Aix*Marseille université
Socialement engagée

Conceptual Topological theory of Phase Transitions

(Just an overview by a non-expert)

presented by: Abdelhamid Haddad

Supervised by: *Pr. Marco Pettini*

Outline

- **Phase transitions overview :**
  - *Introduction.*
  - *Leading theory.*
  - *Limits and diverse issues.*

- **BKT transition and XY model.**

- **Topological point of view :**
  - *Topological hypothesis.*
  - *Validations and additional issues.*

- **Summary.**

What is a phase transition ?

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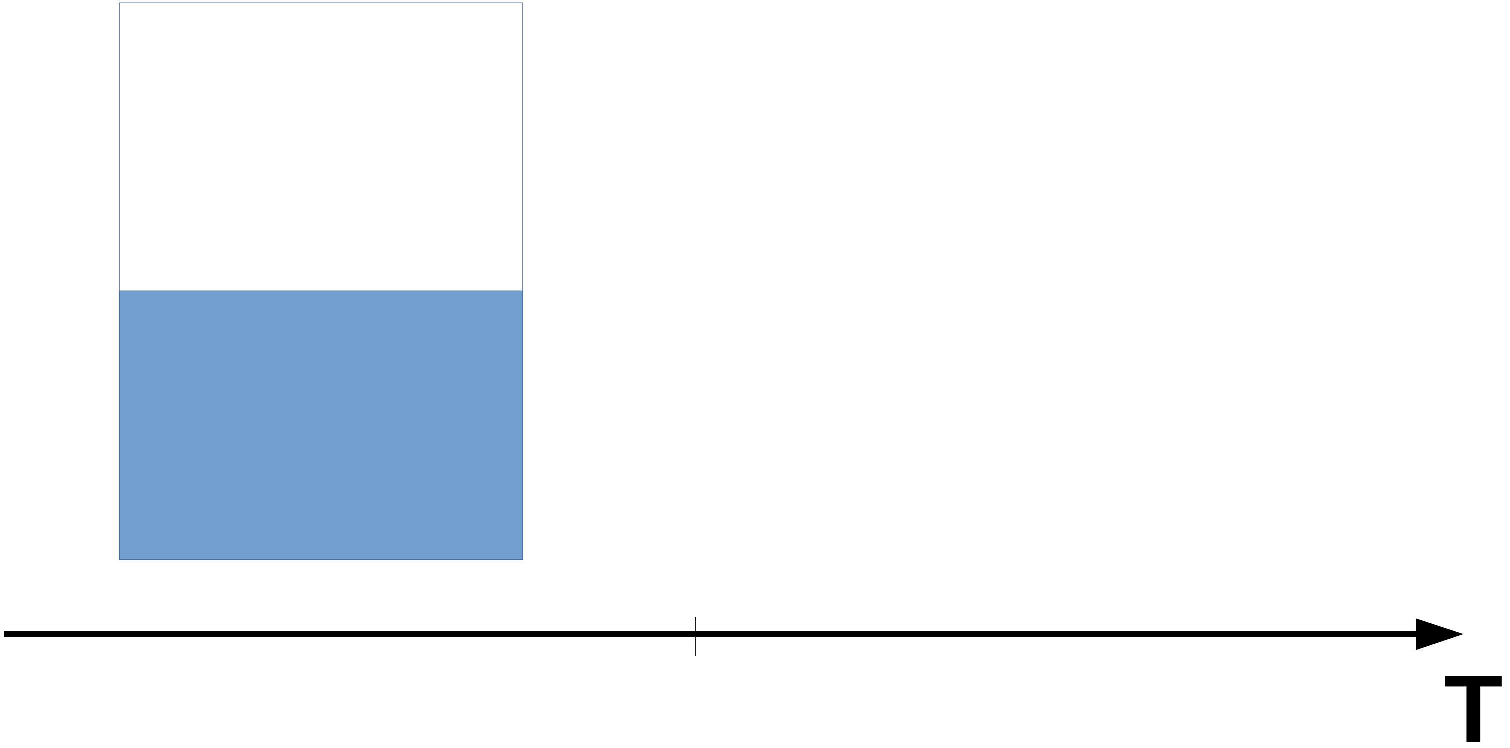

