A complex network approach of the Ising model

Alma Mater Studiorum - Bologna University

Master in Theoretical Physics

Complex Networks

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Contents

Abstract

The goal of this article is to analyze the Ising model through a network approach. We have implemented the Ising model and we have compared the theoretical predictions of the main observables (internal energy, magnetisation, entropy and specific heat) with the numerical ones. We have both analysed the $1d$ Ising model and the $2d$ Ising model, in particular we have considered three different geometries for the $2d$ Ising model: square, hexagonal and triangular. The $2d$ Ising models display a phase transition, hence we have calculated the critical temperatures and we have compared them with the theoretical ones. To analyze the behaviour of the $2d$ Ising model we have created a complete weighted graph, where the matrix of the weights is the two point correlation function. Using this approach it has been possible to describe the behaviour of the $2d$ Ising models near the critical temperatures through some network parameters (disparity, betweenness centrality, density, clustering coefficient, geodesic distance and diameter).

1 Theory of the Ising model

In this section we will introduce the main quantities and ideas of the Ising model. We follow the books of Mussardo [5], the notes of [3] and the book of Ercolessi [4].

1.1 Introduction

The $2d$ Ising model is the simplest statistical model that has a phase transition. The aim of the model is to describe a d-dimensional magnetic solid. We consider a lattice with dimension 1 and 2 in which each lattice site is associated with a variable $\sigma_j = \pm 1$. We suppose that each site can only interact with his neighbours. In this way, the Hamiltonian of the system is

$$H = -\frac{J}{2} \sum_{nn} \sigma_i \sigma_j - B \sum_i \sigma_i, \tag{1}$$

where J is a coupling constant, B is a constant electromagnetic field and the first summation is intended to be only with the nearest neighbours. We observe that if $B = 0$ then the Hamiltonian has a Z_2 symmetry.

1.2 Observables

We will use the canonical approach to derive the observables of the Ising model. The canonical partition function is

$$Z = \sum_{\{\sigma\}} e^{-\beta H}. \tag{2}$$

The free energy F can be obtained from the partition function using

$$\frac{F}{N} = f = -\frac{1}{\beta} \log(Z), \tag{3}$$

while the internal energy can be calculating by

$$\frac{E}{N} = \epsilon = -\frac{\partial \log Z}{\partial \beta}. \tag{4}$$

Other important quantities that can be calculated are the specific heat

$$\frac{C}{N} = c = -\beta^2 \frac{\partial E}{\partial \beta} \tag{5}$$

and the susceptibility

$$\chi = \frac{1}{N} \frac{\partial^2 F}{\partial B^2}. \tag{6}$$

The most important parameter of the Ising model is the magnetization per spin, which is the average value of σ over all the sites of the lattice, namely

$$m = \langle \sigma_i \rangle = -\frac{\partial f}{\partial B}. \tag{7}$$

Entropy can be calculate in two ways. One way is via the microcanonical approach (we have used Stirling approximation)

$$S = N \log N - N_+ \log(N_+) - N_- \log(N_-) + O(\log(N)), \tag{8}$$

where N_+ is the number of spins up and N_- are the number of spins down. The other way is using the canonical approach, from which the entropy is simple correlated to the free energy and the internal energy

$$S = \frac{E - F}{T}. \tag{9}$$

Finally the two-point correlation function is defined as

$$G_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = \langle (\sigma_i - \langle \sigma_j \rangle)(\sigma_i - \langle \sigma_j \rangle) \rangle. \tag{10}$$

1.3 Fluctuations

It is useful to define the generating function

$$W = \beta F,$$

from which we can calculate

$$\frac{\partial^2 W}{\partial B^2} = \beta^2 (\langle M^2 \rangle - \langle M \rangle^2) = \beta N \frac{\partial^2 f}{\partial B^2} = \chi \beta N,$$

where we used the definition of the susceptibility (6). From the previous equation we can observe that the role of χ is to be the magnetic variance

$$\chi T = \frac{1}{N} (\langle M^2 \rangle - \langle M \rangle^2). \tag{11}$$

The specific heat is instead the energy variance

$$\frac{\partial^2 W}{\partial \beta^2} = \langle E^2 \rangle - \langle E \rangle^2 = \frac{C}{\beta^2} \quad \Rightarrow \quad cT^2 = \frac{1}{N} (\langle E^2 \rangle - \langle E \rangle^2). \tag{12}$$

We observe that the two-point correlation function can be calculated using W

$$G_{ij} = \frac{1}{\beta^2} \frac{\partial^2 W}{\partial B_i \partial B_j},$$

hence the two point correlation function should be somehow related to the susceptibility because they are both derived from W by derivatives to the magnetic field. Indeed the following relation holds

$$\sum_{ij} G_{ij} = NT\chi,$$

from the previous equation we can see that is convenient to change variable $|i - j| = r$ so the two-point correlation faction is related to the susceptibility as

$$\sum_r G(r) = NT\chi.$$

1.4 Phase transition

A phase transition is a physical concept to describe the change of the physical properties of a system. The occurrence of a phase transition is the result of two competitive instances, the first tends to minimize the energy, while the second tends to maximize the entropy.

In our case we will show that for the $2d$ Ising model there exists a temperature, called critical temperature T_c, such that, for temperature below T_c, the system has a magnetization different from zero. This happens because at low temperature the minimization of the energy wins and the Hamiltonian (1) reaches the minimum when all the spins are aligned. Instead, as the temperature increases, the entropy grows, so at high temperature the maximization of the entropy wins, hence the spins are randomly distributed.

m is called order parameter; it is very useful because observing its value, it is possible to recognize if a phase transition is happening or not. We also observe that because $m \neq 0$ at temperature $T < T_c$ then we have broken the Z_2 symmetry. This phenomenon is known as spontaneous symmetry breaking.

We observe that in a phase transition we expect that the fluctuations (11), (12) are large and both the susceptibility and specific heat will diverge.

1.5 1d Ising model

In the case of a 1 dimensional model the system is simply a ring (we suppose that $\sigma_1 = \sigma_{N+1}$), hence every node can have only 2 neighbours. To solve this model we will follow Baxter approach [1] The canonical partition function (2) becomes

$$Z = \sum_{\{\sigma\}} e^{-\frac{1}{T}H} = \sum_{\{\sigma\}} e^{\frac{1}{T}(J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} + \frac{B}{2}\sum_{i=1}^{N}(\sigma_i+\sigma_{i+1}))} = \sum_{\{\sigma\}}\prod_{i=1}^{N} e^{\frac{1}{T}(J\sigma_i\sigma_{i+1}+\frac{B}{2}(\sigma_i+\sigma_{i+1}))}.$$

We can define the transfer matrix T as

$$T = \begin{pmatrix} e^{\frac{1}{T}(J+B)} & e^{-\frac{J}{T}} \\ e^{-\frac{J}{T}} & e^{\frac{1}{T}(J-B)} \end{pmatrix}$$

where the components T_{ij} are the possible results of the factors in the partition function. We have

$$Z = \sum_{\{\sigma\}} T^N = \lambda_+^N + \lambda_-^N,$$

where λ_+ and λ_- are the eigenvalues of the transfer matrix. The eigenvalues are

$$\lambda_\pm = e^{\frac{J}{T}}\left(cosh\left(\frac{B}{T}\right) \pm \sqrt{\sinh^2\left(\frac{B}{T}\right) + e^{-4\frac{J}{T}}}\right).$$

