Matteo Zandi

On general relativity 1:

gravity

December 8, 2023

Theoretical Physics

Contents

Part I

Einstein's field equations

Chapter 1

Principles

Principle 1.1 (of Galileian relativity)
The laws of Newtonian mechanics are the same for all inertial observers and time is absolute.

Principle 1.2 (of special relativity)
The laws of physics are the same for all inertial observers and the speed of light in vacuum is invariant.

Principle 1.3 (of general relativity)
The laws of Newtonian mechanics are the same for all observers (in all reference frames).

1.1 Equivalence principles

Principle 1.4 (weak equivalence)
For all physics objects, the gravitational charge mass m_g equals the inertial mass m_i.

Principle 1.5 (Einstein's equivalence)
Motion in a uniform gravitational field cannot be distinguished from free fall.

Principle 1.6 (strong equivalence)
There always exists a local reference frame in which all gravitational effects vanish.

1.2 General covariance

Principle 1.7 (weak equivalence)
The laws of physics in a general frame are obtained from the laws of special relativity by replacing tensor quantities of the Lorentz group with tensor quantities of the spacetime manifold.

Chapter 2

Equations

In this chapter, we would like to answer to two questions: what are the gravitational test particles? What are the gravitational sources?

2.1 Gravitational test particles

By the Einstein's equivalence principle, we can associate inertial observers to freely falling objects. Consider a test-particle with 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ subjected only to gravity. It must move along a straight line $x^i = x^i_0 + u^i_0 t$ which satisfies Newton's law

$$\frac{d^2 x^i}{dt^2} = 0 \ .$$

which can be put in a covariant form since $\frac{d^2 t}{dt^2} = 0$

$$\frac{d^2 x^\alpha}{d\tau^2} = \gamma^2 \frac{d^2 x^\alpha}{dt^2} = 0 \ .$$

In a local inertial frame $\Gamma^\alpha_{\mu\nu} = 0$, therefore

$$0 = \frac{d^2 x^\mu}{d\alpha^2} = \frac{d^2 x^\mu}{d\alpha^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = u^\mu \nabla_\mu u^\alpha \ ,$$

which is the geodesic equation. Notice that this happens only at a point in which the test particle and the freely falling observer trajectories cross. However, it is a frame-independent result and it can be generalised by saying that test particles follows geodesic, called world-lines as well. Furthermore, if we consider the inertial observer a test particle at rest, even it itself follows a geodesic. In a different frame, $\Gamma^\alpha_{\mu\nu} \neq 0$ and we can consider the metric g as the gravitational interactions potential

$$\Gamma^\alpha_{\mu\nu} \sim g_{\mu\nu,\beta} \ .$$

The same reasoning can be applied to massless light. Infact the modulus of the parallely transported 4-velocity $g(u, u)$ is conserved along a geodesic, since

$$0 = \nabla_u g(u, u) = 2u^\nu u^\mu \nabla_\nu u_\mu = 2u^\mu (u^\nu \nabla_\nu u_\mu) \ .$$

Therefore, by the principle of general covariance, in any reference frame, the modulus

$$g(u, u) = \begin{cases} -1 & \text{for massive particles} \\ 0 & \text{for light} \end{cases}$$

is conserved along a geodesic. However, for the light, there is no affine parsameter identified as proper time. This means that the metric encodes information about the causal structure of spacetime, because it governs the propagation of signals, like light.

2.2 Gravitational sources

The metric g is a 4×4 symmetric matrix that contains 10 indepenent components. Hence, we need 10 equation to completely indentify it, at most second order partial differential equations. The most reasonable guess for a tensor which contains the second derivative of the metric is the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \ .$$

Notice that, by the second Bianchi identity, i.e.

$$G^\mu{}_{\nu,\mu} = 0 \ ,$$

which means that is covariantly conserved and there are only 6 independent components.

The source tensor must have the same properties: it is a symmetric, covariantly conserved $(0, 2)$ tensor, containing information about matter. By the weak equivalence principle, the strength of gravity interaction is measured by the proper mass, and in special relativity, mass and energy are equivalent. Hence, the reasonable guess it that the source of gravity is encoded in the energy-momentum tensor.

For a perfect fluid with 4-velocity u, the energy-momentum tensor is

$$T = \rho u \otimes u + p(g^{-1} + u \otimes u) = (\rho + p) u \otimes u + p g^{-1} \ ,$$

or, in components,

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \ ,$$

where ρ is the proper dentity and p is the proper pressure, both measured by an observer comoving with the fluid, therefore it is a true scalar.

Notice that the pressure part is orthogonal to u

$$(g^{\mu}\nu + u^{\mu}u^{\nu})u_{\nu} = (g^{\mu\nu} + g^{\mu\nu}\underbrace{u^{\mu}u_{\nu}}_{-1})u_{\nu} = g^{\mu\nu} - g^{\mu\nu} = 0 \ .$$

Infact, in the comoving frame, $u = (1, 0, 0, 0)$ and

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 \\ 0 & pg^{ij} \end{bmatrix} \ ,$$

or, equivalently,

$$T^{\mu}{}_{\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \ .$$

TO COMPLETE.

In a general reference frame, the continuity equation is written as

$$nabla_{\mu}T^{\mu\nu} = 0 \ ,$$

which enforces the natural candidate as the source of gravity.

2.3 Einstein's field equation

The Einstein's field equations are therefore

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu} \ , \tag{2.1}$$

where $k = 8\pi G_N$, due to Newtonian regime.

Proof. In fact, $[G^{\mu\nu}] = L^{-2}$ whereas $[G^{\mu\nu}] = ML^{-3}$. This means that we require a coupling constant $[G_N] = \frac{L}{M}$. q.e.d.

2.4 Geodesic equation

Straight lines are replaced by geodesic lines. However, the geodesic equation is a second order differential equation and we can see the connection term as a force. Hence, gravity is geometry. The Einstein's field equation are non-lineare, therefore the effects of two gravitational sources is not their sum. Furthermore, for dust ($p = 0$), geodesic motion follows from the covariant conservation of the energy-momentum tensor.

