

Matteo Zandi

On path integral:
an alternative way to quantise?

December 12, 2023

Theoretical Physics

Contents

	Page
Contents	ii
I Path integrals in configuration and phase space	1
1 Introduction	2
1.1 Two slit experiment	2
1.2 Schroedinger equation	4
2 Path integrals in phase space	6
3 Path integrals in configuration space	8
4 Free particle	9
5 Wick rotation	11
6 Correlation functions	13
7 Gaussian integrals	15
8 Hypercondensed notation	16
II Free theory	18
9 Free theory	19
10 Harmonic oscillator	21
10.1 Harmonic oscillator in Euclidean time	22
11 Klein-Gordon theory	24

<i>CONTENTS</i>	iii
12 Perturbation expansion	25
12.1 Vacuum diagrams	25
List of Theorems	29

Part I

Path integrals in configuration and phase
space

Chapter 1

Introduction

In quantum mechanics, there are two equivalent methods to quantise: operatorial formalism and path integrals. The latter has been introduced by Feynman and it is useful for two reasons: quantisation of (non-abelian) gauge theories in the standard model and relation between quantum field theory and statistical mechanics.

1.1 Two slit experiment

Consider an electron, created by a source, that passes through a barrier with two slits and reach a detector. The standard way to study such system is to highlight the wave behaviour of the electron and calculate the interference pattern with the Huygens principle. However, Feynman proposes an alternative way. The electron is a particle which accomplishes all the possible paths with an associated amplitude. Therefore, for each path c_i , there is an amplitude $A(c_i)$ and the total amplitude $A_{tot} = \sum_i A(c_i)$ is related to the probability to find the particle in a point of the detector screen $p = |A_{tot}|^2$. To each amplitude, we associated a unit norm and a phase equal to $\frac{S}{\hbar}$, where S is the action of the particle,

$$A(c_i) = \exp\left(\frac{i}{\hbar}S(c_i)\right)$$

and the total amplitude is

$$A_{tot} = \sum_i \exp\left(\frac{i}{\hbar}S(c_i)\right) .$$

Heuristical free particle

Consider a free particle with associated action

$$S[q] = \int_0^T dt \frac{m}{2} \dot{q}^2 .$$

We suppose that the difference between the two path is $d \ll D$ and we evaluate the action for the path c_1

$$S(c_1) = \frac{m}{2} \frac{D^2}{T^2} T = \frac{m}{2} \frac{D^2}{T}$$

and for the path c_2

$$S(c_2) = \frac{m}{2} \frac{(D+d)^2}{T^2} T = \frac{m}{2T} (D^2 + 2Dd + O(d^2)) = \frac{m}{2} \frac{D^2}{T} + \frac{mDd}{T} O(d^2) = \frac{m}{2} \frac{D^2}{T} + pd + O(d^2) = S(c_1) + p$$

where we roughly estimate $p \sim \frac{mD}{T}$. Therefore, the total amplitude is

$$A_{tot} = A(c_1) + A(c_2) = \exp\left(\frac{i}{\hbar} S(c_1)\right) + \exp\left(\frac{i}{\hbar} S(c_2)\right) = \exp\left(\frac{i}{\hbar} S(c_1)\right) \left(1 + \exp\left(\frac{i}{\hbar} pd\right) + O(d^2)\right).$$

We notice that the maximum probability is given when

$$\exp\left(\frac{i}{\hbar} pd\right) = 1 \quad \Rightarrow \quad \frac{pd}{\hbar} = 2\pi n ,$$

where $n \in \mathbb{Z}$. We recover quantum mechanics, since we recognise the Compton wavelength of the electron

$$\lambda = \frac{\hbar}{p}$$

and we find the relation

$$\frac{d}{\lambda} = n .$$

We can generalise by increasing the number of slits and intermediary screens to have all the possible paths between the initial point (source) and the final point (detector). The action becomes

$$S[q] = \int_{t_i}^{t_f} dt L(q, \dot{q})$$

and the transition amplitude in the continuum limit

$$A = \sum_i \exp\left(\frac{i}{\hbar} S(c_i)\right) = \int \mathcal{D}q \exp\left(\frac{i}{\hbar} S[q]\right) ,$$

where $S[q]$ is a functional, A is an integral functional and $\mathcal{D}q$ is the measure in the path space.

We can recover the classical limit by noticing that, for macroscopic systems, the rate $\frac{S}{\hbar}$ is very big and small variations $\frac{\delta S}{\hbar}$ are bigger than $i\pi$. Therefore, the amplitudes of nearby paths cancel by destructive interference, unless it is the real classical path, since $\delta S = 0$.

1.2 Schroedinger equation

Consider a 1-dimensional particle. The lagrangian action in the configuration space

$$S[x(t)] = \int dt \left(\frac{m}{2} \dot{x}^2 - V(x) \right) .$$

The momentum is defined as

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} .$$

By a Legendre transformation, we can introduce the hamiltonian

$$H = p\dot{x} - L = p\frac{p}{m} + \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} .$$

The hamiltonian action in the phase space

$$S[x(t)] = \int dt \left(p\dot{x} - \frac{p^2}{2m} - V(x) \right) .$$

Given 2 function in phase space f and g , the Poisson brackets are

$$\{x, x\} = \{p, p\} = 0 , \quad \{x, p\} = 1 .$$

The canonical quantisation consists in promoting position and momentum to operators \hat{x} and \hat{p} such that they satisfy the canonical commutation relations

$$[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0 , \quad [\hat{x}, \hat{p}] = i\hbar .$$

Therefore, also the hamiltonian is promoted to an operator acting on the Hilbert space \mathcal{H}

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + V(\hat{x}) .$$

The Schroedinger equation reads as

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle ,$$

where $|\psi\rangle \in \mathcal{H}$ is a state in the Hilbert space.

