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# On relativistic quantum mechanics:

first quantisation of free theories.

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# Contents

Page

# Abstract

In these notes, we will study.

# Part I

Klein-Gordon theory

# Chapter 1

# Klein-Gordon equation

In this chapter, after recalling some notions of standard quantum mechanics like the Schroedinger equation, we will study how to find the Klein-Gordon equation.

## 1.1 Schroedinger equation

At the beginning of quantum theory, there were no ideas about how to describe quantum systems. Planck introduced the idea of quanta for the black body radiation. Einstein supposed that electromagnetic radiation is made of particles, called photons, of energy $E = h\nu = \hbar\omega$. De Broglie used the inverse procedure for particles and he associated to them a wavefunction

$$\psi(t, \mathbf{x}) \sim \exp(2\pi i(\mathbf{k} \cdot \mathbf{x} - \nu t)) \ .$$

Assuming that the phase is Lorentz-invariant, we can extend the relation between wave quantities $\mathbf{k}, \nu$ and particle ones $\mathbf{p}, E$ by means of the Einstein-De Broglie relations

$$E = h\nu \ , \quad \mathbf{p} = \hbar\mathbf{k} \ ,$$

or equivalently in covariant formalism

$$p^\mu = hk^\mu \ ,$$

where $k^\mu = (\nu/c, \mathbf{k})$ and $p^\mu = (E/c, \mathbf{p})$. Therefore, the wavefunction becomes

$$\psi(t, \mathbf{x}) \sim \exp(2\pi i(\mathbf{k} \cdot \mathbf{x} - \nu t)) = \exp(\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - Et)) \ .$$

Schroedinger obtained the equation that describes how this wavefunction evolves in time

$$i\hbar\frac{\partial}{\partial t}\psi(t, \mathbf{x}) = E\psi(t, \mathbf{x}) \ , \tag{1.1}$$

The construction is made by the substitutions

$$E \to i\hbar\frac{\partial}{\partial t} \ , \quad \mathbf{p} \to -i\hbar\boldsymbol{\nabla} \ . \tag{1.2}$$

**Example 1.1.** Consider a non-relativistic free particle. Its energy is

$$E = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2 \ ,$$

so that the time evolution can be obtained by solving the Schroedinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(t,\mathbf{x}) \ .$$

In order to have a probabilistic interpretation, the wavefunction must be normalised (for infinite space plane waves, we should consider wave packets). In this way, we can interpret $\rho(t,\mathbf{x}) = |\psi(t,\mathbf{x})|^2$ as the density probability to find the particle in $\mathbf{x}$ at time $t$. In particular, $\rho$ satisfies a continuity equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0 \ ,$$

where $\mathbf{J} = \frac{\hbar}{2im}(\psi^*\boldsymbol{\psi} - \psi\boldsymbol{\nabla}\psi^*)$ is the current density. Physically, it means that particles do not disappear.

*Proof.* Multiplying (1.1) by its complex conjugate

$$\begin{aligned} 0 &= \psi^*(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2)\psi - \psi(-i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2)\psi^* \\ &= i\hbar(\psi^*\frac{\partial}{\partial t}\psi + \psi\frac{\partial}{\partial t}\psi^*) + \frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) \\ &= i\hbar\frac{\partial\psi^*\psi}{\partial+}\frac{\hbar^2}{2m}\boldsymbol{\nabla}\cdot(\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi) \ . \end{aligned}$$

Hence,

$$\frac{\partial}{\partial t}(\psi^*\psi) - \boldsymbol{\nabla}\cdot\left(\frac{\hbar}{2im}(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)\right) \ .$$

q.e.d.

## 1.2 Klein-Gordon equation

From now on, we will use $\hbar = c = 1$. So far, we have work with non-relativistic free particles satisfy the Schroedinger equation. Now, we take into account also relativistic effects. We start from the relativistic relation between energy and momentum

$$E^2 = p^2 + m^2 \ , \tag{1.3}$$

where $p$ is the 4-momentum, such that its norm is constant

$$p^\mu p_\mu = -E^2 + |\vec{p}|^2 = -m^2 \ , \tag{1.4}$$

This last relation is called the mass-shell condition. There are two way to promote the energy-momentum relation to operator. The first one is to use the square-root

$$\hat{E} = \sqrt{\hat{p}^2 + m^2}$$

and then put it inside the Schroedinger equation

$$i\frac{\partial}{\partial t}\phi(t,\vec{x}) = \hat{H}\phi(t,\vec{x}) = \sqrt{\hat{p}^2 + m^2}\phi(t,\vec{x}) \ .$$

However, the square root of an operator could lead us to a non-local theory and this approach was abandoned. The second approach is the one used by Klein and Gordon, keeping the square. This leads us to the Klein-Gordon equation and the energy-momentum relation becomes

$$(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2)\phi(t,\vec{x}) = 0 \ ,$$

which in covariant notation is

$$(\partial^\mu \partial_\mu - m^2)\phi(x) = (\Box - m^2)\phi(x) = 0 \ . \tag{1.5}$$

*Proof.* In fact, using (1.2)

$$E^2 = (i\frac{\partial}{\partial t})^2 = -\frac{\partial^2}{\partial t^2}$$

and

$$p^2 = (-i\boldsymbol{\nabla})^2 = -\nabla^2 \ ,$$

we find

$$-\frac{\partial^2}{\partial t^2} = -\nabla^2 + m^2 \ .$$

Hence,

$$(-\frac{\partial^2}{\partial t^2} + \nabla^2\psi(x) - m^2)\psi(x) = 0 \ .$$

q.e.d.

Notice that we could have started from the covariant relation

$$p^\mu p_\mu + m^2 = 0 \ ,$$

and used the substitution

$$p^\mu = -i\partial_\mu \ .$$

# Chapter 2

# Solutions

In this chapter, after recalling some notions of standard quantum mechanics like the Schroedinger equation, we will study more deeply the Klein-Gordon theory: its solutions as plane waves, the probability interpretation, its Green function and its action.

## 2.1 Plane waves

Plane waves are solutions of the Klein-Gordon equation by construction. Consider a plane waves ansatz

$$\phi(x) = \exp(ip^\mu x_\mu) \ ,$$

where for now $p^\mu$ is arbitrary. However, using (1.5)

$$-(p^\mu p_\mu + m^2) \exp(ip_\mu x^\mu) = 0 \ ,$$

we find that plane wave is a solution only if $p^\mu$ satisfies the mass-shell condition

$$(p^0)^2 = \mathbf{p}^2 + m^2 \ , \quad p^0 = \pm\sqrt{\mathbf{p}^2 + m^2} = \pm E_{\mathbf{p}} \ .$$

Since $E_p \geq 0$, notice that negative energies are allowed as solutions. This shows that the theory is not stable, since there is no energy limitation from below. They cannot be neglected because interactions lead to negative energy transitions. Therefore, we indicate positive energy solutions as

$$\phi_p^+ = \exp(i(-E_p t + \mathbf{p}\cdot\mathbf{x})) \ ,$$

whereas the negative energy solutions are

$$\phi_p^- = \exp(i(E_p t - \mathbf{p}\cdot\mathbf{x})) \ .$$

A general solution is a linear combination of plane waves

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(a(\mathbf{p})\exp(i(-E_p t + \mathbf{p}\cdot\mathbf{x})) + b^*(\mathbf{p})\exp(i(E_p t - \mathbf{p}\cdot\mathbf{x})))$$

and its complex conjugate

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(a^*(\mathbf{p})\exp(i(-E_pt+\mathbf{p}\cdot\mathbf{x})) + b(\mathbf{p})\exp(i(E_pt-\mathbf{p}\cdot\mathbf{x})))\ ,$$

where $a(\mathbf{p})$ and $b(\mathbf{p})$ are the Fourier coefficients and the factor $1/2E_\mathbf{p}$ is conventional to keep Lorentz invariant measure. If the field is real, i.e. $\phi = \phi^*$, then $a(\mathbf{p}) = b(\mathbf{p})$.

## 2.2 Continuity equation

The density current associated to the Klein-Gordon equation is

$$J^\mu = \frac{1}{2im}(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)\ , \tag{2.1}$$

such that it satisfies the continuity equation

$$\partial_\mu J^\mu = 0\ .$$

*Proof.* Multiplying (1.5) by its complex conjugate

$$\begin{aligned} 0 &= \phi^*(\Box - m^2)\phi - \phi(\Box - m^2)\phi^* \\ &= \phi^*(\partial_\mu\partial^\mu - m^2)\phi - \phi(\partial_\mu\partial^\mu - m^2)\phi^* \\ &= \partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)\ . \end{aligned}$$

Hence, the current is

$$J^\mu = (\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)\ .$$

We choose a normalisation constant to make it real and to match $\mathbf{J}$ with the Schroedinger one

$$J^\mu = \frac{1}{2im}(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)\ . \tag{2.2}$$

q.e.d.

The time component, that corresponds to the probability density, is

$$J^0 = \frac{1}{2im}(\phi^*\partial^t\phi - \phi\partial^t\phi^*) = \frac{i}{2m}(\phi^*\partial_t\phi - \phi\partial_t\phi^*)\ ,$$

which is not positive, even though it is real. This is because it can have negative or positive values by the initial condition choice. Recall that Klein-Gordon equation is second-order in time and we must choose both $\phi$ and $\dot{\phi}$ as initial conditions.

**Example 2.1.** For plane waves, we have

$$J^0(\phi^{\pm}_{\mathbf{p}}) = \pm\frac{E_p}{m} \ .$$

In fact

$$\begin{aligned} J^0(\phi^{\pm}_{\mathbf{p}}) &= \frac{i}{2m}(\exp(\mp ip^\mu x_\mu)\partial_t \exp(\pm ip^\mu x_\mu) - \exp(\pm ip^\mu x_\mu)\partial_t \exp(\mp ip^\mu x_\mu)) \\ &= \frac{i}{2m}(\pm E_p \pm E_p) = \pm\frac{E_p}{m} \ . \end{aligned}$$

To conclude, there is no probability interpretation, since we do not know what a negative probability is.

## 2.3 Green functions

The Green function is a solution of the Klein-Gordon equation in presence of a point-like instantaneous source. For convenience, we put it at the origin. Mathematically, it satisfies the equation

$$(-\Box + m^2)G(x) = \delta^4(x) \ .$$

It is useful to determine the solution of a general source $J(x)$

$$(-\Box + m^2)\phi(x) = J(x) \ ,$$

which are related by

$$\phi(x) = \phi_0(x) + \int d^4y \ G(x-y)J(y) \ ,$$

where $\phi_0(x)$ is a solution of the associated inhomogeneous equation.

*Proof.* In fact,

$$\begin{aligned} (-\Box + m^2)\phi(x) &= (-\Box + m^2)\Big(\phi_0(x) + \int d^4y \ G(x-y)J(y)\Big) \\ &= \underbrace{(-\Box + m^2)\phi_0(x)}_{0} + (-\Box + m^2)\int d^4y \ G(x-y)J(y) \\ &= \int d^4y \ \underbrace{(-\Box + m^2)G(x-y)}_{\delta^4(x-y)} J(y) \\ &= \int d^4y \ \delta^4(x-y)J(y) \\ &= J(x) \end{aligned}$$

q.e.d.

Now, we calculate the Green function, using the Fourier transform

$$G(x) = \int \frac{d^4p}{(2\pi)^4} \exp(ip_\mu x^\mu)\tilde{G}(p) \ ,$$

where $\tilde{G}$ is evaluated to be

$$\tilde{G}(p) = \frac{1}{p^2 + m^2} \ .$$

Hence,

$$G(x) = \int \frac{d^4p}{(2\pi)^4} \frac{\exp(ip_\mu x^\mu)}{p^2 + m^2}$$

*Proof.* In fact

$$\begin{aligned}(-\Box + m^2)G(x) &= (-\Box + m^2) \int \frac{d^4p}{(2\pi)^4} \exp(ip_\mu x^\mu)\tilde{G}(p) \\ &= (-\Box + m^2) \int \frac{d^4p}{(2\pi)^4} \exp(ip_\mu x^\mu)(p^\mu p_\mu + m^2)\tilde{G}(p) \ ,\end{aligned}$$

which is equal to

$$\delta^4(x) = \int \frac{d^4p}{(2\pi)^4} \exp(ip_\mu x^\mu) \ .$$

Hence,

$$1 = (p^\mu p_\mu + m^2)\tilde{G}(p) \ ,$$

and

$$\tilde{G}(p) = \frac{1}{p^2 + m^2} \ .$$

q.e.d.

In general, the Green function is not unique, but it depends on boundary conditions at the infinity. We use the Feynman-Stueckelberg prescription which gives the correct quantum interpretation, i.e. negative energy solutions are antiparticles. This means that positive energy particles propagate forward in time and negative energy particles propagate backward in time. It describes also real particles, i.e. whose satisfy the mass-shell condition (1.4) and can travel macroscopical distances, and virtual particles, i.e. whose do not and are hidden by the uncertainty principle. Mathematically, it means that we shift the poles on the complex plane: one upward and one downward. This means that we add a factor $\epsilon \ll 1$

$$G(x) = \int \frac{d^4p}{(2\pi)^4} \frac{\exp(ip_\mu x^\mu)}{p^2 + m^2 - i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \frac{\exp(ip_\mu x^\mu)}{(p^0 + E_p - i\epsilon)(p^0 + E_p + i\epsilon)} \ ,$$

where $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$.

The propagator is the Green function which is the amplitude of propagation from a point y to another point x

$$\Delta(x-y) = -iG(x-y)\ .$$

In our case, it becomes

$$\Delta(x-y) = -iG(x-y) = \int \frac{d^3p}{(2\pi)^3} \exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))\frac{\exp(-iE_p|x^o - y^0|)}{2E_p}\ . \tag{2.3}$$

*Proof.* In fact

$$\begin{aligned}
-iG(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{-i\exp(ip_\mu x^\mu)}{(p^0+E_p-i\epsilon)(p^0+E_p+i\epsilon)} \\
&= \int \frac{d^3p}{(2\pi)^3}\exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))\int\frac{dp^0}{2\pi}\exp(-ip^0(x^0-y^0))\frac{i}{(p^0+E_p-i\epsilon)(p^0+E_p+i\epsilon)} \\
&= \int \frac{d^3p}{(2\pi)^3}\exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))\Big(\theta(x^0-y^0)\frac{\exp(-iE_p(x^0-y^0))}{2E_p} + \theta(y^0-x^0)\frac{\exp(-iE_p(y^0-}{2E_p}
\end{aligned}$$

q.e.d.

Yukawa suggested from the Klein-Gordon equation, a theory of nuclear interactions.

## 2.4 Action

It is possible to formulate the Klein-Gordon theory in terms of an action principle. In fact, the Klein-Gordon equation is the Euler-Lagrange equation of the action for a complex scalar field $\phi(x)$

$$\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\ . \tag{2.4}$$

*Proof.* We apply the Euler-Lagrange equation, first for $\phi$

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = -m^2\phi^* + \partial_\mu\partial^\mu\phi^* = (\Box - m^2)\phi^*\ ,$$

which is the Klein-Gordon equation (1.5) for $\phi^*$, and similarly for $\phi^*$

$$0 = \frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*} = -m^2\phi + \partial_\mu\partial^\mu\phi = (\Box - m^2)\phi\ ,$$

which is the Klein-Gordon equation (1.5) for $\phi$. q.e.d.

Instead, the Klein-Gordon equation for a real scalar field $\phi = \phi^*$ can be obtained by the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \ . \tag{2.5}$$

*Proof.* We apply the Euler-Lagrange equation

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = -m^2\phi + \partial_\mu\partial^\mu\phi = (\Box - m^2)\phi \ ,$$

which is the Klein-Gordon equation (1.5). q.e.d.

The complex scalar field can be seen as a linear combination of two real fields of the same mass

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \ , \quad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}} \ ,$$

and the related Lagrangian (2.4) becomes (2.5)

$$\mathcal{L} = -\partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi = -\frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 - \frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) \ .$$

The Lagrangian formalism is useful to study symmetries of the symmetry. The free Klein-Gordon complex field has two global symmetries: one associated to the Poincaré group, which gives rise that to the energy-momentum tensor, and a $U(1)$ symmetry, which gives rise to the probability current.

