Quantum Field Theory
2 - Feynman's rules

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April 6, 2024

Abstract

In this notes, we will deduce the Feynamn's rule about vertices, internal lines (propagators) and external lines.

Contents

1 Feynman's rules: vertices

Feynman's rules for vertices can be derived from the Lagrangian of the theory.

1.1 Scalar theory

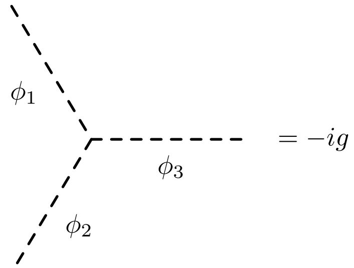
The Lagrangian of a scalar theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \ .$$

The first interaction vertex

$$-\frac{g}{3!}\phi^3$$

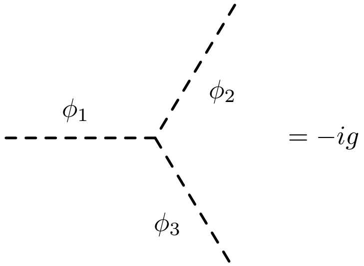
gives two Feynman's diagram:



Proof. In the vertex, $i\mathcal{L}_{int}$ becomes

$$-i\frac{g}{3!}\phi_1\phi_2\phi_3 = \phi_3(-i\frac{g}{3!})\phi_1\phi_2 \ ,$$

which means that, since the final states on the left is ϕ_3, the initial state on the right is $\phi_1\phi_2$ and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3$, the vertex contribution is $-ig$. q.e.d.



Proof. In the vertex, $i\mathcal{L}_{int}$ becomes

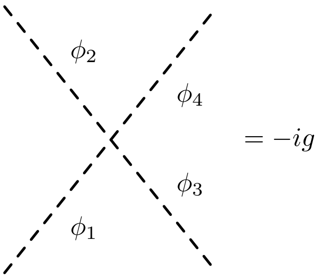
$$-i\frac{g}{3!}\phi_1\phi_2\phi_3 = \phi_1\phi_2(-i\frac{g}{3!})\phi_3 \ ,$$

which means that, since the final states on the left is $\phi_1\phi_2$, the initial state on the right is ϕ_3 and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3$ for a total of 3! times, the vertex contribution is $-ig$. q.e.d.

The second interaction vertex

$$-\frac{\lambda}{4!}\phi^4$$

gives one Feynman's diagram:



Proof. In the vertex, $i\mathcal{L}_{int}$ becomes

$$-i\frac{\lambda}{4!}\phi_1\phi_2\phi_3\phi_4 = \phi_3\phi_4(-i\frac{\lambda}{4!})\phi_1\phi_2 \ ,$$

which means that, since the final states on the left is $\phi_3\phi_4$, the initial state on the right is $\phi_1\phi_2$ and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ for a total of 4! times, the vertex contribution is $-ig$. q.e.d.

1.2 Scalar QED

The Lagrangian of scalar quantum electrodynamics is

$$\begin{aligned}
\mathcal{L}_{sQED} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - m^2\phi^*\phi \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi\partial^\mu\phi^* - m^2\phi^*\phi \\
&\quad - ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + e^2A_\mu A^\mu\phi^*\phi \ .
\end{aligned}$$

The first interaction vertex

$$-ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$$

gives four different Feynman's diagrams:

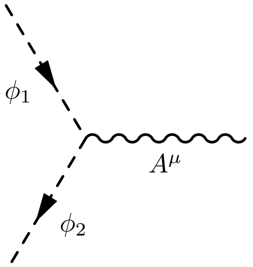
Proof. In the vertex, there is an annihilated scalar ϕ_1 and an annihilated anti-scalar ϕ_2, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \ , \quad \phi_2^* \sim \hat{b}e^{-ip_2x} \ ,$$

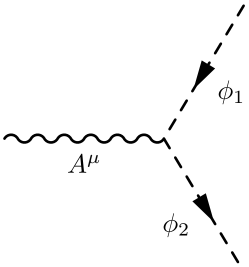
hence, $i\mathcal{L}_{int}$ becomes

$$\begin{aligned}
eA_\mu(\phi_2^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2^*) &= eA_\mu\Big(\phi_2^*(-ip_1^\mu)\phi_1 - \phi_1(-ip_2^\mu)\phi_2^*\Big) \\
&= A^\mu\Big(-ie(p_1^\mu - p_2^\mu)\Big)\phi_1\phi_2^* \ ,
\end{aligned}$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\phi_1\phi_2^*$, the vertex contribution is $-ie(p_1^\mu - p_2^\mu)$. q.e.d.



$$= -ie(p_1^\mu - p_2^\mu)$$



$$= -ie(-p_1^\mu + p_2^\mu)$$

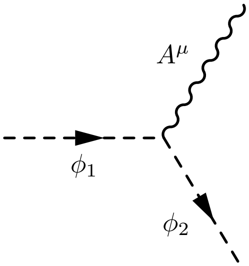
Proof. In the vertex, there is a created scalar ϕ_2 and a created antiscalar ϕ_1, so that

$$\phi_1 \sim \hat{b}^\dagger e^{ip_1 x} \ , \quad \phi_2^* \sim \hat{a}^\dagger e^{ip_2 x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$\begin{aligned} eA_\mu(\phi_2^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2^*) &= eA_\mu\Big(\phi_2^*(ip_1^\mu)\phi_1 - \phi_1(ip_2^\mu)\phi_2^*\Big) \\ &= A^\mu\Big(-ie(-p_1^\mu + p_2^\mu)\Big)\phi_1\phi_2^* \ , \end{aligned}$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\phi_1\phi_2^*$, the vertex contribution is $-ie(-p_1^\mu + p_2^\mu)$. q.e.d.



