Contents

Chapter 1

Introduction to differential equations

1.1 Preamble

The differential equations arose in order to study and describe physical phenomena.
A cite that sums up this is that one who said the mathematician Henry Poincaré when was asked bout what is a physical law. He said:

"*A physical law is ... that relates a phenomenon of today with one of tomorrow*"

And this refers to the concept of differential equation.

1.2 Definitions

Definition 1.2.1. Let $y(x)$ be an unknown function. Then, a *differential equation* is an equation where x, $y(x)$, and its derivatives intervene. We write this equation in the following ways.

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0 \qquad F\left(x, y(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) \tag{1.1}$$

Definition 1.2.2. Let $F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$ be a differential equation. Then, we define the *order of the equation* as the order of the maximum derivative in the equation.

Definition 1.2.3. Let $F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$ be a differential equation. Then, we define the *degree of the equation* as the number of the exponent in the highest order derivative.

Definition 1.2.4. If there are more than one differential equations, we say that we have a *system of differential equations.*

Definition 1.2.5. If the function y that intervenes in the differential equation has more than one variable, we say that we have a *partial differential equation*, a *differential equation in partial derivative*, or simply *PDE*. If not, we say we have a *ordinary differential equation*, a *differential equation in ordinary derivative*, or simply *ODE*.

Definition 1.2.6. Let $F = 0$ be a differential equation. We define *finding the solution to the differential equation* as finding the function y that satisfies this equation.

Example 1.2.1. Let be the following differential equation.

$$y'' + y = 0$$

It is a differential equation of second order in ordinary derivative. We can see some solutions to this equations are the functions $y_1 = \sin x$ and $y_2 = \cos x$.

Example 1.2.2. Let be the following differential equation.

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x + y \tag{1.2}$$

This is a differential equation of first order in partial derivative. We can see the function $z = xy$ is a solution.

Example 1.2.3. Let be the following differential equations.

$$\begin{cases} \frac{dx}{dt} = 2y + x \\ \frac{dy}{dt} = 3y + 4x \end{cases}$$

This is a system of two differential equations of first order in ordinary derivative. We an see the functions $x = e^{-t}$ and $y = -e^{-t}$ form a solution.

Sometimes the solution is described as an implicit expression, that is, the function y is not isolated.

Example 1.2.4. Let be the following differential equation.

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

In this case, the solution is the relation $x^2 = 2y^2 \ln y$.

1.3 Solutions

All differential equations have infinite solutions.

Example 1.3.1. A common case of differential equation is the integration of a function. We know that the derivative function has some expression of x, and we want to find y. If we proceed as always, we get

$$\frac{dy}{dx} = f(x) \Rightarrow y = F(x) + C$$

Since C can have every real value, we obtain infinite solutions.

Definition 1.3.1. In a differential equation, we define the *general solution* as the set of all the solutions to the equation.

Definition 1.3.2. In a differential equation, we define a *particular solution* as an element of the general solution.

Example 1.3.2. In the previous example, $F(x) + C$ was the general solution, whereas $F(x) + 1$ would be a particular solution.

The same that happened in this examples will occur in any other differential equation, since in the process of finding the solution there is always a step where we eliminate the derivatives by integrating (and obtaining and arbitrary constant with infinite values). Therefore, the fact that there are infinite primitives of a function follows that exist infinite solutions.

Proposition 1.3.1. *Let be a differential equation of the form $f(x)dx = g(y)dy$. Then, a function $y(x)$ is a solution of the differential equation if and only if it is a solution for*

$$\int f(x)dx = \int g(y)dy \tag{1.3}$$

.

Proof. In the onenote [?]. ■

Proposition 1.3.2. *Let be a differential equation of the form $f(x)dx = g(y)dy$, with the initial condition $y(a) = b$. Then, a function $y(x)$ is a solution of the differential equation and the satisfies the initial condition if and only if it satisfies*

$$\int_a^x f(x)dx = \int_b^y g(y)dy. \tag{1.4}$$

Proof. In the onenote [?]. ■

Example 1.3.3. Let be the following differential equation.

$$\frac{dy}{dx} + \frac{1}{y} = \frac{x}{y}$$

What we will do is to separate the variables to obtain an expression of the form $f(y)dy = g(x)dx$, and then integrate both sides.

$$\frac{dy}{dx} + \frac{1}{y} = \frac{x}{y} \Rightarrow \frac{dy}{dx} = \frac{x-1}{y} \Rightarrow ydy = (x-1)dx \Rightarrow \int ydy = \int x - 1dx \Rightarrow$$
$$\frac{y^2}{2} + C_1 = \frac{x^2}{2} - x + C_2 \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} - x + C$$

We can see that it is a differential equation of first order with ordinary derivative. Note that this equation has a general solution and one free constant.

Example 1.3.4. Let be the following differential equation.

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{1}{x^2}$$

We will first integrate both sides and then proceed as the previous example.

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{1}{x^2}\Rightarrow\int x\frac{d^2y}{dx^2}+\frac{dy}{dx}dx=\int\frac{1}{x^2}dx\Rightarrow\int x\frac{d^2y}{dx^2}dx+\int\frac{dy}{dx}=\int\frac{1}{x^2}dx\Rightarrow$$
$$x\frac{dy}{dx}-\int 1\cdot\frac{dy}{dx}+\int\frac{dy}{dx}=\int\frac{1}{x^2}dx\Rightarrow x\frac{dy}{dx}=-\frac{1}{x}+C_1\Rightarrow\frac{dy}{dx}=-\frac{1}{x^2}+\frac{C_1}{x}\Rightarrow$$
$$\int\frac{dy}{dx}dx=-\int\frac{1}{x^2}dx+\int\frac{C_1}{x}dx\Rightarrow y=\frac{1}{x}+C_1\ln x+C_2$$

It is a differential equation of second order with ordinary variable. Note that, as in the previous example, the order of the equation coincides with the number of free constants. In fact, this coincidence is always true by certain conditions, and we will prove it later [ ].

1.4 Picard method of successive approximations

1.4.1 Introduction

If we have a differential equation of the form $dy/dx = f(x, y)$, we can find a function y that satisfies the problem of initial values $y(a) = b$ [ ] through the method of successive approximations.
The first step is to replace the initial value problem with an integral equation, which, although it may seem more complicated, supports an iterative solution technique. We must first see that the above problem is equivalent to solving:

$$y(x)=b+\int_a^x f\left(t,y(x)\right)dt \tag{1.5}$$

Where t ia an auxiliary variable of integration. To see the equivalence, let us derive the equation with respect to x.

$$\frac{d}{dx}y(x)=\frac{db}{dx}+\frac{d}{dx}\int_a^x f\left(t,y(x)\right)dt=f\left(t,y(x)\right) \tag{1.6}$$

If we consider that $y(x)$ is the solution to the problem of initial values, then by construction this equality will be true. We can see that the inverse reasoning is also true.
The advantage of this new form of the equation is that it admits a iterative solution method. To do that, we will consider an initial function $y_0(x) = b$ (we could have chosen another one but this is easy to integrate) as a first approximation. Now, we will proceed as the right side of the equation 1.6 and obtain another function $y_1(x)$.

$$y_1(x)=b+\int_a^x f\left(t,y_0(t)\right)dt$$

We check if this new function is a solution and if not, we do the same process to obtain a new function $y_2(x)$ and check if it works. We will do this successively and, in general, we have the relation

$$y_n(x)=b+\int_a^x f\left(t,y_{n-1}(t)\right)dt \tag{1.7}$$

It is possible that for some n we find the solution, but if not, it turns out the sequence of functions $\{y_n(x)\}$ converges to the solution of the differential equation and satisfies the problem of initial values. Clearly, we can do this only if the function $f\left(t, y_n(t)\right)$ is integrable for all n. Let us now see an example of this statement.

Example 1.4.1. Let be the following differential equation and its initial conditions.

$$\frac{dy}{dx} = x + y \qquad x_0 = 0, y\left(x_0\right) = 0$$

Following the Picard's method, we will begin with $y_0(x) = 0$ and then use the relation 1.7.

$$y_0(x) = 0$$

$$y_1(x) = \int_0^x f\left(t, y_0\right) dt = \int_0^x t + y_0 dt = \int_0^x t dt = \frac{x^2}{2 \cdot 1}$$

$$y_2(x) = \int_0^x f\left(t, y_1\right) dt = \int_0^x t + y_1 dt = \int_0^x t + \frac{t^2}{2 \cdot 1} dt = \frac{x^2}{2} + \frac{x^3}{3 \cdot 2 \cdot 1}$$

$$y_3(x) = \int_0^x f\left(t, y_2\right) dt = \int_0^x t + y_2 dt = \int_0^x t + \frac{t^2}{2!} + \frac{t^2}{3!} dt = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

From this we can imagine that our term $y_n(x)$ will have the form

$$y_n(x) = \sum_{k=2}^{n+1} \frac{x^k}{k!}$$

And we will prove it by induction. We have seen already the basis cases, so we only need to prove that it will be true for $y_{n+1}(x)$.

$$y_{n+1}(x) = \int_0^x f\left(t, y_n(t)\right) dt = \int_0^x t + y_n dt = \int_0^x t + \sum_{k=2}^{n+1} \frac{x^k}{k!} dt = \int_0^x t dt + 0 \sum_{k=2}^{n+1} \int_0^x \frac{t^k}{k!} =$$

$$\frac{x^2}{2!} + \sum_{k=2}^{n+1} \frac{x^{k+1}}{(k+1)!} = \frac{x^2}{2!} + \sum_{k=3}^{n+2} \frac{x^k}{k!} = \sum_{k=2}^{n+2} \frac{x^k}{k!}$$

Therefore, it is true and now we can calculate the function that satisfies the differential equation.

$$y(x) = \lim_{n \to \infty} y_n(x) = \sum_{k=2}^{\infty} \frac{x^k}{k!} = -\frac{x^0}{0!} - \frac{x^1}{1!} + \sum_{k=0}^{\infty} \frac{x^k}{k!} = -1 - x + e^x$$

In this example we could use other methods than reach the solution faster (we will see them later [ ]), but although that this method very important for two reasons. The firs tone is that is a simple method (not necessary fast) that works for all the equations were there is not an exact method, and the second one is that this process will allow us to prove the theorem of existence [ ].

1.4.2 Previous concepts

Definition 1.4.1. Let $f(x, y) : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say *the function satisfies the Lipschitz condition with respect to the variable* y if it is true that

$$\forall (x, y_1), (x, y_2)\ \exists k \mid \left|f(x, y_1) - f(x, y_2)\right| \leq k \left|y_1 - y_2\right| \tag{1.8}$$

And we call the constant k as *the Lipschitz constant of the function* [ ].

We say it satisfies the L condition in S if it is true in S.
We say it satisfies the L condition uniformily if k does not depend.
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Proposition 1.4.1. *Every function such that* $\frac{df}{dy} \neq \infty$ *in* S *satisfies Lipschitz Condition. It's enough to take* $A \geq \max_S |\frac{\partial f}{\partial y}|$. *Furthermost, in most cases the theorem fulfills* $\forall x$, *given that there's no limit for* h, h'.

Proposition 1.4.2. *If* $\exists!$ *a solution such that* $y = b$ *when* $x = a$, *the general solution to the first order differential equation is a function* $y(x)$ *with one trivial constant: the value of* y *when* $x = a$, *i.e.* b

1.5 Existence theorem of a first order differential equation's general solution

Theorem 1.5.1. *Let $S \subset \mathbb{R}^2$ be a closed set defined by $|x-a| \leq h$ and $|y-b| \leq k$, and let $f(x,y,)$ a continuous function in S which satisfies Lipschitz Condition and which is bounded by real number M. Then, the differential equation $y' = f(x,y)$ has a unique solution such that $y = b$ when $x = a$, in the interval $|x-a| \leq h'$, where $h' = \min(h, k/M)$*

Proof. First, we will prove that the sequence of approximations presented by the equation 1.7 converge to a certain limit. For that, notice y_n can be expressed as

$$y_n(x) = y_1(x) + [y_2(x) - y_1(x)] + \cdots + [y_n(x) - y_{n-1}(x)]. \tag{1.9}$$

Besides,

$$y_{n+1}(x) - y_n(x) = \int_a^x f(t, y_n(t))dt - \int_a^x f(t, y_{n-1}(t))dt = \int_a^x [f(t, y_n(t)) - f(t, y_{n-1}(t))]dt \Rightarrow$$

$$|y_{n+1}(x) - y_n(x)| = \left| \int_a^x f(t, y_n) - f(t, y_{n-1})dt \right| \leq$$

$$\int_a^x |f(t, y_n) - f(t, y_{n-1})|dt. \tag{1.10}$$

And by the Lipschitz condition, there exists a Lipschitz constant A such that

$$|y_{n+1}(x) - y_n(x)| \leq \int_a^x |f(t, y_n) - f(t, y_{n-1})|dt \leq \int_a^x A|t_n(t) - y_{n-1}(t)|dt \leq$$

$$A \int_a^x |y_n(t) - y_{n-1}(t)||dt|$$

Next we are going to apply this conclusion to all n values, but first we need to prove that for every $n \in \mathbb{N}$ we will have that $(x, y_n) \in S$.
For that, let $(x,y) \in S$ be a point, M the bound of f (that we know exists because the function is continuous in a closed set), and $h' = \min\{h, k/M\}$. We will prove this by induction, starting with (x, y_0). By hypothesis, this needs to be true, so it only remains that one case implies the next one. If $(x, y_{n-1}) \in S$ with $|x-a| < h'$, we have that

$$|y_n - b| = \left| \int_a^x f(t, y_{n-1}(t))dt \right| \leq \int_a^x |f(t, y_{n-1}(t))|dt \leq \int_a^x M dt = M \int_a^x dt \leq M(x-a) \leq$$

$$M|x-a| \leq Mh' \leq k$$

Therefore, if $|x-a| < h'$, it is always true that $7y_n - b| < k$ and, by definition, that $y_n \in S$. Now, let y_1, y_2 such that $|y_2 - y_1| < N$ (since they are in a finite set, their subtraction can be bounded by a finite number). If we apply this last relation,

$$|y_3 - y_2| \leq A \int_a^x |y_2 - y_1||dt| \leq A \int_a^x N|dt| = AN|x-a|$$

$$\Rightarrow |y_4 - y_3| \leq A \int_a^x |y_3 - y_2||dt| \leq A \int_a^x AN|t-a||dt| \leq A^2 N \frac{|x-a|^2}{2}$$

$$\Rightarrow |y_{n+1} - y_n| \leq A^{n-1} N \frac{|x-a|^{n-1}}{(n-1)!} \leq A^{n-1} N \frac{h'^{n-1}}{(n-1)!}.$$

Going back to the formula 1.10,

$$y_n(x) = y_1(x) + \sum_{k=1}^{n} y_{k+1}(x) - y_k(x) \leq y_1(x) + \sum_{k=1}^{n} N\frac{(Ah')^{k-1}}{(k-1)!} = y_1(x) + N\sum_{k=0}^{n-1}\frac{(Ah')^k}{k!}$$

If we apply the limit to this relation, we get an upper bound for y_n.

$$\lim_{n\to\infty} y_n(x) \leq y_1(x) + N\lim_{n\Rightarrow\infty}\sum_{k=0}^{n-1}\frac{(Ah')^k}{k!} = y_1(x) + N\sum_{k=0}^{\infty}\frac{(Ah')^k}{k!} = y_1(x) + Ne^{Ah'}$$

The series converge given that it's fitted and $|y_{n+1} - y_n|$ gets smaller every time n increases. ■

Proof. Now we need to prove that the limit $Y(x) = \lim_{n\to\infty} y_n(x)$ is solution of the differential equation $y' = f(x, y)$ and that satisfies the initial conditions problem with $y(a) = b$. For that, we will see that

$$Y(x) = b + \int_a^x f(t, Y)dt,$$

since as we discussed before this is an equivalent equation. Hence,

$$|Y - b - \int_a^x f(x, Y)dx| = |Y - y_n + \int_a^x [f(x, y_{n-1}) - f(x, Y)]dx|$$

$$\leq |Y - y_n| + A\int_a^x |y_{n-1} - Y||dx| \leq \epsilon + A\epsilon|x - a| \leq \epsilon(1 + Ah')$$

If we take n big enough, ϵ can be as small as we want. ■

Proof. Now we need to prove that there are no more solutions. For that, we will suppose that there is another solution Y_1 and conclude that $Y_1 = Y$. Since Y_1 is defined in S, f will satisfy the Lipschitz condition, and therefore, given that $|Y_1 - Y| < C$ (they are bounded because belong to the same finite set),

$$Y_1(x) = b + \int_a^x f(t, Y_1)dt \Rightarrow Y_1 - Y = \int_a^x [f(t, Y_1) - f(t, Y)]dt \Rightarrow |Y_1 - Y| \leq$$

$$A\int_a^x |Y_1 - Y|dt \leq A\int_a^x Cdt \leq CA|x - a|.$$

Applying this successively,

$$|Y_1 - Y| \leq CA^2\frac{|x-a|^2}{2} \Rightarrow \ldots |Y_1 - Y| \leq CA^n\frac{|x-a|^n}{n!} \leq CA^n\frac{h'^n}{n!}.$$

Making n larger $\ldots Y_1(x) \equiv Y(x)$

■

The Existence theorem of a n-order differential equation's general solution (n first order differential equations system) can be proven similarly.
Note that this theorem is not a double implication. Therefore, the fact that there is only one solution does not imply the conditions of the theorem are satisfied. Similarly, if these conditions are not satisfied, we can't say there are more than one solution.

