ANNOUNCEMENTS : HW4b IS UP.
IT'S LONG, BUT MOSTLY COPY-PASTA

Q: last time: $\dot{x} = -x^3 + 0.1$ ... what if $+x^3 + 0.1$?

Last time : ODE $\longrightarrow$ (1) EULER : world's most obvious ODE solver.

$$x_{i+1} = x_i + \Delta t \; f(x_i, t_i)$$

$$\uparrow$$

$$x(t + dt) = x(t) + dt \left.\frac{dx}{dt}\right|_t$$

(2) RUNGE - KUTTA

why? $\longrightarrow$ RAISON D'ETRE : $\sum^N \mathcal{O}(\Delta t^2) \sim \mathcal{O}(\Delta t)$
SO EULER GIVES A "BIG" ERROR THAT ACCUMULATES.

RUNGE - KUTTA : messy :

(RK2) $$x_{i+1} = x_i + \underbrace{\Delta t \; f\Big(x_i + \tfrac{1}{2} \underbrace{\Delta t \, f(x_i, t_i)}_{= K_1} , \; t_i + \tfrac{\Delta t}{2} \Big)}_{= K_2 \text{ for convenience}} + \mathcal{O}(\Delta t^3)$$

!

SO THAT WE MAY WRITE
eg np.arange(...)

```
for t in tlist :
    K1 = Δt f(xi, ti)
    K2 = Δt f(xi + K1, t + Δt/2)
    x  = x + K2
```

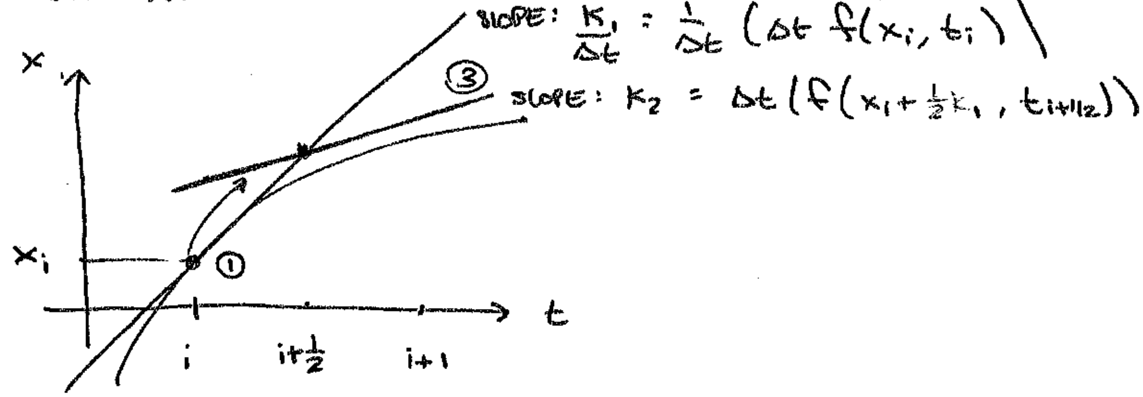
← just a useful intermediate step (USEFUL FOR DEBUGGING)

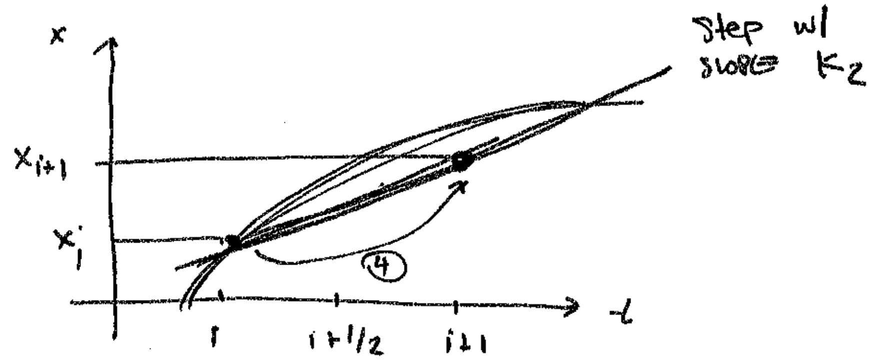
or $x \mathrel{+}= K_2$

K_1 & K_2 are Δx's. slope × displacement

SLOPE @ x_i ESTIM. SLOPE @ $x_{i+1/2}$

HOW DID WE DERIVE?





