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On differential geometry:

manifolds and all that

October 26, 2023

Theoretical Physics

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Manifolds

Chapter 1

Manifolds

In physics, we often find problems which involve quantities (and operations on them) defined in a continuous space. Moreover, this space is not always flat, but it could be curved, like in general relativity. In this chapter, we will study how we can mathematically define what a space is with the notion of differential manifold.

1.1 Differentiable Manifolds

A differential manifold $\mathcal{M}$ is a topological space which locally looks like $\mathbb{R}^n$. This means that if you zoom in to any patch, you find $\mathbb{R}^n$.

Definition 1.1 (Topological space)
A topological space $(\mathcal{M}, \{A_i\})$ is a set of points $\mathcal{M}$ in which is defined a tolopology, i.e. a family of open sets $\{A_i\}$ such that

1. *the empty set and the whole set are open sets, i.e.*

$$\mathcal{M}, \emptyset \in \{A_i\} \ ,$$

2. *the intersection of a finite number of open sets is an open sets, i.e.*

$$\bigcap_{i<\infty} U_i \in \{A_i\} \ ,$$

3. *the union of an arbitrary number of open sets is an open sets, i.e.*

$$\bigcap_{i} U_i \in \{A_i\} \ .$$

Definition 1.2 (Hausdorff space)
An Hausdorff space is a topological space which has the additional property

$$\forall P, Q \in \mathcal{M} \quad \exists U \ni P, V \ni Q \quad : \quad U \cap V = \emptyset \ .$$

In a topological space, the notions of contiguity and continuity are well defined. Two points are contiguous if they belong to the same open set, called neighbourhood. Intuitively, it means that one is next to the other. A map is an application $\phi\colon D \subset \mathcal{M} \to \mathbb{R}^n$. It is continuous if it maps open sets into open sets.

Definition 1.3 (Chart)
A chart is a pair $(A,\ \phi)$, where A is an open set of a point $P \in \mathcal{M}$, i.e. $A \subseteq \mathcal{M}$, and ϕ is an invertible and continuous map $\phi\colon A \to \mathbb{R}^n$ which associates a set of n real coordinates $x^i = \phi$ for the neighbourhood U of P. See Figure 1.1.

Definition 1.4 (Atlas)
An atlas $\mathcal{A}$ is a collection of charts that covers entirely the manifold, i.e.

$$\mathcal{A} = \{\{(A_i, \phi_i)\}\colon\ \cup_i A_i \supseteq \mathcal{M}\}\ .$$

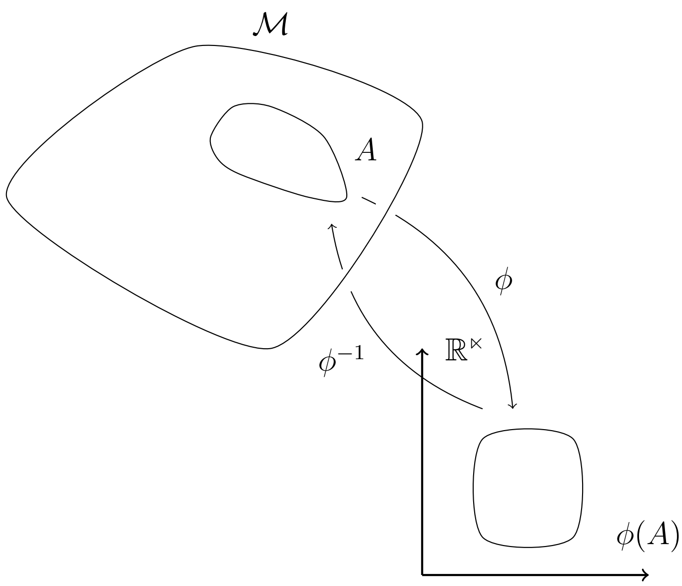

Figure 1.1: A coordinate chart A of a neighbourhood U of a manifold $\mathcal{M}$

In order to include all the points of the manifold $\mathcal{M}$, neighbourhoods must overlap each others. In the intersection of them, which is open, there are two different coordinate systems that describe it and there must be a relation between them.

Definition 1.5 (Consistency map)
A consistency map between two charts ϕ_1 and ϕ_2, over a point $P \in A_1 \cap A_2$, is an invertible map $\phi\colon \phi(A_2) \subseteq \mathbb{R}^n \to \psi(A_2) \subseteq \mathbb{R}^n$ such that

$$\psi(\phi_1(P)) = \phi_2(P) \quad \text{or} \quad (\phi_2^{-1} \circ \psi \circ \phi_1) = \mathbb{I}$$

or, equivalently,

$$\psi^{-1}(\phi_2(P)) = \phi_1(P) \quad \text{or} \quad (\phi_1^{-1} \circ \psi \circ \phi_2) = \mathbb{I}$$

See Figure 1.2.

The consistency map ψ can be seen as a change of coordinates in $\mathbb{R}^n$. Since it is invertible, it follows that the dimension n must be the same for all charts and can be defined as the dimension of the manifold. If $\psi \in C^p(\mathbb{R}^n)$, the manifold is a p-differentiable manifold.

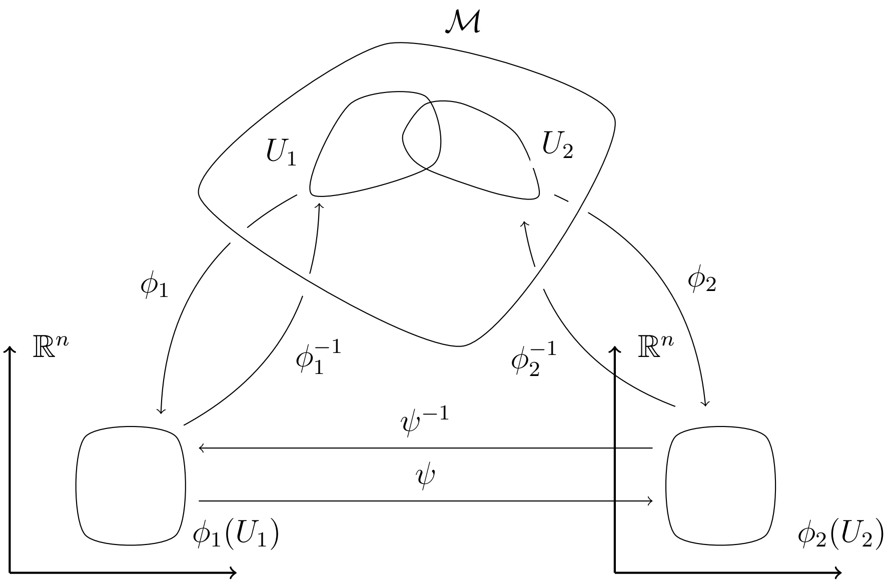

Figure 1.2: A consistency map ψ between two charts ϕ_1 and ϕ_2

Definition 1.6 (Manifold)
A manifold is an equivalence class of atlases, where two atlases are equivalent if there exists a bijective correspondence between them.

Examples are the n-dimensional Euclidean space $\mathbb{R}^n$, the n-dimensional sphere S^n, the n-dimensional torus T^n, the lagrangian configuration space or the hamiltonian phase space.