In the position representation, the eigenstates of the position are

$$\hat{x}|x\rangle = x|x\rangle ,$$

such that they satisfy

$$\langle x|x'\rangle = \delta(x - x') , \quad \mathbb{I} = \int dx |x\rangle \langle x| .$$

The momentum operator in this representation is

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} .$$

Hence, the Schroedinger equation is

$$i\hbar \frac{\partial}{\partial t} \psi(t, x) = \left(\frac{1}{2m} \hat{p}^2 + V(\hat{x}) \right) \psi(t, x) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} V(x) \right) \psi(t, x) .$$

However, for any time-independent hamiltonian, we can solve the Schroedinger equation introducing an evolution operator

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \hat{H} t\right) |\psi_i\rangle .$$

Proof. In fact,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = i\hbar \left(-\frac{i}{\hbar} \hat{H} \right) |\psi(t)\rangle = \hat{H} |\psi(t)\rangle .$$

q.e.d.

The transition amplitude to find the system from the initial state ψ_i to the final state ψ_f is

$$A(x_i, x_f, T) = \langle \psi_f | \psi(t) \rangle = \langle \psi_f | \exp\left(-\frac{i}{\hbar} \hat{H} T\right) | \psi_i \rangle .$$

Proof. In fact,

$$\begin{aligned} \langle \psi_f | \psi(t) \rangle &= \langle \psi_f | \exp\left(-\frac{i}{\hbar} \hat{H} T\right) | \psi_i \rangle \\ &= \langle \psi_f | \mathbb{I} \exp\left(-\frac{i}{\hbar} \hat{H} T\right) \mathbb{I} | \psi_i \rangle \\ &= \int dx_f \underbrace{\langle \psi_f | x_i \rangle}_{\psi_f(x_f)} \exp\left(-\frac{i}{\hbar} \hat{H} T\right) \int dx_i |x_i\rangle \underbrace{\langle x_i | \psi_i \rangle}_{\psi_i(x_i)} \\ &= \int dx_i \int dx_f \psi_f^*(x_f) \psi_i(x_i) \langle x_f | \exp\left(-\frac{i}{\hbar} \hat{H} T\right) | x_i \rangle , \end{aligned}$$

which shows that it is a matrix element between position eigenstates of the evolution operator. q.e.d.

Chapter 2

Path integrals in phase space

The path integral in phase space is

$$\begin{aligned} A &= \lim_{N \rightarrow \infty} \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^N \frac{dp_k}{2\pi\hbar} \right) \exp\left(\frac{i\epsilon}{\hbar} \sum_{k=1}^N \left(p_k \frac{x_k - x_{k-1}}{\epsilon} - H(x_{k-1}, p_k) \right)\right) \\ &= \int \mathcal{D}x \mathcal{D}p \exp\left(\frac{i}{\hbar} S[x, p]\right), \end{aligned}$$

where $\epsilon = \frac{T}{N}$.

Proof. In fact

$$\begin{aligned} A(x_i, x_f, T) &= \langle x_f | \exp\left(-\frac{i}{\hbar} \hat{H} T\right) | x_i \rangle \\ &= \langle x_f | \exp\left(-\frac{i}{\hbar} \hat{H} \underbrace{\frac{T}{N} N}_{\epsilon}\right) | x_i \rangle \\ &= \langle x_f | \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right)^N | x_i \rangle \\ &= \langle x_f | \mathbb{I} \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right) \mathbb{I} \dots \mathbb{I} \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right) \mathbb{I} | x_i \rangle \\ &= \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^N \langle x_k | \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right) | x_{k-1} \rangle \right) \\ &= \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^N \langle x_k | \mathbb{I} \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right) | x_{k-1} \rangle \right) \\ &= \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^N \frac{dp_k}{2\pi\hbar} \right) \prod_{k=1}^N \underbrace{\langle x_k | p_k \rangle}_{\exp\left(\frac{i}{\hbar} p_k x_k\right)} \langle p_k | \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right) | x_{k-1} \rangle \\ &= \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^N \frac{dp_k}{2\pi\hbar} \right) \prod_{k=1}^N \exp\left(\frac{i}{\hbar} p_k x_k\right) \langle p_k | \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right) | x_{k-1} \rangle. \end{aligned}$$

So far, we compute the exact development. Now, we go infinitesimally with $N \rightarrow \infty$

$$\begin{aligned}
A &= \lim_{N \rightarrow \infty} \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^N \frac{dp_k}{2\pi\hbar} \right) \prod_{k=1}^N \exp\left(\frac{i}{\hbar} p_k x_k\right) \langle p_k | \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right) | x_{k-1} \rangle \\
&= \lim_{N \rightarrow \infty} \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^N \frac{dp_k}{2\pi\hbar} \right) \prod_{k=1}^N \exp\left(\frac{i}{\hbar} (p_k(x_k - x_{k-1}) - H(x_{k-1}, p_k) \epsilon)\right) \\
&= \lim_{N \rightarrow \infty} \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^N \frac{dp_k}{2\pi\hbar} \right) \prod_{k=1}^N \exp\left(\frac{i}{\hbar} \underbrace{\epsilon}_{dt} \left(\underbrace{p_k}_p \underbrace{\frac{x_k - x_{k-1}}{\epsilon}}_{\dot{x}} - \underbrace{H(x_{k-1}, p_k)}_H \right)\right) \\
&= \int \mathcal{D}x \mathcal{D}p \exp\left(\frac{i}{\hbar} S[x, p]\right),
\end{aligned}$$

where we have used

$$\langle p_k | \exp\left(-\frac{i}{\hbar} \hat{H} \epsilon\right) | x_{k-1} \rangle = \langle p_k | x_{k-1} \rangle \exp\left(-\frac{i}{\hbar} H(x_{k-1}, p_k) \epsilon\right).$$

q.e.d.

