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On relativistic quantum mechanics:

how to firstly quantise?

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Theoretical Physics

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Part I

Klein-Gordon equation

Chapter 1

Derivation

To find the equation of motion which governs a relativistic quantum particle, we start with the relation between energy and momentum. We define the 4-momentum as

$$p^\mu = (E, \vec{p})$$

such that its norm is constant

$$p^\mu p_\mu = -E^2 + |\vec{p}|^2 = -m^2$$

which is called the mass-shell condition. The energy-momentum relation is then

$$E^2 = p^2 + m^2 \tag{1.1}$$

Now, we can use this energy and promote it to an operator

$$E = \sqrt{p^2 + m^2}$$

and then put it inside the Schroedinger equation

$$i\frac{\partial}{\partial t}\phi(t, \vec{x}) = E\phi(t, \vec{x}) = \sqrt{p^2 + m^2}\phi(t, \vec{x})$$

The square root of an operator could lead us to a non-local theory and this approach was abandoned.

Klein and Gordon used instead the square relation between energy and momentum, but uses the operator substitution

$$E \to i\frac{\partial}{\partial t} \quad \vec{p} \to -i\vec{\nabla} \tag{1.2}$$

and the energy-momentum relation becomes

$$(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2)\phi(t, \vec{x}) = 0$$

which in covariant notation is

$$(\partial^\mu \partial_\mu - m^2)\phi(x) = (\Box - m^2)\phi(x) = 0 \tag{1.3}$$

Chapter 2

Solutions

2.1 Plane waves

Plane waves are solutions. Let us consider a plane waves ansatz

$$\phi(x) = \exp(ip^\mu x_\mu)$$

where p^μ is arbitrary. Using (1.3)

$$-(p^\mu p_\mu + m^2)\exp(ip_\mu x^\mu) = 0$$

Hence, the plane wave is a solution only if p^μ satistfies the mass-shell condition

$$(p^0)^2 = \vec{p}^2 + m^2 \Rightarrow p^0 = \pm\sqrt{\vec{p}^2 + m^2} = \pm E_p$$

Since $E_p geq 0$, notice that negative energy are allowed as solutions. This shows that the theory is not stable, since there is no energy limitation from below.

Positive energy solutions are

$$\phi_p^+ = \exp(i(-E_p t + \vec{p}\cdot\vec{x}))$$

while the negative energy solutions are

$$\phi_p^- = \exp(i(E_p t - \vec{p}\cdot\vec{x}))$$

Hence, a general solution is a linear combination of plane waves

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(a(\vec{p})\exp(i(-E_p t + \vec{p}\cdot\vec{x})) + b^*(\vec{p})\exp(i(E_p t - \vec{p}\cdot\vec{x})))$$

and its complex conjugate

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(a^*(\vec{p})\exp(i(-E_p t + \vec{p}\cdot\vec{x})) + b(\vec{p})\exp(i(E_p t - \vec{p}\cdot\vec{x})))$$

where $a(\vec{p})$ and $b(\vec{p})$ are the Fourier coefficients. If the field is real, i.e. $\phi 0 \phi^*$, then $a(\vec{p}) = b(\vec{p})$.

2.2 Continuity equation

Starting from (1.3) and multiplying by the complex conjugation

$$0 = \phi^*(\Box - m^2)\phi - \phi(\Box - m^2)\phi^* = \partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$$

Hence, the current is

$$J^\mu = \frac{1}{2im}(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$$

such that it satisfies the continuity equation

$$\partial_\mu J^\mu = 0$$

The time component is

$$J^0 = \frac{1}{2im}(\phi^*\partial^t\phi - \phi\partial^t\phi^*) = \frac{i}{2m}(\phi^*\partial_t\phi - \phi\partial_t\phi^*)$$

which is not positive, even though it is real. This is because it can be negative or positive by the initial condition choice, e.g. for plane waves

$$J^0(\phi^\pm_{\vec{p}}) = \pm\frac{E_p}{m}$$

Hence, there is no probability intepretation (what is a negative probability?). Dirac suggested a particle intepretation, lead to a electromagnetism comparison.

2.3 Green functions

The Green function is a solution of the Klein-Gordon equation in presence of a point-like istantaneous source. For convenience, we put it at the origin. Mathematically, it satisfies the equation

$$(-\Box + m^2)G(x) = \delta^4(x)$$

It is useful to determine the solution of a general source $J(x)$

$$(-\Box + m^2)\phi(x) = J(x)$$

which are related by

$$\phi(x) = \phi_0(x) + \int d^4y\ G(x-y)J(y)$$

wherw $\phi_0(x)$ is a solution of the associated inhomogeneous equation.

Proof.

$$
\begin{aligned}
(-\Box + m^2)\phi(x) &= (-\Box + m^2)\Big(\phi_0(x) + \int d^4y\ G(x-y)J(y)\Big) \\
&= \underbrace{(-\Box + m^2)\phi_0(x)}_{0} + (-\Box + m^2)\int d^4y\ G(x-y)J(y) \\
&= \int d^4y\ \underbrace{(-\Box + m^2)G(x-y)}_{\delta^4(x-y)}\, J(y) \\
&= \int d^4y\ \delta^4(x-y)J(y) \\
&= J(x)
\end{aligned}
$$

q.e.d.

In general, the Green function is not unique, but it depends on boundary conditions chosen at the infinity. We use the Feynman-Stueckelberg which gives the correct quantum intepretation, i.e. negative energies are antiparticles. This means that positive energy particle are propagated forward in time and negative energy particle are propagated backward in time. It describes also real particle, i.e. whose satisfy the mass-shell condition, and virtal particle, i.e. whose do not. The latter ones are not visible at large distances.

Using the Fourier transform

$$G(x) = \int \frac{d^4p}{(2\pi)^4} \exp(ip_\mu x^\mu)\tilde{G}(p)$$

which could be written as

$$G(x) = \int \frac{d^4p}{(2\pi)^4} \exp(ip_\mu x^\mu) p^2 + m^2 - i\epsilon$$

where $\epsilon \ll 1$ is a positive infinitesimal number.

The propagator is the Green function which is the amplitude of propagation from a point y to another point x

$$\Delta(x-y) = -iG(x-y)$$

In our case, it becomes

$$\Delta(x-y) = -iG(x-y) = \tag{2.1}$$

2.4 Yukawa potential

Yukawa suggested from the Klein-Gordon equation, a theory of nuclear interactions.

2.5 Action

The Klein-Gordon equation for a complex scalar field $\phi(x)$ can be obtained by the following lagrangian

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{2.2}$$

Proof. We apply the Euler-Lagrange equation, first for ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = -m^2 \phi^* + \partial_\mu \partial^\mu \phi^* = (\Box - m^2)\phi^* = 0$$

which is the Klein-Gordon equation (1.3) for ϕ^*, and similarly for ϕ^*

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} = -m^2 \phi + \partial_\mu \partial^\mu \phi = (\Box - m^2)\phi = 0$$

which is the Klein-Gordon equation (1.3) for ϕ. q.e.d.

The Klein-Gordon equation for a real scalar field $\phi = \phi^*$ can be obtained by the following lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 \tag{2.3}$$

Proof. We apply the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = -m^2 \phi + \partial_\mu \partial^\mu \phi = (\Box - m^2)\phi = 0$$

which is the Klein-Gordon equation (1.3). q.e.d.

