# The Harmonic Oscillator

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# Outline

1. Introduction.
2. Hooke's law. Position, velocity, and acceleration.
3. The differential equation.
4. Potential and kinetic energy.
5. Angular velocity.
6. Momentum. The Hamiltonian formulation.

- In order to introduce the oscillator, we'll visualize it as a block attached to a spring.
- This is an idealized situation, in which there is no friction or air resistance. Once set in motion, the block will continue to move back and forth forever.
- The harmonic oscillator is extremely important in classical physics, because all kinds of (non idealized) motions can be approximated to a useful degree by harmonic motion. We'll be looking at a couple of examples. We'll get to the significance of the oscillator in quantum theory later on.
- Why is it called "harmonic?"

  Apparently it goes back to harmonic functions being involved in musical overtones (harmonics). Here's Wikipedia's explanation: Etymology of the term "harmonic."

- **Position:** We define the position as being how far the block is from its rest or equilibrium point (the point where the spring is not exerting any force on the block). In our picture, $x = 0$ at the rest point and $x$ goes positive to the right.
- **Force:** Hooke's law: $F = -kx$.
- **Acceleration:** From Newton's 2nd law, acceleration is equal to the force applied to an object divided by the object's mass, or $a = F/m$. In our case, this is: $a = -\frac{k}{m}x$. Writing this as a function of time, we have: $a(t) = -\frac{k}{m}x(t)$.
- **Velocity:** We'll calculate the exact velocity in a moment, but you can see that when we first release the block it will be moving very slowly. The speed will increase as the spring pulls on it, reaching a maximum when the block is at the center. Then it will begin to slow down as the spring pushes back on it, eventually stopping for an instant at the left-hand maximum point, then turning around and repeating the cycle.

- We already know that $a(t) = -\frac{k}{m}x(t)$.
- But acceleration is just the 2nd derivative of position (wrt time), so:

$$x''(t) = -\frac{k}{m}x(t).$$

- This is a *2nd degree* (two derivatives) *ordinary* (no partial derivatives) differential equation. Abbreviated "ODE."
- Solving the ODE

  Either sines or cosines could be solutions: $\sin\left(\sqrt{\frac{k}{m}}\,t\right), \cos\left(\sqrt{\frac{k}{m}}\,t\right)$

  But since we define $t = 0$ to be the time when we initially release the block (which we have pulled out to $x = x_0$), We must have: $x(0) = x_0$. So the solution we need is:

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

  To get the velocity we simply take the first derivative of the position:

$$v(t) = -\sqrt{\frac{k}{m}}\,x_0 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

- To summarize what we know so far:

  We pull the block to the right, to the position $x = x_0$.

  We let it go at the time $t = 0$.

  $x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

  $v(t) = -\sqrt{\frac{k}{m}}\, x_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

  $a(t) = -\frac{k}{m} x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

- **Kinetic Energy**

Kinetic energy is the energy associated with the motion of a mass:

$$K = \frac{1}{2} m v^2$$

In our oscillator, we know what the velocity of the block is at any given time:

$$v(t) = -\sqrt{\frac{k}{m}}\, x_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

From that, we know the kinetic energy as a function of time:

$$K(t) = \frac{1}{2} k\, x_0^2 \sin^2\left(\sqrt{\frac{k}{m}}\, t\right)$$

The block has maximum kinetic energy at the center point, when $x(t) = 0$, since this is when its velocity is maximum. Solving for $x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) = 0$, we need $\cos(\sqrt{\frac{k}{m}}\, t)$ to be zero, this happens when $t = \sqrt{\frac{m}{k}}\frac{\pi}{2}$. At this point $\sin\left(\sqrt{\frac{k}{m}}\, t\right) = 1$, so the velocity is: $v(t) = -\sqrt{\frac{k}{m}} x_0$, and the kinetic energy is:

$$K = \frac{1}{2} k x_0^2$$

- **Potential Energy**

  Potential energy is the result of a force acting on an object. In our oscillator, we inject potential energy into the system when we pull the oscillator away from the center position. As we hold the block with the spring stretched out, that energy has the *potential* to make the block move. When we release the block, that potential energy starts being converted into kinetic energy. It's customary to use $U$ or $V$ to represent potential energy. I'll use $U$ in what follows.