PUZZLE: *how do we do better?*

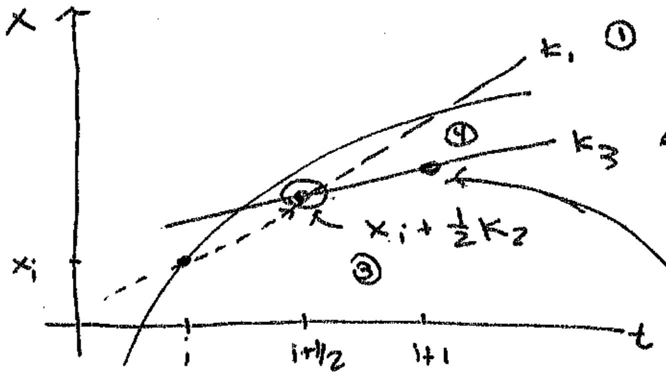
RK2: each step to $\mathcal{O}(\Delta t^3)$

can we beat this?

(analogous to Trapezoidal $\rightarrow$ Simpson)

RK4 adds additional steps:

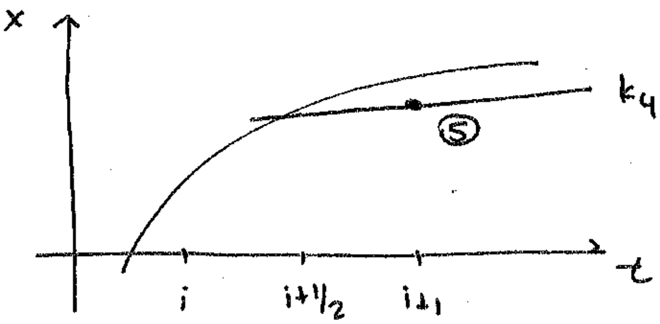
$$K_3 = \Delta t\ f\left(x_i + \tfrac{1}{2}K_2,\ t_{i+1/2}\right)$$
$$K_4 = \Delta t\ f\left(x_i + K_3,\ t_{i+1/2}\right)$$
$$x_{i+1} = x_i + \tfrac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$



, then get ② as before

← another estimate of the slope @ $x_{i+1/2}$

estimate of x_i



estimate of slope @ x_{i+1}

↑ need this

slopes @ i, i+1

slope est @ i+1/2

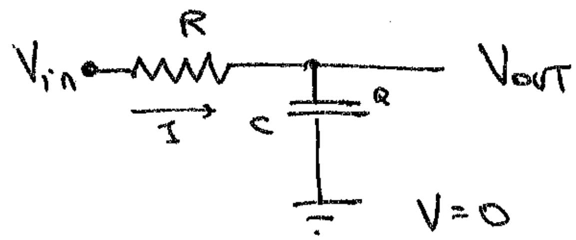
$$X_{i+1} = X_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

Pf that this combo cancels errors to $O(\Delta t^5)$?
→ exercise. (it's a little tedious)

HINT: math.stackexchange.com/.../528856/

$$x(t+\Delta t) = x(t) + \underbrace{\int_t^{t+\Delta t} f(x(t), t)\, dt}_{\sum_{j=1}^{N} w_j \underbrace{f\left(x(t+\Delta t_j), t+\Delta t_j\right)\Delta t}_{k_j}}$$

