Quantum Field Theory

3 - scattering processes

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Abstract

In this note, we will study all the important scattering processes for ϕ^3, sQED, Yukawa and QED.

Contents

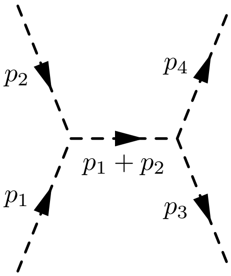
1 Scalar

1.1 2 to 2

Consider for ϕ^3 theory the scattering

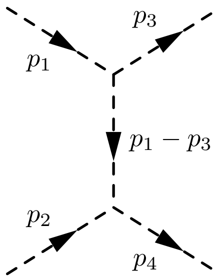
$$\phi\phi \to \phi\phi$$

at tree level. There are three possible Feynman's diagram:



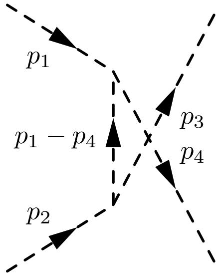
which gives

$$i\mathcal{M} = (-i\lambda)\frac{i}{(p_1+p_2)^2 - m^2}(-i\lambda) = -\frac{i\lambda^2}{s-m^2} \ .$$



which gives

$$i\mathcal{M} = (-i\lambda)\frac{i}{(p_1-p_3)^2 - m^2}(-i\lambda) = -\frac{i\lambda^2}{t-m^2} \ .$$



which gives

$$i\mathcal{M} = (-i\lambda)\frac{i}{(p_1 - p_4)^2 - m^2 + i\epsilon}(-i\lambda) = -\frac{i\lambda^2}{u - m^2} \ .$$

Putting everything together, we find

$$i\mathcal{M} = -i\lambda^2\Big(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2}\Big) \ ,$$

hence

$$|\mathcal{M}|^2 = \lambda^4\Big(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2}\Big)^2 \ .$$

For example, the differential cross section in the center of mass is

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^4}{64\pi^2 s}\Big(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2}\Big)^2 \ .$$

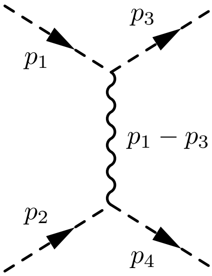
2 sQED

2.1 Moller scattering

Consider for sQED theory the scattering

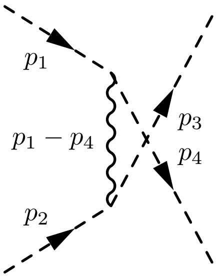
$$\phi\phi \to \phi\phi$$

at tree level. There are two possible Feynman's diagram:



which gives, using Mandelstam's variables properties in (1),

$$\begin{aligned}
i\mathcal{M} &= (-ie)(p_1^\mu + p_3^\mu)\frac{-ig_{\mu\nu}}{(p_1 - p_3)^2}(-ie)(p_2^\nu + p_4^\nu) = \frac{ie^2}{t}(p_1 + p_3)(p_2 + p_4) \\
&= \frac{ie^2}{t}(p_1p_2 + p_1p_4 + p_3p_2 + p_3p_4) \\
&= \frac{ie^2}{t}\Big(\frac{s - m_1^2 - m_2^2}{2} - \frac{u - m_1^2 - m_4^2}{2} - \frac{u - m_2^2 - m_3^2}{2} + \frac{s - m_3^2 - m_4^2}{2}\Big) \\
&= \frac{ie^2}{t}\Big(\frac{s}{2} - \frac{u}{2} - \frac{u}{2} + \frac{s}{2}\Big) = \frac{ie^2}{t}(s - u) \ .
\end{aligned}$$



which gives, using Mandelstam's variables properties in (1),

$$
\begin{aligned}
i\mathcal{M} &= (-ie)(p_1^\mu + p_4^\mu)\frac{-ig_{\mu\nu}}{(p_1-p_4)^2}(-ie)(p_2^\nu + p_3^\nu) = \frac{ie^2}{u}(p_1+p_4)(p_2+p_3) \\
&= \frac{ie^2}{u}(p_1p_2 + p_1p_3 + p_4p_2 + p_4p_3) \\
&= \frac{ie^2}{u}\Big(\frac{s-m_1^2-m_2^2}{2} - \frac{t-m_1^2-m_3^2}{2} - \frac{t-m_2^2-m_4^2}{2} + \frac{s-m_3^2-m_4^2}{2}\Big) \\
&= \frac{ie^2}{u}\Big(\frac{s}{2} - \frac{t}{2} - \frac{t}{2} + \frac{s}{2}\Big) = \frac{ie^2}{u}(s-t)\ .
\end{aligned}
$$