The complex scalar field can be seen as a linear combination of two real fields of the same mass

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \qquad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$$

and the related lagrangian (2.2) becomes (2.3)

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi = -\frac{1}{2}\partial^\mu \phi_1 \partial_\mu \phi_1 - \frac{1}{2}\partial^\mu \phi_2 \partial_\mu \phi_2 - \frac{m^2}{2}(\phi_1^2 + \phi_2^2)$$

Noether's theorem

The free Klein-Gordon complex field is globally invariant under the $U(1)$ symmetry

$$\phi'(x) = \exp(i\alpha)\phi(x) \quad (\phi')^*(x) = \exp(-i\alpha)\phi^*(x)$$

or, infinitesimally,

$$\delta_\alpha\phi(x) = i\alpha\phi(x) \quad \delta_\alpha\phi^*(x) = -i\alpha\phi^*(x)$$

Proof. Maybe in the future. q.e.d.

We apply the Noether's theorem and we find associated Noether's current

$$J^\mu = i\phi^*\partial^\mu\phi - i(\partial^\mu\phi^*)\phi$$

which satisfies the continuity equation

$$\partial^\mu J_\mu = 0$$

and its related conserved charge is

$$Q = \int d^3x \ J^0 = -i\int d^3x \ (\phi^*\partial_0\phi - i(\partial_0\phi^*)\phi)$$

Proof. Maybe in the future. q.e.d.

Energy-momentum tensor

The free Klein-Gordon complex field is globally invariant under the Poincaré group

$$(xi)^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \phi'(x') = \phi(x) \quad (\phi')^*(x') = \phi^*(x)$$

In particular, we are interested in the spacetime translations a^μ, which infinitesimally look like

$$\delta_\alpha\phi(x) = -a^\mu\partial_\mu\phi(x) \quad \delta_\alpha\phi^*(x) = -a^\mu\partial_\mu\phi^*(x)$$

Proof. Maybe in the future. q.e.d.

We apply the Noether's theorem and we find associated energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi + \eta^{\mu\nu}\mathcal{L}$$

which satisfies the conserved equation

$$\partial_\mu T^{\mu\nu} = 0$$

and its related conserved charges are the total 4-momentum carried by the field

$$P^\mu = \in d^3x \; T^{0\mu}$$

In particular, the energy density is

$$\mathcal{E} = T^{00} = \partial_0\phi^*\partial_0\phi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi$$

and the total energy is

$$E = \int d^3x \; \mathcal{E} = \int d^3x \; (\partial_0\phi^*\partial_0\phi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)$$

Proof. Maybe in the future. q.e.d.

Part II

Dirac theory

Chapter 3

Dirac equation

3.1 Derivation

We are looking for a quantum equation that describes $\frac{1}{2}$-spin particles, but unlike the Klein-Gordon equation, it allows a probabilistic interpretation. The problem with the Klein-Gordon equation is the presence of second-order terms in time, therefore we need an hamiltonian which is linear but at the same time recover the energy-momentum relation (1.1). The first guess is

$$E = c\mathbf{p} \cdot \boldsymbol{\alpha} + mc^2\beta \ , \tag{3.1}$$

where α and β are hermitian matrices such that satisfies the Clifford algebra

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}\mathbb{I} \ , \quad \{\beta, \beta\} = 2\mathbb{I} \ , \quad \{\alpha^i, \beta\} = 0 \ . \tag{3.2}$$

Proof. Infact, we compute the square of (3.1)

$$\begin{aligned}
E^2 &= (cp^i\alpha^i + mc^2\beta)^2 \\
&= (cp^i\alpha^i + mc^2\beta)(cp^j\alpha^j + mc^2\beta) \\
&= c^2\alpha^i p^i \alpha^j p^j + \beta^2 m^2 c^4 + mc^3 p^i \alpha^i \beta + mc^3 p^j \beta \alpha^j \\
&= c^2 p^i p^j \underbrace{\alpha^i \alpha^j}_{\frac{\alpha^i\alpha^j + \alpha^j\alpha^i}{2}} + \beta^2 m^2 c^4 + mc^3 p^i(\alpha^i\beta + \beta\alpha^i) \\
&= c^2 p^i p^j \frac{\alpha^i\alpha^j + \alpha^j\alpha^i}{2} + \beta^2 m^2 c^4 + mc^3 p^i(\alpha^i\beta + \beta\alpha^i) \ ,
\end{aligned}$$

where in the fourth row, we exploit the symmetry of $p^i p^j$ to symmetrise $\alpha^i\alpha^j$. We compare it with (1.1)

$$E^2 = c^2 p^i p^j \underbrace{\frac{\alpha^i\alpha^j + \alpha^j\alpha^i}{2}}_{\delta^{ij}} + \underbrace{\beta^2}_{1} m^2 c^4 + mc^3 p^i \underbrace{(\alpha^i\beta + \beta\alpha^i)}_{0} = p^2c^2 + m^2c^4 \ .$$

Hence

$$\{\alpha^i, \alpha^j\} = \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij} \ ,$$
$$\{\beta, \beta\} = \beta^2 + \beta^2 = 2\beta^2 = 2 \ ,$$
$$\{\alpha^i, \beta\} = \alpha^i\beta + \beta\alpha^i = 0 \ .$$

q.e.d.

The minimal solutions for the set of algebraic equation (3.2) are 4×4 traceless matrices $\boldsymbol{\alpha}$ and β such that

$$\alpha^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \ , \quad \beta = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \ ,$$

where σ^i are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ , \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ ,$$

and they satisfy the relation

$$\sigma^i\sigma^j = \delta^{ij}\mathbb{I} + i\epsilon^{ijk}\sigma^k \ .$$

It is called the Dirac representation and it is the only irreducible representation of the Clifford algebra up to others that are unitarily equivalent (by a change of basis) to the Dirac one or that are higher dimensional and thus reducible.

The hamiltonian form of the Dirac equation becomes

$$i\hbar\frac{\partial}{\partial t}\psi(t, \mathbf{x}) = (-i\hbar c\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta mc^2)\psi(t, \mathbf{x}) = H_D\psi(t, \mathbf{x}) \ , \tag{3.3}$$

while in covariant form it becomes

$$(\gamma^\mu\partial_\mu + m)\psi(x) = (\not{\partial} + m)\psi(x) = 0 \ , \tag{3.4}$$

where $\psi(t, \mathbf{x})$ is a matrix

$$\psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix} \ ,$$

and γ^μ are the matrices

$$\gamma^0 = -i\beta \ , \quad \gamma^i = -i\beta\alpha^i \ ,$$

such that they satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ . \tag{3.5}$$

Explicitly, they are

$$\gamma^0 = -i\begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix} \ , \quad \gamma^i = \begin{bmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{bmatrix} \ .$$

Proof. Infact, we compute operator substitution (1.2) on (3.1)

$$\underbrace{E}_{i\hbar\frac{\partial}{\partial t}}\psi(t,\mathbf{x}) = (c\underbrace{\mathbf{p}}_{-i\hbar\boldsymbol{\nabla}}\cdot\boldsymbol{\alpha} + mc^2\beta)\psi(t,\mathbf{x}) \ .$$

Hence

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{x}) = (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)\psi(t,\mathbf{x}) = H_D\psi(t,\mathbf{x}) \ .$$

In order to write it in covariant form, we compute

$$\begin{aligned}\frac{\beta}{\cancel{\hbar}c}i\cancel{\hbar}\frac{\partial}{\partial t}\psi(t,\mathbf{x}) &= \frac{\beta}{\hbar c}(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)\psi(t,\mathbf{x}) \\ &= \Big(-\frac{\beta}{\cancel{\hbar}\cancel{c}}i\cancel{\hbar}\cancel{c}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{mc}{\hbar}\underbrace{\beta^2}_{1}\Big)\psi(t,\mathbf{x}) \ .\end{aligned}$$