Proof. The energy-momentum tensor becomes

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} \ .$$

Hence,

$$\rho u^{\mu} \nabla_{\mu} u^{\nu} = -u^{\nu} \nabla_{\mu}(\rho u^{\mu}) \propto u^{\nu}$$

and

$$\rho u^{\mu}(u_{\nu} \nabla_{\mu} u^{\nu}) \propto \nabla_{\mu}(u_{\nu} u^{\nu}) = 0 = \nabla_{\mu}(\rho u^{\mu}) \ ,$$

where the affine parameter is the proper time. q.e.d.

Unlike electromagnetism, we do not need a Lorentz-like force to completeley determine gravity.

2.5 Einstein-Hilbert action

Chapter 3

Linearised

3.1 Diffeomorphisms

Consider a point P in two overlapping charts x^μ and y^μ, a change of coordinates is a diffeomorphism if it reduces smoothly to the identity

$$y^\mu(P) = x^\mu(P) + \epsilon\xi^\mu(P) \ . \tag{3.1}$$

where ϵ is a parameter such that when $\epsilon = 0$ the diffeomorphism reduces to $y^\mu(P) = x^\mu(P)$.

If we see ξ^μ as the components of a smooth vector field in the basis $\frac{\partial}{\partial x^\mu}$, i.e.

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} \ ,$$

the bases $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial y^\mu}$ are related by

$$\frac{\partial}{\partial x^\mu} = \underbrace{\frac{\partial y^\nu}{\partial x^\mu}}_{\delta^\nu{}_\mu + \epsilon\frac{\partial \xi^\nu}{\partial x^\mu}} \frac{\partial}{\partial y^\nu} = \Big(\delta^\nu{}_\mu + \epsilon\frac{\partial \xi^\nu}{\partial x^\mu}\Big)\frac{\partial}{\partial y^\mu}$$

3.2 Einstein's field equations in linearised regime

In the weak field limit, local curvature is small. This means that there exists a reference frame in which the components of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \ .$$

The linearised Einstein's field equations in first order in ϵ are

$$-\Box h_{\mu\nu} + \eta_{\mu\nu}\Box h + \partial_\mu\partial^\lambda + \partial_\nu\partial^\lambda h_{\lambda\mu} - \eta_{\mu\nu}\partial^\lambda\partial^\rho h_{\lambda\rho} - \partial_\mu\partial_\nu h = 16\pi G_N T_{\mu\nu} \ .$$

Proof. The Christoffel symbols in first order in ϵ is

$$\begin{aligned}
\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2} g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \\
&= \frac{1}{2} g^{\alpha\beta}(\cancel{\eta_{\mu\beta,\nu}} + \epsilon h_{\mu\beta,\nu} + \cancel{\eta_{\nu\beta,\mu}} + \epsilon h_{\nu\beta,\mu} - \cancel{\eta_{\mu\nu,\beta}} - \epsilon h_{\mu\nu,\beta}) \\
&= \frac{\epsilon}{2} g^{\alpha\beta}(h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) \\
&\simeq \frac{\epsilon}{2}(\eta^{\alpha\beta} - \epsilon h^{\alpha\beta})(h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) \\
&\simeq \frac{1}{2}(\epsilon\eta^{\alpha\beta} - \cancel{\epsilon^2 h^{\alpha\beta}})(h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) \\
&\simeq \frac{\epsilon}{2}\eta^{\alpha\beta}(h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) \ ,
\end{aligned}$$

where we have Taylor expanded the inverse of the metric

$$g^{\alpha\beta} = (g_{\alpha\beta})^{-1} = (\eta_{\alpha\beta} + \epsilon h_{\alpha\beta})^{-1} \simeq \eta_{\alpha\beta} - \epsilon h_{\alpha\beta} \ .$$

The Riemann tensor in first order in ϵ is

$$\begin{aligned}
R^{\mu}{}_{\nu\alpha\beta} &= \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\lambda}_{\nu\beta}\Gamma^{\mu}_{\lambda\alpha} - \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\mu}_{\lambda\beta} \\
&\simeq \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} \\
&= \partial_\alpha\left(\frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\nu\lambda,\beta} + h_{\beta\lambda,\nu} - h_{\nu\beta,\lambda})\right) - \partial_\beta\left(\frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\nu\lambda,\alpha} + h_{\alpha\lambda,\nu} - h_{\nu\alpha,\lambda})\right) \\
&= \frac{\epsilon}{2}\cancel{\eta^{\mu\lambda}{}_{,\alpha}}(h_{\nu\lambda,\beta} + h_{\beta\lambda,\nu} - h_{\nu\beta,\lambda}) + \frac{\epsilon}{2}\eta^{\mu\lambda}(\cancel{h_{\nu\lambda,\beta\alpha}} + h_{\beta\lambda,\nu\alpha} - h_{\nu\beta,\lambda\alpha}) \\
&\quad - \frac{\epsilon}{2}\cancel{\eta^{\mu\lambda}{}_{,\beta}}(h_{\nu\lambda,\alpha} + h_{\alpha\lambda,\nu} - h_{\nu\alpha,\lambda}) - \frac{\epsilon}{2}\eta^{\mu\lambda}(\cancel{h_{\nu\lambda,\alpha\beta}} + h_{\alpha\lambda,\nu\beta} - h_{\nu\alpha,\lambda\beta}) \\
&= \frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\beta\lambda,\nu\alpha} - h_{\nu\beta,\lambda\alpha}) - \frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\alpha\lambda,\nu\beta} - h_{\nu\alpha,\lambda\beta}) \\
&= \frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\beta\lambda,\nu\alpha} - h_{\nu\beta,\lambda\alpha} - h_{\alpha\lambda,\nu\beta} + h_{\nu\alpha,\lambda\beta}) \ .
\end{aligned}$$