<u>EXAMPLE</u> :



ex 8.1

LOW-PASS FILTER

CIRCUIT ANALYSIS

① $V_{in} - V_{out} = IR$

② $I = \dot{Q}$

③ $Q = C V_{out}$

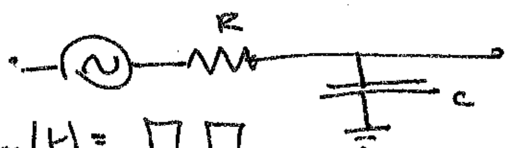
$\dot{V}_{out} = \frac{1}{C}\dot{Q} = \frac{1}{C} I$

①

$$\boxed{\dot{V}_{OUT} = \frac{1}{C}\frac{(V_{in} - V_{out})}{R}}$$

1^{st} Ø ODE $\qquad f(V_{out}, t)$

DRIVEN SYSTEM :



$V_{in}(t) =$ ⎍⎍⎍...

$$\frac{dV_{out}}{dt} = \frac{1}{RC}\left(V_{in}(t) - V_{out}(t)\right)$$

turns out this is a <u>low-pass filter</u>

↳ let's see why [DEMO]

LESSON: BUILD IT, THEN <u>BREAK IT</u>

understand where the <u>model</u> breaks down.

<u>PUZZLE</u>: nonlinear pendulum

$$m\ell\ddot{\theta} = -mg\sin\theta$$

$$\boxed{\frac{d^2\theta}{dt^2}}$$

$$\left.\begin{aligned} \dot{\omega} &= -\frac{g}{\ell}\sin\theta \\ \omega &\equiv \dot{\theta} \end{aligned}\right] \text{ two 1st O DIFF EQ.}$$

BOOK: eg. 8.6

$$\left.\begin{aligned} g &= 9.81 \\ \ell &= 0.1 \end{aligned}\right\} \text{ mindful of units}$$

```
def f(θ, ω, t):
    return ω             ← θ̇ = f

def g(θ, ω, t):
    return -(g/ℓ) sin θ  ← ω̇ = g
```

$\boxed{\text{RK2}}$ ~~k1 = dt~~

→ what GOES WRONG?

(demo)

$$E = \tfrac{1}{2} m l^2 \omega^2 + mgl(1-\cos\theta)$$

$$\approx \tfrac{1}{2} m l^2 (\omega^2 + \underbrace{\tfrac{g}{l}}_{\omega_0^2} \theta^2) + \mathcal{O}(\theta^4)$$

$$E_i \approx \underbrace{\tfrac{1}{2} m l^2}_{\text{const}} (\omega_i^2 + \omega_0^2 \theta_i^2)$$

$$\theta_{i+1} = \theta_i + \omega_i \Delta t$$

$$\omega_{i+1} = \omega_i - \omega_0^2 \theta_i \Delta t$$

$$E_{i+1} = \tfrac{1}{2} m l^2 (\omega_{i+1}^2 + \omega_0^2 \theta_{i+1}^2)$$

$$= \tfrac{1}{2} m l^2 \big(\omega_i^2 - 2\omega_0^2 \theta_i \omega_i \Delta t + \omega_0^4 \theta_i^2 \Delta t^2$$

$$+ \underbrace{\omega_0^2 \theta_i^2}_{\text{OLD TERM}} + \underbrace{2\omega_0^2 \theta_i \omega_i \Delta t}_{= 0} + \underbrace{\omega_0^2 \omega_i^2 \Delta t^2}_{\text{DRIFT!}} \big)$$

$$\Delta E = m l^2 \omega_0^2 (\underbrace{\omega_i^2 + \omega_0^2 \theta_i^2}_{E_i}) \Delta t^2$$

$$> 0 \; !$$

I CAN LIVE W/ ERRORS LIKE ROUNDING ...
ESSENTIALLY NOISE ...

BUT THIS IS A SYSTEMATIC, ACCUMULATING VIOLATION OF A DEEP PHYSICAL PRINCIPLE.