Putting everything together, we find

$$
i\mathcal{M} = ie^2\Big(\frac{s-u}{t} + \frac{s-t}{u}\Big)\ ,
$$

hence

$$
|\mathcal{M}|^2 = e^4\Big(\frac{s-u}{t} + \frac{s-t}{u}\Big)^2\ .
$$

For example, the differential cross section in the center of mass is

$$
\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2}\Big(\frac{s-u}{t} + \frac{s-t}{u}\Big)^2\ .
$$

Crossing Moller scattering

Furthermore, by crossing symmetry $p_2 \mapsto -p_4$ and $p_4 \mapsto -p_2$, so that

$$
\begin{aligned}
s = (p_1+p_2)^2 &\mapsto (p_1-p_4)^2 = u\ , \\
t = (p_1-p_3)^2 &\mapsto (p_1-p_3)^2 = t\ , \\
u = (p_1-p_4)^2 &\mapsto (p_1+p_2)^2 = s\ ,
\end{aligned}
$$

we can evaluate the scattering

$$
\phi\phi^* \to \phi\phi^*
$$

at tree level, which becomes

$$
i\mathcal{M} = ie^2\Big(\frac{u-s}{t} + \frac{u-t}{s}\Big) = -ie^2\Big(\frac{s-u}{t} + \frac{t-u}{s}\Big)\ .
$$

hence

$$
|\mathcal{M}|^2 = e^4\Big(\frac{s-u}{t} + \frac{t-u}{s}\Big)^2
$$

and

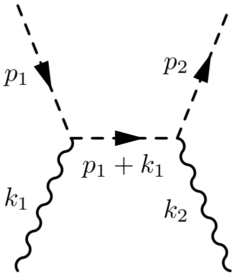
$$
\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2}\Big(\frac{s-u}{t} + \frac{t-u}{s}\Big)^2\ .
$$

2.2 Compton scattering

Consider for sQED theory the scattering

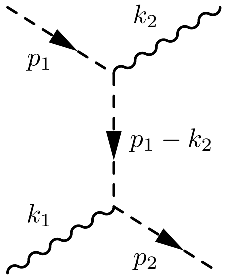
$$\gamma\phi \to \gamma\phi$$

at tree level. There are three possible Feynman's diagram:



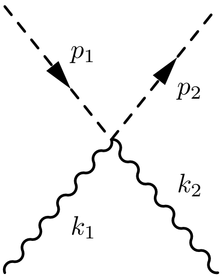
which gives

$$\begin{aligned} i\mathcal{M} &= (-ie)\epsilon_{1\mu}(p_1^\mu + p_1^\mu + k_1^\mu)\frac{i}{(p_1+k_1)^2 - m^2}(-ie)(p_2^\nu + p_2^\nu + k_2^\nu)\epsilon_{2\nu}^* \\ &= -\frac{ie^2}{m^2 + 2p_1k_1 - m^2}(2p_1^\mu + k_1^\mu)(2p_2^\nu + k_2^\nu)\epsilon_{1\mu}\epsilon_{2\nu}^* \\ &= -ie^2\frac{(2p_1^\mu + k_1^\mu)(2p_2^\nu + k_2^\nu)}{2p_1k_1}\epsilon_{1\mu}\epsilon_{2\nu}^* \ . \end{aligned}$$



which gives

$$\begin{aligned} i\mathcal{M} &= (-ie)\epsilon_{1\mu}(p_2^\mu + p_2^\mu - k_1^\mu)\frac{i}{(p_1-k_2)^2 - m^2}(-ie)(p_1^\nu + p_1^\nu - k_2^\nu)\epsilon_{2\nu}^* \\ &= -\frac{ie^2}{m^2 - 2p_1k_2 - m^2}(2p_2^\mu - k_1^\mu)(2p_1^\nu - k_2^\nu)\epsilon_{1\mu}\epsilon_{2\nu}^* \\ &= ie^2\frac{(2p_2^\mu - k_1^\mu)(2p_1^\nu - k_2^\nu)}{2p_1k_2}\epsilon_{1\mu}\epsilon_{2\nu}^* \ . \end{aligned}$$



