

Matteo Zandi

On general relativity 1:

gravity

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Theoretical Physics

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Part I

Einstein's field equations

Chapter 1

Principles

Principle 1.1 (of Galileian relativity)

The laws of Newtonian mechanics are the same for all inertial observers and time is absolute.

Principle 1.2 (of special relativity)

The laws of physics are the same for all inertial observers and the speed of light in vacuum is invariant.

Principle 1.3 (of general relativity)

The laws of Newtonian mechanics are the same for all observers (in all reference frames).

1.1 Equivalence principles

Principle 1.4 (weak equivalence)

For all physics objects, the gravitational charge mass m_g equals the inertial mass m_i .

Principle 1.5 (Einstein's equivalence)

Motion in a uniform gravitational field cannot be distinguished from free fall.

Principle 1.6 (strong equivalence)

There always exists a local reference frame in which all gravitational effects vanish.

1.2 General covariance

Principle 1.7 (weak equivalence)

The laws of physics in a general frame are obtained from the laws of special relativity by replacing tensor quantities of the Lorentz group with tensor quantities of the spacetime manifold.

Chapter 2

Equations

In this chapter, we would like to answer to two questions: what are the gravitational test particles? What are the gravitational sources?

2.1 Gravitational test particles

By the Einstein's equivalence principle, we can associate inertial observers to freely falling objects. Consider a test-particle with 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ subjected only to gravity. It must move along a straight line $x^i = x_0^i + u_0^i t$ which satisfies Newton's law

$$\frac{d^2 x^i}{dt^2} = 0 .$$

which can be put in a covariant form since $\frac{d^2 t}{dt^2} = 0$

$$\frac{d^2 x^\alpha}{d\tau^2} = \gamma^2 \frac{d^2 x^\alpha}{dt^2} = 0 .$$

In a local inertial frame $\Gamma_{\mu\nu}^\alpha = 0$, therefore

$$0 = \frac{d^2 x^\mu}{d\alpha^2} = \frac{d^2 x^\mu}{d\alpha^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = u^\mu \nabla_\mu u^\alpha ,$$

which is the geodesic equation. Notice that this happens only at a point in which the test particle and the freely falling observer trajectories cross. However, it is a frame-independent result and it can be generalised by saying that test particles follows geodesic, called world-lines as well. Furthermore, if we consider the inertial observer a test particle at rest, even it itself follows a geodesic. In a different frame, $\Gamma_{\mu\nu}^\alpha \neq 0$ and we can consider the metric g as the gravitational interactions potential

$$\Gamma_{\mu\nu}^\alpha \sim g_{\mu\nu,\beta} .$$

The same reasoning can be applied to massless light. Infact the modulus of the parallelly transported 4-velocity $g(u, u)$ is conserved along a geodesic, since

$$0 = \nabla_u g(u, u) = 2u^\nu u^\mu \nabla_\nu u_\mu = 2u^\mu (u^\nu \nabla_\nu u_\mu) .$$

Therefore, by the principle of general covariance, in any reference frame, the modulus

$$g(u, u) = \begin{cases} -1 & \text{for massive particles} \\ 0 & \text{for light} \end{cases}$$

is conserved along a geodesic. However, for the light, there is no affine parameter identified as proper time. This means that the metric encodes information about the causal structure of spacetime, because it governs the propagation of signals, like light.

2.2 Gravitational sources

The metric g is a 4×4 symmetric matrix that contains 10 independent components. Hence, we need 10 equation to completely indentify it, at most second order partial differential equations. The most reasonable guess for a tensor which contains the second derivative of the metric is the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} .$$

Notice that, by the second Bianchi identity, i.e.

$$G^\mu{}_{\nu,\mu} = 0 ,$$

which means that is covariantly conserved and there are only 6 independent components.

The source tensor must have the same properties: it is a symmetric, covariantly conserved $(0, 2)$ tensor, containing information about matter. By the weak equivalence principle, the strength of gravity interaction is measured by the proper mass, and in special relativity, mass and energy are equivalent. Hence, the reasonable guess is that the source of gravity is encoded in the energy-momentum tensor.

For a perfect fluid with 4-velocity u , the energy-momentum tensor is

$$T = \rho u \otimes u + p(g^{-1} + u \otimes u) = (\rho + p)u \otimes u + pg^{-1} ,$$

or, in components,

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} ,$$

where ρ is the proper density and p is the proper pressure, both measured by an observer comoving with the fluid, therefore it is a true scalar.