Hence

$$i\frac{\beta}{c}\frac{\partial}{\partial t}\psi(t,\mathbf{x}) = \Big(-i\beta\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{mc}{\hbar}\Big)\psi(t,\mathbf{x}) \ ,$$

$$\Big(\underbrace{-i\beta}_{\gamma^0}\frac{1}{c}\frac{\partial}{\partial t}\underbrace{-i\beta\boldsymbol{\alpha}}_{\boldsymbol{\gamma}}\cdot\boldsymbol{\nabla} + \frac{mc}{\hbar}\Big)\psi(t,\mathbf{x}) = 0 \ ,$$

$$\Big(\gamma^0\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + \frac{mc}{\hbar}\Big)\psi(t,\mathbf{x}) = 0 \ ,$$

and in covariant form, we obtain

$$(\gamma^\mu\partial_\mu + \mu)\psi(x) = 0 \ ,$$

where $\mu = \frac{mc}{\hbar}$ is the inverse reduced Compton wavelength. In natural units, it becomes

$$(\gamma^\mu\partial_\mu + m)\psi(x) = 0 \ .$$

Finally, they satisfy the Clifford algebra

$$\{\gamma^0,\gamma^0\} = 2\eta^{00} = 2 \ ,$$

$$\{\gamma^i,\gamma^j\} = 2\eta^{ij} = 2\delta^{ij}$$

and

$$\{\gamma^0,\gamma^i\} = 2\eta^{0i} = 0 \ .$$

q.e.d.

3.2 Continuity equation

The continuity equation associated to the Dirac equation is

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + \boldsymbol{\nabla}\cdot(c\psi^\dagger\boldsymbol{\alpha}\psi) = 0$$

where the density charge is positive defined $\psi^\dagger\psi > 0$ and it's compatible with the probabilistic intepretation.

Proof. Infact, we multiply by $\psi^\dagger$ and subtract the hermitian conjugate on (3.3)

$$\begin{aligned}
0 &= \psi^\dagger(i\hbar\frac{\partial}{\partial t}\psi - (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta)\psi) - (\psi^\dagger(i\hbar\frac{\partial}{\partial t}\psi - (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta)\psi))^\dagger \\
&= \psi^\dagger(i\hbar\frac{\partial}{\partial t}\psi - (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta)\psi) - (-i\hbar\frac{\partial}{\partial t}\psi^\dagger - (i\hbar c\underbrace{\boldsymbol{\alpha}^\dagger}_{\boldsymbol{\alpha}}\cdot\boldsymbol{\nabla} + mc^2\underbrace{\beta^\dagger}_{\beta})\psi^\dagger)\psi \\
&= i\hbar\psi^\dagger\frac{\partial}{\partial t}\psi + i\hbar c\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi - \cancel{mc^2\beta\psi^\dagger\psi} + i\hbar\psi\frac{\partial}{\partial t}\psi^\dagger + i\hbar c\psi\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi^\dagger + \cancel{mc^2\beta\psi^\dagger\psi} \\
&= \cancel{i\hbar}\underbrace{\left(\psi^\dagger\frac{\partial}{\partial t}\psi + \psi\frac{\partial}{\partial t}\psi^\dagger\right)}_{\frac{\partial}{\partial t}(\psi^\dagger\psi)} + \cancel{i\hbar}\underbrace{(c\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \psi\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi^\dagger)}_{\boldsymbol{\nabla}\cdot(c\psi^\dagger\boldsymbol{\alpha}\psi)} \\
&= \frac{\partial}{\partial t}(\psi^\dagger\psi) + \boldsymbol{\nabla}\cdot(c\psi^\dagger\boldsymbol{\alpha}\psi)\ .
\end{aligned}$$

q.e.d.

3.3 Gamma matrices

As we said before, α^i and β are hermitian whicle γ^0 is antihermitian and γ^i is hermitian

$$(\gamma^0)^\dagger = -\gamma^0\ , \quad (\gamma^i)^\dagger = \gamma^i\ .$$

which can be written in the following way

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 = -\beta\gamma^\mu\beta\ .$$

This means that the Clifford algebra is valid for $(\gamma^\mu)^\dagger$, intepreted as a change of basis

$$\{-(\gamma^\mu)^\dagger, -(\gamma^\nu)^\dagger\} = 2\eta^{\mu\nu}\ .$$

Proof. Infact, by the hermiticity of α^i and β

$$(\gamma^0)^\dagger = (-i\beta)^\dagger = i\beta = -\gamma^0$$

and

$$\begin{aligned}(\gamma^i)^\dagger = (-i\beta\alpha^i)^\dagger = (\gamma^0\alpha^i)^\dagger = \alpha^i \underbrace{(\gamma^0)^\dagger}_{\gamma^0} \\ = -\alpha^i\gamma^0 = -i\underbrace{\alpha^i\beta}_{\beta\alpha^i} = -i\beta\alpha^i = \gamma^i \ .\end{aligned}$$

Furthermore,

$$(\gamma^0)^\dagger = \underbrace{\gamma^0\gamma^0}_{-1}\gamma^0 = -\gamma^0$$

and

$$(\gamma^i)^\dagger = \gamma^0\underbrace{\gamma^i\gamma^0}_{-\gamma^0\gamma^i} = -\gamma^0\gamma^0\gamma^i = -\underbrace{(\gamma^0)^2}_{-1}\gamma^i = \gamma^i \ .$$

Finally, using (3.5)

$$\begin{aligned}\{-(\gamma^\mu)^\dagger, -(\gamma^\nu)^\dagger\} &= (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger + (\gamma^\nu)^\dagger(\gamma^\mu)^\dagger \\ &= \gamma^0\gamma^\mu\underbrace{\gamma^0\gamma^0}_{-1}\gamma^\nu\gamma^0 + \gamma^0\gamma^\nu\underbrace{\gamma^0\gamma^0}_{-1}\gamma^\mu\gamma^0 \\ &= -\underbrace{\gamma^0\gamma^\mu}_{-\gamma^\mu\gamma^0}\underbrace{\gamma^\nu\gamma^0}_{-\gamma^0\gamma^\nu} - \underbrace{\gamma^0\gamma^\nu}_{-\gamma^\nu\gamma^0}\underbrace{\gamma^\mu\gamma^0}_{-\gamma^0\gamma^\mu} \\ &= -\gamma^\mu\underbrace{\gamma^0\gamma^0}_{-1}\gamma^\nu - \gamma^\nu\underbrace{\gamma^0\gamma^0}_{-1}\gamma^\mu = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \ .\end{aligned}$$

q.e.d.

As we said before, α^i and β are traceless and so γ^μ are

$$\operatorname{tr}\gamma^\mu = 0 \ .$$

Proof. Infact, by the linearity and cyclic property of the trace and (3.2)

$$\operatorname{tr}\gamma^0 = \operatorname{tr}(-i\beta) = -i\underbrace{\operatorname{tr}\beta}_{0} = 0$$

and

$$\begin{aligned}\operatorname{tr}(\gamma^i) = \operatorname{tr}(\mathbb{I}\gamma^i) = \operatorname{tr}((\gamma^j)^2\gamma^i) = \operatorname{tr}(\gamma^j\underbrace{\gamma^j\gamma^i}_{-\gamma^i\gamma^j}) \\ = -\operatorname{tr}(\gamma^j\gamma^i\gamma^j) = -\operatorname{tr}(\gamma^i\gamma^j\gamma^j) = -\operatorname{tr}(\gamma^i(\gamma^j)^2) = -\operatorname{tr}(\gamma^i) \ .\end{aligned}$$

q.e.d.