1.6 Existence theorem of a n-order differential equation's general solution

Theorem 1.6.1. *If a system of n first order differential equations* $y_1' = f_1(x, y_1, y_n), y_2' = f_2(x, y_1, y_n) \dots y_n' = f_n(x, y_1, y_n)$*, the functions* $f_1 \dots f_n$ *are continuous in the* S *region defined by* $|x-a| \leq h$*,* $|y_1 - b_1| \leq k_1 \dots |y_n - b_n| \leq k_n$*, and if the satisfy (in S) Lipschitz Condition* $|f_i(x, y_1 \dots y_n) - f_i(x, z_1, z_n)| \leq A_1|y_1 - z_1| + \dots + A_n|y_n - z_n|$ *then in the* I *interval,* $|x - a| \leq h' = min(h, \frac{k_i}{M})$ *where* $M \geq |f_i(x, y_1 \dots y_n)$*, there exists a unique set of continuous functions* $y_1(x) \dots y_n(x)$ *with continuous derivatives inside* I*, which are solution to the system of differential equations and satisfy* $y_i(a) = b_i$

Proposition 1.6.2. *Given that a differential equation of n order in* y*,* $\frac{d^n y}{dx^n} = f(x, y, \frac{dy}{dx} \dots \frac{d^{n-1}y}{dx^{n-1}})$ *is equivalent to n first order differential equations in* $y, y_1 \dots y_{n-1}$ *which are* $\frac{dy}{dx} = y_1, \frac{dy_1}{dx} = y_2 \dots \frac{dy_{n-2}}{dx} = y_{n-1}, \frac{dy_{n-1}}{dx} = f(x, y, y_1 \dots y_{n-1})$*, a differential equation of order* n *has a unique solution with* $y(x_0) = b_0, y'(x_0) = b_1 \dots y^{(n-1)}(x_0) = b_{n-1}$ *if the previous conditions are fulfilled.*

1.7 Summary

Picard's method

$$\frac{dy}{dx} = f(x, y) \Rightarrow y_n(x) = b + \int_a^x f(t, y_{n-1}(t))dt$$

Bibliography

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Chapter 2

First order differential equations

2.1 Family of curves to a parameter in the plane

Definition 2.1.1. Every family is expressed by a relation $F(x, y, C)$ and they're the geometric representation of a differential equation's general solution.

To find said differential equation (inverse problem) we have to get rid of the C constant among $F, dF/dx$.

Example 2.1.1. The circumferences' family centered in the origin $x^2 + y^2 = 0$ represents the general solution to $x + y\frac{dy}{dx} = 0$.

Example 2.1.2. The family of elipses centered in the origin with semiaxis a $x^2/a^2 + y^2 = 1$ represents the general solution to $xy\frac{dy}{dx} = y^2 - 1$

2.2 Orthogonal trajectories

Definition 2.2.1. Given a family of curves $F(x, y, C) = 0$ we find out its differential equation $f(x, y, dy/dx) = 0$. The differential equation of the trajectories will be $f(x, y, -dx/dy) = 0$.

This last differential equation comes up from the fact that two lines are orthogonals between them must satisfy:

$$m_1 = -\frac{1}{m_2}$$

Generally, expanding the concept beyond two lines, two orthogonal trajectories must satisfy:

$$y_1' = \frac{dy_1}{dx} = -\frac{dx}{dy_2} = -\frac{1}{y_2'}$$

If we are able to solve it, we'll obtain the trajectories. This is connection between the lines of force (electrical, gravitational) and equipotential lines.

Example 2.2.1. Orthogonal trajectories to the lines which pass through the origin, the family defined by $y = Cx$. Its differential equation is $y = x\frac{dy}{dx}$. The orthogonal trajectories' differential equation is $y = x(\frac{-dx}{dy}) \rightarrow xdx + ydy = 0$. The trajectories, then, are $x^2 + y^2 = R^2$, which is the family of circumferences centered in the origin.

2.3 Clairaut's Equation

The family of non-parallel lines $y = Cx + f(C)$ represents the general solution of the Clairaut's Equation.

Definition 2.3.1. The Clairaut's Equation is defined by

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

Proposition 2.3.1. *The family of non-parallel lines $y = Cx + f(C)$ represents the general solution of the Clairaut's Equation.*

Proof. If we start by computing the derivative of the equation, we obtain

$$\frac{dy}{dx} = \frac{dy}{dx} + x\frac{d^2y}{dx^2} + f'\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} \Rightarrow \left[1 + f'\left(\frac{dy}{dx}\right)\right]\frac{d^2y}{dx^2} = 0,$$

with leads to two possible results. If $y'' = 0$,

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{dy}{dx} = c \Rightarrow y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) = cx + f(c).$$

This is a general solution of the differential equation that has the form we wanted to prove. However, there is still one result. If the first component is zero, we get

$$x + f'\left(\frac{dy}{x}\right) = 0,$$

which gives a singular solution. Therefore, the general solution is only expressed by the equation we $y = cx + f(c)$, as we wanted to prove. ∎

We will discuss this singular solution later, referring to it as the *envelope*.

Example 2.3.1. The general solution to $y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2$ is $y = Cx - \frac{C^2}{4}$

Example 2.3.2. The general solution to $y = x\frac{dy}{dx} - \frac{dx}{dy}$ is $y = Cx - \frac{1}{C}$

Example 2.3.3. The general solution to $\left(y - x\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$ is $y = Cx \pm \sqrt{C^2 - C}$

2.4 Singular Solutions

Besides the general solution, the Clairaut's equation can have a singular solution.

Definition 2.4.1. A singular solution is an additional solution not included in the general one (it doesn't come up with any C value).

2.5 Envelope's Properties

Definition 2.5.1. An envelope is a curve which is tangent in every of its points to the general solution $y = Cx + f(C)$

If it exists an envelope then it's easy to see that $\forall$ its points $x, y, dy/dx$ coincides with any line. If the envelope $\exists$, its equation is a solution to the differential equation.

2.5.1 Finding the envelope

The envelope, if exists, must obey in all of its points that $y = Cx + f(C)$, for a given value C, as it coincides with one of the lines. If we vary C this relation will continue to be true as the envelope will coincide with another line. This implies that $\frac{\partial}{\partial C}$ must equal to 0, so $x = -\frac{\partial f}{\partial C}$.

Proposition 2.5.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and let y be another function of the form $y(x) = cx + f(c)$. Then, the envelope satisfies that*

$$x = -\frac{\partial f}{\partial c} \tag{2.1}$$

Proof. Let $(x_1, y_1), (x_2, y_2)$ two points of the envelope. As we said before, there always exist values of c that satisfy $y = cx + f(c)$. If we apply it to these points and compute the derivatives of the relations, we get

$$\begin{aligned} y_1 &= c_1 x_1 + f(c_1), & \frac{dy}{dx}(x_1) &= c_1 \\ y_2 &= c_2 x_2 + f(c_2), & \frac{dy}{dx}(x_2) &= c_2 \end{aligned}$$

Subtracting the first one to the second one,

$$y_2 - y_1 = c_2(x_2 - x_1) + x_1(c_2 - c_1) + f(c_2) - f(c_1) \Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = c_2 + \frac{c_2 - c_1}{x_2 - x_1}\left[x_1 + \frac{f(c_2) - f(c_1)}{c_2 - c_1}\right]$$

Before dealing with the fraction of $f(c)$, we will see what happens with the other components when $x_2 \to x_1$.

$$\lim_{x_2\to x_1} \frac{y_2 - y_1}{x_2 - x_1} = \frac{dy}{dx}(x_1) = c_2, \qquad \lim_{x_2\to x_1} c_2 = c_1$$

Note that the second limit is true if we deal with c as an independent variable or as dy/dx (sin $y(x)$ is a line, if it is of class C^∞ and its derivatives are continuous). With that, we conclude that at the limit

$$\lim_{x_2\to x_1} \frac{c_2 - c_1}{x_2 - x_1}\left[x_1 + \frac{f(c_2) - f(c_1)}{c_2 - c_1}\right] = 0.$$

This can only be true if the first component is zero or the second. In the first case,

$$0 = \lim_{x_2\to x_1} \frac{c_2 - c_1}{x_2 - x_1} = \lim_{x_2\to x_1} \frac{1}{x_2 - x_1}\left[\frac{dy}{dx}(x_2) - \frac{dy}{dx}(x_1)\right] = \frac{d^2y}{dx^2}(x_1),$$

which shows us that y is a line. This expression forms part of the general solution of lines, so it must be the second condition.

$$0 = \lim_{x_2\to x_1} x_1 + \frac{f(c_2) - f(c_1)}{c_2 - c_1} = x_1 + \lim_{x_2\to x_1} \frac{f(c_2) - f(c_1)}{c_2 - c_1} = x_1 + \frac{\partial f}{\partial c}(c_1) \Rightarrow x_1 = -\frac{\partial f}{\partial c}(c_1)$$

This is what needs to satisfy a curve to be tangent at all points to one of the lines without being itself a line. Therefore, this is what it needs to satisfy the envelope, and since the election of x_1 is arbitrary, this relation is true for every value of x. ■

Proposition 2.5.2. *If the envelope exists, it must satisfy:*

$$y = Cx + f(C)$$
$$x = -\frac{\partial f}{\partial C}$$

If the envelope must satisfy the previous two conditions, then we can find it getting rid of the C constant.

Example 2.5.1. The general solution to $y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2$ is $y = Cx - \frac{C^2}{4}$. By the second condition we find out $x = C/2$. Removing C, we find the envelope's equation $y = x^2$.

Example 2.5.2. The general solution to $y = x\frac{dy}{dx} - \frac{dx}{dy}$ is $y = Cx - \frac{1}{C}$. By the second condition, $x = -1/C^2$, then the envelope's equation is the parabola $y^2 = -4x$

Example 2.5.3. The general solution to $\left(y - x\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$ is $y = Cx \pm \sqrt{C^2 - C}$. The envelope's equation is the next hyperbole $xy - y^2 = 1/4$

Important Observation: this method only uses necessary but not sufficient conditions. We're finding candidates to a valid envelope, we must prove they're the correct ones by substituting into the original differential equation.

Example 2.5.4. The general solution to $y = x\frac{dy}{dx} - \frac{1}{3}\left(\frac{dy}{dx}\right)^2$ is $y = Cx - \frac{C^3}{3}$ and it has two different envelopes, two differential singular solutions $y = \pm\frac{2}{3}x^{3/2}$

2.5.2 On the Existence and Unicity Theorem

It's obvious that if a singular solution exists, then the theorem does not fulfil. Let's see it with an example

$$\frac{dy}{dx} = 2x \pm 2\sqrt{x^2 - y}$$

If $y < x^2$ the function $f(x, y)$ is continuous and the theorem fulfils
If $x^2 < y$, $\nexists \frac{dy}{dx}$ real that satisfies the relation then $\nexists$ real solution

If $x^2 = y$ the function $f(x, y)$ isn't continuous and the theorem doesn't fulfil

We must remember that the example we used are two equations, that's why two solutions can pass through one point in the $y < x^2$ region (1 solution to each equation, the Existence Theorem continues to fulfil). Nevertheless, two solutions pass through one point in the $y = x^2$ but they both belong to the same equation, so it doesn't satisfy the Unicity property.

2.6 Linear Equations

Definition 2.6.1. A linear differential equation is an equation such that:

$$A(x)\frac{dy}{dx} + B(x)y + C(x) = 0 \Leftrightarrow \frac{dy}{dx} + P(x)y = Q(x)$$

Proposition 2.6.1. *The general solution to a linear differential equation is*

$$y = C\exp\left\{\left[-\int P(x)dx\right]\right\} + \exp\left\{\left[-\int P(x)dx\right]\right\}\int Q(x)\exp\left\{\left[\int P(x)dx\right]\right\}dx$$

Proof. We're going to use the advanced concept of integrating factor.

$$\exp\left\{\left[\int P(x)dx\right]\right\}\left[\frac{dy}{dx} + P(x)y\right] = Q(x)\exp\left\{\left[\int P(x)dx\right]\right\}$$
$$y \cdot exp\left[\int P(x)dx\right] = \int Q(x)exp\left[\int P(x)dx\right]dx$$

as $$\frac{d}{dx}\left[y \cdot exp\left[\int P(x)dx\right]\right] = \exp\left\{\left[\int P(x)dx\right]\right\}\frac{dy}{dx} + P(x)y$$

■

Example 2.6.1. $y' + y/x = sinx \Rightarrow y = Cexp(-\ln x) + \exp\{(-\ln x)\}\int \sin x\exp\{(\ln x)\}dx$
$\Rightarrow y = C/x + 1/x\int x\sin xdx \Rightarrow y = 1/x[C - x\cos x + \sin x])$

Definition 2.6.2. A linear differential is defined as reduced if $Q(x) = 0$

$$\frac{dy}{dx} + P(x)y = 0$$

Proposition 2.6.2. *The general solution to a reduced linear differential equation is*

$$y = D\exp\left\{[-\int P(x)dx]\right\}$$

Proof.

$$\frac{dy}{y} = -P(x)x \Rightarrow \ln y = -\int P(x)dx + C \Rightarrow y = \exp\left\{\left[-\int P(x)dx + C\right]\right\}$$
$$= \exp\left\{\left[-\int P(x)dx\right]\right\}\exp\{C\} = D\exp\left\{\left[-\int P(x)dx\right]\right\}$$

■

Example 2.6.2. $y' + y/x = 0 \Rightarrow P(x) = 1/x \Rightarrow y = D\exp\{(-lnx)\} \Rightarrow y = D/x$

Proposition 2.6.3. *Let y_1 be a particular solution to the reduced differential equation, then the general soluton is $y = Cy_1$*

Proposition 2.6.4. *Let y_1 be a particular solution to a reduced differential equation and y_2 to the complete differential equation, then the general solution to the complete differential equation is $y = Cy_1 + y_2$*

Example 2.6.3. Let be the following differential equation.

$$y' + y = 10$$

Let $y_1 = e^{-x}$ be a particular solution to the reduced form and $y_2 = 10$ to the complete form.. Then the general solution is

$$y = Ce^{-x} + 10$$

Example 2.6.4. Let be the following differential equation.

$$y' + \frac{y}{x^2} = 2x + 1$$

Let $y_1 = e^{-x}$ be a particular solution to the reduced form and $y_2 = x^2$ to the complete form.. Then the general solution is

$$y = Ce^{-x} + x^2$$

2.7 Bernoulli equation

Definition 2.7.1. The Bernoulli's equation is defined as:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$\forall n \in \mathbb{R}$, being linear for $n = 0, n = 1$

Proposition 2.7.1. *In the rest of cases, it can be reduced to a linear differential equation using the substitution $z = y^{1-n}$*

Proof. Consider the following change in variables $z = y^{1-n}$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Then Bernoulli's equation transforms into:

$$\frac{dz}{dx}\frac{y^n}{(1-n)} + P(x)zy^n = Q(x)y^n$$

Eventually getting a linear equation we are able to solve with methods described in previous sections.