First, the free Klein-Gordon complex field is globally invariant under the Poincaré group

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \ , \quad \phi'(x') = \phi(x) \ , \quad (\phi')^*(x') = \phi^*(x) \ .$$

In particular, we are interested in spacetime translations $a^\mu$, which infinitesimally look like

$$\delta_\alpha\phi(x) = -a^\mu\partial_\mu\phi(x) \ , \quad \delta_\alpha\phi^*(x) = -a^\mu\partial_\mu\phi^*(x) \ .$$

We apply the Noether's theorem and we find the associated energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi + \eta^{\mu\nu}\mathcal{L} \ ,$$

which satisfies the conserved equation

$$\partial_\mu T^{\mu\nu} = 0 \ .$$

Its related conserved charges are the total 4-momentum carried by the field

$$P^\mu = \int d^3x \ T^{0\mu} \ ,$$

the energy density is

$$\mathcal{E} = T^{00} = \partial_0\phi^*\partial_0\phi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi$$

and the total energy is

$$E = \int d^3x\ \mathcal{E} = \int d^3x\ (\partial_0\phi^*\partial_0\phi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)\ .$$

*Proof.* The action is invariant under a translation. In fact

$$\mathcal{L}' = -\partial^\mu(\phi')^*\partial_\mu\psi' - m^2(\phi')^*\phi' = -\partial^\mu\phi^*\partial_\mu\psi - m^2\phi^*\phi = \mathcal{L}\ .$$

We apply the Noether's theorem, we find the boundary term

$$\begin{aligned} 0 = \delta L &= \partial_\mu\delta\phi\partial^\mu\phi^* + \partial_\mu\phi\partial^\mu\delta\phi^* + m^2\delta\phi\phi^* + m^2\phi\delta\phi^* \\ &= \epsilon_\nu\partial_\mu\partial^\nu\phi\partial^\mu\phi^* + \epsilon_\nu\partial_\mu\phi\partial^\mu\partial^\nu\phi^* + m^2\epsilon_\nu\partial^\nu\phi^*\phi + m^2\epsilon_\nu\phi^*\partial^\nu\phi \\ &= \epsilon_\nu\partial^\nu(\partial_\mu\phi\partial^\mu\phi^* + m^2\phi^*\phi)\ , \end{aligned}$$

which is

$$G = \partial_\mu\phi\partial^\mu\phi^* + m^2\phi^*\phi\ ,$$

then we find the current

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\delta\phi^* + \epsilon_\nu\eta^{\mu\nu}G \\ &= (-\partial^\mu\phi^*)(-\epsilon_\nu\partial^\nu\phi) + (-\partial^\mu\phi)(-\epsilon_\nu\partial^\nu\phi^*) + \epsilon_\nu\eta^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi^* + \epsilon_\nu\eta^{\mu\nu}m^2\phi^*\phi \\ &= \epsilon_\nu(\partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi + \eta^{\mu\nu}\mathcal{L})\ . \end{aligned}$$

Hence, the energy-momentum tensor is

$$T^{\mu\nu} = \partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi + \eta^{\mu\nu}\mathcal{L}\ .$$

q.e.d.

Second, the free Klein-Gordon complex field is globally invariant under the an the $U(1)$ symmetry

$$\phi'(x) = \exp(i\alpha)\phi(x)\ , \quad (\phi')^*(x) = \exp(-i\alpha)\phi^*(x)\ ,$$

which infinitesimally looks like

$$\delta_\alpha\phi(x) = i\alpha\phi(x)\ , \quad \delta_\alpha\phi^*(x) = -i\alpha\phi^*(x)\ .$$

We apply the Noether's theorem and we find associated Noether's current

$$J^\mu = i\phi^*\partial^\mu\phi - i(\partial^\mu\phi^*)\phi\ ,$$

which satisfies the continuity equation

$$\partial^\mu J_\mu = 0 \ .$$

Its related conserved charge is

$$Q = \int d^3x \ J^0 = -i \int d^3x \ (\phi^* \partial_0 \phi - i\partial_0 \phi^* \phi) \ .$$

Notice that it is the same as (2.1) up to a constant.

*Proof.* The action is invariant under an $U(1)$ rotation. In fact

$$\begin{aligned}
\mathcal{L}' &= -\partial^\mu (\phi')^* \partial_\mu \psi' - m^2 (\phi')^* \phi' \\
&= -\partial^\mu \phi^* \exp(-i\alpha) \partial_\mu \psi \exp(i\alpha) - m^2 \phi^* \exp(-i\alpha) \phi \exp(i\alpha) \\
&= -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi = \mathcal{L} \ .
\end{aligned}$$

We apply the Noether's theorem, we find the boundary term

$$\begin{aligned}
0 = \delta L &= \partial_\mu \delta\phi \partial^\mu \phi^* + \partial_\mu \phi \partial^\mu \delta\phi^* + m^2 \delta\phi\phi^* + m^2 \phi\delta\phi^* \\
&= \partial_\mu i\alpha\phi \partial^\mu \phi^* - \partial_\mu \phi \partial^\mu i\alpha\phi^* + m^2 i\alpha\phi\phi^* - m^2 \phi i\alpha\phi^* \\
&= 0 \ ,
\end{aligned}$$

which is

$$G = 0 \ ,$$

then we find the current

$$\begin{aligned}
J^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \delta\phi^* \\
&= (-\partial^\mu \phi^*)(i\alpha\phi) + (-\partial^\mu \phi)(-i\alpha\phi^*) = \alpha(i\phi^* \partial^\mu \phi - i\partial^\mu \phi^* \phi) \ .
\end{aligned}$$

q.e.d.

# Part II

# Dirac theory

# Chapter 3

# Dirac equation

## 3.1 Derivation

We are looking for a quantum equation that describes $\frac{1}{2}$-spin particles, but unlike the Klein-Gordon equation, it allows a probabilistic interpretation. The problem with the Klein-Gordon equation is the presence of second-order terms in time, therefore we need an hamiltonian which is linear but at the same time recover the energy-momentum relation (1.3). The first guess is

$$E = c\mathbf{p} \cdot \boldsymbol{\alpha} + mc^2\beta \ , \tag{3.1}$$

where $\alpha$ and $\beta$ are hermitian matrices such that satisfies the Clifford algebra

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}\mathbb{I} \ , \quad \{\beta, \beta\} = 2\mathbb{I} \ , \quad \{\alpha^i, \beta\} = 0 \ . \tag{3.2}$$

*Proof.* Infact, we compute the square of (3.1)

$$\begin{aligned}
E^2 &= (cp^i\alpha^i + mc^2\beta)^2 \\
&= (cp^i\alpha^i + mc^2\beta)(cp^j\alpha^j + mc^2\beta) \\
&= c^2\alpha^i p^i \alpha^j p^j + \beta^2 m^2 c^4 + mc^3 p^i \alpha^i \beta + mc^3 p^j \beta \alpha^j \\
&= c^2 p^i p^j \underbrace{\alpha^i\alpha^j}_{\frac{\alpha^i\alpha^j + \alpha^j\alpha^i}{2}} + \beta^2 m^2 c^4 + mc^3 p^i(\alpha^i\beta + \beta\alpha^i) \\
&= c^2 p^i p^j \frac{\alpha^i\alpha^j + \alpha^j\alpha^i}{2} + \beta^2 m^2 c^4 + mc^3 p^i(\alpha^i\beta + \beta\alpha^i) \ ,
\end{aligned}$$

where in the fourth row, we exploit the symmetry of $p^i p^j$ to symmetrise $\alpha^i\alpha^j$. We compare it with (1.3)

$$E^2 = c^2 p^i p^j \underbrace{\frac{\alpha^i\alpha^j + \alpha^j\alpha^i}{2}}_{\delta^{ij}} + \underbrace{\beta^2}_{1} m^2c^4 + mc^3p^i \underbrace{(\alpha^i\beta + \beta\alpha^i)}_{0} = p^2c^2 + m^2c^4 \ .$$

Hence

$$\{\alpha^i, \alpha^j\} = \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij} \ ,$$
$$\{\beta, \beta\} = \beta^2 + \beta^2 = 2\beta^2 = 2 \ ,$$
$$\{\alpha^i, \beta\} = \alpha^i\beta + \beta\alpha^i = 0 \ .$$

q.e.d.

The minimal solutions for the set of algebraic equation (3.2) are $4 \times 4$ traceless matrices $\boldsymbol{\alpha}$ and $\beta$ such that

$$\alpha^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \ , \quad \beta = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \ ,$$

where $\sigma^i$ are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ , \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ ,$$

and they satisfy the relation

$$\sigma^i\sigma^j = \delta^{ij}\mathbb{I} + i\epsilon^{ijk}\sigma^k \ . \tag{3.3}$$

It is called the Dirac representation and it is the only irreducible representation of the Clifford algebra up to others that are unitarily equivalent (by a change of basis) to the Dirac one or that are higher dimensional and thus reducible.

The hamiltonian form of the Dirac equation becomes

$$i\hbar\frac{\partial}{\partial t}\psi(t, \mathbf{x}) = (-i\hbar c\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta mc^2)\psi(t, \mathbf{x}) = H_D\psi(t, \mathbf{x}) \ , \tag{3.4}$$

while in covariant form it becomes

$$(\gamma^\mu\partial_\mu + m)\psi(x) = (\not{\partial} + m)\psi(x) = 0 \ , \tag{3.5}$$

where $\psi(t, \mathbf{x})$ is a matrix

$$\psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix} \ ,$$

and $\gamma^\mu$ are the matrices

$$\gamma^0 = -i\beta \ , \quad \gamma^i = -i\beta\alpha^i \ ,$$

such that they satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ . \tag{3.6}$$

Explicitly, they are

$$\gamma^0 = -i\begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix} \ , \quad \gamma^i = \begin{bmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{bmatrix} \ .$$

*Proof.* Infact, we compute operator substitution (??) on (3.1)

$$\underbrace{E}_{i\hbar\frac{\partial}{\partial t}}\psi(t,\mathbf{x}) = (c\underbrace{\mathbf{p}}_{-i\hbar\boldsymbol{\nabla}}\cdot\boldsymbol{\alpha} + mc^2\beta)\psi(t,\mathbf{x}) \ .$$

Hence

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{x}) = (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)\psi(t,\mathbf{x}) = H_D\psi(t,\mathbf{x}) \ .$$

In order to write it in covariant form, we compute

$$\begin{aligned}\frac{\beta}{\cancel{\hbar}c}i\cancel{\hbar}\frac{\partial}{\partial t}\psi(t,\mathbf{x}) &= \frac{\beta}{\hbar c}(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)\psi(t,\mathbf{x}) \\ &= \Big(-\frac{\beta}{\cancel{\hbar}\cancel{c}}i\cancel{\hbar}\cancel{c}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{mc}{\hbar}\underbrace{\beta^2}_{1}\Big)\psi(t,\mathbf{x}) \ .\end{aligned}$$

Hence

$$i\frac{\beta}{c}\frac{\partial}{\partial t}\psi(t,\mathbf{x}) = \Big(-i\beta\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{mc}{\hbar}\Big)\psi(t,\mathbf{x}) \ ,$$

$$\Big(\underbrace{-i\beta}_{\gamma^0}\frac{1}{c}\frac{\partial}{\partial t}\underbrace{-i\beta\boldsymbol{\alpha}}_{\boldsymbol{\gamma}}\cdot\boldsymbol{\nabla} + \frac{mc}{\hbar}\Big)\psi(t,\mathbf{x}) = 0 \ ,$$

$$\Big(\gamma^0\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + \frac{mc}{\hbar}\Big)\psi(t,\mathbf{x}) = 0 \ ,$$

and in covariant form, we obtain

$$(\gamma^\mu\partial_\mu + \mu)\psi(x) = 0 \ ,$$

where $\mu = \frac{mc}{\hbar}$ is the inverse reduced Compton wavelength. In natural units, it becomes

$$(\gamma^\mu\partial_\mu + m)\psi(x) = 0 \ .$$

Finally, they satisfy the Clifford algebra

$$\{\gamma^0,\gamma^0\} = 2\eta^{00} = 2 \ ,$$

$$\{\gamma^i,\gamma^j\} = 2\eta^{ij} = 2\delta^{ij}$$

and

$$\{\gamma^0,\gamma^i\} = 2\eta^{0i} = 0 \ .$$

q.e.d.

## 3.2 Continuity equation

The continuity equation associated to the Dirac equation is

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + \boldsymbol{\nabla}\cdot(c\psi^\dagger\boldsymbol{\alpha}\psi) = 0$$

where the density charge is positive defined $\psi^\dagger\psi > 0$ and it's compatible with the probabilistic intepretation.

*Proof.* Infact, we multiply by $\psi^\dagger$ and subtract the hermitian conjugate on (3.4)

$$\begin{aligned}
0 &= \psi^\dagger(i\hbar\frac{\partial}{\partial t}\psi - (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta)\psi) - (\psi^\dagger(i\hbar\frac{\partial}{\partial t}\psi - (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta)\psi))^\dagger \\
&= \psi^\dagger(i\hbar\frac{\partial}{\partial t}\psi - (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta)\psi) - (-i\hbar\frac{\partial}{\partial t}\psi^\dagger - (i\hbar c\underbrace{\boldsymbol{\alpha}^\dagger}_{\boldsymbol{\alpha}}\cdot\boldsymbol{\nabla} + mc^2\underbrace{\beta^\dagger}_{\beta})\psi^\dagger)\psi \\
&= i\hbar\psi^\dagger\frac{\partial}{\partial t}\psi + i\hbar c\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi - \cancel{mc^2\beta\psi^\dagger\psi} + i\hbar\psi\frac{\partial}{\partial t}\psi^\dagger + i\hbar c\psi\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi^\dagger + \cancel{mc^2\beta\psi^\dagger\psi} \\
&= \cancel{i\hbar}\underbrace{\left(\psi^\dagger\frac{\partial}{\partial t}\psi + \psi\frac{\partial}{\partial t}\psi^\dagger\right)}_{\frac{\partial}{\partial t}(\psi^\dagger\psi)} + \cancel{i\hbar}\underbrace{(c\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \psi\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi^\dagger)}_{\boldsymbol{\nabla}\cdot(c\psi^\dagger\boldsymbol{\alpha}\psi)} \\
&= \frac{\partial}{\partial t}(\psi^\dagger\psi) + \boldsymbol{\nabla}\cdot(c\psi^\dagger\boldsymbol{\alpha}\psi)\ .
\end{aligned}$$

q.e.d.

## 3.3 Gamma matrices

As we said before, $\alpha^i$ and $\beta$ are hermitian whicle $\gamma^0$ is antihermitian and $\gamma^i$ is hermitian

$$(\gamma^0)^\dagger = -\gamma^0\ , \quad (\gamma^i)^\dagger = \gamma^i\ .$$

which can be written in the following way

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 = -\beta\gamma^\mu\beta\ . \tag{3.7}$$

This means that the Clifford algebra is valid for $(\gamma^\mu)^\dagger$, intepreted as a change of basis

$$\{-(\gamma^\mu)^\dagger, -(\gamma^\nu)^\dagger\} = 2\eta^{\mu\nu}\ .$$

*Proof.* Infact, by the hermiticity of $\alpha^i$ and $\beta$

$$(\gamma^0)^\dagger = (-i\beta)^\dagger = i\beta = -\gamma^0$$

and

$$\begin{aligned}(\gamma^i)^\dagger = (-i\beta\alpha^i)^\dagger = (\gamma^0\alpha^i)^\dagger = \alpha^i \underbrace{(\gamma^0)^\dagger}_{\gamma^0} \\ = -\alpha^i\gamma^0 = -i \underbrace{\alpha^i\beta}_{\beta\alpha^i} = -i\beta\alpha^i = \gamma^i \ .\end{aligned}$$

Furthermore,

$$(\gamma^0)^\dagger = \underbrace{\gamma^0\gamma^0}_{-1}\gamma^0 = -\gamma^0$$

and

$$(\gamma^i)^\dagger = \gamma^0 \underbrace{\gamma^i\gamma^0}_{-\gamma^0\gamma^i} = -\gamma^0\gamma^0\gamma^i = -\underbrace{(\gamma^0)^2}_{-1}\gamma^i = \gamma^i \ .$$

Finally, using (3.6)

$$\begin{aligned}\{-(\gamma^\mu)^\dagger, -(\gamma^\nu)^\dagger\} &= (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger + (\gamma^\nu)^\dagger(\gamma^\mu)^\dagger \\ &= \gamma^0\gamma^\mu \underbrace{\gamma^0\gamma^0}_{-1}\gamma^\nu\gamma^0 + \gamma^0\gamma^\nu \underbrace{\gamma^0\gamma^0}_{-1}\gamma^\mu\gamma^0 \\ &= -\underbrace{\gamma^0\gamma^\mu}_{-\gamma^\mu\gamma^0}\underbrace{\gamma^\nu\gamma^0}_{-\gamma^0\gamma^\nu} - \underbrace{\gamma^0\gamma^\nu}_{-\gamma^\nu\gamma^0}\underbrace{\gamma^\mu\gamma^0}_{-\gamma^0\gamma^\mu} \\ &= -\gamma^\mu\underbrace{\gamma^0\gamma^0}_{-1}\gamma^\nu - \gamma^\nu\underbrace{\gamma^0\gamma^0}_{-1}\gamma^\mu = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \ .\end{aligned}$$

q.e.d.

As we said before, $\alpha^i$ and $\beta$ are traceless and so $\gamma^\mu$ are

$$\operatorname{tr}\gamma^\mu = 0 \ .$$

*Proof.* Infact, by the linearity and cyclic property of the trace and (3.2)

$$\operatorname{tr}\gamma^0 = \operatorname{tr}(-i\beta) = -i\underbrace{\operatorname{tr}\beta}_{0} = 0$$

and

$$\begin{aligned}\operatorname{tr}(\gamma^i) = \operatorname{tr}(\mathbb{I}\gamma^i) = \operatorname{tr}((\gamma^j)^2\gamma^i) = \operatorname{tr}(\gamma^j \underbrace{\gamma^j\gamma^i}_{-\gamma^i\gamma^j}) \\ = -\operatorname{tr}(\gamma^j\gamma^i\gamma^j) = -\operatorname{tr}(\gamma^i\gamma^j\gamma^j) = -\operatorname{tr}(\gamma^i(\gamma^j)^2) = -\operatorname{tr}(\gamma^i) \ .\end{aligned}$$

q.e.d.