γ^5

We introduce another matrix γ^5, called the chirality matrix

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

such that it satisfies the gamma-matrix properties

1. anticommutator, i.e.
$$\{\gamma^5, \gamma^\mu\} = 0 \ ,$$
2. the square is the identity, i.e.
$$(\gamma^5)^2 = \mathbb{I} \ , \tag{3.6}$$
3. hermiticity, i.e.
$$(\gamma^5)^\dagger = \gamma^5 \ ,$$
4. traceless, i.e.
$$\mathrm{tr}(\gamma^5) = 0 \ .$$

Proof. For the anticommutator property

$$\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu - i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \ .$$

Now, consider $\mu = 0$

$$\begin{aligned}\{\gamma^5, \gamma^0\} &= -i\gamma^0\gamma^1\gamma^2 \underbrace{\gamma^3\gamma^0}_{-\gamma^0\gamma^3} - i \underbrace{\gamma^0\gamma^0}_{-1} \gamma^1\gamma^2\gamma^3 \\ &= i\gamma^0\gamma^1 \underbrace{\gamma^2\gamma^0}_{-\gamma^0\gamma^2} \gamma^3 + i\gamma^1\gamma^2\gamma^3 \\ &= -i\gamma^0 \underbrace{\gamma^1\gamma^0}_{-\gamma^0\gamma^1} \gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3 \\ &= i \underbrace{\gamma^0\gamma^0}_{-1} \gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3 \\ &= -i\gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3 = 0\end{aligned}$$

and similarly for $\mu = 1, 2, 3$.

For the square property

$$(\gamma^5)^2 = (-i\gamma^0\gamma^1\gamma^2\gamma^3)^2 = -\underbrace{(\gamma^0)^2}_{-\mathbb{I}} \underbrace{(\gamma^1)^2}_{\mathbb{I}} \underbrace{(\gamma^2)^2}_{\mathbb{I}} \underbrace{(\gamma^3)^2}_{\mathbb{I}} = \mathbb{I} \ .$$

For the hermiticity property

$$
\begin{aligned}
(\gamma^5)^\dagger &= (-i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger \\
&= i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \\
&= i\gamma^0\gamma^3\underbrace{\gamma^0\gamma^0}_{-1}\gamma^2\underbrace{\gamma^0\gamma^0}_{-1}\gamma^1\underbrace{\gamma^0\gamma^0}_{-1}\underbrace{\gamma^0\gamma^0}_{-1} \\
&= i\gamma^0\gamma^3\underbrace{\gamma^2\gamma^1}_{\gamma^1\gamma^2} \\
&= -i\gamma^0\underbrace{\gamma^3\gamma^1}_{-\gamma^3\gamma^1}\gamma^2 \\
&= i\gamma^0\gamma^1\underbrace{\gamma^3\gamma^2}_{\gamma^2\gamma^3} \\
&= -i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \ .
\end{aligned}
$$

For the traceless property

$$
\begin{aligned}
\mathrm{tr}(\gamma^5) &= \mathrm{tr}(-i\gamma^0\gamma^1\gamma^2\gamma^3) \\
&= -i\,\mathrm{tr}(\underbrace{\gamma^0\gamma^1}_{-\gamma^1\gamma^0}\gamma^2\gamma^3) \\
&= i\,\mathrm{tr}(\gamma^1\underbrace{\gamma^0\gamma^2}_{-\gamma^2\gamma^0}\gamma^3) \\
&= -i\,\mathrm{tr}(\gamma^1\gamma^2\underbrace{\gamma^0\gamma^3}_{-\gamma^3\gamma^0}) \\
&= i\,\mathrm{tr}(\gamma^1\gamma^2\gamma^3\gamma^0) \\
&= i\,\mathrm{tr}(\gamma^0\gamma^1\gamma^2\gamma^3) \\
&= -\,\mathrm{tr}(\gamma^5) \ ,
\end{aligned}
$$

where we have used the cyclic property of the trace. q.e.d.

Explicitly, in the Dirac representation, it becomes

$$
\gamma^5 = \begin{bmatrix} 0 & -\mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{bmatrix} \ .
$$

Proof. Infact

$$\begin{aligned}
\gamma^5 &= -i\gamma^0\gamma^1\gamma^2\gamma^3 = (-i)^5 \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix} \\
&= -i \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & -\sigma^1\sigma^2 \\ -\sigma^1\sigma^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix} \\
&= -i \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & -\underbrace{\sigma^1\sigma^2}_{i\sigma^3}\sigma^3 \\ \underbrace{\sigma^1\sigma^2}_{i\sigma^3}\sigma^3 & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & -\underbrace{(\sigma^3)^2}_{\mathbb{I}_2} \\ \underbrace{(\sigma^3)^2}_{\mathbb{I}_2} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{bmatrix} .
\end{aligned}$$

q.e.d.

It adds another dimension to the 4-dimensional Clifford algebra. Infact, we can define a 5-dimensional Clifford algebra by the anticommutator relations

$$\{\gamma^M, \gamma^N\} = 2\eta^{MN} ,$$

where $M, N = 0, 1, 2, 3, 5$ with Minkovski metric $\eta^{MN} = diag(-++++)$.

It can be used to define the projection operators on chiral (Weyl) spinors

$$P_L = \frac{\mathbb{I} - \gamma^5}{2} , \quad P_L = \frac{\mathbb{I} - \gamma^5}{2} ,$$

such that they satisfy the following properties

1. nilpotent, i.e.
$$P_L^2 = P_L , \quad P_R^2 = P_R ,$$
2. orthogonality, i.e.
$$P_L P_R = 0 , \quad P_L + P_R = \mathbb{I} .$$

Proof. For the nilpotent property, using (3.6)

$$P_L^2 = \Big(\frac{\mathbb{I} - \gamma^5}{2}\Big)^2 = \frac{1}{4}(\mathbb{I}^2 - 2\gamma^5 + \underbrace{(\gamma^5)^2}_{\mathbb{I}}) = \frac{2\mathbb{I} - 2\gamma^5}{4} = \frac{\mathbb{I} - \gamma^5}{2} = P_L$$

and

$$P_R^2 = \Big(\frac{\mathbb{I} + \gamma^5}{2}\Big)^2 = \frac{1}{4}(\mathbb{I}^2 + 2\gamma^5 + \underbrace{(\gamma^5)^2}_{\mathbb{I}}) = \frac{2\mathbb{I} + 2\gamma^5}{4} = \frac{\mathbb{I} + \gamma^5}{2} = P_R .$$

For the orthogonality property, using (3.6)

$$P_L P_R = \left(\frac{\mathbb{I} - \gamma^5}{2}\right)\left(\frac{\mathbb{I} + \gamma^5}{2}\right) = \frac{1}{4}(\mathbb{I}^2 - \underbrace{(\gamma^5)^2}_{\mathbb{I}}) = \frac{\mathbb{I} - \mathbb{I}}{4} = 0$$

and

$$P_L + P_R = \left(\frac{\mathbb{I} - \cancel{\gamma^5}}{2}\right) + \left(\frac{\mathbb{I} + \cancel{\gamma^5}}{2}\right) = \frac{\mathbb{I} + \mathbb{I}}{2} = \mathbb{I}\ .$$

q.e.d.

