Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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Neutron Stars (TOV equation)

Matteo Zandi

11 October 2024

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

What is a neutron star?

A neutron star is the remnant of a star with initial mass greater than 8 $M_{\odot}$.

Solver
Equation of state: Fermi gas
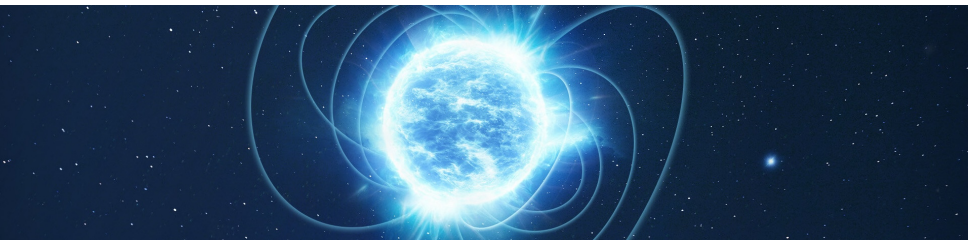Equation of state: CompOSE

What is a neutron star?

A neutron star is the remnant of a star with initial mass greater than 8 $M_\odot$. After the fusion of all heavier elements up to iron, the outer layers are expelled in a supernova explosion and, if the remaining core has a mass of about 2 to 3 $M_\odot$, it becomes a neutron star.

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Equation of state: Fermi gas
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Equation of state: CompOSE
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Therefore, we need a model to describe the matter content of a neutron star and a framework to get the equations to solve. The results will be the radius and the mass of the neutron star.

❶ Solver

❷ Equation of state: Fermi gas

❸ Equation of state: CompOSE

The TOV equation

The framework is general relativity. Consider a spherical symmetric astrophysical object such as a neutron star. Its radius and its mass can be determined by solving a system of ordinary differential equations composed by

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Equation of state: Fermi gas
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Equation of state: CompOSE
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Solver
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\epsilon = \epsilon(p) \\
\frac{dm}{dr} = \frac{4\pi r^2 \epsilon(r)}{c^2} \\
\frac{dp}{dr} = \frac{G\epsilon(r)m(r)}{c^2 r^2}\left(1 + \frac{p(r)}{\epsilon(r)}\right)\left(1 + \frac{4\pi r^3 p(r)}{m(r)c^2}\right)\left(1 - \frac{2Gm(r)}{c^2 r}\right)
\end{cases}, \quad (1)
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p(r = R_{object}) = 0 \, ,
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\end{cases}
, \quad (1)
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with boundary conditions

$$
p(r = R_{object}) = 0 , \quad p(r = 0) = p_0 ,
$$

Solver
Equation of state: Fermi gas
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\end{cases}
, \quad (1)
$$

with boundary conditions

$$
p(r = R_{object}) = 0 , \quad p(r = 0) = p_0 , \quad m(r = 0) = 0 .
$$

So, one parameter of the model is the initial central pressure p_0.
Our goal is to numerically implement

Solver.py

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Solver.py

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Solver.py

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- numerical integration, which results into the whole radius evolution of mass and pressure;

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

Solver.py

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Solver
Equation of state: Fermi gas
Equation of state: CompOSE

Solver.py

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- the mass and the radius of the neutron star.

It is achieved by means of four classes.

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Equation of state: Fermi gas
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Equation of state: CompOSE
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EquationOfState

Class to represent an equation of state, i.e. energy density as function of pressure.

EquationOfState

Class to represent an equation of state, i.e. energy density as function of pressure.

The main methods are

```python
def load_from_file(file_path):
    ...
    data = np.loadtxt(file_path, delimiter=",", skiprows=1)
    self.number_densities = np.append(self.number_densities, data[:, 0])
    self.pressures = np.append(self.pressures, data[:, 1])
    self.energy_densities = np.append(self.energy_densities, data[:, 2])

def interpolate():
    ...
    eos = CubicSpline(self.pressures, self.energy_densities)
    return eos
```

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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TOVSystem

Class to represent the system of ordinary differential equations.

TOVSystem

Class to represent the system of ordinary differential equations.
The main methods are

```python
def dmdr(r, p, m):
    ...
    e = self.eos(p)
    dmdr = 4 * np.pi * r**2 * e / c**2
    return dmdr

def dpdr(r, p, m):
    ...
    e = self.eos(p)

    if self.relativity_corrections:
        first_term = G * e * m / c**2 / r**2
        second_term = 1 + p / e
        third_term = 1 + 4 * np.pi * r**3 * p / m / c**2
        fourth_term = (1 - 2 * G * m / c**2 / r) ** (-1)
        dpdr = -first_term * second_term * third_term * fourth_term
    else:
        dpdr = -G / c**2 / r**2 * m * e

    return dpdr
```

SolverTOVSinglePressure

Class to integrate numerically the system of 2 ordinary differential equations (mass and pressure equations) given a single initial central pressure. Implement the 4th-order Runge-Kutta algorithm with a breaking condition when the pressure becomes zero or negative, because it means that the surface of the object is reached.