Notice that the pressure part is orthogonal to u

$$(g^{\mu\nu} + u^{\mu}u^{\nu})u_{\nu} = (g^{\mu\nu} + \underbrace{g^{\mu\nu}u^{\mu}u_{\nu}}_{-1})u_{\nu} = g^{\mu\nu} - g^{\mu\nu} = 0 .$$

In fact, in the comoving frame, $u = (1, 0, 0, 0)$ and

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 \\ 0 & pg^{ij} \end{bmatrix} ,$$

or, equivalently,

$$T^{\mu}_{\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} .$$

TO COMPLETE.

In a general reference frame, the continuity equation is written as

$$\nabla_{\mu} T^{\mu\nu} = 0 ,$$

which enforces the natural candidate as the source of gravity.

2.3 Einstein's field equation

The Einstein's field equations are therefore

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu} ,$$

where $k = 8\pi G_N$, due to Newtonian regime.

Proof. In fact, $[G^{\mu\nu}] = L^{-2}$ whereas $[G^{\mu\nu}] = ML^{-3}$. This means that we require a coupling constant $[G_N] = \frac{L}{M}$. q.e.d.

2.4 Geodesic equation

Straight lines are replaced by geodesic lines. However, the geodesic equation is a second order differential equation and we can see the connection term as a force. Hence, gravity is geometry. The Einstein's field equation are non-linear, therefore the effects of two gravitational sources is not their sum. Furthermore, for dust ($p = 0$), geodesic motion follows from the covariant conservation of the energy-momentum tensor.

Proof. The energy-momentum tensor becomes

$$T^{\mu\nu} = \rho u^\mu u^\nu .$$

Hence,

$$\rho u^\mu \nabla_\mu u^\nu = -u^\nu \nabla_\mu (\rho u^\mu) \propto u^\nu$$

and

$$\rho u^\mu (u_\nu \nabla_\mu u^\nu) \propto \nabla_\mu (u_\nu u^\nu) = 0 = \nabla_\mu (\rho u^\mu) ,$$

where the affine parameter is the proper time.

q.e.d.

Unlike electromagnetism, we do not need a Lorentz-like force to completely determine gravity.

2.5 Einstein-Hilbert action

Chapter 3

Linearised

3.1 Einstein's field equations in linearised regime

In the weak field limit, local curvature is small. This means that there exists a reference frame in which the components of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} .$$

The linearised Einstein's field equations in first order in ϵ are

$$-\square h_{\mu\nu} + \eta_{\mu\nu} \square h + \partial_\mu \partial^\lambda h_{\lambda\nu} + \partial_\nu \partial^\lambda h_{\lambda\mu} - \eta_{\mu\nu} \partial^\lambda \partial^\rho h_{\lambda\rho} - \partial_\mu \partial_\nu h = 16\pi G_N T_{\mu\nu} .$$

Proof. The Christoffel symbols in first order in ϵ is

$$\begin{aligned} \Gamma_{\mu\nu}^\alpha &= \frac{1}{2} g^{\alpha\beta} (g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \\ &= \frac{1}{2} g^{\alpha\beta} (\cancel{\eta_{\mu\beta,\nu}} + \epsilon h_{\mu\beta,\nu} + \cancel{\eta_{\nu\beta,\mu}} + \epsilon h_{\nu\beta,\mu} - \cancel{\eta_{\mu\nu,\beta}} - \epsilon h_{\mu\nu,\beta}) \\ &= \frac{\epsilon}{2} g^{\alpha\beta} (h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) \\ &\simeq \frac{\epsilon}{2} (\eta^{\alpha\beta} - \epsilon h^{\alpha\beta}) (h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) \\ &\simeq \frac{1}{2} (\epsilon \eta^{\alpha\beta} - \epsilon^2 h^{\alpha\beta}) (h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) \\ &\simeq \frac{\epsilon}{2} \eta^{\alpha\beta} (h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) , \end{aligned}$$

where we have Taylor expanded the inverse of the metric

$$g^{\alpha\beta} = (g_{\alpha\beta})^{-1} = (\eta_{\alpha\beta} + \epsilon h_{\alpha\beta})^{-1} \simeq \eta^{\alpha\beta} - \epsilon h^{\alpha\beta} .$$

