \frac{I_3}{I_1}\right)\Omega_3 \qquad (7.16)$$

Misael's equations

$$\sin\theta = \frac{\Omega_1}{\sqrt{\Omega_1^2 + (\Omega_3 - \omega)^2}}, \qquad \cos\theta = \frac{\Omega_3 - \omega}{\sqrt{\Omega_1^2 + (\Omega_3 - \omega)^2}}, \qquad \tan\theta = \frac{\Omega_1}{\Omega_3 - \omega} \qquad (7.17)$$

$$\Omega_1^2 + (\Omega_3 - \omega)^2 = \Omega_{\rm prec}^2, \qquad \Omega_1 = \sqrt{\Omega_{\rm prec}^2 - \frac{\omega^2}{(I_1/I_3 - 1)^2}} \qquad (7.18)$$

$$\sin\theta = \frac{\Omega_1}{\Omega_{\rm prec}}, \qquad \cos\theta = \frac{\omega}{\Omega_{\rm prec}}\frac{1}{I_1/I_3 - 1}, \qquad \tan\theta = \sqrt{\left(\frac{\Omega_p}{\omega}\right)^2\left(\frac{I_1}{I_3} - 1\right)^2 - 1}, \qquad (7.19)$$

$$\frac{\Omega_1}{\Omega_3} = \sqrt{\left(\frac{\Omega_{\rm prec}}{\omega}\right)^2\left(1 - \frac{I_3}{I_1}\right)^2 - \left(\frac{I_3}{I_1}\right)^2} \qquad (7.20)$$

Angles α and β

$$\Omega_{\rm prec}\sin\beta = \omega_{\rm rot}\sin\alpha, \qquad \tan\alpha = \frac{\sin\theta}{\omega_{\rm rot}/\Omega_{\rm prec} + \cos\theta}, \qquad \tan\beta = \frac{\sin\theta}{\Omega_{\rm prec}/\omega_{\rm rot} + \cos\theta} \qquad (7.21)$$

The movement can be decomposed in two cones. The first one has the fixed axis $\mathbf{L}$ with angle β. The second one is a moving cone that rotates over the surface of the first cone without sliding. If $I_1 > I_3$ the second cone rotates over the exterior of the first cone, and if $I_1 < I_3$ over the interior.

7.2.6   Eulerian angles

$$\theta = \arg(\mathbf{e}_z, \mathbf{e}_3'), \theta \in [0, \pi]$$
$$\phi = \arg(\mathbf{e}_x, \mathbf{e}_n), \phi \in [0, 2\pi] \qquad (7.22)$$
$$\psi = \arg(\mathbf{e}_n, \mathbf{e}_1'), \psi \in [0, 2\pi]$$

$$(\mathbf{e}_\phi)_P = \begin{pmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{pmatrix}, \quad (\mathbf{e}_\theta)_P = \begin{pmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{pmatrix}, \quad (\mathbf{e}_\psi)_P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (7.23)$$

$$\mathbf{\Omega} = \dot{\phi}\mathbf{e}_\phi + \dot{\theta}\mathbf{e}_\theta + \dot{\psi}\mathbf{e}_\psi \Rightarrow (\mathbf{\Omega})_B = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix} \qquad (7.24)$$

1. $\Omega_1^2 + \Omega_2^2 = \dot{\psi}^2\sin^2\theta + \dot{\theta}^2$

2. For symmetric top: $\dot{\theta} = 0 \Rightarrow \dot{\phi} = \Omega_{\rm prec}, \dot{\psi} = \omega$

3. For an asymmetric top there are three possible movements: precession ($\dot{\phi}$), intrinsic rotation ($\dot{\psi}$), and nutation ($\dot{\theta}$).

$$\mathbf{R}(\phi, \theta, \psi) = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi\cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \quad (7.25)$$

Kinetic energy and angular momentum as functions of Eulerian angles

$$L_z = I_1(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\sin\theta\sin\psi + I_2(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\sin\theta\cos\psi + I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta \tag{7.26}$$

$$L_Z(\text{s.top}) = I_1\dot{\phi}\sin^2\theta + I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta = I_1\dot{\phi}\sin^2\theta + L_3\cos\theta \tag{7.27}$$

$$T_{\text{rot}}(\text{s.top}) = \frac{I_1}{2}(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{I_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 = \frac{I_1}{2}\frac{(L_z - L_3\cos\theta)^2}{I_1^2\sin^2\theta} + \frac{I_1}{2}\dot{\theta}^2 + \frac{L_3^2}{2I_3} \tag{7.28}$$

$$\dot{\phi}\sin^2\theta = \frac{L_z}{I_1} - \frac{L_3}{I_1}\cos\theta, \qquad \dot{\psi} = \frac{L_3}{I_3} - \dot{\phi}\cos\theta \tag{7.29}$$

Euler equations for rigid body

$$\dot{\mathbf{L}} + \mathbf{\Omega} \times \mathbf{L} = \boldsymbol{\tau}, \qquad \mathbf{I}\dot{\mathbf{\Omega}} + \mathbf{\Omega} \times (\mathbf{I}\mathbf{\Omega}) = \boldsymbol{\tau} \tag{7.30}$$

In basis with principal axis:

$$\begin{cases} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = \tau_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \tau_3 \end{cases} \tag{7.31}$$

7.2.7   Rotation of symmetric top under the action of gravity

Notation: now $\mathbf{r}$ is the vector of position from the fixed point and $\mathbf{R}$ is the vector of CM from the fixed point.

$$T_{\text{rot}} = \frac{1}{2}\mathbf{\Omega}^t\mathbf{I}\mathbf{\Omega} = \frac{1}{2}\langle\mathbf{\Omega}, \mathbf{L}\rangle_I, \qquad I_{ij} = \int \|\mathbf{r}\|^2\delta_{ij} - x_ix_j\,\mathrm{d}m\,, \qquad L_3 = I_1a, L_z = I_1b \tag{7.32}$$

$$\mathbf{F} = -Mg\mathbf{e}_z, \qquad \boldsymbol{\tau} \perp \mathbf{e}_3, \mathbf{e}_z, \qquad U = MgR\cos\theta \tag{7.33}$$

$$T_{\text{rot}} = \frac{I_1}{2}\frac{(L_z - L_3\cos\theta)^2}{I_1^2\sin^2\theta} + \frac{I_1}{2}\dot{\theta}^2 + \frac{L_3^2}{2I_3} = \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1^2a^2}{2I_3} \tag{7.34}$$

$$I_1\ddot{\theta} + I_1\frac{a\sin^3\theta(b - a\cos\theta) - (b - a\cos\theta)^2\sin\theta\cos\theta}{\sin^4\theta} - MgR\sin\theta = 0 \tag{7.35}$$

$$I_1\ddot{\theta} = I_1\dot{\phi}^2\sin\theta\cos\theta - L_3\dot{\phi}\sin\theta + MgR\sin\theta \tag{7.36}$$

$$\dot{\phi}\sin^2\theta = b - a\cos\theta, \qquad \dot{\psi} = \frac{I_1a}{I_3} - \dot{\phi}\cos\theta \tag{7.37}$$

Possible movements with constant angle

If $\theta = ctt$ then $\dot{\phi} = ctt = \Omega_{\text{prec}}$

$$\Omega_{\text{prec}} = \frac{L_3 \pm \sqrt{L_3^2 - 4I_1MgR\cos\theta}}{2I_1\cos\theta} \tag{7.38}$$

$$L_3^2 \gg I_1MgR(T_{\text{rot}} \gg U) \Rightarrow \Omega_{\text{p,fast}} = \frac{L_3}{I_1\cos\theta}, \qquad \Omega_{\text{p,slow}} = \frac{MgR}{L_3} \tag{7.39}$$

If $\theta < \pi/2$ both precessions have the same direction and if $\theta > \pi/2$ they have the opposite.