Using relation (3) we obtain the free energy

$$F = -T\left(N\log(\lambda_+) + \log\left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right)\right)$$

in the thermodynamic limit ($N \to \infty$) only the contribution from λ_+ survives ($\lambda_+ > \lambda_-$) and so the free energy per spin is given by

$$f = -T\log\left(e^{\frac{J}{T}}\cosh\left(\frac{B}{T}\right) + \sqrt{\sinh^2\left(\frac{B}{T}\right) + e^{-4\frac{J}{T}}}\right),$$

which, in the case $B = 0$, becomes

$$f = -T\log\left(2\cosh\frac{J}{T}\right).$$

The magnetization per spin (7) is

$$m = \frac{\sinh\left(\frac{B}{T}\right)}{\sqrt{\sinh^2\left(\frac{B}{T}\right) + e^{-4\frac{J}{T}}}}, \tag{13}$$

which is zero if $B = 0$, hence we have no spontaneous magnetization. The susceptibility can be explicitly calculated from equation (6), which yields

$$\chi = \frac{\frac{1}{T}e^{-4\frac{J}{T}\cosh\left(\frac{B}{T}\right)}}{\left(\sinh^2\left(\frac{B}{T}\right) + e^{-4\frac{J}{T}}\right)^{\frac{3}{2}}},$$

while the energy per spin (4) if $B = 0$ is

$$\epsilon = -J tanh\frac{J}{T}. \tag{14}$$

Finally we can also calculate the specific heat (5)

$$c = \left(\frac{J}{T}\right)^2 sech^2\frac{J}{T}, \tag{15}$$

we observe that we have not a phase transition since neither χ nor c diverges for some value of T.

We want to compute the correlation function using equation (10); using again the transfer matrix it is possible to show that

$$G(r) = \langle\sigma_i\sigma_{i+r}\rangle = e^{-\frac{r}{\xi}},$$

where ξ is the correlation length

$$\xi(T, 0) = \frac{1}{\log tanh\left(\frac{J}{T}\right)}.$$

In conclusion we have seen that the $1d$ Ising model doesn't display a phase transition because the specific heat is regular while the magnetization is always zero.

1.6 2d Ising model

In this section we will focus on the $2d$ dimensional Ising model, which admits an exact solution but it is not easy to find it as it has been for the $1d$ Ising model. Due to the complexity of the problem we will only give the relevant results that were found by L.Onsager in 1943 [6].

We will focus on three lattices: square, triangular and hexagonal. For every lattices we give the free energy

$$f(T, B = 0) = -\frac{p}{T}\log 2 - \frac{T}{2}\int d\Pi \log P(T, \boldsymbol{k}),$$

where the argument of the logarithm is

$$P_S(T, \boldsymbol{k}) = \cosh^2\left(2\frac{J}{T}\right) - \sinh\left(\frac{2J}{T}\right)(\cos k_1 + \cos k_2)$$
$$P_T(T, \boldsymbol{k}) = \cosh^3\left(2\frac{J}{T}\right) + \sinh^3\left(2\frac{J}{T}\right) - \sinh\left(\frac{2J}{T}\right)(\cos k_1 + \cos k_2 + \cos k_3)$$
$$P_H(T, \boldsymbol{k}) = \frac{1}{2}\left(1 + \cosh^3\left(2\frac{J}{T}\right) - \sinh^2\left(2\frac{J}{T}\right)(\cos k_1 + \cos k_2 + \cos k_3)\right),$$

$\boldsymbol{k}$ is the Fourier variable and it is two dimensional for the square lattice and three dimensional for triangular and hexagonal lattices. p is the number of sites in a unit cell of the lattice, which is one for the square and triangular lattices and two for the hexagonal lattice, while the integral measure is

$$\Pi_S = \frac{d^2k}{(2\pi)^2},$$
$$\Pi_T = \Pi_H = \frac{d^3k}{(2\pi)^3}.$$

It is possible to show that the argument of the logarithm, which is $P(T,\boldsymbol{k})$ is always positive except for $\boldsymbol{k}=0$. Hence it exist a critical temperature $T=T_c$ for which the logarithm diverges. If we impose $\boldsymbol{k}=0$ we have the following equations

$$\cosh^2 2\frac{J}{T} - 2\sinh 2\frac{J}{T} = 0 \Rightarrow T_c^S = \frac{2J}{\log\left(1+\sqrt{2}\right)} = 2.26919J, \tag{16}$$

$$\cosh^3 2\frac{J}{T} + \sinh^3 2\frac{J}{T} - 3\sinh 2\frac{J}{T} = 0 \Rightarrow T_c^T = \frac{2J}{\log\sqrt{3}} = 3.64096J, \tag{17}$$

$$1 + cosh^3 2\frac{J}{T} - \sinh^2 2\frac{J}{T} = 0 \Rightarrow T_c^H = \frac{2J}{\log\left(2+\sqrt{3}\right)} = 1.51865J. \tag{18}$$

The energy and the specific heat are not analytical function, but it is possible to calculate them solving an integral numerically, the expression for the internal energy per spin is

$$\epsilon(T,0) = \frac{T^2}{2}\int d\Pi \frac{1}{P(T,\boldsymbol{k})}\frac{\partial P(T,\boldsymbol{k})}{\partial T} \tag{19}$$

and the specific heat is

$$c(T,0) = 2\epsilon\frac{1}{T} + \frac{T^2}{2}\int d\Pi \frac{1}{P(T,\boldsymbol{k})}\left(\frac{\partial^2 P(T,\boldsymbol{k})}{\partial^2 T} - \frac{1}{P(T,\boldsymbol{k})}\frac{\partial P(T,\boldsymbol{k})}{\partial T}\right). \tag{20}$$

It is possible to show that for $T=T_c$ the specific heat diverges. The magnetization can be expressed in analytically, we have the following results

$$m_S(T,0) = \left(\frac{(1+e^{4\frac{J}{T}})^2(1-6e^{4\frac{J}{T}}+e^{8\frac{J}{T}})}{(1-e^{4\frac{J}{T}})^4}\right)^{\frac{1}{8}}, \tag{21}$$

$$m_T(T,0) = \left(\frac{(1+e^{4\frac{J}{T}})^3(3-e^{4\frac{J}{T}})}{(1-e^{4\frac{J}{T}})^3(3+e^{4\frac{J}{T}})}\right)^{\frac{1}{8}}, \tag{22}$$

$$m_H(T,0) = \left(\frac{(1+e^{4\frac{J}{T}})^3(1+e^{4\frac{J}{T}}-4e^{2\frac{J}{T}})}{(1-e^{2\frac{J}{T}})^6(1+e^{2\frac{J}{T}})^2}\right)^{\frac{1}{8}}. \tag{23}$$

The previous expressions are valid for $T<T_c$, for the square lattice it is also possible to find a simpler equivalent form

$$m_S(T,0) = \left(1 - \frac{1}{\sinh^4(2\frac{J}{T})}\right)^{\frac{1}{8}}.$$

The correlation length can be expressed as

$$\xi(T, \boldsymbol{k}) = \sqrt{\frac{P(T, \frac{\pi}{2}, \frac{\pi}{2},) - P(T, 0, 0, 0)}{P(T, 0, 0, 0)}}.$$

In conclusion we have proven that all the $2d$ Ising models admit a phase transition, hence the specific heats diverge for the critical temperatures and the magnetization is different from 0 below the critical temperatures.

2 Network of Ising model

In this section we will analyze the Ising model from networks perspective, in particular we focus on two networks, the clustering network and the correlation network.

2.1 Clustering network

This network is just a visualization of the Ising model: we create a matrix C_{ij} which has value

$$C_{ij} = 1 \, if \ aligned,$$
$$C_{ij} = 0 \, elsewhere.$$

The important feature of this network is that at $T \simeq T_c$, there are regions of spins (sometimes called islands) with the same values.

