Quantum Field Theory
Feynman's rules

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Abstract

Ready to sail? We are about to depart for the magical world of quantum field theory, where quantum mechanics wizards try to defeat special relativity army.

Contents

1 S-matrix

In this section, we will define the S-matrix and we will relate its elements to physical quantities, like cross sections and decay rates.

1.1 Transition amplitudes

In quantum mechanics, experimentally measureable quantities are related to transition ampllitudes.

Definition 1.1 (Transition amplitude)
Let $|a\rangle$ be a generic initial state and $|b\rangle$ a generic final state. Then, in the most generic case in which states are not normalised, the probability of the transition between the initial and the final state is given by

$$\mathcal{P}(a \to b) = \frac{|\langle b|a\rangle|^2}{|\langle b|b\rangle|^2 |\langle a|a\rangle|^2} \ .$$

In Schroedinger picture, states depend on time while operators do not.

Definition 1.2 (Transition amplitude in Schroedinger picture)
Let $|i, t_i\rangle$ be a initial state at time t_i, $|f, t_f\rangle$ be a final state at time t_f. Then the probability of the transition between the initial and the final state is

$$\mathcal{P}(i, t_i \to f, t_f) = \frac{|\langle f, t_f|i, t_i\rangle|^2}{|\langle f, t_f|f, t_f\rangle|^2 |\langle i, t_i|i, t_i\rangle|^2} \ .$$

In Heisenberg picture, states are time-independent while operators do not. Braket products in different pictures are related by

$$\langle f, t_f|i, t_i\rangle_S = \langle f|\hat{S}|i\rangle_H \ ,$$

where S is an operator that carries information about time evolution, called the S-matrix.

Definition 1.3 (Transition amplitude in Heisenberg picture)
Let $|i\rangle$ be a initial state, $|f\rangle$ a final state, $\hat{S}$ the time evolution operator. Then the probability of the transition between the initial and the final state is

$$\mathcal{P}(i \to f) = \frac{|\langle f|\hat{S}|i\rangle|^2}{|\langle f|f\rangle|^2 |\langle i|i\rangle|^2} \ .$$

1.2 Cross section

Definition 1.4 (Cross section)
Consider a scattering experiment. Let N_{in} and N_{out} be respectively the number of incoming and outgoing particles, T the time of the experiment, $\Phi = N_{in}|\mathbf{v}|/V$ the flux of the incoming beam, where V is the volume and $\mathbf{v}$ the velocity of the beam. Then the classical cross section is defined by

$$\sigma = \frac{N_{out}}{T\Phi} = \frac{V}{|\mathbf{v}|T} \frac{N_{out}}{N_{in}} \ .$$

Introducing the probability $\mathcal{P} = N_{out}/N_{in}$, its quantum mechanical counterpart is

$$\sigma = \frac{V}{|\mathbf{v}|T}\mathcal{P} = \frac{N_{in}}{T\Phi}\mathcal{P} = \frac{1}{T\Phi}\mathcal{P} \ ,$$

where we have redefined $\Phi = \Phi/N_{in}$ as the normalised one-particle flux. The differential cross section is

$$d\sigma = \frac{V}{|\mathbf{v}|T}d\mathcal{P} \ ,$$

differential with respect to solid angle $d\Omega$ or energy dE. It has the dimension of an area, i.e. $[\sigma] = [L]^2$.

1.3 2 to n process

Consider a scattering experiment in which two incoming particle interact to produce n outgoing particles

$$p_1 + p_2 \to \{p_j\}_{j=1}^n \ .$$

In perturbative theory, the S-matrix can be decomposed into

$$\hat{S} = \hat{1} + i\hat{T} \ ,$$

where the identity $\hat{1}$ represents no interactions, i.e. when $|i\rangle = |f\rangle$, and $\hat{T}$ describes deviations from it. Furthermore, since 4-momentum is conserved, we can extract a delta from $\hat{T}$ to obtain

$$i\hat{T} = (2\pi)^4\delta^4(p_1 + p_2 - \textstyle\sum_j p_j)i\hat{\mathcal{M}} \ ,$$

where $\hat{\mathcal{M}}$ is the scattering amplitude.