Chapter 3

Path integrals in configuration space

The path integral in configuration space is

$$A = \lim_{N \rightarrow \infty} \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{\frac{N}{2}} \exp\left(\frac{i\epsilon}{\hbar} \sum_{k=1}^N \left(\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1}) \right)\right) = \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[x]\right).$$

Proof. With the use of the gaussian integral,

$$\int_{-\infty}^{\infty} dp \exp\left(-\frac{\alpha}{2} p^2 + \beta p\right) = \sqrt{\frac{2\pi}{\alpha}} \exp\left(\frac{\beta^2}{2\alpha}\right),$$

where in our case they are

$$\alpha = \frac{i\epsilon}{\hbar m}, \quad \beta = \frac{i}{\hbar} (x_k - x_{k-1}),$$

we have

$$\begin{aligned} A &= \lim_{N \rightarrow \infty} \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\frac{1}{2\pi \hbar} \left(\frac{2\pi \hbar m}{i\epsilon} \right)^{\frac{N}{2}} \exp\left(\frac{i\epsilon}{\hbar} \sum_{k=1}^N \epsilon \left(\frac{m(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1}) \right)\right) \right) \\ &= \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} dx_k \left(\frac{m}{2\pi \hbar i \epsilon} \right)^{\frac{N}{2}} \exp\left(\frac{i\epsilon}{\hbar} \sum_{k=1}^N \epsilon \left(\frac{m(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1}) \right)\right) \\ &= \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[x]\right). \end{aligned}$$

q.e.d.

Chapter 4

Free particle

Consider a particle with $N = 1$ and $T = \epsilon$, the path integral is

$$A = \sqrt{\frac{m}{2\pi\hbar iT}} \exp\left(\frac{i}{\hbar} \frac{m(x_f - x_i)^2}{2T}\right) .$$

Proof. In fact, heuristically for $x_{cl}(t) = x_i + \frac{x_f - x_i}{T}t$

$$S[x_{cl}] = \int_0^T \frac{m}{2} \frac{(x_f - x_i)^2}{T^2} = T \frac{m}{2} \frac{(x_f - x_i)^2}{T^2} = \frac{m(x_f - x_i)^2}{2T} .$$

Moreover, considering $x(t) = x_{cl}(t) + q(t)$

$$\begin{aligned} A &= \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[x(t)]\right) \\ &= \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[x_{cl}(t) + q(t)]\right) \\ &= \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[x_{cl}(t)] + S[q(t)]\right) \\ &= \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[x_{cl}(t)]\right) \underbrace{\int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[q(t)]\right)}_N \\ &= N \exp\left(\frac{i}{\hbar} S[x_{cl}(t)]\right) , \end{aligned}$$

where we have used for the mixed term

$$S[x] \propto \int dt \frac{d}{dt}(x_{cl} + q) \frac{d}{dt}(x_{cl} + q) = \int dt 2\dot{x}\dot{q} = \int dt \underbrace{\ddot{x}}_0 q = 0$$

q.e.d.

It is a semiclassical approximation, since it solves the Schroedinger equation

$$i\hbar \frac{\partial}{\partial t} A(x_i, x_f, T) = -\frac{\hbar^2}{2m} A(x_i, x_f, T) .$$

Chapter 5

Wick rotation

Quantum mechanics can be related to statistical mechanics by an analytic continuation $T \rightarrow i\beta$, called a Wick rotation.

The Schroedinger equation becomes a diffusion equation

$$\frac{\partial}{\partial \beta} A = \frac{1}{2m} \frac{\partial^2}{\partial x_f^2} A .$$

Proof. In fact

$$-\frac{1}{2m} \frac{\partial^2}{\partial x_f^2} A = -\frac{1}{2m} \frac{1}{i} \frac{\partial}{\partial T} = -\frac{1}{2m} \frac{1}{i} \frac{\partial^2}{\partial x_f^2} A = \frac{1}{2m} \frac{\partial^2}{\partial x_f^2} A .$$

q.e.d.

The fundamental solution of the heat equation is

$$\begin{cases} A(x_i, x_f, \beta) = \sqrt{\frac{m}{2\pi\beta}} \exp\left(-\frac{m(x_f - x_i)^2}{2\beta}\right) \\ A(x_i, x_f, \beta = 0) = \delta(x_i - x_f) \end{cases} .$$

Through a Wick rotation, we can obtain an euclidean metric

$$ds^2 = -dt^2 + dx^2 \rightarrow ds^2 dt^2 + dx^2 .$$

Therefore, we have an euclidean action

$$iS[x] = i \int_0^T dt \left(\frac{m}{2} \dot{x}^2 - V(x) \right) = \int_0^\beta da \left(-\frac{m}{2} \left(\frac{dx}{da} \right)^2 - V(x) \right) = - \int_0^\beta da \left(\frac{m}{2} \left(\frac{dx}{da} \right)^2 + V(x) \right) = -S$$

with an euclidean path integral

$$\int \mathcal{D}x \exp(-S_E[x]) .$$

To summarise

$$\begin{aligned} A(x_i, x_f, T) &= \langle x_f | \exp(-it\hat{H}) | x_i \rangle = \int \mathcal{D}x \exp(itS[x]) \\ &\rightarrow A(x_i, x_f, T) = \langle x_f | \exp(-\beta\hat{H}) | x_i \rangle = \int \mathcal{D}x \exp(-S_E[x]) \end{aligned}$$

and

$$i \frac{\partial}{\partial t} A = \hat{H} A \rightarrow \frac{\partial}{\partial \beta} A = -\hat{H} A .$$

The connection with statistical mechanics can be made by computing the canonical partition function, in quantum mechanics

$$Z = \text{tr} \exp(-i\hat{H}t) = \sum_n \exp(-iE_n t) = \int dx \langle x | \exp(-i\hat{H}t) | x \rangle = \int_{PBC} \mathcal{D}x \exp(iS[x]) ,$$

where we have used a discrete energy spectrum (otherwise there would have been an integral), we have used position eigenstates and the path integral is on a circle with periodic boundary condition $x(T) = x(0)$; and in statistical mechanics

$$Z_E = \text{tr} \exp(-\beta\hat{H}) = \sum_n \exp(-\beta E_n) = \int dx \langle x | \exp(-\beta\hat{H}) | x \rangle = \int_{PBC} \mathcal{D}x \exp(-S_E[x]) ,$$

where the path integral is on a circle with periodic boundary condition $x(\beta) = x(0)$.