2.2 Correlation network

This network is a complete graph with $C_{ij} = G_{ij}$, where G_{ij} is the two point correlation function at fixed temperature (10). The aim of this network is to describe how the spins interact with each other. In order to analyze the network behaviour we will introduce some network measurements. We will follow the idea of [7]. In particular we define

$$\begin{aligned}
&\textit{average disparity} && \langle Y_i \rangle = \frac{1}{N} \sum_i \frac{\sum_j G_{ij}^2}{(\sum_j G_{ij})^2} \\
&\textit{average betweenness centrality} && \langle B_i \rangle = \frac{1}{N} \sum_i \sum_{j,k \neq i} \frac{N_{jk}^i}{N_{jk}}, \\
&\textit{average density} && \langle d_i \rangle = \frac{1}{N} \sum_i \frac{\sum_j G_{ij}}{N-1}, \\
&\textit{clustering coefficient} && C = \frac{\sum_{i \neq j \neq k} G_{ij} G_{jk} G_{ki}}{\sum_k \sum_{i \neq j \neq k} G_{ik} G_{jk}}, \\
&\textit{average geodesic distance} && \langle D \rangle = \frac{2 \sum_{ij} D_{ij}}{N(N-1)}, \\
&\textit{diameter} && D_{max} = \max_{i,j} D_{ij}.
\end{aligned}$$

where D_{ij} is the geodesic distance (i.e., length of the shortest path) between nodes i and j, with the distance of a direct path from i to j defined as $1/G_{ij}$. N_{jk}^i is the number of geodesic paths from j to k via i and N_{jk} is the number of geodesic paths from j to k.

The meaning of the disparity is to quantify how dissimilar a node's connections are, hence we expect that that the disparity starts to increase as $T \simeq T_c$. The reason is that if $T < T_c$ all the spins are aligned (due to the phase transition), hence the connection between neighbours nodes are identical; instead when $T \simeq T_c$ the spins tend to form islands hence the disparity increases. Finally when $T >> T_c$ we expect that the disparity decreases due to isotropic behaviour of the lattice at high temperature.

The betweenness centrality measure the importance of node i to the connectivity of other parts of the network to each other. A node i has a high betweeness centrality if removing it distances many other parts of the graph from each other. Again we expect that, when all the spins are aligned, if we remove a node the distance doesn't change too much and also this should happen at high temperature. But for temperature a little bit higher than T_c removing a node can create an appreciable change in the distance between nodes, while if $T \gg T_c$ we expect that the betwenness centrality because at high temperature the spins are randomly distributed.

The density d_i quantifies the importance of node i as the sum of the link weights connected to it and hence it should be 1 for $T \ll 1$ and 0 at high temperature, while it decreases around $T = T_c$.

The clustering coefficient of a network measures cohesiveness of a network and it should be very similar to the density.

The average geodesic distance $\langle D \rangle$ is the average of the shortest path between all pairs of nodes, hence it should start to increase when $T \simeq T_c$.

The diameter is the geodesic distance between the most distant pair of nodes, hence it should behave very similar to the density.

3 Implementation

In this section we describe our implementation of the Ising model.

3.1 Metropolis algorithm

In the following section we briefly describe the main tools that have been used to implement the Ising model: the Metropolis algorithm. We followed the approach of [2] to implement the Ising model.

Monte Carlo simulations work by creating a set of lattice configurations that together approximate an ensemble with Boltzmann weighting. The probability of appearance of a given configuration a is proportional to its Boltzmann weight

$$P(a) = e^{\frac{-E_a}{T}}.$$

There are some similarities with the microcanonical ensemble, where time average is taken to equal ensemble average. However, Monte Carlo time, associated with the production of a stream of configurations, is not real time, and the "time" evolution is only loosely related to real time evolution.

Suppose we have an initial configuration $a = 0$, from which we will produce $a = 1, 2, ...$ sequentially, with each configuration determined by the previous one and some Monte Carlo algorithm. Conceptually, considering an ensemble of initial configurations, each evolving to produce a stream of configurations, is useful. We can then associate with a given configuration a a probability $P(a)$ of appearing at a Monte Carlo time t. It is common to model the evolution of $P(a)$ by a master equation

$$\frac{\mathrm{d}P(a)}{\mathrm{d}t} = \sum_b r(a \leftarrow b)P(b) - r(b \leftarrow a)P(a),$$

where $r(a \leftarrow b)$ is the rate at which a configuration a becomes a configuration b. A time-independent equilibrium solution is given by detailed balance

$$\frac{P(a)}{P(b)} = \frac{e^{\frac{-E_a}{T}}}{e^{\frac{-E_b}{T}}}.$$

This only specifies the ratio of weights, not their value. This behavior is given by the Metropolis algorithm, for which some of the transition probabilities are one. For the Metropolis algorithm, we define

$$r(b \leftarrow a) = min\left(1, e^{\frac{E_a - E_b}{T}}\right)$$

so that $r(b \leftarrow a) = 1$ if $E_b > E_a$, but is less than one if $E_b > E_a$. The essential algorithm is this: consider a change from a to b, if the change lowers the energy, always make the change. If it raises the energy, make the change with a probability given by

$$R = e^{\frac{E_a - E_b}{T}}.$$

A simple algorithm is to compute R, if R is greater than a random number distributed uniformly between 0 and 1, make the change. Of course, if $R > 1$, there is no need to bother generating a random number.

3.2 Source code

The code of the program can be found here, `https://github.com/PhysicsZandi/IsingModel`

4 Analysis of the Ising model

The main idea of our program is to reproduce the Ising model, in particular we are interest in

- compare the graph of the theoretical physical observables with the numerical one;
- find the critical temperatures of the models;
- reproduce the correct behaviour around $T \simeq 0$, $T \simeq T_c$ and $T \gg T_c$.

In order to compare the theoretical prediction with the one that we have obtained in our model we calculate the physics observables (internal energy, magnetization, entropy and specific heat). In particular

- internal energy can be calculated numerically using (1) (with $B = 0$) and we compare it with the equations (14),(19) respectively in case of $1d$ or $2d$;
- to calculate the magnetization we used the definition (7) and equations (21),(22),(23) for $2d$ Ising models, while (7) for $1d$ case;
- entropy can be calculated both with the microcanonical approach (8) and with the canonical approach (9), where the free energy is known from the theoretical point of view and the Stirling approximation is quite good for $N = 100$;

- specific heat is the variance of the energy (12), we can also compute it via (15) in the $1d$ Ising model and (20) in the $2d$ case.

In particular one of our goal is to ensure that all the $2d$ Ising model that we have reproduced (square, triangular and hexagonal) have a phase transition at critical temperatures (16),(17),(18). To do so we will look at the graphics of the magnetization and specific heat versus the temperature. We expect that the specific heat diverges for $T = T_c$ and the magnetization stats decrease around $T = T_c$. We also expect that $1d$ Ising model doesn't have a phase transition.