which gives

$$i\mathcal{M} = 2ie^2 \epsilon_{1\mu} \eta^{\mu\nu} \epsilon^*_{2\nu} \ .$$

Putting everything together, we find

$$\begin{aligned} i\mathcal{M} = &-ie^2 \frac{(2p_1^\mu + k_1^\mu)(2p_2^\nu + k_2^\nu)}{2p_1 k_1} \epsilon_{1\mu} \epsilon^*_{2\nu} \\ &+ ie^2 \frac{(2p_2^\mu - k_1^\mu)(2p_1^\nu - k_2^\nu)}{2p_1 k_2} \epsilon_{1\mu} \epsilon^*_{2\nu} + 2ie^2 \epsilon_{1\mu} \eta^{\mu\nu} \epsilon^*_{2\nu} \\ = &\, ie^2 \Big(- \frac{(2p_1^\mu + k_1^\mu)(2p_2^\nu + k_2^\nu)}{2p_1 k_1} + \frac{(2p_2^\mu - k_1^\mu)(2p_1^\nu - k_2^\nu)}{2p_1 k_2} + 2\eta^{\mu\nu} \Big) \epsilon_{1\mu} \epsilon^*_{2\nu} \ . \end{aligned}$$

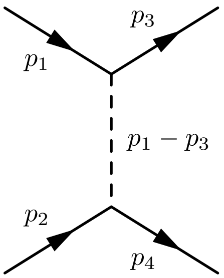
3 Yukawa

3.1 Moller scattering

Consider for Yukawa theory the massless-fermion scattering

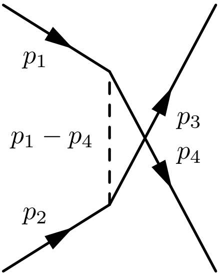
$$\psi\psi \to \psi\psi$$

at tree level. There are two possible Feynman's diagram:



which gives

$$i\mathcal{M} = (-ig) u_1^a \overline{u}_3^a \frac{i}{t - m^2} (-ig) u_2^b \overline{u}_4^b \ .$$



which gives

$$i\mathcal{M} = -(-ig)u_1^c\overline{u}_4^c\frac{i}{u-m^2}(-ig)u_2^d\overline{u}_3^d \ .$$

Putting everything together, we find

$$i\mathcal{M} = -ig^2\Big(\frac{u_1^a\overline{u}_3^a u_2^b\overline{u}_4^b}{t-m^2} - \frac{u_1^c\overline{u}_4^c u_2^d\overline{u}_3^d}{u-m^2}\Big) \ ,$$

$$-i\mathcal{M}^* = ig^2\Big(\frac{\overline{u}_1^e u_3^e\overline{u}_2^f u_4^f}{t-m^2} - \frac{\overline{u}_1^g u_4^g\overline{u}_2^h u_3^h}{u-m^2}\Big) \ ,$$

hence

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{g^4}{4}\sum_{s_1,s_2,s_3,s_4}\Big(\frac{\overline{u}_1^e u_3^e\overline{u}_2^f u_4^f}{t-m^2} + \frac{\overline{u}_1^g u_4^g\overline{u}_2^h u_3^h}{u-m^2}\Big) \times\Big(\frac{u_1^a\overline{u}_3^a u_2^b\overline{u}_4^b}{t-m^2} + \frac{u_1^c\overline{u}_4^c u_2^d\overline{u}_3^d}{u-m^2}\Big) \ .$$

The first term gives, using Mandelstam's variables properties in (1) (massless) and traces properties in (2)

$$\frac{1}{(t-m^2)^2}\sum_{s_1}u_1^a\overline{u}_1^e\sum_{s_2}u_2^b\overline{u}_2^f\sum_{s_3}u_3^e\overline{u}_3^a\sum_{s_4}u_4^f\overline{u}_4^b$$
$$= \frac{1}{(t-m^2)^2}\not{p}_{1ae}\not{p}_{2bf}\not{p}_{3ea}\not{p}_{4fb} = \frac{p_1^\mu p_2^\alpha p_3^\nu p_4^\beta}{(t-m^2)^2}\operatorname{tr}(\gamma^\mu\gamma^\nu)\operatorname{tr}(\gamma^\alpha\gamma^\beta)$$
$$= \frac{p_1^\mu p_2^\alpha p_3^\nu p_4^\beta}{(t-m^2)^2}4\eta^{\mu\nu}4\eta^{\alpha\beta} = \frac{16}{(t-m^2)^2}(p_1\cdot p_3 p_2\cdot p_4) = \frac{4t^2}{(t-m^2)^2} \ .$$