### $\gamma^5$

We introduce another matrix $\gamma^5$, called the chirality matrix

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

such that it satisfies the gamma-matrix properties

1. anticommutator, i.e.
$$\{\gamma^5, \gamma^\mu\} = 0 \ ,$$
2. the square is the identity, i.e.
$$(\gamma^5)^2 = \mathbb{I} \ , \tag{3.8}$$
3. hermiticity, i.e.
$$(\gamma^5)^\dagger = \gamma^5 \ ,$$
4. traceless, i.e.
$$\mathrm{tr}(\gamma^5) = 0 \ .$$

*Proof.* For the anticommutator property

$$\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu - i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \ .$$

Now, consider $\mu = 0$

$$\begin{aligned}
\{\gamma^5, \gamma^0\} &= -i\gamma^0\gamma^1\gamma^2 \underbrace{\gamma^3\gamma^0}_{-\gamma^0\gamma^3} - i \underbrace{\gamma^0\gamma^0}_{-1} \gamma^1\gamma^2\gamma^3 \\
&= i\gamma^0\gamma^1 \underbrace{\gamma^2\gamma^0}_{-\gamma^0\gamma^2} \gamma^3 + i\gamma^1\gamma^2\gamma^3 \\
&= -i\gamma^0 \underbrace{\gamma^1\gamma^0}_{-\gamma^0\gamma^1} \gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3 \\
&= i \underbrace{\gamma^0\gamma^0}_{-1} \gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3 \\
&= -i\gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3 = 0
\end{aligned}$$

and similarly for $\mu = 1, 2, 3$.

For the square property

$$(\gamma^5)^2 = (-i\gamma^0\gamma^1\gamma^2\gamma^3)^2 = -\underbrace{(\gamma^0)^2}_{-\mathbb{I}}\underbrace{(\gamma^1)^2}_{\mathbb{I}}\underbrace{(\gamma^2)^2}_{\mathbb{I}}\underbrace{(\gamma^3)^2}_{\mathbb{I}} = \mathbb{I} \ .$$

For the hermiticity property

$$
\begin{aligned}
(\gamma^5)^\dagger &= (-i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger \\
&= i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \\
&= i\gamma^0\gamma^3\underbrace{\gamma^0\gamma^0}_{-1}\gamma^2\underbrace{\gamma^0\gamma^0}_{-1}\gamma^1\underbrace{\gamma^0\gamma^0}_{-1}\underbrace{\gamma^0\gamma^0}_{-1} \\
&= i\gamma^0\gamma^3\underbrace{\gamma^2\gamma^1}_{\gamma^1\gamma^2} \\
&= -i\gamma^0\underbrace{\gamma^3\gamma^1}_{-\gamma^3\gamma^1}\gamma^2 \\
&= i\gamma^0\gamma^1\underbrace{\gamma^3\gamma^2}_{\gamma^2\gamma^3} \\
&= -i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \ .
\end{aligned}
$$

For the traceless property

$$
\begin{aligned}
\mathrm{tr}(\gamma^5) &= \mathrm{tr}(-i\gamma^0\gamma^1\gamma^2\gamma^3) \\
&= -i\,\mathrm{tr}(\underbrace{\gamma^0\gamma^1}_{-\gamma^1\gamma^0}\gamma^2\gamma^3) \\
&= i\,\mathrm{tr}(\gamma^1\underbrace{\gamma^0\gamma^2}_{-\gamma^2\gamma^0}\gamma^3) \\
&= -i\,\mathrm{tr}(\gamma^1\gamma^2\underbrace{\gamma^0\gamma^3}_{-\gamma^3\gamma^0}) \\
&= i\,\mathrm{tr}(\gamma^1\gamma^2\gamma^3\gamma^0) \\
&= i\,\mathrm{tr}(\gamma^0\gamma^1\gamma^2\gamma^3) \\
&= -\,\mathrm{tr}(\gamma^5) \ ,
\end{aligned}
$$

where we have used the cyclic property of the trace. q.e.d.

Explicitly, in the Dirac representation, it becomes

$$
\gamma^5 = \begin{bmatrix} 0 & -\mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{bmatrix} \ .
$$

*Proof.* Infact

$$\begin{aligned}\gamma^5 &= -i\gamma^0\gamma^1\gamma^2\gamma^3 = (-i)^5 \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix} \\ &= -i \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & -\sigma^1\sigma^2 \\ -\sigma^1\sigma^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix} \\ &= -i \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & -\underbrace{\sigma^1\sigma^2}_{i\sigma^3}\sigma^3 \\ \underbrace{\sigma^1\sigma^2}_{i\sigma^3}\sigma^3 & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & -\underbrace{(\sigma^3)^2}_{\mathbb{I}_2} \\ \underbrace{(\sigma^3)^2}_{\mathbb{I}_2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{bmatrix} .\end{aligned}$$

q.e.d.

It adds another dimension to the 4-dimensional Clifford algebra. Infact, we can define a 5-dimensional Clifford algebra by the anticommutator relations

$$\{\gamma^M, \gamma^N\} = 2\eta^{MN} ,$$

where $M, N = 0, 1, 2, 3, 5$ with Minkovski metric $\eta^{MN} = diag(-++++)$.

It can be used to define the projection operators on chiral (Weyl) spinors

$$P_L = \frac{\mathbb{I} - \gamma^5}{2} , \quad P_L = \frac{\mathbb{I} - \gamma^5}{2} , \tag{3.9}$$

such that they satisfy the following properties

1. nilpotent, i.e.
$$P_L^2 = P_L , \quad P_R^2 = P_R ,$$

2. orthogonality, i.e.
$$P_L P_R = 0 , \quad P_L + P_R = \mathbb{I} .$$

*Proof.* For the nilpotent property, using (3.8)

$$P_L^2 = \left(\frac{\mathbb{I} - \gamma^5}{2}\right)^2 = \frac{1}{4}(\mathbb{I}^2 - 2\gamma^5 + \underbrace{(\gamma^5)^2}_{\mathbb{I}}) = \frac{2\mathbb{I} - 2\gamma^5}{4} = \frac{\mathbb{I} - \gamma^5}{2} = P_L$$

and

$$P_R^2 = \left(\frac{\mathbb{I} + \gamma^5}{2}\right)^2 = \frac{1}{4}(\mathbb{I}^2 + 2\gamma^5 + \underbrace{(\gamma^5)^2}_{\mathbb{I}}) = \frac{2\mathbb{I} + 2\gamma^5}{4} = \frac{\mathbb{I} + \gamma^5}{2} = P_R .$$

For the orthogonality property, using (3.8)

$$P_L P_R = \left(\frac{\mathbb{I} - \gamma^5}{2}\right)\left(\frac{\mathbb{I} + \gamma^5}{2}\right) = \frac{1}{4}(\mathbb{I}^2 - \underbrace{(\gamma^5)^2}_{\mathbb{I}}) = \frac{\mathbb{I} - \mathbb{I}}{4} = 0$$

and

$$P_L + P_R = \left(\frac{\mathbb{I} - \cancel{\gamma^5}}{2}\right) + \left(\frac{\mathbb{I} + \cancel{\gamma^5}}{2}\right) = \frac{\mathbb{I} + \mathbb{I}}{2} = \mathbb{I}\ .$$

q.e.d.

They allow to divide a Dirac spinor into two components $\psi = \psi_L + \psi_R$, where $\psi_L = P_L\psi$ is the left-handed one and $\psi_R = P_R\psi$ is the right-handed one. Infact a Dirac spinor can be written as $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. Spinors live in a 4-dimensional complex linear space $\psi(x) \in \mathbb{C}^4$. Therefore, the gamma matrices are an example of 4-dimensional matrices act on this space. A complete basis of linear operators must have 16 of them and we can choose $(\mathbb{I}, \gamma^5, \Sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5)$ where $\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ with $\mu > \nu$. They are indeed respectively $1+4+6+4+1 = 16$ linearly independent matrices.

# Chapter 4

# Non-relativistic limit

In order to study the non-relativistic limit for $c \to \infty$, we insert back the $\hbar$ and $c$.

## 4.1 Free Pauli equation

Starting from (3.4), we decompose the Dirac spinor wave function into two spinor wave functions

$$\psi(t, \mathbf{x}) = \begin{bmatrix} \varphi(t, \mathbf{x}) \\ \chi(t, \mathbf{x}) \end{bmatrix} \exp(-\frac{i}{\hbar} mc^2 t) \tag{4.1}$$

where we have brought out the time dependence from the energy-mass (similarly to the rest-frame wave plane).

In the non-relativistic limit, we recove the free Pauli equation

$$i\hbar \frac{\partial}{\partial t} \varphi = \frac{p^2}{2m} \varphi$$

where $\varphi$ is a 2-dimensional spinor, bringing information about the spin of the particle.

*Proof.* We insert (4.1) into (3.4)

$$i\hbar \frac{\partial}{\partial t} ( \begin{bmatrix} \varphi \\ \chi \end{bmatrix} \exp(-\frac{i}{\hbar} mc^2 t)) = (c\boldsymbol{\alpha} \cdot \mathbf{p} + mc^2 \beta) \begin{bmatrix} \varphi \\ \chi \end{bmatrix} \exp(-\frac{i}{\hbar} mc^2 t) \ ,$$

$$i\hbar ( \begin{bmatrix} \dot{\varphi} \\ \dot{\chi} \end{bmatrix} - \frac{i}{\hbar} mc^2 \begin{bmatrix} \varphi \\ \chi \end{bmatrix} ) \cancel{\exp(-\frac{i}{\hbar} mc^2 t)} = (c \begin{bmatrix} 0 & \boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \end{bmatrix} + mc^2 \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix} ) \begin{bmatrix} \varphi \\ \chi \end{bmatrix} \cancel{\exp(-\frac{i}{\hbar} mc^2 t)} \ ,$$

$$\begin{bmatrix} i\hbar\dot{\varphi} + mc^2 \varphi \\ i\hbar\dot{\chi} + mc^2 \chi \end{bmatrix} = c \begin{bmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} \chi \\ \boldsymbol{\sigma} \cdot \mathbf{p} \varphi \end{bmatrix} + \begin{bmatrix} mc^2 \varphi \\ -mc^2 \chi \end{bmatrix} \ .$$

This is a system of 2 equations

$$\begin{cases} i\hbar\dot{\varphi} + \cancel{mc^2\varphi} = c\boldsymbol{\sigma}\cdot\mathbf{p}\chi + \cancel{mc^2\varphi} \\ i\hbar\dot{\chi} + mc^2\chi = c\boldsymbol{\sigma}\cdot\mathbf{p}\varphi - mc^2\chi \end{cases} ,$$

$$\begin{cases} i\hbar\dot{\varphi} = c\boldsymbol{\sigma}\cdot\mathbf{p}\chi \\ i\hbar\dot{\chi} + 2mc^2\chi = c\boldsymbol{\sigma}\cdot\mathbf{p}\varphi \end{cases} .$$

Now, we go into the non-relativistic limit for $c\to\infty$ which means that $i\hbar\dot{\chi} \ll 2mc^2\chi$ and we obtain

$$\begin{cases} i\hbar\dot{\varphi} = c\boldsymbol{\sigma}\cdot\mathbf{p}\chi \\ \underbrace{\cancel{i\hbar\dot{\chi}}}_{c\to\infty} + 2mc^2\chi = c\boldsymbol{\sigma}\cdot\mathbf{p}\varphi \end{cases} \quad \Rightarrow \quad \begin{cases} i\hbar\dot{\varphi} = c\boldsymbol{\sigma}\cdot\mathbf{p}\chi \\ 2mc^2\chi = c\boldsymbol{\sigma}\cdot\mathbf{p}\varphi \end{cases} .$$

We solve this algebraic equation, starting with the second

$$\chi = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2mc}\varphi$$

and putting into the first

$$i\hbar\dot{\varphi} = c\boldsymbol{\sigma}\cdot\mathbf{p}\chi = \cancel{c}\boldsymbol{\sigma}\cdot\mathbf{p}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m\cancel{c}}\varphi = \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}{2m}\varphi .$$

We notice that, using (3.3)

$$(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \sigma^i p^i \sigma^j p^j = p^i p^j \sigma^i \sigma^j = p^i p^j (\delta^{ij} + i\epsilon^{ijk}\sigma^k) = p^i p^j \delta^{ij} + \cancel{i \underbrace{p^i p^j}_{\text{symm}} \underbrace{\epsilon^{ijk}}_{\text{anti}}} \sigma^k = p^i p^i ,$$

and we obtain the free Pauli equation

$$i\hbar\dot{\varphi} = \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}{2m}\varphi = \frac{p^2}{2m}\varphi .$$

q.e.d.

## 4.2 External electromagnetic field

We study how particles, in this case an electron with charge $e<0$, couple with the electromagnetic field through the minimal substitution

$$E \mapsto E - q\phi , \quad \mathbf{p} \mapsto \boldsymbol{\pi} = \mathbf{p} - \frac{e}{c}\mathbf{A}$$

or in covariant form

$$p_\mu \mapsto \pi_\mu = p_\mu - \frac{e}{c}A_\mu \; , \tag{4.2}$$

where the 4-momentum is $p^\mu = (\frac{E}{c}, \mathbf{p})$ and the 4-potential is $A^\mu = (\phi, \mathbf{A})$. In this way we are also able to study the effects due to the spin.

The non-relativistic Dirac equation for an electron is

$$i\hbar\dot{\varphi} = \Big(\frac{\pi^2}{2m} - \frac{e}{mc}\mathbf{B}\cdot\mathbf{S} + e\phi\Big)\varphi$$

where $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ is the spin of the electron.

*Proof.* We use the minimal substitution in (3.4)

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{x}) = (c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2)\psi(t,\mathbf{x}) \mapsto (c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta mc^2 + e\phi)\psi(t,\mathbf{x}) \; ,$$

where

$$\boldsymbol{\pi} = -i\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A} = -i\hbar\Big(\boldsymbol{\nabla} - \frac{ie}{\hbar c}\mathbf{A}\Big)$$

is the covariant derivative, covariant because of the gauge transformation.

Using the same procedure as before, we decompose $\psi$ with (4.1)

$$i\hbar\frac{\partial}{\partial t}(\begin{bmatrix}\varphi\\ \chi\end{bmatrix}\exp(-\frac{i}{\hbar}mc^2t)) = (c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + mc^2\beta + e\phi)\begin{bmatrix}\varphi\\ \chi\end{bmatrix}\exp(-\frac{i}{\hbar}mc^2t) \; ,$$

$$i\hbar(\begin{bmatrix}\dot\varphi\\ \dot\chi\end{bmatrix} - \frac{i}{\hbar}mc^2\begin{bmatrix}\varphi\\ \chi\end{bmatrix})\cancel{\exp(-\frac{i}{\hbar}mc^2t)} = (c\begin{bmatrix}0 & \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi} & 0\end{bmatrix} + mc^2\begin{bmatrix}\mathbb{I}_2 & 0\\ 0 & -\mathbb{I}_2\end{bmatrix} + e\phi)\begin{bmatrix}\varphi\\ \chi\end{bmatrix}\cancel{\exp(-\frac{i}{\hbar}mc^2t)} \; ,$$

$$\begin{bmatrix}i\hbar\dot\varphi + mc^2\varphi\\ i\hbar\dot\chi + mc^2\chi\end{bmatrix} = c\begin{bmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi\\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\varphi\end{bmatrix} + \begin{bmatrix}(mc^2 + e\phi)\varphi\\ (-mc^2 + e\phi)\chi\end{bmatrix} \; .$$

This is a system of 2 equations

$$\begin{cases} i\hbar\dot\varphi + \cancel{mc^2\varphi} = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi + \cancel{mc^2\varphi} + e\phi\varphi \\ i\hbar\dot\chi + mc^2\chi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\varphi - mc^2\chi + e\phi\chi \end{cases} \; ,$$

$$\begin{cases} i\hbar\dot\varphi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi + e\phi\varphi \\ i\hbar\dot\chi + 2mc^2\chi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\varphi + e\phi\chi \end{cases} \; .$$

Now, we go into the non-relativistic limit for $c \to \infty$ which means that $i\hbar\dot\chi \ll 2mc^2\chi$ and $e\phi\chi \ll 2mc^2\chi$, thus we obtain

$$\begin{cases} i\hbar\dot\varphi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi + e\phi\varphi \\ \underbrace{\cancel{i\hbar\dot\chi}}_{c\to\infty} + 2mc^2\chi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\varphi + \underbrace{\cancel{e\phi\chi}}_{c\to\infty} \end{cases} \Rightarrow \begin{cases} i\hbar\dot\varphi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi + e\phi\varphi \\ 2mc^2\chi = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\varphi \end{cases} \; .$$

We solve this algebraic equation, starting with the second

$$\chi = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\pi}}{2mc}\varphi$$

and putting into the first

$$i\hbar\dot{\varphi} = c\boldsymbol{\sigma} \cdot \boldsymbol{\pi}\chi + e\phi\varphi = \cancel{c}\boldsymbol{\sigma} \cdot \boldsymbol{\pi}\frac{\boldsymbol{\sigma} \cdot \boldsymbol{\pi}}{2m\cancel{c}}\varphi + e\phi\varphi = \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2}{2m}\varphi + e\phi\varphi \ .$$

We notice that, using (3.3)

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 &= \sigma^i\pi^i\sigma^j\pi^j \\
&= \pi^i\pi^j\sigma^i\sigma^j \\
&= \pi^i\pi^j(\delta^{ij} + i\epsilon^{ijk}\sigma^k) \\
&= \pi^i\pi^j\delta^{ij} + i\epsilon^{ijk}\sigma^k \underbrace{\pi^i\pi^j}_{\frac{\pi^i\pi^j - \pi^j\pi^i}{2}} \\
&= \pi^2 + i\epsilon^{ijk}\sigma^k\frac{\pi^i\pi^j - \pi^j\pi^i}{2} \\
&= \pi^2 + \frac{i}{2}\epsilon^{ijk}\sigma^k[\pi^i, \pi^j] \ ,
\end{aligned}$$

where we exploit the antisymmetry of $\epsilon^{ijk}$ to antisymmetrise $\pi^i\pi^j$. We compute the commutator

$$\begin{aligned}
[\pi^i, \pi^j] &= [p^i - \frac{e}{c}A^i, p^j - \frac{e}{c}A^j] \\
&= \underbrace{[p^i, p^j]}_{0} - \frac{e}{c}[p^i, A^j] - \frac{e}{c}\underbrace{[A^i, p^j]}_{-[p^j, A^i]} + \frac{e^2}{c^2}\underbrace{[A^i(x), A^j(x)]}_{0} \\
&= -\frac{e}{c}([p^i, A^j] - [p^j, A^i]) \\
&= -\frac{e}{c}(p^iA^j - A^jp^i - p^jA^i + p^iA^j) \\
&= \frac{i\hbar e}{c}(\partial^iA^j - A^j\partial^i - \partial^jA^i + A^i\partial^j) \\
&= \frac{i\hbar e}{c}(\partial^iA^j - \partial^jA^i) \\
&= \frac{2i\hbar e}{c}\partial^iA^j
\end{aligned}$$

where we used the canonical commutation relation, $[A^i(x), A^j(x)] = 0$ because $A(x)$

is a function of $x$, (**??**) and the fact that second partial derivatives commute. Hence

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 &= \pi^2 + \frac{i\hbar e}{2c} i\epsilon^{ijk}\underbrace{\left(\partial^i A^j - \partial^j A^i\right)}_{2\partial^i A^j} \\
&= \pi^2 + \frac{i\hbar e}{\cancel{2}c}\cancel{2}i\,\underbrace{\epsilon^{ijk}\partial^i A^j}_{B^k}\sigma^k \\
&= \pi^2 - \frac{\hbar e}{c} B^k \sigma^k \\
&= \pi^2 - \frac{2e}{c}\mathbf{B}\cdot\underbrace{\frac{1}{2}\boldsymbol{\sigma}}_{\mathbf{S}} \\
&= \pi^2 - \frac{2e}{c}\mathbf{B}\cdot\mathbf{S}
\end{aligned}
$$

where we exploit the antisymmetry of $\epsilon^{ijk}$ to antisymmetrise $\partial^i A^j$. Finally we obtain the non-relativistic Dirac equation for the electron

$$
i\hbar\dot{\varphi} = \Big(\frac{1}{2m}(\pi^2 - \frac{2e}{mc}\mathbf{B}\cdot\mathbf{S}) + e\phi\Big)\varphi = \Big(\frac{\pi^2}{2m} - \frac{e}{mc}\mathbf{B}\cdot\mathbf{S} + e\phi\Big)\varphi\ .
$$

q.e.d.