The Ricci tensor in first order in ϵ is

$$\begin{aligned}
R_{\nu\beta} &= R^{\mu}{}_{\nu\mu\beta} \\
&= \frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\beta\lambda,\nu\mu} - h_{\nu\beta,\lambda\mu} - h_{\mu\lambda,\nu\beta} + h_{\nu\mu,\lambda\beta}) \\
&= \frac{\epsilon}{2}\eta^{\mu\lambda}(\partial_\nu\partial_\mu h_{\beta\lambda} - \partial_\lambda\partial_\mu h_{\nu\beta} - \partial_\nu\partial_\beta h_{\mu\lambda} + \partial_\lambda\partial_\beta h_{\nu\mu}) \\
&= \frac{\epsilon}{2}(\partial_\nu\partial^\mu h_{\beta\mu} - \partial^\mu\partial_\mu h_{\nu\beta} - \partial_\nu\partial_\beta h^{\mu}{}_{\mu} + \partial^\mu\partial_\beta h_{\nu\mu}) \\
&= \frac{\epsilon}{2}(\partial^\mu\partial_\beta h_{\nu\mu} + \partial^\mu\partial_\nu h_{\beta\mu} - \partial_\nu\partial_\beta h - \Box h_{\nu\beta}) \ ,
\end{aligned}$$

where we have set $\lambda = \mu$.

The Ricci scalar in first order in ϵ is

$$
\begin{aligned}
R &= R^{\nu}{}_{\nu} \\
&= \eta^{\nu\beta} R_{\nu\beta} \\
&= \frac{\epsilon}{2}\eta^{\nu\beta}(\partial^{\mu}\partial_{\beta}h_{\nu\mu} + \partial^{\mu}\partial_{\nu}h_{\beta\mu} - \partial_{\nu}\partial_{\beta}h - \Box h_{\nu\beta}) \\
&= \frac{\epsilon}{2}(\partial^{\mu}\partial^{\nu}h_{\nu\mu} + \partial^{\mu}\partial^{\beta}h_{\beta\mu} - \partial_{\nu}\partial^{\nu}h - \Box h^{\nu}{}_{\nu}) \\
&= \frac{\epsilon}{2}(2\partial^{\mu}\partial^{\nu}h_{\nu\mu} - 2\Box h) \\
&= \epsilon(\partial^{\mu}\partial^{\nu}h_{\nu\mu} - \Box h) \ ,
\end{aligned}
$$

where we have set $\beta = \nu$.

Finally, the Einstein tensor in first order in ϵ is

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \\
&= \frac{\epsilon}{2}(\partial^{\beta}\partial_{\nu}h_{\mu\beta} + \partial^{\beta}\partial_{\mu}h_{\nu\beta} - \partial_{\mu}\partial_{\nu}h - \Box h_{\mu\nu}) - \frac{1}{2}\epsilon(\partial^{\alpha}\partial^{\beta}h_{\alpha\beta} - \Box h)g_{\mu\nu} \\
&= \frac{\epsilon}{2}(\eta_{\mu\nu}\Box h - \Box h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h - \eta_{\mu\nu}\partial^{\alpha}\partial^{\beta}h_{\alpha\beta} + \partial^{\beta}\partial_{\nu}h_{\mu\beta} + \partial^{\beta}\partial_{\mu}h_{\nu\beta}) \ .
\end{aligned}
$$

The linearised Einstein's field equations are

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \ ,$$

$$\frac{\epsilon}{2}(\eta_{\mu\nu}\Box h - \Box h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h - \eta_{\mu\nu}\partial^{\alpha}\partial^{\beta}h_{\alpha\beta} + \partial^{\beta}\partial_{\nu}h_{\mu\beta} + \partial^{\beta}\partial_{\mu}h_{\nu\beta}) = 8\pi G_N \epsilon\epsilon T_{\mu\nu} \ ,$$

$$\eta_{\mu\nu}\Box h - \Box h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h - \eta_{\mu\nu}\partial^{\alpha}\partial^{\beta}h_{\alpha\beta} + \partial^{\beta}\partial_{\nu}h_{\mu\beta} + \partial^{\beta}\partial_{\mu}h_{\nu\beta} = 16\pi G_N \epsilon T_{\mu\nu} \ ,$$

where we have expanded $T'_{\mu\nu} = T^{(0)}_{\mu\nu} + \epsilon T_{\mu\nu}$ and noticed that $T^{(0)}_{\mu\nu} = 0$, because the Minkovski metric solves Einstein's equations. q.e.d.

The second Bianchi identity allows us to define a condition, called gauge fixing, on the h beacuse there are only 6 independent components and not 10. Using the De Donder gauge

$$2\partial^{\mu}h_{\mu\nu} = \partial_{\nu}h \ ,$$

the linearised Einstein's field equations becomes

$$-\Box h_{\mu\nu} = 16\pi G_N(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T) \ .$$

Proof. Infact, the right side

$$\begin{aligned}
G_{\mu\nu} &= \eta_{\mu\nu}\Box h - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta} \\
&= \eta_{\mu\nu}\partial^\alpha\partial_\alpha h - \partial^\alpha\partial_\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta} \\
&= \eta_{\mu\nu}\partial^\alpha\partial_\alpha h - \partial^\alpha\partial_\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha \underbrace{\partial^\beta h_{\beta\alpha}}_{\frac{1}{2}\partial_\alpha h} + \partial_\nu \underbrace{\partial^\beta h_{\beta\mu}}_{\frac{1}{2}\partial_\mu h} + \partial_\mu \underbrace{\partial^\beta h_{\beta\nu}}_{\frac{1}{2}\partial_\nu h} \\
&= \eta_{\mu\nu}\partial^\alpha\partial_\alpha h - \partial^\alpha\partial_\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial_\alpha h + \frac{1}{2}\partial_\mu h + \frac{1}{2}\partial_\mu\partial_\nu h \\
&= \eta_{\mu\nu}\partial^\alpha\partial_\alpha h - \partial^\alpha\partial_\alpha h_{\mu\nu} - \cancel{\partial_\mu\partial_\nu h} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial_\alpha h + \cancel{\frac{1}{2}\partial_\nu}\partial_\mu h + \cancel{\frac{1}{2}\partial_\mu}\partial_\nu h \\
&= -\Box h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box h \ .
\end{aligned}$$

Hence

$$\Box h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box h = 16\pi G_N T_{\mu\nu} \ .$$

