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On quantum field theory I:

how to secondly quantise?

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Theoretical Physics

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Part I

Classical field theory

Chapter 1

Action

A field is a physical quantity $\phi(t, \mathbf{x})$ which is defined ad every point in spacetime. The dynamics of a field is governed by an action, which is a functional that associates a real number to each field configuration for a fixed time interval $[t_1, t_2]$

$$S[\phi_i(x), \partial_\mu \phi_i(x)] = \int_{t_1}^{t_2} dt\ L = \int_{t_1}^{t_2} dt\ \int d^3x\ \mathcal{L} = \int d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i)\ , \tag{1.1}$$

where $\mathcal{L}$ is the lagrangian density, defined by

$$L = \int d^3x\ \mathcal{L}\ .$$

In natural units, the dimensional analysis is

$$[S] = 0\ [d^4x] = -4\ [\mathcal{L}] = 4\ .$$

1.1 The principle of stationary action

The dynamics of the system can be determined by the principle of stationary action.

Principle 1.1

The system evolve from an initial configuration at time t_1 to a final configuration at time t_2 along a path in configuration space which extremises the action (1.1), i.e.

$$\delta S = 0\ . \tag{1.2}$$

with the additional conditions

1. *fields vanish at spatial infinity*

$$\phi_i(t, \mathbf{x}) \to 0 \quad |\mathbf{x}| \to \infty\ ,$$

hence

$$\delta\phi_i(t, \infty) = 0\ , \tag{1.3}$$

2. *fields vanish at time extremes*

$$\delta\phi_i(t_1, \mathbf{x}) = \delta\phi_i(t_2, \mathbf{x}) = 0 \ . \tag{1.4}$$

The equation of motion of the system are the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} = 0 \ . \tag{1.5}$$

Proof. The variation of the action is

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\partial_\mu \phi_i \right) = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial_\mu \delta\phi_i \right) ,$$

where

$$\delta\phi_i = {\phi'}_i(x) - \phi_i(x) \ ,$$

and

$$\delta\partial_\mu\phi_i(x) = \partial_\mu {\phi'}_i - \partial_\mu \phi(x) = \partial_\mu({\phi'}_i(x) - \phi_i(x)) = \partial_\mu \delta\phi(x) \ .$$

By integration by parts, we obtain

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial_\mu \delta\phi_i \right) = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) + \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) .$$

Notice that the last term is a total derivative and it vanishes at the boundary by the condition (1.4) and (1.3)

$$\int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) = 0 \ .$$

Hence, we find

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) ,$$

and, by the principle of stationary action (1.2)

$$\int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i \right) = 0 \ .$$

Finally, since $\delta_i \phi$ is arbitrary, we obtain (1.5)

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} = 0 \ .$$

q.e.d.

In order to quantise the theory, we need the hamiltonian formalism.

Definition 1.1
The conjugate field $\phi^i(x)$ associated to the field ϕ_i is

$$\phi^i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(x)}$$

The hamiltonian density is given by the Legendre transformation

$$\mathcal{H} = \Phi^i \dot{\phi}_i - \mathcal{L}$$

where the hamiltonian is

$$H = \int d^3x \ \mathcal{H}$$

Chapter 2

Noether's theorem

Symmetries are fundamental in quantum field theory and they can be classified into

1. spacetime
   a) global
      i. continuous (Poincarè)
      ii. discrete (Parity, time reversal)
   b) local
      i. continuous (General relativity)
      ii. discrete (Parity coordinate dependent)
2. internal
   a) global
      i. continuous (Flavour)
      ii. discrete ($\mathbb{Z}_2$)
   b) local
      i. continuous ($SU(3) \times SU(2) \times U(1)$)
      ii. discrete ($\mathbb{Z}_2(x)$)

Through the Noether's theorem, we can associate conserved quantities to continuous symmetries.