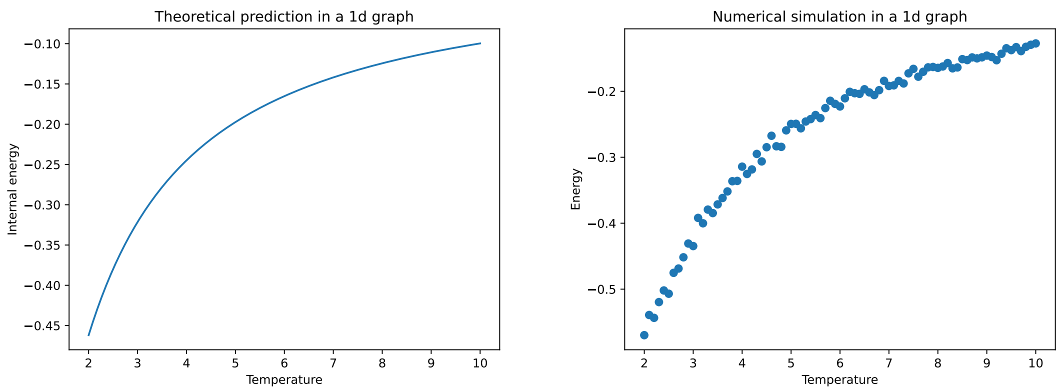
To analyze the behaviour of the Ising model we have used two types of networks, the clustering network and the correlation network. We expect that the clustering network shows all spins align for $T << T_c$, while for $T \simeq T_c$ the system should display distinct areas where the spins are aligned. Finally, when the temperature increase, these areas disappear, hence the system is isotropic because the spins are randomly distributed. The clustering network is purely qualitative. The previous considerations are reflected in the correlation network. To analyze it we have used several parameters (diameter, betweenness centrality, average geodesic distance, average density, average disparity and clustering coefficient) that should behave as we describe them in the previous section.

5 Results

In this section we systematically give the results that we have presented in the previous section.

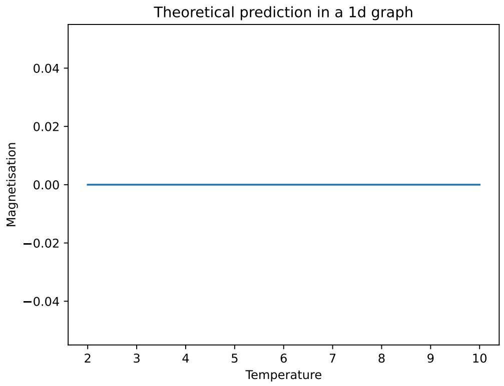
5.1 1d Ising model

The main observables of the $1d$ Ising model are given in figure 1, 2, 3 and 4.

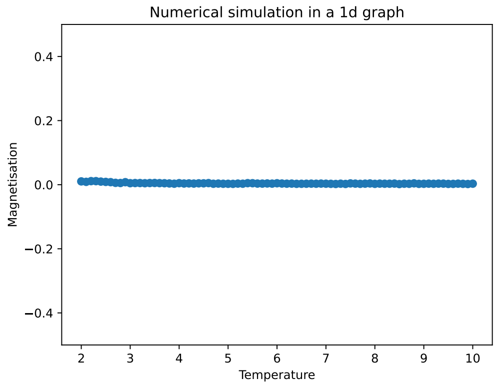


(a) Exact solution. **(b)** Numerical result.

Figure 1: Internal energy of $1d$ Ising model.

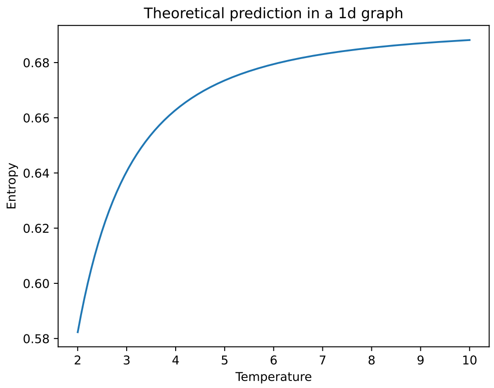


(a) Exact solution.

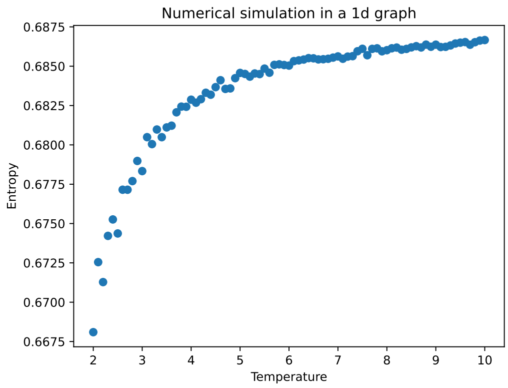


(b) Numerical result.

Figure 2: Magnetisation of $1d$ Ising model, it is always zero due to the fact that the system doesn't exhibit any phase transition.

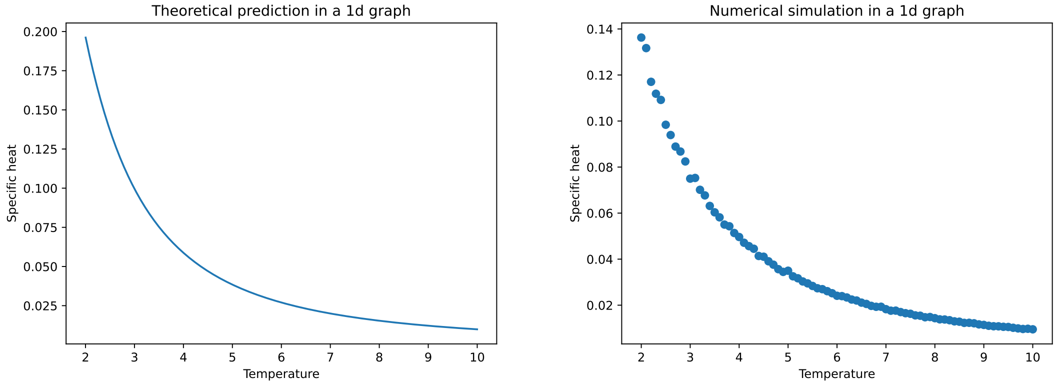


(a) Exact solution.



(b) Numerical result.

Figure 3: Entropy of the $1d$ Ising model.



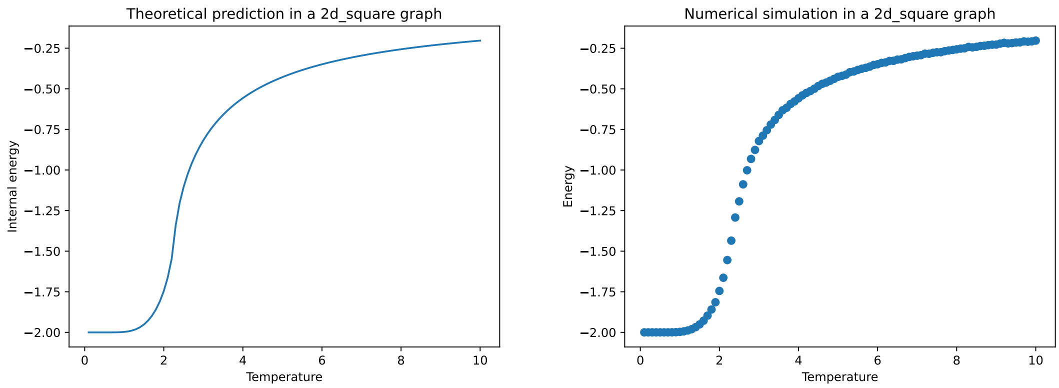
(a) Exact solution. **(b)** Numerical result.

Figure 4: Specific heat of the $1d$ Ising model, as we expect the function is regular.

As we can see there no phase transition occur in the $1d$ Ising model.

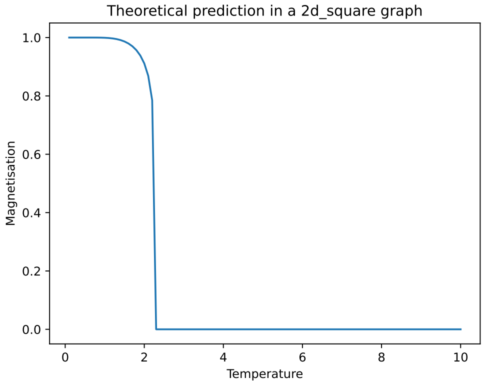
5.2 2d Ising model

The numerical result and their theoretical prediction of the observables of the $2d$ Ising models (square, hexagonal and triangular) are shown in fig 5,6,7,8,9,10,11,12,13,14,15,16.

