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On mathematics for physics:

logic and what follows from that

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Part I

Logic and set theory

Chapter 1

Logic

In this chapter, we will study propositional and predicate logic. Furthermore, we will introduce axiomatic systems.

1.1 Propositional logic

Definition 1.1
A proposition is a variable that can take true T or false F values.

Logical operators build new propositions from old ones. There are four unary operators (Table 1.1) and sixteen binary operators (Tables 1.2 1.3 1.4).

p	$\neg p$	$\operatorname{id} p$	$\top p$	$\bot p$
T	F	T	T	F
F	T	F	T	F

Table 1.1: All four possible unary operators: identity id, negation $\neg$, tautology $\top$, contradiction $\bot$.

p	q	$\operatorname{id} p$	$\operatorname{id} q$	$\neg p$	$\neg q$	$\top pq$	$\bot qp$
T	T	T	T	F	F	T	F
T	F	T	F	F	T	T	F
F	T	F	T	T	F	T	F
F	F	F	F	T	T	T	F

Table 1.2: Six of all possible binary operators: identity id, negation $\neg$, tautology $\top$, contradiction $\bot$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee q$	$\neg(p \vee q)$	$p \veebar q$	$\neg(p \veebar q)$
T	T	T	F	T	F	F	T
T	F	F	T	T	F	T	F
F	T	F	T	T	F	T	F
F	F	F	T	F	T	F	T

Table 1.3: Other six of all possible binary operators: and $\wedge$, or $\vee$, xor $\veebar$.

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$p \Leftarrow q$	$\neg(p \Leftarrow q)$
T	T	T	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	F	T	F

Table 1.4: Remaining four of all possible binary operators: implications $\Rightarrow$ and $\Leftarrow$.

Theorem 1.1
All operators can be built from the nand operator.

Theorem 1.2
Let p, q be propositions. Then $p \Rightarrow q$ is equivalent to $\neg p \Rightarrow \neg q$.

Proof. It can be simply proved by looking at the truth table: q.e.d.

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$	$p \Rightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Theorem 1.3
Let p, q be propositions. Then $p \Leftrightarrow q$ is equivalent to $\neg(p \veebar q)$.

Proof. It can be simply proved by looking at the truth table: q.e.d.

p	q	$\neg(p \veebar q)$	$p \Leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

1.2 Predicate logic

Definition 1.2
A predicate $P(x)$ is a function of propositions. A predicate of two variables is called a relation.

Definition 1.3
Let $P(x)$ a predicate. Then the universal quantifier $\forall x\colon P(x)$ means that it is true if $P(x)$ is independent of x.

Definition 1.4
Let $P(x)$ a predicate. Then the existential quantifier $\exists x\colon P(x)$ is defined by

$$\exists x\colon P(x) \Leftrightarrow \neg(\forall x\colon \neg P(x)) \ .$$

Definition 1.5
Let $P(x)$ a predicate. Then the unique existential quantifier $\exists ! x\colon P(x)$ is defined by

$$\exists ! x\colon P(x) \Leftrightarrow \exists x\colon \forall y\colon (P(y) \Leftrightarrow x = y) \ .$$

1.3 Axiomatic systems

Definition 1.6
A sequence of propositions (axioms) $a_1, \dots a_N$ is called an axiomatic system.

Definition 1.7
Let $a_1, \dots a_N$ be an axiomatic system. Then the proof of a proposition p is a finite sequence of propositions $q_1, \dots q_M = p$ such that either

1. *q_i is an axiom,*
2. *q_i a tautology or*
3. *$\exists n, m \in [1, i]$ such that $q_m \wedge q_n \Rightarrow q_i$ is true.*

In formula

$$a_1, \dots a_N \vdash p \ .$$

Propositional logic is an axiomatic system without axioms (all propositions are tautologies).

Definition 1.8
An axiomatic system is consistent if

$$\exists q\colon \neg(a_1, \ldots a_N \vdash q) \ .$$

Intuitively, it means that there is no contradiction, since it is possible to prove anything from contradictions.

Proof. Suppose there is a contradiction in the axioms $a_1, \ldots a, \neg a, \ldots a_N$. Therefore, an arbitrary proposition q can be proven by a, $\neg a$ and q. The first two steps are axioms and the last follows from the tautology $(a \wedge \neg a) \Rightarrow q$. q.e.d.