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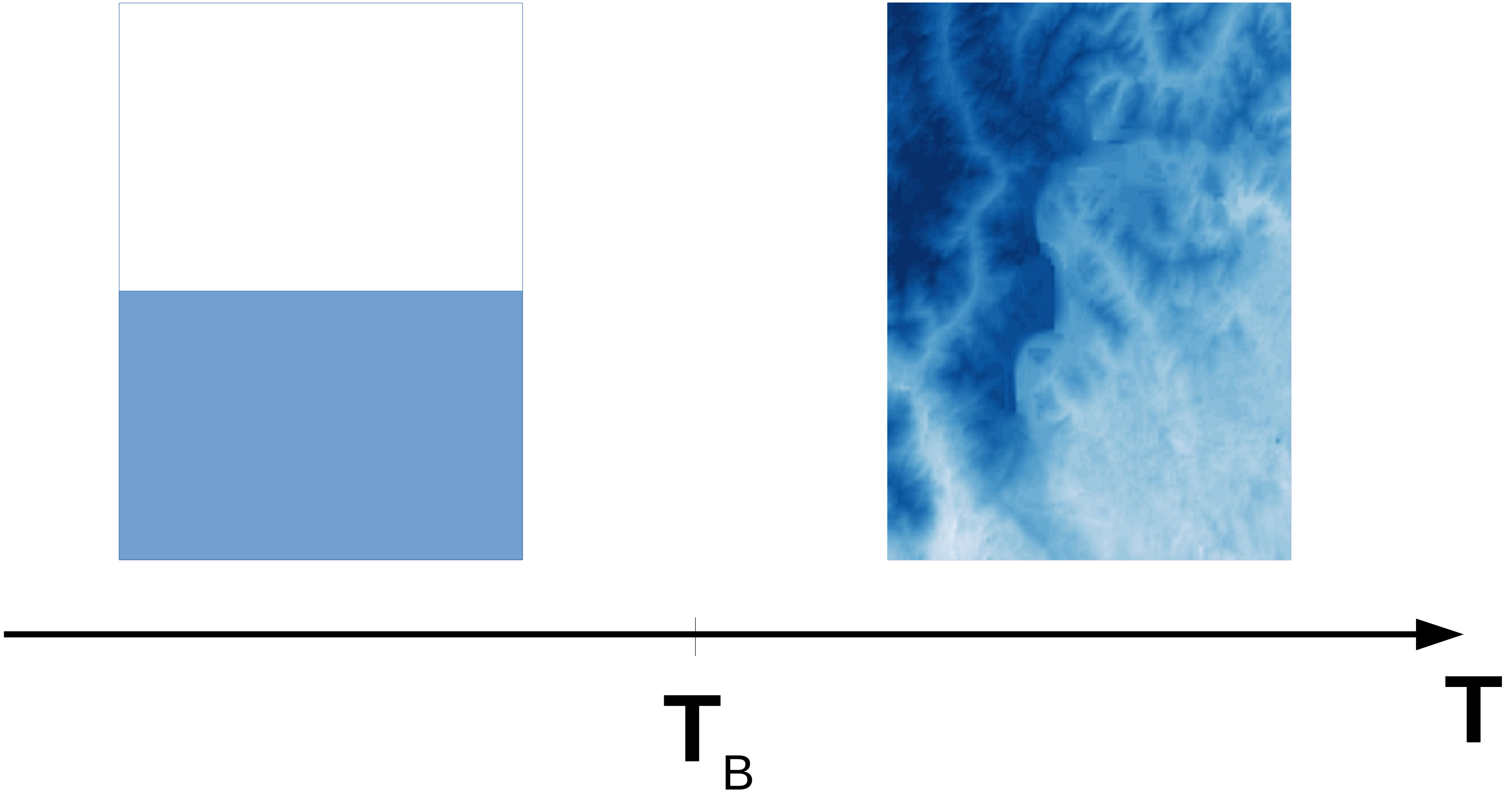

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Paramagnetic to ferromagnetic transition

Superconductivity

Clustering transitions

Bose-Einstein condensation

...... and many others!

Physical description of a PT

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A general and complete theory is not yet available.

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Associates a PT with a spontaneous symmetry breaking

At higher temperatures correspond to a "higher symmetry" disordered phase;

at lower temperatures, the phases are of ordered "lower symmetry".

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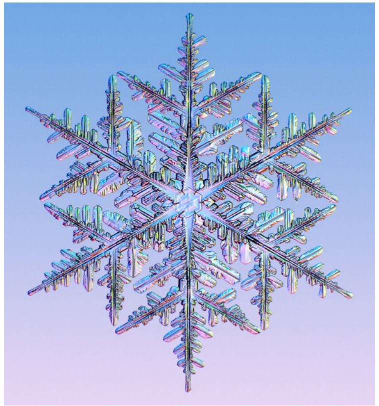
(To describe the symmetry of the phases Landau introduced a thermodynamic variable, the order parameter ϕ, which vanishes above the transition and, below the transition, it is nonzero.)

Limitations



275 water molecules

- Traditionally, phase transitions are defined in the thermodynamic limit only !

(Thermodynamic limit dogma)

How to characterize PT in small systems ?