$$= -ie(p_1^\mu + p_2^\mu)$$

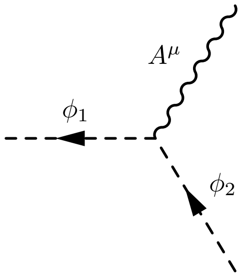
Proof. In the vertex, there is an annihilated scalar ϕ_1 and a created scalar ϕ_2, so that

$$\phi_1 \sim \hat{a} e^{-ip_1 x} \ , \quad \phi_2^* \sim \hat{a}^\dagger e^{ip_2 x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$\begin{aligned} eA_\mu(\phi_2^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2^*) &= eA_\mu\Big(\phi_2^*(-ip_1^\mu)\phi_1 - \phi_1(ip_2^\mu)\phi_2^*\Big) \\ &= A^\mu\Big(-ie(p_1^\mu + p_2^\mu)\Big)\phi_1\phi_2^* \ , \end{aligned}$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\phi_1\phi_2^*$, the vertex contribution is $-ie(p_1^\mu + p_2^\mu)$. q.e.d.



$$= -ie(-p_1^\mu - p_2^\mu)$$

Proof. In the vertex, there is an annihilated antiscalar ϕ_1 and a created antiscalar ϕ_2, so that

$$\phi_1 \sim \hat{b}^\dagger e^{ip_1x} \ , \quad \phi_2^* \sim \hat{b}e^{-ip_2x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

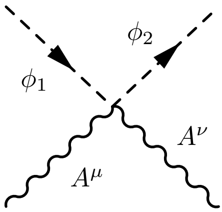
$$\begin{aligned} eA_\mu(\phi_2^*\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2^*) &= eA_\mu\Big(\phi_2^*(ip_1^\mu)\phi_1 - \phi_1(-ip_2^\mu)\phi_2^*\Big) \\ &= A^\mu\Big(-ie(-p_1^\mu - p_2^\mu)\Big)\phi_1\phi_2^* \ , \end{aligned}$$

which means that, since the final states on the left is A^μ and the initial state on the right is $\phi_1\phi_2^*$, the vertex contribution is $-ie(-p_1^\mu - p_2^\mu)$. q.e.d.

The second interaction vertex

$$e^2 g^{\mu\nu} A_\mu A_\nu \phi^*\phi$$

gives one Feynman's diagram



$$= 2ie^2 g^{\mu\nu}$$

Proof. In the vertex, there is an annihilated scalar ϕ_1 and a created scalar ϕ_2, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \ , \quad \phi_2^* \sim \hat{a}^\dagger e^{ip_2x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$ie^2 g^{\mu\nu} A_\mu A_\nu \phi_2^*\phi_1 = A_\nu\phi_2^*(ie^2 g^{\mu\nu})A_\mu\phi_1 \ ,$$

which means that, since the final states on the left is $A_\nu\phi_2^*$, the initial state on the right is $A_\mu\phi_1$ and we can exchange $\mu \leftrightarrow \nu$ for a total of 2! times, the vertex contribution is $2ie^2 g^{\mu\nu}$. q.e.d.

Derivative coupling

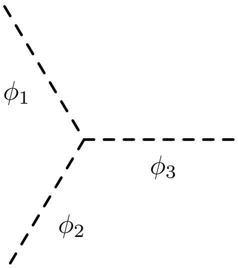
In order to understand the behaviour of derivative in the interaction Lagrangian, suppose to have a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi\partial_\mu\phi\partial^\mu\phi \; .$$

The interaction vertex

$$-\frac{\lambda}{3!}\phi\partial_\mu\phi\partial^\mu\phi$$

gives two different Feynman's diagram:



$$= -i\lambda p_2^\mu p_3^\mu$$

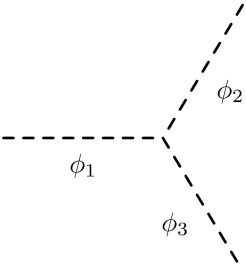
Proof. In the vertex, there are two annihilated scalar ϕ_1, ϕ_2 and a created scalar ϕ_2, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \;, \quad \phi_2 \sim \hat{a}e^{-ip_2x} \;, \quad \phi_3 \sim \hat{a}^\dagger e^{ip_3x} \;,$$

hence, $i\mathcal{L}_{int}$ becomes

$$\begin{aligned} -i\frac{\lambda}{3!}\phi_1\partial_\mu\phi_2\partial^\mu\phi_3 &= -i\frac{\lambda}{3!}\phi_1(-ip_2^\mu)\phi_2(ip_3^\mu)\phi_2 \\ &= \phi_3(-i\frac{\lambda}{3!}p_2^\mu p_3^\mu)\phi_1\phi_2 \;, \end{aligned}$$

which means that, since the final states on the left is ϕ_3, the initial state on the right is $\phi_1\phi_2$ and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3$ for a total of 3! times, the vertex contribution is $-i\lambda p_2^\mu p_3^\mu$. q.e.d.



$$= i\lambda p_2^\mu p_3^\mu$$

Proof. In the vertex, there are two created scalar ϕ_2, ϕ_3 and an annihilated scalar ϕ_1, so that

$$\phi_1 \sim \hat{a}e^{-ip_1x} \;, \quad \phi_2 \sim \hat{a}^\dagger e^{ip_2x} \;, \quad \phi_3 \sim \hat{a}^\dagger e^{ip_3x} \;,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-i\frac{\lambda}{3!}\phi_1\partial_\mu\phi_2\partial^\mu\phi_3 = -i\frac{\lambda}{3!}\phi_1(ip_2^\mu)\phi_2(ip_3^\mu)\phi_2$$
$$= \phi_2\phi_3(i\frac{\lambda}{3!}p_2^\mu p_3^\mu)\phi_1 \ ,$$

which means that, since the final states on the left is $\phi_2\phi_3$, the initial state on the right is ϕ_1 and we can exchange $1 \leftrightarrow 2 \leftrightarrow 3$ for a total of 3! times, the vertex contribution is $i\lambda p_2^\mu p_3^\mu$. q.e.d.