SolverTOVSinglePressure

The main method is

```
def solve(step_r, p0):
    ...

    pressure_is_negative = False

    while not pressure_is_negative:
        r = self.r_values[-1]
        p = self.p_values[-1]
        m = self.m_values[-1]

        p_new, m_new = self.runge_kutta_4th_step(r, step_r, p, m)
        r_new = r + step_r

        if p_new <= 0:
            pressure_is_negative = True

        self.r_values = np.append(self.r_values, r_new)
        self.p_values = np.append(self.p_values, p_new)
        self.m_values = np.append(self.m_values, m_new)
```

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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SolverTOVRangePressure

Class to integrate numerically the system of 2 ordinary differential equations (mass and pressure equations) given a range of initial central pressures. It uses the class SolverTOVSinglePressure for each initial central pressure.

SolverTOVRangePressure

Class to integrate numerically the system of 2 ordinary differential
equations (mass and pressure equations) given a range of initial
central pressures. It uses the class SolverTOVSinglePressure for
each initial central pressure.

The main method is

```python
def solve(step_r, initial_pressures):
    ...
    for p0 in initial_pressures:
        self.solver_single_p = SolverTOVSinglePressure(
            self.eos, self.relativity_corrections
        )
        self.solver_single_p.solve(step_r, p0)
        r, m, p = self.solver_single_p.get()
        self.masses = np.append(self.masses, m[-1])
        self.radii = np.append(self.radii, r[-1])
```

The full code is in solver.py.

Choices

There are three critical choices:

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Equation of state: Fermi gas
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Equation of state: CompOSE
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Choices

There are three critical choices:

- the use of NumPy arrays;

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

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- the use of NumPy arrays;

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- the not use of solve_ivp;

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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Choices

There are three critical choices:

- the use of NumPy arrays;

- the use of CubicSpline;

- the not use of solve_ivp;

- the not use of performance improvement.

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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❶ Solver

❷ Equation of state: Fermi gas

❸ Equation of state: CompOSE

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

Equation of state

The equation of state depends on the matter content and the framework in which is computed. In this chapter, we will consider protons, neutrons and electrons in statistical mechanics and nuclear physics. In the next chapter, also advanced methods will be considered, such as particle physics and QCD.

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Equation of state: Fermi gas
Equation of state: CompOSE
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Fermi gas of neutrons

Consider a degenerate ($T = 0$) ideal (non-interacting) Fermi gas of neutrons. In the non-relativistic limit ($k_F \ll mc$), the equation of state is

$$p = \frac{\hbar^2}{15\pi^2 m_n}\Big(\frac{3\pi^2}{m_n c^2}\Big)^{5/3}\epsilon^{5/3} = K_{nonrel}\epsilon^{5/3};$$

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whereas in the generic case, the equation of state can be computed from

$$p(x) = \frac{\epsilon_0}{24}\Big((2x^3 - 3x)(1 + x^2)^{1/2} + 3\sinh^{-1}(x)\Big),$$

$$\epsilon(x) = nm_n c^2 + \frac{\epsilon_0}{8}\Big((2x^3 + x)(1 + x^2)^{1/2} - \sinh^{-1}(x)\Big).$$

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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The main functions are

```
def compute_pressure_gen(x, p):
    ...
    p_calculated = (
        e_0_n / 24 * ((2 * x**3 - 3 * x) * (1 + x**2) ** (1 / 2)
        + 3 * np.arcsinh(x))
    )
    return p_calculated - p

def compute_x_gen(p):
    ...
    x = opt.toms748(
        compute_pressure_gen, 1e-4, 1e2, args=(p,)
    )
    return x

def compute_energy_density_gen(p):
    ...
    x = compute_x_gen(p)
    e = compute_energy_density_from_x_gen(x)
    return e
```

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

Fermi gas of neutrons, protons and electrons

If we add also electrons and protons, the equation of state can be computed from

$$p_{tot} = \sum_i p_i \ , \quad p_i = \frac{\epsilon_0}{24}\Big((2x_i^3 - 3x_i)(1 + x_i^2)^{1/2} + 3\sinh^{-1}(x_i)\Big) \ ,$$

$$\epsilon_{tot} = \sum_i \epsilon_i \ , \quad \epsilon_i = n_i m_i c^2 + \frac{\epsilon_0}{8}\Big((2x_i^3 + x_i)(1 + x_i^2)^{1/2} - \sinh^{-1}(x_i)\Big) \ ,$$

with constraint

$$\sqrt{k_n^2 c^2 + m_n^2 c^4} = \sqrt{k_p^2 c^2 + m_p^2 c^4} + \sqrt{k_e^2 c^2 + m_e^2 c^4} \ .$$

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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The main functions are

```python
def beta_equilibrium_condition(k_p, k_n):
    ...
    mu_p = (k_p**2 * c**2 + m_p**2 * c**4) ** (
        1 / 2
    )
    mu_e = (k_p**2 * c**2 + m_e**2 * c**4) ** (
        1 / 2
    )
    mu_n = (k_n**2 * c**2 + m_n**2 * c**4) ** (
        1 / 2
    )
    return mu_n - mu_p - mu_e

def compute_k_p(k_n):
    ...
    k_p = opt.toms748(
        beta_equilibrium_condition, 1e-17, 1e2, args=(k_n,)
    )
    return k_p
```