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

∎

Example 2.7.1. Let be the following differential equation:

$$\frac{dy}{dx} + \frac{y}{x} = y^2\frac{x\cos x - \sin x}{x}$$

We can identify Bernoulli's equation with $n = 2$. Realizing this change of variables $z = y^{-1}$. We are left with the next linear differential equation

$$\frac{dz}{dx} - \frac{z}{x} = \frac{\sin x - x\cos x}{x}$$

We can solve it using integrating factors:

$$P(x) = -\frac{1}{x} \Rightarrow \mu = \exp\left\{\left[-\int P(x)dx\right]\right\} = x$$

Multiplying both sides by the integrating factors (keep in mind we aren't changing anything in the differential equation, this step continues to be valid)

$$x\frac{dz}{dx} - z = \sin x - x\cos x \Rightarrow z = Cx + x\int \frac{\sin x - x\cos x}{x^2}dx$$

$$z = Cx - \sin x \Rightarrow y = \frac{1}{Cx - \sin x}$$

2.7.1 Logistic differential equation

2.8 Riccati equation

Definition 2.8.1. The Riccati's equation is defined as:

$$y' = P(x)y^2 + Q(x)y + R(x)$$

Generally, it can't be solved, but given a particular solution we can find the general one.

Proposition 2.8.1. *Given a particular solution*

$$y_p = P(x)y_p^2 + Q(X)y_P + R(x)$$

We can transform Riccati's equation into Bernoulli's equation with $n = 2$, realizing a change of variables we eventually obtain a linear equation.

Proof. Let $z = y - y_p$

$$z' = y' - y_p' = P(x)y^2 + Q(x)y + R(x) - P(x)y_p^2 - Q(x)y_p - R(x) = P(x)(y^2 - y_p^2) + Q(x)(y - y_p)$$

$$z' = P(x)(z^2 + y_p^2 - 2zy_p - y_p^2) + Q(x)z = P(x)z(z - 2y_p) + Q(x)z$$

$$z' - (2y_pP(x) + Q(x))z = Pz^2$$

Eventually obtaining a solvable Bernoulli's equation

■

Example 2.8.1. Let be the next differential equation

$$y' = x^3(y - x)^2 + \frac{y}{x}$$

We identify the polynomials and one particular solution (trial and error):

$$y_p = x, P(x) = x^3, Q(x) = -2^4 + \frac{1}{x}$$

Realizing change of variables

$$z = y - y_p = y - x \Rightarrow z' - \frac{z}{x} = x^3z^2$$

In the Bernoulli's equation:

$$P(x) = \frac{1}{x}, Q(x) = x^3, n = 2 \Rightarrow y' + \frac{u}{x} = -x^3$$

In the linear equation:

$$P(x) = \frac{1}{x}, Q(x) = -x^3 \Rightarrow u = \frac{C}{x} - \frac{x^4}{5}$$

$$y = x + \frac{5x}{5C - x^5}$$

We must remember C is an arbitrary constant which can become any value, including $\pm\infty$, that's why some Riccati's general solutions can seem not to include the particular one we proposed in the beginning.

In fact, singular solutions can also be proposed as the first solution to start with since they satisfy the conditions we imposed.

Proposition 2.8.2. *Given two particular solutions to a Riccati's equation we can find the general solution.*

Proof. Let y_1 and y_2 be two particular solutions.

$$y' - y_1' = P(x)(y - y_1)^2 + Q(x)(y - y_1) = P(x)(y - y_1)(y + y_1) + Q(x)(y - y_1)$$

Then

$$\frac{y' - y_1'}{y - y_1} = P(x)(y + y_1) + Q(x)$$

$$\frac{y' - y_2'}{y - y_2} = P(x)(y + y_2) + Q(x)$$

Substracting and integrating

$$\frac{y' - y_1'}{y - y_1} - \frac{y' - y_2'}{y - y_2} = P(x)(y_1 - y_2)$$

$$\ln \frac{y - y_1}{y - y_2} = \int P(x)(y_1 - y_2)dx$$

$$\frac{y - y_1}{y - y_2} = C \exp\left\{\left[\int P(x)(y_1 - y_2)dx\right]\right\}$$

■

Example 2.8.2. Let be the following differential equation

$$y' = x^3(y^2 - x^2) + \frac{y}{x}$$

Given two particular solutions

$$y_1 = x, y_2 = -x$$

Then the general solution is

$$y = x + \frac{2Cxe^{\frac{2}{5}x^5}}{1 - Ce^{\frac{2}{5}x^5}}$$

2.9 Homogeneous equations

Definition 2.9.1. $F(x, y)$ is an nth-grade homogeneous equation if and only if

$$F(tx, ty) \equiv t^n F(x, y) \quad \forall t \in \mathbb{R}$$

Proposition 2.9.1. *If $F(x, y)$ is an homogeneous equation of grade n, $G(x, y)$ m-th grade homogeneous, then $FG, F/G$ are homogeneous equations of grade $n + m$, $n - m$, respectively.*

Proof. If $F(x, y)$ and $G(x, y)$ are homogeneous then they satisfy

$$F(tx, ty) \equiv t^n F(x, y) \quad \forall t \in \mathbb{R}$$

$$G(tx, ty) \equiv t^m G(x, y) \quad \forall t \in \mathbb{R}$$

Then

$$F(x, y)G(x, y) = t^n F(x, y) t^m G(x, y) = t^{n+m} F(x, y)G(x, y)$$

$$\frac{F(x, y)}{G(x, y)} = \frac{t^n F(x, y)}{t^m G(x, y)} = t^{n-m} \frac{F(x, y)}{G(x, y)}$$

■

Proposition 2.9.2. *If $F(x, y)$ is an homogeneous equation of grade 0, then $F(x, y)$ is only a function of y/x*

Proof. Let the following change in variables $t = 1/x$. If its grade is 0, then it must satisfy

$$F(tx, ty) = t^0 F(x, y) = F(x, y)$$

Then we have the condition

$$F(tx, ty) = F(x, y)$$

The only way this can happen is $F(x, y)$ being a function of y/x

$$F(x, y) = G(y/x)$$

■

Definition 2.9.2. An homogeneous differential equation is one such that

$$M(x, y)dx + N(x, y)dy = 0$$

Being M, N homogeneous functions both with same grade

Proposition 2.9.3. *An homogeneous differential equation can be resolved with an easy change of variables rigged by $y = vx$ or $x = yv$, depending on whether we want the solution to be a function of x or a function of y $(x(y), y(x))$*

Proof. We first can solve it by separation of variables

$$\frac{dy}{dx} = -\frac{M}{N} = f\left(\frac{x}{y}\right)$$

Using $y = vx$

$$v + x\frac{dv}{dx} = f(v) \Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v)}$$

$$\ln|x| = \frac{dv}{f(v) - v)}$$

■

2.9.1 Graphic representation

Definition 2.9.3. An homothecy is a linear transformation of a point x, y to a point kx, ky

Proposition 2.9.4. *The curves representing solutions can be transform among themselves between homothecy*

Proof. Indeed, given two homogeneous functions M, N of the same grade

$$M(kx,ky)dx + N(kx,ky)dy = k^n \left[M(x,y)dx + N(x,y)dy\right] = 0$$

∎

Example 2.9.1. Consider the following homogeneous differential equation

$$(2x+3y)dx + (3x+2y)dy = 0$$

Then

$$f(v) = \frac{2+3v}{-3-2v}$$

$$v = \frac{y}{x} \Rightarrow \frac{dx}{x} = \frac{dv(3+2v)}{-2-6v-2v^2} = -\frac{1}{2}\frac{dv(3+2v)}{v^2+3v+1}$$

$$\ln x + \ln\left(v^2+3v+1\right)^{1/2} = \ln A \Rightarrow x^2(v^2+3v+1) = A^2$$

$$x^2 + 3xy + y^2 = A^2$$

If we transform one of the solutions by an homothecy, we obtain another curve belonging to the hyperbole family (in this case)

$$(kx,ky) = (\overline{x},\overline{y}) \Rightarrow \overline{x}^2 + 3\overline{xy} + \overline{y}^2 = k^2A^2 = A'^2$$

2.10 Exact equations

Definition 2.10.1. An exact differential equation is an homogeneous differential equation such that

$$M(x,y)dx + N(x,y)dy = 0 \;\; \text{exact} \quad \Leftrightarrow \exists u(x,y) \| du \equiv M(x,y)dx + N(x,y)dy$$

Proposition 2.10.1. *Exact differential equations can be solved using*

$$du = 0 \Rightarrow u(x,y) = C$$

Proposition 2.10.2.

$$M(x,y)dx + N(x,y)dy = 0 \;\; \textit{exact} \quad \Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof. Proof that it's a necessary condition

$$\exists u \mid du \equiv Mdx + Ndy \Rightarrow \exists u \mid \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = Mdx + Ndy$$

$$\Rightarrow \exists u \mid \frac{\partial u}{\partial x} = M, \frac{\partial u}{\partial y} = N \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

∎

Proof. Proof that it's a sufficient condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

Let

$$u = \int_a^x M(x,y)\partial x + \int N(a,y)dy$$

Then

$$\frac{\partial u}{\partial x} = M(x,y)$$

$$\frac{\partial u}{\partial y} = \int_a^x \frac{\partial M(x,y)}{\partial y}\partial x + N(a,y) = \int_a^x \frac{\partial N(x,y)}{\partial x}\partial x + N(a,y) = N(x,y) - N(a,y) + N(a,y) = N(x,y)$$

■

You must remember a is an arbitrary constant, so that it doesn't bother the general solution as it can be combined with the integration constant into another constant. However, terms containing a constant tend to cancel between themselves, just be sure you are only left with constants.

Example 2.10.1. Let be the following differential equation

$$\left(\frac{y}{x} + y^3\right) dx + (\ln x + 3y^2 x + 4y)dy = 0$$

Firstly we check if it's homogeneous

$$\frac{\partial M}{\partial y} = \frac{1}{x} + 3y^2 = \frac{\partial N}{\partial x}$$

Let u be

$$u = \int_a^x (y/x + y^3)\partial x + \int (\ln a + 3y^2 a + 4y)dy$$

Then

$$u = y \ln x y^3 x - y \ln a - y^3 a + y \ln a + a y^3 + 2y^2$$
$$u = y \ln x + y^3 x + 2y^2$$

The general solution is

$$y \ln x + y^3 x + 2y^2 = C$$

2.11 Integrating factors

Definition 2.11.1. A non-null function $\mu(x,y)$ is an integrating factor of the differential equation $M(x,y)dx + N(x,y)dy = 0$ if and only if

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is an exact differential equation

Example 2.11.1. Let be the following differential equation

$$2ydx + xdy = 0$$

which indeed isn't exact. Let μ be the following integrating factor

$$\mu = x$$

Then

$$2xydx + x^2 dy = 0$$

becomes exact and consequently solvable if and only if $d(x^2 y) = 0$

Proposition 2.11.1. *There $\exists$ ∞ integrating factors.*

Proof. Given that exists a solution $u(x,y) = C$ from an homogeneous differential equation, $\mu = \frac{\partial u/\partial x}{M} = \frac{\partial u \partial y}{N}$ is an integrating factor. Then $\mu F(u)$ is also an integrating factor $\forall F \Rightarrow \exists \infty$. Indeed, if $\mu M dx + \mu N dy = du \Rightarrow \mu F(u) M dx + \mu F(u) N dy = F(u) du = d[\int F(u) du]$

■

Example 2.11.2. $\forall F, xF(x^2 y)$ is an integrating factor of $2ydx + xdy = 0$

2.11.1 Finding integrating factors

2.11.2 Integrating factors differential equation

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \Rightarrow \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

Then we obtain the differential equation of the integrating factors

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = N\frac{\partial \ln \mu}{\partial x} - M\frac{\partial \ln \mu}{\partial y}$$

Nevertheless, this differential equation is more difficult to solve than the original but we only require one particular solution, not the general solution. Moreover, if we find another second integrating factor

$$N\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = N\frac{\partial \ln \nu}{\partial x} - M\frac{\partial \ln \nu}{\partial y}$$

Then

$$N\frac{\partial \ln \nu}{\partial x} - M\frac{\partial \ln \nu}{\partial y} = N\frac{\partial \ln \mu}{\partial x} - M\frac{\partial \ln \mu}{\partial y}$$

$$\frac{1}{M}\frac{\partial}{\partial x}\ln(\mu/\nu) = \frac{1}{N}\ln(\mu/\nu)$$

We know by definition that $\ln(\mu/\nu)$ isn't a constant, so the general solution is

$$\ln(\mu/\nu) = C,\ \ \mu/\nu = D$$

2.11.3 Search of integrating factors only function of x

Proposition 2.11.2. *Integrating factors depending only on x can be found*

$$\mu(x) = \exp\left\{\left[\int \frac{M_y - N_x}{N} dx\right]\right\}$$

Proof. Let $\mu(x)$ be an integrating factors which is a function of x. Let

$$Mdx + Ndy = 0$$

be a non-exact differential equation. Then, by definition of the intengrating factor:

$$M\mu(x)dx + N\mu(x)dy = 0$$

By definition of an exact differential equation:

$$\frac{\partial M\mu(x)}{\partial y} = \frac{\partial N\mu(x)}{\partial x}$$

$$\mu(x)M_x = \mu(x)N_x + \mu'(x)N$$

Rearranging terms

$$\frac{\mu'(x)}{\mu(x)} = \frac{M_y - N_x}{N} = P(x)$$

By separation of variables

$$\frac{d\mu}{\mu} = P(x)dx \Rightarrow \ln\mu = \int P(x)dx \Rightarrow \mu = \exp\left\{\left[\int \frac{M_y - N_x}{N}dx\right]\right\}$$

■

Example 2.11.3. Let be the following non-exact differential equation

$$(4x^2 + y)dx - xdy = 0$$

Then we try to obtain an integrating factors dependant on x or y using the methods described previously

$$P(x) = \frac{1-(-1)}{-x} \Rightarrow \mu = \exp\left\{\left[\int \frac{-2}{x}dx\right]\right\} = \exp\{[-2\ln x]\} = \frac{1}{x^2}$$

Now the following differential equation is exact

$$x^2(4x^2 + y)dx - x^3dy = 0$$

Finding u by the methods explained in the exact differential equations part

$$u = 4x - \frac{y}{x} = C \Rightarrow y = 4x^2 - Cx$$

2.11.4 Search of integrating factors only function of y

Searching for integration factors dependant only on y is analogous to searching for integration factors dependant on x so that proofs and propositions are also analogous.