1.3 Yukawa theory

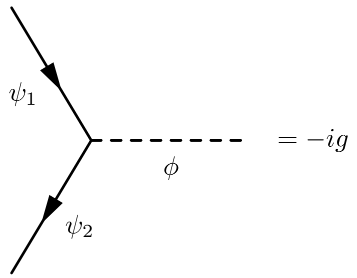
The Lagrangian of Yukawa theory is

$$\mathcal{L}_Y = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi - g\overline{\psi}\phi\psi \ .$$

The interaction vertex

$$-g\overline{\psi}\phi\psi$$

gives two different Feynman's diagram:



Proof. In the vertex, there is an annihilated fermion ψ_1, an annihilated antifermion $\overline{\psi}_2$ and a created scalar, so that

$$\phi \sim \hat{a}^\dagger e^{ipx} \ , \quad \psi_1 \sim \hat{a}_s u_s e^{-ip_1x} \ , \quad \overline{\psi}_2 \sim \hat{b}_s\overline{v}_s e^{-ip_2x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ig\overline{\psi}_2\phi\psi_1 = \phi(-ig)\overline{\psi}_2\psi_1 \ ,$$

which means that, since the final states on the left is ϕ and the initial state on the right is $\overline{\psi}_2\psi_1$, the vertex contribution is $-ig$. q.e.d.

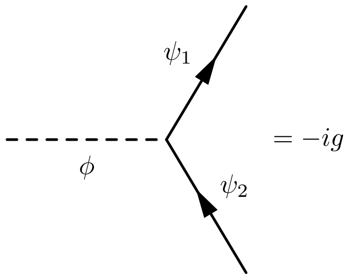
Proof. In the vertex, there is a created fermion ψ_1, a created antifermion ψ_2 and an annihilated scalar, so that

$$\phi \sim \hat{a}e^{-ipx} \ , \quad \overline{\psi}_1 \sim \hat{a}_s^\dagger\overline{u}_s e^{ip_1x} \ , \quad \psi_2 \sim \hat{b}_s^\dagger\overline{v}_s e^{ip_2x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ig\overline{\psi}_2\phi\psi_1 = \overline{\psi}_2\psi_1(-ig)\phi \ ,$$

which means that, since the final states on the left is $\overline{\psi}_2\psi_1$ and the initial state on the right is ϕ, the vertex contribution is $-ig$. q.e.d.



1.4 QED

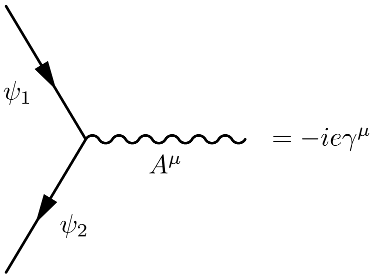
The Lagrangian of quantum electrodynamics is

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) - m)\psi$$
$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{\partial} - m)\psi - e\overline{\psi}\gamma^{\mu}\psi A_{\mu} \ .$$

The interaction vertex

$$-e\overline{\psi}\gamma^{\mu}\psi A_{\mu}$$

gives four different Feynman's diagrams:



Proof. In the vertex, there is an annihilated fermion ψ_1 and an annihilated antifermion ψ_2, so that

$$\psi_1 \sim \hat{a}_s u_s e^{-ip_1 x} \ , \quad \overline{\psi}_2 \sim \hat{b}_s \overline{v}_s e^{-ip_2 x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ie\overline{\psi}_2\gamma^{\mu}\psi_1 A_{\mu} = A_{\mu}(-ie\gamma^{\mu})\overline{\psi}_2\psi_1 \ ,$$

which means that, since the final states on the left is A^{μ} and the initial state on the right is $\overline{\psi}_2\psi_1$, the vertex contribution is $-ie\gamma^{\mu}$. q.e.d.

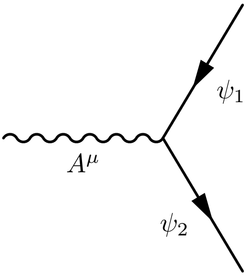
Proof. In the vertex, there is a created fermion ψ_2 and a created antifermion ψ_1, so that

$$\psi_1 \sim \hat{b}_s^{\dagger} v_s e^{ip_1 x} \ , \quad \overline{\psi}_2 \sim \hat{a}_s^{\dagger} \overline{u}_s e^{ip_2 x} \ ,$$

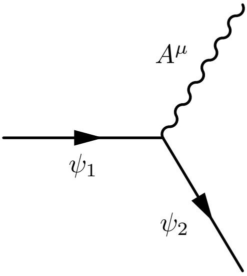
hence, $i\mathcal{L}_{int}$ becomes

$$-ie\overline{\psi}_2\gamma^{\mu}\psi_1 A_{\mu} = \overline{\psi}_2\psi_1(-ie\gamma^{\mu})A_{\mu} \ ,$$

which means that, since the final states on the left is $\overline{\psi}_2\psi_1$ and the initial state on the right is A^{μ}, the vertex contribution is $-ie\gamma^{\mu}$. q.e.d.



$= -ie\gamma^\mu$



$= -ie\gamma^\mu$

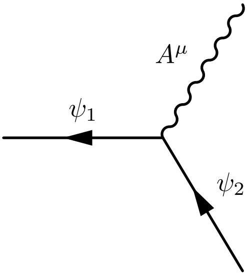
Proof. In the vertex, there is an annihilated fermion ψ_1 and a created fermion ψ_2, so that

$$\psi_1 \sim \hat{a}_s u_s e^{-ip_1 x} \ , \quad \overline{\psi}_2 \sim \hat{a}_s^\dagger \overline{u}_s e^{ip_2 x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ie\overline{\psi}_2 \gamma^\mu \psi_1 A_\mu = A_\mu \overline{\psi}_2 (-ie\gamma^\mu) \psi_1 \ ,$$

which means that, since the final states on the left is $A^\mu \overline{\psi}_2$ and the initial state on the right is ψ_1, the vertex contribution is $-ie\gamma^\mu$. q.e.d.