Theorem 1.4
Propositional logic is consistent.

Proof. Propositional logic permits proving only tautologies. Indeed, there are no axioms, so in the proof only tautologies are allowed (even the third possibilities (1.7) imply tautologies because q_m and q_n are so). Therefore, given a proposition q, we cannot prove the contradiction $q \wedge \neg q$. q.e.d.

We can now state Godel's theorem.

Theorem 1.5
Any axiomatic system that contains elementary arithmetic is either inconsistent or there is a proposition that cannot be proven or disproven.

The continuum hypothesis (there is no set with cardinality between $\mathbb{Z}$ and $\mathbb{R}$) is the proposition for the Zermelo-Fraenkel axiomatic system.

Chapter 2

Set theory

In this chapter, we will study the ZFC (Zermelo-Fraenkel-Choice) axiomatic system. Suppose we have a relation ϵ, called belonging, such that

1. $x \in y$ is equivalent to $\epsilon(x, y)$,
2. $x \notin y$ is equivalent to $\neg(x \in y)$,
3. $x \subseteq y$ is equivalent to $\forall a\colon (a \in x \Rightarrow a \in y)$,
4. $x = y$ is equivalent to $(x \subseteq y) \wedge (y \subseteq x)$,
5. $x \subset y$ is equivalent to $(x \subseteq y) \wedge \neg(y = x)$.

2.1 Axiom of belonging relation

Axiom 2.1 (Belonging relation)
$x \in y$ is a proposition $\Leftrightarrow$ both x, y are sets. In formula

$$\forall x\colon \forall y\colon (x \in y) \veebar \neg(x \in y) \ .$$

Intuitively, if $x \in y$ is not true, x, y are not sets, because the xor in this case is a tautology (see Table (2.1)).

The first axiom defines what is a set. To better understand this, consider the Russell's paradox: suppose we have a set that contains all the set that are not contained in themselves

$$\exists u\colon \forall x\colon (x \notin x \Leftrightarrow x \in u) \ .$$

For reduction ad absurdum, suppose $u \in u$ is true. Therefore, $\neg(u \notin u)$ is true and u contains itself, but we have a contradiction

$$u \in u \Rightarrow \neg(u \in u) \ .$$

p	$\neg p$	$p \veebar \neg p$
T	F	T
T	F	T
F	T	T
F	T	T

Table 2.1: Xor binary operator for the first axiom, where $p = x \in y$.

On the other hand, suppose $u \notin u$ is true. Therefore, $\neg(u \in u)$ is true and u does not contain itself, but we have a contradiction

$$u \notin u \Rightarrow \neg(u \notin u) \ .$$

Therefore, $u \in u$ is not a proposition and u is not a set.

Sometimes, the first axiom is replaced by the axiom of extensionality.

Axiom 2.2 (Extensionality)
Two sets are equal if they have the same elements. In formula

$$\forall x\colon \forall y\colon \forall z\colon (z \in x \Leftrightarrow z \in y) \Rightarrow x = y \ .$$

2.2 Axiom of empty set

Axiom 2.3 (Empty set)
There exists an empty set. In formula

$$\exists y\colon \forall x\colon x \notin y \ .$$

Theorem 2.1
The empty set is unique.

Proof. Suppose we have two empty sets x and x', in axioms

$$a_1 \Leftrightarrow \forall y\colon u \notin x \ , \quad a_2 \Leftrightarrow \forall y\colon u \notin x' \ .$$

We begin with a tautology

$$q_1 \Leftrightarrow y \notin x \Rightarrow \forall y\colon (y \in x \Rightarrow u \in x') \ ,$$

then using a_1, we have

$$q_2 \Leftrightarrow \forall y\colon y \notin x \ ,$$

and by the previous propositions q_1 and q_2

$$q_3 \Leftrightarrow (\forall y \colon (y \in x \Rightarrow u \in x')) \Leftrightarrow x \subseteq x' \ .$$

Similarly,

$$q_4 \Leftrightarrow y \notin x' \Rightarrow \forall y \colon (y \in x' \Rightarrow u \in x) \ ,$$

then using a_1, we have

$$q_5 \Leftrightarrow \forall y \colon y \notin x' \ ,$$

and by the previous propositions q_4 and q_5

$$q_3 \Leftrightarrow (\forall y \colon (y \in x' \Rightarrow u \in x)) \Leftrightarrow x' \subseteq x \ .$$

Finally, using q_3 and q_6

$$q_7 \Leftrightarrow ((x \subseteq x') \wedge (x' \subseteq x)) \Leftrightarrow x = x' \ .$$

q.e.d.