Definition 1.7 (Homeomorphism)
An homeomorphism between two manifold $\mathcal{M}$ and $\mathcal{N}$ is an injective, surjective and continuous with also inverse continuous map between them. Two manifolds are homeomorphic if it exists such map.

Definition 1.8 (Diffeomorphism)
An diffeomorphism between two manifold $\mathcal{M}$ and $\mathcal{N}$ is a smooth (differentiable) homeomorphism. Two manifolds are diffeomorphic if it exists such map.

Since locally all manifolds of the same dimension n look like $\mathbb{R}^n$, they can be divide up into classes given by their global properties: two manifolds which are diffeomorphic (hence they have the same dimension) can be deformed one into each other.

1.2 Curves

A curve is a continuous map $\gamma\colon I \subseteq \mathbb{R} \to \mathcal{M}$. Introducing a chart $\phi \circ \gamma\colon I \subseteq \mathbb{R} \to \mathbb{R}^n$, or $x^i = x^i(\lambda)$, where λ is a real parameter. If $x^i(\lambda) \in C^p(\mathbb{R})$, then *gamma* is p-differentiable. A reparameterization $\gamma' = \gamma'(\gamma)$ defines a different curve, although the images of the curves coincide.

1.3 Scalars

A function is a map $f\colon \mathcal{M} \to \mathbb{R}$. Introducting a chart $f \circ \phi^{-1}\colon \mathbb{R}^n \to \mathbb{R}$, or $f = f(x^i)$. If ϕ' is another chart, then $f'(x'(P)) = f(x(P))$, showing that it is indeed a scalar.

1.4 Vectors

A vector at a point $P \in \mathcal{M}$ is a map that associates to the derivative to a function defined in a neighbourhood of P $v_\gamma\colon f \to v_\gamma(f) = \frac{df}{d\lambda}\Big|_{\lambda_P} \in \mathbb{R}$, where $\gamma(\lambda_P) = P$. Introducing a chart

$$\begin{aligned}
v_{\gamma,\ P}(f) &= \frac{d(f \circ \gamma)}{d\lambda}\Big|_{\lambda_P} \\
&= \frac{d}{d\lambda}(f \circ \mathbb{I} \circ \gamma)\Big|_{lambda_P} \\
&= \frac{d}{d\lambda}(f \circ \phi^{-1} \circ \phi \circ \gamma)\Big|_{lambda_P} \\
&= \frac{d}{d\lambda}(f(x^i) \circ x^i(\lambda))\Big|_{\lambda_P} \\
&= \frac{d}{d\lambda} f(x^i(\lambda))\Big|_{\lambda_P} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda}
\end{aligned}$$

and since it is true $\forall f$

$$v_\gamma = dv\lambda = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} \tag{1.1}$$

which means that a vector is the tangent to a curve γ at a point P.

By definition a vector is a linear functional

$$v_\gamma(af+bg) = \frac{d}{d\lambda}(af+bg) = a\frac{df}{d\lambda} + b\frac{dg}{d\lambda}$$

From (1.1)

$$v = \underbrace{\frac{dx^i}{d\lambda}}_{v_i}\underbrace{\frac{\partial}{\partial x^i}}_{e_i} = v^i e_i$$

where v^i are the components and e_i are the coordinate basis vectors, whose vectors tangent to the coordinate line defined by constant x^j for $i \neq j$.

Under a change of coordinates

$$y^i = y^i(x^j)$$

components transform by

$$v^i = \frac{dx^i}{d\lambda} = \frac{\partial x^i}{\partial y^{j'}}\frac{dy^{j'}}{d\lambda} = \frac{\partial x^i}{\partial y^{j'}}v^{j'}$$

and basis vectors transform by

$$e_i = \frac{\partial}{\partial x^i} = \frac{\partial y^{j'}}{\partial x^i}\frac{\partial}{\partial y^{j'}} = \frac{\partial y^{j'}}{\partial x^i}e_{j'}$$

Remember that components transform but basis change too, inversely, then the vector remains the same.

A vector field in an open set $U \subseteq \mathcal{M}$ is map from each point $P \in U$ into a vector $v(P)$. Introducing a chart, $v(x^i) = v \circ \phi^{-1}$.

The coordinate vectors $e_i = \frac{\partial}{\partial x^i}$ form a basis of a linear space composed by all the vectors tangent to a point P, called the tangent space T_P.

Proof. First, every tangent vector can be expressed as linear combination.

Consider two curves, parametrized by λ and σ, across a point P which generate two vectors $v = \frac{\partial}{\partial\lambda}$ and $w = \frac{d}{d\sigma}$. Hence, a generic linear combination of them

$$av+bw = a\frac{d}{d\lambda}+b\frac{d}{d\sigma} = a\frac{\partial x^i}{\partial\lambda}\frac{\partial}{\partial x^i}+b\frac{dx^i}{d\sigma}\frac{\partial}{\partial x^i} = \left(a\frac{\partial x^i}{\partial\lambda}+b\frac{dx^i}{d\sigma}\right)\frac{\partial}{\partial x^i} = \left(a\frac{\partial x^i}{\partial\lambda}+b\frac{dx^i}{d\sigma}\right)e_i$$

Since there are n coordinates x^i, we have n indipendent curves.

Second, the coordinate basis vectors are linearly independent.

The determinant of the jacobian matrix of $y^i = y^i(x^j)$ must not vanish

$$\det J = \det\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

hence, there are n columns (or rows) which are linearly independent and also n basis vector

$$e_i = \frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

q.e.d.

It is important to remark that coordinate basis vectors at different points belong to different tangent space and cannot be linearly combined.

1.5 Fiber bundles

A tangent bundle is the set of all tangent space at each point together with the manifold itself $\mathcal{TM} = \{\mathcal{M},\ \{T_P \colon \forall P \in \mathcal{M}\}\}$. It can be shown that $\mathcal{TM}$ is a manifold too.

1.6 Exponential map

An integral curve $\gamma = \gamma(\lambda)$ of a vector field V is the curve which as tangent vector $\frac{d}{d\lambda}$ has the element of V in $P \in \gamma$, i.e.

$$V = \frac{d}{d\lambda}$$

Introducing a point P_0 and a chart x^i

$$\begin{aligned} V^i(\lambda) &= \frac{dx^i(\lambda)}{d\lambda} \\ x^i(P_0) &= x^i(\lambda_0) \end{aligned} \tag{1.2}$$

which are a system of n Cauchy problems and the components of V at an arbitrary point $P = \phi^{-1}(x^i(\lambda))$ are $V(P) = V^i(x^j(\lambda))\frac{\partial}{\partial x^i} = V^i(\lambda)\frac{\partial}{\partial x^i}$.

Theorems of calculus in $\mathbb{R}^n$ ensure that locally tha solution of (1.2) always exists, which is indeed the integral curve $\gamma(\lambda)$.

Formally, the solution of (1.2) is the exponential map

$$x^i(\lambda) = \exp((\lambda - \lambda_0)V)x^i\Big|_{\lambda_0}$$

which describes the flow of V in a neighbourhood of P.