$= -ie\gamma^\mu$

Proof. In the vertex, there is an annihilated antifermion ψ_1 and a created antifermion ψ_2, so that

$$\psi_1 \sim \hat{b}_s^\dagger v_s e^{ip_1 x} \ , \quad \overline{\psi}_2 \sim \hat{b}_s \overline{v}_s e^{-ip_2 x} \ ,$$

hence, $i\mathcal{L}_{int}$ becomes

$$-ie\overline{\psi}_2 \gamma^\mu \psi_1 A_\mu = A_\mu \overline{\psi}_2 (-ie\gamma^\mu) \psi_1 \ ,$$

which means that, since the final states on the left is $A^\mu \overline{\psi}_2$ and the initial state on the right is ψ_1, the vertex contribution is $-ie\gamma^\mu$. q.e.d.

2 Feynman's rules: internal lines

Feynman's rules for internal lines can be derived by computing the propagator.

2.1 Scalar propagator

The scalar propagator is given by

$$D_F(x,y) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2-m^2+i\epsilon}\ .$$

Proof. Recall that a real scalar field is expanded in creation and annihilation operators by

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(\hat{a}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx}\right)\ .$$

By definition, we need to compute

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \langle 0|\phi(x)\phi(y)|0\rangle\theta(x_0-y_0) + \langle 0|\phi(y)\phi(x)|0\rangle\theta(y_0-x_0)\ .$$

First, we compute this quantity without the time-ordering operator

$$\begin{aligned}
&\langle 0|\phi(x)\phi(y)|0\rangle\\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|\left(\hat{a}_p e^{-ipx}+\hat{a}_p^\dagger e^{ipx}\right)\left(\hat{a}_q e^{-iqy}+\hat{a}_q^\dagger e^{iqy}\right)|0\rangle\\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|\hat{a}_p\hat{a}_q^\dagger|0\rangle e^{i(qy-px)}\\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle e^{i(qy-px)}\\
&= \int\frac{d^3p}{(2\pi)^3 2E_p}e^{-ip(x-y)}\ ,
\end{aligned}$$

then, we have

$$\begin{aligned}
&\langle 0|T\{\phi(x)\phi(y)\}|0\rangle\\
&= \int\frac{d^3p}{(2\pi)^3 2E_p}e^{-ip(x-y)}\theta(x_0-y_0) + \int\frac{d^3p}{(2\pi)^3 2E_p}e^{ip(x-y)}\theta(y_0-x_0)\\
&= \int\frac{d^3p}{(2\pi)^3 2E_p}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}e^{-iE_p(x_0-y_0)}\theta(x_0-y_0)\\
&\quad + \int\frac{d^3p}{(2\pi)^3 2E_p}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}e^{iE_p(x_0-y_0)}\theta(y_0-x_0)\\
&= \int\frac{d^3p}{(2\pi)^3 2E_p}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\\
&\quad\times\left(e^{-iE_p(x_0-y_0)}\theta(x_0-y_0) + e^{iE_p(x_0-y_0)}\theta(-(x_0-y_0))\right)\ ,
\end{aligned}$$

where we have made a change $\mathbf{p}\leftrightarrow -\mathbf{p}$. Now, we use the identity in (1)

$$e^{-iE_p(x_0-y_0)}\theta(x_0-y_0) + e^{iE_p(x_0-y_0)}\theta(-(x_0-y_0)) = -\frac{2E_p}{2\pi i}\int dp_0\frac{e^{ip_0(x_0-y_0)}}{p_0^2-E_p^2+i\epsilon}\ ,$$

to obtain

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big(-\frac{2E_p}{2\pi i}\int dp_0 \frac{e^{ip_0(x_0-y_0)}}{p_0^2 - E_p^2 + i\epsilon}\Big)$$

$$= i\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip_0(x_0-y_0)}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon} = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2 - m^2 + i\epsilon} ,$$

where we have used $E_p = \sqrt{|\mathbf{p}|^2 + m^2|} \neq p_0$ since it is only a integration variable.

This propagator is valid for a complex field as well. Recall that a complex scalar fields are expanded in creation and annihilation operators by

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\Big(\hat{a}_p e^{-ipx} + \hat{b}_p^\dagger e^{ipx}\Big) ,$$

$$\hat{\phi}^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\Big(\hat{b}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx}\Big) .$$

By definition, we need to compute

$$\langle 0|T\{\phi(x)\phi^\dagger(y)\}|0\rangle = \langle 0|\phi(x)\phi^\dagger(y)|0\rangle\theta(x_0 - y_0) + \langle 0|\phi^\dagger(y)\phi(x)|0\rangle\theta(y_0 - x_0) .$$

First, we compute these quantities without the time-ordering operator

$$\begin{aligned}
&\langle 0|\phi(x)\phi^\dagger(y)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|\Big(\hat{a}_p e^{-ipx} + \hat{b}_p^\dagger e^{ipx}\Big)\Big(\hat{b}_q e^{-iqy} + \hat{a}_q^\dagger e^{iqy}\Big)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|\hat{a}_p\hat{a}_q^\dagger|0\rangle e^{i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle e^{i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{ip(y-x)} ,
\end{aligned}$$

$$\begin{aligned}
&\langle 0|\phi^\dagger(y)\phi(x)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|\Big(\hat{b}_q e^{-iqy} + \hat{a}_q^\dagger e^{iqy}\Big)\Big(\hat{a}_p e^{-ipx} + \hat{b}_p^\dagger e^{ipx}\Big)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|\hat{b}_p\hat{b}_q^\dagger|0\rangle e^{-i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle e^{-i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{ip(x-y)} ,
\end{aligned}$$

but we observe that they are exactly the same as the real case, so the following computations would be identical. q.e.d.