The Riemann tensor in first order in ϵ is

$$\begin{aligned}
R^\mu_{\nu\alpha\beta} &= \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\mu_{\lambda\alpha} - \Gamma^\lambda_{\nu\alpha}\Gamma^\mu_{\lambda\beta} \\
&\simeq \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} \\
&= \partial_\alpha\left(\frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\nu\lambda,\beta} + h_{\beta\lambda,\nu} - h_{\nu\beta,\lambda})\right) - \partial_\beta\left(\frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\nu\lambda,\alpha} + h_{\alpha\lambda,\nu} - h_{\nu\alpha,\lambda})\right) \\
&= \frac{\epsilon}{2}\cancel{\eta^{\mu\lambda}}_{,\alpha}(h_{\nu\lambda,\beta} + h_{\beta\lambda,\nu} - h_{\nu\beta,\lambda}) + \frac{\epsilon}{2}\eta^{\mu\lambda}(\cancel{h_{\nu\lambda,\beta\alpha}} + h_{\beta\lambda,\nu\alpha} - h_{\nu\beta,\lambda\alpha}) \\
&\quad - \frac{\epsilon}{2}\cancel{\eta^{\mu\lambda}}_{,\beta}(h_{\nu\lambda,\alpha} + h_{\alpha\lambda,\nu} - h_{\nu\alpha,\lambda}) - \frac{\epsilon}{2}\eta^{\mu\lambda}(\cancel{h_{\nu\lambda,\alpha\beta}} + h_{\alpha\lambda,\nu\beta} - h_{\nu\alpha,\lambda\beta}) \\
&= \frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\beta\lambda,\nu\alpha} - h_{\nu\beta,\lambda\alpha}) - \frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\alpha\lambda,\nu\beta} - h_{\nu\alpha,\lambda\beta}) \\
&= \frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\beta\lambda,\nu\alpha} - h_{\nu\beta,\lambda\alpha} - h_{\alpha\lambda,\nu\beta} + h_{\nu\alpha,\lambda\beta}) .
\end{aligned}$$

The Ricci tensor in first order in ϵ is

$$\begin{aligned}
R_{\nu\beta} &= R^\mu_{\nu\mu\beta} \\
&= \frac{\epsilon}{2}\eta^{\mu\lambda}(h_{\beta\lambda,\nu\mu} - h_{\nu\beta,\lambda\mu} - h_{\mu\lambda,\nu\beta} + h_{\nu\mu,\lambda\beta}) \\
&= \frac{\epsilon}{2}\eta^{\mu\lambda}(\partial_\nu\partial_\mu h_{\beta\lambda} - \partial_\lambda\partial_\mu h_{\nu\beta} - \partial_\nu\partial_\beta h_{\mu\lambda} + \partial_\lambda\partial_\beta h_{\nu\mu}) \\
&= \frac{\epsilon}{2}(\partial_\nu\partial^\mu h_{\beta\mu} - \partial^\mu\partial_\mu h_{\nu\beta} - \partial_\nu\partial_\beta h^\mu_\mu + \partial^\mu\partial_\beta h_{\nu\mu}) \\
&= \frac{\epsilon}{2}(\partial^\mu\partial_\beta h_{\nu\mu} + \partial^\mu\partial_\nu h_{\beta\mu} - \partial_\nu\partial_\beta h - \square h_{\nu\beta}) ,
\end{aligned}$$

where we have set $\lambda = \mu$.

The Ricci scalar in first order in ϵ is

$$\begin{aligned}
R &= R^\nu_\nu \\
&= \eta^{\nu\beta}R_{\nu\beta} \\
&= \frac{\epsilon}{2}\eta^{\nu\beta}(\partial^\mu\partial_\beta h_{\nu\mu} + \partial^\mu\partial_\nu h_{\beta\mu} - \partial_\nu\partial_\beta h - \square h_{\nu\beta}) \\
&= \frac{\epsilon}{2}(\partial^\mu\partial^\nu h_{\nu\mu} + \partial^\mu\partial^\beta h_{\beta\mu} - \partial_\nu\partial^\nu h - \square h^\nu_\nu) \\
&= \frac{\epsilon}{2}(2\partial^\mu\partial^\nu h_{\nu\mu} - 2\square h) \\
&= \epsilon(\partial^\mu\partial^\nu h_{\nu\mu} - \square h) ,
\end{aligned}$$

where we have set $\beta = \nu$.