## Angular momentum and spin

Recall that a magnetic dipole with magnetic moment $\boldsymbol{\mu}$ couples with an external magnetic field $\mathbf{B}$ through the hamiltonian

$$
H = -\boldsymbol{\mu}\cdot\mathbf{B}\ .
$$

In particular, a charge $e$ in motion with angular momentum $\mathbf{L}$ has magnetic moment proportional to it

$$
|\boldsymbol{\mu}| = \frac{e}{2mc} g|\mathbf{L}|
$$

where the gyromagnetic factor $g = 1$ for the angular momentum of the electron and $g = 2$ for its spin.

**Example 4.1.** Consider a constant magnetic field **B** and its associated vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$. Hence

$$\begin{aligned}
\pi^2 &= (\mathbf{p} - \frac{e}{c}\mathbf{A})(\mathbf{p} - \frac{e}{c}\mathbf{A}) \\
&= p^2 - \frac{e}{c}(\mathbf{A}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{A}) + \frac{e^2}{c^2}A^2 \\
&= p^2 - \frac{2e}{c}(\mathbf{A}\cdot\mathbf{p}) + \frac{e^2}{c^2}A^2 \\
&= p^2 - \frac{2e}{c}(\frac{1}{2}\mathbf{B}\times\mathbf{r}\cdot\mathbf{p}) + \frac{e^2}{c^2}A^2 \\
&= p^2 - \frac{2e}{c}(\frac{1}{2}\mathbf{B}\cdot\underbrace{\mathbf{r}\times\mathbf{p}}_{\mathbf{L}}) + \frac{e^2}{c^2}A^2 \\
&= p^2 - \frac{2e}{c}(\frac{1}{2}\mathbf{B}\cdot\mathbf{L}) + \frac{e^2}{c^2}A^2 \\
&= p^2 - \frac{e}{c}\mathbf{B}\cdot\mathbf{L} + \frac{e^2}{c^2}A^2 \ ,
\end{aligned}$$

where we have used the fact that $\mathbf{p}\cdot\mathbf{A}$ commutes for constant **B** and the cyclic property for the mixed product. This means that the gyromagnetic factors associated to angular momentum and spin are

$$i\hbar\dot{\varphi} = \Big(\frac{\pi^2}{2m} - \frac{e}{2mc}\mathbf{B}\cdot(\underbrace{1}_{g=1}\mathbf{L} + \underbrace{2}_{g=2}\mathbf{S}) + \frac{e^2}{2mc^2}A^2 + e\phi\Big)\varphi \ .$$

By similarity with the non-relativistic limit $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ and $\chi \propto \varphi$, the spin operator of the Dirac spinor in the Dirac representation is

$$\mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\begin{bmatrix}\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}\end{bmatrix} \ .$$

In index notation

$$\Sigma^i = -\frac{i}{2}\epsilon^{ijk}\alpha^j\alpha^k \ .$$

Introducing the angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, we have the following commutation relations

$$[H_D, L^i] = -i\epsilon^{ijk}\alpha^j p^k \ ,$$

$$[H_D, S^i] = -i\epsilon^{ijk}p^j\alpha^k$$

and

$$[H_D, J^i] = 0 \ .$$

*Proof.* For the angular momentum commutation relation, (3.2)

$$
\begin{aligned}
[H_D, L^i] &= [\alpha^l p_l + \beta m, \epsilon^{ijk} x^j p^k] \\
&= [\alpha^l p_l, \epsilon^{ijk} x^j p^k] + \underbrace{[\beta m, \epsilon^{ijk} x^j p^k]}_{0} \\
&= [\alpha^l p_l, \epsilon^{ijk} x^j p^k] \\
&= \alpha^l \underbrace{[p_l, x^j]}_{-i\delta_l{}^j} \epsilon^{ijk} p^k + \alpha^l \underbrace{[p_l, p^k]}_{0} \epsilon^{ijk} x^j \\
&= -i\alpha^l \delta_l{}^j \epsilon^{ijk} p^k \\
&= -i\epsilon^{ijk} \alpha^j p^k \ .
\end{aligned}
$$

For the spin commutation relation, (3.2)

$$
\begin{aligned}
[H_D, S^i] &= [\alpha^l p_l + \beta m, -\frac{i}{4}\epsilon^{ijk}\alpha^j\alpha^k] \\
&= [\alpha^l p_l, -\frac{i}{4}\epsilon^{ijk}\alpha^j\alpha^k] + [\beta m, -\frac{i}{4}\epsilon^{ijk}\alpha^j\alpha^k] \\
&= -\frac{i}{4} p_l \epsilon^{ijk} [\alpha^l, \alpha^j\alpha^k] - \frac{i}{4} m \epsilon^{ijk} [\beta, \alpha^j \alpha^k] \\
&= -\frac{i}{4} p_l \epsilon^{ijk} (\underbrace{\{\alpha^l, \alpha^j\}}_{2\eta^{lj}} \alpha^k - \alpha^j \underbrace{\{\alpha^l, \alpha^k\}}_{2\eta^{lk}}) - \frac{i}{4} m \epsilon^{ijk} (\underbrace{\{\beta, \alpha^j\}}_{0} \alpha^k - \alpha^j \underbrace{\{\beta, \alpha^k\}}_{0}) \\
&= -\frac{i}{4} p_l \epsilon^{ijk} (2\eta^{lj}\alpha^k - \alpha^j 2\eta^{lk}) \\
&= -i\epsilon^{ijk} p^j \alpha^k \ ,
\end{aligned}
$$

where we exploit the antysymmetry of $\epsilon^{ijk}$ to antisymmetrise $\alpha^j \eta^{lk}$ and we have used the identity

$$
[AB, C] = ABC - CAB = ABC + ACB - CAB - ACB = A\{B, C\} - \{A, C\}B \ .
$$

For the total momentum commutation relation

$$
\begin{aligned}
[H_D, J^i] = [H_D, L^i + S^i] &= [H_D, L_i] + [H_D, S^i] = -i\epsilon^{ijk}\alpha^j p^k - i\epsilon^{ijk} p^j \alpha^k \\
&= -i\epsilon^{ijk}\alpha^j p^k + i\epsilon^{ikj} p^j \alpha^k = 0 \ .
\end{aligned}
$$

q.e.d.

An experimental test for the Dirac theory is the prediction of the energy levels of the hydrogen atom, via perturbation solutions. Schoredinger solution is

$$
E_{nl} = -\frac{m_e \alpha^2}{2n^2} \ ,
$$

Klein-Gordon solution is

$$E_{nl} = m_e\Big(1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{n^4}\Big(\frac{n}{2l+1} - \frac{3}{8}\Big) + O(\alpha^6)\Big)$$

and Dirac solution is

$$E_{nl} = m_e\Big(1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{n^4}\Big(\frac{n}{2j+1} - \frac{3}{8}\Big) + O(\alpha^6)\Big) \ .$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant. Notice that the Schoredinger solution is degenerate in $l$, Klein-Gordon solution lifts this degeneracy but it is not in agree with experiments, since $2l+1$ is odd and it must be even, and finally Dirac solution is the right one, since $2j+1$ is even.

# Chapter 5

# Covariance

## 5.1 Dirac spinor representation

Now, we verify that the Dirac equation is Lorentz invariant, i.e it is covariant under a generic transformation of $SO^+(1,3)$. Recall that given a Lorentz transformation $\Lambda \in SO^+(1,3)$, the coordinates transform as

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu$$

and the partial derivatives transform as

$$\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu \ .$$

Therefore, the Dirac spinor transform as

$$\psi'(x') = S(\Lambda)\psi(x)$$

where $S(\Lambda)$ is linear representation of the proper orthochronous group of spinors and the Dirac equation is covariant

$$(\gamma^\mu \partial'_\mu + m)\psi'(x') = 0 \ .$$

The infinitesimal Lorentz transformation $S(\Lambda)$ is

$$S = \mathbb{I} + \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}$$

where $\Sigma^{\mu\nu}$ are a set of 6 antisymmetric $4 \times 4$ matrices that act on spinors

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] \ , \tag{5.1}$$

such that they satisfy the commutator relations

$$[\Sigma^{\mu\nu}, \gamma^\rho] = i(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu) \ .$$

*Proof.* We transform with a Lorentz transformation every components in the Dirac equation

$$0 = (\gamma^\mu \partial'_\mu + m)\psi'(x') = (\gamma^\mu {\Lambda_\mu}^\nu \partial_\nu + m)S(\Lambda)\psi(x)\ .$$

Hence

$$\begin{aligned} 0 &= S^{-1}(\Lambda)(\gamma^\mu {\Lambda_\mu}^\nu \partial_\nu + m)S(\Lambda)\psi(x) \\ &= (S^{-1}(\Lambda)\gamma^\mu {\Lambda_\mu}^\nu S(\Lambda)\partial_\nu + m\,\underbrace{S^{-1}(\Lambda)S(\Lambda)}_{1})\psi(x) \\ &= (S^{-1}(\Lambda)\gamma^\mu {\Lambda_\mu}^\nu S(\Lambda)\partial_\nu + m)\psi(x)\ . \end{aligned}$$

We compare it with (3.5)

$$0 = (S^{-1}(\Lambda)\gamma^\mu {\Lambda_\mu}^\nu S(\Lambda)\partial_\nu + m)\psi(x) = (\gamma^\nu \partial_\nu + m)\psi(x)$$

and we find

$$S^{-1}(\Lambda)\gamma^\mu {\Lambda_\mu}^\nu S(\Lambda) = \gamma^\nu$$

or, equivalently,

$$\begin{aligned} S^{-1}(\Lambda)\gamma^\mu \underbrace{{\Lambda^\rho}_\nu {\Lambda_\mu}^\nu}_{{\delta^\rho}_\mu} S(\Lambda) &= {\Lambda^\rho}_\nu \gamma^\nu\ , \\ S^{-1}(\Lambda)\underbrace{\gamma^\mu {\delta^\rho}_\mu}_{\gamma^\rho} S(\Lambda) &= {\Lambda^\rho}_\nu \gamma^\nu\ , \\ S^{-1}(\Lambda)\gamma^\rho S(\Lambda) &= {\Lambda^\rho}_\nu \gamma^\nu\ . \end{aligned} \tag{5.2}$$

Now, we consider an infinitesimal Lorentz transformation

$${\Lambda^\mu}_\nu = {\delta^\mu}_\nu + {\omega^\mu}_\nu\ ,$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, which induces an infinitesimal Lorentz transformation on the spinor

$$S(\Lambda) = \mathbb{I} + \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\ ,$$

where $\Sigma^{\mu\nu} = -\Sigma^{\mu\nu}$. Substituting in (5.2), we find

$$\Big(\mathbb{I} - \frac{i}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}\Big)\gamma^\rho\Big(\mathbb{I} + \frac{i}{2}\omega_{\sigma\lambda}\Sigma^{\sigma\lambda}\Big) = ({\delta^\rho}_\nu + {\omega^\rho}_\nu)\gamma^\nu\ .$$

and we only keep first order terms in $\omega$

$$\begin{aligned} \cancel{\gamma^\rho} - \frac{i}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}\gamma^\rho + \frac{i}{2}\gamma^\rho\omega_{\sigma\lambda}\Sigma^{\sigma\lambda} &= \cancel{\gamma^\rho} + {\omega^\rho}_\nu\gamma^\nu\ , \\ -\frac{i}{2}\omega_{\alpha\beta}\underbrace{(\Sigma^{\alpha\beta}\gamma^\rho - \gamma^\rho\Sigma^{\alpha\beta})}_{[\Sigma^{\alpha\beta},\gamma^\rho]} &= {\omega^\rho}_\nu\gamma^\nu\ , \end{aligned}$$

$$-\frac{i}{2}\omega_{\alpha\beta}[\Sigma^{\alpha\beta},\gamma^\rho]=\omega^\rho{}_\nu\gamma^\nu \ ,$$

where we have exchanged $\sigma=\alpha$ and $\lambda=\beta$. Hence

$$\cancel{\omega_{\alpha\beta}}[\Sigma^{\alpha\beta},\gamma^\rho]=2i\omega^\rho{}_\beta\gamma^\beta=\omega_{\alpha\beta}2i\underbrace{\eta^{\rho\alpha}\gamma^\beta}_{\frac{\eta^{\rho\alpha}\gamma^\beta-\eta^{\rho\beta}\gamma^\alpha}{2}}=\cancel{\omega_{\alpha\beta}}i(\eta^{\rho\alpha}\gamma^\beta-\eta^{\rho\beta}\gamma^\alpha) \ ,$$

where we have exchanged $\nu=\beta$ and we exploit the antysymmetry of $\omega_{\alpha\beta}$ to anti-symmetrise $\eta^{\rho\alpha}\gamma^\beta$. Thus

$$[\Sigma^{\alpha\beta},\gamma^\rho]=i(\eta^{\rho\alpha}\gamma^\beta-\eta^{\rho\beta}\gamma^\alpha) \ .$$

The solution of this algebraic commutation equation is

$$\Sigma^{\alpha\beta}=-\frac{i}{4}[\gamma^\alpha,\gamma^\beta] \ .$$

Infact, using (3.6)

$$\begin{aligned}
[\Sigma^{\alpha\beta},\gamma^\mu]&=-\frac{i}{4}[\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha,\gamma^\mu]\\
&=-\frac{i}{4}[\gamma^\alpha\gamma^\beta,\gamma^\mu]-\frac{i}{4}[\gamma^\beta\gamma^\alpha,\gamma^\mu]\\
&=-\frac{i}{4}(\gamma^\alpha\{\gamma^\beta,\gamma^\mu\}-\{\gamma^\alpha,\gamma^\mu\}\gamma^\beta-\gamma^\beta\{\gamma^\alpha,\gamma^\mu\}+\{\gamma^\beta,\gamma^\mu\}\gamma^\alpha)\\
&=-\frac{i}{4}(\gamma^\alpha\underbrace{\{\gamma^\beta,\gamma^\mu\}}_{2\eta^{\beta\mu}}-\underbrace{\{\gamma^\alpha,\gamma^\mu\}}_{2\eta^{\alpha\mu}}\gamma^\beta-\gamma^\beta\underbrace{\{\gamma^\alpha,\gamma^\mu\}}_{2\eta^{\alpha\mu}}+\underbrace{\{\gamma^\beta,\gamma^\mu\}}_{2\eta^{\beta\mu}}\gamma^\alpha)\\
&=-\frac{i}{2}(\gamma^\alpha\eta^{\beta\mu}-\eta^{\alpha\mu}\gamma^\beta-\gamma^\beta\eta^{\alpha\mu}+\eta^{\beta\mu}\gamma^\alpha)\\
&=-\frac{i}{2}(\eta^{\beta\mu}\gamma^\alpha-\eta^{\alpha\mu}\gamma^\beta-\eta^{\alpha\mu}\gamma^\beta+\eta^{\beta\mu}\gamma^\alpha)\\
&=-i(\eta^{\mu\beta}\gamma^\alpha-\eta^{\mu\alpha}\gamma^\beta) \ ,
\end{aligned}$$

where we have used the fact that the $\eta$ is symmetric, it commutes with the $\gamma$'s and the identity

$$[AB,C]=ABC-CAB=ABC+ACB-CAB-ACB=A\{B,C\}-\{A,C\}B \ .$$

q.e.d.

