Matteo Zandi

# On quantum field theory I:

second quantisation and all that

November 6, 2023

Theoretical Physics

# Contents

Page

# Part I

Classical field theory

# Chapter 1

# Action

A field is a physical quantity $\phi(t, \mathbf{x})$ which is defined ad every point in spacetime. The dynamics of a field is governed by an action, which is a functional that associates a real number to each field configuration for a fixed time interval $[t_1, t_2]$

$$S[\phi_i(x), \partial_\mu \phi_i(x)] = \int_{t_1}^{t_2} dt\ L = \int_{t_1}^{t_2} dt\ \int d^3x\ \mathcal{L} = \int d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i)\ , \tag{1.1}$$

where $\mathcal{L}$ is the lagrangian density, defined by

$$L = \int d^3x\ \mathcal{L}\ .$$

In natural units, the dimensional analysis is

$$[S] = 0\ [d^4x] = -4\ [\mathcal{L}] = 4\ .$$

## 1.1 The principle of stationary action

The dynamics of the system can be determined by the principle of stationary action.

**Principle 1.1**

*The system evolve from an initial configuration at time $t_1$ to a final configuration at time $t_2$ along a path in configuration space which extremises the action (1.1), i.e.*

$$\delta S = 0\ . \tag{1.2}$$

*with the additional conditions*

1. *fields vanish at spatial infinity*

$$\phi_i(t, \mathbf{x}) \to 0 \quad |\mathbf{x}| \to \infty\ ,$$

*hence*

$$\delta \phi_i(t, \infty) = 0\ , \tag{1.3}$$

2. *fields vanish at time extremes*

$$\delta\phi_i(t_1, \mathbf{x}) = \delta\phi_i(t_2, \mathbf{x}) = 0 \ . \tag{1.4}$$

The equation of motion of the system are the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} = 0 \ . \tag{1.5}$$

*Proof.* The variation of the action is

$$\delta S = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\partial_\mu \phi_i \right) = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial_\mu \delta\phi_i \right) \ ,$$

where

$$\delta\phi_i = {\phi'}_i(x) - \phi_i(x) \ ,$$

and

$$\delta\partial_\mu\phi_i(x) = \partial_\mu {\phi'}_i - \partial_\mu \phi(x) = \partial_\mu({\phi'}_i(x) - \phi_i(x)) = \partial_\mu \delta\phi(x) \ .$$

By integration by parts, we obtain

$$\delta S = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial_\mu \delta\phi_i \right) = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) + \int d^4x \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) .$$

Notice that the last term is a total derivative and it vanishes at the boundary by the condition (1.4) and (1.3)

$$\int d^4x \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) = 0 \ .$$

Hence, we find

$$\delta S = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) \ ,$$

and, by the principle of stationary action (1.2)

$$\int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) = 0 \ .$$

Finally, since $\delta_i\phi$ is arbitrary, we obtain (1.5)

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} = 0 \ .$$

q.e.d.

In order to quantise the theory, we need the hamiltonian formalism.

**Definition 1.1**
*The conjugate field $\phi^i(x)$ associated to the field $\phi_i$ is*

$$\phi^i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(x)}$$

The hamiltonian density is given by the Legendre transformation

$$\mathcal{H} = \Phi^i \dot{\phi}_i - \mathcal{L}$$

where the hamiltonian is

$$H = \int d^3x \; \mathcal{H}$$

# Chapter 2

# Noether's theorem

Symmetries are fundamental in quantum field theory and they can be classified into

1. spacetime
   a) global
      i. continuous (Poincarè)
      ii. discrete (Parity, time reversal)
   b) local
      i. continuous (General relativity)
      ii. discrete (Parity coordinate dependent)
2. internal
   a) global
      i. continuous (Flavour)
      ii. discrete ($\mathbb{Z}_2$)
   b) local
      i. continuous ($SU(3) \times SU(2) \times U(1)$)
      ii. discrete ($\mathbb{Z}_2(x)$)

Through the Noether's theorem, we can associate conserved quantities to continuous symmetries.