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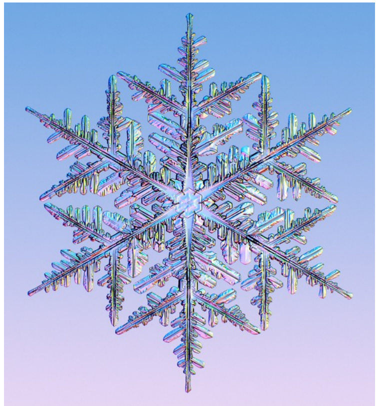

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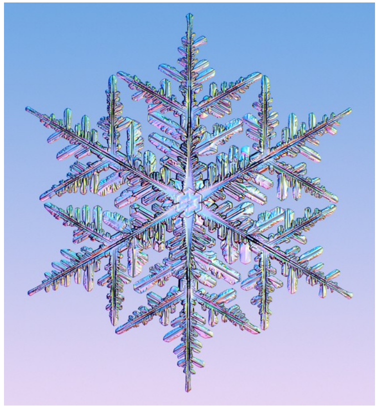

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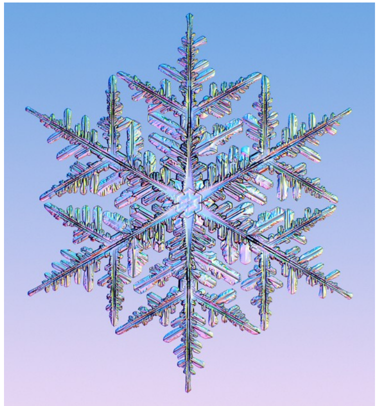

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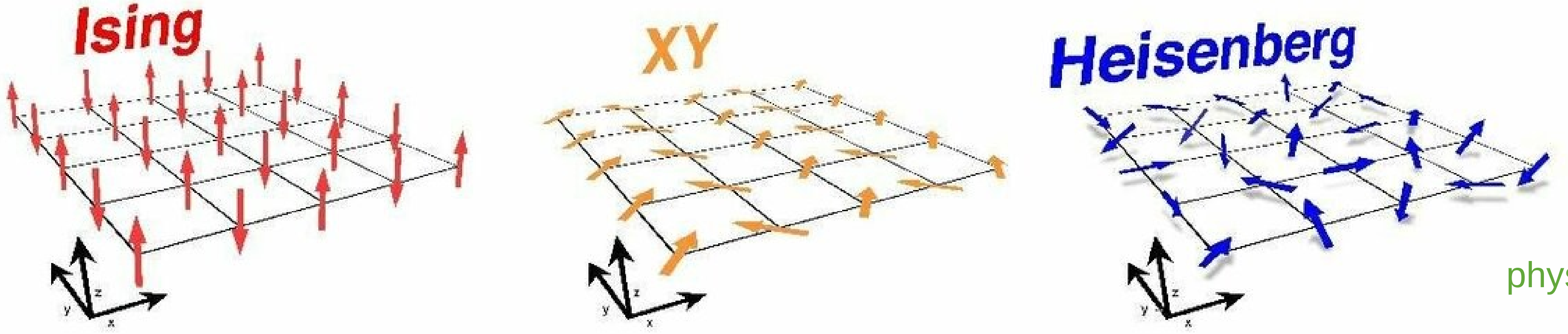
How to characterize PT in small systems ?

- Even WORSE :
  - There are systems undergoing PT in the absence of symmetry-breaking.
  - Systems in d ≤ 2 with continuous symmetry cannot be broken spontaneously at any finite T and then No PT (Mermin–Wagner theorem).
  - systems with local symmetries (gauge theories) undergo PT in the absence of an order parameter (Elitzur theorem).

XY model
(Rotator model)



phys.org/

$$H = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j = -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j).$$

XY model : Correlation function

In condensed matter physics, we classify matter in terms of the decay of their correlation functions:

Johannes Walcher lecture notes, heidelberg university.

- Long range ordered state (LRO) : $G(x,x') \longrightarrow c \neq 0$
- Disordered state : $G(x,x') \xrightarrow{\propto exp(|x-x'|)} 0$ (Exponential decay of the correlation function)
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$$g(\vec{r}_x) = \langle \vec{s}_0 \cdot \vec{s}_x \rangle = \frac{1}{Z} \int D\theta \, \cos(\theta_0 - \theta_x) \exp\left[\beta \sum_{\langle l,k \rangle} \cos(\theta_l - \theta_k)\right]$$

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At low temperatures (large β) one expects neighboring spins to point into similar directions. (spin-wave approximation)

$$g(|\vec{r}_x|) \approx \left|\frac{\vec{r}_x}{r_0}\right|^{\frac{1}{2\pi\beta}}$$

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At high temperatures (small β) the exponential in the partition function can be expanded in β and higher order terms may be neglected

$$g(|\vec{r}|) \propto 2\pi\beta^{|\vec{r}|} = e^{\ln(2\pi\beta)|\vec{r}|}$$

The correlations decay exponentially.