Proposition 2.11.3. *Integrating factors depending only on x can be found*

$$\mu(x) = \exp\left\{\left[\int \frac{N_x - M_y}{M}dx\right]\right\}$$

Proof. Analogous to [**?**] ■

2.12 Line integrals

Proposition 2.12.1. *Let $\vec{V} = (V_x, V_y, 0)$ be a vector on a plane, such that*

$$V_x = M(x,y),\ V_y = N(x,y)$$

Then

$$\int \vec{V}\cdot d\vec{l} \ \textit{ independant of the path} \Leftrightarrow \nabla\times\vec{V} = 0 \Leftrightarrow (\nabla\times\vec{V})_z = 0 \Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Leftrightarrow Mdx+Ndy = 0 \ \textit{ exact}$$

Proof. Indeed, if $Mdx + Ndy = 0$ is exact, then

$$\int_{(a,b)}^{(c,d)} \vec{V}\cdot d\vec{l} = \int_{(a,b)}^{(c,d)} Mdx + Ndy = \int_{(a,b)}^{(c,d)} du(x,y) = u(c,d) - u(a,b)$$

■

2.13 Second grade differential equations solved by first grade methods

Differentials equations of the form

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

can easily be decomposed into two first-order differential equations in two different ways

- Case $f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$

  Realizing the next change of variables

$$v = \frac{dy}{dx} \Rightarrow f(x, v, \frac{dv}{dx}) = 0 \Rightarrow F(x, v, C) = 0 \Rightarrow F(x, \frac{dy}{dx}, C) = 0$$

$$\phi(x, y, C, D) = 0$$

- Case $f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$

  Realizing the same change of variables

$$v = \frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v \Rightarrow f(y, v, v\frac{dv}{dy}) = 0 \Rightarrow F(y, v, C) = 0 \Rightarrow F(y, \frac{dy}{dx}, C)$$

$$\phi(x, y, C, D) = 0$$

Example 2.13.1. Let be the following differential equation

$$yy'' = 2y'^2$$

Realizing the change of variables

$$v = y', y'' = v\frac{dv}{dy} \Rightarrow vy\frac{dv}{dy} = 2v^2$$

The first possibility is $v = 0$, if not

$$y\frac{dv}{dy} = 2v \Rightarrow 2\frac{dy}{y} = \frac{dv}{v} \Rightarrow 2\ln y = \ln v + A \Rightarrow \ln y^2 = \ln Bv$$

$$y^2 = B\frac{dy}{dx} \Rightarrow B\frac{dy}{y^2} = dx \Rightarrow y = \frac{B}{C - x} = \frac{1}{C_1x + C_2}$$

The particular solution $v = 0 \Rightarrow y = cte$ is already part of the general solution when $c_1 = 0$.

2.14 Summary

Orthogonal trajectories

$$F(x,y,dy/dx)=0 \Rightarrow F(x,y,-dx/dy)=0$$

Clairaut's equation

$$y=x\frac{dy}{dx}=f\left(\frac{dy}{dx}\right) \Leftrightarrow y=cx+f(c)$$

Envelope

$$\text{Necessary:} \qquad \exists c | y=cx+f(c), \qquad x=-\frac{\partial f}{\partial c}$$

Sufficient: satisfies the equation

Linear equation

$$\frac{dy}{dx}+P(x)y=Q(x)$$

$$y=C\exp\left\{\left[-\int P(x)dx\right]\right\}+\exp\left\{\left[-\int P(x)dx\right]\right\}\int Q(x)\exp\left\{\left[\int P(x)dx\right]\right\}dx$$

Bernouilli equation

$$\frac{dy}{dx}+P(x)y=Q(x)y^n \overset{z=y^{1-n}}{\Rightarrow} \frac{dz}{dx}+(1-n)P(x)z=(1-n)Q(x)$$

Ricatti's equation

$$y'=P(x)y^2+Q(x)y+R(x)$$

$$\text{One solution:} \qquad z=y-y_p \Rightarrow z'-(2y_pP(x)+Q(x))z=Pz^2$$

$$\text{Two solutions:} \qquad \frac{y-y_1}{y-y_2}=C\exp\left\{\left[\int P(x)(y_1-y_2)dx\right]\right\}$$

$$\text{Three solutions:} \qquad \frac{y-y_1}{y-y_3}\frac{y_3-y_2}{y_3-y_1}=B$$

Equation of the pdf

$$\frac{dy}{dx}=f(ax+by+c) \Rightarrow u=ax+by+c$$

Homogeneous equations with M and N of grade n

$$M(x,y)dx+N(x,y)dy=0 \overset{y=ux}{\Rightarrow} \frac{1}{x}dx=-\frac{N(1,u)}{M(1,u)+uN(1,u)}du$$

Exact equations

$$M(x,y)dx+N(x,y)dy=0=df \Leftrightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$$

$$y=\int M(x,y)dx+\int N(x,y)dy+\int\left[\frac{d}{dy}\int M(x,y)dx\right]dy$$

Integrating factor

$$\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N) \Rightarrow \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x}-M\frac{\partial \mu}{\partial y}\right)=\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}$$

$$\mu=\mu(x) \Rightarrow \mu(x)=\exp\left\{\left[\int \frac{M_y-N_x}{N}dx\right]\right\}$$

If μ is an integrating factor, $F(f)\mu$ is an integrating factor.

Some 2nd order DEs

$$f(x,y',y'')=0 \overset{v=y'}{\Rightarrow} f(x,y',y'')=0$$

$$f(y,y',y'')=0 \Rightarrow v=\frac{dy}{dx}=\frac{dv}{dy}v \Rightarrow f\left(y,v,v\frac{dv}{dy}\right)=0$$

Chapter 3

Linear equations

3.1 Preamble

Definition 3.1.1. A nth-order linear differential equation is:

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = R(x)$$

If $R(x) = 0$ the differential equation is reduced, otherwise is complete.
We've always solved this equation for $n = 1$. Let's study the case where $n \geq 2$, with particular uses on $n = 2$ case.
We have already previously seen:

- The general solution of a nth-differential equation is a function $y(x)$ with n arbitrary constants (which can be $\pm\infty$)
- $\exists!$ $y(x)$ with certain values of $y(x_0), y'(x_0) \ldots y^{(n-1)}(x_0)$

3.2 Reduced and complete equations

Solutions to reduced and complete differential equations satisfy the next proprieties

Proposition 3.2.1. *Let a solution to the reduced equation be cancelled, as they ordinary derivatives until $(n-1)$ in some x_0 point, then the solution is $y(x) \equiv 0$*

Proof. content... ∎

Proposition 3.2.2. *Let $u_1 \ldots u_k$ be solutions to the reduced diff. eq., then a lineal combinations of those ones $c_1u_1 + \ldots c_ku_k$ are also solution.*

Proof. content... ∎

Proposition 3.2.3. *Let y_1 and y_2 be solutions to the complete equation, $y_1 - y_2$ is solution to the reduced one.*

Proof. content... ∎

Proposition 3.2.4. *Let y_1 be solution to the complete equation and u_1 to the reduced one, then $y_1 + u_1$ is solution to the complete one.*

Proof. content... ∎

Theorem 3.2.5. *The general solution to a complete differential equation can be obtained adding the general solution of the reduced one to the particular solution of the complete one.*

Proof. content... ∎

3.3 Wronskians

Definition 3.3.1. The wronskian of n functions is $W(u_1 \ldots u_n)$ defined as

$$W(u_1 \dots u_n) = \begin{vmatrix} u_1 & \dots & u_n \\ u_1' & \dots & u_n' \\ \vdots & & \vdots \\ u_1^{(n-1)} & \dots & u_n^{(n-1)} \end{vmatrix}$$

$u_1 \dots u_n$ linearly dependant $\Leftrightarrow \exists c_1 \dots c_n$ not-null so that $c_1u_1 + \dots + c_nu_n \equiv 0$

Theorem 3.3.1. *Let be n linearly dependant functions and exists their derivatives until $(n-1)$, then its wronskian $\equiv 0$*

Proof. Columns are a linear combination of the rest, then W gets cancelled as it's a sum of null determinants by column repetition.

■

Theorem 3.3.2. *Let the wronskian of n solutions to a reduced differential equation get cancelled in a point, then the solutions are linearly dependant and its wronskian gets cancelled $\forall$ points.*

Proof. Let $u_1(x) \dots u_n(x)$ be the solutions. Let $W(x_0) = 0$. Then the equations $c_1u_1(x_0) + \dots + c_nu_n(x_o) = 0 = \dots c_1u_1^{(n-1)}(x_0) + \dots + c_nu_n^{(n-1)}(x_o) = 0$ have a null determinant $\Rightarrow$ they have $c_1 \dots c_n$ not all null as roots.
The solution $c_1u_1(x) + \dots + c_nu_n(x) = 0$ with not all null $c_1 \dots c_n$ gets cancelled (with each of its derivatives) in $x_0 \Rightarrow$ is identically null $\Rightarrow u_1 \dots u_n$ are linearly dependant $\Rightarrow W(x) = 0$

■

Theorem 3.3.3. *Every solution to the reduced equation can be expressed as a linear combination of n linearly independent solutions*

Proof. Let $\exists u_1 \dots u_n$ be linearly independent solutions. Let y be one solution. To satisfy $y = c_1u_1 + \dots + c_nu_n$, $c_1 \dots c_n$ must satisfy that

$$y(x_0) = c_1u_1(x_0) + \dots + c_nu_n(x_0) \dots\dots u^{(n-1)}(x_0) = c - 1u_1^{(n-1)}(x_0) + \dots + c_nu_n^{(n-1)(x_0)}$$

As $W(x_0) \neq 0$, there exists a unique solution $c_1 \dots c_n \Rightarrow c_1u_1 + \dots + c_nu_n$ coincides with y in one point (as well as its derivatives) $\Rightarrow$ coincides with y in all points.
This n linearly independent solutions we've supposed actually exist so that there always exist solutions $u_1 \dots u_n$ so that each u_i has certain values $u_i(x_0) \dots u_i^{(n-1)}(x_0)$. Enforcing values which don't cancel the wronskian $W(x_0)$ they are linearly independent.

■

In conclusion, to obtain the general solution to the reduced equation we must find out n linearly independent particular solutions (proving them through the wronskian if they are so)

All this resolution method (properties of the solutions to reduced and complete equations, Wronskians theorems) is only valid inside an interval $a < x < b$ where functions $P_1(x) \dots P_n(x)$ are continuous. This is because only in that acse the existence theorem we've already used many times (solution's unicity) in proofs.

3.4 Reduced equation with constant coefficients (second order)

Definition 3.4.1. A reduced second order differential equation with constant coefficients is one such

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

Let us try $y = e^{mx}$ as a solution. Then we obtain the equation

$$(m^2 + pm + q)e^{mx} = 0$$

$y = e^{mx}$ is solution to the reduced second order differential equation with constant coefficients if is a root from $m^2 + pm + q = 0$

Definition 3.4.2. The auxiliary equation is the reduced equation written as a polynomial.

$$m^2 + pm + q = 0$$

The two roots of the auxiliary equation can be: real and different, both real and equal, conjugate complex roots.

3.4.1 Real and distinct roots

This case happens when the discriminant $p^2 - 4q > 0$ is positive.
$e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent solutions to the reduced equation as $W = e^{m_1 x}e^{m_2 x}(m_2 - m_1) \neq 0$. The general solution is

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

3.4.2 Double real roots

This case happens when the discriminant $p^2 - 4q = 0$ is equal to zero.
Then $m = -p/2$ and e^{nx} is solution. Let's see xe^{mx} is also solution.

$$(xe^{mx})' = (xm + 1)e^{mx}$$
$$xe^{mx})'' = (xm^2 + 2m)e^{mx}$$

$$y'' + p' + qy = (xm^2 + 2m + pxm + p + qx)e^{mx} = 0$$
$$m = -p/2 \Rightarrow 2m + p = 0$$

Then the wronskian

$$W = (xe^{mx})(me^{mx}) - (xm + 1)e^{mx}e^{mx} = -e^{2mx} \neq 0$$

Solutions are linearly independent so the general solution is

$$y = Ae^{mx} + Bxe^{mx}$$

3.4.3 Complex conjugate roots

This case happens when the discriminant $p^2 - 4q < 0$ is less than 0, then we're getting two complex conjugate roots $m_1 = \alpha + ibeta, m_2 = \alpha - i\beta$. Then $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ are solutions. Moreover, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are solution, as they are linear combinations of the previous ones. They're linearly independent between them as:

$$W = e^{2\alpha x}\left[\cos\beta x(\alpha\sin\beta x + \beta\cos\beta x) - \sin\beta x(\alpha\cos\beta x) - \beta\sin\beta x)\right]$$
$$W = e^{2\alpha x}\beta(\cos^2\beta x + \sin^2\beta x) = \beta e^{2\alpha x} \neq 0$$

Then the general solution is

$$y = e\alpha x(A\cos\beta x + B\sin\beta x)$$

Example 3.4.1. Let be the following differential equation

$$y'' + 4' + 3y = 0 \Rightarrow m^2 + 4m + 3 = 0$$

The roots are

$$m_1 = -1, m_2 = -3$$

Then the general solution is

$$y = Ae^{-x} + Be^{-3x}$$

Example 3.4.2. Let be the next differential equation

$$y'' + 3y' + 3y = 0 \Rightarrow m^2 + 3m + 3 = 0$$

Then the roots are

$$m = \frac{-3 \pm i\sqrt{3}}{2}$$

So the general solution is

$$y = e^{-3x/2}(A \cos \frac{x\sqrt{3}}{2} + B \sin \frac{x\sqrt{3}}{2})$$

Example 3.4.3. Let be the following differential equation

$$y'' + 4y' + 4y = x + 1 \Rightarrow m^2 + 4m + 4 = 0$$

The double root is

$$m = -2$$

And the general equation to the reduced equation is

$$y = e^{-2x}(A + Bx)$$

Since the initial diff. eq. is complete, we need to find a particular solution to the complete one in order to get the general solution to it (remember, the general solution to a second order differential equation with constant coefficients is a linear combination of the general solution to the reduced one and a particular one to the general one). By trial and error we find

$$y_p = \frac{x}{4}$$

Then the general solution to the complete is

$$y = e^{-2x}(A + Bx) + \frac{x}{4}$$

3.5 Reduced differential equation with constant coefficients (nth-order)

Definition 3.5.1. A reduced nth-order differential equation with constant coefficients is one such

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0$$

There exists solutions of the form $e = mx$, being m root of the auxiliary equation