Theorem 1.1 (Relation between cross section and S-matrix)
In the approximation that interaction happens at finite time, the differential cross section of a $2 \to n$ process is

$$\begin{aligned} d\sigma &= \frac{|\mathcal{M}|^2}{4E_1E_2|\mathbf{v}_2 - \mathbf{v}_1|}d\Pi_n \\ &= \frac{|\mathcal{M}|^2}{4E_1E_2|\mathbf{v}_2 - \mathbf{v}_1|}\prod_j \frac{d^3p_j}{(2\pi)^3 2E_j}(2\pi)^4\delta^4(p_1 + p_2 - \textstyle\sum_j p_j) \ . \end{aligned}$$

Proof. q.e.d.

1.4 2 to 2 scattering

Consider the particular case in which there are two outgoing particles

$$p_1 + p_2 \to p_3 + p_4 \ .$$

In the center of mass frame, the differential cross section is

$$d\sigma = \frac{1}{64\pi^2 E_{cm}}^2 \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}|\mathcal{M}|^2 d\Omega \ ,$$

where $|\mathbf{p}_i| = |\mathbf{p}_1| = |\mathbf{p}_2|$ and $|\mathbf{p}_f| = |\mathbf{p}_3| = |\mathbf{p}_4|$.

Proof. q.e.d.

In the rest frame of particle 1, the differential cross section is

$$d\sigma = \frac{1}{64\pi^2 E_{cm}} \left[E_4 + E_3 \left(1 - \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \cos\theta \right) \right]^{-1} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} |\mathcal{M}|^2 d\Omega \ .$$

Proof. q.e.d.

1.5 Decay rates

Definition 1.5 (Decay rate)
Consider a decay experiment. Let $\mathcal{P}$ be the probablity that a particle decays with mean lifetime τ and T the time of the experiment. Then the decay rate is defined by

$$\Gamma = \frac{1}{\tau} = \frac{\mathcal{P}}{T} \ .$$

The differential decay rate is

$$d\Gamma = \frac{1}{T} d\mathcal{P} \ ,$$

differential with respect to solid angle $d\Omega$ or energy dE. It has the dimension of an inverse time, i.e. $[\Gamma] = [T]^{-1}$.

1.6 1 to n process

Consider a decay experiment in which a particle decays to produce n outgoing particles

$$p_1 \to \{p_j\}_{j=1}^n \ .$$

Theorem 1.2 (Relation between decay rate and S-matrix)
In the approximation that interaction happens at finite time, the differential decay rate of a $1 \to n$ process is

$$d\Gamma = \frac{|\mathcal{M}|^2}{2E_1} d\Pi_n = \frac{|\mathcal{M}|^2}{2E_1} \prod_j \frac{d^3 p_j}{(2\pi)^3 2E_j} (2\pi)^4 \delta^4(p_1 - \textstyle\sum_j p_j) \ .$$

Proof. q.e.d.

Propagators

In this section, we will relate S-matrix elements to time-ordered product of fields applied to interacting vacuum states.

1.7 LSZ reduction formula

Theorem 1.3 (LSZ reduction formula)
In the approximation that interaction happens at finite time, so that initial and final states are (asymptotic) free theory states, the S-matrix is given by

$$\langle f|\hat{S}|i\rangle = i\int dx_1 \exp(-ip_1x_1)(\Box + m^2)\dots i\int dx_1 \exp(ip_nx_n)(\Box + m^2)$$
$$\times \langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle \ ,$$

where $|\Omega\rangle \neq |0\rangle$ is the interacting vacuum, $-i$ in the exponent for initial states, $+i$ in the exponent for final states and T is the time ordering operator which sorts all the operators in order to have time increasing from right to left.

Proof. q.e.d.