They allow to divide a Dirac spinor into two components $\psi = \psi_L + \psi_R$, where $\psi_L = P_L\psi$ is the left-handed one and $\psi_R = P_R\psi$ is the right-handed one. Infact a Dirac spinor can be written as $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. Spinors live in a 4-dimensional complex linear space $\psi(x) \in \mathbb{C}^4$. Therefore, the gamma matrices are an example of 4-dimensional matrices act on this space. A complete basis of linear operators must have 16 of them and we can choose $(\mathbb{I}, \gamma^5, \Sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5)$ where $\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ with $\mu > \nu$. They are indeed respectively $1+4+6+4+1 = 16$ linearly independent matrices.

Chapter 4

Non-relativistic limit

Chapter 5

Covariance

5.1 Dirac spinor representation

Now, we verify that the Dirac equation is Lorentz invariant, i.e it is covariant under a generic transformation of $SO^+(1,3)$. Recall that given a Lorentz transformation $\Lambda \in SO^+(1,3)$, the coordinates transform as

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu$$

and the partial derivatives transform as

$$\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu \ .$$

Therefore, the Dirac spinor transform as

$$\psi'(x') = S(\Lambda)\psi(x)$$

where $S(\Lambda)$ is linear representation of the proper orthochronous group of spinors and the Dirac equation is covariant

$$(\gamma^\mu \partial'_\mu + m)\psi'(x') = 0 \ .$$

The infinitesimal Lorentz transformation $S(\Lambda)$ is

$$S = \mathbb{I} + \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}$$

where $\Sigma^{\mu\nu}$ are a set of 6 antisymmetric 4×4 matrices that act on spinors

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] \ , \tag{5.1}$$

such that they satisfy the commutator relations

$$[\Sigma^{\mu\nu}, \gamma^\rho] = i(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu) \ .$$

Proof. We transform with a Lorentz transformation every components in the Dirac equation

$$0 = (\gamma^\mu \partial'_\mu + m)\psi'(x') = (\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu + m)S(\Lambda)\psi(x) \ .$$

Hence

$$\begin{aligned} 0 &= S^{-1}(\Lambda)(\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu + m)S(\Lambda)\psi(x) \\ &= (S^{-1}(\Lambda)\gamma^\mu \Lambda_\mu{}^\nu S(\Lambda)\partial_\nu + m\, \underbrace{S^{-1}(\Lambda)S(\Lambda)}_{1})\psi(x) \\ &= (S^{-1}(\Lambda)\gamma^\mu \Lambda_\mu{}^\nu S(\Lambda)\partial_\nu + m)\psi(x) \ . \end{aligned}$$

We compare it with (3.4)

$$0 = (S^{-1}(\Lambda)\gamma^\mu \Lambda_\mu{}^\nu S(\Lambda)\partial_\nu + m)\psi(x) = (\gamma^\nu \partial_\nu + m)\psi(x)$$

and we find

$$S^{-1}(\Lambda)\gamma^\mu \Lambda_\mu{}^\nu S(\Lambda) = \gamma^\nu$$

or, equivalently,

$$\begin{aligned} S^{-1}(\Lambda)\gamma^\mu \underbrace{\Lambda^\rho{}_\nu \Lambda_\mu{}^\nu}_{\delta^\rho{}_\mu} S(\Lambda) &= \Lambda^\rho{}_\nu \gamma^\nu \ , \\ S^{-1}(\Lambda)\underbrace{\gamma^\mu \delta^\rho{}_\mu}_{\gamma^\rho} S(\Lambda) &= \Lambda^\rho{}_\nu \gamma^\nu \ , \\ S^{-1}(\Lambda)\gamma^\rho S(\Lambda) &= \Lambda^\rho{}_\nu \gamma^\nu \ . \end{aligned} \tag{5.2}$$

Now, we consider an infinitesimal Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \ ,$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, which induces an infinitesimal Lorentz transformation on the spinor

$$S(\Lambda) = \mathbb{I} + \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu} \ ,$$

where $\Sigma^{\mu\nu} = -\Sigma^{\mu\nu}$. Substituting in (5.2), we find

$$\left(\mathbb{I} - \frac{i}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}\right)\gamma^\rho\left(\mathbb{I} + \frac{i}{2}\omega_{\sigma\lambda}\Sigma^{\sigma\lambda}\right) = (\delta^\rho{}_\nu + \omega^\rho{}_\nu)\gamma^\nu \ .$$

and we only keep first order terms in ω

$$\begin{aligned} \cancel{\gamma^\rho} - \frac{i}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}\gamma^\rho + \frac{i}{2}\gamma^\rho\omega_{\sigma\lambda}\Sigma^{\sigma\lambda} &= \cancel{\gamma^\rho} + \omega^\rho{}_\nu\gamma^\nu \ , \\ -\frac{i}{2}\omega_{\alpha\beta}\underbrace{(\Sigma^{\alpha\beta}\gamma^\rho - \gamma^\rho\Sigma^{\alpha\beta})}_{[\Sigma^{\alpha\beta},\gamma^\rho]} &= \omega^\rho{}_\nu\gamma^\nu \ , \end{aligned}$$

$$-\frac{i}{2}\omega_{\alpha\beta}[\Sigma^{\alpha\beta},\gamma^\rho]=\omega^\rho{}_\nu\gamma^\nu\ ,$$

where we have exchanged $\sigma=\alpha$ and $\lambda=\beta$. Hence

$$\cancel{\omega_{\alpha\beta}}[\Sigma^{\alpha\beta},\gamma^\rho]=2i\omega^\rho{}_\beta\gamma^\beta=\omega_{\alpha\beta}2i\underbrace{\eta^{\rho\alpha}\gamma^\beta}_{\frac{\eta^{\rho\alpha}\gamma^\beta-\eta^{\rho\beta}\gamma^\alpha}{2}}=\cancel{\omega_{\alpha\beta}}i(\eta^{\rho\alpha}\gamma^\beta-\eta^{\rho\beta}\gamma^\alpha)\ ,$$

where we have exchanged $\nu=\beta$ and we exploit the antysymmetry of $\omega_{\alpha\beta}$ to anti-symmetrise $\eta^{\rho\alpha}\gamma^\beta$. Thus

$$[\Sigma^{\alpha\beta},\gamma^\rho]=i(\eta^{\rho\alpha}\gamma^\beta-\eta^{\rho\beta}\gamma^\alpha)\ .$$

