PHYS 177
winter 2019

lecture 6: Euler's method

many physical principles described by ODEs → local dynamics

learning goals
- understand basic principles of numerical integration of ODEs
- apply Euler's method numerically

simple example:

$$\frac{dx}{dt} = \frac{2x}{t}$$ ← how to solve it?

separation of variables: $\frac{1}{2}\frac{dx}{x} = \frac{dt}{t}$, $\frac{1}{2}\log\frac{x}{x_0} = \log\frac{t}{t_0}$

$$\log x = \log\left(\frac{x_0}{t_0^2}t^2\right), \quad x(t) = \left(\frac{x_0}{t_0^2}\right)t^2 \rightarrow \frac{dx}{dt} = \frac{2x}{t}$$

but what about:

$$\frac{dx}{dt} = \frac{2x}{t} + \frac{3x^2}{t^3}$$ ← how to solve it? I don't know! nonlinear

to do this computationally, use perturbation theory (Euler's method)

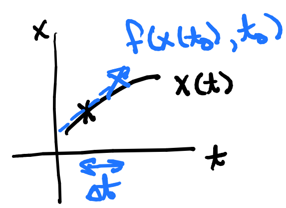
$$\frac{dx}{dt} = f(x,t)$$ ← how can we approach this?

assume we have $x(t_0)$ and we want $x(t)$, do an expansion

$$x(t) = x(t_0) + \underbrace{(t-t_0)}_{\Delta t}x'(t_0) + \frac{1}{2}(t-t_0)^2 x''(t_0) + \ldots$$

$$x(t_0+\Delta t) = x(t_0) + \Delta t\, f(x(t_0),t_0) + O(\Delta t^2)$$

choose Δt small and go along step by step



note: for higher order derivatives we can do the same thing

$$\frac{d^2x}{dt^2} = f(x,t) \rightarrow \frac{dx}{dt} = y, \quad \frac{d^2x}{dt^2} = \frac{dy}{dt} = f(x,y,t)$$

* notebook example with plot, look @ dependence on time step

for a bad time, choose [0,50] w/ 10 steps

error analysis:

we can estimate error by summing dropped terms

$$\epsilon \approx \frac{1}{2}\Delta t^2 \sum_i \frac{d^2x}{dt^2} \approx \frac{1}{2}\Delta t \int_a^b dt\, \frac{df}{dt} = \frac{1}{2}\Delta t\,[f(x(b),b) - f(x(a),a)]$$

error is <u>linear</u> in Δt and depends on difference of derivatives