The last term gives, using Mandelstam's variables properties in (1) (massless) and traces properties in (2)

$$\frac{1}{(u-m^2)^2}\sum_{s_1}u_1^c\overline{u}_1^g\sum_{s_2}u_2^d\overline{u}_2^h\sum_{s_3}u_3^h\overline{u}_3^d\sum_{s_4}u_4^g\overline{u}_4^c$$
$$= \frac{1}{(t-m^2)^2}\not{p}_{1cg}\not{p}_{2dh}\not{p}_{3hd}\not{p}_{4gc} = \frac{p_1^\mu p_2^\alpha p_3^\beta p_4^\nu}{(t-m^2)^2}\operatorname{tr}(\gamma^\mu\gamma^\nu)\operatorname{tr}(\gamma^\alpha\gamma^\beta)$$
$$= \frac{p_1^\mu p_2^\alpha p_3^\beta p_4^\nu}{(t-m^2)^2}4\eta^{\mu\nu}4\eta^{\alpha\beta} = \frac{16}{(t-m^2)^2}(p_1\cdot p_4 p_2\cdot p_3) = \frac{4u^2}{(t-m^2)^2} \ .$$

The double term gives, using Mandelstam's variables properties in (1) (massless) and traces properties in (2)

$$
\begin{aligned}
&-\frac{2}{(t-m^2)(u-m^2)}\sum_{s_1} u_1^c\overline{u}_1^e\sum_{s_2} u_2^d\overline{u}_2^f\sum_{s_3} u_3^e\overline{u}_3^d\sum_{s_4} u_4^f\overline{u}_4^c \\
&= -\frac{2}{(t-m^2)(u-m^2)}\not{p}_{1ce}\not{p}_{2df}\not{p}_{3ed}\not{p}_{4fc} = \frac{p_1^\mu p_2^\alpha p_3^\nu p_4^\beta}{(t-m^2)(u-m^2)}\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) \\
&= -\frac{2p_1^\mu p_2^\alpha p_3^\nu p_4^\beta}{(t-m^2)(u-m^2)}4(\eta^{\mu\nu}\eta^{\alpha\beta}-\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}) \\
&= -\frac{8}{(t-m^2)(u-m^2)}(p_1\cdot p_3 p_2\cdot p_4 - p_1\cdot p_2 p_3\cdot p_4 + p_1\cdot p_4 p_2\cdot p_3) \\
&= -\frac{2}{(t-m^2)(u-m^2)}(t^2-s^2+u^2) = \frac{4ut}{(t-m^2)(u-m^2)}\ ,
\end{aligned}
$$

since

$$s+t+u=0\ ,\quad t^2+u^2-s^2=-2ut\ .$$

Hence

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = g^4\Big(\frac{t^2}{(t-m^2)^2}+\frac{u^2}{(u-m^2)^2}+\frac{ut}{(t-m^2)(u-m^2)}\Big)\ .$$

Crossing Moller scattering

Furthermore, by crossing symmetry $p_2\mapsto -p_4$ and $p_4\mapsto -p_2$, so that

$$s=(p_1+p_2)^2\mapsto(p_1-p_4)^2=u\ ,$$

$$t=(p_1-p_3)^2\mapsto(p_1-p_3)^2=t\ ,$$

$$u=(p_1-p_4)^2\mapsto(p_1+p_2)^2=s\ ,$$

we can evaluate the scattering

$$\psi\overline{\psi}\to\psi\overline{\psi}$$

at tree level, which becomes

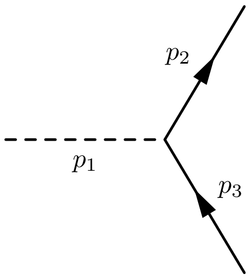
$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = g^4\Big(\frac{t^2}{(t-m^2)^2}+\frac{s^2}{(s-m^2)^2}+\frac{st}{(t-m^2)(s-m^2)}\Big)\ .$$