The Berezinsky, Kosterlitz, Thouless PT

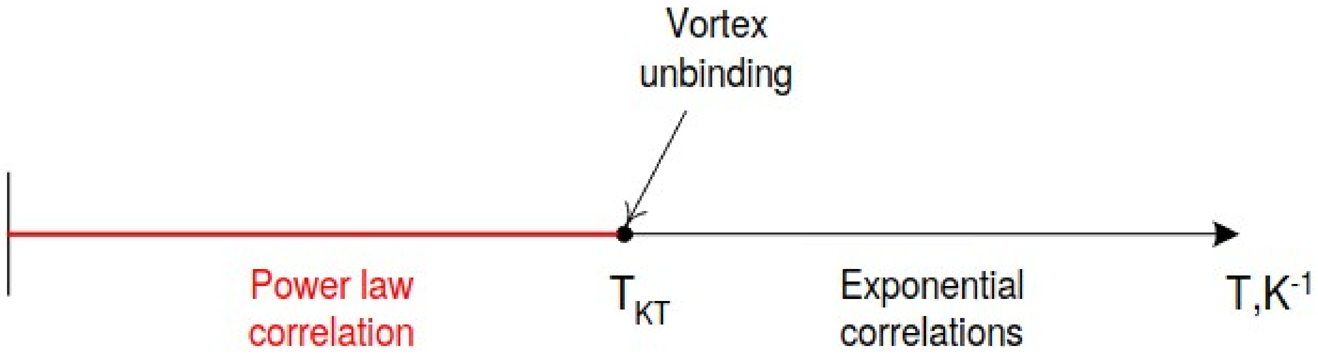

Figure 1: Schematic of the Kosterlitz-Thouless transition.

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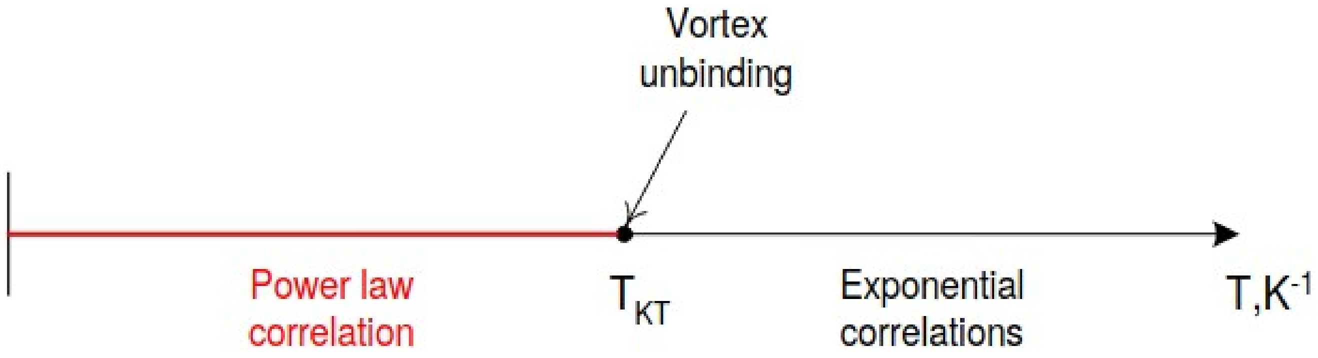


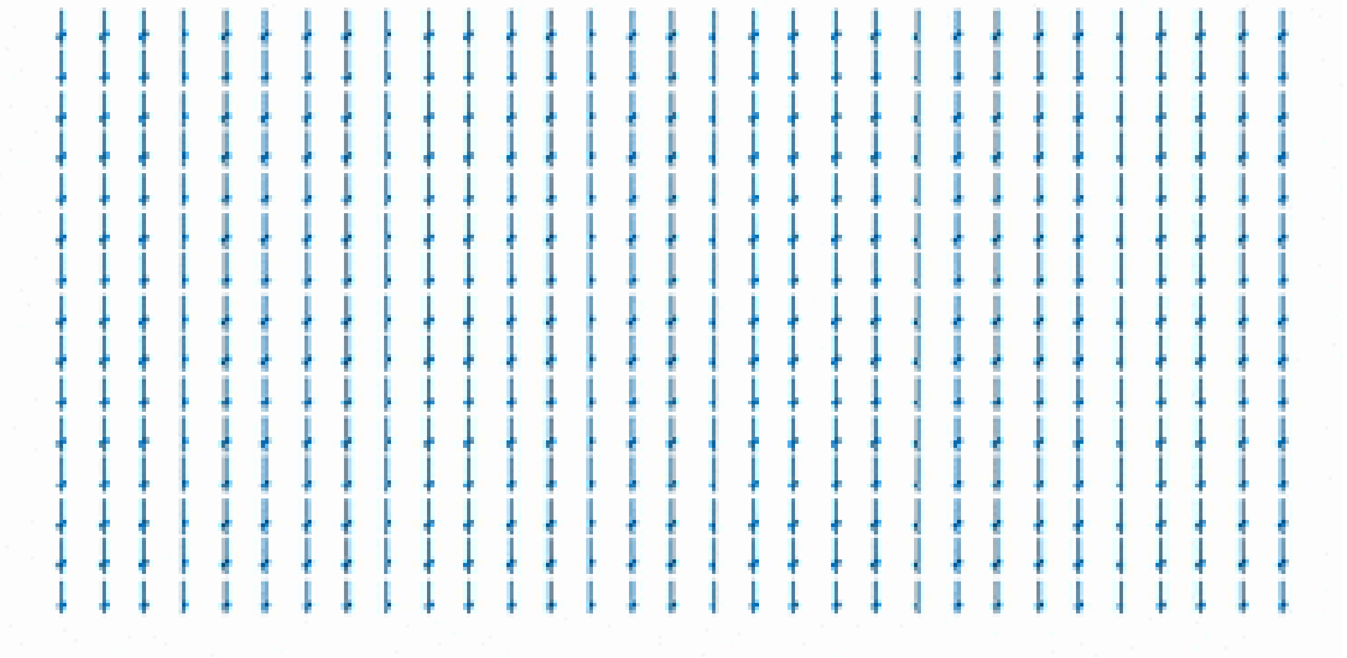
Figure 1: Schematic of the Kosterlitz-Thouless transition.

