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On quantum field theory 2:

interactions.

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Abstract

In these notes, we will study

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Chapter 1

Introduction

Quantum field theory is a mathematical framework with a highly promising goal: describe physical phenomena that must be described with both quantum mechanics and special relativity formalisms.

Part II

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Chapter 2

Cross section

2.1 S-matrix

Let $|i, t_i\rangle$ be a initial state at time t_i and $|f, t_f\rangle$ be a final state at time t_f. In Schroedinger picture, the probability of this transition is $|\langle f, t_f|i, t_i\rangle|^2$. Suppose that the scattering interaction happens at short spacetime scales, so that we can approximate $t_i \to -\infty$, $t_f \to +\infty$ and $|i, t_i\rangle, |f, t_f\rangle$ are free theory (asymptotic) states. In Heisenberg picture, the probability of this transition becomes

$$\langle f|\hat{S}|i\rangle_H = \langle f, +\infty|i, -\infty\rangle \ ,$$

where S is the so-called S-matrix.

2.2 Cross section

Consider a scattering experiment. Let T be the time of experiment, N_{in} and N_{out} the number of incoming and outgoing particle, Φ be the incoming flux ($N_{in}|\mathbf{v}|/V$, where $|\mathbf{v}|$ is the velocity of the beam). Then the classical cross section is

$$\sigma = \frac{N_{out}}{T\Phi} = \frac{N_{out}}{N_{in}}\frac{V}{|\mathbf{v}|T} \ .$$

In quantum mechanics, we can define the probability of interaction $P = \frac{N_{out}}{N_{in}}$ so that

$$\sigma = \frac{1}{T\Phi}P = \frac{V}{|\mathbf{v}|T}P \ ,$$

or we can define the luminosity $L = T\Phi = \frac{|\mathbf{v}|T}{V}N_{in}$ so that

$$N_{out} = L\sigma \ .$$

Furthermore, the differential cross section is given by

$$d\sigma = \frac{V}{|\mathbf{v}|T} dP \ . \tag{2.1}$$

We can differentiate this quantity over a generic differential volume df is the space of final states to get

$$\frac{d\sigma}{df} = \frac{V}{|\mathbf{v}|T}\frac{dP}{df} \ ,$$

e.g. energy dE or solid angle $d\Omega$.

Consider a scattering experiment $2 \to n$, in which two incoming particles interacts and form n outgoing particles $p_1 + p_2 \to \{p_j\}_{j=1}^n$. In a generic inertial frame, since we have two particles with different velocities that collide, the flux becomes

$$\Phi = \frac{N_{in}|\mathbf{v}_1 - \mathbf{v}_2|}{V} \ .$$

Moreover, the probability of interaction is given by

$$dP = \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} d\Pi = |\langle f|\hat{S}|i\rangle|^2 d\Pi \ ,$$

where $d\Pi$ is a volume differential of the phase (momentum) space

$$d\Pi = \prod_j \frac{V d^3 p_j}{(2\pi)^3} \ .$$

Hence, (2.13) becomes

$$d\sigma = \frac{V}{|\mathbf{v}_1 - \mathbf{v}_2|T}\frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}\prod_j \frac{V d^3 p_j}{(2\pi)^3} \ . \tag{2.2}$$

Now, using the property

$$\delta^3(0) = \int \frac{d^3x}{(2\pi)^3} \exp(ipx)\Big|_{p=0} = \frac{1}{(2\pi)^3}\int d^3x = \frac{V}{(2\pi)^3} \ ,$$

we find

$$\langle p|p\rangle = (2\pi)^3\delta^3(0) = (2\pi)^3 2E_p \frac{V}{(2\pi)^3} = 2E_p V \ .$$

In particular, taking $|i\rangle = |p_1, p_2\rangle$ and $|f\rangle = |\prod_j p_j\rangle$, we obtain

$$\langle i|i\rangle = 4E_1E_2V^2 \ , \quad \langle f|f\rangle = \prod_j 2E_j V \ . \tag{2.3}$$