3.2 Decay

Consider for Yukawa theory the decay

$$\phi\to\psi\overline{\psi}$$

at tree level. There is one Feynman's diagram:



which gives

$$i\mathcal{M} = (-ig)\overline{u}_2^a v_3^a \ .$$

hence, using traces properties in (2)

$$\begin{aligned}
\sum_{s_2,s_3} |\mathcal{M}|^2 &= g^2 \sum_{s_2,s_3} (\overline{u}_2^a u_2^b \overline{v}_3^b v_3^a) = \frac{g^2}{4} (\sum_{s_2} \overline{u}_2^a u_2^b \sum_{s_3} \overline{v}_3^b v_3^a) \\
&= g^2 (\not{p}_2 + m)^{ab} (\not{p}_3 - m)^{ba} = g^2 \operatorname{tr} \Big((\not{p}_2 + m)(\not{p}_3 - m) \Big) \\
&= g^2 \Big(p_2^\mu p_3^\nu \operatorname{tr}(\gamma^\mu \gamma^\nu) - m^2 \operatorname{tr} \mathbb{I} \Big) = g^2 \Big(4 p_2^\mu p_3^\nu \eta^{\mu\nu} - 4m^2 \Big) \\
&= 4g^2 (p_2 p_3 - m^2) = 4g^2 \Big(\frac{M^2}{2} - 2m^2 \Big) = 2M^2 g^2 \Big(1 - 4\frac{m^2}{M^2} \Big) \ ,
\end{aligned}$$

where

$$p_1 = p_2 + p_3 \ , \quad 2p_3 p_4 = M^2 - 2m^2 \ .$$

For example, the differential decay rate is

$$\begin{aligned}
\frac{d\Gamma}{d\Omega} &= \frac{|\mathbf{p}_f|}{32\pi^2 M^2} |\mathcal{M}|^2 = \frac{1}{32\pi^2 M^2} \frac{M}{2} \sqrt{1 - 4\frac{m^2}{M^2}} 2M^2 g^2 \Big(1 - 4\frac{m^2}{M^2} \Big) \\
&= \frac{M}{32\pi^2} \Big(1 - 4\frac{m^2}{M^2} \Big)^{3/2} \ ,
\end{aligned}$$

where

$$|\mathbf{p}_3| = \sqrt{E^2 - m^2} = \sqrt{\frac{M^2}{2} - m^2} = \frac{M}{2} \sqrt{1 - 4\frac{m^2}{M^2}} \ .$$

Hence,

$$\Gamma = \frac{M}{8\pi} \Big(1 - 4\frac{m^2}{M^2} \Big)^{3/2} \ .$$

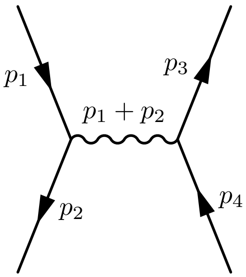
4 QED

4.1 Bhabha scattering

Consider for Yukawa theory the massless-fermion scattering

$$\psi\psi \to \psi\psi$$

at tree level. There is one Feynman's diagram:



which gives

$$i\mathcal{M} = (-ie\gamma^{\mu}_{ab})u^a_1\overline{v}^b_2 \frac{-ig_{\mu\nu}}{s}(-ie\gamma^{\nu}_{cd})\overline{u}^c_3 v^d_4 = -\frac{ie^2}{s}\gamma^{\mu}_{ab}u^a_1\overline{v}^b_2\gamma^{cd}_{\mu}\overline{u}^c_3 v^d_4 \ ,$$

$$-i\mathcal{M}^* = \frac{ie^2}{s}\gamma^{\mu}_{ef}\overline{u}^e_1 v^f_2\gamma^{gh}_{\mu}u^g_3\overline{v}^h_4 \ ,$$

hence, using traces properties in (2)

$$\begin{aligned}
&\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\sum_{s_1,s_2,s_3,s_4}\left(\gamma^{\mu}_{ef}\overline{u}^e_1 v^f_2\gamma^{gh}_{\mu}u^g_3\overline{v}^h_4\right)\left(\gamma^{\nu}_{ab}u^a_1\overline{v}^b_2\gamma^{cd}_{\nu}\overline{u}^c_3 v^d_4\right) \\
&= \frac{e^4}{4s^2}\gamma^{\mu}_{ef}\gamma^{gh}_{\mu}\gamma^{\nu}ab\gamma^{cd}_{\nu}\sum_{s_1}u^a_1\overline{u}^e_1\sum_{s_2}v^f_2\overline{v}^b_2\sum_{s_3}u^g_3\overline{u}^c_3\sum_{s_4}v^d_4\overline{v}^h_4 \\
&= \frac{e^4}{4s^2}\gamma^{\mu}_{ef}\gamma^{gh}_{\mu}\gamma^{\nu}_{ab}\gamma^{cd}_{\nu}\not{p}^{ae}_1\not{p}^{fb}_2\not{p}^{gc}_3\not{p}^{dh}_4 = \frac{e^4}{4s}\,\mathrm{tr}\left(\not{p}_1\gamma^{\mu}\not{p}_2\gamma^{\nu}\right)\mathrm{tr}\left(\not{p}_3\gamma_{\mu}\not{p}_4\gamma_{\nu}\right) \\
&= \frac{e^4}{4s^2}\left(p^{\mu}_1 p^{\alpha}_2\,\mathrm{tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta})\right)\left(p^{\mu}_3 p^{\alpha}_4\,\mathrm{tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta})\right) \\
&= \frac{4e^4}{s^2}p^{\mu}_1 p^{\alpha}_2(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha})p^{\mu}_3 p^{\alpha}_4(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) \\
&= \frac{8e^4}{s^2}(p_1\cdot p_3 p_2\cdot p_4 + p_2\cdot p_3 p_1\cdot p_4) = \frac{2e^4}{s^2}(t^2+u^2) \ .
\end{aligned}$$