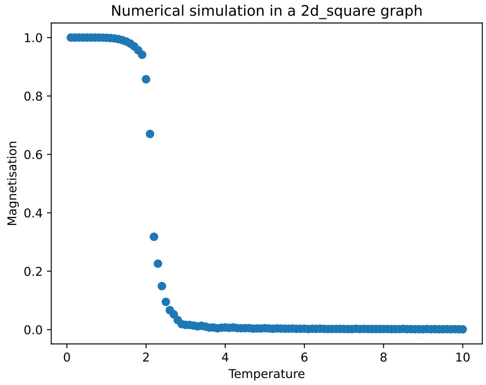


(a) Exact solution. **(b)** Numerical result.

Figure 5: Internal energy of $2d$ square Ising model.

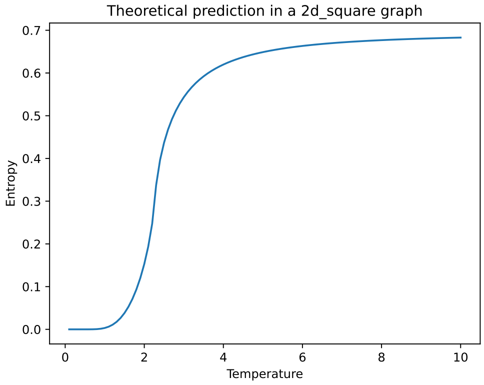


(a) Exact solution.

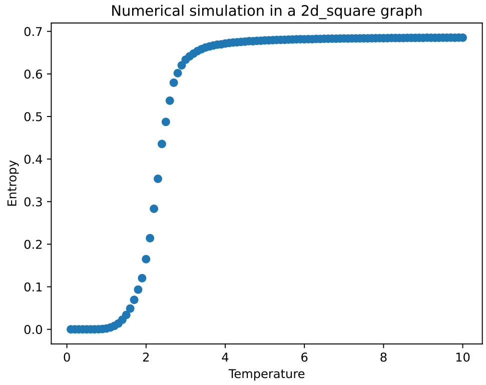


(b) Numerical result.

Figure 6: Magnetisation of $2d$ square Ising model, at the critical temperature $T_c^S = 2,26919$ it goes to zero.

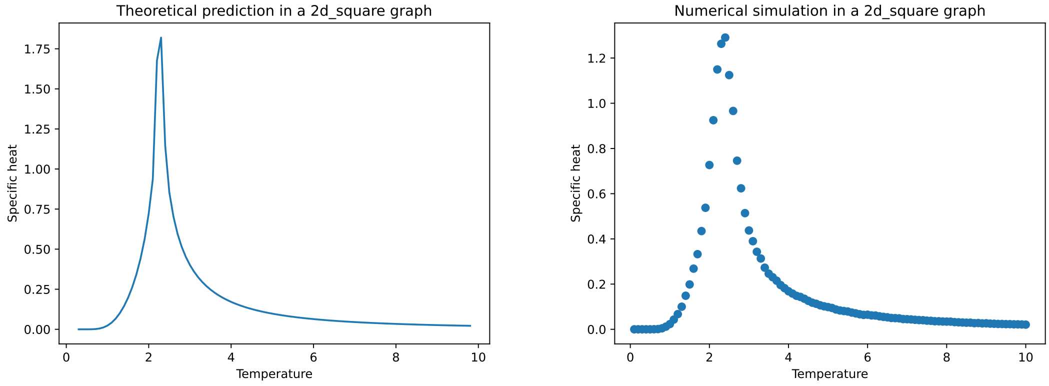


(a) Exact solution.



(b) Numerical result.

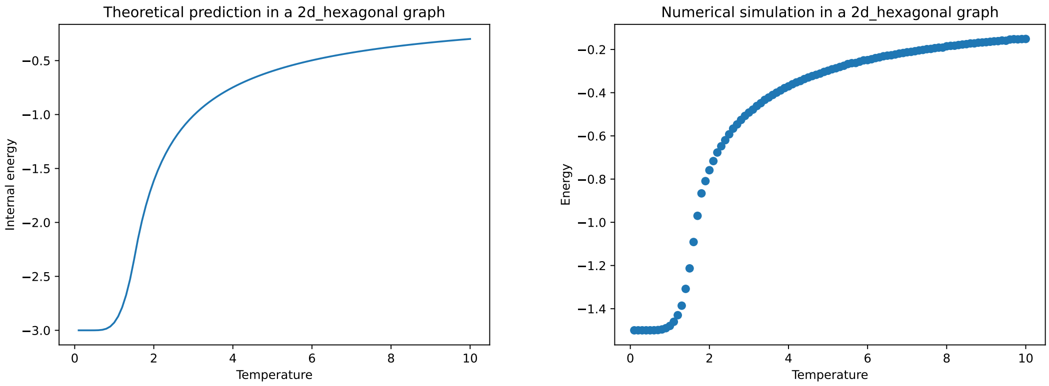
Figure 7: Entropy of the $2d$ square Ising model.



(a) Exact solution.

(b) Numerical result.

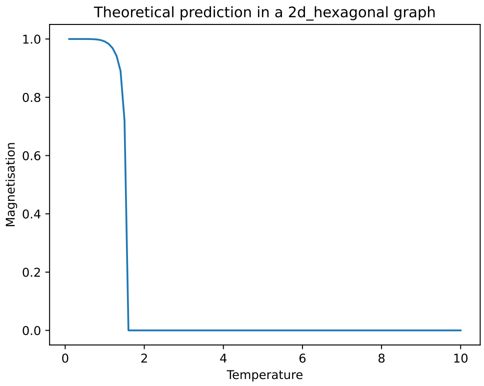
Figure 8: Specific heat of the $2d$ square Ising model, as we expect it diverges for the critical temperature $T_c^S = 2,26919$.



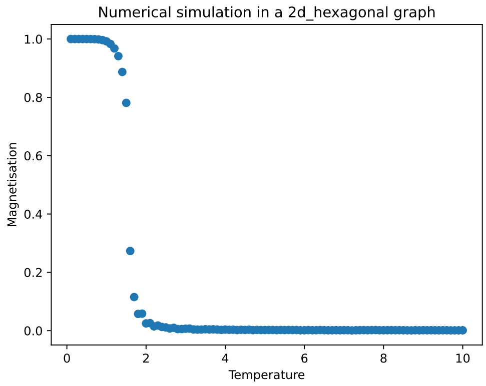
(a) Exact solution.

(b) Numerical result.

Figure 9: Internal energy of $2d$ hexagonal Ising model.

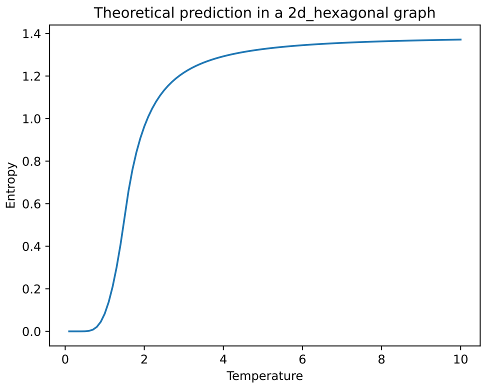


(a) Exact solution.