A generic Lorentz transformation is obtained by iterating intinitesimal ones via exponential map

$$S(\Lambda)=\exp(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})=\exp(\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu) \tag{5.3}$$

*Proof.* Infact, using (5.1)

$$\begin{aligned} S(\Lambda) &= \exp(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}) \\ &= \exp(\frac{i}{2}\omega_{\mu\nu}(-\frac{i}{4}[\gamma^\mu,\gamma^\nu])) \\ &= \exp(\frac{1}{8}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu - \underbrace{\gamma^\nu\gamma^\mu}_{-\gamma^\mu\gamma^\nu})) \\ &= \exp(\frac{1}{8}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu)) \\ &= \exp(\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu)\ . \end{aligned}$$

q.e.d.

## 5.2 Application on rotations and boosts

**Example 5.1** (Rotation around the z-axis)**.** Consider a rotation around the z-axis. The infinitesimal Lorentz transformation is parametrised by

$$\omega_{\mu\nu} = \begin{cases} \varphi & (\mu,\nu) = (1,2) \\ -\varphi & (\mu,\nu) = (2,1) \\ 0 & otherwise \end{cases}\ .$$

Therefore, the infinitesimal $\omega$ matrix is

$$\omega^\mu{}_\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 \\ 0 & -\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and a finite Lorentz transformation can be found by the exponential map

$$\Lambda^\mu{}_\nu = (\exp(\omega))^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\ .$$

*Proof.* Recall the Taylor expansions of the sine and cosine functions

$$\cos\varphi = 1 - \frac{\varphi^2}{2} + \dots\ , \quad \sin\varphi = \varphi - \frac{\varphi^3}{3!} + \dots\ .$$

We Taylor expand the exponential and find

$$\exp(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 \\ 0 & -\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \mathbb{I}_4 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 \\ 0 & -\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\varphi^2 & 0 & 0 \\ 0 & 0 & -\varphi^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi^3 & 0 \\ 0 & \varphi^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{\varphi^2}{2} + \dots & \varphi - \frac{\varphi^3}{3!} + \dots & 0 \\ 0 & -\varphi + \frac{\varphi^3}{3!} + \dots & 1 - \frac{\varphi^2}{2} + \dots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

q.e.d.

Moreover, a generic Lorents transformation on a Dirac spinor is

$$S(\Lambda) = \exp(\frac{i\varphi}{2} \begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix}) = \begin{bmatrix} \exp(\frac{i\varphi}{2}) & 0 & 0 & 0 \\ 0 & \exp(\frac{-i\varphi}{2}) & 0 & 0 \\ 0 & 0 & \exp(\frac{i\varphi}{2}) & 0 \\ 0 & 0 & 0 & \exp(\frac{-i\varphi}{2}) \end{bmatrix}.$$

*Proof.* Infact, using (5.3)

$$S(\Lambda) = \exp(\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu) = \exp(\frac{1}{4}(\underbrace{\omega_{12}}_{\varphi}\gamma^1\gamma^2 + \underbrace{\omega_{21}}_{-\varphi}\gamma^2\gamma^1)) = \exp(\frac{\varphi}{4}(\gamma^1\gamma^2 - \underbrace{\gamma^2\gamma^1}_{-\gamma^1\gamma^2}))$$

$$= \exp(\frac{\varphi}{2}\underbrace{\gamma^1}_{-i\beta\alpha^1}\underbrace{\gamma^2}_{-i\beta\alpha^3}) = \exp(\frac{\varphi}{2}\underbrace{\beta^2}_{1}\alpha^1\alpha^2) = \exp(\frac{\varphi}{2}\begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix})$$

$$= \exp(\frac{\varphi}{2}\begin{bmatrix} \underbrace{\sigma^1\sigma^2}_{i\sigma^3} & 0 \\ 0 & \underbrace{\sigma^1\sigma^2}_{i\sigma^3} \end{bmatrix}) = \exp(\frac{i\varphi}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix})$$

$$= \exp(\begin{bmatrix} \frac{i\varphi}{2} & 0 & 0 & 0 \\ 0 & -\frac{i\varphi}{2} & 0 & 0 \\ 0 & 0 & \frac{i\varphi}{2} & 0 \\ 0 & 0 & 0 & -\frac{i\varphi}{2} \end{bmatrix}) = \begin{bmatrix} \exp(\frac{i\varphi}{2}) & 0 & 0 & 0 \\ 0 & \exp(\frac{-i\varphi}{2}) & 0 & 0 \\ 0 & 0 & \exp(\frac{i\varphi}{2}) & 0 \\ 0 & 0 & 0 & \exp(\frac{-i\varphi}{2}) \end{bmatrix},$$

where we used the property of the exponential of a diagonal matrix. q.e.d.

Notice that it is a unitary representation $S^\dagger(\Lambda) = S^{-1}(\Lambda)$. It is also a double-valued representation, since a rotation of $\varphi = 2\pi$ is represented by $S(\Lambda) = -\mathbb{I}$. Only with a rotation of $\varphi = 4\pi$, we find the identity again.

**Example 5.2** (Generic rotation)**.** A generic rotation of an angle $\varphi$ around an axis $\ltimes$ is represented by

$$S(\Lambda) = \begin{bmatrix} \exp(\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}) & 0 \\ 0 & \exp(\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}) \end{bmatrix} .$$

This means that a decomposition like the non-relativistic limit make a rotation on both the wave functions indipendently.

**Example 5.3** (Boost along the x-axis)**.** Consider a boost around the x-axis. The infinitesimal Lorentz transformation is parametrised by

$$\omega_{\mu\nu} = \begin{cases} -w & (\mu,\nu) = (0,1) \\ -w & (\mu,\nu) = (1,0) \\ 0 & otherwise \end{cases} .$$

Therefore, the infinitesimal $\omega$ matrix is

$$\omega^\mu{}_\nu = \begin{bmatrix} 0 & -w & 0 & 0 \\ -w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and a finite Lorentz transformation can be found by the exponential map

$$\Lambda^\mu{}_\nu = (\exp(\omega))^\mu{}_\nu = \begin{bmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

where we defined the rapidity $w$ in terms of the Lorentz factors

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh w \ , \quad \beta = v = \tanh w \ , \quad \beta\gamma = \sinh w \ . \tag{5.4}$$

*Proof.* Recall the Taylor expansions of the hyperbolic sine and hyperbolic cosine functions

$$\cosh w = 1 + \frac{w^2}{2} + \dots \ , \quad \sinh w = w + \frac{w^3}{3!} + \dots \ .$$

We Taylor expand the exponential and find

$$
\begin{aligned}
\exp(\omega) &= \sum_{k=0}^{\infty} \frac{\omega^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & -w & 0 & 0 \\ -w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \mathbb{I}_4 + \begin{bmatrix} 0 & -w & 0 & 0 \\ -w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} w^2 & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & -w^3 & 0 & 0 \\ -w^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \dots \\
&= \begin{bmatrix} 1 + \frac{w^2}{2} + \dots & -w - \frac{w^3}{3!} + \dots & 0 & 0 \\ -w - \frac{w^3}{3!} + \dots & 1 + \frac{w^2}{2} + \dots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .
\end{aligned}
$$

q.e.d.

Moreover, a generic Lorents transformation on a Dirac spinor is

$$S(\Lambda) = \cosh \frac{w}{2} \mathbb{I} - \sinh \frac{w}{2} \alpha^1 \ . \tag{5.5}$$

*Proof.* Infact, using (5.3)

$$S(\Lambda) = \exp(\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu) = \exp(\frac{1}{4}(\underbrace{\omega_{01}}_{w}\gamma^0\gamma^1 + \underbrace{\omega_{10}}_{w}\gamma^1\gamma^0)) = \exp(\frac{w}{4}(\gamma^0\gamma^1 + \underbrace{\gamma^0\gamma^1}_{\gamma^1\gamma^0}))$$

$$= \exp(\frac{w}{2}\underbrace{\gamma^0}_{-i\beta}\underbrace{\gamma^1}_{-i\beta\alpha^1}) = \exp(-\frac{w}{2}\underbrace{\beta^2}_{1}\alpha^1) = \exp(-\frac{w}{2}\begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix})$$

$$= \exp(\begin{bmatrix} 0 & 0 & 0 & -\frac{w}{2} \\ 0 & 0 & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & 0 & 0 \\ -\frac{w}{2} & 0 & 0 & 0 \end{bmatrix}) = \sum_{k=0}^{\infty}\frac{1}{k!}\begin{bmatrix} 0 & 0 & 0 & -\frac{w}{2} \\ 0 & 0 & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & 0 & 0 \\ -\frac{w}{2} & 0 & 0 & 0 \end{bmatrix}^k$$

$$= \mathbb{I} + \begin{bmatrix} 0 & 0 & 0 & -\frac{w}{2} \\ 0 & 0 & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & 0 & 0 \\ -\frac{w}{2} & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{w^2}{4} & 0 & 0 & 0 \\ 0 & \frac{w^2}{4} & 0 & 0 \\ 0 & 0 & \frac{w^2}{4} & 0 \\ 0 & 0 & 0 & \frac{w^2}{4} \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & 0 & 0 & -\frac{w^3}{8} \\ 0 & 0 & -\frac{w^3}{8} & 0 \\ 0 & -\frac{w^3}{8} & 0 & 0 \\ -\frac{w^3}{8} & 0 & 0 & 0 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 + \frac{w^2}{8} + \dots & 0 & 0 & -\frac{w}{2} - \frac{w^3}{48} + \dots \\ 0 & 1 + \frac{w^2}{8} + \dots & -\frac{w}{2} - \frac{w^3}{48} + \dots & 0 \\ 0 & -\frac{w}{2} - \frac{w^3}{48} + \dots & 1 + \frac{w^2}{8} + \dots & 0 \\ -\frac{w}{2} - \frac{w^3}{48} + \dots & 0 & 0 & 1 + \frac{w^2}{8} + \dots \end{bmatrix}$$

$$= \begin{bmatrix} \cosh\frac{w}{2} & 0 & 0 & -\sinh\frac{w}{2} \\ 0 & \cosh\frac{w}{2} & -\sinh\frac{w}{2} & 0 \\ 0 & -\sinh\frac{w}{2} & \cosh\frac{w}{2} & 0 \\ -\sinh\frac{w}{2} & 0 & 0 & \cosh\frac{w}{2} \end{bmatrix}$$

$$= \cosh\frac{w}{2}\mathbb{I} - \sinh\frac{w}{2}\alpha^1 \ ,$$

where we used the Taylor expansion of the exponential. q.e.d.

Notice that it is a not unitary representation $S^\dagger(\Lambda) \neq S^{-1}(\Lambda)$, since there is a theorem that states that in a non-compact group, like the boost because they are not upper-bounded in velocity, the only irreducible representations are infinite-dimensional. However, it satisfies $S^\dagger(\Lambda) = S(\Lambda)$.

In terms of mass, momentum and energy, a finite Lorentz transformation on a Dirac spinor becomes

$$S(\Lambda) = \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} - \frac{\alpha^1|\mathbf{p}|}{m+E}\Big) \ .$$

*Proof.* Infact, using the hyberbolic trigonometry identities

$$\tanh\frac{w}{2} = \frac{\sinh w}{1+\cosh w}$$

and

$$\cosh\frac{w}{2} = \sqrt{\frac{1+\cosh w}{2}}$$

Using the rapidity relations (5.4), we can rewrite (5.5) as

$$\begin{aligned} S(\Lambda) &= \cosh\frac{w}{2}\left(\mathbb{I} - \alpha^1\tanh\frac{w}{2}\right) \\ &= \sqrt{\frac{1}{2}}(1+\underbrace{\cosh w}_{\gamma})^{\frac{1}{2}}\left(\mathbb{I} + \alpha^1\frac{\overbrace{\sinh w}^{\beta\gamma}}{1+\underbrace{\cosh w}_{\gamma}}\right) \\ &= \sqrt{\frac{1+\gamma}{2}}\left(\mathbb{I} + \alpha^1\frac{\beta\gamma}{1+\gamma}\right) . \end{aligned}$$

Now, we use the 4-momentum $(E, p) = (m\gamma, m\gamma\beta)$ and we reverse to find

$$\gamma = \frac{E}{m} , \quad \beta\gamma = \frac{|\mathbf{p}|}{m} .$$

Putting together, we obtain

$$S(\Lambda) = \sqrt{\frac{1+\gamma}{2}}\left(\mathbb{I}+\alpha^1\frac{\beta\gamma}{1+\gamma}\right) = \sqrt{\frac{1+\frac{E}{m}}{2}}\left(\mathbb{I}+\alpha^1\frac{\frac{\mathbf{p}}{m}}{1+\frac{E}{m}}\right) = \sqrt{\frac{m+E}{2m}}\left(\mathbb{I}+\frac{\alpha^1|\mathbf{p}|}{m+E}\right) .$$

q.e.d.

**Example 5.4** (Generic boost)**.** A generic boost of rapidity $w$ along an axis $\ltimes$ is represented by

$$S(\Lambda) = \sqrt{\frac{m+E}{2m}}\left(\mathbb{I} + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\right)$$

## Pseudo-unitary

Dirac spinorial representation is pseudo-unitary

$$S^\dagger(\Lambda) = \beta S^{-1}(\Lambda)\beta .$$

*Proof.* Infact, using (3.7)

$$
\begin{aligned}
(\Sigma^{\mu\nu})^\dagger &= (-\frac{i}{4}[\gamma^\mu,\gamma^\nu])^\dagger \\
&= \frac{i}{4}[\underbrace{(\gamma^\nu)^\dagger}_{-\beta\gamma^\nu\beta},\underbrace{(\gamma^\mu)^\dagger}_{-\beta\gamma^\mu\beta}] \\
&= \frac{i}{4}[\beta\gamma^\nu\beta,\beta\gamma^\mu\beta] \\
&= \frac{i}{4}(\beta\gamma^\nu\underbrace{\beta\beta}_{1}\gamma^\mu\beta-\beta\gamma^\mu\underbrace{\beta\beta}_{1}\gamma^\nu\beta) \\
&= \beta\frac{i}{4}[\gamma^\nu,\gamma^\mu]\beta \\
&= -\beta\underbrace{(-\frac{i}{4}[\gamma^\mu,\gamma^\nu])}_{\Sigma^{\mu\nu}}\beta \\
&= -\beta\Sigma^{\mu\nu}\beta\ .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S^\dagger(\Lambda) &= \exp(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})^\dagger \\
&= \exp(-\frac{i}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})^\dagger) \\
&= \exp(-\frac{i}{2}\omega_{\mu\nu}\beta\Sigma^{\mu\nu}\beta) \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}(-\frac{i}{2}\omega_{\mu\nu}\beta\Sigma^{\mu\nu}\beta)^k \\
&= \beta\sum_{k=0}^{\infty}\frac{1}{k!}(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})^k\beta \\
&= \beta\exp(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\beta \\
&= \beta S^{-1}(\Lambda)\beta\ .
\end{aligned}
$$

q.e.d.

## 5.3 Fermionic bilinears

Since the Dirac spinor representatio is pseudo-unitary, we define the Dirac conjugate by

$$\overline{\psi}(x) = \psi^\dagger(x)\beta\ ,$$

such that it transforms as

$$\overline{\psi}'(x') = \overline{\psi}(x) S^{-1}(\Lambda) \ .$$

*Proof.* Infact

$$\overline{\psi}'(x') = (\underbrace{\psi'}_{S(\Lambda)\psi})^\dagger(x')\beta = (S^(\Lambda)\psi(x))^\dagger = \psi^\dagger(x) \underbrace{S^\dagger(\Lambda)}_{\beta S^{-1}(\Lambda)\beta} \beta = \underbrace{\psi^\dagger(x)\beta}_{\overline{\psi}(x)} S^{-1}(\Lambda) \underbrace{\beta\beta}_{1} = \overline{\psi}(x) S^{-1}(\Lambda) \ .$$

q.e.d.

With the Dirac spinor and its conjugate we can build a scalar, which is

$$\overline{\psi}(x)\psi(x) \ ,$$

but not this

$$\psi^\dagger(x)\psi(x) \ .$$

*Proof.* For the scalar

$$\overline{\psi}'(x')\psi'(x') = \overline{\psi}(x) \underbrace{S^{-1}(\Lambda)S(\Lambda)}_{1} \psi(x) = \overline{\psi}(x)\psi(x) \ .$$

For the non-scalar

$$(\psi')^\dagger(x')\psi'(x') = (S(\Lambda)\psi)^\dagger(x) S(\Lambda)\psi(x) = \psi^\dagger(x) \underbrace{S^\dagger(\Lambda)}_{\beta S^{-1}\beta} S(\Lambda)\psi(x) = \psi^\dagger(x)\beta S^{-1}\beta S(\Lambda)\psi(x) \neq \psi^\dagger(x)\psi($$

q.e.d.

Actually, it is the 0-th component of a four-vector

$$J^\mu = (J^0, \mathbf{J}) = (\psi^\dagger\psi, \psi^\dagger \boldsymbol{\alpha}\psi) = i\overline{\psi}(x)\gamma^\mu\psi(x)$$

which is the current that appears in the continuity equation.

*Proof.* Infact

$$(J')^\mu(x') = i\overline{\psi}'(x')\gamma^\mu\psi'(x') = i\overline{\psi}(x) \underbrace{S^{-1}(\Lambda)\gamma^\mu S(\Lambda)}_{\Lambda^\mu{}_\nu\gamma^\nu} \psi(x) = \Lambda^\mu{}_\nu \underbrace{i\overline{\psi}(x)\gamma^\nu\psi(x)}_{J^\mu(x)} = \Lambda^\mu{}_\nu J^\mu(x) \ .$$

q.e.d.