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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```python
def compute_pressure_from_x_npe(x, e_0):
    ...
    p = e_0 / 24 * ((2 * x**3 - 3 * x) * (1 + x**2) ** (1 / 2)
    + 3 * np.arcsinh(x))
    return p

def compute_pressure_npe(k_n):
    ...
    k_p = compute_k_p(k_n)
    x_e = k_p / m_e / c
    x_p = k_p / m_p / c
    x_n = k_n / m_n / c
    p_e = compute_pressure_from_x_npe(x_e, e_0_e)
    p_n = compute_pressure_from_x_npe(x_n, e_0_n)
    p_p = compute_pressure_from_x_npe(x_p, e_0_p)
    p_tot = p_n + p_p + p_e
    return p_tot

def compute_k_n(n):
    ...
    k_n = hbar * (3 * np.pi**2 * n) ** (1 / 3)
    return k_n
```

```python
def equation_of_state_npe(number_densities):
    ...
    for k in neutron_fermi_momenta:
        p_below = compute_pressure_pe(k)
        p_above = compute_pressure_npe(k)

        if p_below < 3.038e24:
            pressures_npe = np.append(pressures_npe, p_below)
            e_below = compute_energy_density_pe(k)
            energy_densities_npe = np.append(energy_densities_npe, e_below)
        else:
            pressures_npe = np.append(pressures_npe, p_above)
            e_above = compute_energy_density_npe(k)
            energy_densities_npe = np.append(energy_densities_npe, e_above)
    ...
```

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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Empirical interactions

If we add empirical interactions

$$\frac{\epsilon(n)}{n} = m_n c^2 + <E_0> u^{2/3} + \frac{A}{2} u + \frac{B}{\sigma+1} u^\sigma \ ,$$

the equation of state can be computed from

$$p(n) = n_0 \Big(\frac{2}{3} <E_0> u^{5/3} + \frac{A}{2} u^2 + \frac{B\sigma}{\sigma+1} u^{\sigma+1} +$$
$$(2^{2/3} - 1) <E_0> (\frac{2}{3} u^{5/3} - u^2) + S_0 u^2 \Big) \ ,$$

$$\epsilon(n) = n \Big(m_n c^2 + 2^{2/3} <E_0> u^{2/3} + \frac{A}{2} u + \frac{B}{\sigma+1} u^\sigma +$$
$$(S_0 - (2^{2/3} - 1) <E_0>) u \Big) \ .$$

Skyrme interactions

If we add Skyrme interactions

$$V(x, y) = \delta^3(x - y)\left(\frac{t_3 n}{6} - t_0\right) ,$$

the equation of state can be computed from

$$p(n) = \frac{2(3\pi^2\hbar^3)^{2/3}}{10m_n}n^{5/3} + \frac{t_3}{12}n^3 - \frac{t_0}{4}n^2 ,$$

$$\epsilon(n) = m_n n c^2 + \frac{3(3\pi^2\hbar^3)^{2/3}}{10m_n}n^{5/3} + \frac{t_3}{24}n^3 - \frac{t_0}{4}n^2 .$$

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

For the last two, the same procedure as the generic Fermi gas has been implemented.

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

Full code

The full code is in fermi_eos.py and fermi_eos_examples.ipynb.

Solver
Equation of state: Fermi gas
Equation of state: CompOSE
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Choices

There are two critical choices:

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

Choices

There are two critical choices:

- the use of NumPy arrays;

Solver
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Equation of state: Fermi gas
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Equation of state: CompOSE
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Choices

There are two critical choices:

- the use of NumPy arrays;
- the use of toms748 algorithm;

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Equation of state: Fermi gas
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Equation of state: CompOSE
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Choices

There are two critical choices:

- the use of NumPy arrays;

- the use of toms748 algorithm;

- to either use root finding or give a range of number densities as input.

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Equation of state: Fermi gas
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Equation of state: CompOSE
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① Solver

② Equation of state: Fermi gas

③ Equation of state: CompOSE

We can also use solver.py for more realistic equations of state,
present in the CompOSE archive.

compose.py

Sound speed, gravitational redshift and Newtonian moment of inertia are added as well

```python
def compute_sound_speed(range_p, range_e):
    ...
    p_prime = numerical_derivative(range_e, range_p)
    sound_speeds = np.array([])
    sound_speeds = np.maximum(p_prime, 0)
    return sound_speeds

def compute_redshift(r, m):
    ...
    r = r * km
    m = m * m_sun
    z = (1 - 2 * G * m / (r * c**2)) ** (-1 / 2) - 1
    return z

def compute_moment_inertia(r, m):
    ...
    r = r * km
    m = m * m_sun
    I = (2 / 5) * m * r**2
    return I
```

Solver
Equation of state: Fermi gas
Equation of state: CompOSE

Full code

The full code is in compose.py and compose_examples.ipynb.

The end.