Proof. Let $V = \frac{d}{d\lambda}$ be a vector fields with integral curve $\gamma = \gamma(\lambda)$. Introducing a chart x^i and Taylor expanding around P_0 along γ

$$\begin{aligned} x^i(\lambda_0 + \epsilon) &= x^i(\lambda_0) + \epsilon \frac{dx^i}{d\lambda}\Big|_{\lambda_0} + \frac{\epsilon^2}{2}\frac{d^2x^i}{d\lambda^2}\Big|_{\lambda_0} + \dots \\ &= \left(1 + \epsilon \frac{d}{d\lambda}\Big|_{\lambda_0} + \frac{\epsilon^2}{2}\frac{d^2}{d\lambda^2}\Big|_{\lambda_0} + \dots\right) x^i(\lambda_0) \\ &= \exp(\epsilon \frac{d}{d\lambda})\ x^i\Big|_{\lambda_0} \\ &= \exp(\epsilon V)\ x^i\Big|_{\lambda_0} \end{aligned}$$

q.e.d.

For an arbitrary function f in a neighbourhood of P

$$f(\lambda_0 + \epsilon) = \exp\left(\epsilon \frac{d}{d\lambda}\right) f\Big|_{\lambda_0} = \exp(\epsilon V)\ f\Big|_{\lambda_0}$$

1.7 Lie brackets

Introducing a chart x^i, the Lie brackets of two vector fields $V = \frac{d}{d\lambda} = v^i \frac{\partial}{\partial x^i}$ and $W = \frac{d}{d\mu} = w^i \frac{\partial}{\partial x^i}$ are

$$\begin{aligned} [V,\ W] &= \frac{d}{d\lambda}\frac{d}{d\mu} - \frac{d}{d\mu}\frac{d}{d\lambda} \\ &= v^i \frac{\partial}{\partial x^i}\left(w^j \frac{\partial}{\partial x^j}\right) - w^i \frac{\partial}{\partial x^i}\left(v^j \frac{\partial}{\partial x^j}\right) \\ &= \cancel{v^i w^j \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}} + v^i \frac{\partial w^j}{\partial x^i}\frac{\partial}{\partial x^j} - \cancel{v^j w^i \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}} - w^i \frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial x^j} \\ &= \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} \end{aligned}$$

where it is used the facf that the partial derivatives commute. Hence the commutator of two vectors is still a vector.

The geometrical meaning of the commutator is the following: consider two vector fields $V = \frac{d}{d\lambda}$ and $W = \frac{d}{d\mu}$. Using the exponential map, the coordinates of A, moving before along V and then along W, are

$$x^i(A) = \exp\left(\epsilon_2 \frac{d}{d\mu}\right) \exp\left(\epsilon_1 \frac{d}{d\lambda}\right)\ x^i\Big|_P$$

whereas the coordinates of B, moving before along W and then along Y, are

$$x^i(B) = \exp\left(\epsilon_1 \frac{d}{d\lambda}\right) \exp\left(\epsilon_2 \frac{d}{d\mu}\right)\ x^i\Big|_P$$

Computing the difference

$$x^i(B) - x^i(A) = \epsilon_1\epsilon_2 \left[\frac{d}{d\lambda},\ \frac{d}{d\mu}\right] x^i\Big|_P + O(\epsilon^3)$$

Hence, if the commutator does not vanish, the final points are different $A \neq B$ and the path $PA \cup PB$ does not close.

A sufficient and necessary condition for a set of fields to be coordinate vectors is that their commutator vanishes.

Proof. First, the sufficient condition. Consider two coordinate vector fields $V = \frac{\partial}{\partial x^1}$ and $W = \frac{\partial}{\partial x^2}$. Then $v^i = \delta^i{}_1$, $w^j = \delta^j{}_2$ and the commutator vanishes, since the derivative of a constant is so.

Second, the necessary condition. Consider two commuting vector fields $V = \frac{\partial}{\partial x^1}$ and $W = \frac{\partial}{\partial x^2}$. Suppose also that they are linearly independent

$$aV(P) + bW(P) = 0 \qquad \Longleftrightarrow \qquad a = b = 0. \tag{1.3}$$

Introducing a chart x^i, moving from P along V by $\Delta\lambda = \alpha$ to a point R

$$x^(R) = \exp\left(\alpha\frac{d}{d\lambda}\right) x^i\Big|_P$$

and then along W by $\Delta\mu = \beta$ to a point Q

$$x^(Q) = \exp\left(\beta\frac{d}{d\mu}\right)\exp\left(\alpha\frac{d}{d\lambda}\right) x^i\Big|_P \tag{1.4}$$

If α and β are coordinates, the corresponding basis vectors are $\frac{\partial}{\partial\alpha} = \frac{\partial x^i}{\partial\alpha}$ and $\frac{\partial}{\partial\beta} = \frac{\partial x^i}{\partial\beta}$. Hence, using (1.4)

$$\begin{aligned}
\frac{\partial x^i}{\partial\alpha} &= \frac{\partial}{\partial\alpha}\left(\exp\left(\beta\frac{d}{d\mu}\right)\exp\left(\alpha\frac{d}{d\lambda}\right) x^i\Big|_P\right)\\
&= \exp\left(\beta\frac{d}{d\mu}\right)\frac{\partial}{\partial\alpha}\left(\exp\left(\alpha\frac{d}{d\lambda}\right) x^i\Big|_P\right)\\
&= \exp\left(\beta\frac{d}{d\mu}\right)\exp\left(\alpha\frac{d}{d\lambda}\right) \frac{dx^i}{d\lambda}\Big|_P
\end{aligned}$$

and, similarly,

$$\frac{\partial x^i}{\partial\beta} = \exp\left(\beta\frac{d}{d\mu}\right)\exp\left(\alpha\frac{d}{d\lambda}\right) \frac{dx^i}{d\mu}\Big|_P$$

This shows that $\frac{\partial}{\partial\alpha}$ and $\frac{\partial}{\partial\beta}$ are respectively the vector fields $\frac{d}{d\lambda}$ and $\frac{\partial}{\partial\mu}$ evaluated in Q, using the fact that the commutator is vanishing. Then the determinant of the Jacobians is not zero

$$\det J = \det\begin{bmatrix}\frac{\partial x^1}{\partial\alpha} & \frac{\partial x^2}{\partial\alpha}\\ \frac{\partial x^1}{\partial\beta} & \frac{\partial x^2}{\partial\beta}\end{bmatrix} \neq 0$$

since we assumed the linearly independence of $\frac{d}{d\lambda}$ and $\frac{d}{d\mu}$. q.e.d.

1.8 1-forms

A 1-form is a linear functional w acting on a vector $w\colon T_P \to \mathbb{R}$ such that $w(\alpha v + \beta u) = \alpha w(v) + \beta w(u)$ and $(\alpha w + \alpha z)(v) = \alpha w(v) + \beta z(v)$. Linearity implies that the action of a 1-form is completely determined by the action on a basis of T_P. 1-forms acting on the same T_P form a linear space T_P^*, called the cotangent space. A cotangent bundle is the set of all cotangent space at each point together with the manifold itself $\mathcal{T}^*\mathcal{M} = \{\mathcal{M},\ \{T_P^*\colon \forall P \in \mathcal{M}\}\}$. A 1-form field is a map associates a 1-form of T^*P to each point $P \in \mathcal{M}$.