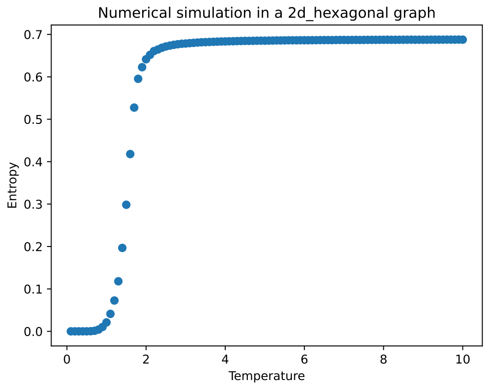


(b) Numerical result.

Figure 10: Magnetisation of $2d$ hexagonal Ising model, at the critical temperature $T_c^H = 1,51865$ it goes to zero.

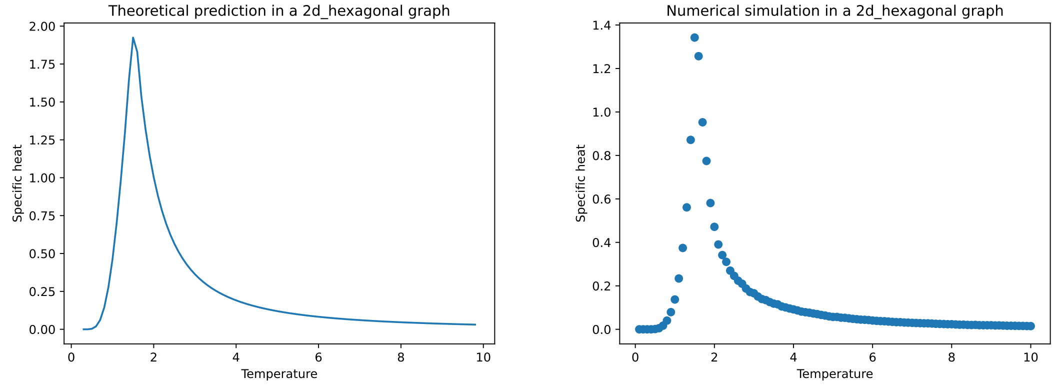


(a) Exact solution.



(b) Numerical result.

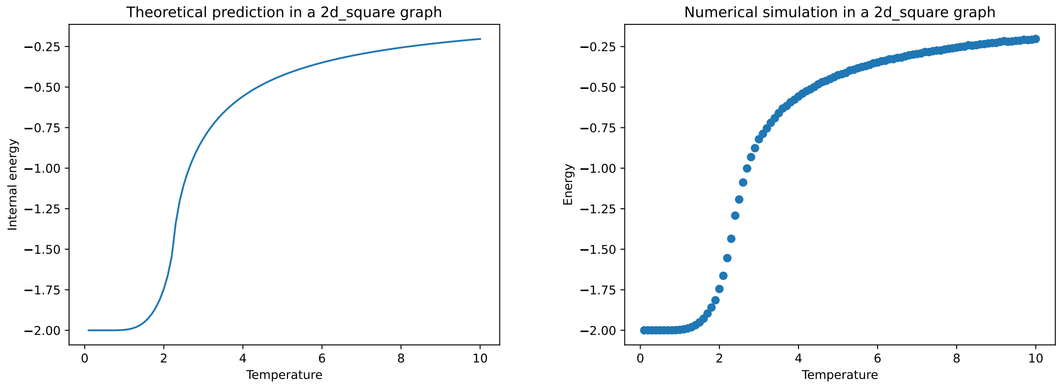
Figure 11: Entropy of the $2d$ hexagonal Ising model.



(a) Exact solution.

(b) Numerical result.

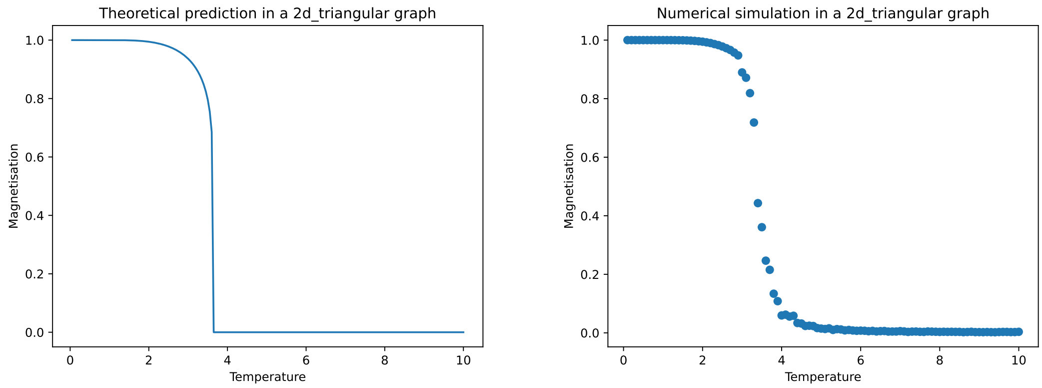
Figure 12: Specific heat of the $2d$ hexagonal Ising model, as we expect it diverges for the critical temperature $T_c^H = 1,51865$.



(a) Exact solution.

(b) Numerical result.

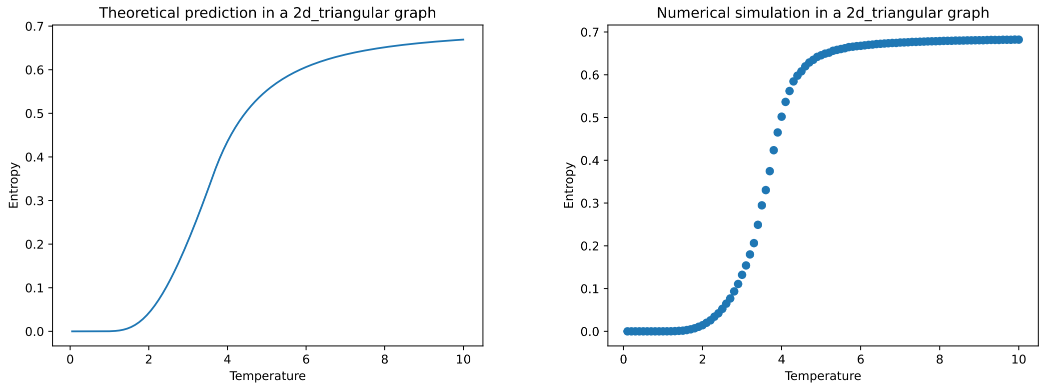
Figure 13: Internal energy of $2d$ triangular Ising model.



(a) Exact solution.

(b) Numerical result.

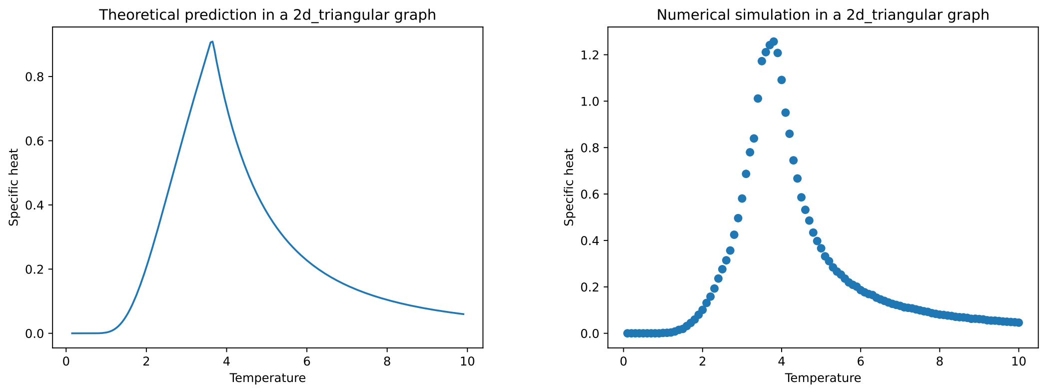
Figure 14: Magnetisation of $2d$ triangular Ising model, at the critical temperature $T_c^T = 3,64096$ it goes to zero.