A fermionic bilinear is a useful quantity to describe physical properties. We can costruct them starting with the basis

$$\Gamma^A = (\mathbb{I}, \gamma^5, \Sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5) \ , \tag{5.6}$$

such that

$$\overline{\psi}\Gamma^a\psi \ .$$

Notice that under Lorentz transformation, they all five transform in a good way: $\mathbb{I}$ and $\gamma^5$ are scalars, $\gamma^5$ and $\gamma^\mu\gamma^5$ are 4-vectors, $\Sigma^{\mu\nu}$ is a 2-tensor. However, under parity transformation, the last two of (5.6) are an axial-vector and pseudo-scalar.

*Proof.* Maybe in the future. q.e.d.

Recall from group theory, the Dirac spinor representation of the Lorentz group satisfies the Lie algebra

$$[\Sigma, \Sigma]$$

and the gamma matrices $\gamma^\mu$ are invariant tensors (Clebsh-Gordan coefficients).

*Proof.* Maybe in the future. q.e.d.

This means that the covariance of the Dirac equation can be understood in the usual way: contraction of tensor indices that makes the spinor $\xi(x) = (\gamma^\mu\partial_\mu + m)\psi(x)$ manifestly covariant and equals zero in any inertial frame.

# Chapter 6

# Wave plane solutions

In order to find a solution of the Dirac equation (3.5), we propose a plane wave ansatz

$$\psi_P(x) = w(p)\exp(ip_\mu x^\mu) \ ,$$

where $\exp(ip_\mu x^\mu)$ is the propagation in space-time, $p^\mu$ is arbitrary and $w(p)$ is the polarisation

$$w(p) = \begin{bmatrix} w_1(p) \\ w_2(p) \\ w_3(p) \\ w_4(p) \end{bmatrix} \ .$$

such that the polarisation satisfies the algebraic equation

$$(i\gamma^\mu p_\mu + m)w(p) = 0 \ , \tag{6.1}$$

and $p^\mu$ satisfies the mass-shell relation for a relativistic particle

$$p^\mu p_\mu + m^2 = 0 \ .$$

*Proof.* Infact, inserting the ansatz in (3.5)

$$0 = (\gamma^\mu\partial_\mu + m)\psi(x) = (\gamma^\mu\partial_\mu + m)w(p)\exp(ip_\mu x^\mu) = \underbrace{(i\gamma^\mu p_\mu + m)w(p)}_{0}\underbrace{\exp(ip_\mu x^\mu)}_{\neq 0} \ .$$

Hence

$$(i\gamma^\mu p_\mu + m)w(p) = (i\not{p} + m)w(p) = 0 \ .$$

Furthermore, we have

$$0 = (-i\not{p} + m)(i\not{p} + m)w(p) = \underbrace{(\not{p}^2 + m^2)}_{0}\underbrace{w(p)}_{\neq 0} \ ,$$

and we notice, using (3.6)

$$\not{p}^2 = \gamma^\mu p_\mu \gamma^\nu p_\nu = p_\mu p_\nu \underbrace{\gamma^\mu \gamma^\nu}_{\frac{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}{2}} = p_\mu p_\nu \frac{1}{2} \underbrace{(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)}_{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}} = p_\mu p_\nu \eta^{\mu\nu} = p_\mu p^\mu = p^2 \ ,$$

where we exploit the symmetry of $p^\mu p^\nu$ to symmetrise $\gamma^\mu \gamma^\nu$. Hence

$$p^2 + m^2 = 0 \ .$$

q.e.d.

## 6.1 Plane wave at rest

**Example 6.1** (Rest-frame)**.** Consider a particle at rest, which means with $p^\mu = (E, 0, 0, 0)$. Then, we substitute in (6.1)

$$0 = (i\not{p} + m)w(p) = (i\gamma^0 p_0)w(p) = (-\underbrace{i\gamma^0}_{\beta}\underbrace{p^0}_{E} + m)w(p) = (-\beta E + m)w(p) \ .$$

Hence

$$0 = \beta(-\beta E + m)w(P) = (-\beta^2 E + \beta m)w(p) = (-E + \beta m)w(P)$$

and

$$Ew(p) = \beta m w(p)$$

Recalling the matrix representation of $\beta$, we obtain

$$\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{bmatrix} \begin{bmatrix} w_1(p) \\ w_2(p) \\ w_3(p) \\ w_4(p) \end{bmatrix} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \end{bmatrix} \begin{bmatrix} w_1(p) \\ w_2(p) \\ w_3(p) \\ w_4(p) \end{bmatrix} \ .$$

This means that we have four different solutions: two are with positive energy $E = m$ which can be intepreted electrons with spin-up and spin-down

$$\psi_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \exp(-imt) \ , \quad \psi_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \exp(-imt) \ ,$$

and two with negative energy $E = -m$ which can be intepreted positrons with spin-up and spin-down

$$\psi_3(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \exp(imt) \ , \quad \psi_4(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \exp(imt) \ .$$

## 6.2 Moving plane wave

In order to find general solutions with arbitrary momentum, we apply a Lorentz transformation to the rest-frame solutions. A generic boost trasforms the rest-frame plane wave into

$$\psi_1(x) = \sqrt{\frac{m+E}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p_3}{m+E} \\ \frac{p_+}{m+E} \end{bmatrix} \exp(ip_\mu x^\mu t) \ , \quad \psi_2(x) = \sqrt{\frac{m+E}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p_-}{m+E} \\ -\frac{p_3}{m+E} \end{bmatrix} \exp(ip_\mu x^\mu t) \ ,$$
$$\psi_3(x) = \sqrt{\frac{m+E}{2m}} \begin{bmatrix} \frac{p_3}{m+E} \\ \frac{p_+}{m+E} \\ 1 \\ 0 \end{bmatrix} \exp(-ip_\mu x^\mu t) \ , \quad \psi_4(x) = \sqrt{\frac{m+E}{2m}} \begin{bmatrix} \frac{p_-}{m+E} \\ -\frac{p_3}{m+E} \\ 0 \\ 1 \end{bmatrix} \exp(-ip_\mu x^\mu t) \ ,$$

where $p_\pm = p_1 \pm ip_2$.

*Proof.* Firsly, we compute

$$\alpha^1 = \begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \ ,$$

$$\alpha^2 = \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \ ,$$

$$\alpha^3 = \begin{bmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \ .$$

Hence

$$\boldsymbol{\alpha}\cdot\mathbf{p} = \alpha^1 p_1 + \alpha^2 p_2 + \alpha^3 p_3 = \begin{bmatrix} 0 & 0 & p_3 & p_1 - ip_2 \\ 0 & 0 & p_1 + ip_2 & -p_3 \\ p_3 & p_1 - ip_2 & 0 & 0 \\ p_1 + ip_2 & -p_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix} \ .$$

Now, we compute $\psi_1$

$$
\begin{aligned}
S(\Lambda)\psi_1(x) &= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)\psi_1(x) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} + \frac{1}{m+E}\begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix}\Big)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\exp(ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \frac{1}{m+E}\begin{bmatrix}0\\0\\p_3\\p_+\end{bmatrix}\Big)\exp(ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\begin{bmatrix}1\\0\\\frac{p_3}{m+E}\\\frac{p_+}{m+E}\end{bmatrix}\exp(ip_\mu x^\mu t)\ ,
\end{aligned}
$$

we compute $\psi_2$

$$
\begin{aligned}
S(\Lambda)\psi_2(x) &= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)\psi_2(x) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I} + \frac{1}{m+E}\begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix}\Big)\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\exp(ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \frac{1}{m+E}\begin{bmatrix}0\\0\\p_-\\-p_3\end{bmatrix}\Big)\exp(ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\begin{bmatrix}0\\1\\\frac{p_-}{m+E}\\-\frac{p_3}{m+E}\end{bmatrix}\exp(ip_\mu x^\mu t)\ ,
\end{aligned}
$$

we compute $\psi_3$

$$
\begin{aligned}
S(\Lambda)\psi_3(x) &= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)\psi_3(x) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{1}{m+E}\begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix}\Big)\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\exp(-ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+\frac{1}{m+E}\begin{bmatrix}p_3\\p_+\\0\\0\end{bmatrix}\Big)\exp(-ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\begin{bmatrix}\frac{p_3}{m+E}\\ \frac{p_+}{m+E}\\ 1\\ 0\end{bmatrix}\exp(-ip_\mu x^\mu t)
\end{aligned}
$$

and we compute $\psi_4$

$$
\begin{aligned}
S(\Lambda)\psi_4(x) &= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m+E}\Big)\psi_4(x) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\mathbb{I}+\frac{1}{m+E}\begin{bmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{bmatrix}\Big)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\exp(-ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\Big(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}+\frac{1}{m+E}\begin{bmatrix}p_-\\-p_3\\0\\0\end{bmatrix}\Big)\exp(-ip_\mu x^\mu t) \\
&= \sqrt{\frac{m+E}{2m}}\begin{bmatrix}\frac{p_-}{m+E}\\ -\frac{p_3}{m+E}\\ 0\\ 1\end{bmatrix}\exp(-ip_\mu x^\mu t)\ .
\end{aligned}
$$

q.e.d.

# Chapter 7

# Discrete symmetries

In order to recover the whole Lorentz group $O(1,3)$, it is necessary to take into account also discrete transformations, like parity and time reversal, which allows us to go into the disconnected to the identity parts of the group.

## 7.1 Parity

The parity transformation is defined by the revesal of the orientation of the spatial axes

$$(t', \mathbf{x}') = (t, -\mathbf{x}) \ , \quad (x')^\mu = P^\mu{}_\nu x^\nu \ ,$$

where the parity matrix $P$ is

$$P^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \ .$$

It has determinant equals to $-1$, so it is not connected to the identity and it forms a subgroup with the identity, isomorphic to $\mathbb{Z}_2 = \{\mathbb{I}, P\}$.

What is the analoguous parity transformation for a Dirac spinor? We conjecture that exists a linear transformation

$$\psi'(x') = \mathcal{P}\psi(x) \ .$$

It turns out that such a matrix exists and it is

$$\mathcal{P} = \beta \ .$$

*Proof.* Infact

$$0 = (\gamma^\mu \underbrace{\partial'_\mu}_{P_\mu{}^\nu\partial_\nu} + m) \underbrace{\psi'(x')}_{\mathcal{P}\psi(x)} = (\gamma^\mu P_\mu{}^\nu \partial_\nu + m)\mathcal{P}\psi(x) \ .$$

Hence

$$0 = \mathcal{P}^{-1}(\gamma^\mu P_\mu{}^\nu \partial_\nu + m)\mathcal{P}\psi(x) = (\mathcal{P}^{-1}\gamma^\mu P_\mu{}^\nu \mathcal{P}\partial_\nu + m)\psi(x)$$

and to be Lorentz invariant, we have the condition

$$\mathcal{P}^{-1}\gamma^\mu P_\mu{}^\nu \mathcal{P} = \gamma^\nu$$

or equivalently

$$\mathcal{P}^{-1}\gamma^\mu \mathcal{P} = P^\mu{}_\nu \gamma^\nu \ .$$

Since

$$P^\mu{}_\nu \gamma^\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{bmatrix} = \begin{bmatrix} \gamma^0 \\ -\gamma^1 \\ -\gamma^2 \\ -\gamma^3 \end{bmatrix} \ ,$$

we need a matrix that commutes with $\gamma^0$ and anticommutes with $\gamma^i$, which is $\gamma^0$ itself and

$$\mathcal{P} = \eta\gamma^0 \ ,$$

where $\eta$ is an arbitrary phase factor, that we choose to be 1, because it is one of the four solutions of $\mathcal{P}^4 = 1$, which are $\eta = \{\pm 1, \pm i\}$. q.e.d.

Hence, we have under parity these transformations

$$\begin{cases} (x')^\mu = P^\mu{}_\nu x^\nu \\ \psi'(x') = \beta\psi(x) \\ \overline{\psi}'(x') = \overline{\psi}(x)\beta \end{cases} .$$

*Proof.* Infact

$$\overline{\psi}'(x') = (\psi')^\dagger(x')\beta = \beta\psi^\dagger(x)\beta = \psi^\dagger \underbrace{\beta^2}_{1} = \psi^\dagger = \overline{\psi}\beta \ .$$

q.e.d.

From these transformations, we can deduce which are the true scalar or vectors and which are only pseudo, because it appears a minus sign when transformed under parity

$$\begin{cases} \overline{\psi}'(x')\psi'(x') = \overline{\psi}(x)\psi(x) \\ \overline{\psi}'(x')\gamma^5\psi'(x') = -\overline{\psi}(x)\gamma^5\psi(x) \\ \overline{\psi}'(x')\gamma^\mu\psi'(x') = \overline{\psi}(x)P^\mu{}_\nu\gamma^\nu\psi(x) \\ \overline{\psi}'(x')\gamma^\mu\gamma^5\psi'(x') = -\overline{\psi}(x)P^\mu{}_\nu\gamma^\nu\gamma^5\psi(x) \\ \overline{\psi}'(x')\Sigma^{\mu\nu}\psi'(x') = \overline{\psi}(x)P^\mu{}_\alpha P^\nu{}_\beta\Sigma^{\alpha\beta}\psi(x) \end{cases} ,$$

which shows that $\gamma^5$ is indeed a pseudoscalar and $\gamma^\mu\gamma^5$ is indeed an axial vector.

*Proof.* For $\mathbb{I}$,

$$\overline{\psi}'(x')\psi'(x') = \overline{\psi}(x)\underbrace{\beta\beta}_{1}\psi(x) = \overline{\psi}(x)\psi(x) \ .$$

For $\gamma^5$,

$$\overline{\psi}'(x')\gamma^5\psi'(x') = \overline{\psi}(x)\beta\underbrace{\gamma^5\beta}_{-\beta\gamma^5}\psi(x) = -\overline{\psi}(x)\underbrace{\beta\beta}_{1}\gamma^5\psi(x) = -\overline{\psi}(x)\gamma^5\psi(x) \ .$$

For $\gamma^\mu$,

$$\overline{\psi}'(x')\gamma^\mu\psi'(x') = \overline{\psi}(x)\underbrace{\beta\gamma^\mu\beta}_{P^\mu{}_\nu\gamma^\nu}\psi(x) = \overline{\psi}(x)P^\mu{}_\nu\gamma^\nu\psi(x) \ .$$

For $\gamma^\mu\gamma^5$,

$$\overline{\psi}'(x')\gamma^5\psi'(x') = \overline{\psi}(x)\beta\gamma^\mu\underbrace{\gamma^5\beta}_{-\beta\gamma^5}\psi(x) = -\overline{\psi}(x)\underbrace{\beta\gamma^\mu\beta}_{P^\mu{}_\nu\gamma^\nu}\gamma^5\psi(x) = -\overline{\psi}(x)P^\mu{}_\nu\gamma^\nu\gamma^5\psi(x) \ .$$

For $\Sigma^{\mu\nu}$,

$$\begin{aligned}
\overline{\psi}'(x')\Sigma^{\mu\nu}\psi'(x') &= \overline{\psi}(x)\beta\underbrace{\Sigma^{\mu\nu}}_{-\frac{i}{4}[\gamma^\mu,\gamma^\nu]}\beta\psi(x) \\
&= -\frac{i}{4}\overline{\psi}(x)\beta[\gamma^\mu,\gamma^\nu]\beta\psi(x) \\
&= -\frac{i}{4}\overline{\psi}(x)\beta(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\beta\psi(x) \\
&= -\frac{i}{4}\overline{\psi}(x)\beta\gamma^\mu\underbrace{\mathbb{I}}_{\beta\beta}\gamma^\nu\psi(x) + \frac{i}{4}\overline{\psi}(x)\beta\gamma^\nu\underbrace{\mathbb{I}}_{\beta\beta}\gamma^\mu\beta\psi(x) \\
&= -\frac{i}{4}\overline{\psi}(x)\underbrace{\beta\gamma^\mu\beta}_{P^\mu{}_\alpha\gamma^\alpha}\underbrace{\beta\gamma^\nu\beta}_{P^\nu{}_\beta\gamma^\beta}\psi(x) + \frac{i}{4}\overline{\psi}(x)\underbrace{\beta\gamma^\nu\beta}_{P^\nu{}_\beta\gamma^\beta}\underbrace{\beta\gamma^\mu\beta}_{P^\mu{}_\alpha\gamma^\alpha}\psi(x) \\
&= -\frac{i}{4}\overline{\psi}(x)P^\mu{}_\alpha\gamma^\alpha P^\nu{}_\beta\gamma^\beta\psi(x) + \frac{i}{4}\overline{\psi}(x)P^\nu{}_\beta\gamma^\beta P^\mu{}_\alpha\gamma^\alpha\psi(x) \\
&= \overline{\psi}(x)P^\mu{}_\alpha P^\nu{}_\beta\Big(-\frac{i}{4}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\Big)\psi(x) \\
&= \overline{\psi}(x)P^\mu{}_\alpha P^\nu{}_\beta\underbrace{\Big(-\frac{i}{4}[\gamma^\alpha,\gamma^\beta]\Big)}_{\Sigma^{\alpha\beta}}\psi(x) \\
&= \overline{\psi}(x)P^\mu{}_\alpha P^\nu{}_\beta\Sigma^{\alpha\beta}\psi(x) \ .
\end{aligned}$$

q.e.d.

## Weyl spinors

The projectors (3.9) allow us to show that the Dirac representation is reducible into two spin $\frac{1}{2}$ representations of the Lorentz group. It follows from the fact that $\Sigma^{\mu\nu}$ commutes with $P_L$ and $P_R$

$$[\Sigma^{\mu\nu}, P_L] = 0 \ , \quad [\Sigma^{\mu\nu}, P_R] = 0 \ ,$$

which means that an infinitesimal Lorentz transformation acts on a Weyl spinor and it does not change its chirality.