1.8 Feynman's propagator

Definition 1.6 (Feynman's propagator in momentum space)
Let $\phi_0(x)$ be a free scalar field, x_1, x_2 two spacetime points. Then the Feynman's propagator or two-points Green's function is

$$D_F(x_2, x_2) = \langle 0|T\{\phi_0(x_1)\phi_0(x_2)\}|0\rangle = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x_1-x_2)}}{k^2 - m^2 + i\epsilon} \ ,$$

where $k_0 \neq \sqrt{|\mathbf{k}|^2 + m^2}$. It has a pole at $k^2 = m^2$.

Proof. q.e.d.

Definition 1.7 (Feynman's propagator in position space)
Let $\phi_0(x)$ be a free scalar field, x_1, x_2 two spacetime points. Then the Feynman's propagator or two-points Green's function is

$$D_F(x_2, x_2) = \langle 0|T\{\phi_0(x_1)\phi_0(x_2)\}|0\rangle = -\frac{1}{4\pi^2}\frac{1}{(x_1 - x_2)^2 - i\epsilon}.$$

Proof. q.e.d.

1.9 Interaction picture

In Heisenberg picture, the dynamics is governed by the Hamiltonian $\hat{H}$. Fields evolve in time with the Heisenberg equation of motion

$$i\partial_t\hat{\phi}(t,\mathbf{x}) = [\hat{\phi}(t,\mathbf{x}), \hat{H}(t)] \ .$$

Its solution is

$$\hat{\phi}(t,\mathbf{x}) = \hat{S}^\dagger(t,t_0)\hat{\phi}(\mathbf{x})\hat{S}(t,t_0) \ ,$$

where $\hat{S}(t,t_0)$ is the time evolution operator that satisfies the Schroedinger equation

$$i\partial_t\hat{S}(t,t_0) = \hat{H}(t)\hat{S}(t,t_0) \ .$$

Proof. q.e.d.

Now, suppose that the Hamiltonian can be perturbatively decomposed into two pieces

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t) \ ,$$

where $\hat{H}_0$ is exactly solved and $\hat{V}(t)$ is small. In interaction picture, operators evolve with $\hat{H}_0$, so that

$$\hat{\phi}_0(t,\mathbf{x}) = e^{i\hat{H}_0(t-t_0)}\hat{\phi}(\mathbf{x})e^{-i\hat{H}_0(t-t_0)}$$

where t_0 is a time in which Schroedinger and Heisenberg picture field coincide. Therefore

$$\phi(t,\mathbf{x})$$

1.10 Vacuum matrix elements

Theorem 1.4 (Relation between interacting and free vacuum matrix elements)

$$\begin{aligned}\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle &= \frac{\langle 0|T\{\phi_0(x_1)\ldots\phi_0(x_n)\exp(-i\int_{-\infty}^{\infty}dt\ V_I(t))\}|0\rangle}{\langle 0|T\{\exp(-i\int_{-\infty}^{\infty}dt\ V_I(t))\}|0\rangle} \\ &= \frac{\langle 0|T\{\phi_0(x_1)\ldots\phi_0(x_n)\exp(i\int d^4x\ \mathcal{L}_{int}[\phi_0])\}|0\rangle}{\langle 0|T\{\exp(i\int d^4x\ \mathcal{L}_{int}[\phi_0])\}|0\rangle} \ .\end{aligned}$$

Proof. q.e.d.

2 Wick's theorem

3 Feynman's rules: vertices

Feynman's rules for vertices can be derived from the Lagrangian of the theory.

3.1 Scalar theory

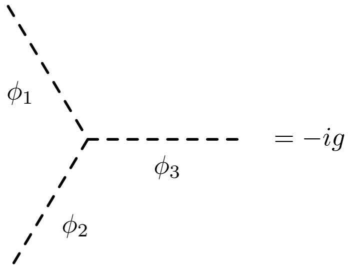
The Lagrangian of scalar theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \ .$$

The first interaction vertex

$$-\frac{g}{3!}\phi^3$$

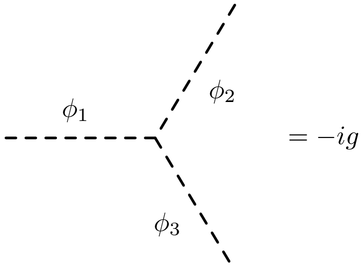
gives two Feynman's diagram:



Proof. In the vertex, $i\mathcal{L}_{int}$ becomes

$$-i\frac{g}{3!}\phi_1\phi_2\phi_3 = \phi_3(-i\frac{g}{3!})\phi_1\phi_2 \ ,$$

which means that, since the final states on the left is ϕ_3, the initial state on the right is $\phi_1\phi_2$ and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3$, the vertex contribution is $-ig$. q.e.d.