Determination of movement of the angle through effective potential

$$U_{\text{eff}}(\theta) = MgR\cos\theta + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} \tag{7.40}$$

The movement consists of a variation of θ between two extreme angles θ_1, θ_2. We see $\dot{\phi}$ will change of sign only if $L_z < L_3$. If $\dot{\phi}$ does not get zero in $[\theta_1, \theta_2]$, the shape of the oscillations are like a sinusoidal wave. If it gets null in (θ_1, θ_2), there will be a change in direction at some points and the shape will be like closed curves. Finally, if it gets null at θ_1 we will see a shape similar to $-|\sin x|$. This last case appears when $\dot{\phi} = \dot{\theta} = 0, \theta = \theta_0, \dot{\psi} \neq 0$, which are the common conditions for a toy top.
Stability of vertical top (with $\mathbf{e}_3 = \mathbf{e}_L$)

$$U_{\text{eff}}(\theta) = MgR\cos\theta + \frac{L_3^2}{2I_1}\tan^2\frac{\theta}{2} \approx MgR + \frac{1}{2}\left(\frac{L_3^2}{2I_1} - MgR\right)\theta^2 \tag{7.41}$$

The point $\theta = 0$ is stable if and only if $\theta_3^2 > \theta_0^2 = 4I_1 MgR/I_3^2$. Otherwise, it will fall until $\cos\theta/2 = \Omega_3/\Omega_0$. If $\Omega_3 \to 0$, $\theta_2 \to \pi$ and the top becomes a pendulum.

7.2.8   Euler equations

Application of Euler equations to Symmetric top

$$\Omega_1(t) = A\cos(-\omega_{\text{rot}}t + \varphi_0), \qquad \Omega_2(t) = A\sin(-\omega_{\text{rot}}t + \varphi_0), \qquad \Omega_3 = \text{ctt} \tag{7.42}$$

$$\Omega_1^2 + \Omega_2^2 = \text{ctt}, \qquad \arg(\mathbf{L}, \mathbf{e}_3) = \theta = \text{ctt}, \qquad \dot{\psi} = \omega_{\text{rot}}, \qquad \dot{\phi} = \Omega_{\text{prec}} = \frac{L}{I_1} \tag{7.43}$$

Stability of rotation with respect a principal axis

There can only be a stationary rotation without external forces if it is around a principal axis. This rotation will be stable if

$$(I_3 - I_2)(I_3 - I_1) > 0. \tag{7.44}$$

Chapter 8

Special relativity

8.1 Introduction

8.1.1 Problems in classical mechanics

Theorem 8.1.1. *If Maxwell's equations*

$$\langle \boldsymbol{\nabla}, \mathbf{E} \rangle_I = \frac{\rho}{\epsilon_0}, \tag{8.1}$$

$$\langle \boldsymbol{\nabla}, \mathbf{B} \rangle_I = 0, \tag{8.2}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{8.3}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \tag{8.4}$$

are invariant under Galileo transformations, then $\mathbf{E} = \mathbf{B} = \mathbf{0}$.

Proof.

$$\begin{cases} x^1 = x'^1 + wt' \\ x^2 = x'^2 \\ x^3 = x'^3 \\ t = t' \end{cases}$$

From this,

$$\frac{\partial}{\partial t'} = \frac{\partial x^1}{\partial t'} \frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial t'} \frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial t'} \frac{\partial}{\partial x^3} + \frac{\partial t}{\partial t'} \frac{\partial}{\partial t'} = w \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial x'^i} = \frac{\partial}{\partial x^i}.$$

Notice we have omitted the dependence of x'^i of t' since they act (in these relations) as independent variables. ∎

8.1.2 First attempt of solving the problem

Definition 8.1.1 (Reference system)**.** We define a *reference system* S as a set of three axis and one origin over which we have determined an orientation. We will suppose we have selected a unit of length and that in each point a in the immobile space with respect the axis there is a clock q_a such that the clocks q_a and q_b corresponding to two different points a and b immobile with respect these axis are synchronized [1].

Definition 8.1.2. Let S and S' two reference systems. We say S' *is moving in a uniform straight motion* if and only if there exists a vector $\mathbf{v} \in \mathbb{R}^3$ such that every particle a immobile with respect to S' moves in each instant t of the time of S to the point of coordinates $(a^1(t), a^2(t), a^3(t))$ with respect the axis of S, such that $\mathrm{d}a^i / \mathrm{d}t = v^i$ for $i = 1, 2, 3$ [1].

Definition 8.1.3. Let S, S' two reference systems. We say they are *equivalent* if and only if they are immobile one with respect the other and use the same unit of length and scale of time, or satisfy the following conditions [1].

1. Each one of them is moving with respect to the other with a uniform line motion with $\mathbf{v} \neq \mathbf{0}$.

2. Two events occur in an immobile point with respect to s are seen from S' with the same temporal order as that seen from S. The same happens exchanging the roles of S and S'.

3. We make the following experiment: from S' two events are observed in an immobile point a with respect to S and separated by an interval of temps of S of length 1. Since S is moving with respect to S', these events will be seen in two different positions from S'. From S' it is measured the distance between these two points. The condition we are enunciating is the result is the same exchanging the roles of S and S'.

The two postulates we will use are stated as follows.

Axiom 1 (Relativity principle). *There exists a privileged class of reference systems, each one called inertial system, that satisfies 4 properties [1].*

1. *If S is inertial, it is inertial every other system S' immobile with respect to S obtained from S by a displacement (in a mathematical sense) of the axis and a change of origin of time.*

2. *If S is inertial, and P is a point that moves with respect to S in a uniform line motion and velocity **v**, there exists a reference system that has P as the origin.*

3. *Two inertial systems S and S' are always equivalent.*

4. *The observers that travel in any inertial reference system can state the physical laws in the same way, independently of the system where we are working.*

Axiom 2 (Invariance of light velocity). *Let S be a reference system [1]. Then,*

1. *If an immobile observer with respect to an inertial reference system S contemplates the propagation of light in the vacuum emitted by any focus, the observer will see it propagates with a uniform line motion.*

2. *The speed of such ray of light is constant, independent on the inertial reference system S. We denote by c this speed.*

Notice the last part of the relativity principle is what makes a distinction from the previous framework, where the electromagnetic equations change depending on the reference system.

Lemma 8.1.2. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms lines that are not contained in hyperplanes of the form $t = $ ctt to lines that are not contained in hyperplanes of the form $t' = $ ctt.*

Lemma 8.1.3. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms planes that are not contained in hyperplanes of the form $t = $ ctt to planes that are not contained in hyperplanes of the form $t' = $ ctt.*

Lemma 8.1.4. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f transforms hyperplanes that are not of the form $t = $ ctt to hyperplanes that are not of the form $t' = $ ctt.*

Theorem 8.1.5. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f is an affine transformation.*

Proof. We use the fundamental theorem of affine geometry. ■

Corollary 8.1.6. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation and H an hyperplane of the form $t = $ ctt. Then, $f(H)$ is an hyperplane of the form $t' = $ ctt.*

We can use both axioms to obtain the Lorentz transformations.

$$\begin{cases} x'^1 = \gamma_v x^1 - \gamma_v v t \\ x'^2 = x^2 \\ x'^3 = x^3 \\ t' = -\gamma_v \dfrac{v}{c^2} x^1 + \gamma_v t \end{cases} \tag{8.5}$$

Theorem 8.1.7. *Let S, S' be two inertial reference systems. We can make orthogonal changes (isometries) of axis to S and S' and a change of origin of time such thate the Lorentz transformation has the form of the equation 8.5.*

We see there is an interpretation of v, the velocity of S'. However, we have imposed that $v < c$, which is a contradiction to the statement 2 of the first axiom. Then, we need to remake the postulates.

