Superdense Coding

"Alice sends Bob two classical bits encoded in a single quantum bit"

1. Alice and Bob share a pair of qbits entangled in the first Bell basis state and Alice has two classical bits she wants to send to Bob.

   Alice and Bob share: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

   Alice has $|\text{cbits}\rangle$ which are equal to either $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$,

2. Alice changes the entangled state she shares with Bob to one of the four Bell basis states, corresponding to the state of her two classical bits. She can do this by operating on her qbit with X and Z gates.

   $|\text{cbits}\rangle = |00\rangle$: (Alice doesn't do anything)

   $|\text{cbits}\rangle = |01\rangle$: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \to X \to \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

   $|\text{cbits}\rangle = |10\rangle$: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \to Z \to \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$

   $|\text{cbits}\rangle = |11\rangle$: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \to X \to Z \to \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$

3. Alice ships her qbit to Bob.

4. Bob "does a measurement in the Bell basis" telling him which of the four possibilities was selected.

   What this means in practice is that Bob operates on the two entangled bits with a CNOT and then with a Hadamard. This transforms whichever Bell basis state he has back to the corresponding classical basis state.

Note:

This is the operator which transforms the Bell basis into the classical basis:

$$\to CNOT \to H \quad == \quad (H \otimes I)(\text{CNOT}) \quad == \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

Discussion:

How does this relate to what Alice does in the quantum teleportation algorithm?