Few comments are to be made. First, we need some (dimensional) regularisation, for instance we need an imaginary time to preserve gauge symmetry. Second, since so far we only have compute the 1-dimensional case, we need to generalise for the finite or infinite degrees of freedom. For instance, in the 3-dimensional particle

$$\int_{\mathbf{R}^3} \mathcal{D}x \exp(iS[x]) = \lim_{N \rightarrow \infty} \int \left(\prod_{k=1}^{N-1} d^3 x_k \right) \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{\frac{3N}{2}} \exp \left(\frac{i}{\hbar} \sum_{k=1}^N \epsilon \left(\frac{m}{2} \frac{(\mathbf{x}_k - \mathbf{x}_{k-1})^2}{\epsilon^2} - V(\mathbf{x}_{k-1}) \right) \right) .$$

Third, there is an ambiguity in choosing the argument of the potential: the prepoint discretisation chooses $V(x_{k-1})$, the midpoint discretisation chooses $V(\frac{x_k + x_{k-1}}{2})$ and the postpoint discretisation chooses $V(x_k)$. For gauge theory, the midpoint prescription is chosen. However all the quantisation methods (among ordering) are equivalent, since there are the same ambiguities in all methods.

Chapter 6

Correlation functions

We define the normalised n -point correlation function as

$$\begin{aligned}\langle x(t_1), \dots x(t_n) \rangle &= \frac{\int \mathcal{D}x \, x(t_1) \dots x(t_n) \exp(\frac{i}{\hbar} S[x])}{\int \mathcal{D}x \, \exp(\frac{i}{\hbar} S[x])} \\ &= \frac{1}{Z} \int \mathcal{D}x \, x(t_1) \dots x(t_n) \exp(\frac{i}{\hbar} S[x]) ,\end{aligned}$$

where $x(t)$ is a dynamical variable.

The generating functional of correlation functions is

$$Z[J] = \int \mathcal{D}x \, \exp \left(\frac{i}{\hbar} S[x] + \frac{i}{\hbar} \int_0^T dt \, J(t)x(t) \right) ,$$

where the functional $J(t)$ is called the source.

We can write the normalised n -point correlation function in terms of the generating functional in the following way

$$\langle x(t_1), \dots x(t_n) \rangle = \frac{1}{Z[0]} \left(\frac{\hbar}{i} \right)^n \frac{\delta^n Z[J]}{\delta J(t_1) \dots \delta J(t_n)} \Big|_{J=0} .$$

The 0-point function is

$$\langle 1 \rangle = \frac{Z}{Z} = 1 .$$

If you know $Z[J]$, you know all the correlation functions.

In the Schroedinger picture, the correlation function is

$$\langle x(t_1) \dots x(t_n) \rangle = \frac{1}{Z} \langle x_f | \exp(-\frac{i}{\hbar} \hat{H}(t_f - T_n) \hat{x} \dots \hat{x} \exp(-\frac{i}{\hbar} \hat{H}(T_1 - t_i))) | x_i \rangle ,$$

where we have ordered to start from the earliest T_1 to the latest T_n . In the Heisenberg picture, the correlation function is

$$\langle x(t_1) \dots x(t_n) \rangle = \frac{1}{Z} \langle x_f, t_f | T \hat{h}_H(t_1) \dots \hat{h}_H(t_n) | x_i, t_i \rangle ,$$

where T is the time-ordered prescription. For example, a 2-point correlation function is

$$\langle 0|T\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle \ .$$

Chapter 7

Gaussian integrals

We make a list of useful gaussian integrals

$$\int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \exp(-\frac{1}{2}K\phi^2) = \frac{1}{\sqrt{K}} ,$$

with $K \geq 0$.

$$\int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \exp(-\frac{1}{2}K\phi^2 + J\phi) = \frac{1}{\sqrt{K}} \exp(\frac{1}{2K}J^2) \quad K \geq 0 .$$

$$\int_{\mathbb{R}^n} \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}} \exp(-\frac{1}{2}\phi^i K_{ij} \phi^j) = \frac{1}{\sqrt{\det k_{ij}}} ,$$

where K_{ij} is symmetric, real, positive defined and $\det K_{ij}$ is the product of the eigenvalues.

$$\int_{\mathbb{R}^n} \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}} \exp(-\frac{1}{2}\phi^i K_{ij} \phi^j + J_i \phi^i) = \frac{1}{\sqrt{\det k_{ij}}} \exp(\frac{1}{2}J_i G^{ij} J_j) ,$$

where G^{ij} is the inverse of K_{ij} such that $K_{ij} G^{jl} = \delta_i^l$.

By analitic continuation, we have

$$\int_{\mathbb{R}^n} \frac{d^n\phi}{(-i2\pi)^{\frac{n}{2}}} \exp(-\frac{i}{2}\phi^i K_{ij} \phi^j + iJ_i \phi^i) = \frac{1}{\sqrt{\det k_{ij}}} \exp(\frac{i}{2}J_i G^{ij} J_j) ,$$

but in order to converge, we must have $K_{ij} - i\epsilon\delta_{ij}$, so that a damping exponential compares.