Furthermore, the trace of the Einstein tensor is

$$\begin{aligned}
\eta^{\mu\nu}G_{\mu\nu} &= \eta^{\mu\nu}(\eta_{\mu\nu}\Box h - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta}) \\
&= \underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_{4}\Box h - \Box \underbrace{\eta^{\mu\nu}h_{\mu\nu}}_{h} - \eta^{\mu\nu}\partial_\mu\partial_\nu h - \underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_{4}\partial^\alpha\partial^\beta h_{\alpha\beta} + \eta^{\mu\nu}\partial^\beta\partial_\nu h_{\mu\beta} + \eta^{\mu\nu}\partial^\beta\partial_\mu h_{\nu\beta} \\
&= 4\Box h - \Box h - \partial^\nu\partial_\nu h - 4\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial^\mu h_{\mu\beta} + \partial^\beta\partial^\nu h_{\nu\beta} \\
&= 2\Box h - 2\partial^\nu \underbrace{\partial^\mu h_{\mu\nu}}_{\frac{1}{2}\partial_\nu h} \\
&= \Box h \ ,
\end{aligned}$$

which is equal to

$$16\pi G_N \underbrace{\eta^{\mu\nu}T_{\mu\nu}}_{T} = 16\pi G_N T \ .$$

Putting together

$$-\Box h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box h = 16\pi G_N T_{\mu\nu} \ ,$$

$$-\Box h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}(16\pi G_N T) = 16\pi G_N T_{\mu\nu} \ ,$$

$$-\Box h_{\mu\nu} = 16\pi G_N \Big(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\Big) \ .$$

q.e.d.

Moreover, in the Newtonian regime where the matter source is static

$$T^{\mu\nu} \simeq T_{00} = -T \ ,$$

we recover the Poisson equation

$$\nabla^2 V_N = 4\pi G_N \rho$$

where $h_{00} = -2V_N$ and G_N is indeed the Newton constant.

Chapter 4

Gravitational waves

4.1 Gauge invariance

Notice that the linearised Einstein's field equations are invariant under a diffeomorphism (3.1), since a variation of the metric

$$h'_{\mu\nu} = h_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}) \ ,$$

leaves unchanged the equations of motion (**??**).

Proof. Maybe in the future. q.e.d.

This frees four degrees of freedom to fullfil. We choose the De Donder gauge

$$2\partial^\mu h'_{\mu\nu} - \partial_\nu h' = 0 \ ,$$

which can be used to determine explicitly the equation which ξ must satisfy in order to have the De Donder gauge

$$2\Box\xi_\nu = 2\partial^\mu h_{\mu\nu} \ .$$

Proof. Maybe in the future. q.e.d.

In particular, we can define a new metric, called the transverse tensor

$$h^T_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' \ ,$$

which in the De Donder gauge satsfies the trasversality condition

$$\partial^\mu h^T_{\mu\nu} = 0 \ .$$

Proof. Maybe in the future. q.e.d.

Moreover, this tensor is called the trace-inverse

$$h^T = -h' = -h \ .$$

Proof. Maybe in the future. q.e.d.

Finally, we obtain the inhomogeneous linearised Einstein's field equation in the De Donder gauge

$$-\Box h^T_{\mu\nu} = 16\pi G_N T_{\mu\nu} \ .$$

Proof. Maybe in the future. q.e.d.

4.2 Vacuum gauge

In the vacuum, where $T^{\mu\nu} = 0$, we obtain the homogeneous linearised Einstein's field equation in the De Donder gauge

$$-\Box h^T_{\mu\nu} = 0 \ .$$

Furthermore, notice that the De Donder gauge does not uniquely determine the choice of coordinates. In fact, choosing

$$\Box \xi'^{\nu} = 0 \ ,$$

we can see that by a change of metric

$$h'' = h'_{\mu\nu} - ({\xi'}_{\mu,\nu} + \xi_{\nu,\mu}) \ ,$$

we see that we have a residual gauge, since it does not transform the equations of motion.

Proof. Maybe in the future. q.e.d.

Choosing a purely coordinate perturbation, in which $h_{\mu\nu}$,

$$h'_{\mu\nu} = \xi_{\mu,\nu} + \xi_{\nu,\mu} \ ,$$

with the physical meaning that there is no perturbation of the metric but only a change of coordinates, we obtain the same condition of the De Donder gauge

$$\Box \xi_\nu = 0 \ .$$

Part II

Schwarzschild spacetime

Chapter 5

Schwarzschild metric

Consider a spherically symmetric source. We study the region outside the source, in which the spacetime is empty and the energy-momentum tensor vanishes $T_{\mu\nu} = 0$. Einstein's field equation becomes

$$R_{\mu\nu} = 0 \ .$$

Proof. In fact, computing the trace of (2.1) in the vacuum

$$0 = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} \underbrace{g^{\mu\nu} g_{\mu\nu}}_{4} R = R - 2R = -R \ .$$

Therefore $R = 0$ and the Einstein's field equations become

$$0 = G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \underbrace{R}_{0} g_{\mu\nu} = R_{\mu\nu} \ .$$

q.e.d.

The general solution will depend on free parameters which depend on the region inside the source $T_{\mu\nu} \neq 0$. However, we can use spacetime symmetries to reduce to a one-parameter family.

5.1 Isometries

Since the general solution is too difficult to compute, we assume isometries, which are spacetime symmetries, associated to Killing vectors, i.e. vectors K such that

$$\pounds_K g = 0 \ .$$

In our situation, good guesses could be staticity and spherical symmetry.

We require that the metric is static, even though the source loses energy, so that there exists a timelike Killing vector K_t

$$\pounds_{K_t} g = 0 \ , \quad g(K_t, K_t) < 0 \ ,$$

with a coordinate t such that

$$K_t = \frac{\partial}{\partial t} \ .$$

Moreover, we require that the metric is spherical symmetric, even though the source is nnot really a sphere. This means that there exists 3 spacelike Killing vector K_i

$$\pounds_{K_i} g = 0 \ , \quad g(K_i, K_i) > 0 \ ,$$

with a coordinate t such that

$$K_i = \frac{\partial}{\partial \theta_i} \ .$$

where $i = 1, 2, 3$. By a theorem, they form a Lie algebra equals to the rotations around the source center

$$[K_i, K_j] = \epsilon_{ijk} K_k \ .$$

They would also conserve in time, since coordinates does, which means that they commute with K_t

$$[K_i, K_t] = 0 \ .$$

We assume also that they are orthogonal to K_t, i.e. spacelike vectors orthogonal to timelike vectors. In this way, the metric is in diagonal form.