One way of thinking 1-forms is in the following way: given an arbitrary function, a vector field is defined by $V(f) = \frac{df}{d\lambda}$ whereas given an arbitrary vector field, a 1-form is defined by $V(f) = \frac{df}{d\lambda} = df(\frac{d}{d\lambda})$. The difference is the in the former V is fixed and f is arbitrary, whereas in the latter f is fixed and V is arbitrary. Introducing a chart x^i

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda} = \nabla_i f \frac{dx^i}{d\lambda} = df_i \frac{dx^i}{d\lambda}$$

where df_i are the components of the 1-form df, called the gradient of f.

Geometrically, the gradient of a function can be seen as the number of contour lines that a vector V crosses in a neighbourhood of P. Generalizing for 1-forms, they can be seen as a set of level surfaces whose action on a vector is the number of surface the vector crosses.

Let $\{e_i\}$ be a basis of T_P. A basis of T_P^* is not related to it, however it is convenient to choose the dual basis, which completely defined a basis of T^*P by a basis in T_P in the following way

$$e^i(e_j) = \delta^i{}_j \tag{1.5}$$

or, equivalently, applying it to a vector v

$$e^i(v) = e^i(v^j e_j) = v^j e^i(e_j) = v^j \delta^i{}_j = v^i$$

Consequently, $\mathcal{M}$, T_P and T_P^* have the same dimension n. $\{e^i\}$ are actually a basis of T_P^*, since given an arbitrary 1-form q

$$q(v) = q(v^i e_i) = v^i q(e_i) = v^i q_i$$

and

$$v^i q_i = q_i e^i(v)$$

Remember that under a change of coordinates, vectors or 1-forms do not change, only their components and basis change, the latter inversely.

Differentials are the basis 1-form dual to the coordinate basis vectors

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda} = \frac{\partial f}{\partial x^i} dx^i\Big(\frac{dx^j}{d\lambda}\frac{\partial}{\partial x^j}\Big) = \frac{\partial f}{\partial x^i}\frac{dx^j}{d\lambda} dx^i\Big(\frac{\partial}{\partial x^j}\Big) = \frac{\partial f}{\partial x^i}\frac{\partial x^j}{\partial \lambda}\delta^i{}_j = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \lambda}$$

where it has been used the dual basis.

1.9 Tensors

A tensor (n, m) at P is a linear functional that maps n 1-forms and m vectors into a real number

$$T\colon \underbrace{T_P^* \otimes \cdots \otimes T_P^*}_{n\ times} \otimes \underbrace{T_P \otimes \cdots \otimes T_P}_{m\ times} \to \mathbb{R}$$

A tensor can be also seen as the outer product of 1-forms and vectors. A tensor (1, 0) is a vector and a tensor (0, 1) is a 1-form. A tensor (n, m) can be written in terms of the dual basis

$$T = T^{i_1 \cdots i_n}_{j_1 \cdots j_m} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e^{j_1} \otimes \cdots \otimes e^{j_m}$$

where the components are

$$T^{i_1 \cdots i_n}_{j_1 \cdots j_m} = T(e^{i_1},\ \cdots,\ e^{i_n},\ e_{j_1},\ \cdots,\ e^{j_m})$$

A change of basis is determined by a 4×4 non-degenerate matrix $\Lambda \in GL(n)$. On a vector basis, it acts as

$$e'_j = \Lambda^i{}_j e_i \tag{1.6}$$

This transformation has no effects on the dual space, however, in order to keep the duality of the basis, it must induce a transformation with the inverse matrix

$$e'^j = \Lambda^j{}_i e^i$$

Proof. Recalling (1.5), to preserve the duality, also the transformed dual basis must obey

$$e'^i(e'_j) = \delta^i{}_j \tag{1.7}$$

Hence, given an arbitrary transformation matrix,

$$e'^i = M^i{}_k e^k$$

and putting into (1.7), using(1.6)

$$\delta^i{}_j = e'^i(e'_j) = M^i{}_k e^k(\Lambda^l{}_j e_l) = M^i{}_k \Lambda^l{}_j e^k(e_l) = M^i{}_k \Lambda^l{}_j \delta^k{}_l = M^i{}_k \Lambda^k{}_j$$

then, M must satisfy

$$M^i{}_k \Lambda^k{}_j = \delta^i$$

and it is indeed the inverse matrix. q.e.d.

It is possible to perform several operations on tensors at P:

1. scalar multiplication, i.e.

$$S^{(n,m)} = aT^{(n,m)} \quad \forall a \in \mathbb{R}$$

2. addition, i.e.

$$S^{(n,m)} = T^{(n,m)} + Q^{(n,m)}$$

3. outer product, i.e.

$$S^{(n+p,m+q)} = T^{(n,m)} \otimes Q^{(p,q)}$$

4. saturation with 1-forms, i.e.

$$T^{(n-1,m)} = T^{(n,m)}(\ldots,\ w,\ \ldots)$$

5. saturation with vector, i.e.

$$T^{(n,m-1)} = T^{(n,m)}(\ldots,\ v,\ \ldots)$$

The last two can be generalised to an arbitrary saturation of a (n,m) tensor with a $(p < n, q < m)$ tensor.

For a change of basis in the tangent space to correspond a change of coordinates on the manifold, the transformation matrix must obey the condition

$$\frac{\partial \Lambda^j{}_i}{\partial x^k} = \frac{\partial \Lambda^j{}_k}{\partial x^i} \tag{1.8}$$

Proof. Consider two charts x^i and y^i that overlap at P. The transformation matrix between basis is

$$\Lambda^i{}_j = \frac{\partial x^i}{\partial y^j}$$

and the inverse is

$$\Lambda^j{}_i = \frac{\partial y^j}{\partial x^i}$$

If we move continuously to another point Q insider the charts, the matrix transformation will become a field $\Lambda(Q) = \Lambda(x^i(Q)) = \Lambda(y^i(Q))$ and, since the partial derivatives commute

$$\frac{\partial \Lambda^j{}_i}{\partial x^k} = \frac{\partial}{\partial x^k} pdv y^j x^i = \frac{\partial}{\partial x^i} pdv y^j x^k = \frac{\partial \Lambda^j{}_k}{\partial x^i}$$

q.e.d.

1.10 Metric tensor

The notions of lenght and angles on a manifold can be introduced with the metric tensor.

A metric tensor g is a $(2,0)$ tensor which maps two vectors into a real number, satisfying the following properties

1. symmetry, i.e.
$$g(v,w) = g(w,v) = g(v^i e_i, w^j e_j) = g(e_i, e_j) v^i w^j = g_{ij} v^i v^j \quad \forall v,\ w \in T_P$$
2. non-degeneracy, i.i
$$g(v,w) = 0 \quad \forall w \in T_P \qquad \Longleftrightarrow \qquad v = 0$$
or, equivalenlty, if $\det g_{ij} \neq 0$

A metric tensor defines a scalar product
$$g(v,w) = v \cdot w$$
and introduces the notions of norm of a vector
$$v^2 = g(v,v) = v \cdot v = g_{ij} v^i v^j$$
and angle between two vectors
$$g(v,w) = vw \cos\theta$$
Although, the latter only with Riemannian metrics.