2.3 Axiom of pair sets

Axiom 2.4 (Pair sets)
Let x, y be sets. Then there exists the pair set $\{x, y\}$ containing x and y. In formula

$$\forall x \colon \forall y \colon \exists z \colon \forall u \colon (u \in z \Leftrightarrow (u = x \vee u = y)) \ .$$

Theorem 2.2
The pair is unordered, i.e. $\{x, y\} = \{y, x\}$.

Proof. In fact, by definition

$$(a \in \{x, y\} \Rightarrow a \in \{y, x\}) \wedge (a \in \{y, x\} \Rightarrow a \in \{x, y\}) \ ,$$

hence,

$$(\{x, y\} \subseteq \{y, x\}) \wedge (\{y, x\} \subseteq \{x, y\}) \ ,$$

thus $\{x, y\} = \{y, x\}$. q.e.d.

It is possible to define an ordered pair $(x, y) = \{x, \{x, y\}\}$ by

$$(x, y) = (a, b) \Leftrightarrow x = a \wedge y = b \ .$$

Furthermore, the single element set is $\{x\} = \{x, x\}$.

2.4 Axiom of union set

Axiom 2.5 (Union sets)
Let x, y be sets. Then there exists the union set $\cup x$ containing the elements of elements of x. In formula

$$\forall x \colon \exists \cup x \colon \forall y \colon (y \in \cup x \Leftrightarrow \exists s \colon (y \in s \wedge s \in x)) \ .$$

Given $x_1, \dots x_N$ sets, we defined the union set to x_{N+1} as

$$\{x_1, \dots x_{N+1}\} = \bigcup \{\{a_1, \dots a_N\}, \{a_{N+1}\}\} \ .$$

It is a set by the pair set axiom. In the simple case of two sets a, b, we have $\cup x = \{a, b\}$.

Notice that we can only take union of sets, so that the Russell's paradox is left out.

2.5 Axiom of replacement

Definition 2.1
A relation f is a function if

$$\forall x \colon \exists ! y \colon f(x, y) \ .$$

Definition 2.2
The image of f of a set u consists of y such that there is an $x \in u$ for which $f(x, y)$.

Axiom 2.6 (Replacement)
Let x be a set and f a function. Then the image $\mathrm{im}_f(u)$ of u under f is a set.

It exists a weaker form of this axiom, called the principle of restricted comprehension.

Theorem 2.3
Let $P(x)$ be a predicate and u a set. Then the elements $y \in u$ such that $P(y)$ is true are a set $\{y \in u \colon P(y)\}$.

Proof. Let us distinguish two cases. If $\neg(\exists y \in u \colon P(y))$, then $\{y \in m \colon P(y)\} = \emptyset$. If $\exists \tilde{y} \in u \colon P(\tilde{y})$, then we define $f(x, y) = (P(x) \wedge x = y) \vee (\neg P(x) \wedge \tilde{y} = y)$ and $\{y \in m \colon P(y)\} = \mathrm{im}_f(u)$. q.e.d.

The action of P can be written as

1. $\forall xy\colon \in P(x)$ is equivalent to $\forall x\colon (x \in y \Rightarrow P(x))$,
2. $\exists x \in y\colon \in P(x)$ is equivalent to $\neg(\forall x\colon (x \in y \Rightarrow \neg P(x)))$, or $\exists x\colon (x \in y \wedge P(x))$.

Definition 2.3
Let x be a set. Then the intersection of x is

$$\cap x = \{a \in \cup x\colon \forall b \in x\colon a \in b\}\ .$$

If $a, b inx$ and $\cap x = \emptyset$, then a, b are disjoint.

Definition 2.4
Let u, v be sets such that $u \subseteq v$. Then the complement of u of v is

$$u \backslash v = \{x \in v\colon x \notin u\}\ .$$

They are both sets by the axiom of replacement.