Theorem 2.1 (Noether's)
Every continuous symmetry $\delta\phi_i$ of the action (1.1) give rise to a conserved current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi - K^\mu \tag{2.1}$$

such that it satisfies a continuity equation

$$\partial_\mu J^\mu = 0 \tag{2.2}$$

Proof. We consider an infinitesimal transformation for a continuous symmetry of the system

$$\phi'_i = \phi_i + \delta\phi_i$$

which induces a transformation of the lagrangian

$$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$$

In order to be a symmetry of the system, we require that the action is not invariant, but we allow to be up to a boundary term $K^\mu(\phi_i)$, because the dynamics of the system, i.e. the equations of tmotion, do not change with a boundary term. Hence

$$S' = S + \int \partial_\mu K^\mu(\phi_i)$$

but

$$\delta S = \int \partial_\mu K^\mu(\phi_i) \tag{2.3}$$

Explicitly, we obtain

$$\begin{aligned}
\delta S &= \delta \int d^4x \ \mathcal{L} \\
&= \int d^4x \ \Big(\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\partial_\mu\phi_i\Big) \\
&= \int d^4x \ \Big(\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\partial_\mu\delta\phi_i\Big) \\
&= \int d^4x \ \Big(\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i - \partial_\mu\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\phi_i\Big) + \int d^4x \ \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i\Big) \\
&= \int d^4x \ \delta\phi_i \underbrace{\Big(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\Big)}_{0} + \int d^4x \ \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i\Big) \\
&= \int d^4x \ \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i\Big)
\end{aligned}$$

where we used the fact that partial derivatives and symmetries commute, the equation of motions (1.5) and we integrated by parts. Hence, by requiring that it is a symmetry

$$\delta S = \int d^4x \ \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i\Big) = \int d^4x \partial_\mu K^\mu$$

or equivalently

$$\int d^4x \; \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi_i - K^\mu \Big) = 0$$

Since it is for arbitrary integration, the integrand vanishes and

$$\partial_\mu J^\mu = 0$$

with

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta \phi_i - K^\mu$$

q.e.d.

Notice that every conserved current can be related to a conserved quantity Q by

$$Q = \int_{\mathbb{R}^3} d^3x \; J^0$$

This means that Q is conserved locally, i.e. any charge carrier leaving a finite volume V is associated to a flow of current $\mathbf{J}$ out of the volume.

Proof. Infact, by using (2.2)

$$\begin{aligned} \frac{dQ}{dt} &= \frac{d}{dt} \int_{\mathbb{R}^3} d^3x \; J^0 \\ &= \int_{\mathbb{R}^3} d^3x \; \frac{\partial J^0}{\partial t} \\ &= - \int_{\mathbb{R}^3} d^3x \; \nabla \cdot \mathbf{J} = 0 = - \int_{\partial \mathbb{R}^3} d\mathbf{S} \cdot \mathbf{J} = 0 \end{aligned}$$

where we used the Stoke's theorem and the fact that $\mathbf{J} \to 0$ for $|\mathbf{x}| \to 0$. q.e.d.

Chapter 3

Energy-momentum tensor

Spacetime translations give rise to 4 conserved currents, which corresponds to the conservation of energy and momentum. Infact, we consider an infinitesimal spacetime translation

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu}$$

such that fields change by

$$\phi'_i = \phi_i(x + \epsilon) = \phi(x) + \epsilon^{\mu}\partial_{\mu}\phi_i(x)$$

We considered an active transformation, where there is not a change of frame but fields themselves are indeed translated into new fields such that

$$\phi'_i(x') = \phi(x) = \phi(x' + \epsilon)$$

A passive transformation would have acted as

$$\phi'_i = \phi_i(x - \epsilon)$$

Since the lagrangian is a function of the coordinates via fields, we have the following transformation

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \epsilon^{\mu}\partial_{\mu}\mathcal{L} = \epsilon^{\mu}\partial_{\nu}(\delta^{\nu}{}_{\mu}\mathcal{L})$$

