The Harmonic Oscillator

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Outline

1. Introduction.
2. Hooke's law. Position, velocity, and acceleration.
3. The differential equation.
4. Potential and kinetic energy.
5. Momentum. The Hamiltonian formulation.
6. The importance of the oscillator.

- In order to introduce the oscillator, we'll visualize it as a block attached to a spring.
- This is an idealized situation, in which there is no friction or air resistance. Once set in motion, the block will continue to move back and forth forever.
- (Video or picture)
- The harmonic oscillator is extremely important in classical physics, because all kinds (non idealized) motions can be approximated to a useful degree by harmonic motion. We'll be looking at a couple of examples. We'll get to the significance of the oscillator in quantum theory later on.
- Why is it called "harmonic?"

  Apparently it goes back to harmonic functions being involved in musical overtones (harmonics). Here's Wikipedia's explanation: Etymology of the term "harmonic."

- **Position:** We define the position as being how far the block is from its rest or equilibrium point (the point where the spring is not exerting any force on the block). In our picture, $x = 0$ at the rest point and x goes positive to the right.
- **Force:** Hooke's law: $F = -kx$.
- **Acceleration:** From Newton's 2nd law, acceleration is equal to the force applied to an object divided by the object's mass, or $a = F/m$. In our case, this is: $a = -\frac{k}{m}x$. Writing this as a function of time, we have: $a(t) = -\frac{k}{m}x(t)$.
- **Velocity:** We'll calculate the exact velocity in a moment, but you can see that when we first release the block it will be moving very slowly. The speed will increase as the spring pulls on it, reaching a maximum when the block is at the center. Then it will begin to slow down as the spring pushes back on it, eventually stopping for an instant at the left-hand maximum point, then turning around and repeating the cycle.
- Plot position and velocity.

- We already know that $a(t) = -\frac{k}{m}x(t)$.
- But acceleration is just the 2nd derivative of position (wrt time), so:

$$x''(t) = -\frac{k}{m}x(t).$$

- This is a *2nd degree* (two derivatives) *ordinary* (no partial derivatives) differential equation. Abbreviated “ODE.”
- Solving the ODE
  Either sines or cosines could be solutions: $\sin\left(\sqrt{\frac{k}{m}}\,t\right), \cos\left(\sqrt{\frac{k}{m}}\,t\right)$
  But since we define $t = 0$ to be the time when we initially release the block (which we have pulled out to $x = x_0$), We must have: $x(0) = x_0$. So the solution we need is:

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

  To get the velocity we simply take the first derivative of the position:

$$v(t) = -x_0 \frac{k}{m} \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

- So, to summarize what we know so far:

  We pull the block to the right, to the position $x = x_0$.

  We let it go at the time $t = 0$.

  $x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

  $v(t) = -x_0 \frac{k}{m} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

  $a(t) = -x_0 \frac{k}{m} \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

- **Kinetic Energy**

  Kinetic energy is the energy associated with the motion of a mass:

  $$K = \frac{1}{2} m v^2$$

  In our oscillator, we know what the velocity of the block is at any given time:

  $$v(t) = -x_0 \frac{k}{m} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

  From that, we know the kinetic energy as a function of time:

  $$K(t) = \frac{1}{2}\left[m v(t)\right]^2 = \frac{x_0^2 k^2}{2m} \sin^2\left(\sqrt{\frac{k}{m}}\, t\right)$$

- **Potential Energy**

  Potential energy is the result of a force acting on an object. In our oscillator, we inject potential energy into the system when we pull the oscillator away from the enter position. As we hold the block with the string stretched out, that energy has the *potential* to make the block move. When we release the block, that potential energy starts being converted into kinetic energy. It's customary to user U or V to represent potential energy. I'll use U in what follows.

- **Potential energy in the oscillator**

  If you're familiar with calculating work and energy, and the associated integrals, then you already know that we can integrate over the displacement of the block to get the potential energy at time zero:

  $$U_0 = \left| \int_0^{x_0} -kx\,dx \right| = \tfrac{k}{2} x_0^2$$

  But I'm going to go a different route here ...

- **Energy in the oscillator is constant**

  We inject energy into the oscillator by pulling out the block. But once we let it go (given the "ideal" assumptions of no friction and no air resistance) then no energy can escape from the system. The total energy is the sum of the kinetic an potential energy at any given time. This sum is constant.

  When the block is at the center equilibrium point, then there is no tension on the string and therefore no potential energy. So *all* the energy at this moment must be kinetic. If we calculate the kinetic energy at this moment, we'll know how much total energy is in the system.

Energy: Kinetic and Potential

- **When is the block at the center?**

  The block reaches the center when $x(t) = 0$. So we need to solve for $x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) = 0$. We need $\cos(\sqrt{\frac{k}{m}}\, t)$ to be zero, which happens when $t = \sqrt{\frac{m}{k}}\frac{\pi}{2}$. At this point $\sin\left(\sqrt{\frac{k}{m}}\, t\right) = 1$, so the velocity is: $v(t) = -x_0 \frac{k}{m}$, and the kinetic energy (and total energy) are:

  $$K = \frac{k}{2} x_0^2$$

- **Potential energy at any time**

  We know the total energy $\frac{k}{2} x_0^2$ which is constant and the kinetic energy which is $\frac{x_0^2 k^2}{2m} \sin^2\left(\sqrt{\frac{k}{m}}\, t\right)$. So at at any given time the potential energy is just the total energy minus the kinetic energy:

  $$U(t) = \frac{k}{2} x_0^2 - \frac{x_0^2 k^2}{2m} \sin^2\left(\sqrt{\frac{k}{m}}\, t\right) = \frac{k}{2} x_0^2 \left(1 - \frac{k}{m} \sin^2\left(\sqrt{\frac{k}{m}}\, t\right)\right)$$

  I should be able to simplfy that?

- Plot potential and kinetic energy on the time axis
- Another kind of energy plot ...