The metric tensor, under a change of basis Λ, change
$$g' = \Lambda^T g \Lambda$$
where $g'_{ij} = g(e'_i, e'_j)$. Since it is symmetric, it can be always possible to find two matrices $O^{-1} = O^T$ and $D = D^T = diag(\frac{1}{\sqrt{|g_{ii}^{(diag)}|}})$ such that
$$g' = D^T O^T g O D = D g^{(diag)} D$$
and put in canonical form
$$g'_{ij} = \pm\delta_{ij}$$
which defines an orthonormal basis at P, i.e. $g(e_i, e_j) = \pm\delta_{ij}$.

The $\pm$ cannot be eliminated and the sum of the diagonal element is called the signature. A sign inversion does not affect the signature. The diagonal elements can classify the metric in the following way:

1. Riemannian metric, i.e. all of the same sign
2. pseduo-Riemannian metric, i.e. both signs appear (Lorentzian metric if one is of one kind and all the others of the other kind)

Metric tensors define a map between T_P and T_P^*, to lower indices and the inverse to raise them. Infact, a vector $v \in T_P$ can be mapped into a 1-form

$$v_i = v(e_i) = g(v^j e_j, e_i) = v^i g(e_j, e_i) = v^i g_{ij}$$

and a 1-form $w \in T_P^*$ can be mapped into a vector

$$w^i = e^i(w) = g(e^i, w_j e^j) = w_j g(e^i, e^j) = w_j g^{ij}$$

Consequently, at P a vector and a 1-form are equivalent.

The inverse metric tensor in defined by

$$g_{ij}^{-1} = g^{ij} \quad g_{ij} g^{jk} = \delta^k{}_i$$

If the metric is in canonical form, the dual basis will be orthonormal.

A metric tensor field is a map that associates each point of $\mathcal{M}$ into a metric tensor. The manifold becomes a metric manifold $(\mathcal{M},\ g)$. The metric tensor field in terms of coordinate vectors and dual basis is

$$g(x) = g_{ij}(x) dx^i \otimes dx^j$$

which is written as line element

$$ds^2 = g_{ij}(x) dx^i dx^j$$

Consider the integral curve γ of a vector field $v = \frac{d}{d\lambda}$. The scalar infinitesimal displacement along v is

$$ds^2 = dx \cdot dx = g(dx, dx) = g(v d\lambda, v d\lambda) = g(v, v) d\lambda^2$$

Integrating along γ, the length of the path between λ_1 and λ_2 is

$$s(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} d\lambda\ \sqrt{g(v, v)} = \int_{\lambda_1}^{\lambda_2} d\lambda\ \sqrt{g_{ij}(\lambda) v^i(\lambda) v^j(\lambda)}$$

Introducing a chart x^i,

$$s(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} d\lambda\ \sqrt{g_{ij}(\lambda) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$$

It is always possible to find a change of coordinate that put the metric tensor field in the locally canonical form

$$g_{ij}(x) = \pm\delta_{ij} + \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}\bigg|_{x_P} \delta x^k \delta x^l$$

which means to find a locally orthogonal coordinates x^i such that $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \pm\delta_{ij}$. However, this holds only locally, not on the entire manifold.

Proof. Around P, the metric tensor field g_{ij} can be Taylor expanded in $x = x_P + \delta x$

$$g_{ij} = g_{ij}(x_P) + \frac{\partial g_{ij}}{\partial x^k}\Big|_{x_P} \delta x^k + \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}\Big|_{x_P} \delta x^k \delta x^l + \dots \tag{1.9}$$

as well as the transformation matrix

$$\frac{\partial x^i}{\partial y^j}(x) = \frac{\partial x^i}{\partial y^j}(x_P) + \frac{\partial}{\partial x^k}\frac{\partial x^i}{\partial y^j}\Big|_{x_P} \delta x^k + \frac{1}{2}\frac{\partial^2}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial y^j}\Big|_{x_P} \delta x^k \delta x^l + \dots \tag{1.10}$$

and the metric in the new coordinates

$$g'_{ij} = g'_{ij}(y_P) + \frac{\partial g'_{ij}}{\partial y^k}\Big|_{y_P} \delta y^k + \frac{1}{2}\frac{\partial^2 g'_{ij}}{\partial y^k \partial y^l}\Big|_{y_P} \delta y^k \delta y^l + \dots \tag{1.11}$$

Using

$$g'_{ij} = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{kl}$$

then the left-handed side is

$$\begin{aligned}
\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{kl} = &\Big(\frac{\partial x^k}{\partial y^i} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\delta x^a + \frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^e}\frac{\partial x^k}{\partial y^i}\delta x^a \delta x^e + \dots\Big) \\
&\Big(\frac{\partial x^l}{\partial y^j} + \frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}\delta x^b + \frac{1}{2}\frac{\partial^2}{\partial x^b \partial x^f}\frac{\partial x^l}{\partial y^j}\delta x^b \delta x^f + \dots\Big) \\
&\Big(g_{kl} + \frac{\partial g_{kl}}{\partial x^c}\delta x^c + \frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^c \partial x^d}\delta x^c \delta x^d + \dots\Big) \\
= &\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}\delta x^b g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}\delta x^b \frac{\partial g_{kl}}{\partial x^c}\delta x^c + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^c}\delta x^c \\
&+ \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\delta x^a \frac{\partial x^l}{\partial y^j} g_{kl} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\delta x^a \frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}\delta x^b g_{kl} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\delta x^a \frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^c}\delta x^c \\
&+ \frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^e}\frac{\partial x^k}{\partial y^i}\delta x^a \delta x^e \frac{\partial x^l}{\partial y^j} g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{1}{2}\frac{\partial^2}{\partial x^b \partial x^f}\frac{\partial x^l}{\partial y^j}\delta x^b \delta x^f g_{kl} \\
&+ \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^c \partial x^d}\delta x^c \delta x^d + \dots \\
= &\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{kl} + \delta x^a \Big(\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^a}\frac{\partial x^l}{\partial y^j} g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^a} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{kl}\Big) \\
&+ \delta x^a \delta x^b \Big(\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^a}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^b} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j} g_{kl} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^b} \\
&\qquad + \frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^b}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^b}\frac{\partial x^l}{\partial y^j} g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^a \partial x^b}\Big)
\end{aligned}$$

Comparing infinitesimal of the same order

$$\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{kl} = g'_{ij}$$

$$\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^a}\frac{\partial x^l}{\partial y^j}g_{kl}+\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^a}+\frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl}=\frac{\partial {g'}_{ij}}{\partial y^k}$$

$$\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^a}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^b}+\frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}g_{kl}+\frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^b}$$
$$+\frac{1}{2}\frac{\partial^2}{\partial x^a\partial x^b}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl}+\frac{\partial x^k}{\partial y^i}\frac{1}{2}\frac{\partial^2}{\partial x^a\partial x^b}\frac{\partial x^l}{\partial y^j}g_{kl}+\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^a\partial x^b}=\frac{1}{2}\frac{\partial^2 {g'}_{ij}}{\partial y^k\partial y^l}$$

Looking at this system of equations, we find 1 degree of freedom for the first one, n for the second one and n^2 for the third one. Hence, since Λ has n^2-1 degrees of freedom with -1 coming from (1.8), we only have enough degree of freedom to put

$${g'}_{ij}(y_P)=\pm\delta_{ij}$$

and

$$\left.\frac{\partial g_{ij}}{\partial y^k}\right|_{y_P}=0$$

but not enough to put

$$\left.\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}\right|_{y_P}=0$$

q.e.d.