8.1.3 Second attempt of solving the problem

The final form of the special relativity postulates is the following one.

Axiom 1 (Relativity principle and invariance of speed of light)**.** There exists a privileged class of reference systems, each one called *inertial system*, and a physical positive constant c with the dimensions of velocity, that satisfies the following properties [1].

1. If S is inertial, it is inertial every other system S' immobile with respect to S obtained from S by a displacement (in a mathematical sense) of the axis and a change of origin of time.

2. If S is inertial, and P is a point that moves with respect to S in a uniform line motion and velocity $\mathbf{v}$ with $v < c$, there exists a reference system that has P as the origin.

3. Two inertial systems S and S' are always equivalent.

4. The observers that travel in any inertial reference system can state the physical laws in the same way, independently of the system where we are working.

5. If an immobile observer with respect to an inertial reference system S contemplates the propagation of light in the vacuum emitted by any focus, the observer will see it propagates with a uniform line motion.

6. The speed of such ray of light is constant, independent on the inertial reference system S. We denote by c this speed.

Lemma 8.1.8. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation and $r \subseteq \mathbb{R}^4$ a line with a timelike direction vector. Then, f transforms r to a line with a timelike direction vector.*

Definition 8.1.4. Let $\pi \subseteq \mathbb{R}^4$ be a plane (hyperplane). We say it is *admissible* if and only if it has the form $a + H$ with H being a subspace of dimension 2 (3) that admits a basis of vectors of C^+.

Lemma 8.1.9. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation and V and admissible plane (hyperplane). Then, $f(V)$ is an admissible plane (hyperplane).*

Theorem 8.1.10. *Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a Lorentz transformation. Then, f is an affine transformation.*

Now we have a coherent set of assumptions, we can state the transformation theorem (using the suitable set of axis)

Theorem 8.1.11. *(Lorenz transformation) Let S, S' be two reference systems with the same origin such that S' moves with a constant velocity $\mathbf{v} = v\mathbf{e}_x$. Then,*

$$P_{s'} = \Lambda P_s \Leftrightarrow P_{s'}^\nu = \Lambda_\mu^\nu P_s^\mu, \qquad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.6}$$

Knowing this transformation we will proceed to study a structure of the spacetime that will be useful with this new kind of transformations.

8.2 Lorentz-Minkowski space-time

Definition 8.2.1. A Lorentz space is an orthogonal geometry $(L, \langle, \rangle)$ with Sylvester invariants $r_0 = 0$ and $r_- = 1$.

We will take the general case with $L = \mathbb{R}^{n+1}$, with the first component corresponding to the temporal component and the rest the spacial one.

Definition 8.2.2. Let $\mathbf{R} \in L$ be a vector. We define is norm as

$$\|\mathbf{R}\|_m := \sqrt{|\langle \mathbf{R}, \mathbf{R} \rangle_m|}. \tag{8.7}$$

Proposition 8.2.1. *Let $\mathbf{R} \in L$ be a vector and $a \in \mathbb{R}$ a scalar. Then, $\|a\mathbf{R}\|_m = |a|\|\mathbf{R}\|_m$.*

Henceforth we will use an orthonormal basis such that

$$\langle \mathbf{e}_0, \mathbf{e}_0 \rangle_m = 1, \qquad \langle \mathbf{e}_i, \mathbf{e}_i \rangle_m = 1, 1 \leq i \leq n. \tag{8.8}$$

This way, the matrix associated has the form

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}. \tag{8.9}$$

Definition 8.2.3. Let $\mathbf{R} \in L$ be a vector different from $\mathbf{0}_L$.

1. We say $\mathbf{R}$ *is timelike* if and only if $\langle \mathbf{R}, \mathbf{R} \rangle_m < 0$.

2. We say $\mathbf{R}$ *is spacelike* if and only if $\langle \mathbf{R}, \mathbf{R} \rangle_m > 0$.

3. We say $\mathbf{R}$ *is lightlike* if and only if $\langle \mathbf{R}, \mathbf{R} \rangle_m = 0$.

Using the orthogonal basis, these three conditions are translated to the following respective equations.

$$\|\mathbf{r}\| < c^2 t^2, \qquad \|\mathbf{r}\| > c^2 t^2, \qquad \|\mathbf{r}\| = c^2 t^2. \tag{8.10}$$

The set T of timelike vectors can be expressed as the disjoint union of two sets of vectors called *light cones.* The first one is the *future* and the second the *past.*

$$T = C_+ \cup C_-, \qquad C_+ := \{(ct, x_1, \ldots, x_n) \in T \mid t > 0\}, \qquad C_- := \{(ct, x_1, \ldots, x_n) \in T \mid t < 0\}. \tag{8.11}$$

Definition 8.2.4. Let $W \subseteq L$ be a subspace.

1. We say W is *timelike* if and only if $(W, \langle, \rangle)$ is a Lorentz space.

2. We say W is *spacelike* if and only if $(W, \langle, \rangle)$ is an Euclidean space.

3. We say W is *lightlike* if and only if $(W, \langle, \rangle)$ is singular.

Proposition 8.2.2. *Every subspace W of L is either timelike, spacelike, or lightlike. Besides,*

1. *S is timelike $\Leftrightarrow W^\perp$ is spacelike.*

2. *S is spacelike $\Leftrightarrow W^\perp$ is timelike.*

3. *W is lightlike $\Leftrightarrow W^\perp$ is lightlike.*

Some observations

1. A vector $\mathbf{R}$ different from the identity has the same causal character as the subspace $\langle \mathbf{R} \rangle_{\mathbb{R}}$.

2. Every vector different from the identity that is included in a spacelike subspace is spacelike.

3. Lightlike subspaces can contain spacelike vectors.

Proposition 8.2.3. *Two orthogonal vectors different from zero and non spacelike are necessarily lightlike and collinear. In particular, there is not a subspace of dimension 2 where $\langle, \rangle$ is null.*

Corollary 8.2.4. *Let $W \subseteq L$ be a subspace of dimension 2. Then, the following statements are equivalent.*

1. *W is timelike.*

2. *W contains two lightlike vectors that are linearly independent.*

  3. W contains a timelike vector.

Corollary 8.2.5. *Let $W \subseteq L$ be a subspace. Then, the following statements are equivalent.*

  1. W is lightlike.

  2. W contains lightlike vectors and no timelike vectors.

  3. If we add the vector $\mathbf{0}_L$ to the set of isotrop vectors of W, we get a one dimensional subspace.

Proposition 8.2.6. *Let $\mathbf{R}_1, \mathbf{R}_2 \in T$ be two timelike vectors. Then, the following statements are true.*

  1. $|\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m| \geq \|\mathbf{R}_1\|_m \|\mathbf{R}_2\|_m$, and the equality is equivalent to both vectors being collinear.