Proof. In the vertex, $i\mathcal{L}_{int}$ becomes

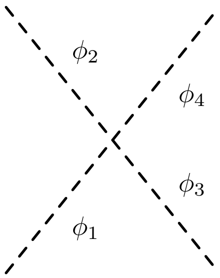
$$-i\frac{g}{3!}\phi_1\phi_2\phi_3 = \phi_1\phi_2(-i\frac{g}{3!})\phi_3 \ ,$$

which means that, since the final states on the left is $\phi_1\phi_2$, the initial state on the right is ϕ_3 and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3$ for a total of 3! times, the vertex contribution is $-ig$. q.e.d.

The second interaction vertex

$$-\frac{\lambda}{4!}\phi^4$$

gives one Feynman's diagram:



$$= -ig$$

Proof. In the vertex, $i\mathcal{L}_{int}$ becomes

$$-i\frac{\lambda}{4!}\phi_1\phi_2\phi_3\phi_4 = \phi_3\phi_4(-i\frac{\lambda}{4!})\phi_1\phi_2 \ ,$$

which means that, since the final states on the left is $\phi_3\phi_4$, the initial state on the right is $\phi_1\phi_2$ and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ for a total of 4! times, the vertex contribution is $-ig$. q.e.d.

3.2 Derivative coupling

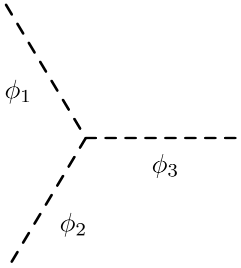
Suppose to have a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi\partial_\mu\phi\partial^\mu\phi \ .$$

The interaction vertex

$$-\frac{\lambda}{3!}\phi\partial_\mu\phi\partial^\mu\phi$$

gives two different Feynman's diagram:



$$= -i\lambda p_2^\mu p_3^\mu$$

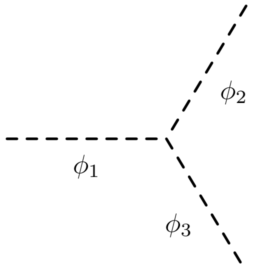
Proof. In the vertex, there are two annihilated scalar ϕ_1, ϕ_2 and a created scalar ϕ_2, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \ , \quad \phi_2 \sim \hat{a}e^{-ip_2x} \ , \quad \phi_3 \sim \hat{a}^\dagger e^{ip_3x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$
\begin{aligned}
-i\frac{\lambda}{3!}\phi_1\partial_\mu\phi_2\partial^\mu\phi_3 &= -i\frac{\lambda}{3!}\phi_1(-ip_2^\mu)\phi_2(ip_3^\mu)\phi_2 \\
&= \phi_3(-i\frac{\lambda}{3!}p_2^\mu p_3^\mu)\phi_1\phi_2 \ ,
\end{aligned}
$$

which means that, since the final states on the left is ϕ_3, the initial state on the right is $\phi_1\phi_2$ and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3$ for a total of 3! times, the vertex contribution is $-i\lambda p_2^\mu p_3^\mu$. q.e.d.



$$= i\lambda p_2^\mu p_3^\mu$$

Proof. In the vertex, there are two created scalar ϕ_2, ϕ_3 and an annihilated scalar ϕ_1, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \ , \quad \phi_2 \sim \hat{a}^\dagger e^{ip_2x} \ , \quad \phi_3 \sim \hat{a}^\dagger e^{ip_3x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$
\begin{aligned}
-i\frac{\lambda}{3!}\phi_1\partial_\mu\phi_2\partial^\mu\phi_3 &= -i\frac{\lambda}{3!}\phi_1(ip_2^\mu)\phi_2(ip_3^\mu)\phi_2 \\
&= \phi_2\phi_3(i\frac{\lambda}{3!}p_2^\mu p_3^\mu)\phi_1 \ ,
\end{aligned}
$$

which means that, since the final states on the left is $\phi_2\phi_3$, the initial state on the right is ϕ_1 and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3$ for a total of 3! times, the vertex contribution is $i\lambda p_2^\mu p_3^\mu$. q.e.d.

