Multiple Classical Oscillators

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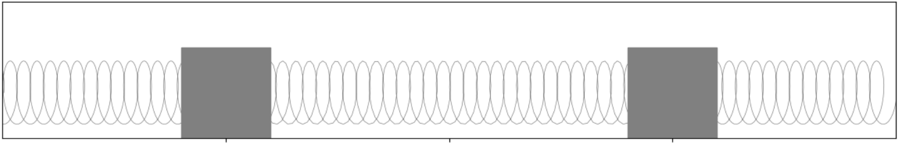
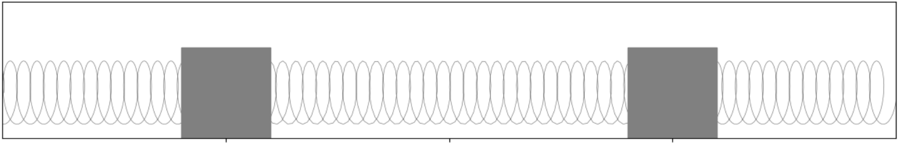


- Both blocks have the same mass, m.
- All three springs have the same constant, k.
- x_1 and x_2 measure the displacement (to the right) of the masses from their respective equilibrium points.

Derive the equations



- The force on either mass depends only on its displacement and the displacement of the other mass.
- $\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$
- $\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$
- Note that $x_1, x_2, \ddot{x}_1, \ddot{x}_2$ are all implicitly functions of time.
- These two ODEs are *coupled*. The solution to $\ddot{x}_1$ depends on x_2 and the solution to $\ddot{x}_2$ depends on x_1.

Express the problem in a vector space

- Define vectors to represent the displacements.
- Let $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represent $x_1 = 1$
- Let $|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represent $x_2 = 1$
- Then $|x(t)\rangle = x_1(t)\,|1\rangle + x_2(t)\,|2\rangle$
- Let $\Omega = \frac{1}{m}\begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}$
- Then $|\ddot{x}(t)\rangle = \Omega|x(t)\rangle$ which comes out to $\begin{bmatrix} -\frac{2k}{m}x_1 + \frac{k}{m}x_2 \\ \frac{k}{m}x_1 - \frac{2k}{m}x_2 \end{bmatrix}$

Change basis to diagonalize the matrix

- The matrix Ω is Hermitian. This means that its eigenvectors form a basis, and that it *can* be diagonalized.
- The eigenvalues are: $-\frac{k}{m}$ and $-\frac{3k}{m}$.
- The normalized eigenvectors are: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- (Recognize the "x basis" from electron spin?)
- The diagonal matrix is: $\Omega' = -\frac{1}{m}\begin{bmatrix} k & 0 \\ 0 & 3k \end{bmatrix}$.

Rewrite the problem in the new basis

- Let's call the new basis vectors $|I\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $|II\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- And define $|x'(t)\rangle = x_I(t)\,|I\rangle + x_{II}(t)\,|II\rangle$.
- Then in the new basis:

$$|\ddot{x}'(t)\rangle = \Omega'|x'(t)\rangle = -\frac{1}{m}\begin{bmatrix} k & 0 \\ 0 & 3k \end{bmatrix}\begin{bmatrix} x_I \\ x_{II} \end{bmatrix} = \begin{bmatrix} -\frac{k}{m}x_I \\ -\frac{3k}{m}x_{II} \end{bmatrix}$$

- The equations are uncoupled.
- But what do x_I and x_{II} represent *physically?*
- x_I represents both masses being pulled to the right.
- x_{II} represents the two masses being pushed together.

Solve the decoupled equations

- Solving the decoupled equations in the x_I, x_{II} basis gives:

$$x_I = A\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$$x_{II} = A'\cos\left(\sqrt{\tfrac{3k}{m}}\,t\right) + B'\sin\left(\sqrt{\tfrac{3k}{m}}\,t\right)$$

- We are assuming an initial velocity of zero for both masses. The derivaties of the position (the velocities) are A sin() and B cos(), so we want to set the Bs to zero, giving:

$$x_I = A\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) \text{ and } x_{II} = A'\cos\left(\sqrt{\tfrac{3k}{m}}\,t\right)$$

- The initial "positions" of the masses at time zero determine the constants:

$$x_I(t) = x_I(0)\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) \text{ and } x_{II}(t) = x_{II}(0)\cos\left(\sqrt{\tfrac{3k}{m}}\,t\right)$$

Take the solution back to the original basis

- We can take the solution back to the original $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ basis like this:

$$|x\rangle = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_I \left(\tfrac{1}{\sqrt{2}}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_{II} \left(\tfrac{1}{\sqrt{2}}\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} x_I + x_{II} \\ x_I - x_{II} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} x_I(0)\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + x_{II}(0)\cos\left(\sqrt{\tfrac{3k}{m}}\,t\right) \\ x_I(0)\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) - x_{II}(0)\cos\left(\sqrt{\tfrac{3k}{m}}\,t\right) \end{bmatrix}$$

- To get the initial conditions in terms of the original basis, do the projections:

$$x_I = \langle I|\, x(0)\rangle = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \tfrac{1}{\sqrt{2}}(\, x_1(0) + x_2(0)\,)$$

$$x_{II} = \langle II|\, x(0)\rangle = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \tfrac{1}{\sqrt{2}}(\, x_1(0) - x_2(0)\,)$$

- And so we end up with:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}}[x_1(0) + x_2(0)] \cos\left(\sqrt{\frac{k}{m}}\, t\right) + \frac{1}{\sqrt{2}}[x_1(0) - x_2(0)] \cos\left(\sqrt{\frac{3k}{m}}\, t\right) \\ \frac{1}{\sqrt{2}}[x_1(0) + x_2(0)] \cos\left(\sqrt{\frac{k}{m}}\, t\right) - \frac{1}{\sqrt{2}}[x_1(0) - x_2(0)] \cos\left(\sqrt{\frac{3k}{m}}\, t\right) \end{bmatrix}$$

- Which simplfies to:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} [x_1(0) + x_2(0)] \cos\left(\sqrt{\frac{k}{m}}\, t\right) + [x_1(0) - x_2(0)] \cos\left(\sqrt{\frac{3k}{m}}\, t\right) \\ [x_1(0) + x_2(0)] \cos\left(\sqrt{\frac{k}{m}}\, t\right) - [x_1(0) - x_2(0)] \cos\left(\sqrt{\frac{3k}{m}}\, t\right) \end{bmatrix}$$

- If you're following this in Shankar, then note that the vector equation above is equivalent to the two equations (1.8.38a) and (1.8.38b) on Shankar page 50.

The Propagator

- Now we'll rewrite the equation above using a matrix multiplying the initial condition vector:

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \frac{1}{2}\left[\begin{array}{cc} \cos\left(\sqrt{\frac{k}{m}}\,t\right) + \cos\left(\sqrt{\frac{3k}{m}}\,t\right) & \cos\left(\sqrt{\frac{k}{m}}\,t\right) - \cos\left(\sqrt{\frac{3k}{m}}\,t\right) \\ \cos\left(\sqrt{\frac{k}{m}}\,t\right) - \cos\left(\sqrt{\frac{3k}{m}}\,t\right) & \cos\left(\sqrt{\frac{k}{m}}\,t\right) + \cos\left(\sqrt{\frac{3k}{m}}\,t\right) \end{array}\right]\left[\begin{array}{c} x_1(0) \\ x_2(0) \end{array}\right]$$

- The matrix above is called the *propagator*. Note that it is a function of time only. We'll encounter the equivalent (but continuous) operator when solving quantum mechanics problems.

The normal modes

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Template

- ...