**Theorem 2.1** (Noether's)
*Every continuous symmetry $\delta\phi_i$ of the action (1.1) give rise to a conserved current*

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi - K^\mu \tag{2.1}$$

*such that it satisfies a continuity equation*

$$\partial_\mu J^\mu = 0 \tag{2.2}$$

*Proof.* We consider an infinitesimal transformation for a continuous symmetry of the system

$$\phi'_i = \phi_i + \delta\phi_i$$

which induces a transformation of the lagrangian

$$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$$

In order to be a symmetry of the system, we require that the action is not invariant, but we allow to be up to a boundary term $K^\mu(\phi_i)$, because the dynamics of the system, i.e. the equations of tmotion, do not change with a boundary term. Hence

$$S' = S + \int \partial_\mu K^\mu(\phi_i)$$

but

$$\delta S = \int \partial_\mu K^\mu(\phi_i) \tag{2.3}$$

Explicitly, we obtain

$$\begin{aligned}
\delta S &= \delta \int d^4x \ \mathcal{L} \\
&= \int d^4x \ \Big( \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\partial_\mu\phi_i \Big) \\
&= \int d^4x \ \Big( \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\partial_\mu\delta\phi_i \Big) \\
&= \int d^4x \ \Big( \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i - \partial_\mu\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\phi_i \Big) + \int d^4x \ \partial_\mu\Big( \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i \Big) \\
&= \int d^4x \ \delta\phi_i \underbrace{\Big( \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \Big)}_{0} + \int d^4x \ \partial_\mu\Big( \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i \Big) \\
&= \int d^4x \ \partial_\mu\Big( \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i \Big)
\end{aligned}$$

where we used the fact that partial derivatives and symmetries commute, the equation of motions (1.5) and we integrated by parts. Hence, by requiring that it is a symmetry

$$\delta S = \int d^4x \ \partial_\mu\Big( \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i \Big) = \int d^4x \partial_\mu K^\mu$$

or equivalently

$$\int d^4x\ \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi_i - K^\mu\right) = 0$$

Since it is for arbitrary integration, the integrand vanishes and

$$\partial_\mu J^\mu = 0$$

with

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i - K^\mu$$

q.e.d.

Notice that every conserved current can be related to a conserved quantity $Q$ by

$$Q = \int_{\mathbb{R}^3} d^3x\ J^0$$

This means that $Q$ is conserved locally, i.e. any charge carrier leaving a finite volume $V$ is associated to a flow of current $\mathbf{J}$ out of the volume.

*Proof.* Infact, by using (2.2)

$$\begin{aligned}\frac{dQ}{dt} &= \frac{d}{dt}\int_{\mathbb{R}^3} d^3x\ J^0\\ &= \int_{\mathbb{R}^3} d^3x\ \frac{\partial J^0}{\partial t}\\ &= -\int_{\mathbb{R}^3} d^3x\ \nabla\cdot\mathbf{J} = 0 = -\int_{\partial\mathbb{R}^3} d\mathbf{S}\cdot\mathbf{J} = 0\end{aligned}$$

where we used the Stoke's theorem and the fact that $\mathbf{J}\to 0$ for $|\mathbf{x}|\to 0$. q.e.d.

# Chapter 3

# Energy-momentum tensor

Spacetime translations give rise to 4 conserved currents, which corresponds to the conservation of energy and momentum. Infact, we consider an infinitesimal spacetime translation

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu}$$

such that fields change by

$$\phi'_i = \phi_i(x+\epsilon) = \phi(x) + \epsilon^{\mu}\partial_{\mu}\phi_i(x)$$

We considered an active transformation, where there is not a change of frame but fields themselves are indeed translated into new fields such that

$$\phi'_i(x') = \phi(x) = \phi(x'+\epsilon)$$

A passive transformation would have acted as

$$\phi'_i = \phi_i(x-\epsilon)$$

Since the lagrangian is a function of the coordinates via fields, we have the following transformation

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \epsilon^{\mu}\partial_{\mu}\mathcal{L} = \epsilon^{\mu}\partial_{\nu}(\delta^{\nu}{}_{\mu}\mathcal{L})$$

Hence, the boundary term is

$$K^{\mu} = \delta^{\mu}{}_{\nu}\mathcal{L}$$

We apply the Noether's theorem (2.1) and find 4 different conserved currents labelled by $\nu$

$$(J^{\mu})_{\nu} = \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_i}\partial_{\nu}\phi_i - \delta^{\mu}{}_{\nu}\mathcal{L}$$

and we define the energy-momentum tensor, or stress-energy tensor,

$$T^{\mu}{}_{\nu} = (J^{\mu})_{\nu}$$

such that

$$\partial_\mu T^\mu{}_\nu = 0$$

In natural units, the dimensional analysis is

$$T^\mu{}_\nu = [\mathcal{L}] = 4$$

The 4 conserved charges are

$$Q_\nu = \int_{\mathbb{R}^3} d^3x\ (J^0)_\nu = \int_{\mathbb{R}^3} d^3x\ T^0{}_\nu$$