They proposed a solution: Topological defects to explain the Phase transition!

The Defects = Vortices

They got (KT and Haldane) the 2016 Nobel prize in physics for their theoretical discovery of Topological PT.

Brian Skinner : https://www.ribbonfarm.com/



At low temperatures, it contain a dilute ‘gas’ of vortex-antivortex pairs. Each vortex will stick to an antivortex, since it takes a lot of energy to separate them

As we heat it up, we get more and more vortex-antivortex pairs, since there’s more energy available to create them up to their unbinnding.

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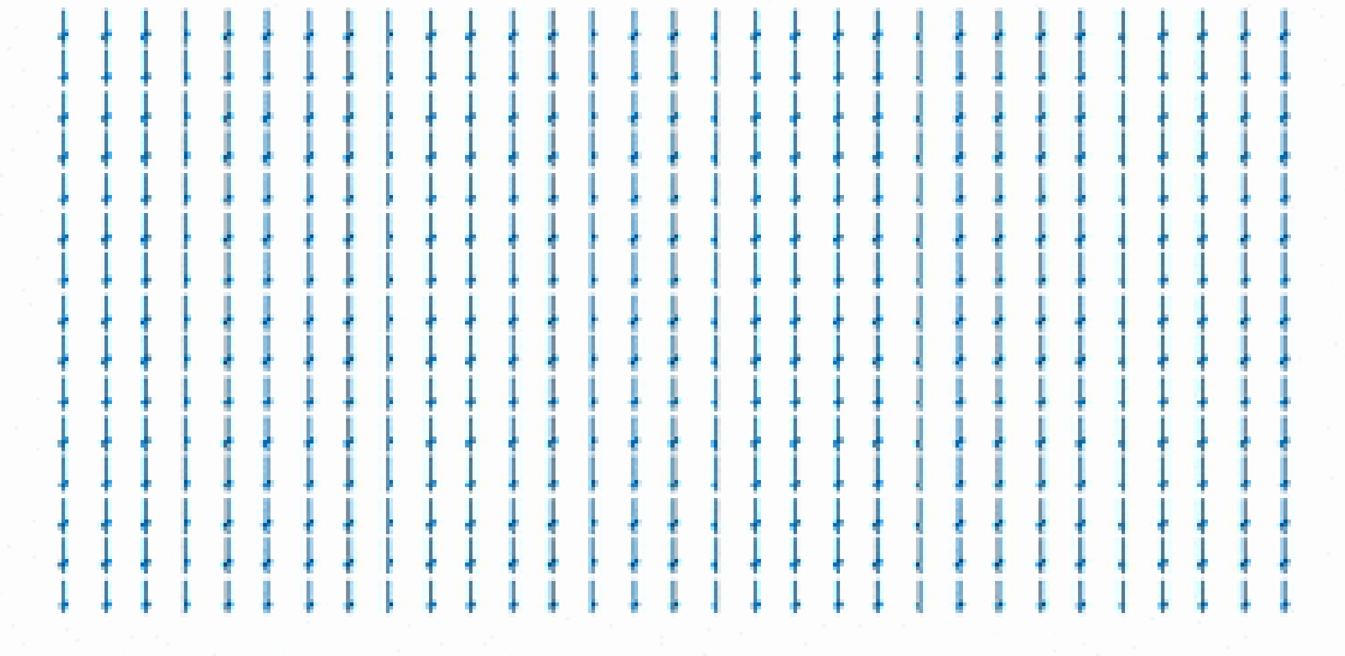