3.5.1 Real and distinct roots

Let $m_1, \dots m_2$ be real and distinct roots of the auxiliary equation. Then the general solution is

$$y = C_1 e^{m_1 x} + \cdots + C_n e^{m_n x}$$

3.5.2 Real, distinct and some multiple roots

Let m_k be a root with multiplicity r. Substituting r

3.5.3 Some complex root

3.5.4 Some complex multiple roots

3.6 Summary

Reduced complete equation with constant coefficients (using the auxiliary equation)

- A root m with one multiplicity: $y = ce^{mx}$
- A root m with multiplicity r: $y = (c_1 + c_2x + \cdots c_rx^{r-1})e^{mx}$
- A pair of complex roots $\alpha \pm i\beta$: $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$
- A pair of complex roots $\alpha \pm i\beta$ with multiplicity r: $y = e^{\alpha x}\left[(c_1 + c_2x + \cdots c_rx^{r-1})\cos(\beta x) + (d_1 + d_2x + \cdots d_2x^{r-1})\sin(\beta x)\right]$

Particular solution for the complete equation: undetermined coefficients' method

- If $R = R_k(x)$: try $P_k(x)$
- If $R = R_k(x)$ but the first r constants of the DE are zero: try $P_{k+r}(x)$ and without the first r components (already included in the h. solution)
- If $R = e^{mx}R_k(x)$ and m is not root of the a.equation: try $e^{mx}P_k(x)$
- $R = e^{mx}R_k(x)$ and m is a root of multiplicity r: try $e^{mx}P_{k+r}(x)$ and without the first r components (already included in the h. solution)
- If $R = e^{\alpha}R_k(x)\sin(\beta x)$ or $R = e^{\alpha}R_k(x)\cos(\beta x)$: try $e^{\alpha x}\left[P_k(x)\sin\beta x + Q_k(x)\cos\beta x\right]$. If $\alpha \pm i\beta$ are root of the a. equation, try P_{k+1} if R has cosine and Q_{k+1} if R has sine (and without the first component)

Particular solution for the complete equation of 2n order: variation of parameters' method

$$y'' + P(x)y' + Q(x)y = R(x) \Rightarrow u_1, u_2 \text{ solutions of the h.equation} \Rightarrow$$
$$y_p = u_1 \int -R\frac{u_2}{W(u_1, u_2)}dx + u_2 \int R\frac{u_1}{W(u_1, u_2)}dx$$

Particular solution for the complete equation: variation of parameters' method

$$\begin{cases} u_1t_1' + \cdots + u_nt_n' = 0 \\ \vdots \\ u_1^{(n-2)}t_1' + \cdots + u_n^{(n-2)}t_n' = 0 \\ u_1^{(n-1)}t_1' + \cdots + u_n^{(n-1)}t_n' = R \end{cases} \Rightarrow \text{solve } t_i = \varphi_i(x) \Rightarrow y_p = \sum_{i=1}^{n} u_i \int \varphi_i(x)dx$$

Symbolic method

$$P[D]y = g \Rightarrow y = \frac{1}{D - m_1} \cdots \frac{1}{D - m_n}g \Rightarrow \frac{1}{D - m}g(x) = e^{mx}\int g(x)e^{-mx}dx$$

Reduction of order (2nd order)

$$y'' + P(x)y' + Q(x)y = R(x) \Rightarrow \text{solution of r.eq } u \Rightarrow y = tu \Rightarrow ut'' + [2u' + Pu]\,t' = R$$

Reduction of order (having p independent solutions)

$$v = \begin{vmatrix} u_1 & \dots & u_p & y \\ \vdots & \ddots & \vdots & \vdots \\ u_1^{(p)} & \dots & u_p^{(p-1)} & y^{(p-1)} \end{vmatrix} \varphi(x) \Rightarrow \begin{cases} v = A(x)y^{(p)} + \cdots\dots \\ \vdots \\ v^{(n-p)} = A(x)y^{(n)} + \cdots \end{cases}$$

- Express $y^{(p)}, \dots y^{(n)}$ in terms of $v, \dots, v^{(n-p)}, x, y, \dots, y^{(p-1)}$. We get

$$Q_0v^{(n-p)} + \cdots + Q_{n-p}v + V = 0 \Rightarrow Q_0v^{(n-p)} + \cdots + Q_{n-p}v = 0$$

- Cauchy-Euler equation

Solved using $x = e^t$ to obtain a secon-order linear differential equation with constant coefficients:

$$x^2 \frac{d^2}{dx^2} + px\frac{dy}{dx} + qy = R(x) \Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} + p\frac{dy}{dt} + qy = R(t)$$

The nth-oder Cauchy-Euler can be reduced with the same change of variables:

$$\sum_{p=0}^{n} a_p x^p \frac{d^p y}{dx^p} = R(x)$$

Chapter 4

Laplace transform

4.1 Laplace transform of a function

4.1.1 Definition

The Laplace Transform named after Pierre-Simon Laplace, is an integral transform that converts a function of a real variable t to a function of a complex variable s (complex frequency). The transform is specially useful when solving differential equations, transforming them into algebraic equations and transforming convolution operation into multiplication.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

It's defined then, in the interval where s makes the integral converge, where s is a complex number frequency parameter with real numbers σ and ω

$$s = \sigma + i\omega$$

4.1.2 Fourier Transform

The Fourier Transform, which we are not going to study, is directly related to the Laplace transform. While the Fourier Transform of a function is a complex function of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable.
The only difference between them is the missing real number σ in the Fourier Transform, being $s = i\omega$ now

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_0^\infty e^{-i\omega t} f(t) dt$$

4.1.3 Bilateral Laplace Transform

When one says "the Laplace transform" without qualification, the unilateral or one-sided transform is usually intended. However, the Laplace Transform can be alternatively defined as the bilateral Laplace transform, by extending the limits of integration to be the entire real axis.
If that is done, the common unilateral transform simply becomes a special case of the bilateral transform, where the definition of the function being transformed is multiplied by the Heaviside step function.

$$\mathcal{L}\{f(t)\} = F(s) = \int_{-\infty}^\infty e^{-st} f(t) dt$$

4.1.4 Inverse Laplace Transform

The Inverse Laplace Transform of a function $F(s)$ will often be found remembering the transforms of commons functions. Nevertheless, it's important to remember there exists the Bromwich integral, Fourier-Mellin integral or Mellin's inverse formula to find the Inverse Laplace Transform of a function, given by

$$f(t) = \mathcal{L}^{-1}\{F(t)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s) ds$$

Time domain $f(t) = \mathcal{L}^{-1}\{f(t)\}$	s domain $F(s) = \mathcal{L}\{f(t)\}$
$f(t) = a$	$F(s) = \frac{a}{s}$
$f(t) = \frac{t^{n-1}}{(n-1)}$	$F(s) = \frac{1}{s^n}$
$f(t) = \frac{\sin(at)}{a}$	$F(s) = \frac{1}{s^2+a^2}$

4.1.5 Some unilateral transforms

4.1.6 Properties

Proposition 4.1.1. *Let $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$*

Proof.

$$\mathcal{L}\{f(t)e^{at}\} = \int_0^\infty e^{-st}f(t)e^{at}dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$$

■

Proposition 4.1.2. *Let $\mathcal{L}\{f_1(t)\} = F_1(s)$, $\mathcal{L}\{f_2(t)\} = F_2(s)$, then $\mathcal{L}\{c_1f_1(t)+c_2f_2(t)\} = c_1F_1(s)+c_2F_2(s)$*

Proof.

$$\mathcal{L}\{c_1f_1(t)+c_2f_2(t)\} = \int_0^\infty e^{-st}(c_1f_1(t)+c_2f_2(t))dt = c_1\int_0^\infty e^{-st}f_1(t)+c_2\int_0^\infty e^{-st}f_2(t) = c_1F_1(s)+c_2F_2(s)$$

■

Proposition 4.1.3. *Let $F(s) = s_0$ converge for $s = s_0$, then it converges $\forall s > s_0$*

Proof. content... ■

Proposition 4.1.4. *For any continuous function $f(t)$ in the interval $0 \leq t \leq \infty$ (or at least discontinuous in a finite number of points) which satisfies $\mid f(t) \mid Me^{at}$, there exists the Laplace Transform $\forall s > a$.*

Proof. content... ■

4.2 Laplace Transform of a differential equation

4.2.1 Laplace Transform of derivatives

If $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt$. Then the Transform of its derivative is

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

This can be easily proven with integration by parts

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt = -f(0) + sF(s)$$

In the case of the second derivative

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st}f''(t)dt = e^{-st}f'(t)\Big|_0^\infty + se^{-st}f(t)\Big|_0^\infty + s^2\int_0^\infty e^{-st}f(t)dt = -f'(0) - sf(0) + s^2F(s)$$

We can repeat successively to find the nth-derivative, as we do in the next proposition.

Proposition 4.2.1. *The Laplace Transform the nth-derivative of a function is*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)(0)}$$

Proof. This proposition can be proved by induction over n ∎

This property is extremely useful for solving differential equations as it transforms derivatives into algebraic expressions.

4.2.2 Linear differential equations with constant coefficients

Let be the following differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = r(t) \qquad y = y(t) \tag{4.1}$$

Taking the Laplace Transform of every member we end up with

$$R(s) = (a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n) Y(s) - \sum_{j=1}^{n} a_{n-j} \sum_{k=0}^{j-1} y^{(k)}(0) s^{j-1-k}$$

$$R(s) = \int_0^\infty e^{-st} r(t) dt \qquad Y(s) = \int_0^\infty e^{-st} y(t) dt$$

The solution to the initial differential equation (4.1) then is, isolating the function $Y(s)$

$$Y(s) = \frac{R(s) + \sum_{j=1}^{n} a_{n-j} \sum_{k=0}^{j-1} y^{(k)}(0) s^{j-1-k}}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

This method is useful for solving differential equations if we know the required boundary conditions $y(0), y'(0), \ldots y^{n-1}(0)$ (unique solution by The Existence and Unicity Theorem) as then $Y(s)$ can be computed and we can find $y(t)$ as long as we can identify $Y(s)$ with the corresponding Inverse Laplace Transform.

Example 4.2.1. Let be the following differential equation

$$y''' - y' = 2\cos(t) \qquad y(0) = y'(0) = y''(0) = 0$$

Then the solution in terms of s is

$$Y(s) = \frac{2s}{(1+s^2)(s^3 - s)}$$

Decomposing the function $Y(s)$ to partial fractions

$$Y(s) = \frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{1}{s^2+1}$$

Identifying the Inverse Laplace Transform

$$y(t) = \frac{e^t}{2} - \frac{e^{-t}}{2} - \sin(t) = \sinh(t) - \sin(t)$$

Example 4.2.2. Let be the following differential equation

$$y''' - y' = 2\cos(t) \qquad y(0) = 3, y'(0) = 2, y''(0) = 1$$

Then

$$Y(s) = \frac{\frac{2s}{1+s^2} + (-3 + 3s^2 + 2s + 1)}{s^3 - s} = \frac{2s + (3s^2 + 2s - 2)(s^2 + 1)}{(s^2+1)s(s-1)(s+1)}$$

$$Y(s) = \frac{2}{s} + \frac{2}{s-1} - \frac{1}{s+1} - \frac{1}{s^2+1}$$

The solution is

$$y(t) = 2 + 2e^t - e^{-t} - \sin(t)$$

4.3 Unicity of the Inverse Laplace Transform

4.3.1 Prelims of the Theorem

Proposition 4.3.1. *$\forall \varepsilon > 0$ arbitrarily small, if $u(x)$ is a continuous function, we can find a polynomial $P(x) \mid |u(x) - P(x)| \leq \varepsilon, \forall x$.*

Lemma 4.3.2. *If $u(x)$ is continuous in $[0,1]$ and $\int_0^1 x^n u(x)dx = 0$ $\forall n \in \mathbb{N}*$, then $u(x) = 0$ in $[0,1]$.*

Proof. Obviously

$$\int_0^1 u(x)P(x)dx = 0$$

$$\int_0^1 u^2(x)dx = \int_0^1 u(x)[u(x) - P(x)]dx \leq \varepsilon |u(x)|dx$$

Then

$$\int_0^1 u^2(x)dx \leq A\varepsilon = \varepsilon' \ A \text{ finite} \Rightarrow u(x) = 0 \text{ in } [0,1]$$

■

4.3.2 Unicity Theorem

Theorem 4.3.3. *The Laplace Transform $F(s)$ of a continuous function $f(t)$ isn't the Transform of any other continuous function.*

Proof. Let $f(t)$ and $g(t)$ have the same Laplace Transform for $s \geq s_0$, and let

$$h(t) = f(t) - g(t) \qquad \mathcal{L}\{h(t)\} \equiv 0 \tag{4.2}$$

Then, for $s \geq s_0$

$$\int_0^\infty e^{-st}h(t)dt = 0, \ \forall s \geq s_0$$

Let $s = s_0 + n$, $n \in \mathbb{N}$, then

$$\int_0^\infty e^{-nt}e^{-s_0 t}h(t)dt = 0$$

Integrating by parts:

$$e^{-nt}\int_0^t e^{-s_0\tau}h(\tau)d\tau\Big|_0^\infty + n\int_0^\infty e^{-nt}\left[\int_0^t e^{-s_0\tau}h(\tau)d\tau\right]dt = 0$$

The first term is 00, as $e^{-nt} = 0$ when $t = \infty$, and the integral is zero when $t = 0$. With the change of variables $x = e^{-t}$, $dx = -xdt$, then:

$$\int_0^1 x^{n-1}v[-\ln(x)]dx = 0, \ \forall n \in \mathbb{N}$$

According to the previous lemma $v[-\ln(x)] = 0$ if $x \in [0,1]$. Then

$$v(t) \equiv 0 \text{ for } 0 \leq t \leq \infty \Rightarrow v'(t) \equiv 0 \text{ for } 0 \leq t \leq \infty$$

$$e^{-s_0 t}h(t) \equiv 0 \text{ for } 0 \leq t \leq \infty \Rightarrow h(t) \equiv 0 \Rightarrow f(t) \equiv g(t)$$

■

4.4 More properties of the Laplace Transform

Proposition 4.4.1. *The Laplace Transform of $t^n f(t), \forall n \in \mathbb{N}$ is $(-1)^n F^{(n)}(s)$*

Proof. Let $F(s) = \int_0^\infty e^{-st} f(t)dt$, then

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t)dt = (-1)^n \mathcal{L}\{t^n f(t)\}$$

■

Proposition 4.4.2. *The Laplace Transform of $\int_0^t f(\tau)d\tau$ is $F(s)/s$*

Proof. Using the previous property for the Transform of the first derivative of a function. Let $g(t) = \int_0^t f(\tau)d\tau$, then

$$\mathcal{L}\{g'(t)\} = sG(s) - g(0) = s\mathcal{L}\{g(t)\} - \int_0^0 f(\tau)d\tau = s\mathcal{L}\{g(t)\} = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}$$

By the fundamental theorem of Calculus

$$\mathcal{L}\{g'(t)\} = \mathcal{L}\left\{\frac{d}{dt}\int_0^t f(\tau)d\tau\right\} = \mathcal{L}\{f(t)\}$$

Then

$$\mathcal{L}\{f(t)\} = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} \Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

■

Theorem 4.4.3. *Let $F(s)$ and $G(s)$ be the Laplace Transforms of $f(t)$ and $g(t)$, then then product $F(s)G(s)$ is the Laplace Transform of $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$. Both integrals are named the convolution of $f(t), g(t)$.*