The solution of this algebraic commutation equation is

$$\Sigma^{\alpha\beta}=-\frac{i}{4}[\gamma^\alpha,\gamma^\beta]\ .$$

Infact, using (3.5)

$$\begin{aligned}
[\Sigma^{\alpha\beta},\gamma^\mu]&=-\frac{i}{4}[\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha,\gamma^\mu]\\
&=-\frac{i}{4}[\gamma^\alpha\gamma^\beta,\gamma^\mu]-\frac{i}{4}[\gamma^\beta\gamma^\alpha,\gamma^\mu]\\
&=-\frac{i}{4}(\gamma^\alpha\{\gamma^\beta,\gamma^\mu\}-\{\gamma^\alpha,\gamma^\mu\}\gamma^\beta-\gamma^\beta\{\gamma^\alpha,\gamma^\mu\}+\{\gamma^\beta,\gamma^\mu\}\gamma^\alpha)\\
&=-\frac{i}{4}(\gamma^\alpha\underbrace{\{\gamma^\beta,\gamma^\mu\}}_{2\eta^{\beta\mu}}-\underbrace{\{\gamma^\alpha,\gamma^\mu\}}_{2\eta^{\alpha\mu}}\gamma^\beta-\gamma^\beta\underbrace{\{\gamma^\alpha,\gamma^\mu\}}_{2\eta^{\alpha\mu}}+\underbrace{\{\gamma^\beta,\gamma^\mu\}}_{2\eta^{\beta\mu}}\gamma^\alpha)\\
&=-\frac{i}{2}(\gamma^\alpha\eta^{\beta\mu}-\eta^{\alpha\mu}\gamma^\beta-\gamma^\beta\eta^{\alpha\mu}+\eta^{\beta\mu}\gamma^\alpha)\\
&=-\frac{i}{2}(\eta^{\beta\mu}\gamma^\alpha-\eta^{\alpha\mu}\gamma^\beta-\eta^{\alpha\mu}\gamma^\beta+\eta^{\beta\mu}\gamma^\alpha)\\
&=-i(\eta^{\mu\beta}\gamma^\alpha-\eta^{\mu\alpha}\gamma^\beta)\ ,
\end{aligned}$$

where we have used the fact that the η is symmetric, it commutes with the γ's and the identity

$$[AB,C]=ABC-CAB=ABC+ACB-CAB-ACB=A\{B,C\}-\{A,C\}B\ .$$

q.e.d.

A generic Lorentz transformation is obtained by iterating intinitesimal ones via exponential map

$$S(\Lambda)=\exp(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})=\exp(\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu) \tag{5.3}$$

Proof. Infact, using (5.1)

$$\begin{aligned} S(\Lambda) &= \exp(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}) \\ &= \exp(\frac{i}{2}\omega_{\mu\nu}(-\frac{i}{4}[\gamma^\mu, \gamma^\nu])) \\ &= \exp(\frac{1}{8}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu - \underbrace{\gamma^\nu\gamma^\mu}_{-\gamma^\mu\gamma^\nu})) \\ &= \exp(\frac{1}{8}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu)) \\ &= \exp(\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu) \ . \end{aligned}$$

q.e.d.

5.2 Application on rotations and boosts

Example 5.1 (Rotation around the z-axis)**.** Consider a rotation around the z-axis. The infinitesimal Lorentz transformation is parametrised by

$$\omega_{\mu\nu} = \begin{cases} \varphi & (\mu,\nu) = (1,2) \\ -\varphi & (\mu,\nu) = (2,1) \\ 0 & otherwise \end{cases} \ .$$

Therefore, the infinitesimal ω matrix is

$$\omega^\mu{}_\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 \\ 0 & -\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and a finite Lorentz transformation can be found by the exponential map

$$\Lambda^\mu{}_\nu = (\exp(\omega))^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ .$$

Proof. Recall the Taylor expansions of the sine and cosine functions

$$\cos\varphi = 1 - \frac{\varphi^2}{2} + \dots \ , \quad \sin\varphi = \varphi - \frac{\varphi^3}{3!} + \dots \ .$$

We Taylor expand the exponential and find

$$\exp(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 \\ 0 & -\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \mathbb{I}_4 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 \\ 0 & -\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\varphi^2 & 0 & 0 \\ 0 & 0 & -\varphi^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi^3 & 0 \\ 0 & \varphi^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{\varphi^2}{2} + \dots & \varphi - \frac{\varphi^3}{3!} + \dots & 0 \\ 0 & -\varphi + \frac{\varphi^3}{3!} + \dots & 1 - \frac{\varphi^2}{2} + \dots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

q.e.d.

Moreover, a generic Lorents transformation on a Dirac spinor is

$$S(\Lambda) = \exp(\frac{i\varphi}{2} \begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix}) = \begin{bmatrix} \exp(\frac{i\varphi}{2}) & 0 & 0 & 0 \\ 0 & \exp(\frac{-i\varphi}{2}) & 0 & 0 \\ 0 & 0 & \exp(\frac{i\varphi}{2}) & 0 \\ 0 & 0 & 0 & \exp(\frac{-i\varphi}{2}) \end{bmatrix}.$$

Proof. Infact, using (5.3)

$$S(\Lambda) = \exp(\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu) = \exp(\frac{1}{4}(\underbrace{\omega_{12}}_{\varphi}\gamma^1\gamma^2 + \underbrace{\omega_{21}}_{-\varphi}\gamma^2\gamma^1)) = \exp(\frac{\varphi}{4}(\gamma^1\gamma^2 - \underbrace{\gamma^2\gamma^1}_{-\gamma^1\gamma^2}))$$

$$= \exp(\frac{\varphi}{2}\underbrace{\gamma^1}_{-i\beta\alpha^1}\underbrace{\gamma^2}_{-i\beta\alpha^3}) = \exp(\frac{\varphi}{2}\underbrace{\beta^2}_{1}\alpha^1\alpha^2) = \exp(\frac{\varphi}{2} \begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix})$$

$$= \exp(\frac{\varphi}{2} \begin{bmatrix} \underbrace{\sigma^1\sigma^2}_{i\sigma^3} & 0 \\ 0 & \underbrace{\sigma^1\sigma^2}_{i\sigma^3} \end{bmatrix}) = \exp(\frac{i\varphi}{2} \begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix})$$

$$= \exp(\begin{bmatrix} \frac{i\varphi}{2} & 0 & 0 & 0 \\ 0 & -\frac{i\varphi}{2} & 0 & 0 \\ 0 & 0 & \frac{i\varphi}{2} & 0 \\ 0 & 0 & 0 & -\frac{i\varphi}{2} \end{bmatrix}) = \begin{bmatrix} \exp(\frac{i\varphi}{2}) & 0 & 0 & 0 \\ 0 & \exp(\frac{-i\varphi}{2}) & 0 & 0 \\ 0 & 0 & \exp(\frac{i\varphi}{2}) & 0 \\ 0 & 0 & 0 & \exp(\frac{-i\varphi}{2}) \end{bmatrix},$$

where we used the property of the exponential of a diagonal matrix. q.e.d.

Notice that it is a unitary representation $S^\dagger(\Lambda) = S^{-1}(\Lambda)$. It is also a double-valued representation, since a rotation of $\varphi = 2\pi$ is represented by $S(\Lambda) = -\mathbb{I}$. Only with a rotation of $\varphi = 4\pi$, we find the identity again.

Example 5.2 (Generic rotation)**.** A generic rotation of an angle φ around an axis $\ltimes$ is represented by

$$S(\Lambda) = \begin{bmatrix} \exp(\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}) & 0 \\ 0 & \exp(\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}) \end{bmatrix} .$$

This means that a decomposition like the non-relativistic limit make a rotation on both the wave functions indipendently.