3.3 Scalar QED

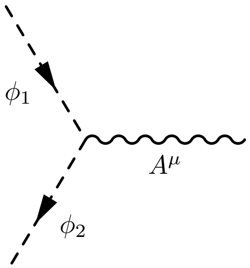
The Lagrangian of scalar quantum electrodynamics is

$$
\begin{aligned}
\mathcal{L}_{sQED} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*D^\mu\phi - m^2\phi^*\phi \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - m^2\phi^*\phi \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi\partial^\mu\phi^* - m^2\phi^*\phi \\
&\quad - ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + e^2A_\mu A^\mu\phi^*\phi \ .
\end{aligned}
$$

The first interaction vertex

$$-ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$$

gives four different Feynman's diagrams:



$$= -ie(p_1^\mu - p_2^\mu)$$

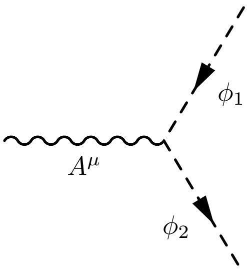
Proof. In the vertex, there is an annihilated scalar ϕ_1 and an annihilated antiscalar ϕ_2, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \ , \quad \phi_2^* \sim \hat{b}e^{-ip_2x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$\begin{aligned} eA_\mu(\phi_2^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2^*) &= eA_\mu\Big(\phi_2^*(-ip_1^\mu)\phi_1 - \phi_1(-ip_2^\mu)\phi_2^*\Big) \\ &= A^\mu\Big(-ie(p_1^\mu - p_2^\mu)\Big)\phi_1\phi_2^* \ , \end{aligned}$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\phi_1\phi_2^*$, the vertex contribution is $-ie(p_1^\mu - p_2^\mu)$. q.e.d.



$$= -ie(-p_1^\mu + p_2^\mu)$$

Proof. In the vertex, there is a created scalar ϕ_2 and a created antiscalar ϕ_1, so that

$$\phi_1 \sim \hat{b}^\dagger e^{ip_1x} \ , \quad \phi_2^* \sim \hat{a}^\dagger e^{ip_2x} \ ,$$

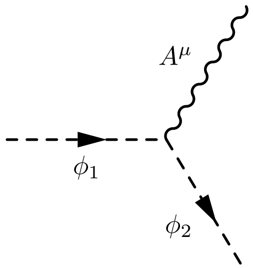
hence, $i\mathcal{L}_{int}$ becomes

$$\begin{aligned} eA_\mu(\phi_2^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2^*) &= eA_\mu\Big(\phi_2^*(ip_1^\mu)\phi_1 - \phi_1(ip_2^\mu)\phi_2^*\Big) \\ &= A^\mu\Big(-ie(-p_1^\mu + p_2^\mu)\Big)\phi_1\phi_2^* \ , \end{aligned}$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\phi_1\phi_2^*$, the vertex contribution is $-ie(-p_1^\mu + p_2^\mu)$. q.e.d.

Proof. In the vertex, there is an annihilated scalar ϕ_1 and a created scalar ϕ_2, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \ , \quad \phi_2^* \sim \hat{a}^\dagger e^{ip_2x} \ ,$$