2.6 Axiom of power sets

Axiom 2.7 (Power sets)
Let u be a set. Then there exists a set $\mathcal{P}(u)$ composed by the subsets of u. In formula

$$\forall x\colon \exists y\colon \forall a\colon (a \in y \Leftrightarrow a \subseteq x)\ .$$

Definition 2.5
The Cartesian product of two sets x, y is the set of all ordered pairs of elements of x, y

$$x \times y \subseteq \mathcal{P}(\mathcal{P}(\cup\{x, y\}))\ .$$

It is a set for the axioms of union, pair set, replacement and power set.

2.7 Axiom of infinity

Axiom 2.8 (Infinity)
There exists a set that contains the empty set and the set itself. In formula

$$\exists x\colon \emptyset \in x \wedge \forall y\colon (y \in x \Rightarrow \{y\} \in x)\ .$$

Theorem 2.4
$\mathbb{N}$ is a set.

Proof. Suppose x is such set. Therefore, $0 = \emptyset \in x$, $1 = \{\emptyset\} \in x$, $= \{\{\emptyset\}\} \in x$ and so on. q.e.d.

$\mathbb{R} = \mathcal{P}(\mathbb{N})$ is also a set.
Another version of this axiom is

Axiom 2.9 (Infinity 2)
There exists a set that contains the empty set and $y \cup \{y\} = \cup\{y, \{y\}\}$, where y is one of its element.

Accordin with this axiom, we have $\mathbb{N} = \{\emptyset, \{\emptyset\}, \{\emptyset\{\emptyset\}\} \dots\}$.

2.8 Axiom of choice

Axiom 2.10 (Choice)
Let x be a set such that it is not empty and its element are mutually disjoint. Then there exists a set y containing exactly one element of each of x. In formula

$$\forall x\colon P(x) \Rightarrow \exists y\colon \forall a \in x\colon \exists! b \in a\colon a \in y \ ,$$

where

$$P(x) \Leftrightarrow (\exists a\colon a \in x) \wedge (\forall a\colon \forall b\colon (a \in x \wedge b \in x) \Rightarrow \cap\{a, b\} = \emptyset) \ .$$

The axiom of choice can be used to prove that every vector space has a basis and that there exists a complete system of representatives of an equivalence relation.

2.9 Axiom of foundation

Axiom 2.11 (Foundation)
Every non-empty set x contains an element y such that it has no elements in common with x. In formula

$$\forall x\colon (\exists a\colon a \in x) \Rightarrow \exists y \in x\colon \ \cap \{x, y\} = \emptyset \ .$$

2.10 ZFC set theory

To summarise, the Zermelo-Fraenkel-Choice axiomatic systems for set theory is

1. belonging relation, i.e. $\forall x\colon \forall y\colon (x \in y) \veebar \neg(x \in y)$,
2. empty set, i.e. $\exists y\colon \forall x\colon x \notin y$,
3. pair set, i.e. $\forall x\colon \forall y\colon \exists z\colon \forall u\colon (u \in z \Leftrightarrow (u = x \vee u = y))$,
4. union set, i.e. $\forall x\colon \exists \cup x\colon \forall y\colon (y \in \cup x \Leftrightarrow \exists s\colon (y \in s \wedge s \in x))$,
5. replacement, i.e. the image $\mathrm{im}_f(u)$ of u under f is a set,
6. power set, i.e. $\forall x\colon \exists y\colon \forall a\colon (a \in y \Leftrightarrow a \subseteq x)$,
7. infinity, i.e. $\exists x\colon \emptyset \in x \wedge \forall y\colon (y \in x \Rightarrow \{y\} \in x)$,
8. choice, i.e. $\forall x\colon P(x) \Rightarrow \exists y\colon \forall a \in x\colon \exists! b \in a\colon a \in y$, where $P(x) \Leftrightarrow (\exists a\colon a \in x) \wedge (\forall a\colon \forall b\colon (a \in x \wedge b \in x) \Rightarrow \cap\{a, b\} = \emptyset)$,
9. foundation, i.e. $\forall x\colon (\exists a\colon a \in x) \Rightarrow \exists y \in x\colon \cap \{x, y\} = \emptyset$.

Chapter 3

Numbers

In this chapter, we will introduce the notion of map between sets and equivalence relations in order to find a construction of number sets: naturals, integers, rationals, reals.

3.1 Maps

Definition 3.1
Let A, B be sets. Then a map $\phi\colon A \to B$ is a relation such that

$$\forall a \in A\colon \exists! b \in B\colon \phi(a,b)\ .$$

The set A is called domain of ϕ, B is called target of ϕ and $\phi(A) = \mathrm{im}_\phi(A)$ is called the image of A under ϕ.