  2. $\mathbf{R}_1, \mathbf{R}_2$ are in the same time cone (C_+ or C_-) if and only if $\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m < 0$. In this case,

  (a) There is a unique $\varphi \in \mathbb{R}$ such that

$$\cosh \varphi = -\frac{\langle \mathbf{R}_1, \mathbf{R}_2 \rangle_m}{\|\mathbf{R}_1\|_m \|\mathbf{R}_2\|_m}. \tag{8.12}$$

  We call this φ the hyperbolic angle.
  (b) $\|\mathbf{R}_1\|_m + \|\mathbf{R}_2\|_m \leq \|\mathbf{R}_1 + \mathbf{R}_2\|_m$.

8.3   Space-time and special relativity

We will consider now a particular case of the Lorenz space, given by $L = \mathbb{R}^4$ and the symmetric bilinear for $\langle, \rangle_m$ with the matrix $M = \eta$. The set of spacelike and timelike vectors are thus

$$E = \left\{ (x, y, z, ct) \in L \,\middle|\, x^2 + y^2 + z^2 > c^2 t^2 \right\}, \qquad T = \left\{ (x, y, z, ct) \in L \,\middle|\, x^2 + y^2 + z^2 < c^2 t^2 \right\}. \tag{8.13}$$

Definition 8.3.1. As a model of space-time, we consider the affine space $\mathbb{M} = \mathbb{R}^4$ over the Lorenz vector space L.

The operation $P + \mathbf{R}$ of the sum of point with vectors is the typical sum used in $\mathbb{R}^4$.

Definition 8.3.2. We define and *event* as a point that has location in $\mathbb{M}$.

Example 8.3.1. A point particle that starts from a point a with a uniform velocity $\mathbf{v}$ follows the trajectory given by

$$p = a + (t - t_0)\mathbf{v}, \qquad (x, y, z, t) = (ct_0, a^1, a^2, a^3) + (t - t_0)(c, v^1, v^2, v^3). \tag{8.14}$$

Note that, since the norm of $\mathbf{v}$ is less than c, the set $P + C_+$ is bounded. Hence we can define the following concept.

Definition 8.3.3. Let $P \in \mathbb{M}$ be an event and C_+, C_- be the past an future cones. Then, we define the *future cone of P* and the *past cone of P* as the sets $P + C_+$ and $P + C_-$.

8.3.1   Lorentz transformations

In order to preserve the two axioms of special relativity, the Galileo transformation must be substituted by another one.

Proposition 8.3.1. *The Lorentz-Minkowski metric (using the proper orthonormal basis) can be expressed by the bilinear form η*

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.15}$$

Proposition 8.3.2. *The transformation in $\mathbb{M} = \mathbb{R}^4$ (using the proper orthonormal basis) can be expressed by the matrix Λ*

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.16}$$

Proposition 8.3.3. *The Lorenz-Minkowski metric is invariant under Lorentz transformations.*

Corollary 8.3.4. *The norm of the vector of the Minkowski space are independent on the inertial reference system.*

Now we have developed the mathematical tools for our study, we can start by analyzing the kinematics.

8.4 Kinematics

8.4.1 Space

Definition 8.4.1. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a parametrization $\gamma(t) : I \longrightarrow \Gamma$ of class $C^2(I)$ and a tangent vector $u \in C^+$ in an inertial reference system S. We define the *4-position vector* as

$$\mathbf{X} = (ct, x, y, z). \tag{8.17}$$

Definition 8.4.2. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a piece-wise differentiable parametrization $\gamma(t) : I \longrightarrow \Gamma$ and tangent vector $u \in C^+$ in an inertial reference system S. We define the *space-time interval* as

$$\Delta s^2 := -\Delta\mathbf{R}^t \boldsymbol{\eta} \Delta\mathbf{R} = -\langle \Delta\mathbf{R}, \Delta\mathbf{R} \rangle_m = -\langle \Delta\mathbf{R}', \Delta\mathbf{R}' \rangle_m, \tag{8.18}$$

or as a differential form,

$$\mathrm{d}s^2 := \mathrm{d}x^\mu \, \eta_{\mu\nu} \, \mathrm{d}x^\nu = c^2 \, \mathrm{d}t^2 - \langle \mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{r} \rangle_I. \tag{8.19}$$

8.4.2 Time

Axiom 3. *Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a piece-wise differentiable parametrization $\gamma(t) : I \longrightarrow \Gamma$ and tangent vector $u \in C^+$ in an inertial reference system S. If the motion starts at an event A and ends at an event B, then the time interval observed by a system S' that moves along p is given by*

$$\tau = \frac{1}{c} \int_\Gamma \sqrt{-\mathrm{d}x^\mu \, \eta_{\mu\nu} \, \mathrm{d}x^\nu} = \int_{t_A}^{t_B} \frac{1}{\gamma(t)} \, \mathrm{d}t \Leftrightarrow c \, \mathrm{d}\tau = \mathrm{d}s. \tag{8.20}$$

Definition 8.4.3. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a piece-wise differentiable parametrization $\gamma(t) : I \longrightarrow \Gamma$ and tangent vector $u \in C^+$ in an inertial reference system S. We define the *proper time* of p as

$$\mathrm{d}\tau := \frac{1}{c} \, \mathrm{d}s = \frac{1}{\gamma} \, \mathrm{d}t. \tag{8.21}$$

8.4.3 Velocity and acceleration

Proposition 8.4.1. *Let S, S' be two inertial reference systems such that the velocity of S' is $\mathbf{w} = w\mathbf{e}_x$ with respect to S. Then,*

$$v_{s'}^1 = \frac{v_s^1 - w}{1 - \beta_v \beta_w}, \qquad v_{s'}^2 = \frac{1}{\gamma_w} \frac{v_s^2}{1 - \beta_v \beta_w}, \qquad v_{s'}^3 = \frac{1}{\gamma_w} \frac{v_s^3}{1 - \beta_v \beta_w}, \tag{8.22}$$

Proposition 8.4.2. *If the system has a general velocity* **w**, *then*

$$\mathbf{v}' = \frac{1}{1 - \langle \boldsymbol{\beta}_v, \boldsymbol{\beta}_2 \rangle_I} \left[\frac{\mathbf{v}}{\gamma_w} - \mathbf{w} + \frac{1}{c^2} \frac{\gamma_w}{1 + \gamma_w} \langle \mathbf{v}, \mathbf{w} \rangle_I \mathbf{w} \right], \tag{8.23}$$

$$\mathbf{v} = \frac{1}{1 + \langle \boldsymbol{\beta}_{v'}, \boldsymbol{\beta}_w \rangle_I} \left[\frac{\mathbf{v}'}{\gamma_w} + \mathbf{w} + \frac{1}{c^2} \frac{\gamma_w}{1 + \gamma_w} \langle \mathbf{w}', \mathbf{w} \rangle_I \mathbf{v} \right]. \tag{8.24}$$

The minimal velocity to consider a motion relativistic is with $\beta = 0,7$ and $\beta = 0,8$. Another convenient condition is with $\gamma \beta^2 = 1$ ($\beta = 0,786$).

Definition 8.4.4. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a parametrization $\gamma(t) : I \longrightarrow \Gamma$ of class $C^2(I)$ and a tangent vector $u \in C^+$ in an inertial reference system S. We define the *4-velocity* as

$$\mathbf{U} := \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau}. \tag{8.25}$$

Proposition 8.4.3. *Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a 4-velocity* **U**. *Then,*

$$\mathbf{U} = (\gamma c, \gamma \mathbf{v}), \qquad -\langle \mathbf{U}, \mathbf{U} \rangle_m = c^2. \tag{8.26}$$

Definition 8.4.5. Let p be a particle moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a parametrization $\gamma(t) : I \longrightarrow \Gamma$ of class $C^2(I)$ and a tangent vector $u \in C^+$ in an inertial reference system S. We define the *4-acceleration* as

$$\mathbf{A} := \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 \mathbf{X}}{\mathrm{d}\tau^2}. \tag{8.27}$$

Proposition 8.4.4. *Let p be a particle of velocity* **U** *and acceleration* **A**. *Then,* $\langle \mathbf{A}, \mathbf{U} \rangle_m = 0$.