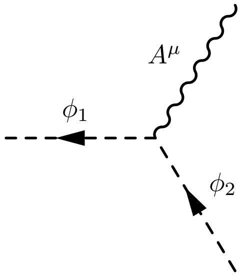


$$= -ie(p_1^\mu + p_2^\mu)$$

hence, $i\mathcal{L}_{int}$ becomes

$$\begin{aligned} eA_\mu(\phi_2^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2^*) &= eA_\mu\Big(\phi_2^*(-ip_1^\mu)\phi_1 - \phi_1(ip_2^\mu)\phi_2^*\Big) \\ &= A^\mu\Big(-ie(p_1^\mu + p_2^\mu)\Big)\phi_1\phi_2^* \ , \end{aligned}$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\phi_1\phi_2^*$, the vertex contribution is $-ie(p_1^\mu + p_2^\mu)$. q.e.d.



$$= -ie(-p_1^\mu - p_2^\mu)$$

Proof. In the vertex, there is an annihilated antiscalar ϕ_1 and a created antiscalar ϕ_2, so that

$$\phi_1 \sim \hat{b}^\dagger e^{ip_1x} \ , \quad \phi_2^* \sim \hat{b}e^{-ip_2x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$\begin{aligned} eA_\mu(\phi_2^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2^*) &= eA_\mu\Big(\phi_2^*(ip_1^\mu)\phi_1 - \phi_1(-ip_2^\mu)\phi_2^*\Big) \\ &= A^\mu\Big(-ie(-p_1^\mu - p_2^\mu)\Big)\phi_1\phi_2^* \ , \end{aligned}$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\phi_1\phi_2^*$, the vertex contribution is $-ie(-p_1^\mu - p_2^\mu)$. q.e.d.

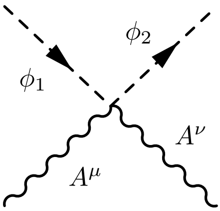
The second interaction vertex

$$e^2 g^{\mu\nu}A_\mu A_\nu \phi^*\phi$$

gives one Feynman's diagram

Proof. In the vertex, there is an annihilated scalar ϕ_1 and a created scalar ϕ_2, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \ , \quad \phi_2^* \sim \hat{a}^\dagger e^{ip_2x} \ ,$$



$$= 2ie^2 g^{\mu\nu}$$

hence, $i\mathcal{L}_{int}$ becomes

$$ie^2 g^{\mu\nu} A_\mu A_\nu \phi_2^* \phi_1 = A_\nu \phi_2^* (ie^2 g^{\mu\nu}) A_\mu \phi_1 \ ,$$

which means that, since the final states on the left is $A_\nu \phi_2^*$, the initial state on the right is $A_\mu \phi_1$ and we can exchange $\mu \leftrightarrow \nu$ for a total of 2! times, the vertex contribution is $2ie^2 g^{\mu\nu}$. q.e.d.

3.4 Yukawa theory

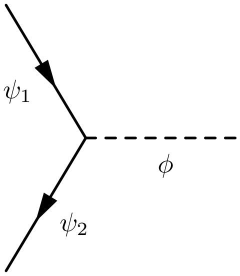
The Lagrangian of Yukawa theory is

$$\mathcal{L}_Y = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi - g\overline{\psi}\phi\psi \ .$$

The interaction vertex

$$-g\overline{\psi}\phi\psi$$

gives two different Feynman's diagram:



$$= -ig$$

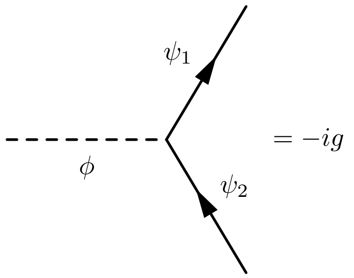
Proof. In the vertex, there is an annihilated fermion ψ_1, an annihilated antifermion $\overline{\psi}_2$ and a created scalar, so that

$$\phi \sim \hat{a}^\dagger e^{ipx} \ , \quad \psi_1 \sim \hat{a}_s u_s e^{-ip_1 x} \ , \quad \overline{\psi}_2 \sim \hat{b}_s \overline{v}_s e^{-ip_2 x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ig\overline{\psi}_2 \phi \psi_1 = \phi(-ig)\overline{\psi}_2 \psi_1 \ ,$$

which means that, since the final states on the left is ϕ and the initial state on the right is $\overline{\psi}_2 \psi_1$, the vertex contribution is $-ig$. q.e.d.