2.2 Fermion propagator

The fermion propagator is given by

$$D_F(x,y) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2-m^2+i\epsilon}(\not{p}+m)\ .$$

Proof. Recall that fermion fields are expanded in creation and annihilation operators by

$$\hat{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(\hat{a}_p^s u_p^s e^{-ipx} + \hat{b}_p^{s\dagger} v_p^s e^{ipx}\right)\ ,$$

$$\hat{\overline{\psi}}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(\hat{b}_p^s \overline{v}_p^s e^{-ipx} + \hat{a}_p^{s\dagger} \overline{u}_p^s e^{ipx}\right)\ ,$$

By definition, we need to compute

$$\langle 0|T\{\psi(x)\overline{\psi}(y)\}|0\rangle = \langle 0|\psi(x)\overline{\psi}(y)|0\rangle\theta(x_0-y_0) - \langle 0|\overline{\psi}(y)\psi(x)|0\rangle\theta(y_0-x_0)\ .$$

First, we compute this quantity without the time-ordering operator

$$\begin{aligned}
&\langle 0|\psi(x)\overline{\psi}(y)|0\rangle \\
&= \sum_{s,r}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}} \\
&\quad \times \langle 0|\left(\hat{a}_p^s u_p^s e^{-ipx} + \hat{b}_p^{s\dagger} v_p^s e^{ipx}\right)\left(\hat{b}_q^r \overline{v}_q^r e^{-iqy} + \hat{a}_q^{r\dagger}\overline{u}_q^r e^{iqy}\right)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\sum_{s,r} u_p^s\overline{u}_q^r\langle 0|\hat{a}_p^s\hat{a}_q^{r\dagger}|0\rangle e^{i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\sum_{s,r} u_p^s\overline{u}_q^r\delta_{rs}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle e^{i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p}e^{ip(y-x)}\sum_s u_p^s\overline{u}_p^s = \int \frac{d^3p}{(2\pi)^3 2E_p}e^{ip(y-x)}(\not{p}+m)\ ,
\end{aligned}$$

$$\begin{aligned}
&\langle 0|\overline{\psi}(y)\psi(x)|0\rangle \\
&= \sum_{s,r}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}} \\
&\quad \times \langle 0|\left(\hat{b}_q^r \overline{v}_q^r e^{-iqy} + \hat{a}_q^{r\dagger}\overline{u}_q^r e^{iqy}\right)\left(\hat{a}_p^s u_p^s e^{-ipx} + \hat{b}_p^{s\dagger} v_p^s e^{ipx}\right)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\sum_{s,r} v_p^s\overline{v}_q^r\langle 0|\hat{b}_p^s\hat{b}_q^{r\dagger}|0\rangle e^{-i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}}\sum_{s,r} v_p^s\overline{v}_q^r\delta_{rs}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle e^{-i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p}e^{ip(y-x)}\sum_s v_p^s\overline{v}_p^s = \int \frac{d^3p}{(2\pi)^3 2E_p}e^{-ip(y-x)}(\not{p}-m)\ ,
\end{aligned}$$

then, we have

$$
\begin{aligned}
&\langle 0|T\{\psi(x)\overline{\psi}(y)\}|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)}(\not{p}+m)\theta(x_0-y_0) \\
&\quad - \int \frac{d^3p}{(2\pi)^3 2E_p} e^{ip(x-y)}(\not{p}-m)\theta(y_0-x_0) \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{-iE_p(x_0-y_0)}(\not{p}+m)\theta(x_0-y_0) \\
&\quad - \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{iE_p(x_0-y_0)}(\not{p}-m)\theta(y_0-x_0) \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{-iE_p(x_0-y_0)}(\not{p}+m)\theta(x_0-y_0) \\
&\quad - \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{iE_p(x_0-y_0)}(-\not{p}-m)\theta(y_0-x_0) \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}(\not{p}+m) \\
&\quad \times \left(e^{-iE_p(x_0-y_0)}\theta(x_0-y_0) + e^{iE_p(x_0-y_0)}\theta(-(x_0-y_0)) \right) ,
\end{aligned}
$$

where we have made a change $\mathbf{p} \leftrightarrow -\mathbf{p}$. Now, we use the identity in (1)

$$
e^{-iE_p(x_0-y_0)}\theta(x_0-y_0) + e^{iE_p(x_0-y_0)}\theta(-(x_0-y_0)) = \frac{2E_p}{2\pi i} \int dp_0 \frac{e^{-ip_0(x_0-y_0)}}{p_0^2 - E_p^2 + i\epsilon} ,
$$

to obtain

$$
\begin{aligned}
&\langle 0|T\{\psi(x)\overline{\psi}(y)\}|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}(\not{p}+m)\left(\frac{2E_p}{2\pi i} \int dp_0 \frac{e^{-ip_0(x_0-y_0)}}{p_0^2 - E_p^2 + i\epsilon} \right) \\
&= i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip_0(x_0-y_0)} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon}(\not{p}+m) \\
&= -i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}(\not{p}+m) ,
\end{aligned}
$$

where we have used $E_p = \sqrt{|\mathbf{p}|^2 + m^2|} \neq p_0$ since it is only a integration variable. q.e.d.

2.3 Photon propagator

The photon propagator is given by

$$
D_F(x,y) = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}\left(-\eta^{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \right) .
$$

Proof. Recall that photon fields are expanded in creation and annihilation operators by

$$\hat{A}^\mu(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \Big(\hat{a}_p^\lambda \epsilon_\lambda^\mu(\mathbf{p}) e^{-ipx} + \hat{a}_p^{\lambda\dagger} \epsilon_\lambda^{*\mu}(\mathbf{p}) e^{ipx} \Big) \ ,$$

By definition, we need to compute

$$\langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle = \langle 0|A^\mu(x)A^\nu(y)\theta(x_0-y_0) + \langle 0|A^\nu(y)A^\mu(x)|0\rangle\theta(y_0-x_0) \ .$$