Chapter 8

Hypercondensed notation

In the hypercondensed notation, we can write

$$\begin{aligned} x(t) &\rightarrow \phi^i, & x &\rightarrow \phi, & t &\rightarrow i, \\ A_\mu(x^\nu) &\rightarrow \phi^i, & A &\rightarrow \phi, & (\mu, x^\nu) &\rightarrow i, \end{aligned}$$

Notice that now i is a continuous index. Furthermore

$$\phi^i \phi_i = \int dt x(t) x(t) = \int dt \int dt' x(t) \delta(t - t') x(t') .$$

or

$$\phi^i \phi_i = \int d^4x A_\mu(x) A^\mu(y) = \int d^4x \int d^4y A_\mu(x) \eta^{\mu\nu} \delta^4(x - y) A_\nu(y) .$$

We can introduce the generating functional of connected correlation functional

$$W[J] = \frac{\hbar}{i} \ln Z[J] ,$$

which implies that

$$Z[J] = \exp\left(\frac{i}{\hbar} W[J]\right) .$$

We can write the normalised n -point correlation function in terms of the generating functional of connected correlation functions in the following way

$$\langle \phi^{i_1} \dots \phi^{i_n} \rangle_c = \left(\frac{\hbar}{i}\right)^{n-1} \frac{\delta^n W[J]}{\delta J_{i_1} \dots \delta J_{i_n}} \Big|_{J=0} .$$

The effective action-generating functional of 1-particle irreducible is

$$\Gamma(\varphi) = \min_J (W[J] - J^i \varphi_i) ,$$

where the procedure to calculate φ is

$$\varphi^i = \frac{\delta W[J]}{\delta J_i} .$$

Proof. The condition of minimum implies

$$\frac{\delta W[J]}{\delta J_i} - \varphi^i = 0 .$$

Therefore

$$\varphi^i = \frac{\delta W[J]}{\delta J_i} ,$$

and we can invert the relation from $\varphi^i = \varphi^i(J)$ to $T_i = T_i(\varphi)$.

q.e.d.

Notice that it is a Legendre transform.

Part II

Free theory

Chapter 9

Free theory

Consider a quadratic action

$$S[\phi] = \frac{1}{2} \phi^i K_{ij} \phi^j ,$$

where K_{ij} is an invertible matrix and the equations of motion are

$$K_{ij} \phi^j = 0 .$$

The generating functional is

$$Z[J] = \int \mathcal{D}x \exp(iS[\phi] + iJ_i \phi^i) = \det^{-1/2} K_{ij} \exp(\frac{1}{2} J_i G^{ij} J_j) ,$$

through which we can obtain all normalised correlation functions. The generating functional of connected correlation functions is

$$W[J] = \frac{1}{2} J_i G^{ij} J_i - \Lambda ,$$

where $\Lambda = \frac{i}{2} \ln \det K_{ij}$.

Proof. In fact

$$W[J] = -i \ln Z[J] = - \underbrace{\frac{i}{2} \ln \det K_{ij}}_{\Lambda} + \frac{1}{2} J_i G^{ij} J_i = \frac{1}{2} J_i G^{ij} J_i - \Lambda .$$

q.e.d.

The effective action is

$$\Gamma[\varphi] = -\frac{1}{2} \varphi^i K_{ij} \varphi^j - \Lambda .$$

Proof. In fact

$$\varphi^i = \frac{\delta W}{\delta J_i} = G^{ij} J_j$$

and inverting it

$$J_i = K_{ij} \varphi^j .$$

Therefore

$$\Gamma[\varphi] = \min_J (W[J] - J_i \varphi^i) = \frac{1}{2} J_i G^{ij} J_j - \Lambda - \underbrace{\varphi^i K_{ij} \varphi^j}_{J_i G^{ij} J_j} = -\frac{1}{2} \varphi^i K_{ij} \varphi^j - \Lambda .$$

q.e.d.

The correlation function is

$$\langle \phi^{i_1} \dots \phi^{i_n} \rangle = \frac{1}{Z} \left(\frac{1}{i} \right)^n \frac{\delta^n Z[J]}{\delta J_{i_1} \dots \delta J_{i_n}} \Big|_{J=0} .$$

Some examples are

$$\begin{aligned} \langle 1 \rangle &= \frac{Z}{Z} = 1 , \\ \langle \phi^i \rangle &= \frac{1}{Z} \frac{1}{i} \frac{\delta Z}{\delta J_i} \Big|_{J=0} = \frac{1}{iZ} G^{ij} J_j \exp\left(\frac{1}{2} J_i G^{ij} J_j\right) \Big|_{J=0} = 0 , \\ \langle \phi^i \phi^j \rangle &= \frac{1}{Z} \left(\frac{1}{i} \right)^2 \frac{\delta^2 Z[J]}{\delta J_i \delta J_j} \Big|_{J=0} = -i G^{ij} . \end{aligned}$$

Notice that 1 point correlation function vanishes since the action is invariant under $\phi' = -\phi$ and it can be generalise for all odd point functions. On the other hand, even point functions can be always decomposed into 2 point ones. For example

$$\langle \phi^1 \phi^2 \phi^3 \phi^3 \rangle = \left(\frac{1}{i} \right)^4 \frac{\delta^4 \exp(\frac{1}{2} J_i G^{ij} J_j)}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} = -G^{12} G^{34} - G^{13} G^{24} - G^{14} G^{23} = \langle \phi^1 \phi^2 \rangle \langle \phi^3 \phi^4 \rangle + \langle \phi^1 \phi^3 \rangle \langle \phi^2 \phi^4 \rangle + \langle \phi^1 \phi^4 \rangle \langle \phi^2 \phi^3 \rangle$$

This result is called the Wick's theorem, which states that in a quadratic theory, odd point functions vanish and even point functions can be expressed by 2 point functions and connect in all possible ways all the points. There are $(n-1)!!$ Wick contractions.