We have found the same situation of a sphere. Therefore, we can use polar coordinates (r, θ, ϕ) on surfaces at constant t. The line element assume a diagonal form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2) \ .$$

where $A(r)$, $B(r)$, $C(r)$ are functions of r and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is an element of solid angle.

Since we have the freedom to rescale r, we reparametrised in order to intepret it as the areal radius. Therefore, we choose $C(r) = r^2$. This can be seen if we restrict to the submanifold (sphere) $\mathcal{S}$ with constant t and r. In fact, its area is

$$4\pi r^2 = A(\mathcal{S}) = \int_0^\pi d\theta \int_0^{2\pi} d\phi \ \sqrt{g_{\theta\phi}} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \ \sin\theta C(r) = 4\pi C(r) \ .$$

Hence $C(r) = r^2$. So far, the line element is

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \ .$$

As a consequence of isometries, we can define the domain of the chart (t, r, θ, ϕ). Time-translation invariance means $t \in (-\infty, \infty)$ and the foliation of spatial volumes into 2-dimensional spheres means $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$.

However, it is important to highlight that Einstein's field equation, unlike Maxwell ones, not only determine the metric tensor but also the manifold on which it exists. Therefore, about the coordinate r and its domain, we cannot say any further, they are defined by those equations.

Notice that the proper length of the radius of a sphere is

$$R(r) = \int_0^r dx \sqrt{g_{rr}} = \int_0^r dx \sqrt{B(x)} \ .$$

This means that if $B(r) \neq 1$, the spacetime outside the source is curved.

5.2 Einstein's field equations

Now, we compute the Einstein's field equations.

The metric is

$$g_{\mu\nu} = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix} ,$$

whereas its inverse is

$$g^{\mu\nu} = \begin{bmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 0 & \frac{1}{B} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix} .$$

The derivative with respect to r is

$$g_{\mu\nu,1} = \begin{bmatrix} A' & 0 & 0 & 0 \\ 0 & B' & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 0 & 0 & 2r \sin^2\theta \end{bmatrix} ,$$

whereas its inverse is

$$g_{\mu\nu,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2r^2 \sin\theta\cos\theta \end{bmatrix} .$$

The Christoffel symbols induced by the metric are

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l}) \ .$$

Proof. The 0-th Christoffel symbols are

$$\Gamma^0_{00} = \frac{1}{2}g^{0l}(g_{0l,0} + g_{0l,0} - g_{00,l}) = \frac{1}{2}g^{00}(g_{00,0} + g_{00,0} - g_{00,0}) = 0 \ ,$$

$$\Gamma^0_{11} = \frac{1}{2}g^{0l}(g_{1l,1} + g_{1l,1} - g_{11,l}) = \frac{1}{2}g^{00}(g_{10,1} + g_{10,1} - g_{11,0}) = 0 \ ,$$

$$\Gamma^0_{22} = \frac{1}{2}g^{0l}(g_{2l,2} + g_{2l,2} - g_{22,l}) = \frac{1}{2}g^{00}(g_{20,2} + g_{20,2} - g_{22,0}) = 0 \ ,$$

$$\Gamma^0_{33} = \frac{1}{2}g^{0l}(g_{3l,3} + g_{3l,3} - g_{33,l}) = \frac{1}{2}g^{00}(g_{30,3} + g_{30,3} - g_{33,0}) = 0 \ ,$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{0l}(g_{1l,0} + g_{0l,1} - g_{10,l}) = \frac{1}{2}\underbrace{g^{00}}_{\frac{1}{A}}(g_{10,0} + \underbrace{g_{00,1}}_{A'} - g_{10,0}) = \frac{A'}{2A} \ ,$$

$$\Gamma^0_{02} = \Gamma^0_{20} = \frac{1}{2}g^{0l}(g_{2l,0} + g_{0l,2} - g_{20,l}) = \frac{1}{2}g^{00}(g_{20,0} + g_{00,2} - g_{20,0}) = 0 \ ,$$

$$\Gamma^0_{03} = \Gamma^0_{30} = \frac{1}{2}g^{0l}(g_{3l,0} + g_{0l,3} - g_{30,l}) = \frac{1}{2}g^{00}(g_{30,0} + g_{00,3} - g_{30,0}) = 0 \ ,$$

$$\Gamma^0_{12} = \Gamma^0_{21} = \frac{1}{2}g^{0l}(g_{2l,1} + g_{1l,2} - g_{21,l}) = \frac{1}{2}g^{00}(g_{20,1} + g_{10,2} - g_{21,0}) = 0 \ ,$$

$$\Gamma^0_{13} = \Gamma^0_{31} = \frac{1}{2}g^{0l}(g_{3l,1} + g_{1l,3} - g_{31,l}) = \frac{1}{2}g^{00}(g_{30,1} + g_{10,3} - g_{31,0}) \ ,$$

$$\Gamma^0_{23} = \Gamma^0_{32} = \frac{1}{2}g^{0l}(g_{3l,2} + g_{2l,3} - g_{32,l}) = \frac{1}{2}g^{00}(g_{30,2} + g_{20,3} - g_{32,0}) = 0 \ .$$

