A few words about infinitesimals

As we begin work with continuous spaces, we will encounter mathematical objects such as:

$$dx, \quad dy, \quad \frac{dy}{dx}, \quad \frac{\partial x}{\partial t},$$

and so on. I am going to treat these objects as *infinitesimals*.

If you're already comfortable with this concept, then you can skip this note.

Infinities

First let's chat for a moment about *infinities*. One way to think about an infinite quantity is that:

(1) It's bigger than any quantity you can specify.

(2) It's not *everything*.

Take the even (positive) integers, for example: $2, 4, 6, 8, ...$

How many of them are there? We say there are an infinite number. Point (1) is pretty clear. No matter what number we name, there are more even integers than that. How about point (2). Well, the even integers are not *all* of the integers. I hope that makes sense.

One infinity can be greater than another. For example the (continuous) infinity of points on the number line (the real numbers) is greater than the (discrete or "countable") infinity of the integers. You'll have to take my word for this, since I don't want to go too far afield.

Infinitesimals

We can think of infinitesimals in a similar way. An infinitesimal:

(1) Is smaller than any number you can specify.

(2) Is not *nothing*.

Furthermore, one infinitesimal can be greater than another.

Infinitesimals vs. limits

To the best of my knowledge, undergraduate analysis (calculus) classes are *always* taught from the perspective of *limits*. If you took calculus, when the derivative dy/dx was first introduced, the professor may have said something like this:

> *The symbols dx and dy have no individual meaning, and dy/dx is certainly NOT a ratio. dy/dx is actually the result of a limiting process ...*

But from the perscpective I'm going to follow, dx and dy do represent individual (infinitesimal) quantities, and dy/dx is in fact a perfectly well-defined ratio.

If you don't know what limits or limiting processes are, that's OK. We're not going to talk about them (at least not much).

The reason I'm taking the perspective of infinitesimals rather than limits is because I believe that, in the context we're working in, infinitesimals are simply much more intuitive to work with and easier to explain than limits.

Finite quantities vs infinitesimals, or Δx vs dx

(Note: The word "finite" is typically taken to mean "not infinite." But it also means "not infinitesimal.")

So what exactly is an infinitesimal quantity such as dx or dy?

Draw a set of x and y axes. Δx is some displacement along the x axis. Δy is some displacement along the y axis. The ratio $\Delta y/\Delta x$ is the ratio of these two numbers and (if you draw things right) is the slope of a line.

Similarly, dx is some infinitesimally small displacement along the x axis. dy is an infinitesimal displacement along the y axis. The ratio dy/dx is the ratio of these two numbers which is not infinitesimal, but finite. It is also a kind of a slope, which we call a derivative.

And finally ...

If you just flat out don't buy the "infinitesimal" concept, you should be able to do everything you need to do simply by thinking of things like dx and dy as really really *really* small numbers – many orders of magnitude smaller than any number you'll encounter in whatever context you're working in.