which correspond to the 4-momentum

$$P^\mu = \int_{\mathbb{R}^3} d^3x\ T^{0\mu}$$

In particular, the 0-th component is the energy

$$\begin{aligned} P^0 &= \int d^3x\ T^{00} \\ &= \int d^3x\ \Big( \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial^0 \phi_i - \delta^{00} \mathcal{L} \Big) \\ &= \int d^3x\ \Big( \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}}_{\pi^i} \dot{\phi}_i - \mathcal{L} \Big) = \int d^3x\ (\underbrace{\pi^i \dot{\phi}_i - \mathcal{L}}_{\mathcal{L}}) = \int d^3x\ \mathcal{H} = H \end{aligned}$$

such that

$$\frac{dH}{dt} = 0$$

and the $j$-th components are the momentum

$$\begin{aligned} P^j &= \int d^3x\ T^{0j} \\ &= \int d^3x\ \Big( \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \underbrace{\partial^j \phi_i}_{-\partial_j \phi_i} - \underbrace{\delta^{0j}}_{0} \mathcal{L} \Big) \\ &= \int d^3x\ \Big( - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}}_{\pi^i} \partial_j \phi_i \Big) \\ &= - \int d^3x\ \pi^i \partial_j \phi_i \end{aligned}$$

such that

$$\frac{dP^i}{dt} = 0$$

# Chapter 4

# An example: electrodynamics

Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot E = \rho \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \tag{4.1}$$

can be written in covariant form

$$\partial_\mu F^{\mu\nu} = J^\nu \quad \partial_\mu F^{*\mu\nu} = 0$$

where $F^{\mu\nu}$ is the electromagnetic tensor and $F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}$ is its dual.

Furthermore, they can be written in terms of the scalar $\phi$ and the vector potentials $\mathbf{A}$, defined by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Maxwell's equations do not change under this transformation.

*Proof.* Maybe in the future. q.e.d.

In covariant form, we can write the electromagnetic tensor as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Maxwell's equations can be seen as the equations of motion of the electromagnetic lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J_\mu A^\mu$$

or, equivalenty written in terms of the 4-potential,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 - A_\mu J^\mu$$

*Proof.* First, we prove that they are equivalent. Maybe in the future.

Second, we prove that it leads to the Maxwell's equations. Maybe in the future.

q.e.d.

In natural units, the dimensional analysis is

$$[F^{\mu\nu}] = 2 \quad [A_\mu] = 1 \quad [J^\mu] = 3$$

The minus sign garanties that the kinetic energy has a positive one

$$-\frac{1}{2}\partial_0 A_i \underbrace{\partial^0}_{\partial_0} \underbrace{A^i}_{-A_i} = \frac{1}{2}\dot{A}_i^2$$

The fourth field $A_0$ is not a dynamical quantity, since there is no kinetic energy in terms of $\dot{A}_0^2$, because the first $-\frac{1}{2}\partial_0 A_0 \partial^0 A^0$ cancels out with $\frac{1}{2}(\partial_0 A_0)^2$. Therefore, there are only 3 degrees of freedom. However, since electrodynamics is a gauge theory, it is possible to restrict to only 2 degrees of freedom, which correspond to the 2 transversal polarisations direction of an electromagnetic wave.

The energy-momentum tensor is

$$T^{\mu\nu} = \partial^\nu A^\mu \partial_\rho A^\rho - \partial^\mu A^\rho \partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{4.2}$$

*Proof.* Maybe in the future. q.e.d.

However, the first term in (4.2) is not symmetric under change $\mu \leftrightarrow \nu$, but in order to take into account general relativity, this tensor must be symmetric, since $R_{\mu\nu}$ and $g_{\mu\nu}$ are so in

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

To do it, we defined a new energy-momentum tensor starting from the old one with the addition of an extra term: the partial derivative of a 3 indices anti-symmetric in the first 2 indices tensor $K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}$

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

This garanties that it is conserved as well

$$\partial_\mu \tilde{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \underbrace{\partial_\mu \partial_\lambda}_{symm} \underbrace{K^{\lambda\mu\nu}}_{anti} = \partial_\mu T^{\mu\nu} = 0$$

In the electromagnetic case, we choose $K$ to be

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$$

and the symmetric energy-momentum tensor becomes

$$\tilde{T}^{\mu\nu} = F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4}\eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

which is called the Belifante-Rosenfeld tensor.

*Proof.* Maybe in the future. q.e.d.

The energy density is

$$\mathcal{E} = \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2)$$

*Proof.* Maybe in the future. q.e.d.

The momentum density is

$$\mathcal{P}^i = (\mathbf{E} \times \mathbf{B})^i$$

*Proof.* Maybe in the future. q.e.d.

# Part II

## Free quantum field theory

# Chapter 5

# Klein-Gordon theory

# Chapter 6

# Dirac theory

# Chapter 7

# Maxwell theory

# List of Theorems