Chapter 2

Lie derivatives

2.1 Active and passive transformation

In the passive interpretation of a diffeomorphism on $\mathcal{M}$, the points remain the same but their coordinates changes. The diffeomorphism does not act on $\mathcal{M}$, but on the coordinates in $\mathbb{R}^n$. It can be sees as a change of coordinates $x' = x'(x, \epsilon)$ where ϵ is a parameter such that $x'(x, 0) = x$. For instance, a function changes in such a way that $\Phi'(x') = \Phi(x)$ where $\Phi(x) = (f ø \phi^{-1})(x)$.

In the active interpretation of a diffeomorphism, the points are actually moved (along the flow of an integral curve). The diffeomorphism do act on $\mathcal{M}$.

The Lie dragged or push forward of a function f from a point P to a point P' is a new function such that $f^*(P') = f(P)$.

2.2 Congruence

A congruence of a vector field V is a set of integral curve which start from a curve Σ_0, that is an hypersurface of dimension $n-1$ and uniquely cover a portion of $\mathcal{M}$. A Lie dragging or a push-forward $\phi_{\Delta\lambda} \colon \mathcal{M} \to \mathcal{M}$ is the motion of a point $P(\lambda_0)$ in $P(\lambda_0 + \Delta\lambda)$ such that $\phi_{\Delta\lambda}$ is continuous and invertible. If $V \in C^\infty$, the push-forward becomes a diffeomorphism and form a group. Infact, $\phi_{\lambda_1} ø \phi_{\lambda_2} = \phi_{\lambda_1 + \lambda_2}$, $\phi_\lambda^{-1} = \phi_{-\lambda}$ and $\phi_{\lambda=0} = \mathbb{I}$.

The push-forward of a function f along a congruence of $V = \frac{d}{d\lambda}$ is a map f^* from a point P to a point $Q = \phi_{\Delta\lambda}(P)$ such that

$$f^*_{\Delta\lambda}(Q) = f(P)$$

If it is true $\forall Q$ along the integral curve of V, $f^*_{\Delta\lambda}$ is constant and $\frac{df}{d\lambda} = 0$.

The push-forward of a vector field $W = \frac{d}{d\mu}$ along a congruence of $V = \frac{d}{d\lambda}$ is a map f^* from a point P to a point $Q = \phi_{\Delta\lambda}(P)$ such that

$$W^*_{\Delta\lambda}(f^*_{\Delta\lambda})|_Q = W(f)|_P$$

where f is an arbitrary function. It can be also written as

$$\frac{df^*}{d\mu}\Big|_{\lambda_0+\Delta\lambda} = \frac{df}{d\mu}$$

where $\lambda(P) = \lambda_0$ and $\lambda(Q) = \lambda_0 + \Delta\lambda$.

Furthermore, the commutator between V and W^* vanishes

$$[V, W^*] = [\frac{d}{d\lambda}, \frac{d}{d\mu^*}] = 0 \tag{2.1}$$

Proof. Fixing f and varying $\Delta\lambda$, $\frac{df^*}{d\mu^*}$ is constant along the congruences of V. Mapping the initial curve Σ_0 into a new curve $\Sigma_{\Delta\lambda}$ and since λ is constant, it can be used as a coordinate.

Since W^* is tangent to $\Sigma_{\Delta\lambda}$ its parameter μ^* is constant along the congruences of V. Hence there are two coordinates (λ, μ^*) are coordinates and their coordinate vectors commute. q.e.d.

2.3 Lie derivatives

The Lie derivative of a function f along a vector field $V = \frac{d}{d\lambda}$ is

$$\pounds_V f|_{\lambda_0} = \lim_{\Delta\lambda\to 0} \frac{f^*_{-\Delta\lambda}(\lambda_0) - f(\lambda_0)}{\Delta\lambda} = \lim_{\Delta\lambda\to 0} \frac{f(\lambda_0+\Delta\lambda) - f(\lambda_0)}{\Delta\lambda} = \frac{df}{d\lambda}\Big|_{\lambda_0} = V(f)$$

where it has been used the push-back $\phi_{-\Delta\lambda}(P(\lambda_0 + \Delta\lambda)) = P(\lambda_0)$ and $f^*_{-\Delta\lambda}(\lambda_0) = f(\lambda_0 + \Delta\lambda)$. If f is constant along the congruences, then $\pounds_V f = 0$.

The Lie derivative of a vector field $W = \frac{d}{d\mu}$ along a vector field $V = \frac{d}{d\lambda}$ is

$$\pounds_V W(f)|_{\lambda_0} = \lim_{\Delta\lambda\to 0} \frac{W^*_{-\Delta\lambda} - W}{\Delta\lambda}(f)\Big|_{\lambda_0}$$

or it can be written as

$$\pounds_V W = [V, W]$$

in components

Proof. Taylor expanding around $\lambda_0 + \Delta\lambda$

$$W^*_{-\Delta\lambda}(f)\Big|_{\lambda_0} = \frac{df}{d\mu^*}\Big|_{\lambda_0} = \frac{df}{d\mu^*}\Big|_{\lambda_0+\Delta\lambda} - \Delta\lambda\Big(\frac{d}{d\lambda}\frac{d}{d\mu^*}f\Big)\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2)$$

Using (2.1) and at first order $\frac{d}{d\mu} = \frac{d}{d\mu^*}$,

$$\begin{aligned}
\frac{df}{d\mu^*}\Big|_{\lambda_0} &= \frac{df}{d\mu}\Big|_{\lambda_0+\Delta\lambda} - \Delta\lambda\Big(\frac{d}{d\lambda}\frac{d}{d\mu^*}f\Big)\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2) \\
&= \frac{df}{d\mu}\Big|_{\lambda_0} + \Delta\lambda\frac{d}{d\lambda}\frac{d}{d\mu^*}f\Big|_{\lambda_0} - \Delta\lambda\Big(\frac{d}{d\lambda}\frac{d}{d\mu^*}f\Big)\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2) \\
&= \frac{df}{d\mu}\Big|_{\lambda_0} + \Delta\lambda\Big(\frac{d}{d\lambda}\frac{d}{d\mu^*}f - \frac{d}{d\mu^*}\frac{d}{d\lambda}f\Big)\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2) \\
&= \frac{df}{d\mu}\Big|_{\lambda_0} + \Delta\lambda\Big[\frac{d}{d\lambda},\frac{d}{d\mu}\Big]f\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2)
\end{aligned}$$

Hence

$$W^*_{-\Delta\lambda}(f)\Big|_{\lambda_0} = W(f)\Big|_{\lambda_0} + \Delta\lambda\Big[\frac{d}{d\lambda},\frac{d}{d\mu}\Big]f\Big|_{\lambda_0} + O(\Delta\lambda^2)$$

and

$$\pounds_V W(f)|_{\lambda_0} = \lim_{\Delta\lambda\to 0}\frac{\Delta\lambda\Big[\frac{d}{d\lambda},\frac{d}{d\mu}\Big]f\Big|_{\lambda_0}(f)\Big|_{\lambda_0} + O(\Delta\lambda^2)}{\Delta\lambda} = \Big[\frac{d}{d\lambda},\frac{d}{d\mu}\Big]f\Big|_{\lambda_0}$$

q.e.d.