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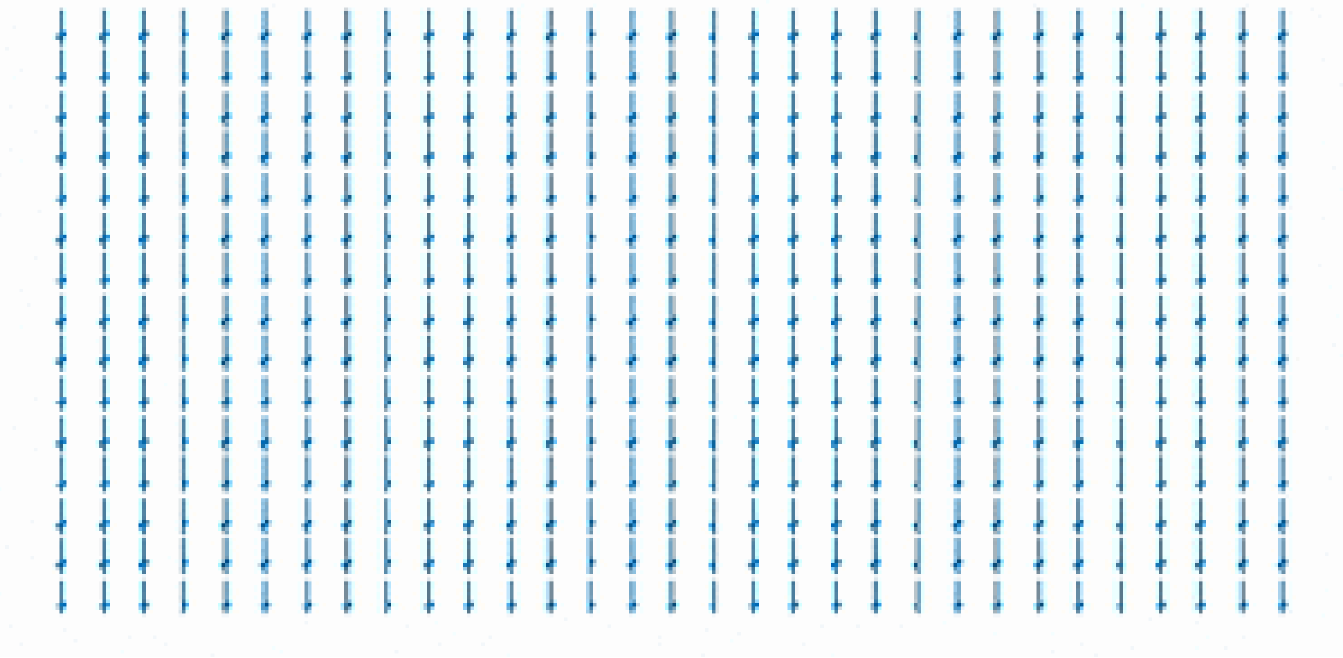

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Hamiltonian dynamics

$$\mathcal{H}(\{\varphi,\pi\}) = \frac{1}{2}\sum_i \pi_i^2 + V(\{\varphi_i\})$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\ ,$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\ , \qquad i = 1, 2, \ldots, N$$

The coordinates (q, p) are canonical variables, and the space of canonical variables is known as phase space.

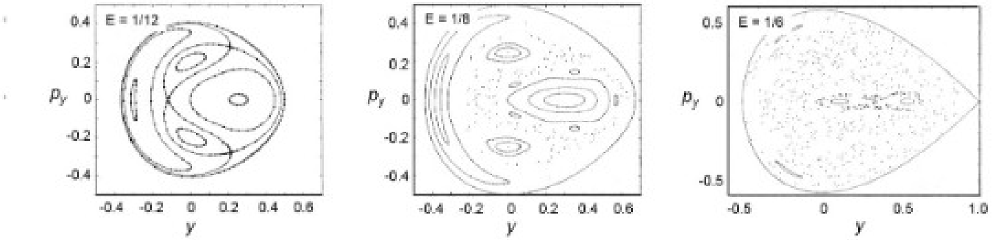

Figure 2: Poincare cross sections of the plane $x = 0$ for three values of parameter E: regular motion at $E = 1/24$, mix of regular and irregular motion at $E = 1/8$, and chaotic motion at $E = 1/6$. The particles are placed with the initial $y = 0$. The dots, which appear at random for E = 1/8 and E = 1/6, are generated by a single particle trajectory [1].