Proof. Let $F(s) = \int_0^\infty e^{-su} f(u)du$ and $G(s) = \int_0^\infty e^{-sv} g(v)dv$. Then

$$F(s)G(s) = \int\int e^{-s(u+v)} f(u)g(v)dudv$$

Is an integral in the (u, v) plane which extends over a certain region. Let $u + v = t, v = \tau$, then

$$F(s)G(s) = \int\int e^{-st} f(t-\tau)g(\tau)dtd\tau$$

Is an integral in the (t, τ) plane which below a line region, the line $\tau = t - u$ with slope $m = 1$

$$F(s)G(s) = \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]dt = \mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} = \mathcal{L}\{(f * g)(t)\}$$

We can prove the case $F(s)G(s) = \mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\}$ similarly by taking $u + v = t$ and $u = \tau$. ■

Proposition 4.4.4. *The Laplace Transform of $\mathcal{L}\{f(t-a)\theta(t-a)\}$, where $\theta(t)$ is the Heaviside step function, is $e^{-as}\mathcal{L}\{f(t)\}$*

Proof. Recalling the Heaviside step function

$$\theta(t-a) = \begin{cases} 1 & \text{if } t \leq a \\ 0 & \text{if } t < a \end{cases}$$

Then the Laplace Transform is

$$\mathcal{L}\{f(t-a)\theta(t-a)\} = \int_0^\infty e^{-st} f(t-a)\theta(t-a)dt = \int_a^\infty e^{-st} f(t-a)dt$$

Let $\tau = t - a$

$$\mathcal{L}\{f(t-a)\theta(t-a)\} = e^{-as} \int_a^\infty e^{-s\tau} f(\tau)d\tau = e^{-as}\mathcal{L}\{f(t)\}$$

■

Example 4.4.1. Remember that $\mathcal{L}\{1\} = 1/s$ and $\mathcal{L}\{t\} = 1/s^2$

$$\mathcal{L}\{t\theta(t-2)\} = \mathcal{L}\{(t-2+2)\theta(t-2)\} = \mathcal{L}\{(t-2)\theta(t-2)\} + 2\mathcal{L}\{\theta(t-2)\} = \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}$$

4.5 Summary

- **Definition**

Being defined only in the interval of s where the integral converges for.

$$f(t) \to \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

- **Some transforms**

- **Main properties**

⋆ $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{f(t)e^{at}\} = F(s-a)$
$\mathcal{L}\{f_1(t)\} = F_1(s)$, $\mathcal{L}\{f_2(t)\} = F_2(s)$, then $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$

⋆ If $F(s)$ converges for $s = s_0$, then it converges $\forall s > s_0$

⋆ For every continuous function $f(t)$ in $0 \leq t \leq \infty$ (or at least discontinuous on a finite number of points) that is upper bounded ($|f(t)| < Me^{at}$), there exists the Laplace transform for $s > a$.

- **More properties**

⋆ $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$
$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}$

⋆ $\mathcal{L}\{f(t-a)\theta(t-a)\} = e^{-as}\mathcal{L}\{f(t-a)\}$

- **Differential equation solving**

$$\mathcal{L}\{f(t)\} = F(s)$$
$$\mathcal{L}\{f(t)'\} = sF(s) - f(0)$$
$$\mathcal{L}\{f(t)''\} = s^2 F(s) - s(f0) - f'(9)$$
$$\mathcal{L}\{f(t)^{(n)}\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$$

- **Convolution theorem**

Let $F(s)$, $G(s)$ be the Laplace transforms of the functions $f(t)$, $g(t)$, then:

$$F(s)G(s) = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\}$$

Chapter 5

Power series

5.1 Introduction

In this have chapter we are going to study some differential equations of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{5.1}$$

which require the construction of power series to find the solution.

The most important solutions are *hypergeometric equations* (or Gauss equations), *Legendre equations* (particular case of the previous one), *Bessel equations*, *Laguerre equations* and *Hermite equations.*

These equations are key in some physics fundamental problems:

- **Electromagnetism:** In the Laplace equation $\nabla^2 u = 0$, the radial part in cylindrical coordinates is a Bessel equation, the angular part in spherical coordinates is a Legendre equation.
- **Quantum Mechanics:** To determine the radial part of the wave function associated to a free particle we'll have Bessel equations. To determine the eigen values of the angular momentum operator we have Legendre equations. To determine the eigen values of the energy in an harmonic oscillator potential we have Hermite equations. In the case of Coulomb's potential (e.g. and hydrogen atom) we have Legendre equations.

5.2 Ordinary points

5.2.1 Definitions

Definition 5.2.1. Let $f(x)$ be a function, then $f(x)$ is *analytical* in a if it is an infinitely differentiable function such that the Taylor series at a converges pointwise to $f(x)$ for a in a neighbourhood of x.

Definition 5.2.2. In equations of the form (5.1), a is an *ordinary point* if $P(x)$, $Q(x)$ are analytical in a.

Definition 5.2.3. If one the functions isn't analytical in a, but both of them are analytical in a neighbourhood of a, then a is a *singular point.*

5.2.2 Series expansion

If a is an ordinary point, the general solution to (5.1) can be found with power series expansion around $(x-a)$, convergent until the closest singular point. Outside this interval mar or may not converge.

Proof. Let $a = 0$. Generally, the process is the same but instead of x the powers are $x - a$, and let

$$P(x) = \sum_{n=0}^{\infty} p_n x^n \qquad Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

Let

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \qquad y'(x) = \sum_{n=0}^{\infty} (n+1)c_{n+1} x^n \qquad y'(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

Substituting in (5.1) using

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(a_{n-k}b_k)\right)x^n$$

Then

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)c_{n+2}+\left[\sum_{k=0}^{n}(k+1)p_{n-k}c_{k+1}+q_{n-k}c_k\right]\right)x^n=0$$

Isolating

$$c_{n+2}=\frac{-\sum_{k=0}^{\infty}(k+1)p_{n-k}c_{k+1}+q_{n-k}c_k}{(n+1)(n+2)}$$

Then every coefficient is determined by the previous ones. There's two arbitrary constants c_0 and c_1, leading to the general solution of the differential equation (5.1). ■

Example 5.2.1. Let be the following differential equation

$$(x^2-1)y''+(x+3)y'-y=0 \text{ around } x=0$$

We know that

$$y(x)=\sum_{n=0}^{\infty}c_nx^n \Rightarrow y'(x)=\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n \Rightarrow xy'(x)=\sum_{n=0}^{\infty}nc_nx^n$$
$$y''(x)=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n \Rightarrow x^2y''(x)=\sum_{n=0}^{\infty}n(n-1)c_nx^n$$

Substituting

$$n(n-1)c_n-(n+2)(n+1)c_{n+2}+nc_n+3(n+1)c_{n+1}-c_n=0$$
$$c_n(n^2-1)+3c_{n+1}(n+1)-c_{n+2}(n+2)(n+1)=0$$

$$n=0 \Rightarrow -c_0+3c_1-2c_2=0 \Rightarrow c_0,c_1 \text{ arbitraty constants which determine } c_2$$
$$n=1 \Rightarrow 6c_2-6c_3=0 \Rightarrow c_2=c_3$$
$$n>1 \Rightarrow c_{n+2}=\frac{c_n(n^2-1)+3c_{n+1}(n+1)}{(n+1)(n+2)} \text{ then if } c_n=c_{n+1} \Rightarrow c_{n+2}=c_{n+1}=c_n$$

So $c_2=c_3=c_4\dots$ being

$$-c_0+3c_1-2c_2=0 \Rightarrow c_2=\frac{3c_1-c_0}{2}$$

Then the general solution is

$$y=\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+\left(\frac{3c_1-c_0}{2}\right)(x^2+x^3+x^4+x^5+\dots)$$

Which can be written as

$$y=\frac{A}{1-x}+B(x+3)$$

The solution $\frac{1}{1-x}=1+x+x^2+x^3+\dots$ corresponds to $c_0=c_1=1$.
The solution $x+3$ corresponds to $c_0=3$, $c_1=1$.

The general solution must be valid from the ordinary point $x=0$ until the closest singular points $x=\pm1$. Nevertheless, in this example the solution is also valid outside this interval.

5.3 Regular singular points

5.3.1 Definition

Definition 5.3.1. Consider the previous (5.1) equation

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

a singular point a is *regular singular* if $(x-a)P(x)$, $(x-a)^2Q(x)$ are analytical, thus they allow Taylor Series expansion

5.3.2 Series expansion (Frobenius Method)

Let be the following differential equation

$$(x-a)^2\frac{d^2y}{dx^2}+(x-a)p(x)\frac{dy}{dx}+q(x)y=0$$

and

$$\begin{aligned}p(x)&=(x-a)P(x)=p_0+p_1(x-a)+p_2(x-a)^2+\dots\\q(x)&=(x-a)^2Q(x)=q_0+q_1(x-a)+q_2(x-a)^2+\dots\end{aligned}$$

Let's prove there exits a power series solution of the form

$$y=(x-a)^r+c_1(x-a)^{r+1}+c_2(x-a)^{r+2}+\dots$$

Proof. Let $a=0$. Generally, the process is the same but instead of x the powers are $x-a$. So the differential equation is

$$x^2\frac{d^2y}{dx^2}+xp(x)\frac{dy}{dx}+q(x)y=0$$

So let

$$p(x)=\sum_{n=0}^{\infty}p_nx^n\qquad q(x)=\sum_{n=0}^{\infty}q_nx^n\qquad y(x)=\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$xy'(x)=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}\qquad x^2y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r}$$

Substituting in (5.1) using

$$\left(\sum_{n=0}^{\infty}a_nx^n\right)\left(\sum_{n=0}^{\infty}b_nx^n\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(a_{n-k}b_k)\right)x^n$$

Then

$$\sum_{n=0}^{\infty}\left((n+r)(n+r-1)c_n+\left[\sum_{k=0}^{n}(k+r)p_{n-k}c_k+q_{n-k}c_k\right]\right)x^{n+r}=0$$

Isolating the n-th term of the k summations

$$\sum_{n=0}^{\infty}\left([(n+r)(n+r-1)+(n+r)p_0+q_0]c_n+\sum_{k=0}^{n-1}[(k+r)p_{n-k}+q_{n-k}]c_k\right)x^{n+r}=0$$

Defining $F(t)=t(t-1)+tp_0+q_0$, it can be proven that every coefficient is determined by the previous ones as long as F doesn't cancel.

The value of r is determined for having to be null when $n=0$. This gives us the *indicial equation*

Definition 5.3.2. The indicial equation in a series expansion with Frobenius' Method is

$$r(r-1)+rp_0+q_0=F(r)=0$$

Being p_0 and q_0 the zero-order terms in Taylor expansions of $p(x)$ and $q(x)$.

Knowing r and letting $c_0=1$, the following recurrence relation gives us the rest of coefficients

$$c_n=\frac{-\sum_{k=0}^{n-1}[(k+r)p_{n-k}+q_{n-k}]c_k}{F(n+r)}\tag{5.2}$$

■

Distinguishing 3 different cases of the indicial equation

- **Case 1:** distinct r_1, r_2, they don't differ by an integer

  If F is zero for r_1, r_2, it isn't null for any $n + r_1$, $n + r_2$. Nothing avoids finding two series (one for every root). The linear combination of them is the general solution

- **Case 2:** Same roots, $r_1 = r_2$

  Then we only have one series as if $y(r,x)\Big|_{r=r_1}$ is solution, then $\frac{\partial}{\partial r}y(r,x)\Big|_{r=r_1}$ where x and r are independent variables. In effect, substituting $y(r,x)$ $\forall r$ into the equation, each term is cancelled due to the recurrence relation except for the $n = 0$ term. We finally have $x^r F(r)$ where in this case $F(r) = (r - r_1)^2$.
  For $\frac{\partial}{\partial r}y(r,x)$ we have $x^r(r - r_1)[(r - r_1)\ln(x) + 2]$ which indeed is zero if $r = r_1$.

  Eventually, if the first solution has the form of $\sum_{n=0}^{\infty} c_n x^{n+r_1}$, the second solution will be

$$y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} c'_n x^{n+r_1}$$

  Where

$$c'_n = \frac{\partial c_n}{\partial r}\Big|_{r=r_1}$$

  It's recurrence relation is determined by (5.2) or by direct substitution into the equation. As $c_0 = 1$, the dependence of c_n on r resets for $n = 1$.

- **Case 3:** distinct r_1, r_2 differing by an integer

  We already have one solution of the form $\sum_{n=0}^{\infty} c_n x^{n+r_1}$. To obtain the second one, we can apply the same idea treating r as a variable, like in the previous case, to get

$$y_2 = b_N(r_2) y_1 \ln(x) + \sum_{n=0}^{\infty} b'_n(r_2) x^{n+r_2}$$

  Being

$$b_n = (r - r_2)c_n \qquad b'_n(r) = \frac{\partial}{\partial r} b_n(r) \qquad b'_n(r_2) = c_n(r_2) \text{ if } n < N$$

  While $b_N(r_2)$ can be zero, $b'_0(r_2) \neq 0$ can't be.
  b_N and b'_n are determined by (5.2) or by direct substitution into the equation.

- **Alternatives for Cases 2 and 3**

  In Case 3, if with the r_2 series we end up with $c_N = 0/0$, we can take c_N so that the series doesn't match with r_1 series, then we have this second solution.
  In both Cases 2 and 3, we can also find the second solution using the studied method of "Finding the general solution from a solution to the reduced one". In many cases it'll be the easiest and shortest option to choose.

Example 5.3.1. Let be the following differential equation

$$2xy'' + y' + y = 0 \qquad y'' + P(x)y' + Q(x) = 0 \Rightarrow P(x) = \frac{1}{2x}, Q(x) = \frac{1}{2x}$$

Which can also be rewritten as the following to better find the polynomials

$$x^2y'' + \frac{x}{2}y' + \frac{x}{2}y = 0 \Leftarrow x^2y'' + xp(x)y' + q(x)y = 0$$

To solve it around $x = 0$ by series expansion (Frobenius method) we'll have to identify the polynomials

$$p(x) = xP(x) = \frac{1}{2} \Rightarrow p_0 = \frac{1}{2} \qquad q(x) = x^2 Q(x) = \frac{x}{2} \Rightarrow q_0 = 0, q_1 = \frac{x}{2}$$

The indicial equation then will be

$$F(r) = r(r-1) + \frac{r}{2} = r\left(r - \frac{1}{2}\right) = 0$$

With roots $r = 0$ and $r = 1/2$.

- Series for $r = 0$

  The solution must be of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad c_0 = 1$$

  Substituting into the original differential equation we end up with

$$\sum_{n=0}^{\infty} 2c_n n(n-1)x^{n-1} + nc_n x^{n-1} + c_n x^n = 0$$

  Redefining indices to factor out x^n

$$\sum_{n=0}^{\infty} (c_{n+1}[2n(n+1) + (n+1)] + c_n)x^n = 0$$

  The recurrence relation will be

$$c_{n+1} = \frac{-c_n}{(n+1)(2n+1)} \Rightarrow c_n = \frac{(-1)^n 2^n}{(2n)!}$$

  Which satisfies that $c_0 = 1$. Later on it will be provided some tricks and formulas to identify the recurrence relation with the sequence c_n without any term depending on the previous ones. So far, our solution for $r = 0$ is

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n = \begin{cases} \cos\left(\sqrt{2x}\right) & x > 0 \\ \cosh\left(\sqrt{-2x}\right) & x < 0 \end{cases}$$

  Which is valid $\forall x$ except for the regular singular point $x = 0$, as it would be a trivial solution for the equation with $p(x)$ and $q(x)$ and a different differential equation in case of the original problem.

