Quantum Field Theory
3 - scattering processes

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Abstract

In this note, we will study the mathematical devolopment to compute the S-matrix.

Contents

1 S-matrix

In this section, we will define the S-matrix and we will relate its elements to physical quantities, like cross sections and decay rates.

1.1 Transition amplitudes

In quantum mechanics, experimentally measureable quantities are related to transition ampllitudes.

Definition 1.1 (Transition amplitude)
Let $|a\rangle$ be a generic initial state and $|b\rangle$ a generic final state. Then, in the most generic case in which states are not normalised, the probability of the transition between the initial and the final state is given by

$$\mathcal{P}(a \to b) = \frac{|\langle b|a\rangle|^2}{|\langle b|b\rangle|^2 |\langle a|a\rangle|^2} \ .$$

In Schroedinger picture, states depend on time while operators do not.

Definition 1.2 (Transition amplitude in Schroedinger picture)
Let $|i, t_i\rangle$ be a initial state at time t_i, $|f, t_f\rangle$ be a final state at time t_f. Then the probability of the transition between the initial and the final state is

$$\mathcal{P}(i, t_i \to f, t_f) = \frac{|\langle f, t_f|i, t_i\rangle|^2}{|\langle f, t_f|f, t_f\rangle|^2 |\langle i, t_i|i, t_i\rangle|^2} \ .$$

In Heisenberg picture, states are time-independent while operators do not. Braket products in different pictures are related by

$$\langle f, t_f|i, t_i\rangle_S = \langle f|\hat{S}|i\rangle_H \ ,$$

where S is an operator that carries information about time evolution, called the S-matrix.

Definition 1.3 (Transition amplitude in Heisenberg picture)
Let $|i\rangle$ be a initial state, $|f\rangle$ a final state, $\hat{S}$ the time evolution operator. Then the probability of the transition between the initial and the final state is

$$\mathcal{P}(i \to f) = \frac{|\langle f|\hat{S}|i\rangle|^2}{|\langle f|f\rangle|^2 |\langle i|i\rangle|^2} \ .$$

1.2 Cross section

Definition 1.4 (Cross section)
Consider a scattering experiment. Let N_{in} and N_{out} be respectively the number of incoming and outgoing particles, T the time of the experiment, $\Phi = N_{in}|\mathbf{v}|/V$ the flux of the incoming beam, where V is the volume and $\mathbf{v}$ the velocity of the beam. Then the classical cross section is defined by

$$\sigma = \frac{N_{out}}{T\Phi} = \frac{V}{|\mathbf{v}|T} \frac{N_{out}}{N_{in}} \ .$$

Introducing the probability $\mathcal{P} = N_{out}/N_{in}$, its quantum mechanical counterpart is

$$\sigma = \frac{V}{|\mathbf{v}|T}\mathcal{P} = \frac{N_{in}}{T\Phi}\mathcal{P} = \frac{1}{T\Phi}\mathcal{P} \ ,$$

where we have redefined $\Phi = \Phi/N_{in}$ as the normalised one-particle flux. The differential cross section is

$$d\sigma = \frac{V}{|\mathbf{v}|T}d\mathcal{P} \ ,$$

differential with respect to solid angle $d\Omega$ or energy dE. It has the dimension of an area, i.e. $[\sigma] = [L]^2$.

1.3 2 to n process

Consider a scattering experiment in which two incoming particle interact to produce n outgoing particles

$$p_1 + p_2 \to \{p_j\}_{j=1}^{n} \ .$$

In perturbative theory, the S-matrix can be decomposed into

$$\hat{S} = \hat{1} + i\hat{T} \ ,$$

where the identity $\hat{1}$ represents no interactions, i.e. when $|i\rangle = |f\rangle$, and $\hat{T}$ describes deviations from it. Furthermore, since 4-momentum is conserved, we can extract a delta from $\hat{T}$ to obtain

$$i\hat{T} = (2\pi)^4\delta^4(p_1 + p_2 - \textstyle\sum_j p_j)i\hat{\mathcal{M}} \ ,$$

where $\hat{\mathcal{M}}$ is the scattering amplitude.

Theorem 1.1 (Relation between cross section and S-matrix)
In the approximation that interaction happens at finite time, the differential cross section of a $2 \to n$ process is

$$\begin{aligned} d\sigma &= \frac{|\mathcal{M}|^2}{4E_1E_2|\mathbf{v}_2 - \mathbf{v}_1|}d\Pi_n \\ &= \frac{|\mathcal{M}|^2}{4E_1E_2|\mathbf{v}_2 - \mathbf{v}_1|}\prod_j \frac{d^3p_j}{(2\pi)^3 2E_j}(2\pi)^4\delta^4(p_1 + p_2 - \textstyle\sum_j p_j) \ . \end{aligned}$$

Proof. q.e.d.