As E increases, the dynamics look increasingly chaotic

This dynamical approach brings us new observables such as the Lyapounov eponent and the curvature properties

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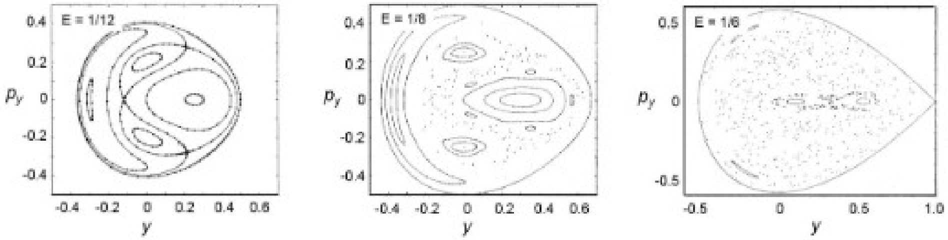

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$$H(\boldsymbol{p},\boldsymbol{\theta}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{p_{(i,j)}^2}{2} + J\left[\cos(\theta_{(i,j)} - \theta_{(i,j+1)}) + \cos(\theta_{(i,j)} - \theta_{(i+1,j)}) - 2\right]$$

Topological hypothesis

Pettini et al. *Geometry of Dynamics, Lyapunov Exponents, and Phase Transitions, P H Y S I C A L R E V I E W L E T T E R s (1997)*

"*A thermodynamic transition might be related to a change in the topology of the configuration space*,

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(Numerical) confirmations

Pettini et al. *Geometry of Dynamics, Lyapunov Exponents, and Phase Transitions, P H Y S I C A L R E V I E W L E T T E R s (1997)*

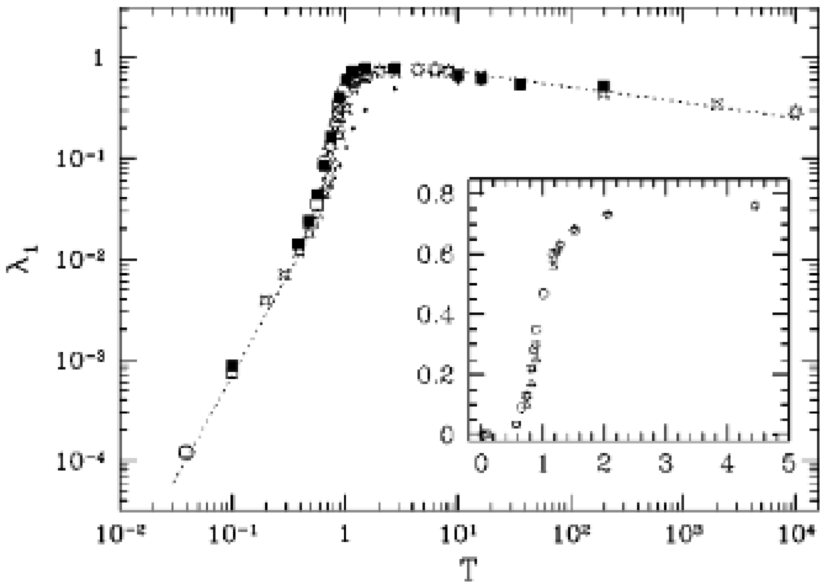

FIG. 1. Lyapunov exponent λ_1 vs temperature T for the $d = 2$ case. Numerical results correspond to lattice size: $N = 10^2$ (starred open squares), $N = 20^2$ (open triangles), $N = 40^2$ (open stars), $N = 50^2$ (open squares), and $N = 100^2$ (open circles). Full squares are analytic results according to Eq. (4); dots are analytic results without correction (see text). In the inset symbols have the same meaning.