Hence, the boundary term is

$$K^{\mu} = \delta^{\mu}{}_{\nu}\mathcal{L}$$

We apply the Noether's theorem (2.1) and find 4 different conserved currents labelled by ν

$$(J^{\mu})_{\nu} = \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_i}\partial_{\nu}\phi_i - \delta^{\mu}{}_{\nu}\mathcal{L}$$

and we define the energy-momentum tensor, or stress-energy tensor,

$$T^{\mu}{}_{\nu} = (J^{\mu})_{\nu}$$

such that

$$\partial_\mu T^\mu{}_\nu = 0$$

In natural units, the dimensional analysis is

$$T^\mu{}_\nu = [\mathcal{L}] = 4$$

The 4 conserved charges are

$$Q_\nu = \int_{\mathbb{R}^3} d^3x\ (J^0)_\nu = \int_{\mathbb{R}^3} d^3x\ T^0{}_\nu$$

which correspond to the 4-momentum

$$P^\mu = \int_{\mathbb{R}^3} d^3x\ T^{0\mu}$$

In particular, the 0-th component is the energy

$$\begin{aligned} P^0 &= \int d^3x\ T^{00} \\ &= \int d^3x\ \Big(\frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial^0 \phi_i - \delta^{00} \mathcal{L}\Big) \\ &= \int d^3x\ \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}}_{\pi^i} \dot{\phi}_i - \mathcal{L}\Big) = \int d^3x\ (\underbrace{\pi^i \dot{\phi}_i - \mathcal{L}}_{\mathcal{L}}) = \int d^3x\ \mathcal{H} = H \end{aligned} \tag{3.1}$$

such that

$$\frac{dH}{dt} = 0$$

and the j-th components are the momentum

$$\begin{aligned} P^j &= \int d^3x\ T^{0j} \\ &= \int d^3x\ \Big(\frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \underbrace{\partial^j \phi_i}_{-\partial_j \phi_i} - \underbrace{\delta^{0j}}_{0} \mathcal{L}\Big) \\ &= \int d^3x\ \Big(- \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}}_{\pi^i} \partial_j \phi_i\Big) \\ &= - \int d^3x\ \pi^i \partial_j \phi_i \end{aligned} \tag{3.2}$$

such that

$$\frac{dP^i}{dt} = 0$$

Chapter 4

An example: electrodynamics

Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot E = \rho \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \tag{4.1}$$

can be written in covariant form

$$\partial_\mu F^{\mu\nu} = J^\nu \quad \partial_\mu F^{*\mu\nu} = 0$$

where $F^{\mu\nu}$ is the electromagnetic tensor and $F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}$ is its dual.

Furthermore, they can be written in terms of the scalar ϕ and the vector potentials $\mathbf{A}$, defined by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Maxwell's equations do not change under this transformation.

Proof. Maybe in the future. q.e.d.

In covariant form, we can write the electromagnetic tensor as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Maxwell's equations can be seen as the equations of motion of the electromagnetic lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J_\mu A^\mu$$

or, equivalenty written in terms of the 4-potential,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 - A_\mu J^\mu$$

Proof. First, we prove that they are equivalent. Maybe in the future.

Second, we prove that it leads to the Maxwell's equations. Maybe in the future.
q.e.d.

In natural units, the dimensional analysis is

$$[F^{\mu\nu}] = 2 \quad [A_\mu] = 1 \quad [J^\mu] = 3$$

The minus sign garanties that the kinetic energy has a positive one

$$-\frac{1}{2}\partial_0 A_i \underbrace{\partial^0}_{\partial_0} \underbrace{A^i}_{-A_i} = \frac{1}{2}\dot{A}_i^2$$

The fourth field A_0 is not a dynamical quantity, since there is no kinetic energy in terms of $\dot{A}_0^2$, because the first $-\frac{1}{2}\partial_0 A_0 \partial^0 A^0$ cancels out with $\frac{1}{2}(\partial_0 A_0)^2$. Therefore, there are only 3 degrees of freedom. However, since electrodynamics is a gauge theory, it is possible to restrict to only 2 degrees of freedom, which correspond to the 2 transversal polarisations direction of an electromagnetic wave.

The energy-momentum tensor is

$$T^{\mu\nu} = \partial^\nu A^\mu \partial_\rho A^\rho - \partial^\mu A^\rho \partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{4.2}$$

Proof. Maybe in the future. q.e.d.