Definition 3.2
Let $\phi\colon A \to B$ be a map. Then ϕ is injective if

$$\forall x, y \in A\colon \phi(x) = \phi(y) \Rightarrow x = y\ ,$$

surjective if

$$\mathrm{im}_\phi(A) = B\ ,$$

bijective if it is both injective and surjective.

Definition 3.3
Two sets A, B are isomorphic $A \simeq B$ if there exists a bijection $\phi\colon A \to B$.

Definition 3.4
Let A be a set. Then it is infinite if

$$\exists B \subset A\colon B \simeq A\ ,$$

in particular, it is countably infinite if $A \simeq \mathbb{N}$, otherwise uncountably infinite. It is finite if it is not infinite, where $A \simeq \{1, 2, \ldots N\}$ and N is the cardinality of A.

Definition 3.5
The composition of two maps $\phi \colon A \to B$ and $\psi \colon B \to C$ is defined by

$$\begin{tikzcd} & A \arrow[dl, "\phi"'] \arrow[dr, "\phi\circ\psi"] & \\ B \arrow[rr, "\phi"'] & & C \end{tikzcd}$$

Theorem 3.1
Let $\phi \colon A \to B$, $\psi \colon B \to C$ and $\rho \colon C \to D$ maps. Then the composition of maps is associative.

Proof. In fact,

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho\circ\psi\circ\phi"'] \arrow[dr, "\psi\circ\phi"'] & B \arrow[d, "\psi"] \arrow[dl, "\rho\circ\psi"] \\ C & D \arrow[l, "\rho"] \end{tikzcd}$$

q.e.d.

Definition 3.6
Let $\phi \colon A \to B$ be a bijection. Then the inverse ϕ^{-1} of ϕ is defined by

$$\phi^{-1} \circ \phi = \mathrm{id}_A \quad , \quad \phi \circ \phi^{-1} = \mathrm{id}_B \quad .$$

3.2 Equivalence relation

Definition 3.7
Let A be a set and $\sim$ a relation, called an equivalence relation, such that it satisfies the following properties

1. *reflexivity, i.e. $\forall x \in A \colon x \sim x$,*
2. *symmetry, i.e. $\forall x, y \in A \colon x \sim y \Leftrightarrow y \sim x$,*
3. *transitivity, i.e. $\forall x, y, z \in A \colon (x \sim y \wedge y \sim z) \Rightarrow x \sim z$.*

Definition 3.8
Let $\sim$ be an equivalence relation and $x \in A$. Then the equivalence class of x is defined by
$$[x] = \{y \in A \colon x \sim y\} \ .$$

Definition 3.9
Let $\sim$ be an equivalence relation. Then the quotient set of A by $\sim$ is

$$A/\sim = \{[x] \in \mathcal{P}(A) \colon x \in A\} \ .$$

It is a set by the power set axiom. Due to the choice axiom, there exists a complete system of representatives, i.e. a set $R \simeq M/\sim$.

For example, we take $A = \mathbb{Z}$ and define

$$x \sim y \Leftrightarrow n - m \in 2\mathbb{Z} \ .$$

Notice that
$$[0] = [2] = [-2] = \dots \ , \quad [1] = [3] = [-1] = \dots \ .$$
Therefore, $\mathbb{Z}/\sim = \{[0], [2]\}$. Furthermore, we define the addition

$$\oplus \colon \mathbb{Z}/\sim \times \mathbb{Z} \to \mathbb{Z}/\sim \ , \quad [x] \oplus [y] = [x + y] \ .$$

We need to check that it is independent by the choice

$$[x] \oplus [y] = [x'] \oplus [y'] \ .$$

In fact, $[x] = [x']$ means $x - x' = 2n$ and $[y] = [y']$ means $y - y' = 2m$, with $n, m \in \mathbb{Z}$. Hence,

$$[x' + y'] = [x - 2n + y - 2m] = [x + y - 2(n + m)] \ .$$

Now, by definition

$$x + y - 2(n + m) - (x + y) = -2(n + m) \in 2\mathbb{Z} \ ,$$

implies that
$$[x' + y'] = [x + y] \ .$$

3.3 Number sets