(a) Exact solution.

(b) Numerical result.

Figure 15: Entropy of the $2d$ triangular Ising model.



(a) Exact solution.

(b) Numerical result.

Figure 16: Specific heat of the $2d$ triangular Ising model, as we expect it diverges for the critical temperature $T_c^T = 3,64096$.

All $2d$ Ising models display a phase transition at their critical temperature and our numerical calculation well fit with the theoretical predictions.

The clustering network is shown in fig 17,18,19.

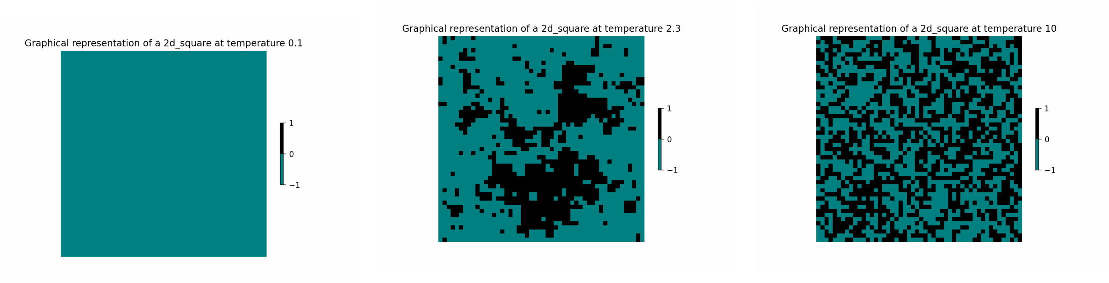
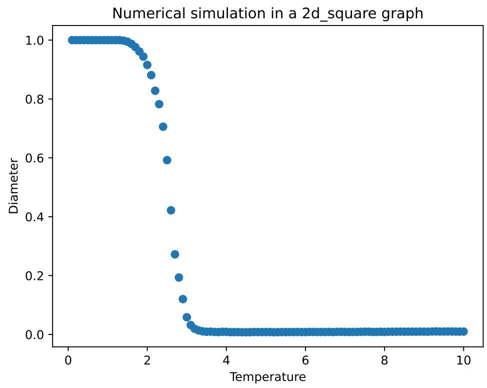


Figure 17: Clustering network of the $2d$ square Ising model for $T \simeq 0$. All the spins are aligned.

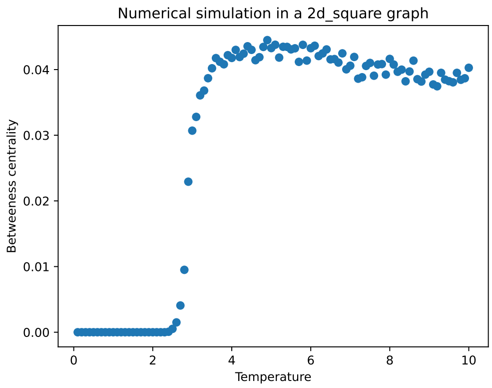
Figure 18: Clustering network of the $2d$ square Ising model for $T \simeq T_c^S$. As we expect there are islands of spins.

Figure 19: Clustering network of the $2d$ square Ising model for $T \gg T_c^S$. As we expect the spins are randomly distributed.

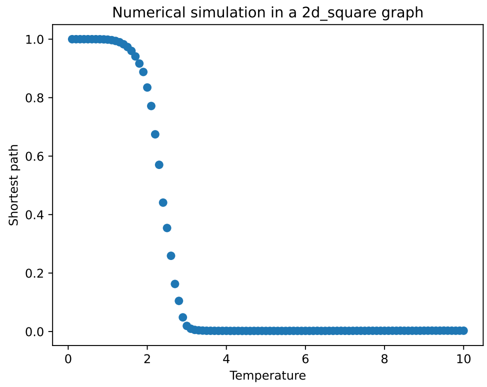
The network analysis of the $2d$ Ising models is display in fig.20,21,22 respectively for square,hexagonal and triangular Ising models.



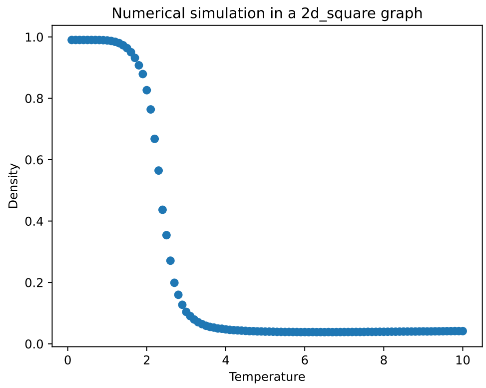
(a) Diameter of the $2d$ square Ising model.



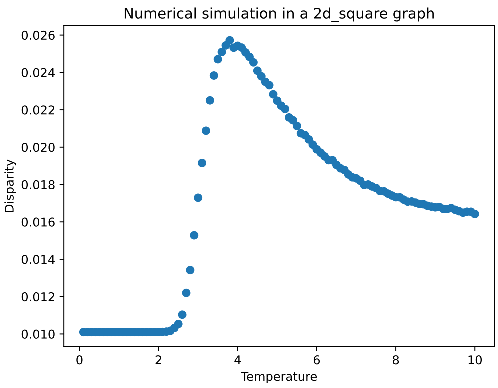
(b) Betweenness centrality of the $2d$ square Ising model.



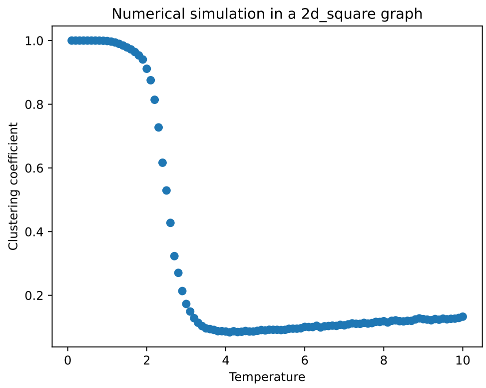
(c) Shortest paths (average geodesic distance) of the $2d$ square Ising model.



(d) Density of the $2d$ square Ising model.

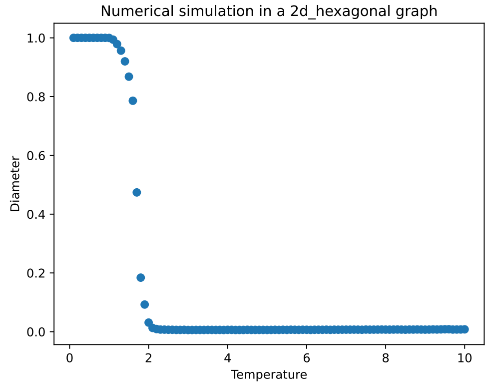


(e) Disparity of the $2d$ square Ising model.

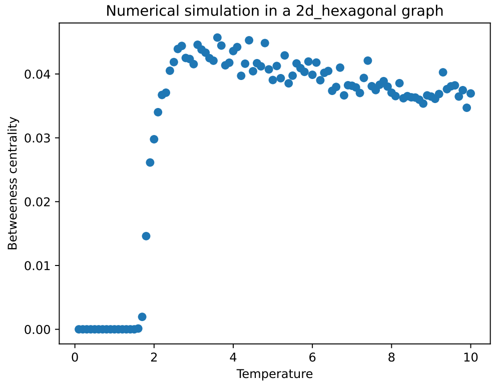


(f) Clustering coefficient of the $2d$ square Ising model.