Proof. In the vertex, there is a created fermion ψ_1, a created antifermion ψ_2 and an annihilated scalar, so that

$$\phi \sim \hat{a}e^{-ipx} \ , \quad \overline{\psi}_1 \sim \hat{a}_s^\dagger \overline{u}_s e^{ip_1x} \ , \quad \psi_2 \sim \hat{b}_s^\dagger \overline{v}_s e^{ip_2x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ig\overline{\psi}_2\phi\psi_1 = \overline{\psi}_2\psi_1(-ig)\phi \ ,$$

which means that, since the final states on the left is $\overline{\psi}_2\psi_1$ and the initial state on the right is ϕ, the vertex contribution is $-ig$. q.e.d.

3.5 QED

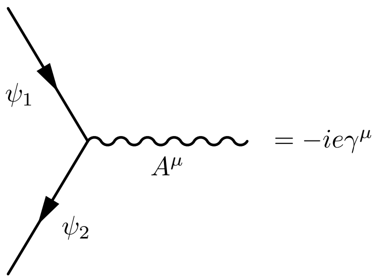
The Lagrangian of quantum electrodynamics is

$$\begin{aligned}\mathcal{L}_{QED} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{\partial} - m)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu \ .\end{aligned}$$

The interaction vertex

$$-e\overline{\psi}\gamma^\mu\psi A_\mu$$

gives four different Feynman's diagrams:



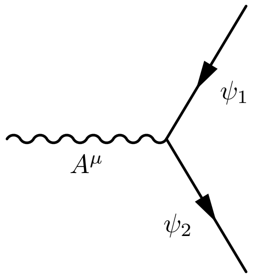
Proof. In the vertex, there is an annihilated fermion ψ_1 and an annihilated antifermion ψ_2, so that

$$\psi_1 \sim \hat{a}_s u_s e^{-ip_1x} \ , \quad \overline{\psi}_2 \sim \hat{b}_s \overline{v}_s e^{-ip_2x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ie\overline{\psi}_2\gamma^\mu\psi_1 A_\mu = A_\mu(-ie\gamma^\mu)\overline{\psi}_2\psi_1 \ ,$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\overline{\psi}_2\psi_1$, the vertex contribution is $-ie\gamma^\mu$. q.e.d.



$$= -ie\gamma^\mu$$

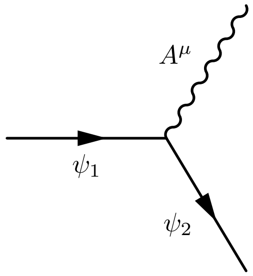
Proof. In the vertex, there is a created fermion ψ_2 and a created antifermion ψ_1, so that

$$\psi_1 \sim \hat{b}_s^\dagger v_s e^{ip_1 x} \ , \quad \overline{\psi}_2 \sim \hat{a}_s^\dagger \overline{u}_s e^{ip_2 x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ie\overline{\psi}_2\gamma^\mu\psi_1 A_\mu = \overline{\psi}_2\psi_1(-ie\gamma^\mu)A_\mu \ ,$$

which means that, since the final states on the left is $\overline{\psi}_2\psi_1$ and the initial state on the right is A^μ, the vertex contribution is $-ie\gamma^\mu$. q.e.d.



$$= -ie\gamma^\mu$$

Proof. In the vertex, there is an annihilated fermion ψ_1 and a created fermion ψ_2, so that

$$\psi_1 \sim \hat{a}_s u_s e^{-ip_1 x} \ , \quad \overline{\psi}_2 \sim \hat{a}_s^\dagger \overline{u}_s e^{ip_2 x} \ ,$$

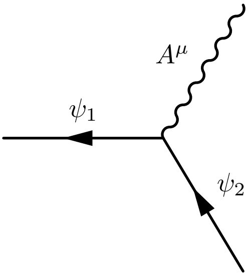
hence, $i\mathcal{L}_{int}$ becomes

$$-ie\overline{\psi}_2\gamma^\mu\psi_1 A_\mu = A_\mu\overline{\psi}_2(-ie\gamma^\mu)\psi_1 \ ,$$

which means that, since the final states on the left is $A^\mu\overline{\psi}_2$ and the initial state on the right is ψ_1, the vertex contribution is $-ie\gamma^\mu$. q.e.d.