However, the first term in (4.2) is not symmetric under change $\mu \leftrightarrow \nu$, but in order to take into account general relativity, this tensor must be symmetric, since $R_{\mu\nu}$ and $g_{\mu\nu}$ are so in

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

To do it, we defined a new energy-momentum tensor starting from the old one with the addition of an extra term: the partial derivative of a 3 indices anti-symmetric in the first 2 indices tensor $K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}$

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

This garanties that it is conserved as well

$$\partial_\mu \tilde{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \underbrace{\partial_\mu \partial_\lambda}_{symm} \underbrace{K^{\lambda\mu\nu}}_{anti} = \partial_\mu T^{\mu\nu} = 0$$

In the electromagnetic case, we choose K to be

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$$

and the symmetric energy-momentum tensor becomes

$$\tilde{T}^{\mu\nu} = F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4}\eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

which is called the Belifante-Rosenfeld tensor.

Proof. Maybe in the future. q.e.d.

The energy density is

$$\mathcal{E} = \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2)$$

Proof. Maybe in the future. q.e.d.

The momentum density is

$$\mathcal{P}^i = (\mathbf{E} \times \mathbf{B})^i$$

Proof. Maybe in the future. q.e.d.

Part II

Klein-Gordon theory

Chapter 5

Canonical or second quantisation

In Schoedinger picture, where states evolve in time while operators do not, recall that standard quantisation from classical mechanics to quantum mechanics works in this way:

1. hamiltonian formalism $H \mapsto$ hamiltonian operator $\hat{H}$,

2. generalised coordinates and conjugate momenta $(q_i, p^i = \frac{\partial L}{\partial \dot{q}_i}) \mapsto$ operators on a Hilbert space $\hat{q}_i$ and $\hat{p}^i$,

3. Poissons brackets $\{q_i, p^j\} = \delta_i^{\ j}$ and $\{p^i, p^j\} = \{q_i, q_j\} = 0 \mapsto$ commutators $[q_i, p^j] = i\delta_i^{\ j}$ and $[p^i, p^j] = [q_i, q_j] = 0$.

Similarly, the second quantisation from classical field theory to quantum field theory works in this way:

1. fields and conjugate fields $(\varphi_i(t, \mathbf{x}), \pi^i(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i}) \mapsto$ operators on a Fock space $\hat{\varphi}_i(t, \mathbf{x})$ and $\hat{\pi}^i(t, \mathbf{x})$,

2. canonical commutation relations $[\hat{\varphi}_i(t, \mathbf{x}), \hat{\pi}^j(t, \mathbf{y})] = i\delta_i^{\ j}\delta^3(\mathbf{x}-\mathbf{y})$ and $[\hat{\varphi}_i(t, \mathbf{x}), \hat{\varphi}_j(t, \mathbf{y})] = [\hat{\pi}^i(t, \mathbf{x}), \hat{\pi}^j(t, \mathbf{y})] = 0$.

States which live in the Fock state $|\psi\rangle$ evolve in time via the Schoedinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

where $|\psi\rangle$ is a wave functional such that its modulus square gives the density probability to find the field in a certain configuration and $\hat{H}(\varphi_i(t, \mathbf{x}), \pi^i(t, \mathbf{x}))$ is an operator, since $\varphi_i(t, \mathbf{x})$ and $\pi^i(t, \mathbf{x})$ are.

In order to solve the theory, we need to find the eigenstates of $\hat{H}$, but it is too difficult expect in the case of a free theory, which the lagrangian is quadratic and the equations od motion are linear and solvable.

5.1 Harmonic oscillator

Recall some feature of the harmonic oscillator.

5.2 Dirac delta

Recall that the integral representation of the Dirac delta is

$$\delta^3(\mathbf{x}-\mathbf{y}) = \int \frac{d^3p}{(2\pi)^3} \exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})) = \int \frac{d^3p}{(2\pi)^3} \exp(-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})) \ . \tag{5.1}$$

Chapter 6

Single real Klein-Gordon field

6.1 Hamiltonian

The simplest relativistic field theory is the Klein-Gordon theory of a single real scalar field chargeless and spinless. Its lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

and its equations of motion are

$$(\Box + m^2)\varphi(x) = 0 \ . \tag{6.1}$$

Proof. Infact, using (1.5)

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} \\ &= \frac{\partial}{\partial\varphi}\Big(\cancel{\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi} - \underbrace{\frac{1}{2}m^2\varphi^2}_{m^2\varphi}\Big) + \partial_\mu\frac{\partial}{\partial\partial_\mu\varphi}\Big(\underbrace{\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi}_{\partial_\mu\partial^\mu\varphi} - \cancel{\frac{1}{2}m^2\varphi^2}\Big) \\ &= \underbrace{\partial_\mu\partial^\mu}_{\Box}\varphi + m^2\varphi \\ &= (\Box + m^2)\varphi \ . \end{aligned}$$

q.e.d.