First, we compute this quantity without the time-ordering operator

$$\begin{aligned}
&\langle 0|A^\mu(x)A^\nu(y)|0\rangle \\
&= \sum_{\lambda,\sigma} \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}} \\
&\quad \times \langle 0| \Big(\hat{a}_p^\lambda \epsilon_\lambda^\mu(\mathbf{p}) e^{-ipx} + \hat{a}_p^{\lambda\dagger} \epsilon_\lambda^{*\mu}(\mathbf{p}) e^{ipx} \Big) \Big(\hat{a}_q^\sigma \epsilon_\sigma^\nu(\mathbf{q}) e^{-iqy} + \hat{a}_q^{\sigma\dagger} \epsilon_\lambda^{*\mu}(\mathbf{q}) e^{iqy} \Big) |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}} \sum_{\lambda,\sigma} \epsilon_\lambda^\mu(\mathbf{p}) \epsilon_\sigma^{*\nu}(\mathbf{q}) \langle 0|\hat{a}_p^\lambda \hat{a}_q^{\sigma\dagger}|0\rangle e^{i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_q}} \sum_{\lambda,\sigma} \epsilon_{p\lambda}^\mu \epsilon_{q\sigma}^{*\nu} \delta_{\lambda\sigma} \langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle e^{i(qy-px)} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)} \sum_\lambda \epsilon_{p\lambda}^\mu \epsilon_{p\lambda}^{*\nu} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)} \Big(-\eta^{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \Big) \ ,
\end{aligned}$$

then, we have

$$\begin{aligned}
&\langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)} \Big(-\eta^{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \Big) \theta(x_0-y_0) \\
&\quad + \int \frac{d^3p}{(2\pi)^3 2E_p} e^{ip(x-y)} \Big(-\eta^{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \Big) \theta(y_0-x_0) \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{-iE_p(x_0-y_0)} \Big(-\eta^{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \Big) \theta(x_0-y_0) \\
&\quad + \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{iE_p(x_0-y_0)} \Big(-\eta^{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \Big) \theta(y_0-x_0) \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \Big(-\eta^{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \Big) \\
&\quad \times \Big(e^{-iE_p(x_0-y_0)}\theta(x_0-y_0) + e^{iE_p(x_0-y_0)}\theta(-(x_0-y_0)) \Big) \ ,
\end{aligned}$$

where we have made a change $\mathbf{p} \leftrightarrow -\mathbf{p}$. Now, we use the identity in (1)

$$e^{-iE_p(x_0-y_0)}\theta(x_0-y_0) + e^{iE_p(x_0-y_0)}\theta(-(x_0-y_0)) = \frac{2E_p}{2\pi i} \int dp_0 \frac{e^{-ip_0(x_0-y_0)}}{p_0^2 - E_p^2 + i\epsilon} \ ,$$

to obtain

$$
\begin{aligned}
&\langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big(-\eta^{\mu\nu}+(1-\xi)\frac{p_\mu p_\nu}{p^2}\Big) \\
&\quad\times\Big(\frac{2E_p}{2\pi i}\int dp_0 \frac{e^{-ip_0(x_0-y_0)}}{p_0^2-E_p^2+i\epsilon}\Big) \\
&= i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip_0(x_0-y_0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{p_0^2-|\mathbf{p}|^2-m^2+i\epsilon}\Big(-\eta^{\mu\nu}+(1-\xi)\frac{p_\mu p_\nu}{p^2}\Big) \\
&= i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon}\Big(-\eta^{\mu\nu}+(1-\xi)\frac{p_\mu p_\nu}{p^2}\Big)\ ,
\end{aligned}
$$

where we have used $E_p = \sqrt{|\mathbf{p}|^2+m^2|} \neq p_0$ since it is only a integration variable. q.e.d.

3 Feynman's rules: external lines

Feynman's rules for external lines can be derived by the expansions of the fields.

3.1 Scalar external line

External scalar lines gets a factor of 1.

Proof. Recall that complex scalar fields are expanded in creation and annihilation operators by

$$
\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\Big(\hat{a}_p e^{-ipx}+\hat{b}_p^\dagger e^{ipx}\Big)\ ,
$$

$$
\hat{\phi}^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\Big(\hat{b}_p e^{-ipx}+\hat{a}_p^\dagger e^{ipx}\Big)\ .
$$

For an initial scalar, we have

$$
\begin{aligned}
&\langle 0|\hat{\phi}(x)|q\rangle = \langle 0|\hat{\phi}(x)\sqrt{2E_q}\hat{a}_q^\dagger|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\Big(\hat{a}_p e^{-ipx}+\hat{b}_p^\dagger e^{ipx}\Big)\sqrt{2E_q}\hat{a}_{qr}^\dagger|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\langle 0|\hat{a}_p\hat{a}_q^\dagger|0\rangle\sqrt{2E_q}e^{-ipx} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle\sqrt{2E_q}e^{-ipx} \\
&= e^{-ipx}\ .
\end{aligned}
$$

For an final scalar, we have

$$
\begin{aligned}
\langle q|\hat{\phi}^\dagger(x)|0\rangle &= \langle 0|\hat{a}_q\sqrt{2E_q}\hat{\phi}^\dagger|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|\hat{a}_q\left(\hat{b}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx}\right)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|\hat{a}_q\hat{a}_p^\dagger|0\rangle e^{ipx} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle e^{ipx} \\
&= e^{ipx} \ .
\end{aligned}
$$

For an initial antiscalar, we have

$$
\begin{aligned}
\langle 0|\hat{\phi}^\dagger(x)|q\rangle &= \langle 0|\hat{\psi}(x)\sqrt{2E_q}\hat{b}_q^\dagger|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\langle 0|\left(\hat{b}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx}\right)\sqrt{2E_q}\hat{b}_q^\dagger|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\langle 0|\hat{b}_p\hat{b}_q^\dagger|0\rangle\sqrt{2E_q}e^{-ipx} \\
&= \sum_s\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle\sqrt{2E_q}e^{-ipx} \\
&= e^{-ipx} \ .
\end{aligned}
$$

For an final antiscalar, we have

$$
\begin{aligned}
\langle q,r|\hat{\phi}(x)|0\rangle &= \langle 0|\hat{b}_{qr}\sqrt{2E_q}\hat{\phi}(x)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|\hat{b}_q\left(\hat{a}_p e^{-ipx} + \hat{b}_p^\dagger e^{ipx}\right)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|\hat{b}_q\hat{b}_p^\dagger|0\rangle e^{ipx} \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle e^{ipx} \\
&= e^{ipx} \ .
\end{aligned}
$$

The polarisation factors need to to added in the LSZ formula. q.e.d.