- Series for $r = 1/2$

  The solution must be of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+1/2} \qquad c_0 = 1$$

  Substituting into the original differential equation we end up with

$$\sum_{n=0}^{\infty} 2c_n \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) x^{n-1/2} + \left(n + \frac{1}{2}\right) c_n x^{n-1/2} + c_n x^{n+1/2} = 0$$

  Redefining indices to factor $x^{n+1/2}$, we find the first terms of $x^{n-1/2}$ cancel out

$$\sum_{n=0}^{\infty} \left(c_{n+1}\left[2\left(n + \frac{3}{2}\right)\left(n + \frac{1}{2}\right) + \left(n + \frac{3}{2}\right)\right] + c_n\right) x^{n+1/2} + x^{-1/2}\left(-\frac{1}{2} + \frac{1}{2}\right) = 0$$

Thus, the recurrence relation is

$$c_{n+1} = \frac{-c_n}{(2n+3)(n+1)} \Rightarrow c_n = \frac{(-1)2^n}{(2n+1)!}$$

Which also satisfies $c_0 = 1$. The solution must be

$$y = \sum_{n=0}^{\infty} \frac{(-1)2^n}{(2n+1)!} x^{n+1/2} = \begin{cases} \dfrac{1}{\sqrt{2}} \sin(\sqrt{2x}) & x > 0 \\ \dfrac{1}{\sqrt{-2}} \sinh(\sqrt{-2x}) & x < 0 \end{cases}$$

Eventually, if the previous solutions found are linearly independent between them, as in this case, the solution to the original differential equation must be the linear combination of them, as we learnt in Chapter 3

$$y(x) = \begin{cases} A\cos(\sqrt{2x}) + B\sin(\sqrt{2x}) & x > 0 \\ A\cosh(\sqrt{-2x}) + B\sinh(\sqrt{-2x}) & x < 0 \end{cases}$$

Which, as we said before, is valid $\forall x$, as $x = 0$ is the only singularity.

Example 5.3.2. Let be the following differential equation

$$xy'' - y' - 4x^3 y = 0$$

To solve it around $x = 0$ by series expansion (Frobenius method) we'll have to identify the polynomials

$$p(x) = -1 \Rightarrow p_0 = -1 \qquad q(x) = -4x^4 \Rightarrow q_0 = q_1 = q_2 = q_3 = 0, q_4 = -4x^4$$

The indicial equation then will be

$$F(r) = r(r-1) - r = r(r-2) = 0$$

With roots $r = 0$ and $r = 2$.

- Series for $r = 2$
  The solution must be of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+2} \qquad c_0 = 1$$

Substituting into the original differential equation we end up with

$$\sum_{n=0}^{\infty} (n+2)[(n+1)-1]c_n x^{n+1} - 4\sum_{n=0}^{\infty} c_n x^{n+5} = 0$$

Which can be written as

$$3c_1x^2 + 8c_2x^3 + 15c_3x^4 + \sum_{n=4}^{\infty} n(n+2)c_n x^{n+1} - 4\sum_{n=0}^{\infty} c_n x^{n+5} = 0$$

Redefining indices to factor x^{n+5}

$$3c_1x^2 + 8c_2x^3 + 15c_3x^4 + \sum_{n=0}^{\infty} (n+4)(n+6)c_{n+4}x^{n+5} - 4c_n x^{n+5} = 0$$

Thus, the recurrence relation and the first 3 terms are

$$c_{n+4} = \frac{4c_n}{(n+4)(n+6)} \qquad c_1 = c_2 = c_3 = 0$$

Note that all c_n are 0 except for multiples of 4. Let $n = 4m$ and $d_m = c_{4m}$, then

$$c_{4m+4} = \frac{4c_{4m}}{(4m+4)(4m+6)} \Rightarrow d_{m+1} = \frac{d_m}{(2m+2)(2m+3)} \Rightarrow d_m = \frac{1}{(2m+1)!}$$

Satisfying $d_0 = c_0 = 1$. Eventually, the solution must be

$$y = \sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{m=0}^{\infty} d_m x^{4m+2} = \sum_{m=0}^{\infty} \frac{(x^2)^{2m+1}}{(2m+1)!} = \sinh(x^2)$$

- Series for $r = 0$
  The solution must be of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n} \qquad c_0 = 1$$

Substituting into the original differential equation we end up with

$$\sum_{n=0}^{\infty} n[(n-1)-1]c_n x^{n-1} - 4\sum_{n=0}^{\infty} c_n x^{n+3} = 0$$

Which can be written as

$$-c_1 + 3c_3x^2 + \sum_{n=4}^{\infty} n(n-2)c_n x^{n-1} - 4\sum_{n=0}^{\infty} c_n x^{n+3} = 0$$

Redefining indices to factor x^{n+3}

$$-c_1 + 3c_3x^2 + \sum_{n=0}^{\infty} (n+4)(n+2)c_{n+4}x^{n+3} - 4c_n x^{n+3} = 0$$

Thus, the recurrence relation and the c_1, c_3 are

$$c_{n+4} = \frac{4c_n}{(n+4)(n+2)} \qquad c_1 = c_3 = 0$$

Note that all odd c_n are zero, multiples of 2 are undetermined but linked to an arbitrary constant c_2 which we chose to be zero, and multiples of 4 are determined by the recurrence relation. Choosing c_2 and letting $n = 4m$ and $d_m = c_4 m$.

$$c_{4m+4} = \frac{4c_{4m}}{(n+4)(n+2)} \Rightarrow d_{m+1} = \frac{4d_m}{(n+4)(n+2)} \Rightarrow d_m = \frac{1}{(2m)!}$$

Satisfying $d_0 = c_0 = 1$. Eventually, the solution must be

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{m=0}^{\infty} d_m x^{4m} = \sum_{m=0}^{\infty} \frac{(x^2)^{2m}}{(2m)!} = \cosh(x^2)$$

So far, the general solution to the original reduced differential equation must be the linear combination of the previous independent ones

$$y(x) = A \sinh(x^2) + B \cosh(x^2)$$

Which is valid $\forall x$ except for the regular singular point $x = 0$.

Example 5.3.3. Let be the following differential equation

$$xy'' + (2x+2)y' + 4y = 0$$

To solve it around $x = 0$ by series expansion (Frobenius method) we'll have to identify the polynomials

$$p(x) = 2x + 2 \Rightarrow p_0 = 2, p_1 = 2x \qquad q(x) = 4x \Rightarrow q_0 = 0, q_1 = 4x$$

The indicial equation then will be

$$F(r) = r(r-1) + 2r = r(r+1) = 0$$

With roots $r = 0$ and $r = -1$.

- Series for $r = 0$
  The solution must be of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad c_0 = 1$$

Substituting into the original differential equation we end up with

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} 2nc_n x^{n-1} + 4\sum_{n=0}^{\infty} c_n x^n = 0$$

Redefining indices to factor x^n

$$\sum_{n=0}^{\infty}(n+1)nc_{n+1}x^n + \sum_{n=0}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} 2(n+1)c_{n+1}x^n + 4\sum_{n=0}^{\infty} c_n x^n = 0$$

Thus, the recurrence relation is

$$c_{n+1} = \frac{-c_n(2n+4)}{n(n+1)+2(n+1)} = \frac{-c_n(2n+4)}{(n+1)(n+2)} = \frac{-2c_n}{n+1} \Rightarrow c_n = \frac{(-2)^n}{n!}$$

Satisfying $c_0 = 1$. Thus, the solution must be

$$y = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n = e^{-2x}$$

- Series for $r = -1$
  Even though the roots differ by an integer, let'se see what happens if we ignore it. The solution must be of the form
  $$y = \sum_{n=0}^{\infty} c_n x^{n-1} \qquad c_0 = 1$$
  Substituting into the original differential equation we end up with
  $$\sum_{n=0}^{\infty}(n-1)(n-2)c_n x^{n-2} + \sum_{n=0}^{\infty} 2(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty} 2(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} 4c_n x^{n-1} = 0$$
  Redefining indices to factor x^n
  $$\cancel{2c_0x^{-2} - 2c_0x^{-2}} + \sum_{n=0}^{\infty} n(n-1)c_{n+1}x^{n-1} + \sum_{n=0}^{\infty} 2(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty} 2nc_{n+1}x^{n-1} + \sum_{n=0}^{\infty} 4c_n x^{n-1} = 0$$
  Thus, the recurrence relation is
  $$c_{n+1} = \frac{-c_n[2(n-1)+4]}{n(n-1)+2n} = \frac{-c_n(2n+2)}{n(n+1)} = \frac{-2c_n}{n} \Rightarrow c_1 = \infty$$
  This problem can be solved fixing $c_0 = 0$ and $c_1 = 1$, but we would end up with the same series as in the first root, so we wouldn't have two linearly independent solutions to compel the general one. However, the second solution can be found using the "Reduction of order" method, beginning with a change of variables $y = ty_1 = te^{-2x}$. Remembering the original equation
  $$xy'' + (2x+2)y' + 4y = 0 \Rightarrow t''y_1 + t(2y_1' + P(x)y_1) = 0$$
  Defining $v = \frac{dt}{dx}$ and remembering $P(x) = \frac{2}{x} + 2$
  $$y_1v' + (2y_1' + P(x)y_1)v = 0 \Rightarrow v' + \left(\frac{2}{x} - 2\right)v = 0$$
  $$\frac{v'}{v} = 2 - \frac{2}{x} \Rightarrow \ln v = 2x - 2\ln x + \ln A \Rightarrow v = A\frac{e^{2x}}{x^2}$$
  Thus
  $$t = \int v dx = A\int \frac{e^{2x}}{x^2}dx = -A\frac{e^{2x}}{x} + 2A\left(\ln x + \sum_{n=1}^{\infty}\frac{(2x)^n}{n!n}\right) + B$$
  Eventually, the general solution is
  $$y = ty_1 = Be^{-2x} - \frac{A}{x} + 2Ae^{-2x}\left(\ln x + \sum_{n=1}^{\infty}\frac{(2x)^n}{n!n}\right)$$

Example 5.3.4. content...

5.3.3 Notes on the previous examples

- It can always be chosen $c_0 = 1$ as it is a reduced equation.
- Regarding Case 3., we can find the recurrence relation to be $0/0$ (Example 5.3.2) or $\neq 0/0$ (Example 5.3.3)
- Solutions in form of series can often by identified with Taylor series of a common function. Nevertheless, sometimes they doesn't match with any know function and they have to be left as an infinite sum.

- In Example 5.2.4 we fix $c_2 = 0$ as then the series is linearly independent from the first one, which is all it's needed, c_2 is just another arbitrary constant. However, in Example 5.3.3 we try to do something similar, but as $c_0 = 1$, then $c_1 = \infty$, so we try $c_0 = 0$ and $c_1 = 1$, which gives us a correct but repeated solution (it would be the same as the first series).
- In the last two examples, with roots differing by a positive integer, the second solution can be obtained from several ways. Even so, we've chosen the easiest one, constructing the series as it's done with the first solution, and if that ain't possible, constructing the general solution from one solution to the reduced equation (also called Reduction of order).

5.3.4 Recurrence Relation general expression

In all cases, the general expression contains a multiplicative constant which remains undetermined by the recurrence relation. To be determined, we must known at least one of the coefficients of the sequence, say $c_0 = 1$.

1) $c_{n+1} = ac_n \Rightarrow c_n = ka^n$
Note that a negative sign will be transformed into an oscillating $(-1)^n$ in the general expression, as we're going to see in the next examples.

2) $c_{n+1} = (n+q)^p c_n$ q integer $\Rightarrow c_n = k[(n+q-1)!]^p$

3) $c_{n+1} = (n+q)^p c_n$ q half-integer $\Rightarrow c_n = k\left[\frac{(2n+2q-2)!}{2^{2n+2q}(n+q-3/2)!}\right]^p$

4) The quotient c_{n+1}/c_n is the product of some of the previous cases, then the general expression is the product of the expressions in each case (k doesn't multiply by itself, remains unaffected as a constant of first order)

5) $c_{n+1} = \frac{c_n}{(2n+q)(2n+q+1)}$ q integer $\Rightarrow c_n = \frac{k}{(2n+q-1)!}$

These shortcuts can also be obtained finding the first terms of the sequence and proving the continuous relation by mathematical induction.
Remembering the previous examples.

Example 5.3.5.

- First series

$$c_{n+1} = \frac{-c_n}{(n+1)(2n+1)} = \frac{-2c_n}{(2n+2)(2n+1)} \Rightarrow c_n = k\frac{(-2)^n}{(2n)!} \quad c_0 = 1 \to k = 1$$

- Second series

$$c_{n+1} = \frac{-c_n}{(2n+3)(n+1)} = \frac{-2c_n}{(2n+3)(2n+2)} \Rightarrow c_n = k\frac{(-2)^n}{(2n+1)!} \quad c_0 = 1 \to k = 1$$

Example 5.3.6.

- First series

$$d_{m+1} = \frac{d_m}{(2m+2)(2m+3)} \Rightarrow d_m = \frac{k}{(2m+1)!} \quad c_0 = 1 \to k = 1$$

- Second series

$$d_{m+1} = \frac{d_m}{(2m+2)(2m+1)} \Rightarrow d_m = \frac{k}{(2m)!} \quad c_0 = 1 \to k = 1$$

Example 5.3.7.

- First series

$$c_{n+1} = \frac{-2c_n}{n+1} \Rightarrow c_n = \frac{k(-2)^n}{n!} \quad c_0 = 1 \to k = 1$$

- Second series

$$c_{n+1} = \frac{c_n}{n+3} \Rightarrow c_n = \frac{k}{(n+2)!} \quad c_0 = 1 \to k = 2$$

5.4 Hypergeometric differential equation

Definition 5.4.1. The hypergeometric differential equation, also called Gauss' equation is

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0 \tag{5.3}$$

which has 2 regular singular points ($x = 0$ and $x = 1$), the rest of them are just ordinary points.

5.4.1 Origin expansion

To study the differential equation around the origin, let's make it some changes to be easier to work with:

$$x^2y'' + \frac{x}{1-x}[\gamma - (\alpha+\beta+1)x]y' - \frac{\alpha\beta x}{1-x}y = x^2y'' + xp(x)y' + q(x)y = 0$$

where we can identify the polynomials

$$\begin{aligned} p(x) &= \frac{\gamma - (\alpha+\beta+1)x}{1-x} = \gamma + [\gamma - (\alpha+\beta+1)]x + \cdots \Rightarrow p_0 = \gamma \\ q(x) &= -\frac{\alpha\beta x}{1-x} = -\alpha\beta x - \alpha\beta x^2 - \cdots \Rightarrow q_0 = 0 \end{aligned}$$

The indicial equation is, then

$$F(r) = r(r-1) + \gamma r = r(r - [1-\gamma]) = 0$$

with roots $r_1 = 0$ and $r_2 = 1 - \gamma$

First solution

Proposition 5.4.1. *If r_2 isn't a positive integer, namely if γ isn't zero or a negative integer, there exists a first solution*

$$y_1 = \sum_{n=0}^{\infty} c_n x^n = 1 + c_1x + c_2x^2 + c_3x^3 + \dots \quad c_0 = 1$$

Proof. Substituting into the original equation we can factor out x_n and find

$$(n+1)nc_{n+1} - n(n-1)c_n + \gamma(n+1)c_{n+1} - (\alpha+\beta+1)nc_n - \alpha\beta c_n = 0$$

Thus

$$(n+1)(\gamma+n)c_{n+1} = (\alpha+n)(\beta+n)c_n \Rightarrow c_{n+1} = \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}c_n$$

Eventually, the solution is

Definition 5.4.2. The first solution around the origin of the hypergeometric differential equation is

$$y_1 = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!} = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3!\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$$

named the *hypergeometric series* $F(\alpha,\beta,\gamma,x)$ which converges $\forall |x| < 1$. Where $(q)_n = q^{\overline{n}}$ is the rising Pochhammer symbol or commonly named the rising factorial.