Reintroducing \hbar , we obtain

$$\begin{aligned} S[\phi] &= -\frac{1}{2} \phi^i K_{ij} \phi^j , \\ Z[J] &= \det^{-1/2} K_{ij} \exp\left(\frac{1}{2\hbar} J_i G^{ij} J_j\right) , \\ W[J] &= \frac{1}{2} J_i G^{ij} J_j - \hbar \Lambda , \\ \Lambda[\varphi] &= -\frac{1}{2} \varphi^i K_{ij} \varphi^j - \hbar \Lambda = S[\varphi] + \hbar \text{corrections} , \\ \langle \phi^i \phi^j \rangle &= -i \hbar G^{ij} . \end{aligned}$$

Chapter 10

Harmonic oscillator

Consider an harmonic oscillator with mass $m = 1$ and frequency ω . Its action is

$$\begin{aligned} S[x] &= \int_{-\infty}^{\infty} dt \left(\frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 \right) \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} dt \left(\frac{d^2}{dt^2} + \omega^2 \right) x^2 \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' x(t) \left(\frac{d^2}{dt^2} + \omega^2 \right) \delta(t - t') x(t') \\ &= -\frac{1}{2} \phi^i K_{ij} \phi^j , \end{aligned}$$

where we have integrated by parts and we define

$$K_{ij} = \left(\frac{d^2}{dt^2} + \omega^2 \right) \delta(t - t') , \quad \phi^i = x(t) .$$

The Green function representation in Fourier space is

$$G(t, t') = \int \frac{dp}{2\pi} \frac{\exp(-ip(t - t'))}{-p^2 + \omega^2} .$$

Proof. From the definition

$$K(t, t') G(t', t'') = \delta(t - t'') ,$$

we have

$$G(t, t') = \int \frac{dp}{2\pi} \frac{\exp(-ip(t - t'))}{-p^2 + \omega^2} .$$

q.e.d.

Using the Feynman-Stueckelberg prescription, we obtain

$$G(t, t') = \frac{i}{2\pi} \exp(-i\omega|t - t'|) ,$$

which means adding an $\epsilon \rightarrow 0^+$ factor in order to have a damping factor in the path integrals. Therefore, the path integral is

$$Z[J] = \mathcal{N} \exp\left(\frac{i}{2\hbar} \int dt \int dt' J(t) G(t, t') J(t')\right) .$$

Proof. In fact

$$\begin{aligned} Z[J] &= \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[x] + \frac{i}{\hbar} \int dt J(t)x(t)\right) \\ &= \int \mathcal{D}x \exp\left(\int dt \int dt' \left(\frac{1}{2}x(t)K(t, t')x(t') - J(t)\delta(t - t')x(t')\right)\right) \\ &= \int \mathcal{D}x \exp\left(\int dt \int dt' \left(\frac{1}{2}x(t)K(t, t')x(t') - J(t)\delta(t - t')x(t')\right) + \frac{1}{2}J(t)G(t, t')J(t') - \frac{1}{2}J(t)G(t, t')J(t')\right) \\ &= \exp\left(\frac{i}{2\hbar} \int dt \int dt' J(t)G(t, t')J(t')\right) \int \mathcal{D}x \exp\left(-\frac{i}{\hbar} \int dt \int dt' \tilde{x}(t)K(t, t')\tilde{x}(t')\right) \\ &= \exp\left(\frac{i}{2\hbar} \int dt \int dt' J(t)G(t, t')J(t')\right) \underbrace{\int \mathcal{D}\tilde{x} \exp\left(-\frac{i}{\hbar} \int dt \int dt' \tilde{x}(t)K(t, t')\tilde{x}(t')\right)}_{\mathcal{N}} \\ &= \mathcal{N} \exp\left(\frac{i}{2\hbar} \int dt \int dt' J(t)G(t, t')J(t')\right) , \end{aligned}$$

where we have used the translation invariance of the measure and

$$\tilde{x}(t) = x(t) \int dt'' G(t, t'') J(t'') .$$

q.e.d.

The 2 point function is

$$\langle x(t)x(t') \rangle = \frac{\int \mathcal{D}x x(t)x(t') \exp(\frac{i}{\hbar} S[x])}{\int \mathcal{D}x \exp(\frac{i}{\hbar} S[x])} = \frac{1}{Z[0]} \left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J(t) \delta J(t')} = -i\hbar G(t, t') = \frac{\hbar}{2\omega} \exp(-i\omega|t - t'|) .$$

10.1 Harmonic oscillator in Euclidean time

The path integral in Euclidean time, using a discrete diagonalised energy eigenbasis,

$$Z_E = \text{tr} \exp(-\beta \hat{H}) = \sum_n \exp(-\beta E_n) \xrightarrow{\beta \rightarrow \infty} \exp(-\beta E_0) + \text{subleading terms} = \langle 0 | \exp(-\beta \hat{H}) | 0 \rangle ,$$

which is the energy of the vacuum state. Therefore

$$Z_E[x] = \int_{PBC} \mathcal{D}x \exp(-S_E[x] + \int d\tau J(\tau)x(\tau)) \xrightarrow{\beta \rightarrow \infty} ,$$

where

$$S_E[x] = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} m \dot{x}^2 + \frac{\omega^2}{2} x^2 \right) .$$