8.5   Momentum and energy

Definition 8.5.1. Let p be a particle of mass m moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a parametrization $\gamma(t) : I \longrightarrow \Gamma$ of class $C^2(I)$ and a tangent vector $u \in C^+$ in an inertial reference system S. We define the *4-momentum* as

$$\mathbf{P} := m\mathbf{U}. \tag{8.28}$$

Proposition 8.5.1. *Let p be a particle of 4-momentum* **P**. *Then,*

$$\mathbf{P} = (E/c, \mathbf{p}) = (\gamma mc, \gamma m\mathbf{v}). \tag{8.29}$$

Theorem 8.5.2.

$$E^2 = p^2 c^2 + m^2 c^4. \tag{8.30}$$

Proof. Since the norm of the 4-momentum tensor is invariant under Lorentz transformations, we can take two reference frames: where the particle is moving and where not.

$$\|\mathbf{P}\|_m^2 = -\frac{E^2}{c^2} + \|\mathbf{p}\|^2 = (mc^2)^2 \Rightarrow E^2 = m^2 c^4 + \|p\|^2 c^2 \tag{8.31}$$

∎

Proposition 8.5.3. *There are three possible cases: stationary particle with mass, moving particle with mass, particle with no mass.*

$$E = mc^2, \qquad E^2 = m^2 c^4 + \|p\|^2 c^2, \qquad E = pc \tag{8.32}$$

Theorem 8.5.4. *(Work-Energy theorem)*

$$W = \Delta E. \tag{8.33}$$

Proof.

$$W = \int_\Gamma \langle \mathbf{f}, \mathrm{d}\mathbf{r} \rangle_I = \int_\Gamma \langle \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \mathrm{d}\mathbf{r} \rangle_I = \int_{t_0}^{t_\mathrm{f}} \langle \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \rangle_I \mathrm{d}t = \int_{t_0}^{t_\mathrm{f}} \langle \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \mathbf{v} \rangle_I \mathrm{d}t =$$

$$\int_{t_0}^{t_\mathrm{f}} \langle \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m \mathbf{v}), \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \rangle_I \mathrm{d}t = \int_{t_0}^{t_\mathrm{f}} \langle \frac{m\mathbf{v}}{(1-\beta^2)^{3/2}}, \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \rangle_I \mathrm{d}t = \int_{t_0}^{t_\mathrm{f}} \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m c^2) \,\mathrm{d}t = \int_{t_0}^{t_\mathrm{f}} \frac{\mathrm{d}E}{\mathrm{d}t} \,\mathrm{d}t = E_\mathrm{f} - E_0.$$

∎

Theorem 8.5.5. *Let p be a particle of velocity $\mathbf{v}$. Then, the kinetic energy is obtained by the expression*

$$T = \int_\Gamma \langle \mathbf{F}, \mathrm{d}\mathbf{r} \rangle_I = (\gamma(\dot{\mathbf{r}}) - 1)mc^2 \tag{8.34}$$

Theorem 8.5.6 (Compton scattering).

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta). \tag{8.35}$$

Theorem 8.5.7 (Center of momentum). *Let be a system of particles $p_1, \dots, p_n$ with energies $E_1, \dots, E_n$ and momentum $\mathbf{p}_1, \dots, \mathbf{p}_n$. Then, the center of momentum system has a velocity determined by the expression*

$$\mathbf{v}_\mathrm{cp} = \frac{1}{E_t} \sum_{i=1}^{n} \|\mathbf{p}_i\|^2 c^2. \tag{8.36}$$

8.6 Force

Definition 8.6.1. The relativistic *4-force*

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\tau}, \qquad \mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{8.37}$$

Proposition 8.6.1. *Let p be a particle of mass m moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Then,*

$$\langle \mathbf{f}, \mathbf{v} \rangle_I = \frac{\mathrm{d}E}{\mathrm{d}t} = m\gamma^3 \langle \mathbf{v}, \mathbf{a} \rangle_I, \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\gamma\mathbf{a} + m\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v}, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\gamma^3}{c^2}\langle \mathbf{v}, \mathbf{a} \rangle_I \tag{8.38}$$

$$\mathbf{F} = \left(\frac{\gamma}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, \gamma\mathbf{f} \right) = \left(\frac{m\gamma^4}{c}\langle \mathbf{v}, \mathbf{a} \rangle_I, m\gamma^2\mathbf{a} + m\frac{\gamma^4}{c^2}\langle \mathbf{v}, \mathbf{a} \rangle_I \mathbf{v} \right) \tag{8.39}$$

Proposition 8.6.2. *Let p be a particle of mass m moving along a curve $\Gamma \subseteq \mathbb{R}^4$ with a velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Then,*

1. *If $\mathbf{f} \parallel \mathbf{v}$, then*

$$f = m\gamma^3 a, \qquad m\gamma^3 := longitudinal \ mass. \tag{8.40}$$

2. *If $\mathbf{f} \perp \mathbf{v}$, then*

$$f = m\gamma a, \qquad m\gamma := transverse \ mass. \tag{8.41}$$

8.7 Relativistic motion

8.7.1 Movement over constant force

Theorem 8.7.1. *Let p be a particle of mass m with $v_0 = x_0 = t_0 = 0$ on which a constant force F acts. If we denote $a_0 = \gamma^3 a = F/m$ (which is constant), then*

$$x(t) = \frac{c^2}{a_0}\left[\sqrt{1 + \frac{a_0^2 t^2}{c^2}} - 1 \right], \qquad v(t) = \frac{a_0 t}{\sqrt{1 + a_0^2 t^2/c^2}}, \tag{8.42}$$

and in the limit cases,

$$t \to \infty : v(t) \approx c, x(t) \approx ct - \frac{c^2}{a_0}, \tag{8.43}$$

$$a_0 t \ll c : v(t) \approx a_0 t, x(t) \approx \frac{a_0}{2} t^2. \tag{8.44}$$

8.7.2   Motion under Coulomb potential

We will study an attractive Coulomb potential so $U = k/r, F = k/r^2$ with $k < 0$.

Theorem 8.7.2. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$ and $x_0 = -k/mc^2, t_0 = 0$, then*

$$E = \gamma mc^2 + \frac{k}{x} = 0, \qquad \omega^2 x^2 + v^2 = c^2, \tag{8.45}$$

$$\omega = \frac{c}{x_0}, \qquad T = \frac{\pi}{2} \frac{x_0}{c}, \tag{8.46}$$

$$x(t) = x_0 \cos \omega t, \qquad v(t) = -c \sin \omega t, \tag{8.47}$$

while n classical formalism the period would be $\frac{\pi}{2\sqrt{2}} \frac{x_0}{c} < T_{\mathrm{rel}}$, which is normal since in the classical framework the mass can have an infinite velocity while in the relativistic one is bounded by c.