The 1-st Christoffel symbols are

$$\Gamma^1_{00} = \frac{1}{2}g^{1l}(g_{0l,0} + g_{0l,0} - g_{00,l}) = \frac{1}{2}\underbrace{g^{11}}_{\frac{1}{B}}(g_{01,0} + g_{01,0} - \underbrace{g_{00,1}}_{A'}) = -\frac{A'}{2B} \ ,$$

$$\Gamma^1_{11} = \frac{1}{2}g^{1l}(g_{1l,1} + g_{1l,1} - g_{11,l}) = \frac{1}{2}\underbrace{g^{11}}_{\frac{1}{B}}(\underbrace{g_{11,1}}_{B'} + g_{11,1} - g_{11,1}) = \frac{B'}{2B} \ ,$$

$$\Gamma^1_{22} = \frac{1}{2}g^{1l}(g_{2l,2} + g_{2l,2} - g_{22,l}) = \frac{1}{2}\underbrace{g^{11}}_{\frac{1}{B}}(g_{21,2} + g_{21,2} - \underbrace{g_{22,1}}_{2r}) = -\frac{r}{B} \ ,$$

$$\Gamma^1_{33} = \frac{1}{2}g^{1l}(g_{3l,3} + g_{3l,3} - g_{33,l}) = \frac{1}{2}\underbrace{g^{11}}_{\frac{1}{B}}(g_{31,3} + g_{31,3} - \underbrace{g_{33,1}}_{2r\sin^2\theta}) = -\frac{r\sin^2\theta}{B} \ ,$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{2}g^{1l}(g_{1l,0} + g_{0l,1} - g_{10,l}) = \frac{1}{2}g^{11}(g_{11,0} + g_{01,1} - g_{10,1}) = 0 \ ,$$

$$\Gamma^1_{02} = \Gamma^1_{20} = \frac{1}{2}g^{1l}(g_{2l,0} + g_{0l,2} - g_{20,l}) = \frac{1}{2}g^{11}(g_{21,0} + g_{01,2} - g_{20,1}) = 0 \ ,$$

$$\Gamma^1_{03} = \Gamma^1_{30} = \frac{1}{2}g^{1l}(g_{3l,0} + g_{0l,3} - g_{30,l}) = \frac{1}{2}g^{11}(g_{31,0} + g_{01,3} - g_{30,1}) = 0 \ ,$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}g^{1l}(g_{2l,1} + g_{1l,2} - g_{21,l}) = \frac{1}{2}g^{11}(g_{21,1} + g_{11,2} - g_{21,1}) = 0 \ ,$$

$$\Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{2}g^{1l}(g_{3l,1} + g_{1l,3} - g_{31,l}) = \frac{1}{2}g^{11}(g_{31,1} + g_{11,3} - g_{31,1}) = 0 \ ,$$

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2}g^{1l}(g_{3l,2} + g_{2l,3} - g_{32,l}) = \frac{1}{2}g^{11}(g_{31,2} + g_{21,3} - g_{32,1}) = 0 \ .$$

The 2-nd Christoffel symbols are

$$\Gamma^2_{00} = \frac{1}{2}g^{2l}(g_{0l,0} + g_{0l,0} - g_{00,l}) = \frac{1}{2}g^{22}(g_{02,0} + g_{02,0} - g_{00,2}) = 0 \ ,$$

$$\Gamma^2_{11} = \frac{1}{2}g^{2l}(g_{1l,1} + g_{1l,1} - g_{11,l}) = \frac{1}{2}g^{22}(g_{12,1} + g_{12,1} - g_{11,2}) = 0 \ ,$$

$$\Gamma^2_{22} = \frac{1}{2}g^{2l}(g_{2l,2} + g_{2l,2} - g_{22,l}) = \frac{1}{2}g^{22}(g_{22,2} + g_{22,2} - g_{22,2}) = 0 \ ,$$

$$\Gamma^2_{33} = \frac{1}{2}g^{2l}(g_{3l,3} + g_{3l,3} - g_{33,l}) = \frac{1}{2}\underbrace{g^{22}}_{\frac{1}{r^2}}(g_{32,3} + g_{32,3} - \underbrace{g_{33,2}}_{2r^2\sin\theta\cos\theta}) = -\sin\theta\cos\theta \ ,$$

$$\Gamma^2_{01} = \Gamma^2_{10} = \frac{1}{2} g^{2l}(g_{1l,0} + g_{0l,1} - g_{10,l}) = \frac{1}{2} g^{22}(g_{12,0} + g_{02,1} - g_{10,2}) = 0 \ ,$$

$$\Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{2} g^{2l}(g_{2l,0} + g_{0l,2} - g_{20,l}) = \frac{1}{2} g^{22}(g_{22,0} + g_{02,2} - g_{20,2}) = 0 \ ,$$

$$\Gamma^2_{03} = \Gamma^2_{30} = \frac{1}{2} g^{2l}(g_{3l,0} + g_{0l,3} - g_{30,l}) = \frac{1}{2} g^{22}(g_{32,0} + g_{02,3} - g_{30,2}) = 0 \ ,$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{2l}(g_{2l,1} + g_{1l,2} - g_{21,l}) = \frac{1}{2} \underbrace{g^{22}}_{\frac{1}{r^2}} (\underbrace{g_{22,1}}_{2r} + g_{12,2} - g_{21,2}) = \frac{1}{r} \ ,$$

$$\Gamma^2_{13} = \Gamma^2_{31} = \frac{1}{2} g^{2l}(g_{3l,1} + g_{1l,3} - g_{31,l}) = \frac{1}{2} g^{22}(g_{32,1} + g_{12,3} - g_{31,2}) = 0 \ ,$$

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{2} g^{2l}(g_{3l,2} + g_{2l,3} - g_{32,l}) = \frac{1}{2} g^{22}(g_{32,2} + g_{22,3} - g_{32,2}) = 0 \ .$$