The Lie derivative satifies the properties

1. vanishes if the components of W are constant along V
2. Leibniz rule, i.e.
$$\pounds_V(fW) = f\pounds_V(W) + \pounds_V(f)W$$
3. linearity, i.e.
$$\pounds_V + \pounds_W = \pounds_{V+W}$$
4. commutator, i.e.
$$[\pounds_V, \pounds_W] = \pounds_{[V,W]}$$
5. Jacobi identity, i.e.
$$[[\pounds_V, \pounds_W], \pounds_Z] + [[\pounds_W, \pounds_Z], \pounds_X] + [[\pounds_Z, \pounds_V], \pounds_W] = 0$$

The Lie derivative of a 1-form ω along a vector field $V = \frac{d}{d\lambda}$ is

$$(\pounds_V\omega)(W) = \pounds_V(\omega(W)) - \omega(\pounds_V W)$$

or introducing a chart x^i and the sual basis

$$(\pounds_V\omega)_i = V^k\frac{\partial\omega_i}{\partial x^k} + \omega_k\frac{\partial V^k}{\partial x^i}$$

Proof. Using the Leibniz rule,

$$\pounds_V(\omega(W)) = (\pounds_V\omega)(W) + \omega(\pounds_V W)$$

$$(\pounds_V\omega)(W) = \pounds_V(\omega(W)) - \omega(\pounds_V W)$$

Introducing a coordinate basis, the Lie derivative of a scalar and the components of a commutator

$$\begin{aligned}
(\pounds_V\omega)_i &= (\pounds_V\omega)(e_i) \\
&= \pounds_V(\omega(e_i)) - \omega(\pounds_V e_i) = \frac{d\omega(e_i)}{d\lambda} - \omega([V, e_i]) \\
&= V^k\frac{\partial\omega^i}{\partial x^k} + \omega^k\frac{\partial V^k}{\partial x^i}
\end{aligned}$$

q.e.d.

The Lie derivative of a tensor (n, m)

$$T(\omega_1, \ldots, \omega_n, W^1, \ldots, W^m) \colon \mathcal{M} \to \mathbb{R}$$

along a vector field $V = \frac{d}{d\lambda}$ is

$$\begin{aligned}
\pounds_V T(\omega_1, \ldots, \omega_n, W^1, \ldots, W^n) = (\pounds_V T)&(\omega_1, \ldots, \omega_n, W^1, \ldots, W^n) \\
&+ T(\pounds_V\omega_1, \ldots, \omega_n, W^1, \ldots, W^n) + \ldots \\
&+ T(\pounds_V\omega_1, \ldots, \pounds_V\omega_n, W^1, \ldots, W^n) \\
&+ T(\omega_1, \ldots, \omega_n, \pounds_V W^1, \ldots, W^n) + \ldots \\
&+ T(\omega_1, \ldots, \omega_n, W^1, \ldots, \pounds_V W^n)
\end{aligned}$$

2.4 Symmetries

Symmetries are no longer referred to the manifold, but to tensor defined in it. Furthermore, their geometrical meaning is a local feature.

A submanifold is a subset $\mathcal{S} \subset \mathcal{M}$ of dimension $\dim\mathcal{S} \leq \dim\mathcal{M}$ sucin which there exist a chart x^i such that $U \cap \mathcal{S} \subseteq \mathcal{M}$ and $x^{n-m+1} = \ldots = x^n = 0$ for all $P \in mathcalS$.

The tangent space in a point $P \in mathcalS$ has dimension

$$\dim T_P^{(\mathcal{M})} = n \geq \dim T_P^{(\mathcal{M})} = m$$

Curves and vectors in $\mathcal{S}$ maps to $\mathcal{M}$

$$\gamma_{\mathcal{S}} = (x^1(\lambda), \ldots, x^m(\lambda)) \rightleftarrows \gamma_{\mathcal{M}} = (x^1(\lambda), \ldots, x^m(\lambda), 0, \ldots, 0)$$

and

$$V_{\mathcal{S}} = (V^1, \dots, V^m) \rightleftarrows V_{\mathcal{M}} = (V^1, \dots, V^m, 0, \dots, 0)$$

but the inverse is not unique, infact there are infinitely many curves or vectors created putting a different number from 0 in the places with index greater than m.

A 1-form in the submanifold is defined as

$$\omega_{\mathcal{S}}(V) = \omega_{\mathcal{M}}(V, 0, \dots, 0)$$

where $V \in T_P^{(\mathcal{S})}$. Also here, the inverse is not unique, infact there are infinitely many 1-forms created putting a different number from 0 in the places with index greater than m.

A set of vector fields $V^{(k)}$ with $k = 1, \dots, p$ is linearly independent if there exist a_k constants such that

$$\sum_{k=1}^{p} a_k V^k(P) = 0 \quad \forall P \in \mathcal{M}$$

This does not mean that at a given P, they are linearly independent, because the coefficients could depend on point $a_k = a_k(P)$.

Theorem 2.1 (Frobenius)
Let $V^{(k)}$ be a set of linearly independent vector fields with $k = 1, \dots, p$ such that forms a Lie algebra

$$[V^{(i)}, V^{(j)}] = C^{ij}{}_k V^{(k)}$$

where $C^{ij}{}_k \in \mathbb{R}$. Then the integral curves of $V^{(k)}$ form a family of submanifolds or foliations of $\mathcal{M}$ of dimension $m \leq p$.

A vector field V is a symmetry of a tensor field T if

$$\pounds_V T = 0$$

Theorem 2.2
Let $V^{(i)}$ be a set of linearly independent vector fields with $i = 1, \dots, p$ and $T^{(k)}$ be a set of linearly independent vector fields with $k = 1, \dots, q$ such that

$$\pounds_{\sum_i a_i V^{(i)}} \sum_k b_k T^{(k)} = 0$$

Then $V^{(i)}$ form a Lie algebra

Proof. Given a two symmetries $V^{(1)}$ and $V^{(2)}$, using a property of the Lie derivative

$$[\pounds_{V^{(1)}}, \pounds_{V^{(2)}}] = \pounds_{[V^{(1)}, V^{(2)}]} = 0$$

Hence, $[V^{(1)}, V^{(2)}]$ is a symmetry as well. Generalizing for a linear combination $aV^{(1)} + bV^{(2)}$, the only condition to satift the hypothesis is that a and b are independent of P and the structure constant as well. q.e.d.