We can perturbatively expand the S-matrix as $\hat{S} = \mathbb{I} + i\hat{T}$, where $\mathbb{I}$ is the term that describes the no-interaction experiment, whereas the interaction part is encapsulated into $\hat{T}$. From now om, we will study only We can extract a delta function to ensure momentum conservation

$$\hat{T} = (2\pi)^4 \delta^4(p_1 + p_2 - \sum_j p_j)\hat{M} \ ,$$

where $\hat{M}$ is defined by this expression. Hence,

$$|\langle f|\hat{T}|i\rangle|^2 = (2\pi)^8 \Big(\delta^4(p_1 + p_2 - \sum_j p_j)\Big)^2 |\langle f|\hat{M}|i\rangle|^2 = (2\pi)^8 \Big(\delta^4(p_1 + p_2 - \sum_j p_j)\Big)^2 |M|^2 \ ,$$

where we have called $|\mathcal{M}|^2 = |\langle f|\hat{M}|i\rangle|^2$. Since the square of a delta can be written as

$$\Big(\delta^4(p_1 + p_2 - \sum_j p_j)\Big)^2 = \delta^4(p_1 + p_2 - \sum_j p_j)\delta^4(0)$$

and, using the property

$$\delta^4(0) = \int \frac{d^4x}{(2\pi)^4} \exp(ipx)\Big|_{p=0} = \frac{1}{(2\pi)^4} \int d^3x \int dt = \frac{TV}{(2\pi)^4} \ ,$$

we obtain

$$|\langle f|\hat{T}|i\rangle|^2 = (2\pi)^4 TV \delta^4(p_1 + p_2 - \sum_j p_j)|\mathcal{M}|^2 \ . \tag{2.4}$$

Putting everything together (2.14), (2.15) and (2.16), we find

$$\begin{aligned} d\sigma &= \frac{V}{|\mathbf{v}_1 - \mathbf{v}_2|T} (2\pi)^4 TV \delta^4(p_1 + p_2 - \sum_j p_j)|\mathcal{M}|^2 \frac{1}{4E_1E_2V^2} \frac{1}{\prod_j 2E_jV} \prod_j \frac{V d^3p_j}{(2\pi)^3} \\ &= \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2| 4E_1E_2} |\mathcal{M}|^2 \prod_j \frac{d^3p_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 \delta^4(p_1 + p_2 - \sum_j p_j) \\ &= \ , \end{aligned}$$

Therefore, the differential of the cross section is

$$d\sigma \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2| 4E_1E_2} |\mathcal{M}|^2 d\Pi_N \tag{2.5}$$

where we have defined the Lorentz-invariant phase space element $d\Pi_N$ as

$$d\Pi_N = \prod_j \frac{d^3p_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 \delta^4(p_1 + p_2 - \sum_j p_j) \ . \tag{2.6}$$

2.3 $2 \to 2$ scattering in the center of mass

Consider a scattering experiment $2 \to 2$, in which two incoming particles interacts and form 2 outgoing particles $p_1 + p_2 \to p_3 + p_4$. In the reference frame of the center of mass,

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 = 0 \ . \tag{2.7}$$

The Lorentz-invariant phase space (2.6) becomes

$$d\Pi_2 = \frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \tag{2.8}$$

$$= \frac{d^3p_3}{(2\pi)^2}d^3p_4\frac{1}{4E_3E_4}\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(E_1+E_2-E_3-E_4) \tag{2.9}$$

$$= \frac{d^3p_3}{(2\pi)^2}\frac{1}{4E_3E_4}\delta(E_1+E_2-E_3-E_4) = d\Omega\frac{d|\mathbf{p}_3||\mathbf{p}_3|^2}{(2\pi)^2}\frac{1}{4E_3E_4}\delta(E_1+E_2-E_3-E_4) \ . \tag{2.10}$$

Now, we make a change of variable into $x = E_3 + E_4 - E_1 - E_2$ with Jacobian

$$\frac{dx}{d|\mathbf{p}_3|} = \frac{d}{d|\mathbf{p}_3|}(E_3+E_4-E_1-E_2) = \frac{|\mathbf{p}_3|}{E_3} + \frac{|\mathbf{p}_3|}{E_4} = \frac{|\mathbf{p}_3|(E_3+E_4)}{E_3E_4} \ ,$$

where we have used the fact that $|\mathbf{p}_3| = |\mathbf{p}_4|$ in (2.7) and the energies $E_3 = \sqrt{|\mathbf{p}_3|^2 + m_3^2}$ and $E_4 = \sqrt{|\mathbf{p}_4|^2 + m_4^2} = \sqrt{|\mathbf{p}_3|^2 + m_4^2}$. Hence, (2.8) becomes

$$d\Pi_2 = d\Omega\frac{dx}{(2\pi)^2}\frac{E_3E_4|\mathbf{p}_3|^2}{|\mathbf{p}_3|(E_3+E_4)}\delta(x) = d\Omega\frac{1}{16\pi^2}\frac{|\mathbf{p}_3|}{(E_3+E_4)} \ . \tag{2.11}$$