*Proof.* Infact

$$\begin{aligned}
[P_L, \Sigma^{\mu\nu}] &= [\frac{\mathbb{I}-\gamma^5}{2}, -\frac{i}{4}[\gamma^\mu, \gamma^\nu]] \\
&= -\frac{i}{8}[\mathbb{I}-\gamma^5, \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] \\
&= -\frac{i}{8}\underbrace{[\mathbb{I}, \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]}_{0} + \frac{i}{8}[\gamma^5, \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] \\
&= \frac{i}{8}[\gamma^5, \gamma^\mu\gamma^\nu] - \frac{i}{8}[\gamma^5, \gamma^\nu\gamma^\mu] \\
&= \frac{i}{8}\gamma^\mu[\gamma^5, \gamma^\nu] + \frac{i}{8}[\gamma^5, \gamma^\mu]\gamma^\nu - \frac{i}{8}\gamma^\nu[\gamma^5, \gamma^\mu] - \frac{i}{8}[\gamma^5, \gamma^\nu]\gamma^\mu \\
&= \frac{i}{8}\gamma^\mu \underbrace{\{\gamma^5, \gamma^\nu\}}_{0} - 2\frac{i}{8}\gamma^\mu\gamma^\nu\gamma^5 + \frac{i}{8}\underbrace{\{\gamma^5, \gamma^\mu\}}_{0}\gamma^\nu - 2\frac{i}{8}\gamma^\mu \underbrace{\gamma^5\gamma^\nu}_{-\gamma^\nu\gamma^5} \\
&\qquad - \frac{i}{8}\gamma^\nu \underbrace{\{\gamma^5, \gamma^\mu\}}_{0} + 2\frac{i}{8}\gamma^\nu\gamma^\mu\gamma^5 - \frac{i}{8}\underbrace{\{\gamma^5, \gamma^\nu\}}_{0}\gamma^\mu + 2\frac{i}{8}\gamma^\nu \underbrace{\gamma^5\gamma^\mu}_{-\gamma^\mu\gamma^5} \\
&= \cancel{-2\frac{i}{8}\gamma^\mu\gamma^\nu\gamma^5} + \cancel{2\frac{i}{8}\gamma^\mu\gamma^\nu\gamma^5} + \cancel{2\frac{i}{8}\gamma^\nu\gamma^\mu\gamma^5} - \cancel{2\frac{i}{8}\gamma^\nu\gamma^\mu\gamma^5} \\
&= 0 \ ,
\end{aligned}$$

where we have used the identity

$$[A, B] = AB - BA = AB + BA - BA - BA = \{A, B\} - 2BA \ .$$

q.e.d.

Knowing that the Lie algebra of the Lorentz group can be written as $\mathfrak{so}(1,3) = \mathfrak{su}(2) + \mathfrak{su}(2)$, we can label each $SU(2)$ subalgebra with an half-integer $(j, j')$ associated to an irreducible representations. Dirac representation $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ is invariant under parity, while Weyl spinor one is not. Infact under parity, a Weyl spinor get exchanged into the other one

$$(\psi_L)' = (\psi')_R \ , (\psi_R)' = (\psi')_L \ .$$

*Proof.* Infact, Lorentz trasformation does not mix

$$(\psi_L)' = S(\Lambda)\psi_L = \exp(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})P_L\psi = P_L \exp(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\psi = P_L S(\Lambda)\psi = (\psi')_L \ ,$$

while parity transformation does

$$(\psi_L)' = \beta\psi_L = \beta P_L\psi = \beta\frac{\mathbb{I}-\gamma^5}{2}\psi = \frac{\mathbb{I}+\gamma^5}{2}\beta\psi_L = P_R\beta\psi = (\psi')_R \ .$$

q.e.d.

## 7.2 Time reversal

The time reversal transformation is defined by the reversal of the orientation of the time axis

$$(t', \mathbf{x}') = (-t, \mathbf{x}) \ , \quad (x')^\mu = T^\mu{}_\nu x^\nu \ ,$$

where the time reversal matrix $T$ is

$$T^\mu{}_\nu = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ .$$

It has determinant equals to $-1$, so it is not connected to the identity and it forms a subgroup with the identity, isomorphic to $\mathbb{Z}_2 = \{\mathbb{I}, T\}$.

What is the analoguous parity transformation for a Dirac spinor? We conjecture that exists a linear transformation

$$\psi'(x') = \mathcal{T}\psi(x) \ .$$

It turns out that such a matrix does not exist. However, by analogy with the Schoredinger equation in which time reversal is the same as complex conjugate, we can defied as

$$\mathcal{T} = \gamma^1\gamma^3 \ ,$$

with the difference that it transforms its complex conjugate $\psi'(x') = \mathcal{T}\psi^*(x)$.

*Proof.* Infact

$$0 = (\gamma^\mu \underbrace{\partial'_\mu}_{T_\mu{}^\nu\partial_\nu} + m)\underbrace{\psi'(x')}_{\mathcal{T}\psi^*(x)} = (\gamma^\mu T_\mu{}^\nu\partial_\nu + m)\mathcal{T}\psi^*(x) \ .$$

Hence

$$0 = \mathcal{T}^{-1}(\gamma^\mu T_\mu{}^\nu\partial_\nu + m)\mathcal{T}\psi^*(x) = (\mathcal{T}^{-1}\gamma^\mu T_\mu{}^\nu\mathcal{T}\partial_\nu + m)\psi^*(x)$$

and to be Lorentz invariant, we have the condition

$$\mathcal{T}^{-1}\gamma^\mu T_\mu{}^\nu \mathcal{T} = (\gamma^*)^\nu$$

or equivalently

$$\mathcal{T}^{-1}\gamma^\mu \mathcal{T} = T^\mu{}_\nu (\gamma^*)^\nu \ .$$

Since

$$T^\mu{}_\nu (\gamma^*)^\mu = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\gamma^*)^0 \\ (\gamma^*)^1 \\ (\gamma^*)^2 \\ (\gamma^*)^3 \end{bmatrix} = \begin{bmatrix} -(\gamma^*)^0 \\ (\gamma^*)^1 \\ (\gamma^*)^2 \\ (\gamma^*)^3 \\ = \end{bmatrix} = \begin{bmatrix} \gamma^0 \\ -\gamma^1 \\ \gamma^2 \\ -\gamma^3 \end{bmatrix} ,$$

we need a matrix that commutes with $\gamma^0$ and $\gamma^2$ and anticommutes with $\gamma^1$ and $\gamma^3$, which is $\gamma^1\gamma^3$ and

$$\mathcal{T} = \eta\gamma^1\gamma^3 \ ,$$

where $\eta$ is an arbitrary phase factor, that we choose to be 1, because it is one of the four solutions of $\mathcal{T}^4 = 1$, which are $\eta = \{\pm 1, \pm i\}$. q.e.d.

## Hole theory

In order to solve the problem of negative energy solutions, we propose the hole theory. Consider the vacuum state, which is the minimum energy level, defined as configuration in which all negeative energy are occupied by electrons, called the Dirac sea. Pauli's exclusion principle guarantees the stability in which all holes all filled. By definition, the vacuum energy $E_{vac} = 0$ and charge $Q_{vac} = 0$ are zero. An electron state means that a positive energy level is occupied with energy energy $E_{e^-} = E_p > 0$ and charge $Q_{e^-} = e$. Pauli's exclusion principle ensures that it cannot jump in the Dirac sea. If there is a hole, we can interpet it as an antiparticle. Infact, it is equivalent to a configuration with energy $E_{hol} = E_p > 0$ and charge $Q_{hol} = -e$.

*Proof.* Infact

$$E_{hol} + (-E_p) = E_{vac} = 0$$

and

$$Q_{hol} + e = Q_{vac} = 0 \ .$$

q.e.d.

It predicts both the positron and the pair creation, i.e. if a photon interacts with the vacuum, it creates an electron and a positron because it tranfers energy to a state of the Dirac sea and it occupates a positive energy one.

## 7.3 Charge conjugation

The couple with electromagnetism is made through minimal substitution (4.2). For a particle of charge $e$ and mass $m$, it can be written as

$$(\gamma^\mu(\partial_\mu - ieA_\mu) + m)\psi = 0 \ ,$$

while for an antiparticle of charge $-e$ and mass $m$ is

$$(\gamma^\mu(\partial_\mu + ieA_\mu) + m)\psi_c = 0 \ .$$

It is not a symmetry, but they are related by a transformation

$$\mathcal{A}(\gamma^*)^\mu\mathcal{A} = \gamma^\mu \ ,$$

where $A$ can be identify as

$$\mathcal{A} = \mathcal{C}\beta$$

and $\mathcal{C}$ is the charge conjugation matrix, defined as

$$\mathcal{C} = \gamma^0\gamma^2 \ .$$

It is a background symmetry. However, in QED for dynamical $A_\mu$, we have a true symmetry. Notice that $\mathcal{C}$ is antysymmetric and it coincides with its inverse.

*Proof.* Infact, from

$$0 = (\gamma^\mu(\partial_\mu - ieA_\mu) + m)\psi(x)$$

we have

$$0 = ((\gamma^*)^\mu(\partial_\mu + ieA_\mu) + m)\psi^*(x) \ ,$$

and comparing with

$$(\gamma^\mu(\partial_\mu + ieA_\mu) + m)\psi_c = 0 \ ,$$

to be Lorentz invariant, we have the condition

$$\mathcal{A}(\gamma^*)^\mu\mathcal{A} = \gamma^\mu \ ,$$

such that

$$\psi_c = \mathcal{A}\psi^* \ .$$

We choose in the form

$$=\mathcal{C}\beta$$

and in terms of the Dirac conjugate

$$\psi_c = \mathcal{A}\psi^* = \mathcal{C}\beta\psi^* = \mathcal{C}\overline{\psi}^T \ .$$

Hence, using

$$(\gamma^*)^\mu = (-\beta\gamma^\mu\beta)^T = -\beta(\gamma^T)^\mu\beta \ ,$$

we have

$$\gamma^\mu = \mathcal{A}(\gamma^*)^\mu\mathcal{A} = \mathcal{C}\beta(\gamma^*)^\mu\mathcal{C}\beta = -\mathcal{C}\underbrace{\beta\beta}_{1}(\gamma^T)^\mu\beta\mathcal{C}\beta = \ ,$$

or equivalently

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^T)^\mu = \begin{bmatrix} -\gamma^0 \\ \gamma^1 \\ -\gamma^2 \\ \gamma^3 \end{bmatrix} \ ,$$

we need a matrix that commutes with $\gamma^1$ and $\gamma^2$ and anticommutes with $\gamma^0$ and $\gamma^2$, which is $\gamma^0\gamma^2$ and

$$\mathcal{C} = \eta\gamma^0\gamma^2 \ ,$$

where $\eta$ is an arbitrary phase factor, that we choose to be 1, because it is one of the four solutions of $\mathcal{C}^4 = 1$, which are $\eta = \{\pm 1, \pm i\}$. q.e.d.

## Weyl spinors

For Weyl spinors, we have

$$(\psi_L)_c = P_R C(\psi_L)_c \ .$$

*Proof.* Infact

$$\psi_{L,c} = \mathcal{C}(\overline{\psi}_L)^T = (\overline{\psi}_L P_R)^T = \mathcal{C}P_R(\overline{\psi}_L)^T = P_R\mathcal{C}(\overline{\psi}_L)^T = P_R\psi_{L,c} \ .$$

q.e.d.

## CPT

Even though singular discrete symmetries are broken by interaction, the union of the three is always valid for Lorentz invariance theories. It acts as

$$\begin{cases} (x')^\mu = -x^\mu \\ \psi'(x') = \gamma^5\psi(x) \end{cases} \ ,$$

which is a symmetry for the Dirac equation.

*Proof.* Infact

$$(\gamma^\mu\partial'_\mu + m)\psi'(x') = (-\gamma^\mu\partial_\mu + m)\gamma^5\psi(x) = \cancel{\gamma^5}(\gamma^\mu\partial_\mu + m)\psi(x) = 0 \ .$$

q.e.d.

# Chapter 8

# Dirac action

The Dirac action is

$$S[\psi,\overline{\psi}] = \int d^4x \ (-\overline{\psi}(\gamma^\mu \partial_\mu + m)\psi) \ .$$

It is Lorentz invariant, since there is a summation over $\mu$ and a scalar $\overline{\psi}\psi$. The Euler-Lagrange equations are

$$(\gamma^\mu \partial_\mu + m)\psi(x) = 0 \ , \quad \overline{\psi}(x)(\gamma^\mu \overleftarrow{\partial_\mu} - m) = 0 \ .$$

*Proof.* In fact

$$\begin{aligned}\delta S &= \int d^4x \ ( \underbrace{-\overline{\psi}\gamma^\mu \partial_\mu \delta\psi}_{\overline{\psi}\gamma^\mu \overleftarrow{\partial_\mu}\delta\psi - \text{boundary term}} -\overline{\psi}m\delta\psi - \delta\overline{\psi}\gamma^\mu\partial_\mu\psi - \delta\overline{\psi}m\psi) \\ &= \int d^4x \ (\overline{\psi}\gamma^\mu \overleftarrow{\partial_\mu}\delta\psi - \overline{\psi}m\delta\psi - \delta\overline{\psi}\gamma^\mu\partial_\mu\psi - \delta\overline{\psi}m\psi) \ ,\end{aligned}$$

hence

$$\frac{\delta S}{\delta\psi} = \overline{\psi}\gamma^\mu \overleftarrow{\partial_\mu} - \overline{\psi}m = 0$$

and

$$\frac{\delta S}{\delta\overline{\psi}} = -\psi\gamma^\mu\partial_\mu\psi - m\psi = 0 \ .$$

q.e.d.

## 8.1 Noether's theorem

Consider an internal symmetry of a complex rotation $U(1)$, which is

$$\psi'(x) = \exp(i\alpha)\psi(x) \ , \quad \overline{\psi}'(x) = \exp(-i\alpha)\overline{\psi}(x) \ .$$

The Dirac action is indeed invariant.

*Proof.* In fact,

$$\mathcal{L}' = -\overline{\psi}'(\gamma^\mu\partial_\mu + m)\psi' = -\overline{\psi}\cancel{\exp(-i\alpha)}(\gamma^\mu\partial_\mu + m)\psi\cancel{\exp(i\alpha)} = -\overline{\psi}(\gamma^\mu\partial_\mu + m)\psi = \mathcal{L} \ .$$

q.e.d.

The infinitesimal transformation is

$$\delta\psi(x) = i\alpha\psi(x) \ , \quad \psi\overline{\psi}(x) = -i\alpha\overline{\psi}(x)$$

and the Noether's current is

$$J^\mu = i\overline{\psi}\gamma^\mu\psi \ .$$

*Proof.* In fact,

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi + \cancel{\frac{\partial\mathcal{L}}{\partial\partial_\mu\overline{\psi}}}\delta\overline{\psi} = -\overline{\psi}\gamma^\mu i\alpha\psi = -\alpha(i\overline{\psi}\gamma^\mu\psi) \ ,$$

which is defined up to a constant $-\alpha$. q.e.d.

It can be intepreted as a probability density, since its charge is positive definite.

*Proof.* In fact,

$$J^0 = i\overline{\psi}\gamma^0\psi = i\psi^\dagger\beta\gamma^0\psi = i\psi^\dagger i\gamma^0\gamma^0\psi = \psi^\dagger\psi \geq 0 \ .$$

q.e.d.

It can be generalised to $n$ Dirac fermions with the same mass $m$, since it is invariant under the symmetry $U(N) = U(1) \times SU(N)$. In fact, given the fundamental representation ${U^i}_j \in U(N)$

$$Nrep : (\psi')^= {U^i}_j\psi^j \ , \quad \overline{N}rep : (\overline{\psi}')^= \overline{\psi}^j {(U^{-1})^i}_j \ .$$

Writing the generators $U = +i\alpha_a T^a$, we have the infinitesimal transformation $\delta\psi^i = i\alpha_a {(T^a)^i}_j\psi^j$ and, by the Noether's theorem, the Noether current is

$$J^{\mu,a} = i\overline{\psi}_i\gamma^\mu {(T^a)^i}_j\psi^j \ ,$$

where $a = 1, \ldots N^2$.

## 8.2 Action for Weyl spinors

The action in terms of the irreducible chiral components is

$$S[\psi_L, \overline{\psi}_L, \psi_R, \overline{\psi}_R] = \int d^4x(-\psi_L \partial\!\!\!/ \psi_L - \overline{\psi}_R \partial\!\!\!/ \psi_R - m(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L)) \ .$$

The Dirac mass is forbidden by chiral fermions, because setting $\psi_R$ we have

$$S = \int d^4x(-\overline{\psi}_L \gamma^\mu \partial_\mu \psi_L)$$

which does not present mass term. However, there exists another Lorentz invariant mass term: the Maiorana mass, but it breaks the $U(1)$ symmetry, because there is not a term with $\overline{\psi}\psi$ to cancel the phase factor. It is

$$\mathcal{L}_M = \frac{M}{2} \psi^T \mathcal{C}^{-1} \psi + h.c. \ .$$

Lorentz invariance is verified by

$$\delta\psi = \frac{i}{2} \omega_{\mu\nu} \Sigma^{\mu\nu} \psi$$

and

$$\delta\psi^T = \frac{i}{2} \omega_{\mu\nu} \psi^T (\Sigma^{\mu\nu})^T = -\frac{i}{2} \omega_{\mu\nu} \psi^T \mathcal{C}^{-1} \Sigma^{\mu\nu} \mathcal{C} \ ,$$

where we have used $(\gamma^\mu)^T = -\mathcal{C}^{-1}\gamma^\mu \mathcal{C}$ so that $(\Sigma^{\mu\nu})^T = -\mathcal{C}^{-1}\Sigma^{\mu\nu}\mathcal{C}$. Hence $\delta\mathcal{L}_M = 0$.