Theorem 8.7.3. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$ and $x_0 = -k/mc^2, t_0 = 0$, then with respect to the particle reference,*

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\gamma} = \cos \omega t = \frac{x}{x_0}, \qquad \tau = \int \gamma \, \mathrm{d}t = \frac{1}{\omega} \sin \omega t, \tag{8.48}$$

$$x^2 = x_0^2 - c^2 \tau^2, \qquad T' = \frac{x_0}{c}, \qquad \frac{L'}{x_0} = \frac{T}{T'}. \tag{8.49}$$

Theorem 8.7.4. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k < 0$, then*

$$\mathbf{F} \perp \mathbf{v}, \qquad \gamma m \beta^2 c^2 = -\frac{k}{r},$$

$$-1 < \frac{T}{U} = -\frac{\gamma}{\gamma + 1} < -\frac{1}{2}, \qquad E = \frac{mc^2}{\gamma}.$$

Theorem 8.7.5. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k > 0$, then it is not possible falling to the origin, and if $k < 0$, then it is possible if $Lc \leq k$ (in this case $p_r \to \infty$).*

Theorem 8.7.6. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If $k > 0$, then it is always possible to escape to the infinity is always possible, and if $k < 0$, it is possible if $E > mc^2$.*

Theorem 8.7.7. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. Then,*

$$L = \gamma m r^2 \dot{\theta} = \mathrm{ctt}, \qquad E = \gamma mc^2 + \frac{k}{r} = \mathrm{ctt}, \qquad \mathbf{p} = \gamma m (\dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta), \tag{8.50}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (\gamma m \dot{r}) - \frac{L^2}{\gamma m r^3} = \frac{k}{r^2}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \left(\frac{1}{r} \right) + \frac{1}{r} = -\frac{\gamma m k}{L^2}, \tag{8.51}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \left(\frac{1}{r} \right) + (1 - \alpha^2) \frac{1}{r} = -\frac{kE}{L^2 c^2}, \qquad \alpha^2 = \frac{k^2}{L^2 c^2}. \tag{8.52}$$

We will now focus on finding the solution to this equation

Proposition 8.7.8. *Let p be a particle of mass m on which a potential of the form $U = k/r$ acts. If the variation of r is negligible, then $\alpha^2 < 1$.*

Proposition 8.7.9. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If $\alpha^2 <$, $E < mc^2$, and $E > 0$, then the trajectory of p is bounded.*

Theorem 8.7.10. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If $\alpha^2 < 1$, $E < mc^2$, and $E > mc^2\sqrt{1 - \alpha^2}$, then the trajectory of p is determined by the expression*

$$r = \frac{a(1 - e^2)}{1 + e\cos\left(\sqrt{1 - \alpha^2}\,\theta\right)}, \tag{8.53}$$

$$\frac{1}{a} = \frac{E}{k}\left[1 - \frac{m^2c^4}{E^2}\right], \quad e = \frac{1}{\alpha}\sqrt{1 + (\alpha^2 - 1)\frac{m^2c^4}{E^2}}, \tag{8.54}$$

which is an ellipse with a precession $2\pi(1/\sqrt{1 - \alpha^2} - 1)$ per revolution (and $\pi\alpha^2$ if $\alpha^2 \ll 1$).

Theorem 8.7.11. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If p has a closed bounded trajectory, then the average of $\frac{\mathrm{d}\langle \mathbf{r}, \mathbf{p}\rangle_I}{\mathrm{d}t} = 0$ on an interval of nT.*

Proposition 8.7.12. *Let p be a particle of mass m on which a potential of the form $U = k/r$ (with $k < 0$) acts. If p has a closed bounded trajectory, then*

$$E = \left\langle \frac{1}{\gamma} \right\rangle mc^2. \tag{8.55}$$

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Chapter 9

Generalized coordinates

9.1   Introduction

We can characterize the system by n variables $q_1, \ldots, q_n$, which we can see as the n components of a n-dimensional point in a bigger space. This is the *configuration space*, and the movement of this general point represents the behavior of the whole system. This simplifies the problem extremely for some particular cases. For instance, a rigid body needs only six degrees of freedom, so once we determine them as a function of time, a six dimensional point can represent the whole body. We call this point *C-point* and its path on the space *C-curve*.

While changing one set of coordinates to another, the geometry is distorted, but other important properties remain. In particular, properties of space like area, distances, and angles are modified, bu other topological properties are the same: a point remains a point, a curve remains a curve, adjacent curves remain adjacent curves, a neighborhood remains a neighborhood, continuous differentiable curves remain continuous differentiable curves.

This can be seen too as a transformation, where each new coordinate q_i' is a function of all the others from the first set $f_i(x_1, \ldots, x_n)$. These functions must be finite, single valued and with non zero jacobian. This mapping has interesting properties in infinitesimal regions: a straight line remains a straight line, parallel lines remain parallel lines, a parallelepiped remains a parallelepiped. Geometrically, the determinant (the jacobian) represents the ratio between the new differential of volume with respect the original one.

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This is strongly related to Riemannian geometry, which is based on the concept of the line element $\mathrm{d}\bar{s}$, defined as distance between two neighboring points, expressed in terms of coordinate systems. For a general set of coordinates x_1, x_2, x_3 it satisfies

$$\mathrm{d}\bar{s}^2 = \begin{pmatrix} \mathrm{d}x_1 & \mathrm{d}x_2 & \mathrm{d}x_3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{22} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{pmatrix}, \tag{9.1}$$

and it is independent on the coordinates, that is, if we select x_1', x_2', x_3', the value will be the same (g_{ij} are not constant, change in space). This forms a tensor of second order. In particular is an invariant tensor and, if we add that $g_{ij} = g_{ji}$, then we have a symmetric invariant tensor of second order.

9.1.1   Kinetic energy and two approaches

We know

$$T = \sum_{i=1}^{N} \frac{m_i v_i^2}{2}$$

with $\mathbf{v} = \mathrm{d}\bar{s}/dt$. If we consider now a single particle of mass $m = 1$ with the same kinetic energy,

$$T = \frac{1}{2} \left(\frac{\mathrm{d}\bar{s}}{\mathrm{d}t} \right)^2 \Rightarrow \mathrm{d}\bar{s}^2 = 2Tdt = \sum_{i=1}^{N} m_i v_i^2 dt^2 = \sum_{i=1}^{N} m_i (\mathrm{d}x_i{}^2 + \mathrm{d}y_i{}^2 + \mathrm{d}z_i{}^2) =$$

$$\sum_{i=1}^{N} \left(\sqrt{m_i}\, \mathrm{d}x_i \right)^2 + \left(\sqrt{m_i}\, \mathrm{d}y_i \right)^2 + \left(\sqrt{m_i}\, \mathrm{d}z_i \right)^2.$$

If we take each term as a coordinate, we have that $\mathrm{d}\bar{s}$ has 3N components, so the total energy can be seen as the energy of a single particle moving in 3N dimensions. Although the expression of $\mathrm{d}\bar{s}$ is in a Riemannian general form, the geometry is euclidean (even changing the coordinates). If there are now some constraints of the form

$$f_1(x_1, \ldots, z_N) = 0$$

$$\vdots = 0$$

$$f_m(x_1, \ldots, z_N) = 0.$$

Then, there are $n = 3N - m$ independent variables. These form subspaces where the C-point is constrained, and these hypersurfaces are not necessarily euclidean. If we started by expressing the

system directly from the independent variables $q_1, \ldots, q_n$, since these make the C-point line in a non necessarily euclidean space, the differential of length now is expressed in a general geometry (before was euclidean) determined by

$$d\bar{s}^2 = \sum_{i,j=1}^{n} a_{ij}\, dq_i\, dq_j. \tag{9.2}$$

With that, we can transform the problem of a mechanical system to a riemannian geometry problem.