Example 5.3 (Boost along the x-axis)**.** Consider a boost around the x-axis. The infinitesimal Lorentz transformation is parametrised by

$$\omega_{\mu\nu} = \begin{cases} -w & (\mu,\nu) = (0,1) \\ -w & (\mu,\nu) = (1,0) \\ 0 & otherwise \end{cases} .$$

Therefore, the infinitesimal ω matrix is

$$\omega^\mu{}_\nu = \begin{bmatrix} 0 & -w & 0 & 0 \\ -w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and a finite Lorentz transformation can be found by the exponential map

$$\Lambda^\mu{}_\nu = (\exp(\omega))^\mu{}_\nu = \begin{bmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

where we defined the rapidity w in terms of the Lorentz factors

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh w \ , \quad \beta = v = \tanh w \ , \quad \beta\gamma = \sinh w \ . \tag{5.4}$$

Proof. Recall the Taylor expansions of the hyperbolic sine and hyperbolic cosine functions

$$\cosh w = 1 + \frac{w^2}{2} + \dots \ , \quad \sinh w = w + \frac{w^3}{3!} + \dots \ .$$

We Taylor expand the exponential and find

$$
\begin{aligned}
\exp(\omega) &= \sum_{k=0}^{\infty} \frac{\omega^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & -w & 0 & 0 \\ -w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \mathbb{I}_4 + \begin{bmatrix} 0 & -w & 0 & 0 \\ -w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} w^2 & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & -w^3 & 0 & 0 \\ -w^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \dots \\
&= \begin{bmatrix} 1 + \frac{w^2}{2} + \dots & -w - \frac{w^3}{3!} + \dots & 0 & 0 \\ -w - \frac{w^3}{3!} + \dots & 1 + \frac{w^2}{2} + \dots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .
\end{aligned}
$$

q.e.d.

Moreover, a generic Lorents transformation on a Dirac spinor is

$$
S(\Lambda) = \cosh \frac{w}{2} \mathbb{I} - \sinh \frac{w}{2} \alpha^1 \ . \tag{5.5}
$$

Proof. Infact, using (5.3)

$$
\begin{aligned}
S(\Lambda) &= \exp(\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu) = \exp(\frac{1}{4}(\underbrace{\omega_{01}}_{w}\gamma^0\gamma^1 + \underbrace{\omega_{10}}_{w}\gamma^1\gamma^0)) = \exp(\frac{w}{4}(\gamma^0\gamma^1 + \underbrace{\gamma^0\gamma^1}_{\gamma^1\gamma^0})) \\
&= \exp(\frac{w}{2}\underbrace{\gamma^0}_{-i\beta}\underbrace{\gamma^1}_{-i\beta\alpha^1}) = \exp(-\frac{w}{2}\underbrace{\beta^2}_{1}\alpha^1) = \exp(-\frac{w}{2}\begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix}) \\
&= \exp(\begin{bmatrix} 0 & 0 & 0 & -\frac{w}{2} \\ 0 & 0 & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & 0 & 0 \\ -\frac{w}{2} & 0 & 0 & 0 \end{bmatrix}) = \sum_{k=0}^{\infty}\frac{1}{k!}\begin{bmatrix} 0 & 0 & 0 & -\frac{w}{2} \\ 0 & 0 & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & 0 & 0 \\ -\frac{w}{2} & 0 & 0 & 0 \end{bmatrix}^k \\
&= \mathbb{I} + \begin{bmatrix} 0 & 0 & 0 & -\frac{w}{2} \\ 0 & 0 & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & 0 & 0 \\ -\frac{w}{2} & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{w^2}{4} & 0 & 0 & 0 \\ 0 & \frac{w^2}{4} & 0 & 0 \\ 0 & 0 & \frac{w^2}{4} & 0 \\ 0 & 0 & 0 & \frac{w^2}{4} \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & 0 & 0 & -\frac{w^3}{8} \\ 0 & 0 & -\frac{w^3}{8} & 0 \\ 0 & -\frac{w^3}{8} & 0 & 0 \\ -\frac{w^3}{8} & 0 & 0 & 0 \end{bmatrix} + \dots \\
&= \begin{bmatrix} 1 + \frac{w^2}{8} + \dots & 0 & 0 & -\frac{w}{2} - \frac{w^3}{48} + \dots \\ 0 & 1 + \frac{w^2}{8} + \dots & -\frac{w}{2} - \frac{w^3}{48} + \dots & 0 \\ 0 & -\frac{w}{2} - \frac{w^3}{48} + \dots & 1 + \frac{w^2}{8} + \dots & 0 \\ -\frac{w}{2} - \frac{w^3}{48} + \dots & 0 & 0 & 1 + \frac{w^2}{8} + \dots \end{bmatrix} \\
&= \begin{bmatrix} \cosh\frac{w}{2} & 0 & 0 & -\sinh\frac{w}{2} \\ 0 & \cosh\frac{w}{2} & -\sinh\frac{w}{2} & 0 \\ 0 & -\sinh\frac{w}{2} & \cosh\frac{w}{2} & 0 \\ -\sinh\frac{w}{2} & 0 & 0 & \cosh\frac{w}{2} \end{bmatrix} \\
&= \cosh\frac{w}{2}\mathbb{I} - \sinh\frac{w}{2}\alpha^1 \ ,
\end{aligned}
$$

where we used the Taylor expansion of the exponential. q.e.d.

Notice that it is a not unitary representation $S^\dagger(\Lambda) \neq S^{-1}(\Lambda)$, since there is a theorem that states that in a non-compact group, like the boost because they are not upper-bounded in velocity, the only irreducible representations are infinite-dimensional. However, it satisfies $S^\dagger(\Lambda) = S(\Lambda)$.

In terms of mass, momentum and energy, a finite Lorentz transformation on a Dirac spinor becomes

$$
S(\Lambda) = \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} - \frac{\alpha^1|\mathbf{p}|}{m+E}\Big) \ .
$$

Proof. Infact, using the hyberbolic trigonometry identities

$$
\tanh\frac{w}{2} = \frac{\sinh w}{1 + \cosh w}
$$

and

$$\cosh\frac{w}{2} = \sqrt{\frac{1+\cosh w}{2}}$$

Using the rapidity relations (5.4), we can rewrite (5.5) as

$$\begin{aligned}
S(\Lambda) &= \cosh\frac{w}{2}\Big(\mathbb{I} - \alpha^1 \tanh\frac{w}{2}\Big) \\
&= \sqrt{\frac{1}{2}}(1+\underbrace{\cosh w}_{\gamma})^{\frac{1}{2}}\Big(\mathbb{I} + \alpha^1 \frac{\overbrace{\sinh w}^{\beta\gamma}}{1+\underbrace{\cosh w}_{\gamma}}\Big) \\
&= \sqrt{\frac{1+\gamma}{2}}\Big(\mathbb{I} + \alpha^1 \frac{\beta\gamma}{1+\gamma}\Big)\ .
\end{aligned}$$

Now, we use the 4-momentum $(E, p) = (m\gamma, m\gamma\beta)$ and we reverse to find

$$\gamma = \frac{E}{m}\ , \quad \beta\gamma = \frac{|\mathbf{p}|}{m}\ .$$

Putting together, we obtain

$$S(\Lambda) = \sqrt{\frac{1+\gamma}{2}}\Big(\mathbb{I}+\alpha^1\frac{\beta\gamma}{1+\gamma}\Big) = \sqrt{\frac{1+\frac{E}{m}}{2}}\Big(\mathbb{I}+\alpha^1\frac{\frac{\mathbf{p}}{m}}{1+\frac{E}{m}}\Big) = \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{\alpha^1|\mathbf{p}|}{m+E}\Big)\ .$$

q.e.d.