Proof. In the vertex, there is an annihilated antifermion ψ_1 and a created antifermion ψ_2, so that

$$\psi_1 \sim \hat{b}_s^\dagger v_s e^{ip_1 x} \ , \quad \overline{\psi}_2 \sim \hat{b}_s \overline{v}_s e^{-ip_2 x} \ ,$$



$$= -ie\gamma^\mu$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ie\overline{\psi}_2\gamma^\mu\psi_1 A_\mu = A_\mu\overline{\psi}_2(-ie\gamma^\mu)\psi_1 \ ,$$

which means that, since the final states on the left is $A^\mu\overline{\psi}_2$ and the initial state on the right is ψ_1, the vertex contribution is $-ie\gamma^\mu$. q.e.d.

4 Feynman's rules: propagators and external lines

5 Formulae

In this section, we will summarise all the important formulae obtained so far.

Experimental quantities are:

1. cross section for a $p_1 + p_2 \to p_3 + p_4$ in the center of mass frame:

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \ ,$$

   where $E_{cm} = E_1 + E_2 = E_3 + E_4$, $|\mathbf{p}_i| = \mathbf{p}_1| = |\mathbf{p}_2|$, $|\mathbf{p}_f| = \mathbf{p}_3| = \mathbf{p}_4|$, with the addition of 1/2

2. decay rate for a $p \to p_2 + p_3$ in the center of mass frame:

$$\frac{d\Gamma}{d\Omega} = \frac{|\mathcal{M}|^2}{32\pi^2 m^2} |\mathbf{p}_f| \ ,$$

   where m is the mass of the initial particle, $|\mathbf{p}_f| = \mathbf{p}_2| = \mathbf{p}_3|$.

The propagators are

1. scalar field

$$D_F(x,y) = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 - m^2 + i\epsilon} \ ,$$

2. Dirac field

$$D_F(x,y) = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 - m^2 + i\epsilon} (\not{p} + m) \ ,$$

3. photon (Feynman gauge $\xi = 1$ and Lorentz gauge $\xi = 0$)

$$D_F(x,y) = -i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + i\epsilon} \left(g_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right) \ ,$$

The Feynman's rules for scalar field are:

$$\mathcal{L}_{int} = -\frac{\lambda^3}{3!}\phi^3 - \frac{\lambda^4}{4!}\phi^4$$

1. external line gets 1,
2. internal line gets the propagator,
3. vertex gets $i\lambda$.

The Feynman's rules for scalar quantum electrodynamics are:

$$\mathcal{L}_{int} = -ieA_\mu(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) + e^2 A^\mu A_\mu \phi^* \phi$$

1. external line gets 1 for scalar, ϵ_μ for incoming photon and ϵ_μ^* for outgoing photon,
2. internal line gets the propagator,

3. vertex gets $-ie$ times momentum of right-directed arrows minus momentum of left-directed arrows.

The Feynman's rules for Yukawa theory are:

$$\mathcal{L}_{int} = -g\overline{\psi}\phi\psi$$

1. external line gets 1 for scalar, u^s for incoming fermion, $\overline{u}^s$ for outgoing fermion, $\overline{v}^s$ for incoming antifermion and v^s for outgoing antifermion,
2. internal line gets the propagator,
3. vertex gets $-ig$.

The Feynman's rules for quantum electrodynamics are:

$$\mathcal{L}_{int} = -e\overline{\psi}\gamma^\mu\psi A_\mu$$

1. external line gets ϵ_μ for incoming photon, ϵ^*_μ for outgoing photon, u^s for incoming fermion, $\overline{u}^s$ for outgoing fermion, $\overline{v}^s$ for incoming antifermion and v^s for outgoing antifermion,
2. internal line gets the propagator,
3. vertex gets $-ie\gamma^\mu$.

Further observations:

1.

Mandelstam's variables are:

1. $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$,
2. $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$,
3. $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$.