Definition 9.1.1. Let S be a system of particles $p_1, \ldots, p_n$ with masses $m_1, \cdots, m_n$. Then, we say the system has *non stationary holonomic constraints* or *rheonomic constraints* if and only if there is a function $\mathbf{f} : \mathbb{R}^{3n} \times \mathbb{R} \longrightarrow \mathbb{R}^k$ such that

$$\mathbf{f}(\mathbf{r}_1, \ldots, \mathbf{r}_n, t) = \mathbf{0}. \tag{9.3}$$

Definition 9.1.2. Let S be a system of particles $p_1, \ldots, p_n$ with masses $m_1, \cdots, m_n$. Then, we say the system has *stationary holonomic constraints* or *scleronomic constraints* if and only if there is a function $\mathbf{f} : \mathbb{R}^{3n} \longrightarrow \mathbb{R}^k$ such that

$$\mathbf{f}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \mathbf{0}. \tag{9.4}$$

Definition 9.1.3. Let S be a system of n particles. We say the system is *free* if and only if there is no constraints.

If the function $\mathbf{f}$ is differentiable and the jacobian matrix at the points $\mathbf{x}_0$ where $\mathbf{f} = \mathbf{0}$ has rank k, the equation defines a differentiable manifold of dimension $r = 3n - k$ for time independent constraints and $r = 3n + 1 - k$ for time dependent. By the Implicit function theorem, there exists a neighborhood where $\mathbf{x}$ can be expressed by r components or any other set of independent variables. We call them the *degrees of freedom*.

Definition 9.1.4. Let S be a system of n particles and an holonomic constraint $\mathbf{f} = \mathbf{0}$, where $\mathbf{f}$ is a differentiable function. Then, we define the *system positions manifold* as the manifold M determined by the constraint.

Definition 9.1.5. Let S be a system of n particles and M be the system positions manifold of dimension r. We define the *set of possible velocities at the instant t* as the set V that contains $\dot{\mathbf{x}}$, where $\mathbf{x} \in M$.

Proposition 9.1.1. *Let S be a system of n particles with a constraint $\mathbf{f} : \mathbb{R}^{3n} \times \mathbb{R} \longrightarrow \mathbb{R}^k$ and V the set of possible velocities at an instant t. If $\dot{\mathbf{x}} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, then*

$$\sum_{i=1}^{n} \langle \boldsymbol{\nabla}_i f_j, \mathbf{v}_i \rangle_I + \frac{\partial f_j}{\partial t} = 0, \; j = 1, \ldots, k. \tag{9.5}$$

Definition 9.1.6. A *virtual displacement* $\delta\gamma : I \longrightarrow TM$ corresponding to the variation Γ assigns to every $t \in I$ the tangent vector

$$\delta\gamma = (\delta\mathbf{r}_1, \ldots, \delta\mathbf{r}_2) = \left.\frac{d\Gamma(t, \epsilon)}{d\epsilon}\right|_{\epsilon=0} \in R_{\gamma(t)}M. \tag{9.6}$$

Proposition 9.1.2. *If $\gamma, \frac{d\gamma}{dt} \in P(M)$, then*

$$\delta\frac{d\gamma}{dt} = \frac{d}{dt}\,\delta\gamma. \tag{9.7}$$

Proposition 9.1.3. *Let S be a system of n particles with a constraint $\mathbf{f} : \mathbb{R}^{3n} \times \mathbb{R} \longrightarrow \mathbb{R}^k$ and $T_{\gamma(t)}M$ be the set of infinitessimal virtual displacements. Then,*

$$\sum_{i=1}^{n} \langle \boldsymbol{\nabla}_i f_j, \delta\mathbf{r}_i \rangle_I = 0, \; j = 1, \ldots, k. \tag{9.8}$$

Proposition 9.1.4. *Let S be a system of n particles with a stationary constraint $\mathbf{f} : \mathbb{R}^{3n} \longrightarrow \mathbb{R}^k$. Then, $d\mathbf{r}_i = \delta\mathbf{r}_i, \; i = 1, \ldots, n$.*

9.1.2   D'Alembert's principle

Definition 9.1.7. Let S be a system of n particles. Then, we define the *constraint force ϕ_i* over the particle p_i as

$$\phi_i = m_i\mathbf{a}_i - \mathbf{F}_i. \tag{9.9}$$

Proposition 9.1.5. *Let S be a system of n particles. Then, there exists a constraint f if and only if there exists a constraint force $\phi \neq \mathbf{0}$.*

Theorem 9.1.6 (D'Alembert's principle)**.** *Let S be a system of particles. Then,*

$$\boxed{\sum_{i=1}^{n} \langle \mathbf{F}_i - m\mathbf{a}_i, \delta\mathbf{r}_i \rangle_I = 0, \ \forall\, \delta\mathbf{r}_i\,.} \tag{9.10}$$

Corollary 9.1.7. *Let S be a system of particles in equilibrium. Then,*

$$\sum_{i=1}^{n} \langle \mathbf{F}_i, \delta\mathbf{r}_i \rangle_I = 0, \ \forall\, \delta\mathbf{r}_i\,. \tag{9.11}$$

9.2   Generalized coordinates

Definition 9.2.1. Let M be the system positions manifold of dimension r of a system S. Then, we define the *generalized coordinates* $q_1, \ldots, q_r$ as a set of independent variables such that $\mathbf{x}(q_1, \ldots, q_r) \in M$.

Definition 9.2.2. Generalized force

$$Q_j := \sum_{i=1}^{n} \left\langle \mathbf{F}_i, \frac{\partial\mathbf{r}_i}{\partial q_j} \right\rangle_I. \tag{9.12}$$

Proposition 9.2.1. *Let S be a system of particles Then,*

$$\delta W = \sum_{j=1}^{d} Q_j\, \delta q_j\,. \tag{9.13}$$

Corollary 9.2.2. *Let S be a system of particles in equilibrium. Then, $Q_j = 0$ for all j.*

9.2.1   Euler-Lagrange equations

Proposition 9.2.3.
$$\frac{\partial\mathbf{v}_i}{\partial\dot{q}_j} = \frac{\partial\mathbf{r}_i}{\partial q_j}. \tag{9.14}$$

Theorem 9.2.4. *Let S be a system of n particles with generalized coordinates $q^1, \ldots, q^r$. Then,*

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j, \ j = 1, \ldots, d.} \tag{9.15}$$

And if the forces $\mathbf{F}_i$ are derived from a potential Φ, then

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_j} = 0, \ j = 1, \ldots, d.} \tag{9.16}$$

Proposition 9.2.5. *If the forces are derived from a potential Φ, then the generalized forces Q_j are derived from the same potential Φ. The converse is not generally true.*

Proposition 9.2.6. *Let S be a system of particles with Lagrangian L. Then, $L' = L + \frac{\mathrm{d}F}{\mathrm{d}t}$, with $F(\mathbf{q}, t)$ an arbitrary function, is a Lagrangian.*

Definition 9.2.3. Generalized momentum

$$p_j := \frac{\partial L}{\partial \dot{q}_j}. \tag{9.17}$$

Theorem 9.2.7. *Let S be a system of n particles and $Q_j = Q_j^c + Q_j^{nc}$ the generalized forces, where Q_j^c represents the part derives from a potential Φ and Q_j^{nc} the other part. Then,*

$$\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} = Q_j^{nc}, \ j = 1, \ldots, d. \tag{9.18}$$

Definition 9.2.4. *Let S be a system of n particles and $\mathbf{q} = (q_1, \ldots, q_d)$ a set of generalized coordinates, with $d \le n$. We say q_j is a cyclic coordinate of the system if and only if*

$$\frac{\partial L}{\partial q_j} = 0. \tag{9.19}$$

Proposition 9.2.8. *Let S be a system of n particles with no rheonomic constraints and $\mathbf{q} = (q_1, \ldots, q_d)$ a set of generalized coordinates. Then, q_j is a cyclic coordinate if and only if its generalized momentum p_j is constant over time.*