Corollary 2.1
$V^{(i)}$ define a submanifold of dimension $m \leq p$.

An isometries is a symmetry of the metric tensor

$$\pounds_V g = 0$$

where V is called the Killing vector. Hence, congruences along a Killing vector preserves lengths and angles.

In special relativity, inertial observers can be seen as coordinate frames along Killing vectors, using the Minkovski metric $g = \eta$.

Chapter 3

Integrals and forms

3.1 p-forms

A p-form is an antysymmetric tensor $(0, p)$ in the tangent T_P. p-forms form a linear space.

A 2-form ω is

$$\omega_{[ij]} = \frac{1}{2!}(\omega_{ij} - \omega_{ji})$$

A 3-form ω is

$$\omega_{[ijk]} = \frac{1}{3!}(\omega_{ijk} + \omega_{jki} + \omega_{kij} - \omega_{ikj} - \omega_{kji} - \omega_{jki})$$

A general p-form ω is

$$\omega_{[i_1 \ldots i_p]} = \frac{1}{p!}(\omega_{i_1 \ldots i_p} + \text{permutations})$$

The number of independent components of a p-form is the binomial coefficient

$$\binom{n}{p}$$

with the condition $\sum_p \binom{n}{p} = n^2$.

Introducing the wedge product

$$\omega = \omega_{i_1 \ldots i_p} e^{i_1} \wedge \ldots \wedge e^{i_p} = \omega_{i_1 \ldots i_p} \frac{1}{p!}(e^{i_1} \otimes \ldots \otimes e^{i_p} + \text{permutations})$$

Moreover, the wedge product can be used to compose a p-form and a q-form into a (p+q)-form

$$\text{p-form} \wedge \text{q-form} = \text{(p+q)-form}$$

and to contract a p-form with a vector to obtain a (p-1)-form

$$\begin{aligned} p(V,\ldots) &= (\omega_{i_1\ldots i_p} e^{i_1} \wedge \ldots \wedge e^{i_p})(V^k e_k) \\ &= \frac{1}{p!}(\omega_{i_1\ldots i_p} e^{i_1}(e_k) \otimes \ldots \otimes e^{i_p} + \text{permutations}) \\ &= V^k \omega_{i_1\ldots i_p} e^{i_2} \wedge \ldots \wedge e^{i_p} \end{aligned}$$

3.2 Volume

A polyhedron in $\mathcal{M}$ is defined by n linearly independent vectors and its volume is a number. Therefore it is natural to associate an n-form, given the additional antysymmetric property, i.e. to vanish if two vectors are linearly dependent. In a coordinate basis, the n vectors are

$$\Delta x_k = dx^i_{(k)} \frac{\partial}{\partial x^i}$$

and the n-form is

$$\omega = f e^1 \wedge \ldots \wedge e^n$$

Putting together, the volume of an infinitesimal polyhedron is

$$\omega(\Delta x_{(1)}, \ldots, \Delta x_{(n)}) = f e^i(\Delta x_{(1)}) \ldots e^n(\Delta x_{(n)}) + \text{permutations}$$

Choosing coordinate basis,

$$\begin{aligned} \omega(\Delta x_{(1)}, \ldots, \Delta x_{(n)}) &= f dx^1_{(1)} \ldots dx^n_{(n)} + 0 + \ldots + 0 \\ &= f dx^1_{(1)} \ldots dx^n_{(n)} \\ &= dV \end{aligned}$$

Introducing a lattice of charts, the volume of a region $U \subseteq \mathcal{M}$ is the integral

$$V(U) = \int_{\phi(U)} f dx^1 \ldots dx^n = \int_U \omega$$

It is a scalar, since under a change of coordinates $y^i(x^i)$

$$V = \int_U \omega = \int_{\phi'(U)} f(y) J(y) d^n y$$

where J is the determinant of the jacobian, which shows that the volume is coordinate independent.

3.3 Area

In a submanifold of dimension $n-1$, the infinitesimal area of an hypersurface uses a (n-1)-form, taken by contracting the n-form of the volume with a vector $v \in T_P^{(\mathcal{M})} \notin T_P^{(\mathcal{S})}$ which is $A = \omega(v, \ldots)$.

The infinitesimal area, choosing coordinate basis, is

$$\begin{aligned}\omega(v, \omega_{(1)}, \ldots, \omega_{(n-1)}) &= A(\omega_{(1)}, \ldots, \omega_{(n-1)}) \\ &= vfe^1(\omega_{(1)}) \wedge \ldots \wedge e^{n-1}(\omega_{(n-1)}) \\ &= vfdx^1 \ldots dx^{n-1} \\ &= dA\end{aligned}$$

and the area of a portion $\Sigma \subseteq \mathcal{S}$ is

$$A(\Sigma) = \int_{\phi(\Sigma)} fvdx^1 \ldots dx^{n-1} = \int_\Sigma A$$

It is a scalar, since under a change of coordinates $y^i(x^i)$

$$A' = J^{(n-1)}A$$

where $J^{(n-1)}$ is the determinant of the jacobian restricted to the image of Σ.

3.4 Integrating with the metric

In a point P, the metric can be put in canonical form

$$g_{ij}(P) = \pm\delta_{ij}$$

The natural volume n-form is

$$\omega_g = e^1 \wedge \ldots \wedge e^n$$

Under a local change of coordinates $y^i(x^i)$

$$\omega_g = J\omega'_g = Jdy^1 \wedge \ldots \wedge dy^n$$

Using $g' = \Lambda^T g \Lambda$

$$\det g' = \det(\Lambda^T g\Lambda) = \det g \overset{2}{\det} \Lambda = \det g J^2 = \pm J^2$$

where $J = \sqrt{|\det g'|}$.

Hence, the volume of U becomes

$$V(U) = \int_U \omega_g = \int_{\phi(U)} \sqrt{|\det g'|} dy^1 \ldots dy^n$$

Similarly, the natural area (n-1)-form is

$$A_g = \omega_g(\ldots, e_n) = e^1 \wedge \ldots \wedge e^{n-1}$$

and the area of a portion Σ is

$$A = \int_\Sigma A_g = \int_{\phi(\Sigma)} \sqrt{|\det g^{(n-1)}|} dx^1 \ldots dx^{n-1}$$

where the metric locally is

$$g_{ij} = \begin{bmatrix} g_{ij}^{(n-1)} & 0 \\ 0 & \pm 1 \end{bmatrix}$$

3.5 Differential forms

The exterior derivative of a p-form

$$\omega = \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

is

$$d\omega = (\partial_k \omega_{i_1 \ldots i_p}) dx^k \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

It satisfies the following properties

1. addition, i.e.
$$d(\omega + \sigma) = d\omega + d\sigma$$

2. wedge product, i.e.
$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^p \omega \wedge d\sigma$$

3. vanishing boundary, i.e.
$$d(d\omega) = 0$$

For a function, this is the differential

$$df = \partial_i f dx^i = df$$

such that

$$d(df) = \partial_i \partial_j f dx^i \wedge dy^j = 0$$

List of Theorems