3.2 Fermion external line

External fermion lines gets a factor of u for an incoming fermion, $\overline{u}$ for an outgoing fermion, $\overline{v}$ for an incoming antifermion and v for an outgoing antifermion.

Proof. Recall that fermion fields are expanded in creation and annihilation operators by

$$
\hat{\psi}(x) = \sum_s\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(\hat{a}_p^s u_p^s e^{-ipx} + \hat{b}_p^{s\dagger} v_p^s e^{ipx}\right) ,
$$

$$\hat{\overline{\psi}}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(\hat{b}^s_p \overline{v}^s_p e^{-ipx} + \hat{a}^{s\dagger}_p \overline{u}^s_p e^{ipx} \right) ,$$

For an initial fermion, we have

$$\begin{aligned}
&\langle 0|\hat{\psi}(x)|q,r\rangle = \langle 0|\hat{\psi}(x)\sqrt{2E_q}\hat{a}^\dagger_{qr}|0\rangle \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \langle 0| \left(\hat{a}_{ps} u_{ps} e^{-ipx} + \hat{b}^\dagger_{ps} v_{ps} e^{ipx} \right) \sqrt{2E_q}\hat{a}^\dagger_{qr}|0\rangle \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \langle 0|\hat{a}_{ps}\hat{a}^\dagger_{qr}|0\rangle u_{ps}\sqrt{2E_q} e^{-ipx} \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{rs}|0\rangle u_{ps}\sqrt{2E_q} e^{-ipx} \\
&= u_{qr} e^{-ipx} \ .
\end{aligned}$$

For an final fermion, we have

$$\begin{aligned}
&\langle q,r|\hat{\overline{\psi}}(x)|0\rangle = \langle 0|\hat{a}_{qr}\sqrt{2E_q}\hat{\psi}(x)|0\rangle \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sqrt{2E_q}\langle 0|\hat{a}_{qr} \left(\hat{b}^s_p \overline{v}^s_p e^{-ipx} + \hat{a}^{s\dagger}_p \overline{u}^s_p e^{ipx} \right) |0\rangle \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sqrt{2E_q}\langle 0|\hat{a}_{qr}\hat{a}^\dagger_{ps}|0\rangle \overline{u}_{ps} e^{ipx} \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sqrt{2E_q}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{rs}|0\rangle \overline{u}_{ps} e^{ipx} \\
&= \overline{u}_{qr} e^{ipx} \ .
\end{aligned}$$

For an initial antifermion, we have

$$\begin{aligned}
&\langle 0|\hat{\overline{\psi}}(x)|q,r\rangle = \langle 0|\hat{\psi}(x)\sqrt{2E_q}\hat{b}^\dagger_{qr}|0\rangle \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \langle 0| \left(\hat{b}^s_p \overline{v}^s_p e^{-ipx} + \hat{a}^{s\dagger}_p \overline{u}^s_p e^{ipx} \right) \sqrt{2E_q}\hat{b}^\dagger_{qr}|0\rangle \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \langle 0|\hat{b}_{ps}\hat{b}^\dagger_{qr}|0\rangle \overline{v}_{ps}\sqrt{2E_q} e^{-ipx} \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{rs}|0\rangle \overline{v}_{ps}\sqrt{2E_q} e^{-ipx} \\
&= \overline{v}_{qr} e^{-ipx} \ .
\end{aligned}$$

For an final antifermion, we have

$$
\begin{aligned}
\langle q, r|\hat{\psi}(x)|0\rangle &= \langle 0|\hat{b}_{qr}\sqrt{2E_q}\hat{\psi}(x)|0\rangle \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|\hat{b}_{qr}\left(\hat{a}_p^s u_p^s e^{-ipx} + \hat{b}_p^{s\dagger} v_p^s e^{ipx}\right)|0\rangle \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|\hat{b}_{qr}\hat{b}_{ps}^\dagger|0\rangle v_{ps}e^{ipx} \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{rs}|0\rangle v_{ps}e^{ipx} \\
&= v_{qr}e^{ipx} \ .
\end{aligned}
$$

The polarisation factors need to to added in the LSZ formula. q.e.d.

3.3 Photon external line

External photon lines gets a factor of ϵ_μ for an incoming photon and ϵ_μ^* for an outgoing photon.

Proof. Recall that photon fields are expanded in creation and annihilation operators by

$$
\hat{A}^\mu(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(\hat{a}_p^\lambda \epsilon_\lambda^\mu(\mathbf{p})e^{-ipx} + \hat{a}_p^{\lambda\dagger}\epsilon_\lambda^{*\mu}(\mathbf{p})e^{ipx}\right) .
$$

For an initial particle, we have

$$
\begin{aligned}
\langle 0|\hat{A}^\mu(x)|q,\sigma\rangle &= \langle 0|\hat{A}^\mu(x)\sqrt{2E_q}\hat{a}_{q\sigma}^\dagger|0\rangle \\
&= \sum_\lambda \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\langle 0|\left(\hat{a}_{p\lambda}\epsilon_\lambda^\mu(\mathbf{p})e^{-ipx} + \hat{a}_{p\lambda}^\dagger\epsilon_\lambda^{*\mu}(\mathbf{p})e^{ipx}\right)\sqrt{2E_q}\hat{a}_{q\sigma}^\dagger|0\rangle \\
&= \sum_\lambda \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\langle 0|\hat{a}_{p\lambda}\hat{a}_{q\sigma}^\dagger|0\rangle\epsilon_{p\lambda}^\mu\sqrt{2E_q}e^{-ipx} \\
&= \sum_\lambda \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{\lambda\sigma}|0\rangle\sqrt{2E_q}\epsilon_{p\lambda}^\mu e^{-ipx} = \epsilon_{q\sigma}^\mu e^{-iqx} \ .
\end{aligned}
$$