The left-handed neutrino, the one which weakly interacts, has no Maiorana mass, but it is possible that the right-handed neutrino does have it.

## 8.3 Action for Maiorana spinors

The action with the Maiorana mass becomes

$$\mathcal{L} = -\overline{\psi}(\gamma^\mu \partial_\mu + m)\psi + \frac{M}{2}(\psi^T \mathcal{C}^{-1} \psi + h.c.) \ .$$

Notice that there are two masses: the Dirac mass term that preserves the $U(1)$ symmetry, whereas the Majorana mass term that break this symmetry.

This description is equivalent to a Majorana fermions $\mu_C(x) = \mu(x)$, which is $\mu(x) \sim \psi_L(x) + \psi_{L,c}(x)$. A majorana spinor is analoguous to a real scalar $\varphi^* = \varphi$, while Dirac spinor is analoguous to a complex scalar field, which is equivalent to two real scalar field of equal mass, hence two Majorana spinors of identical mass.

Let's an analogu with a scalar field. Consider two fields with the same mass

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{1}{2} m^2 (\varphi^1 + \varphi^2) \ ,$$

which is invariant under a rotation $SO(2)$ in the $\varphi_1, \varphi_2$ plane, only if $m_1 = m_2 = m$. With a change of variable $\phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ and $\phi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$, we obtain

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \ ,$$

which is invariant under a rotation $U(1)$. However, the addition of the Majorana mass term, it breaks the symmetry

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{M}{2}(\phi\phi + \phi^*\phi^*) \ ,$$

which has positive physical mass term, once we diagonalise,

$$\mu_1 = m^2 + M^2 \ , \quad \mu_2 = m^2 - M^2 \ .$$

## 8.4 Green function and propagator

The Green equation reads

$$(\partial\!\!\!/_x + m)G(x-y) = \delta^4(x-y) \ ,$$

which can be solved by a Fourier transform

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp(ip(x-y))\tilde{G}(p)$$

and

$$\tilde{G}(p) = \frac{1}{ip\!\!\!/ + m} = \frac{-ip\!\!\!/ + m}{p^2 + m^2} \ ,$$

where we can use the Feynman-Stueckelberg prescription. The propagator, which is a particular Green function, is

$$\langle 0|T\hat{\psi}(x)\hat{\bar{\psi}}(y)|0\rangle = \langle \psi(x)\overline{\psi}(y)\rangle = -iG(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp(ip(x-y))\frac{-ip\!\!\!/ + m}{p^2 + m^2} \ .$$

# Part III

Higher spin theory

# Chapter 9

# Pauli-Fierz equations

In this chapter, we will generalise what we have done so far for spin $s \geq 1$ particles. In particular, we will analyse the Pauli-Fierz equations in the case of massive spin 1 (Proca), massless spin 1 (Maxwell) and massless spin 2 (linearised Einstein).

## 9.1 Pauli-Fierz equations

For particles of spin higher than 1, it is straight forward to generalise the Klein-Gordon equation (1.5) for bosonic integer spin particles and the Dirac equation (3.5) for fermionic half-integer spin particles, in the case of interaction-free theories.

First, consider the bosonic integer spin $s$ case. The wavefunction is described by a totally symmetric tensor of rank $s$

$$\phi_{\mu_1 \dots \mu_i \mu_{i+1} \dots \mu_s} = \phi_{\mu_1 \dots \mu_{i+1} \mu_i \dots \mu_s} \ ,$$

such that it has number of independent components equals to

$$\# = \binom{n+s-1}{s} \ , \tag{9.1}$$

which in our case $n = 4$ and $s$ is the spin. However, we know that the number of possible polarisations is $2s+1$. This is due to the fact that spin can only have integer values between $-s$ and $s$ along the $z$-direction, i.e. $|s_z| \leq s$. For example, for $s = 1$, we have $s_z = -1, 0, 1$; for $s = 2$, we have $s_z = -2, -1, 0, 1, 2$, etc. Therefore, other than the equation of motion, we need to add constraints to reduce the degrees of freedom. The set composed by the equation of motion plus the constraints is called Pauli-Fierz equations:

$$\begin{cases} (\Box - m^2)\phi_{\mu_1 \dots \mu_s} = 0 \\ \partial^{\mu_1}\phi_{\mu_1 \mu_2 \dots \mu_s} = 0 \\ \eta^{\mu_1 \mu_2}\phi_{\mu_1 \mu_2 \mu_3 \dots \mu_s} = \phi^{\mu_1}{}_{\mu_1 \mu_3 \dots \mu_s} = 0 \end{cases} . \tag{9.2}$$

To understand them better, consider a wave plane solution ansatz

$$\phi_{\mu_1\dots\mu_s}(x) \propto \epsilon_{\mu_1\dots\mu_s}(p)\exp(ip_\mu x^\mu) \ ,$$

where $\epsilon_{\mu_1\dots\mu_s}(p)$ is the spin polarisation tensor, which describes the spin orientation. Imposing (9.2), we find that

1. $p$ satisfies the mass-shell condition $p^2+m^2=0$,
2. the only non-vanishing component of $\epsilon$ must be spacelike $\epsilon_{i_1\dots i_s}\neq 0$,
3. $\epsilon$ belong to the irreducible representations of $SO(3)$, which are symmetric and traceless tensor, with precisely $2s+1$ possible spin orientations.

*Proof.* For the first condition

$$0=(\Box-m^2)\epsilon_{\mu_1\dots\mu_s}(p)\exp(ipx)=\epsilon_{\mu_1\dots\mu_s}(p)(-p^2-m^2)\exp(ipx) \ ,$$

hence,

$$p^2+m^2=0 \ .$$

For the second condition, consider the rest frame $p^\mu=(m,0,0,0)$, possible because it is massive,

$$0=p^\mu\epsilon_{\mu\mu_1\dots\mu_s}(p)=m\epsilon_{0\mu_1\dots\mu_s}(p) \ ,$$

hence all the time-like components vanish, since it is symmetric. For the third condition, we have that the number of independent components of a symmetric tensor is (9.3) but if we impose this condition for which the trace is null, we have to subtract

$$\#_{tr}=\binom{n+s-3}{s-2} \ .$$

Hence, we find

$$\#-\#_{tr}=\binom{n+s-1}{s}-\binom{n+s-3}{s-2}=\frac{(n+s-1)!}{s!(n-1)!}-\frac{(n+s-3)!}{(s-2)!(n-1)!}$$

which for $n=3$

$$\begin{aligned}\#-\#_{tr}&=\binom{s+2}{s}-\binom{s}{s-2}=\frac{(s+2)!}{s!2!}-\frac{s!}{(s-2)!2!}\\&=\frac{(s+2)(s+1)s!}{s!2!}-\frac{s(s-1)(s-2)!}{(s-2)!2!}=\frac{(s+2)(s+1)}{2}-\frac{s(s-1)}{2}\\&=\frac{s^2+3s+2-s^2+s}{2}=2(s+1) \ .\end{aligned}$$

q.e.d.

Now, consider the fermionic half-integer spin $s$ case. The wavefunction is described by a totally antisymmetric tensor of rank $s$

$$\phi_{\mu_1\ldots\mu_i\mu_{i+1}\ldots\mu_s} = -\phi_{\mu_1\ldots\mu_{i+1}\mu_i\ldots\mu_s} \ ,$$

such that it has number of independent components equals to

$$\# = \binom{n+s-1}{s} \ , \tag{9.3}$$

The Fierz-Pauli equations become

$$\begin{cases} (\gamma^\mu\partial_\mu + m)\psi_{\mu_1\ldots\mu_s} = 0 \\ \partial_1^\mu\psi_{\mu_1\mu_2\ldots\mu_s} = 0 \\ \gamma^{\mu_1}\psi_{\mu_1\mu_2\ldots\mu_s} = 0 \end{cases} .$$

In the massless case, these equations are no longer valid, since we must add gauge symmetries in order to reduce the number of independent components to 2, which are the two possible helicities $h = \pm s$, called the Fronsdal equations.

## 9.2 Proca equation

The Proca equation describes a massive spin 1 particle. The field is encoded in a 4-vector $\psi_\mu = A_\mu$ and the equations of motion are (9.2)

$$\begin{cases} (\Box - m^2)A_\mu = 0 \\ \partial^\mu A_\mu = 0 \end{cases} , \tag{9.4}$$

where the last equation is trivial, since a vector has no trace. Notice that a 4-vector has 4 degrees of freedom, but the second equation reduce them to 3. They can be derived from the Proca action

$$S_p[A_\mu] = \int d^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2A_\mu A^\mu\Big) = \int d^4x\Big(-\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 - \frac{1}{2}m^2A_\mu A^\mu\Big) ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

*Proof.* In fact

$$\begin{aligned} S_p[A_\mu] &= \int d^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2A_\mu A^\mu\Big) \\ &= \int d^4x\Big(-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}m^2A_\mu A^\mu\Big) \\ &= \int d^4x\Big(-\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu - \frac{1}{2}m^2A_\mu A^\mu\Big) \\ &= \int d^4x\Big(-\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 - \frac{1}{2}m^2A_\mu A^\mu\Big) , \end{aligned}$$

where in the last passage, we integrated by parts. q.e.d.

Using the principle of stationary action, we obtain the equations of motion

$$\partial^\mu F_{\mu\nu} = m^2 A_\nu \ .$$

*Proof.* In fact

$$\begin{aligned}0 = \delta S &= \int d^4x\Big(-\frac{1}{2}F_{\mu\nu}\delta F^{\mu\nu} - m^2 A_\mu \delta A^\mu\Big)\\ &= \int d^4x(-F_{\mu\nu}\partial^\mu\delta A^\nu - m^2 A_\mu\delta A^\mu) = \int d^4x\ \delta A^\nu(\partial^\mu F_{\mu\nu} - m^2 A_\nu)\ ,\end{aligned}$$

hence, we find

$$\partial^\mu F_{\mu\nu} = m^2 A_\nu \ .$$

q.e.d.

Notice that the second condition of (9.4) is automatically satisfied.

*Proof.* In fact, for the continuity equation

$$\underbrace{\partial^\mu\partial^\nu}_{\text{symm}}\underbrace{F_{\mu\nu}}_{\text{anti}} = 0 \ ,$$

we find

$$0 = \partial^\mu\partial^\nu F_{\mu\nu} = m^2\partial^\nu A_\mu \ .$$

q.e.d.

The action and the equations of motion are invariant by a Lorentz transformation

$$(x')^\mu = \Lambda^\mu_\nu x^\nu \ , \quad (A')^\mu = \Lambda^\mu_\nu A^\nu \ .$$

Consider a plane wave solution

$$A_\mu(x) = \epsilon_\mu(p) = \exp(ipx) \ ,$$

which satisfies the conditions imposed by (9.4)

$$p^2 + m^2 = 0 \ , \quad p^\mu\epsilon_\mu(p) = 0 \ .$$

The second condition means that there are only 3 polarisations left. To see that, in the rest frame $p^\mu = (m, 0, 0, 0)$, we have $m\epsilon_0 = 0$. This means that polarisation is given by a vector in a 3-dimensional space, since the time component vanishes.

*Proof.* For the first condition

$$(-p^2 + m^2)\epsilon_\mu(p)\exp(ipx) = 0 \ ,$$

hence

$$p^2 + m^2 = 0 \ .$$

For the second condition

$$ip^\mu \epsilon_\mu(p)\exp(ipx) = 0 \ ,$$

hence

$$p^\mu \epsilon_\mu(p) = 0 \ .$$

q.e.d.

In the real case, particles and antiparticles coincide. However, if we consider a complex Proca field, the concept of electric charge arises and particles and antiparticles have opposite one under a $U(1)$ symmetry.

We can rewrite the Proca action in term of a quadratic differential operator $K^{\mu\nu}(\partial)$

$$S_P[A_\mu] = \int d^4x\Big(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 - \frac{1}{2}m^2 A_\mu A^\mu\Big) = \int d^4x\Big(-\frac{1}{2}A_\mu(x)K^{\mu\nu}(\partial)A_\nu(x)\Big)$$

where

$$K^{\mu\nu}(\partial) = (-\Box + m^2)\eta^{\mu\nu} + \partial^\mu\partial^\nu \ .$$

The equations of motion becomes

$$K^{\mu\nu}(\partial)A_\nu(x) = 0 \ .$$

The Green equation is

$$K^{\mu\nu}(\partial)G_{\nu\lambda}(x-y) = \delta^\mu{}_\lambda \delta^4(x-y) \ ,$$

which in momentum space via a Fourier transform is

$$G_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\exp(ip(x-y))\tilde{G}_{\mu\nu}(p) \ ,$$

where

$$\frac{\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}}{p^2 + m^2} \ .$$

Physically, the Proca propagator can be used to describe $W^\pm$ bosons. Notice that the massless case is singular.

## 9.3 Mawxell equation

The free Maxwell action is

$$S_M[A_\mu] = \int d^4x\Big( -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Big) \ .$$

Notice that there is no dependence on the field $A_\mu$ but the only dependence is of $\partial_\mu A_\nu$ via $F^{\mu\nu}$. This means that a gauge symmetry appears. In fact, putting

$$\delta A_\mu(x) = \partial_\mu \alpha(x) \ ,$$

we have

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = \partial_\mu \partial_\nu \alpha(x) - \partial_\nu \partial_\mu \alpha(x) = 0 \ .$$

This means that $F^{\mu\nu}$ is gauge invariant, which means that the parameters of the symmetry are actually arbitrary functions $\alpha(x)$. The gauge symmetry can be written in another way

$$\delta A_\mu(x) = -i \exp(-i\alpha(x))\partial_\mu \exp(i\alpha(x))$$

which highlights that $\exp(i\alpha(x)) \in U(1)$, where $U(1)$ is the gauge group, i.e. for all $x$ there is a $U(1)$ group.

The equations of motion (Maxwell's equations) are

$$\partial^\mu F_{\mu\nu} = 0 \ .$$

Given the gauge invariance, we are allowed to choose a representative of the equivalence class of fields, called gauge fixing. We choose the Lorentz gauge

$$\partial_\mu A^\mu = 0 \ .$$

In this way, the equations of motion becomes

$$0 = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \Box A_\mu - \partial_\nu \partial^\mu A_\mu = \Box A_\mu \ .$$

and the gauge function satisfies

$$\Box \alpha = -\partial^\mu A_\mu \ .$$

*Proof.* In fact,

$$\partial^\mu A'_\mu = \partial^\mu (A_\mu + \partial_\mu \alpha(x)) = 0 \ .$$

q.e.d.

However, there is still a residual gauge trasformation, since this gauge is always valid for gauge functions satisfying

$$\Box\alpha(x) = 0 \ .$$

*Proof.* In fact,

$$\partial^\mu A'_\mu = \partial^\mu A_\mu + \partial^\mu \partial_\mu \alpha(x) \ .$$

q.e.d.

Consider a plane wave ansatz

$$A_\mu(x) = \epsilon(p) \exp(ipx)$$

such that

$$p^2 = 0 \ , \quad p^\mu \epsilon_\mu(p) = 0 \ .$$

However, there is still a non-physical degree of freedom, due to the fact that there are 3 polarisations left but only 2 are physical. To see this, consider the longitudinal polarisation $\epsilon_\mu(p) = p_\mu$, such that $\alpha(x) = -i \exp(px)$ for $p^2 = 0$. This means that this field is equivalent to the vacuum, since

$$A'_\mu = 0 + \partial_\mu \alpha = p_\mu \exp(ipx)$$

and

$$A'_\mu = A_\mu + \partial_\mu \beta = p_\mu \exp(ipx) - p_\mu \exp(ipx) = 0 \ ,$$

where we have used $\Box\beta = 0$ and $\beta(x) = -\alpha(x)$. Hence the longitudinal polarisation is unphysical and there are only two physical polarisation.

## 9.4 Linearised Einstein equation

For a massive spin 2 particle, the Fierz-Pauli equations are

$$\begin{cases} (\Box - m^2)\phi_{\mu\nu} = 0 \\ \partial^\mu \phi_{\mu\nu} = 0 \\ \phi^\mu{}_\nu = 0 \end{cases} ,$$

where the number of indepdendent components are 5, by 10 symmetrical minus the 5 constrains.

For a massless spin 2 particle, like in the linearised Einstein's equation in which the field is the metric

$$\phi_{\mu\nu} = h_{\mu\nu} \ , \quad \eta_{\mu\nu} + h_{\mu\nu} \ .$$

We want a Maxwell-like equation of motion $\Box A_\mu = 0$, to do so we find

$$\Box h_{\mu\nu} - \partial_\mu \partial^\sigma g_{\sigma\mu} + \partial_\mu \partial_\nu h = 0 \ ,$$

where the second term is due to Maxwell's $\partial_\nu \partial?\mu A_\mu$, the third one is for symmetrise the tensor and the last one for make it traceless.

Notice that it is present a gauge symmetry with 4 parameters

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu(x) - \partial_\nu \xi_\mu(x) \ ,$$

such that

$$\Box \delta h_{\mu\nu} - \partial_\mu \partial^\sigma \delta h_{\sigma\mu} + \partial_\mu \partial_\nu \delta h = 0 \ .$$

An useful gauge fixing is the De Donder gauge

$$\partial^\mu h_{\mu\nu} = \frac{1}{2} \partial_\nu h \ ,$$

for which we have

$$\Box h_{\mu\nu} = 0 \ .$$

Observe that there are 2 polarisations, since we have 10 degrees of freedom for the symmetrical property minus 4 for De Donder and minus 4 for the residual gauge $\Box \xi = 0$.

# Bibliography


[1] F. Bastianelli. *Lecture notes taken during the relativistic quantum mechanics and path integrals course.*