Furthermore, inverting $\mathbf{p} = \gamma m$,

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{\mathbf{p}}{E} \ ,$$

we obtain

$$\frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|} = \frac{1}{|\mathbf{p}_1/E_1 - \mathbf{p}_2/E_2|} = \frac{E_1E_2}{(E_1+E_2)|\mathbf{p}_1|} \ , \tag{2.12}$$

where we have used the fact that $\mathbf{p}_1 = -\mathbf{p}_2$ in (2.7).

Putting everything together (2.5), (2.11) and (2.12), we find

$$\begin{aligned} d\sigma &= \frac{1}{4E_1E_2}|\mathcal{M}|^2 d\Omega\frac{1}{16\pi^2}\frac{|\mathbf{p}_3|}{(E_3+E_4)}\frac{E_1E_2}{(E_1+E_2)|\mathbf{p}_1|} \\ &= d\Omega\frac{1}{16\pi^2E_{tot}^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}|\mathcal{M}|^2 \ , \end{aligned}$$

where we have used the energy conservation $E_{tot} = E_1 + E_2 = E_3 + E_4$, relabeled $|\mathbf{p}_i| = |\mathbf{p}_1| = |\mathbf{p}_2|$ and $|\mathbf{p}_f| = |\mathbf{p}_3| = |\mathbf{p}_4|$.

2.4 Decay rates

Consider a scattering experiment. Let T be the time of experiment, N_{in} and N_{out} the number of incoming and outgoing particle, Φ be the incoming flux ($N_{in}|\mathbf{v}|/V$, where $|\mathbf{v}|$ is the velocity of the beam). Then the classical cross section is

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Moreover, the probability of interaction is given by

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where $d\Pi$ is a volume differential of the phase (momentum) space

$$d\Pi = \prod_j \frac{V d^3p_j}{(2\pi)^3} \ .$$

Hence, (2.13) becomes

$$d\sigma = \frac{V}{|\mathbf{v}_1 - \mathbf{v}_2|T} \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} \prod_j \frac{V d^3 p_j}{(2\pi)^3} \ . \tag{2.14}$$

Now, using the property

$$\delta^3(0) = \int \frac{d^3x}{(2\pi)^3} \exp(ipx)\Big|_{p=0} = \frac{1}{(2\pi)^3} \int d^3x = \frac{V}{(2\pi)^3} \ ,$$

we find

$$\langle p|p\rangle = (2\pi)^3 \delta^3(0) = (2\pi)^3 2E_p \frac{V}{(2\pi)^3} = 2E_p V \ .$$

In particular, taking $|i\rangle = |p_1, p_2\rangle$ and $|f\rangle = |\prod_j p_j\rangle$, we obtain

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where we have called $|\mathcal{M}|^2 = |\langle f|\hat{M}|i\rangle|^2$. Since the square of a delta can be written as

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we obtain

$$|\langle f|\hat{T}|i\rangle|^2 = (2\pi)^4 TV \delta^4(p_1 + p_2 - \sum_j p_j)|\mathcal{M}|^2 \ . \tag{2.16}$$

Putting everything together (2.14), (2.15) and (2.16), we find

$$
\begin{aligned}
d\sigma &= \frac{V}{|\mathbf{v}_1 - \mathbf{v}_2|T}(2\pi)^4 TV\delta^4(p_1 + p_2 - \sum_j p_j)|\mathcal{M}|^2 \frac{1}{4E_1E_2V^2}\frac{1}{\prod_j 2E_jV}\prod_j \frac{Vd^3p_j}{(2\pi)^3} \\
&= \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|4E_1E_2}|\mathcal{M}|^2(2\pi)^4\delta^4(p_1 + p_2 - \sum_j p_j)\prod_j \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j} \\
&= \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|4E_1E_2}|\mathcal{M}|^2 d\Pi_N \ ,
\end{aligned}
$$

where we have defined the Lorentz-invariant phase space element $d\Pi_N$ as

$$
d\Pi_N = (2\pi)^4\delta^4(p_1 + p_2 - \sum_j p_j)\prod_j \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j} \ .
$$