The Green function is

$$G_E(\tau, \tau') = \int \frac{dp}{2\pi} \frac{\exp(-ip(t-t'))}{p_E^2 + \omega^2} = \frac{1}{2\omega} \exp(-\omega|\tau - \tau'|) .$$

The 2 point function is

$$\langle x(\tau)x(\tau') \rangle = G_E(\tau, \tau') = \frac{1}{2\omega} \exp(-\omega|\tau - \tau'|) .$$

Notice that in order to preserve the Fourier transform, we need to inverse Wick rotate the momentum, i.e. $t \rightarrow -i\tau$ and $p_M \rightarrow iP_E$. Hence, after a Wick rotation

$$\begin{aligned} \langle x(\tau)x(\tau') \rangle &= G_E(\tau, \tau') = \int \frac{dp_E}{2\pi} \frac{\exp(-ip_E(t-t'))}{p_E^2 + \omega^2} \\ &\xrightarrow{\text{Wick}} -i \int \frac{dp_M}{2\pi} \frac{\exp(-ip_M(t-t'))}{p_E^2 + \omega^2} = -iG(t, t') = \langle x(t)x(t') \rangle \end{aligned}$$

and

$$\frac{1}{2\omega} \exp(-\omega|\tau - \tau'|) \xrightarrow{\text{Wick}} \frac{1}{2\omega} \exp(-i\omega|t - t'|) .$$

Chapter 11

Klein-Gordon theory

Consider a Klein-Gordon field. Its action is

$$\begin{aligned} S[x] &= \int d^4x \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) \\ &= \int d^4x \left(\frac{1}{2} \phi \square \phi - \frac{1}{2} \phi m^2 \phi \right) \\ &= -\frac{1}{2} \int d^4x \int d^4y \phi(x) (-\square_x + m^2) \delta^4(x - y) \phi(y) \\ &= -\frac{1}{2} \phi^i K_{ij} \phi^j , \end{aligned}$$

where we have integrated by parts and we define

$$K_{ij} = \left(-\square_x + m^2 \right) \delta^4(x - y) , \quad \phi^i = \phi(x) .$$

The Green function representation in Fourier space is

$$G(x, y) = \int \frac{d^4p}{(2\pi)^4} \frac{\exp(-ip_\mu(x^\mu - y^\mu))}{p^2 + \omega^2} .$$

We can recover the harmonic oscillator from the Klein-Gordon theory by rescaling ourselves to the situation in which we have only the time 0-dimension. In fact, the Klein-Gordon action becomes

$$\int dt \left(\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 \right)$$

where we can identify $\omega = m$.

Chapter 12

Perturbation expansion

The model that we have in mind is the anharmonic oscillator, which action is

$$S[x] = \int dt \left(\frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 - \frac{g}{3!} x^3 - \frac{\lambda}{4!} x^4 + \dots \right) = S_0 + S_{int}$$

where g and λ are small perturbative quantity, called coupling constants.

The path integral is

$$\begin{aligned} Z[J] &= \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S[x] + \frac{i}{\hbar} \int dt J(t)x(t)\right) \\ &= \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S_{int}[x]\right) \underbrace{\exp\left(\frac{i}{\hbar} S_0[x] + \frac{i}{\hbar} \int dt J(t)x(t)\right)}_{Z_0[J]} \\ &= \int \mathcal{D}x \left(1 + \left(\frac{i}{\hbar} S_{int}[x]\right) + \frac{1}{2} \left(\frac{i}{\hbar} S_{int}[x]\right)^2 + \dots\right) Z_0[J] \\ &= \langle \exp\left(\frac{i}{\hbar} S_{int}[x]\right) \rangle_{0,U} \\ &= \langle 1 \rangle_{0,U} + \langle \frac{i}{\hbar} S_{int}[x] \rangle_{0,U} + \frac{1}{2} \langle \left(\frac{i}{\hbar} S_{int}[x]\right)^2 \rangle_{0,U} . \end{aligned}$$

where 0 means free theory and U not normalised. We can intepret this expression as a differential operator

$$\int \mathcal{D}x \exp\left(\frac{i}{\hbar} S_{int}\left[\frac{\hbar}{i} \frac{\delta}{\delta J}\right]\right) Z_0[J] = \int \mathcal{D}x \left(1 + \left(\frac{i}{\hbar} S_{int}\left[\frac{\hbar}{i} \frac{\delta}{\delta J}\right]\right) + \frac{1}{2} \left(\frac{i}{\hbar} S_{int}\left[\frac{\hbar}{i} \frac{\delta}{\delta J}\right]\right)^2 + \dots\right) Z_0[J] .$$

12.1 Vacuum diagrams

In this section, we will work in Euclidean time. The path integral is

$$Z_E[J] = \int \mathcal{D}x \exp(-S_E[x] + \int d\tau x(\tau)J(\tau)) ,$$

where the Euclidean action is

$$S_E = \lim_{\beta \rightarrow \infty} \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 + \frac{g}{3!} x^3 + \frac{\lambda}{4!} x^4 \right) .$$

Consider the correction to the vacuum energy

$$\begin{aligned} Z_E[0] &= \int \mathcal{D}x \exp(-S_E[x]) \\ &= \langle 1 \rangle_U \\ &= \lim_{\beta \rightarrow \infty} \langle 0 | \exp(-\beta \hat{H}) | 0 \rangle = \lim_{\beta \rightarrow \infty} \exp(-\beta E_0) \\ &= \langle \exp(-S_{E,int}[x]) \rangle_{U,0} \\ &= \lim_{\beta \rightarrow \infty} \exp(-\beta(E_0^{(0)} + \Delta E)) , \end{aligned}$$

where

$$E_0 = E_0^{(0)} + \Delta E .$$

Now we compute the energy corrections ΔE .