Crossing Bhabha scattering

Furthermore, by crossing symmetry $p_2 \mapsto -p_4$ and $p_4 \mapsto -p_2$, so that

$$s = (p_1+p_2)^2 \mapsto (p_1-p_4)^2 = u \ ,$$

$$t = (p_1-p_3)^2 \mapsto (p_1-p_3)^2 = t \ ,$$

$$u = (p_1-p_4)^2 \mapsto (p_1+p_2)^2 = s \ ,$$

we can evaluate the scattering

$$\psi\overline{\psi} \to \psi\overline{\psi}$$

at tree level, which becomes

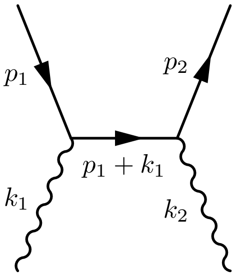
$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{2e^4}{u^2}(t^2+s^2) \ .$$

4.2 Compton scattering

Consider for QED theory the scattering

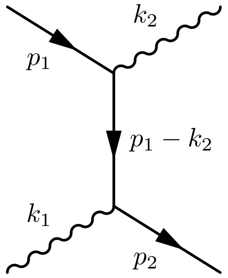
$$\gamma\psi \to \gamma\psi$$

at tree level. There are two possible Feynman's diagram:



which gives

$$\begin{aligned} i\mathcal{M} &= (-ie\gamma^\mu)\overline{u}_1\epsilon_{1\mu}\frac{i(\not{p}_1 + \not{k}_1)}{(p_1+k_1)^2}(-ie\gamma^\nu)u_2\epsilon^*_{2\nu} \\ &= -\frac{ie^2}{s}\epsilon_{1\mu}\epsilon^*_{2\nu}\overline{u}_1 u_2\gamma^\mu\gamma^\nu(\not{p}_1 + \not{p}_2)\ . \end{aligned}$$



which gives

$$\begin{aligned} i\mathcal{M} &= (-ie\gamma^\mu)\overline{u}_1\epsilon_{1\mu}\frac{i(\not{p}_1 - \not{k}_2)}{(p_1-k_2)^2}(-ie\gamma^\nu)u_2\epsilon^*_{2\nu} \\ &= -\frac{ie^2}{t}\epsilon_{1\mu}\epsilon^*_{2\nu}\overline{u}_1 u_2\gamma^\mu\gamma^\nu(\not{p}_1 - \not{k}_2)\ . \end{aligned}$$

Putting everything together, we find

$$i\mathcal{M} = \ .$$

A Useful identities

$$
\begin{aligned}
p_1p_2 &= \frac{s - m_1^2 - m_2^2}{2} = p_3p_4 = \frac{s - m_3^2 - m_4^2}{2} \ , \\
p_1p_3 &= -\frac{t - m_1^2 - m_3^2}{2} = p_2p_4 = -\frac{t - m_2^2 - m_4^2}{2} \ , \\
p_1p_4 &= -\frac{u - m_1^2 - m_4^2}{2} = p_2p_3 = -\frac{u - m_2^2 - m_3^2}{2} \ .
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
&\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \\
&\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) \ ,
\end{aligned}
\tag{2}
$$

We can express Mandelstam's variable in center of mass frame.

References

[1] T. Peraro. *Lecture notes taken during the quantum field theory 2 course.*

[2] M. Schwartz. *Quantum Field Theory and the Standard Model.*