Example 5.4 (Generic boost)**.** A generic boost of rapidity w along an axis $\ltimes$ is represented by

$$S(\Lambda) = \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)$$

Chapter 6

Wave plane solutions

In order to find a solution of the Dirac equation (3.4), we propose a plane wave ansatz

$$\psi_P(x) = w(p) \exp(ip_\mu x^\mu) \ ,$$

where $\exp(ip_\mu x^\mu)$ is the propagation in space-time, p^μ is arbitrary and $w(p)$ is the polarisation

$$w(p) = \begin{bmatrix} w_1(p) \\ w_2(p) \\ w_3(p) \\ w_4(p) \end{bmatrix} \ .$$

such that the polarisation satisfies the algebraic equation

$$(i\gamma^\mu p_\mu + m)w(p) = 0 \ , \tag{6.1}$$

and p^μ satisfies the mass-shell relation for a relativistic particle

$$p^\mu p_\mu + m^2 = 0 \ .$$

Proof. Infact, inserting the ansatz in (3.4)

$$0 = (\gamma^\mu \partial_\mu + m)\psi(x) = (\gamma^\mu \partial_\mu + m)w(p) \exp(ip_\mu x^\mu) = \underbrace{(i\gamma^\mu p_\mu + m)w(p)}_{0} \underbrace{\exp(ip_\mu x^\mu)}_{\neq 0} \ .$$

Hence

$$(i\gamma^\mu p_\mu + m)w(p) = (i\not{p} + m)w(p) = 0 \ .$$

Furthermore, we have

$$0 = (-i\not{p} + m)(i\not{p} + m)w(p) = \underbrace{(\not{p}^2 + m^2)}_{0} \underbrace{w(p)}_{\neq 0} \ ,$$

and we notice, using (3.5)

$$\not{p}^2 = \gamma^\mu p_\mu \gamma^\nu p_\nu = p_\mu p_\nu \underbrace{\gamma^\mu \gamma^\nu}_{\frac{\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu}{2}} = p_\mu p_\nu \frac{1}{2} \underbrace{(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)}_{\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}} = p_\mu p_\nu \eta^{\mu\nu} = p_\mu p^\mu = p^2 \ ,$$

where we exploit the symmetry of $p^\mu p^\nu$ to symmetrise $\gamma^\mu \gamma^\nu$. Hence

$$p^2 + m^2 = 0 \ .$$

q.e.d.

6.1 Plane wave at rest

Example 6.1 (Rest-frame)**.** Consider a particle at rest, which means with $p^\mu = (E, 0, 0, 0)$. Then, we substitute in (6.1)

$$0 = (i\not{p} + m)w(p) = (i\gamma^0 p_0)w(p) = (-\underbrace{i\gamma^0}_{\beta}\underbrace{p^0}_{E} + m)w(p) = (-\beta E + m)w(p) \ .$$

Hence

$$0 = \beta(-\beta E + m)w(P) = (-\beta^2 E + \beta m)w(p) = (-E + \beta m)w(P)$$

and

$$Ew(p) = \beta m w(p)$$

Recalling the matrix representation of β, we obtain

$$\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{bmatrix} \begin{bmatrix} w_1(p) \\ w_2(p) \\ w_3(p) \\ w_4(p) \end{bmatrix} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \end{bmatrix} \begin{bmatrix} w_1(p) \\ w_2(p) \\ w_3(p) \\ w_4(p) \end{bmatrix} \ .$$

This means that we have four different solutions: two are with positive energy $E = m$ which can be intepreted electrons with spin-up and spin-down

$$\psi_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \exp(-imt) \ , \quad \psi_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \exp(-imt) \ ,$$

and two with negative energy $E = -m$ which can be intepreted positrons with spin-up and spin-down

$$\psi_3(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \exp(imt) \ , \quad \psi_4(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \exp(imt) \ .$$

6.2 Moving plane wave

In order to find general solutions with arbitrary momentum, we apply a Lorentz transformation to the rest-frame solutions. A generic boost trasforms the rest-frame plane wave into

$$\psi_1(x) = \sqrt{\frac{m+E}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p_3}{m+E} \\ \frac{p_+}{m+E} \end{bmatrix} \exp(ip_\mu x^\mu t) \ , \quad \psi_2(x) = \sqrt{\frac{m+E}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p_-}{m+E} \\ -\frac{p_3}{m+E} \end{bmatrix} \exp(ip_\mu x^\mu t) \ ,$$
$$\psi_3(x) = \sqrt{\frac{m+E}{2m}} \begin{bmatrix} \frac{p_3}{m+E} \\ \frac{p_+}{m+E} \\ 1 \\ 0 \end{bmatrix} \exp(-ip_\mu x^\mu t) \ , \quad \psi_4(x) = \sqrt{\frac{m+E}{2m}} \begin{bmatrix} \frac{p_-}{m+E} \\ -\frac{p_3}{m+E} \\ 0 \\ 1 \end{bmatrix} \exp(-ip_\mu x^\mu t) \ ,$$

where $p_\pm = p_1 \pm ip_2$.

Proof. Firsly, we compute

$$\alpha^1 = \begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \ ,$$

$$\alpha^2 = \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \ ,$$

$$\alpha^3 = \begin{bmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \ .$$

Hence

$$\boldsymbol{\alpha}\cdot\mathbf{p} = \alpha^1 p_1 + \alpha^2 p_2 + \alpha^3 p_3 = \begin{bmatrix} 0 & 0 & p_3 & p_1 - ip_2 \\ 0 & 0 & p_1 + ip_2 & -p_3 \\ p_3 & p_1 - ip_2 & 0 & 0 \\ p_1 + ip_2 & -p_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix} \ .$$

Now, we compute ψ_1

$$
\begin{aligned}
S(\Lambda)\psi_1(x) &= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)\psi_1(x) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{1}{m+E}\begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix}\Big)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\exp(ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+\frac{1}{m+E}\begin{bmatrix}0\\0\\p_3\\p_+\end{bmatrix}\Big)\exp(ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\begin{bmatrix}1\\0\\\frac{p_3}{m+E}\\\frac{p_+}{m+E}\end{bmatrix}\exp(ip_\mu x^\mu t)\ ,
\end{aligned}
$$

we compute ψ_2

$$
\begin{aligned}
S(\Lambda)\psi_2(x) &= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)\psi_2(x) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{1}{m+E}\begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix}\Big)\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\exp(ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+\frac{1}{m+E}\begin{bmatrix}0\\0\\p_-\\-p_3\end{bmatrix}\Big)\exp(ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\begin{bmatrix}0\\1\\\frac{p_-}{m+E}\\-\frac{p_3}{m+E}\end{bmatrix}\exp(ip_\mu x^\mu t)\ ,
\end{aligned}
$$

we compute ψ_3

$$\begin{aligned}
S(\Lambda)\psi_3(x) &= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)\psi_3(x) \\
&= \sqrt{\frac{m+E}{2m}}\left(\mathbb{I} + \frac{1}{m+E}\begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\exp(-ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{m+E}\begin{bmatrix} p_3 \\ p_+ \\ 0 \\ 0 \end{bmatrix}\right)\exp(-ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\begin{bmatrix} \frac{p_3}{m+E} \\ \frac{p_+}{m+E} \\ 1 \\ 0 \end{bmatrix}\exp(-ip_\mu x^\mu t)
\end{aligned}$$

and we compute ψ_4

$$\begin{aligned}
S(\Lambda)\psi_4(x) &= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)\psi_4(x) \\
&= \sqrt{\frac{m+E}{2m}}\left(\mathbb{I} + \frac{1}{m+E}\begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\exp(-ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{m+E}\begin{bmatrix} p_- \\ -p_3 \\ 0 \\ 0 \end{bmatrix}\right)\exp(-ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\begin{bmatrix} \frac{p_-}{m+E} \\ -\frac{p_3}{m+E} \\ 0 \\ 1 \end{bmatrix}\exp(-ip_\mu x^\mu t)\ .
\end{aligned}$$

q.e.d.

Chapter 7

Discrete symmetries

Chapter 8

Dirac action

List of Theorems