For a final particle, we have

$$
\begin{aligned}
\langle q,\sigma|\hat{A}^\mu(x)|0\rangle &= \langle 0|\hat{a}_{q\sigma}\sqrt{2E_q}\hat{A}^\mu(x)|0\rangle \\
&= \sum_\lambda \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|\hat{a}_{q\sigma}\left(\hat{a}_{p\lambda}\epsilon_\lambda^\mu(\mathbf{p})e^{-ipx} + \hat{a}_{p\lambda}^\dagger\epsilon_\lambda^{*\mu}(\mathbf{p})e^{ipx}\right)|0\rangle \\
&= \sum_\lambda \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|\hat{a}_{q\sigma}\hat{a}_{p\lambda}^\dagger|0\rangle\epsilon_{p\lambda}^{*\mu}e^{ipx} \\
&= \sum_\lambda \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{2E_q}\langle 0|(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{\lambda\sigma}|0\rangle\epsilon_{p\lambda}^{*\mu}e^{ipx} = \epsilon_{q\sigma}^{*\mu}e^{iqx} \ .
\end{aligned}
$$

The polarisation factors need to to added in the LSZ formula. q.e.d.

4 Summary

In this section, we will summarise all the important formulae obtained so far.

The propagators are

1. scalar
$$D_F(x,y) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2-m^2+i\epsilon}\ ,$$

2. fermion
$$D_F(x,y) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2-m^2+i\epsilon}(\not{p}+m)\ ,$$

3. photon (Feynman gauge $\xi = 1$ and Lorentz gauge $\xi = 0$)
$$D_F(x,y) = -i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2+i\epsilon}\left(g_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\ ,$$

The Feynman's rules for scalar field are:

$$\mathcal{L}_{int} = -\frac{\lambda^3}{3!}\phi^3 - \frac{\lambda^4}{4!}\phi^4$$

1. external line gets 1,
2. internal line gets the propagator,
3. vertex gets $i\lambda$.

The Feynman's rules for scalar quantum electrodynamics are:

$$\mathcal{L}_{int} = -ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + e^2A^\mu A_\mu\phi^*\phi$$

1. external line gets 1 for scalar, ϵ_μ for incoming photon and ϵ_μ^* for outgoing photon,
2. internal line gets the propagator,
3. vertex gets $-ie$ times momentum of right-directed arrows minus momentum of left-directed arrows.

The Feynman's rules for Yukawa theory are:

$$\mathcal{L}_{int} = -g\overline{\psi}\phi\psi$$

1. external line gets 1 for scalar, u^s for incoming fermion, $\overline{u}^s$ for outgoing fermion, $\overline{v}^s$ for incoming antifermion and v^s for outgoing antifermion,
2. internal line gets the propagator,
3. vertex gets $-ig$.

The Feynman's rules for quantum electrodynamics are:

$$\mathcal{L}_{int} = -e\overline{\psi}\gamma^\mu\psi A_\mu$$

1. external line gets ϵ_μ for incoming photon, ϵ_μ^* for outgoing photon, u^s for incoming fermion, $\overline{u}^s$ for outgoing fermion, $\overline{v}^s$ for incoming antifermion and v^s for outgoing antifermion,
2. internal line gets the propagator,
3. vertex gets $-ie\gamma^\mu$.

A Useful identities

We need to prove the identity

$$e^{iE_p(y_0-x_0)}\theta(x_0-y_0)+e^{iE_p(x_0-y_0)}\theta(y_0-x_0)=-\frac{2E_p}{2\pi i}\int dp_0\frac{e^{ip_0(x_0-y_0)}}{p_0^2-E_p^2+i\epsilon} \tag{1}$$

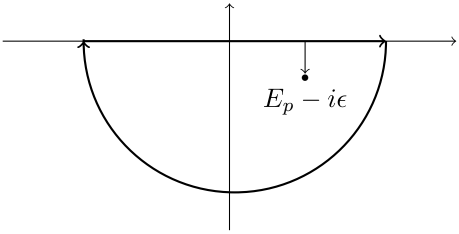
Proof. First, we decompose the product

$$\frac{1}{p_0^2-E_p^2+i\epsilon}=\frac{1}{(p_0-(E_p-i\epsilon))(p_0-(-E_p+i\epsilon))}\ ,$$

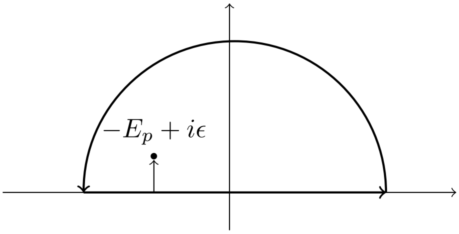
in order to obtain

$$\int dp_0\frac{e^{ip_0(x_0-y_0)}}{p_0^2-E_p^2+i\epsilon}=\int dp_0\frac{e^{ip_0(x_0-y_0)}}{(p_0-(E_p-i\epsilon))(p_0-(-E_p+i\epsilon))}\ .$$

For $x^0<y^0$, we integrate over the countour



whereas for $x^0>y^0$, we integrate over the countour



Therefore, we obtain

$$\begin{aligned}&\int dp_0\frac{e^{ip_0(x_0-y_0)}}{p_0^2-E_p^2+i\epsilon}\\&=2\pi i\Big(-\frac{e^{ip_0(x_0-y_0)}}{2p_0}\Big|_{p_0=E_p}\theta(y_0-x_0)+\frac{e^{ip_0(x_0-y_0)}}{2p_0}\Big|_{p_0=-E_p}\theta(x_0-y_0)\Big)\\&=-\frac{2\pi i}{2E_p}\Big(e^{iE_p(x_0-y_0)}\theta(y_0-x_0)+e^{-iE_p(x_0-y_0)}\theta(x_0-y_0)\Big)\ .\end{aligned}$$

q.e.d.