First case

Consider the case in which $g = 0$ and $\lambda \neq 0$. The path integral is

$$\begin{aligned} Z[0] &= \langle 1 - S_{E,int}[x] + \dots \rangle \\ &= \langle 1 \rangle_{U,0} - \int_{-\beta/2}^{\beta/2} d\tau \frac{\lambda}{4!} \langle x^4(\tau) \rangle_{U,0} \frac{\langle 1 \rangle}{\langle 1 \rangle} + \dots \\ &= \langle 1 \rangle (1 - \int_{-\beta/2}^{\beta/2} d\tau \frac{\lambda}{4!} \langle x(\tau)x(\tau)x(\tau)x(\tau) \rangle_{U,0}) . \end{aligned}$$

Now, we have to compute the 4 point function (which has $(n-1)!! = 3$ terms)

$$\langle x(\tau)x(\tau)x(\tau)x(\tau) \rangle = 3 \langle x(\tau)x(\tau) \rangle^2 ,$$

where we have used the Wick theorem. For the harmonic oscillator is

$$\langle x(\tau)x(\tau)x(\tau)x(\tau) \rangle = \frac{3}{4\omega^2} .$$

Therefore

$$\begin{aligned} Z[0] &= \langle 1 \rangle (1 - \int_{-\beta/2}^{\beta/2} d\tau \frac{\lambda}{4!} \frac{3}{4\omega^2} + \dots) \\ &= \langle 1 \rangle (1 - \frac{\lambda\beta}{32\omega^2} + \dots) \\ &= \langle 1 \rangle \exp(-\frac{\beta\lambda}{32\omega^2}) . \end{aligned}$$

Finally, the energy correction is

$$\Delta E = \frac{\lambda}{32\omega^2} .$$

Second case

Consider the case in which $g \neq 0$ and $\lambda = 0$. The path integral is

$$\begin{aligned} Z[0] &= \langle 1 - S_{E,int}[x] + \frac{1}{2} S_{E,int}^2[x] \dots \rangle \\ &= \langle 1 \rangle (1 - \int_{-\beta/2}^{\beta/2} d\tau \frac{g}{3!} \langle x^3(\tau) \rangle + \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' (\frac{g}{3!})^2 \langle x^3(\tau) x^3(\tau') \rangle) . \end{aligned}$$

Now, we have to compute the 6 point function (which has $(n-1)!! = 15$ terms)

$$\langle x^3(\tau) x^3(\tau') \rangle = 6G_E^3(\tau - \tau') + 9G_E(\tau - \tau) G_E(\tau - \tau') G_E(\tau' - \tau') ,$$

where we have used the Wick theorem. For the harmonic oscillator is

$$\langle x^3(\tau) x^3(\tau') \rangle = \frac{6}{8\omega^3} \exp(-3\omega|\tau - \tau'|) + \frac{9}{8\omega^3} \exp(-\omega|\tau - \tau'|) .$$

Therefore

$$\begin{aligned} Z[0] &= \langle 1 \rangle (1 + \frac{1}{2} (\frac{g}{3!})^2 \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' (\frac{6}{8\omega^3} \exp(-3\omega|\tau - \tau'|) + \frac{9}{8\omega^3} \exp(-\omega|\tau - \tau'|)) + \dots) \\ &= \langle 1 \rangle (1 + \frac{1}{2} (\frac{g}{3!})^2 \frac{1}{8\omega^3} \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' (6 \exp(-3\omega|\tau - \tau'|) + 9 \exp(-\omega|\tau - \tau'|)) + \dots) . \end{aligned}$$

Now, we have to evaluate the integral

$$\int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' (6 \exp(-3\omega|\tau - \tau'|) + 9 \exp(-\omega|\tau - \tau'|)) .$$

We make a change of variable $|\sigma| = |\tau - \tau'|$ and compute the limit $\beta \rightarrow \infty$

$$\begin{aligned} &\underbrace{\int_{-\beta/2}^{\beta/2} d\tau}_{\beta} \int_{-\infty}^{\infty} d\sigma (6 \exp(-3\omega|\sigma|) + 9 \exp(-\omega|\sigma|)) \\ &= \beta (\int_0^{\infty} d\sigma (6 \exp(-3\omega\sigma) + 9 \exp(-\omega\sigma)) + \int_{-\infty}^0 d\sigma (6 \exp(3\omega\sigma) + 9 \exp(\omega\sigma))) . \end{aligned}$$

We make a change of variable $\sigma = -\sigma$ in the second integrand

$$\begin{aligned} &\beta (\int_0^{\infty} d\sigma (6 \exp(-3\omega\sigma) + 9 \exp(-\omega\sigma)) - \int_{-\infty}^0 d\sigma (-6 \exp(3\omega\sigma) + 9 \exp(-\omega\sigma))) \\ &= 2\beta \int_0^{\infty} d\sigma (6 \exp(-3\omega\sigma) + 9 \exp(-\omega\sigma)) \\ &= 2\beta (\frac{6}{3\omega} + \frac{9}{\omega}) . \end{aligned}$$

Therefore,

$$\begin{aligned}
 Z[0] &= \langle 1 \rangle \left(1 + \frac{1}{2} \left(\frac{g}{3!} \right)^2 \frac{1}{8\omega^3} 2\beta \left(\frac{6}{3\omega} + \frac{9}{\omega} \right) + \dots \right) \\
 &= \langle 1 \rangle \left(1 + \left(\frac{g}{3!} \right)^2 \frac{1}{8\omega^3} \beta \frac{11}{\omega} + \dots \right) \\
 &= \langle 1 \rangle \exp \left(\frac{11\beta g^4}{288\omega^4} \right) .
 \end{aligned}$$

Finally, the energy correction is

$$\Delta E = -\frac{11g^4}{288\omega^4} .$$

12.2 Feynman diagrams

List of Theorems