∎

Indeed

$$\frac{c_{n+1}x^{n+1}}{c_nx^n} = \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}x \to \frac{n^2x}{n^2} = x \text{ as } n \to \infty$$

- It's said that the convergence radius of the series around $x = 0$ is $R = 1$ (in both ways).

- If α or β are negative integers or zero, the series is finite and then it converges $\forall x$, so $R = \infty$.

Example 5.4.1. Let $\alpha = 1$, the hypergeometric equation then is

$$x(1-x)y'' + (\gamma - \beta x)y' + \beta y = 0$$

So the first solution is shortened to

$$y_1 = 1 - \frac{\beta x}{\gamma}$$

As the following terms are zero due to α being a negative integer.

Second solution

Proposition 5.4.2. *If r_2 isn't a positive integer, namely if γ isn't zero or a negative integer, there exists a second solution*

$$y_2 = x^{1-\gamma} y_1 = \sum_{n=0}^{\infty} c_n x^{n+1-\gamma} = x^{1-\gamma}(1 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) \quad c_0 = 1$$

Proof. Making the following change of variables

$$\begin{aligned} y &= x^{1-\gamma} w \\ y' &= x^{1-\gamma} w' + (1-\gamma)x^{-\gamma} w \\ y'' &= x^{1-\gamma} w'' + 2(1-\gamma)x^{-\gamma} w' + (1-\gamma)(-\gamma)x^{-\gamma-1} w \end{aligned}$$

The original equation (5.3) multiplied by $x^{\gamma-1}$ becomes

$$\begin{gathered} x(1-x)w'' + 2(1-y)(1-x)w' + \frac{1-x}{x}(1-\gamma)(-\gamma)w + [\gamma - (\alpha+\beta+1)x]w' + \\ \left[\frac{\gamma}{x} - (\alpha+\beta+1)\right] w(1-\gamma) - \alpha\beta w = 0 \\ \Rightarrow x(1-x)w'' + [(2-\gamma) + (2\gamma - \alpha - \beta - 3)x]w' + [(1-\gamma)(\gamma - \alpha - \beta - 1) - \alpha\beta]w = 0 \\ \Rightarrow x(1-x)w'' + [\gamma_1 - (\alpha_1 + \beta_1 + 1)x]w' - \alpha_1\beta_1 w = 0 \end{gathered}$$

where $\alpha_1 = \alpha - \gamma + 1$, $\beta_1 = \beta - \gamma + 1$ and $\gamma_1 = 2 - \gamma$.
As soon as γ isn't a positive integer ($\gamma_1 \neq 0 \Rightarrow \gamma \neq 2$ and $\gamma_1 \neq -n \Rightarrow \gamma \neq 2 - \gamma_1 = 2 + n = 3, 4, 5\dots$). Moreover, $\gamma \neq 1$ as it the second solution must not be a repetition of the first one. So when these conditions apply, the solution is

$$w = F(\alpha_1, \beta_1, \gamma_1, x) = F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x)$$

Thus, the second solution is

$$y_2 = x^{1-\gamma} F(\alpha_1, \beta_1, \gamma_1, x) = x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x)$$

■

So far, the general solution to the hypergeometric differential equation around the origin will be

$$y = Ay_1 + By_2 = AF(\alpha, \beta, \gamma, x) + BF(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x)$$

As long as γ isn't zero or a positive or negative integer ($\gamma \in \mathbb{R} \setminus \mathbb{Z}$).

5.4.2 Expansion around $x = 1$

Taking advantage of the previous result, making the following change of variables

$$t = 1 - x \qquad \frac{dy}{dx} = -\frac{dy}{dt} \qquad \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}$$

The hypergeometric differential equation (5.3) becomes

$$t(1-t)y'' + [\gamma - (\alpha+\beta+1) + (\alpha+\beta+1)t](-y') - \alpha\beta y = 0$$
$$\Rightarrow (1-t)ty'' + [(\alpha+\beta-\gamma+1) - (\alpha+\beta+1)t]y' - \alpha\beta y = 0$$

where now the derivatives are with respect to t. Letting $\gamma_1 = \alpha+\beta-\gamma+1$ we end up with

$$(1-t)ty'' + [\gamma_1 - (\alpha+\beta+1)t]y' - \alpha\beta y = 0$$

as $x = 1 \Leftrightarrow t = 0$, the solutions must be the same as in the expansion around origin ($x = 0$) but with t instead of x, that is

$$y_1 = F(\alpha, \beta, \gamma_1, t) = F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$$
$$y_2 = t^{1-\gamma_1} F(\alpha-\gamma_1+1, \beta-\gamma_1+1, 2-\gamma_1, t) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\beta, \gamma-\alpha, \gamma-\alpha-\beta+1, 1-x)$$

So far, the general solution to the hypergeometric differential equation around $x = 1$ is

$$y = Ay_1 + By_2 = AF(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x) + B(1-x)^{\gamma-\alpha-\beta} F(\gamma-\beta, \gamma-\alpha, \gamma-\alpha-\beta+1, 1-x)$$

which only converges when $|1-x| < 1$, which matches the general solution around the origin when $0 < x < 1$, where both are equally valid.

5.4.3 Differentiation property

Differentiating the hypergeometric differential equation we get

$$x(1-x)y''' + [\gamma+1 - (\alpha+1+\beta+1+1)x]y'' - (\alpha+1)(\beta+1)y' = 0$$

Namely, if y is solution to the hyp. diff. eq. with α, β, γ

- y' is solution to the hyp. diff. eq. with $\alpha+1, \beta+1, \gamma+1$
- y'' is solution to the hyp. diff. eq. with $\alpha+2, \beta+2, \gamma+2$
- ...
- $y^{(n)}$ is solution to the hyp. diff. eq. with $\alpha+n, \beta+n, \gamma+n$

5.5 Legendre's differential equation

Definition 5.5.1. Legendre's differential equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{5.4}$$

with n not necessarily being an integer, $n \in \mathbb{R}$

Thanks to the following change of variables $1 - x = 2t$, the equation becomes

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0 \tag{5.5}$$

where the derivatives are with respect to t. Note that the hypergeometric differential appears with $\gamma = 1$, $\alpha = n+1$ and $\beta = -n$.

5.5.1 First solutions

The first solution around $t = 0 \Leftrightarrow x = 1$ is

$$y_1 = F(\alpha, \beta, \gamma, t) = F\left(n+1, -n, 1, \frac{1}{2}(1-x)\right)$$

Moreover, the first solution around $t = 1 \Leftrightarrow x = -1$ is

$$y_1 = F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-t) = F\left(n+1, -n, 1, \frac{1}{2}(1+x)\right)$$

5.5.2 Second solutions

Since $y = 1$, the indicial equation has a double root $r = 0$.

$$F(r) = r(r-1) + \gamma r = r(r - [1-\gamma]) = r^2 = 0$$

The second solution around $t = 0 \Leftrightarrow x = 1$ is

$$y_2 = y_1 \ln t + \sum_{n=1}^{\infty} b_n t^n$$

which isn't analytical in $t = 0$.
The second solution around $t = 1 \Leftrightarrow x = -1$ is

$$y_2 = y_1 \ln(t-1) + \sum_{n=1}^{\infty} b_n (t-1)^n$$

which isn't analytical in $t = 1$.
We're interested in the interval $-1 \leq x \leq 1$ where both first solutions y_1 are valid, the most general analytical solution i

5.6 Legendre polynomials

Definition 5.6.1. If n is zero or a positive integer, the analytical series solution to (5.4) is finite. Said series is named *Legendre polynomial* of nth-order.

$$P_n(x) = F(n+1, -n, 1, \frac{1}{2}(1-x)) = \sum_{n=0}^{\infty} \frac{(n+1)_n (n)_n}{(1)_n} \frac{((x-1)/2)^n}{n!}$$

where $(q)_n = q^{\overline{n}}$ is the rising factorial.

Example 5.6.1. Let be the following differential equation

$$(1-x^2)y'' + 2xy' + y = 0$$

which is the Legendre's differential equation for $n = 0$, the solution then must be

$$y = P_0(x) = 1$$

5.6.1 Rodrigues' Formula

It can be checked that $y = (t - t^2)^n$ can be solution to the following differential equation

$$t(1-t)y'' + (1-n)(1-2t)y' + 2ny = 0$$

which is the hypergeometrical diff. eq. with $\alpha = 1, \beta = -2n$ and $\gamma = 1 - n$.
Thus, thanks to the differentiation property of the hypergeometrical differential equation

$$y = \frac{d^n}{dt^n}(t - t^2)^n$$

is solution to the diff. eq. with $\alpha = n+1$, $\beta = -n$ and $\gamma = 1$, which is the same as (5.5)
Reversing the change of variables, remembering we transformed (5.4) into (5.5) with $1 - x = 2t$

$$t - t^2 = \frac{1-x^2}{4} \qquad \left(\frac{dx}{dt}\right)^n = (-2)^n \qquad y = \frac{d^n}{dx^n}\left(1-x^2\right)^n \frac{1}{(-2)^n}$$

which is then solution to (5.4). Eventually

$$P_n(x) = C \cdot \frac{d^n}{dx^n}\left(1-x^2\right)^n$$

To determine this arbitrary constant, let's see $P_n(1) = 1$. Let

$$u = (x-1)^n \qquad v = (x+1)^n \qquad \frac{d^n}{dx^n}\left(1-x^2\right)^n = \frac{d^n}{dx^n}(uv)$$

In $x = 1$, all u and its derivatives cancel out until the $(n-1)$-th derivative.

$$P_n(1) = C \cdot v\frac{d^n u}{dx^n} = C \cdot 2^n n! \Rightarrow C = \frac{1}{2^n n!}$$

So far, the alternative expression for Legendre Polynomials is

$$\boxed{P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}\left(x^2-1\right)^n}$$

which is the so called Rodrigues' Formula.

5.6.2 Properties

The term $(x^2-1)^n$ has the powers x^{2n}, x^{2n-2}, $x^{2n-4} \cdots \Rightarrow \frac{d^n}{dx^n}(x^2-1)^n$ has the powers x^n, x^{n-2}, $x^{-n-4} \cdots$ $P_n(x)$ will be an even or odd function depending on n.

Proposition 5.6.1. $P_n(x) = (-1)^n P_n(-x)$

Especially, $P_n(1) = 1$, $P_n(-1) = (-1)^n$.

Example 5.6.2. Using Rodrigues' Formula

$$\begin{aligned}
&P_0(x) = 1 \qquad P_1(x) = x \\
&P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}(12x^2-4) = \frac{3x^2-1}{2} \\
&P_3(x) = \frac{1}{48}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}(120x^3-72x) = \frac{5x^3-3x}{2} \\
&P_4(x) = \frac{1}{384}\frac{d^4}{dx^4}(x^2-1)^4 = \frac{1}{384}(1680x^4-1440x^2+144) = \frac{1}{8}(35x^4-30x^2+3)
\end{aligned}$$

5.6.3 Orthogonality and roots of Legendre Polynomials

Proposition 5.6.2. *Orthogonality of Legendre's Polynomials. If $m \neq n$, then $\forall x \in (-1, 1)$.*

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0$$

Proof. $P_n(x)$ satisfies Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \Rightarrow \frac{d}{dx}\left[(1-x^2)P_n'(x)\right] + n(n+1)P_n(x) = 0 \tag{5.6}$$

Multiplying by $P_m(x)$ and integrating in $(-1, 1)$

$$\int_{-1}^{1} P_m(x)\frac{d}{dx}\left[(1-x^2)P_n'(x)\right]dx + n(n+1)\int_{-1}^{1} P_m(x)P_n(x)dx = 0$$

Integrating by parts

$$\int_{-1}^{1} P_m'(x)\left[(1-x^2)P_n'(x)\right] + \cancelto{0}{\int_{-1}^{1} P_m(x)\left[(1-x^2)P_n'(x)\right]dx} + n(n+1)\int_{-1}^{1} P_m(x)P_n(x)dx = 0$$

Substracting (5.6) to it (with m index instead of n), namely

$$\int_{-1}^{1} P_m'(x)\left[(1-x^2)P_n'(x)\right] + +n(n+1)\int_{-1}^{1} P_m(x)P_n(x)dx = 0$$

$$-\left(\frac{d}{dx}\left[(1-x^2)P_m'(x)\right] + m(m+1)P_m(x)\right) = 0$$

Ending up with

$$[n(n+1) - m(m+1)]\int_{-1}^{1} P_m(x)P_n(x)dx = 0$$

So far, Legendre's Polynomials are orthogonal in $(-1, 1)$ if $m \neq n$ ■

Corollary 5.6.3. *Let $R_m(x)$ be any polynomial of degree $< n$. Then*

$$\int_{-1}^{1} P_n(x)R_m(x)dx = 0$$

Proof. Turning back to Preposition 5.6.2., any polynomial of degree $< n$ (say m-th degree) can be expressed as a linear combination of m Legendre's Polynomials. In other words

$$R_m = \sum_{i=0}^{m} c_i P_i(x)$$

Recalling that $P_i(x)$ is a polynomial of i-th degree. ■

Proposition 5.6.4. *Let $P_n(x)$ be the n-th degree Legendre polynomial, then*

$$\langle |P_n(x)|, |P_n(x)| \rangle_I = \int_{-1}^{1} |P_n(x)|^2 = \frac{2}{2n+1}$$

Proof. According to Rodrigues Formula

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

Let $k = 1/(n!2^n)^2$. Integrating by parts repeatedly

$$\begin{aligned}\int_{-1}^{1}|P_n(x)|^2dx &= k\int_{-1}^{1}\frac{d^n}{dx^n}\left(x^2-1\right)^n\frac{d^n}{dx^n}\left(x^2-1\right)^n dx\\ &= -k\int_{-1}^{1}\frac{d^{n+1}}{dx^{n+1}}\left(x^2-1\right)^n\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\end{aligned}$$

The other integration term gets cancelled. So far,

$$\int_{-1}^{1}|P_n(x)|^2dx = (-1)^n k\int_{-1}^{1}\left[\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\right](x^2-1)^n dx$$

The other integration terms continue to get cancelled, that is because they're always depending on the pairs $d^{(n-1)}, d^{(n)}$, $d^{(n-2)}, d^{(n+1)}$ ... $d^{(0)}, d^{(2n)}$, which themselves depend on $(x^2-1)^n$. That said, $(x^2-1)^n$ is cancelled both at $x=-1$ and $x=1$. The same thing happens with its derivatives from the 0-th derivative to the $n-1$-th derivative, causing them to be 0 anyways. ■