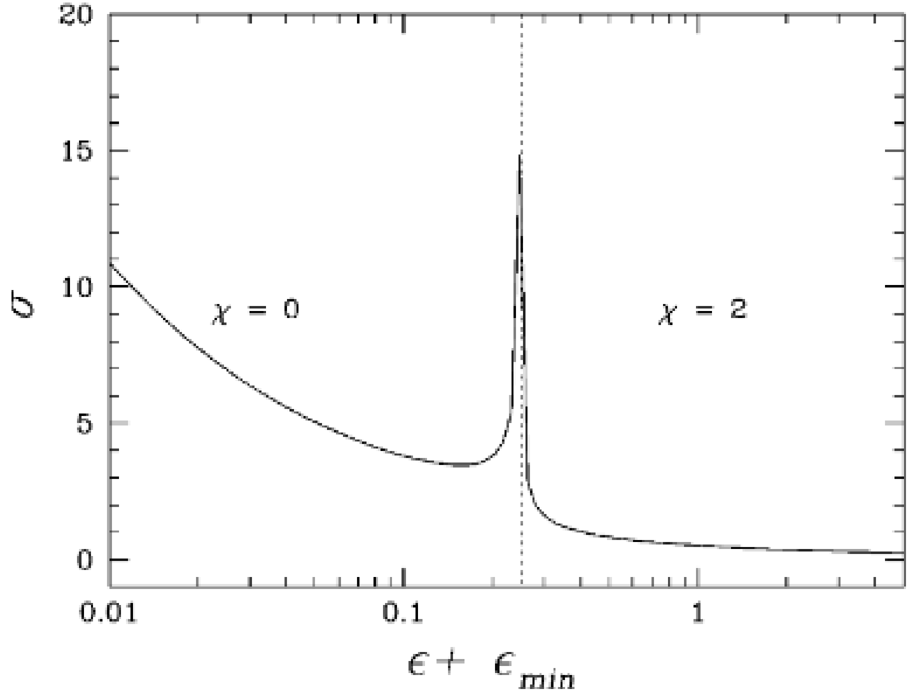

FIG. 5. Fluctuations amplitude, σ, of Gauss curvature of a family of surfaces parametrized by ϵ. For graphical reasons ϵ is shifted by its minimum value $|\epsilon_{\min}| = 0.25$; thus the cusp corresponds to $\epsilon = 0$, the critical value separating two families of different Euler characteristic χ, i.e., of different topology.

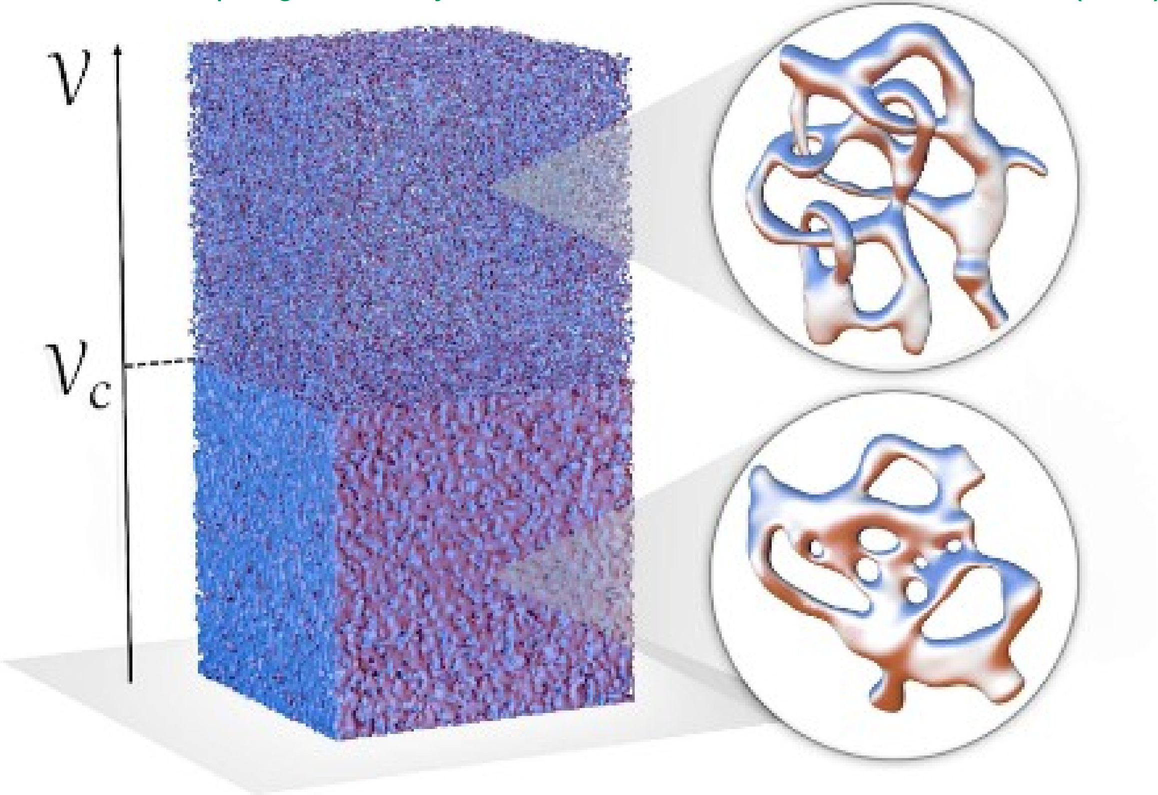

Figure 1. Low-dimensional pictorial representation of the transition between complex topologies as a metaphor of the origin of a phase transition. From the ground level up to the crossover level v_c, the manifolds M_v have a genus which increases with v. Above the crossover level v_c, the manifolds M_v have also a nonvanishing linking number which increases with v

Additional issues

"no topology change in phase space, no phase transition." However, there is at present no theorem that gives a sufficient topological condition for the occurrence of a phase transition.

Summary

- In 2d, continuous spin models cannot have magnetically ordered state. (Mermin-Wagner theorem).
- The XY model, however, has a strange type of phase transition that does not break the symmetry. (BKT transition).
- What kinds of topological transitions are related to phase transitions ?
- What is, from a topological point of view, the difference between various types of phase transitions?

Thanks for your Attention !

Your questions are more than welcomed :)