Proposition 9.2.9. *Let S be a system of n particles and $\mathbf{q} = (q_1, \ldots, q_d)$ a set of generalized coordinates. Then,*

$$\sum_{j=1}^{d} \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T - T_1 - 2T_0. \tag{9.20}$$

Proposition 9.2.10. *Let S be a system of n particles and $\mathbf{q} = (q_1, \ldots, q_d)$ a set of generalized coordinates. Then,*

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(T_1 + 2T_0) - \frac{\mathrm{d}V}{\mathrm{d}t} - \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t} + \sum_{j=1}^{d} Q_j^{nc} \dot{q}_j. \tag{9.21}$$

Theorem 9.2.11. *Let S be a system of n particles and $\mathbf{q} = (q_1, \ldots, q_d)$ a set of generalized coordinates. Then,*

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(T_1 + 2T_0) - \left(\frac{\partial T}{\partial t} - \frac{\partial V}{\partial t}\right) + \sum_{j=1}^{d} Q_j^{nc} \dot{q}_j. \tag{9.22}$$

Theorem 9.2.12. *Let S be a system of n particles. If there are no rheonomic constraints, $\partial_t V = 0$, and $Q_j^{nc} = 0 \ \forall j$, then E is conserved.*

Definition 9.2.5. Let S be a system of n particles. We say the generalized forces are *gyroscopic* if and only if they satisfy

$$\sum_{j=1}^{d} Q_j^{nc} \dot{q}_j = 0. \tag{9.23}$$

Theorem 9.2.13. *Let S be a system of n particles. If there are no rheonomic constraints, $\partial_t V = 0$, the forces are gyroscopic, then E is conserved.*

Proposition 9.2.14. *Let S be a system of n particles. If $\mathbf{F}_i \perp \mathbf{v}_i \ \forall i$ and the constraints are scleronomic, then the generalized forces are gyroscopic.*

9.3 Hamiltonian formulation

Proposition 9.3.1. *Let S be a system of n particles with a constraint $\mathbf{f} = \mathbf{0}$ and $d = 3n - k$ the degrees of freedom. Then, $(t, \mathbf{q}, \dot{q})$ is completely determined if and only if $(t, \mathbf{q}, \mathbf{p})$ is completely determined.*

Definition 9.3.1. Hamiltonian

$$\mathcal{H} := \sum_{j=1}^{d} p_j \dot{q}_j - \mathcal{L}. \tag{9.24}$$

Proposition 9.3.2.

$$\mathcal{H} = E - T_1 - 2T_0. \tag{9.25}$$

Corollary 9.3.3. *Let S be a system of n particles. If there are no rheonomic constraints, then* $\mathcal{H} = E$.

Proposition 9.3.4.

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \sum_{j=1}^{d} Q_j^{\mathrm{nc}} \dot{q}_j - \frac{\partial \mathcal{L}}{\partial t}. \tag{9.26}$$

Proposition 9.3.5. *Let S be a system of n particles with hamiltonian $\mathcal{H}$. If the generelized forces are conservative or gyroscopic and $\partial_t \mathcal{L} = 0$, then $\mathcal{H}$ is conserved.*

9.3.1   Hamilton equations

Theorem 9.3.6. *Let S be a system of n particles with a set of generalized coordinates $\mathbf{q} = (q_1, \ldots, q_d)$. Then,*

$$\boxed{\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j} + Q_j^{\mathrm{nc}}, \qquad j = 1, \ldots, d.} \tag{9.27}$$

Proposition 9.3.7.

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}. \tag{9.28}$$

Theorem 9.3.8 (Hamilton's principle)**.** *Let S be a system of particles and*

$$\mathcal{S}[\mathbf{q}(t)] := \int_{t_0}^{t_1} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \, \mathrm{d}t. \tag{9.29}$$

If there are no rheonomic constraints and the forces are conservatives, then

$$\boxed{\frac{\delta \mathcal{S}}{\delta \mathbf{q}} = 0.} \tag{9.30}$$

9.4   Functional interpretation

9.4.1   Holonomic and non holonomic constraints

Theorem 9.4.1. *Let $J : C^2[x_0, x_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$J(y) = \int_{x_0}^{x_1} f(x, y, y') \, \mathrm{d}x,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_1 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are given real numbers. If $y \in S$ is an extremal for J, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0 \tag{9.31}$$

for all $x \in [a_0, x_1]$.

Theorem 9.4.2 (Lagrange multipliers method for non-holonomic constraints)**.** *If we want to find an extrema having a set of m non-holonomic constraints*

$$
\begin{aligned}
\overline{\delta f_1} &= A_{11}\delta u_1 + \cdots + A_{1n}\delta u_n &&= 0, \\
\vdots &= \ddots &&= \vdots \\
\overline{\delta f_m} &= A_{m1}\delta u_1 + \cdots + A_{mn}\delta u_n &&= 0,
\end{aligned}
\tag{9.32}
$$

Then we can proceed as the original multipliers method by considering every variation as independent and solving the following equation.

$$
\delta F + \lambda_1 \overline{\delta f_1} + \cdots + \lambda_m \overline{\delta f_m} = 0
\tag{9.33}
$$

Theorem 9.4.3. *Let f be a continuous functions with a variation $\delta f = \epsilon \phi$. Then,*

$$
\frac{\mathrm{d}}{\mathrm{d}x}\delta y = \delta \frac{\mathrm{d}}{\mathrm{d}x} y, \qquad \delta \int_a^b f(x)\,\mathrm{d}x = \int_a^b \delta f(x)\,\mathrm{d}x\,.
\tag{9.34}
$$

Theorem 9.4.4. *Let $J : \mathbf{C}^2[t_0, t_1] \longrightarrow \mathbb{R}$ be a functional of the form*

$$
J(\mathbf{q}) = \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}, t)\,\mathrm{d}t\,,
\tag{9.35}
$$

where $\mathbf{q} = (q_1, \ldots, q_n)$, and L has continuous second-order partial derivatives with respect to t, q_k, and $\dot{q}_k$, $k = 1, \ldots, n$. Let

$$
S = \left\{ \mathbf{q} \in \mathbf{C}^2[t_0, t_1] \,\middle|\, \mathbf{q}(t_0) = \mathbf{q}_0, \mathbf{q}(t_1) = \mathbf{q}_1 \right\},
\tag{9.36}
$$

where $\mathbf{q}_0, \mathbf{q}_1 \in \mathbb{R}^n$ are given vectors. If $\mathbf{q}$ is an extremal for J in S then for $k = 1, \ldots, n$

$$
\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} = 0.}
\tag{9.37}
$$

Theorem 9.4.5. *If we have a set of holonomic constraints*

$$
\begin{aligned}
f_1(q_1, \ldots, q_n, t) &= 0, \\
\ldots &= \vdots, \\
f_m(q_1, \ldots, q_n, t) &= 0,
\end{aligned}
\tag{9.38}
$$

then we can treat each variable as independent and search the stationary value of

$$
J' = \int_{t_1}^{t_2} L + \sum_{k=1}^{m} \lambda_m f_m\,\mathrm{d}t\,,
\tag{9.39}
$$

which leads to the equation

$$
\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} + \lambda_1\frac{\partial f_1}{\partial q_k} + \cdots + \lambda_m\frac{\partial f_m}{\partial q_k} = 0.}
\tag{9.40}
$$