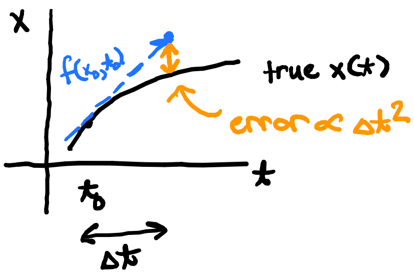
PHYS 177
winter 2019

lecture 7: Runge-Kutta method

learning goals
- understand the principle of interpolation
- apply the second order Runge-Kutta method on practical problems

how to improve integration of ODEs?

return to Euler's method

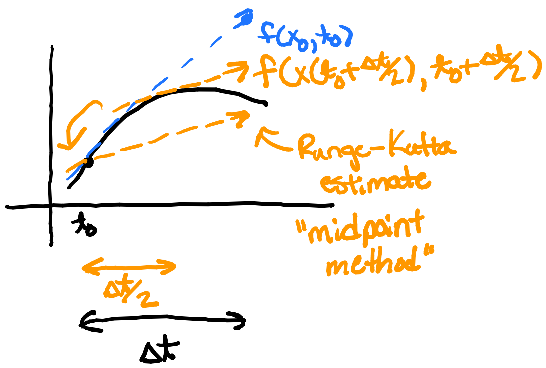


total of $N \propto \frac{1}{\Delta t}$ steps, so

total error is $\propto \frac{\Delta t^2}{\Delta t} = \Delta t$

we could go to 2nd order and include $\frac{d^2x}{dt^2}$ term, but then we would need to know df/dt, which is generally not available

instead, we can interpolate: Runge-Kutta method

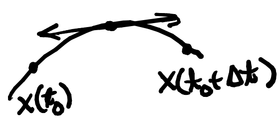


instead of taking derivative @ "boundary" value t_0, take derivative @ "midpoint" $\Delta t/2$ away

is this actually better? yes

expand around $x(t_0+\Delta t/2)$ to get both $x(t_0+\Delta t)$ and $x_0(t)$

$$x(t+\Delta t) = x(t+\Delta t/2) + \frac{\Delta t}{2}\left(\frac{dx}{dt}\right)_{t+\Delta t/2} + \frac{\Delta t^2}{8}\left(\frac{d^2x}{dt^2}\right)_{t+\Delta t/2} + O(\Delta t^3)$$



$$x(t) = x(t+\Delta t/2) - \frac{\Delta t}{2}\left(\frac{dx}{dt}\right)_{t+\Delta t/2} + \frac{\Delta t^2}{8}\left(\frac{d^2x}{dt^2}\right)_{t+\Delta t/2} + O(\Delta t^3)$$

subtracting and rearranging gives

$$x(t+\Delta t) = x(t) + \Delta t\left(\frac{dx}{dt}\right)_{t+\Delta t/2} + O(\Delta t^3)$$

$$= x(t) + \Delta t\, \underbrace{f(x(t+\Delta t/2), t+\Delta t/2)}_{\text{how do we get this? Euler's method}} + O(\Delta t^3)$$

$k_1 = \Delta t\, f(x,t)$
$k_2 = \Delta t\, f(x+\frac{\Delta t}{2}k_1, t+\frac{\Delta t}{2})$
$x(t+\Delta t) = x(t) + k_2$

2nd order Runge-Kutta method

total error $\propto \Delta t^2$

* notebook example