- **Potential energy in the oscillator**

  If you're familiar with calculating work and energy, and the associated integrals, then you already know that we can integrate over the displacement of the block to get the potential energy, which for any given displacement gives:

  $$U = \left| \int_0^x -kx \, dx \right| = \tfrac{1}{2} kx^2$$

  This means that the maximum potential energy, when the block is pulled all the way out at time zero is:

  $$U_{max} = \tfrac{1}{2} kx_0^2$$

- **Energy in the oscillator is constant**

We inject energy into the oscillator by pulling out the block. But once we let it go (given the "ideal" assumptions of no friction and no air resistance) then no energy can escape from the system. The total energy is the sum of the kinetic and potential energy at any given time. This sum is constant.

When the block is at the center equilibrium point, then there is no tension on the spring and therefore no potential energy. So *all* the energy at this moment must be kinetic and equal to the maximum potential energy:

$$K_{max} = \frac{1}{2}kx_0^2$$

This is consistent with what we found above. It also must be the case that for any given position that:

$$E_{total} = K + U$$

$$\frac{1}{2}kx_0^2 = K + \frac{1}{2}kx^2$$

$$K = \frac{1}{2}kx_0^2 - \frac{1}{2}kx^2$$

- Again, a summary:

  We pull the block to the right, to the position $x = x_0$.

  We let it go at the time $t = 0$.

  $x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

  $v(t) = -\sqrt{\frac{k}{m}}\, x_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

  $a(t) = -\frac{k}{m} x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

  Total energy: $E = K + U = \frac{1}{2} k x_0^2$

  $U(x) = \frac{1}{2} k x^2$

  $K(x) = \frac{1}{2} k x_0^2 - \frac{1}{2} k x^2$

- Energy Well plot.

# (Aside) Angular Velocity

Angular velocity, usually denoted by $\omega$ (omega), is the velocity at which an angle changes. For example, picture the position of the oscillator as a vector on a circle of radius $x_0$. At time zero, when the oscillator is at position $x_0$ the vector has an angle of zero (it's pointing straight to the right). As the oscillator moves through one cycle, the vector rotates counter clockwise through the circle, the angle returning to zero as the oscillator returns to $x = x_0$.

The position of the oscillator is $x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$. The radius of the circle comes from the $x_0$ and the angle is $t$ multiplied by $\sqrt{\frac{k}{m}}$. So, for each second that elapses, the angle moves by $\sqrt{\frac{k}{m}}$ radians. This means that $\omega = \sqrt{\frac{k}{m}}$.

Clearly the units of $\omega$ would be radians per second, but by a "definitional trick" the SI standard calls it dimensionless. iopscience.iop.org/article/10.1088/0026-1394/52/1/40

In any event, if you don't want to have $\sqrt{\frac{k}{m}}$ all over the place, then you can say, for example, $x = x_0 \cos(\omega t)$, and (as long as we're using radians) our oscillator will go through one complete cycle in $\frac{2\pi}{\omega}$ seconds.

# 2nd Formulation - The Hamiltonian

- Momentum: $p = mv$.
- Momentum in kinetic energy: $K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$.
- The Hamiltonian: $H = \text{Total energy} = K + U = \frac{p^2}{2m} + U$.
- Hamiltonian for the oscillator: $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$.
- Position and Momentum, from the Hamiltonian

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x} = -kx$$

In other words: $\frac{dx}{dt} = v, \quad \frac{dv}{dt} = a$ ☹

- Why do we care about the Hamiltonian formulation?

It may seem like this didn't accomplish anything. But these equations describe the oscillator in *phase space*. More important, the Hamiltonian will provide the starting point for the calculations in the *quantum* harmonic oscillator (and the Hamiltonian is almost always used in basic quantum mechanical systems).