The 3-rd Christoffel symbols are

$$\Gamma^3_{00} = \frac{1}{2} g^{3l}(g_{0l,0} + g_{0l,0} - g_{00,l}) = \frac{1}{2} g^{33}(g_{03,0} + g_{03,0} - g_{00,3}) = 0 \ ,$$

$$\Gamma^3_{11} = \frac{1}{2} g^{3l}(g_{1l,1} + g_{1l,1} - g_{11,l}) = \frac{1}{2} g^{33}(g_{13,1} + g_{13,1} - g_{11,3}) = 0 \ ,$$

$$\Gamma^3_{22} = \frac{1}{2} g^{3l}(g_{2l,2} + g_{2l,2} - g_{22,l}) = \frac{1}{2} g^{33}(g_{23,2} + g_{23,2} - g_{22,3}) = 0 \ ,$$

$$\Gamma^3_{33} = \frac{1}{2} g^{3l}(g_{3l,3} + g_{3l,3} - g_{33,l}) = \frac{1}{2} g^{33}(g_{33,3} + g_{33,3} - g_{33,3}) = 0 \ ,$$

$$\Gamma^3_{01} = \Gamma^3_{10} = \frac{1}{2} g^{3l}(g_{1l,0} + g_{0l,1} - g_{10,l}) = \frac{1}{2} g^{33}(g_{13,0} + g_{03,1} - g_{10,3}) = 0 \ ,$$

$$\Gamma^3_{02} = \Gamma^3_{20} = \frac{1}{2} g^{3l}(g_{2l,0} + g_{0l,2} - g_{20,l}) = \frac{1}{2} g^{33}(g_{23,0} + g_{03,2} - g_{20,3}) = 0 \ ,$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2} g^{3l}(g_{3l,0} + g_{0l,3} - g_{30,l}) = \frac{1}{2} g^{33}(g_{33,0} + g_{03,3} - g_{30,3}) = 0 \ ,$$

$$\Gamma^3_{12} = \Gamma^3_{21} = \frac{1}{2} g^{3l}(g_{2l,1} + g_{1l,2} - g_{21,l}) = \frac{1}{2} g^{33}(g_{23,1} + g_{13,2} - g_{21,3}) = 0 \ ,$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{3l}(g_{3l,1} + g_{1l,3} - g_{31,l}) = \frac{1}{2} \underbrace{g^{33}}_{\frac{1}{r^2 \sin^2\theta}} (\underbrace{g_{33,1}}_{2r\sin^2\theta} + g_{13,3} - g_{31,3}) = \frac{1}{r} \ ,$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{3l}(g_{3l,2} + g_{2l,3} - g_{32,l}) = \frac{1}{2} \underbrace{g^{33}}_{\frac{1}{r^2 \sin^2\theta}} (\underbrace{g_{33,2}}_{2r^2\sin\theta\cos\theta} + g_{23,3} - g_{32,3}) = \frac{\cos\theta}{\sin\theta} \ .$$

Hence, the non-zero Christoffel symbols are

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{A'}{2A} \ , \quad \Gamma^1_{00} = -\frac{A'}{2B} \ , \quad \Gamma^1_{11} = \frac{B'}{2B} \ , \quad \Gamma^1_{22} = -\frac{r}{B} \ , \quad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B} \ ,$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \ , \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \ , \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \ , \quad \Gamma^3_{23} = -\frac{\cos\theta}{\sin\theta} \ .$$

The derivatives are

$$\Gamma^0_{01,1} = \Gamma^0_{10,1} = \frac{\partial}{\partial r}\Big(\frac{A'}{2A}\Big) = \frac{A''A - (A')^2}{2A^2} \ ,$$

$$\Gamma^1_{00,1} = \frac{\partial}{\partial r}\Big(- \frac{A'}{2B}\Big) = -\frac{A''B - A'B'}{2B^2} \ ,$$

$$\Gamma^1_{11,1} = \frac{\partial}{\partial r}\Big(\frac{B'}{2B}\Big) = \frac{B''B - (B')^2}{2B^2} \ ,$$

$$\Gamma^1_{22,1} = \frac{\partial}{\partial r}\Big(- \frac{r}{B}\Big) = -\frac{B - rB'}{B^2} \ ,$$

$$\Gamma^1_{33,1} = \frac{\partial}{\partial r}\Big(- \frac{r\sin^2\theta}{B}\Big) = -\sin^2\theta \frac{B - rB'}{B^2} \ ,$$

$$\Gamma^1_{33,2} = \frac{\partial}{\partial \theta}\Big(- \frac{r\sin^2\theta}{B}\Big) = -\frac{2r\sin\theta\cos\theta}{B^2} \ ,$$

$$\Gamma^2_{33,2} = \frac{\partial}{\partial \theta}(\sin\theta\cos\theta) = \cos^2\theta - \sin^2\theta \ ,$$

$$\Gamma^2_{12,1} = \Gamma^2_{21,1} = \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2} \ ,$$

$$\Gamma^3_{13,1} = \Gamma^3_{31,1} = \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2} \ ,$$

$$\Gamma^3_{23,2} = \frac{\partial}{\partial \theta}\cot\theta = -\frac{1}{\sin^2\theta} \ .$$

The Ricci tensor is

$$R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^a_{ij}\Gamma^k_{ak} - \Gamma^a_{ik}\Gamma^k_{aj} \ .$$

Its components are

$$\begin{aligned}
R_{00} &= \Gamma^k_{00,k} - \Gamma^k_{0k,0} + \Gamma^a_{00}\Gamma^k_{ak} - \Gamma^a_{0k}\Gamma^k_{a0} \\
&= \Gamma^1_{00,1} + \Gamma^1_{00}\Gamma^0_{10} + \Gamma^1_{00}\Gamma^1_{11} + \Gamma^1_{00}\Gamma^2_{12} + \Gamma^1_{00}\Gamma^3_{13} - \Gamma^0_{01}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^0_{10} \\
&= -\frac{A''B - A'B'}{2B^2} - \frac{A'}{\cancel{2B}}\frac{\cancel{A'}}{2A} - \frac{A'}{2B}\frac{B'}{2B} - \frac{A'}{2rB} - \frac{A'}{2rB} + \frac{A'}{2A}\frac{A'}{2B} + \frac{A'}{\cancel{2B}}\frac{\cancel{A'}}{2A} \\
&= -\frac{A''}{2B} + \frac{A'B'}{4B^2} - \frac{A'}{rB} + \frac{(A')^2}{4AB} \ ,
\end{aligned}$$