Finally, the Einstein tensor in first order in ϵ is

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \\
&= \frac{\epsilon}{2}(\partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta} - \partial_\mu\partial_\nu h - \square h_{\mu\nu}) - \frac{1}{2}\epsilon(\partial^\alpha\partial^\beta h_{\alpha\beta} - \square h)g_{\mu\nu} \\
&= \frac{\epsilon}{2}(\eta_{\mu\nu}\square h - \square h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta}) .
\end{aligned}$$

The linearised Einstein's field equations are

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} ,$$

$$\frac{\epsilon}{2}(\eta_{\mu\nu}\square h - \square h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta}) = 8\pi G_N \epsilon T_{\mu\nu} ,$$

$$\eta_{\mu\nu}\square h - \square h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta} = 16\pi G_N \epsilon T_{\mu\nu} ,$$

where we have expanded $T'_{\mu\nu} = T_{\mu\nu}^{(0)} + \epsilon T_{\mu\nu}$ and noticed that $T_{\mu\nu}^{(0)} = 0$, because the Minkowski metric solves Einstein's equations. q.e.d.

The second Bianchi identity allows us to define a condition, called gauge fixing, on the h because there are only 6 independent components and not 10. Using the De Donder gauge

$$2\partial^\mu h_{\mu\nu} = \partial_\nu h ,$$

the linearised Einstein's field equations becomes

$$-\square h_{\mu\nu} = 16\pi G_N (T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T) .$$

Proof. Infact, the right side

$$\begin{aligned} G_{\mu\nu} &= \eta_{\mu\nu}\square h - \square h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta} \\ &= \eta_{\mu\nu}\partial^\alpha\partial_\alpha h - \partial^\alpha\partial_\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta} \\ &= \eta_{\mu\nu}\partial^\alpha\partial_\alpha h - \partial^\alpha\partial_\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\underbrace{\partial^\alpha\partial^\beta h_{\beta\alpha}}_{\frac{1}{2}\partial_\alpha h} + \underbrace{\partial_\nu\partial^\beta h_{\beta\mu}}_{\frac{1}{2}\partial_\mu h} + \underbrace{\partial_\mu\partial^\beta h_{\beta\nu}}_{\frac{1}{2}\partial_\nu h} \\ &= \eta_{\mu\nu}\partial^\alpha\partial_\alpha h - \partial^\alpha\partial_\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial_\alpha h + \frac{1}{2}\partial_\mu h + \frac{1}{2}\partial_\mu\partial_\nu h \\ &= \eta_{\mu\nu}\partial^\alpha\partial_\alpha h - \partial^\alpha\partial_\alpha h_{\mu\nu} - \cancel{\partial_\mu\partial_\nu h} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial_\alpha h + \cancel{\frac{1}{2}\partial_\mu h} + \cancel{\frac{1}{2}\partial_\mu\partial_\nu h} \\ &= -\square h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\square h . \end{aligned}$$

Hence

$$\square h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\square h = 16\pi G_N T_{\mu\nu} .$$

Furthermore, the trace of the Einstein tensor is

$$\begin{aligned} \eta^{\mu\nu}G_{\mu\nu} &= \eta^{\mu\nu}(\eta_{\mu\nu}\square h - \square h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial_\nu h_{\mu\beta} + \partial^\beta\partial_\mu h_{\nu\beta}) \\ &= \underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_4 \square h - \underbrace{\square \eta^{\mu\nu}h_{\mu\nu}}_h - \underbrace{\eta^{\mu\nu}\partial_\mu\partial_\nu h}_4 - \underbrace{\eta^{\mu\nu}\eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta}}_4 + \eta^{\mu\nu}\partial^\beta\partial_\nu h_{\mu\beta} + \eta^{\mu\nu}\partial^\beta\partial_\mu h_{\nu\beta} \\ &= 4\square h - \square h - \partial^\nu\partial_\nu h - 4\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial^\beta\partial^\mu h_{\mu\beta} + \partial^\beta\partial^\nu h_{\nu\beta} \\ &= 2\square h - 2\partial^\nu \underbrace{\partial^\mu h_{\mu\nu}}_{\frac{1}{2}\partial_\nu h} \\ &= \square h , \end{aligned}$$

which is equal to

$$16\pi G_N \underbrace{\eta^{\mu\nu} T_{\mu\nu}}_T = 16\pi G_N T .$$

Putting together

$$\begin{aligned} -\square h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\square h &= 16\pi G_N T_{\mu\nu} , \\ -\square h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}(16\pi G_N T) &= 16\pi G_N T_{\mu\nu} , \\ -\square h_{\mu\nu} &= 16\pi G_N \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right) . \end{aligned}$$

q.e.d.

Moreover, in the Newtonian regime where the matter source is static

$$T^{\mu\nu} \simeq T_{00} = -T ,$$

we recover the Poisson equation

$$\nabla^2 V_N = 4\pi G_N \rho$$

where $h_{00} = -2V_N$ and G_N is indeed the Newton constant.

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