1.4 2 to 2 scattering

Consider the particular case in which there are two outgoing particles

$$p_1 + p_2 \to p_3 + p_4 \ .$$

In the center of mass frame, the differential cross section is

$$d\sigma = \frac{1}{64\pi^2 E_{cm}}^2\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}|\mathcal{M}|^2 d\Omega \ ,$$

where $|\mathbf{p}_i| = |\mathbf{p}_1| = |\mathbf{p}_2|$ and $|\mathbf{p}_f| = |\mathbf{p}_3| = |\mathbf{p}_4|$.

Proof. q.e.d.

In the rest frame of particle 1, the differential cross section is

$$d\sigma = \frac{1}{64\pi^2 E_{cm}} \left[E_4 + E_3 \left(1 - \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \cos\theta \right) \right]^{-1} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} |\mathcal{M}|^2 d\Omega \ .$$

Proof. q.e.d.

1.5 Decay rates

Definition 1.5 (Decay rate)
Consider a decay experiment. Let $\mathcal{P}$ be the probablity that a particle decays with mean lifetime τ and T the time of the experiment. Then the decay rate is defined by

$$\Gamma = \frac{1}{\tau} = \frac{\mathcal{P}}{T} \ .$$

The differential decay rate is

$$d\Gamma = \frac{1}{T} d\mathcal{P} \ ,$$

differential with respect to solid angle $d\Omega$ or energy dE. It has the dimension of an inverse time, i.e. $[\Gamma] = [T]^{-1}$.

1.6 1 to n process

Consider a decay experiment in which a particle decays to produce n outgoing particles

$$p_1 \to \{p_j\}_{j=1}^n \ .$$

Theorem 1.2 (Relation between decay rate and S-matrix)
In the approximation that interaction happens at finite time, the differential decay rate of a $1 \to n$ process is

$$d\Gamma = \frac{|\mathcal{M}|^2}{2E_1} d\Pi_n = \frac{|\mathcal{M}|^2}{2E_1} \prod_j \frac{d^3 p_j}{(2\pi)^3 2E_j} (2\pi)^4 \delta^4 (p_1 - \textstyle\sum_j p_j) \ .$$

Proof. q.e.d.

Propagators

In this section, we will relate S-matrix elements to time-ordered product of fields applied to interacting vacuum states.

1.7 LSZ reduction formula

Theorem 1.3 (LSZ reduction formula)
In the approximation that interaction happens at finite time, so that initial and final states are (asymptotic) free theory states, the S-matrix is given by

$$\langle f|\hat{S}|i\rangle = i\int dx_1 \exp(-ip_1x_1)(\Box + m^2)\dots i\int dx_1 \exp(ip_nx_n)(\Box + m^2)$$
$$\times \langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle \ ,$$

where $|\Omega\rangle \neq |0\rangle$ is the interacting vacuum, $-i$ in the exponent for initial states, $+i$ in the exponent for final states and T is the time ordering operator which sorts all the operators in order to have time increasing from right to left.

Proof. q.e.d.

1.8 Interaction picture

In Heisenberg picture, the dynamics is governed by the Hamiltonian $\hat{H}$. Fields evolve in time with the Heisenberg equation of motion

$$i\partial_t\hat{\phi}(t,\mathbf{x}) = [\hat{\phi}(t,\mathbf{x}), \hat{H}(t)] \ .$$

Its solution is

$$\hat{\phi}(t,\mathbf{x}) = \hat{S}^\dagger(t,t_0)\hat{\phi}(\mathbf{x})\hat{S}(t,t_0) \ ,$$

where $\hat{S}(t,t_0)$ is the time evolution operator that satisfies the Schroedinger equation

$$i\partial_t\hat{S}(t,t_0) = \hat{H}(t)\hat{S}(t,t_0) \ .$$

Proof. q.e.d.

Now, suppose that the Hamiltonian can be perturbatively decomposed into two pieces

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t) \ ,$$

where $\hat{H}_0$ is exactly solved and $\hat{V}(t)$ is small. In interaction picture, operators evolve with $\hat{H}_0$, so that

$$\hat{\phi}_0(t,\mathbf{x}) = e^{i\hat{H}_0(t-t_0)}\hat{\phi}(\mathbf{x})e^{-i\hat{H}_0(t-t_0)}$$

where t_0 is a time in which Schroedinger and Heisenberg picture field coincide. Therefore

$$\phi(t,\mathbf{x})$$

1.9 Vacuum matrix elements

Theorem 1.4 (Relation between interacting and free vacuum matrix elements)

$$\langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_0(x_1)\dots\phi_0(x_n)\exp(-i\int_{-\infty}^{\infty} dt\ V_I(t))\}|0\rangle}{\langle 0|T\{\exp(-i\int_{-\infty}^{\infty} dt\ V_I(t))\}|0\rangle}$$
$$= \frac{\langle 0|T\{\phi_0(x_1)\dots\phi_0(x_n)\exp(i\int d^4x\ \mathcal{L}_{int}[\phi_0])\}|0\rangle}{\langle 0|T\{\exp(i\int d^4x\ \mathcal{L}_{int}[\phi_0])\}|0\rangle}\ .$$

Proof. q.e.d.

2 Wick's theorem

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