$$\begin{aligned}
R_{11} &= \Gamma^k_{11,k} - \Gamma^k_{1k,1} + \Gamma^a_{11}\Gamma^k_{ak} - \Gamma^a_{1k}\Gamma^k_{a1} \\
&= \cancel{\Gamma^1_{11,1}} - \Gamma^0_{10,1} - \cancel{\Gamma^1_{11,1}} - \Gamma^2_{12,1} - \Gamma^3_{13,1} + \Gamma^1_{11}\Gamma^0_{10} + \cancel{\Gamma^1_{11}\Gamma^1_{11}} \\
&\quad + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^1_{11}\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \cancel{\Gamma^1_{11}\Gamma^1_{11}} - \Gamma^2_{12}\Gamma^2_{21} - \Gamma^3_{13}\Gamma^3_{31} \\
&= -\frac{A''A - (A'^2)}{2A^2} + \cancel{\frac{1}{r^2}} + \cancel{\frac{1}{r^2}} + \frac{B'}{2B}\frac{A'}{2A} + \frac{B'}{2rB} + \frac{B'}{2rB} - \frac{(A')^2}{4A^2} - \cancel{\frac{1}{r^2}} - \cancel{\frac{1}{r^2}} \\
&= -\frac{A''}{2A} + \frac{(A')^2}{4A^2} + \frac{B'A'}{4AB} + \frac{B'}{rB} \ ,
\end{aligned}$$

$$\begin{aligned}
R_{22} &= \Gamma^k_{22,k} - \Gamma^k_{2k,2} + \Gamma^a_{22}\Gamma^k_{ak} - \Gamma^a_{2k}\Gamma^k_{a2} \\
&= \Gamma^1_{22,1} - \Gamma^3_{23,2} + \Gamma^1_{22}\Gamma^0_{10} + \Gamma^1_{22}\Gamma^1_{11} + \cancel{\Gamma^1_{22}\Gamma^2_{12}} \\
&\quad + \Gamma^1_{22}\Gamma^3_{13} - \cancel{\Gamma^1_{22}\Gamma^2_{12}} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{23}\Gamma^3_{32} \\
&= -\frac{B - rB'}{B^2} + \frac{1}{\sin^2\theta} - \frac{r}{B}\frac{A'}{2A} - \frac{r}{B}\frac{B'}{2B} - \frac{r}{\cancel{B}}\frac{\cancel{1}}{r} + \frac{1}{\cancel{r}}\frac{\cancel{r}}{B} - \cot^2\theta \\
&= 1 - \frac{1}{B} + \frac{rB'}{2B^2} - \frac{rA'}{2AB} \ ,
\end{aligned}$$

$$
\begin{aligned}
R_{33} &= \Gamma^{k}_{33,k} - \Gamma^{k}_{3k,3} + \Gamma^{a}_{33}\Gamma^{k}_{ak} - \Gamma^{a}_{3k}\Gamma^{k}_{a3} \\
&= \Gamma^{1}_{33,1} + \Gamma^{2}_{33,2} + \Gamma^{1}_{33}\Gamma^{0}_{10} + \Gamma^{1}_{33}\Gamma^{1}_{11} + \Gamma^{1}_{33}\Gamma^{2}_{12} + \cancel{\Gamma^{2}_{33}\Gamma^{3}_{23}} + \cancel{\Gamma^{1}_{33}\Gamma^{3}_{13}} - \cancel{\Gamma^{1}_{33}\Gamma^{3}_{13}} - \cancel{\Gamma^{2}_{33}\Gamma^{3}_{23}} - \Gamma^{3}_{31}\Gamma^{1}_{33} - \Gamma^{3}_{32}\Gamma^{2}_{3} \\
&= -\sin^2\theta \frac{B - rB'}{B^2} + \cancel{\cos^2\theta} - \sin^2\theta - \frac{r\sin^2\theta}{B}\frac{A'}{2A} - \frac{r\sin^2\theta}{B}\frac{B'}{2B} - \cancel{\frac{r\sin^2\theta}{B}\frac{1}{r}} \\
&\quad - \cancel{\frac{1}{r}\frac{r\sin^2\theta}{B}} - \cancel{\frac{\cos\theta}{\sin\theta}\sin\theta\cos\theta} \\
&= \sin^2\theta - \sin^2\theta\frac{1}{B} + \sin^2\theta\frac{rB'}{2B^2} - \sin^2\theta\frac{rA'}{2AB} \\
&= \sin^2\theta R_{22} \ .
\end{aligned}
$$

Finally, we find a system of differential equations

$$
\begin{aligned}
R_{00} &= -\frac{A''}{2B} + \frac{(A')^2}{4AB} + \frac{A'B'}{4B^2} - \frac{A'}{rB} = 0 \ , \\
R_{11} &= -\frac{A''}{2A} + \frac{(A')^2}{4A^2} + \frac{B'A'}{4AB} + \frac{B'}{rB} = 0 \ , \\
R_{22} &= 1 - \frac{1}{B} + \frac{rB'}{2B^2} - \frac{rA'}{2AB} = 0 \ , \\
R_{33} &= \sin^2\theta R_{22} = 0 \ .
\end{aligned}
$$

We notice that

$$
\begin{aligned}
0 &= \frac{B}{A}R_{00} - R_{11} \\
&= -\cancel{\frac{B}{A}\frac{A''}{2B}} + \cancel{\frac{B}{A}\frac{(A')^2}{4AB}} + \cancel{\frac{B}{A}\frac{A'B'}{4B^2}} - \frac{B}{A}\frac{A'}{rB} + \cancel{\frac{A''}{2A}} - \cancel{\frac{(A')^2}{4A^2}} - \cancel{\frac{B'A'}{4AB}} - \frac{B'}{rB} \\
&= -\frac{A'}{rA} - \frac{B'}{rB} \ .
\end{aligned}
$$

Hence

$$0 = A'B + B'A = (AB)' \ ,$$

which implies, after a time rescaling, that

$$A = -B^{-1} \ .$$

Therefore, the third equation becomes

$$0 = 1 - \frac{1}{B} + \frac{rB'}{2B^2} - \frac{rA'}{2AB} = 1 + A - rA, \ ,$$

or, equivalently,

$$(rA)' = 1 \ .$$

Its solution is

$$A = 1 - \frac{2}{}$$

q.e.d.

List of Theorems