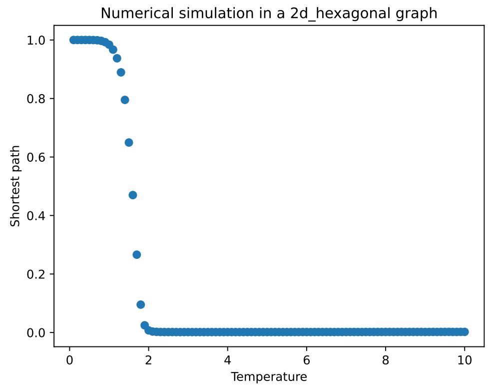
Figure 20: Network parameters of the $2d$ square Ising model.



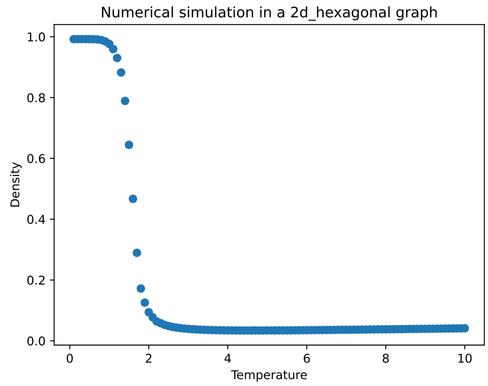
(a) Diameter of the $2d$ hexagonal Ising model.



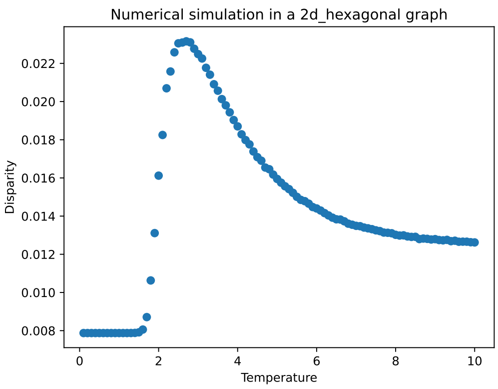
(b) Betweenness centrality of the $2d$ hexagonal Ising model.



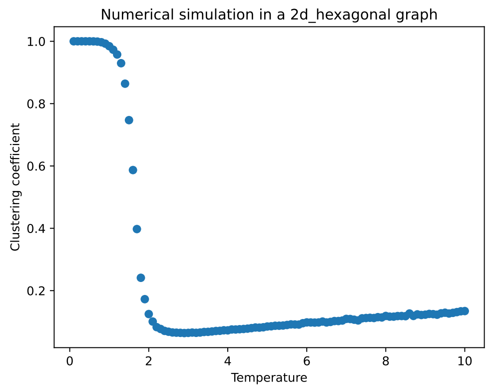
(c) Shortest paths (average geodesic distance) of the $2d$ hexagonal Ising model.



(d) Density of the $2d$ hexagonal Ising model.

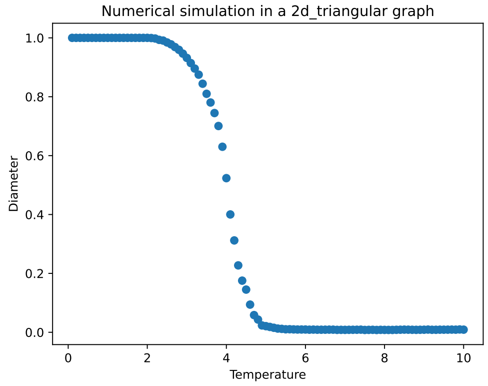


(e) Disparity of the $2d$ hexagonal Ising model.

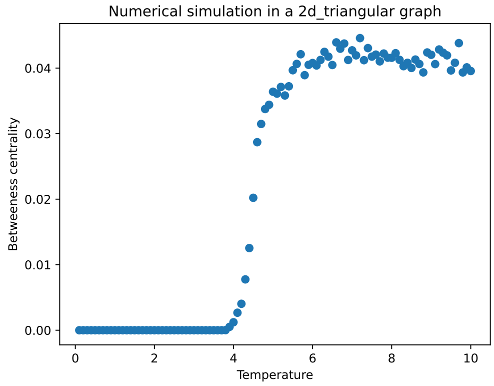


(f) Clustering coefficient of the $2d$ hexagonal Ising model.

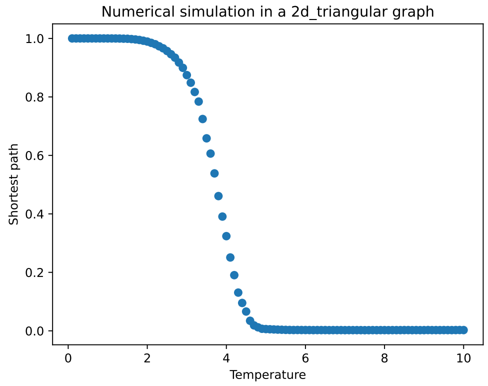
Figure 21: Network parameters of the $2d$ hexagonal Ising model.



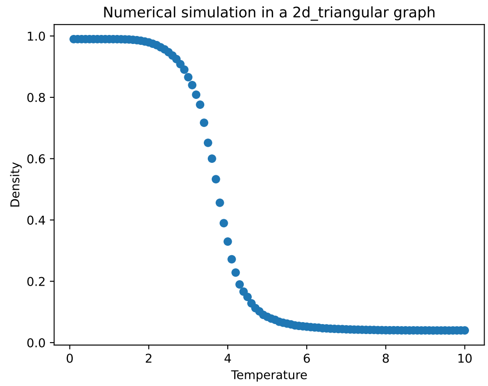
(a) Diameter of the $2d$ triangular Ising model.



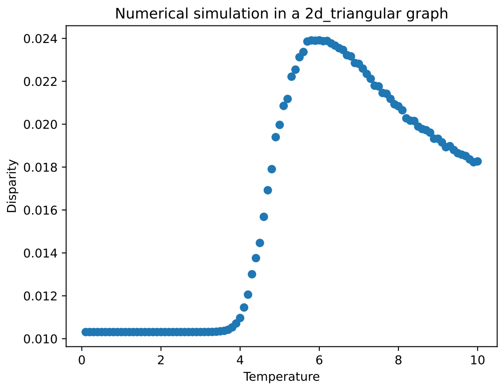
(b) Betweenness centrality of the $2d$ triangular Ising model.



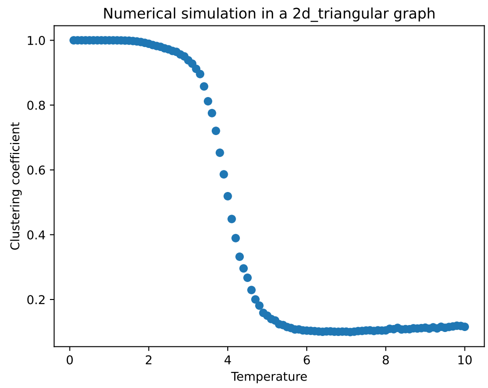
(c) Shortest paths (average geodesic distance) of the $2d$ triangular Ising model.



(d) Density of the $2d$ triangular Ising model.



(e) Disparity of the $2d$ triangular Ising model.



(f) Clustering coefficient of the $2d$ triangular Ising model.

Figure 22: Network parameters of the $2d$ triangular Ising model.

6 Conclusions

We have reproduced successfully the observables of the Ising models, both $1d$ and both $2d$. All the $2d$ Ising models that we have analyzed exhibit a phase transition with critical temperatures that are compatible with the theoretical predictions. Networks parameters behave as we expected but some of them (as the betweenness centrality) are very tough to compute numerically.

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