It is a system of infinitely many degrees of freedom and to decouple them we need to perform a Fourier transform

$$\varphi(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t, \mathbf{p}) \ , \tag{6.2}$$

which in momentum space becomes

$$\Big(\frac{\partial^2}{\partial t^2}+|\mathbf{p}|^2+m^2\Big)\tilde{\varphi}(t,\mathbf{x})=0$$

and its solution is an harmonic oscillator for each $\mathbf{p}$ of frequency

$$\omega_{\mathbf{p}}=\sqrt{|\mathbf{p}|^2+m^2}\ . \tag{6.3}$$

Hence, the most general solution of the Klein-Gordon equation (6.1) is a superposition of simple harmonic oscillators, each vibrating with different frequency and amplitude. To quantise the theory and φ, we need to quantise this set of infinitely decoupled harmonic oscillators.

Proof. We decompose (6.1) into time and space components

$$0=(\Box+m^2)\varphi=(\underbrace{\partial_0}_{\partial^0}\partial^0+\underbrace{\partial_i}_{-\partial^i}\partial^i+m^2)\varphi=((\partial^0)^2-(\partial^i)^2+m^2)\varphi=(\frac{\partial^2}{\partial t^2}-\nabla^2+m^2)\varphi\ ,$$

and we substitute (6.2)

$$\begin{aligned}0&=(\frac{\partial^2}{\partial t^2}-\nabla^2+m^2)\int\frac{d^3p}{(2\pi)^3}\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t,\mathbf{p})\\&=\int\frac{d^3p}{(2\pi)^3}(\frac{\partial^2}{\partial t^2}-\underbrace{\nabla^2}_{-i^2|\mathbf{p}|^2}+m^2)(\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t,\mathbf{p}))\\&=\int\frac{d^3p}{(2\pi)^3}(\frac{\partial^2}{\partial t^2}-i^2|\mathbf{p}|^2+m^2)\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t,\mathbf{p})\\&=\int\frac{d^3p}{(2\pi)^3}(\frac{\partial^2}{\partial t^2}+|\mathbf{p}|^2+m^2)\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t,\mathbf{p})\ ,\end{aligned}$$

where the integrand vanishes with the exponential. Finally, we define the energy (6.3) and we obtain

$$(\frac{\partial^2}{\partial t^2}+\omega_{\mathbf{p}})^2\tilde{\varphi}(t,\mathbf{p})=0\ ,$$

which is indeed the equation of an harmonic oscillator in the form $\ddot{x}+\omega^2x=0$. q.e.d.

By analogy with the simple quantum harmonic oscillator, we define the field operator

$$\hat{\varphi}(\mathbf{x})=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x})+\hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \tag{6.4}$$

and the conjugate operator

$$\hat{\pi}(\mathbf{x})=\int\frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x})-\hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\ , \tag{6.5}$$

such that they satisfies the commutation relations for annihilation and creation operators

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] = 0 \ , \quad [\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \ . \tag{6.6}$$

Therefore, the canonical commutation relations become

$$[\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{y})] = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = 0$$

and

$$[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}) \ .$$

Proof. For the field-field commutator, using (6.6), (6.4) and (5.1)

$$\begin{aligned}
[\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{y})] &= [\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big), \\
&\qquad \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{y}) + \hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{y})\Big)] \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6}\frac{1}{2\sqrt{\omega_{\mathbf{p}}}\omega_{\mathbf{q}}}[\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x}), \\
&\qquad \hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{y}) + \hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{y})] \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6}\frac{1}{2\sqrt{\omega_{\mathbf{p}}}\omega_{\mathbf{q}}}\Big(\underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{0}\exp(i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})) + \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}\exp(i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})) \\
&\qquad + \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p})}\exp(i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})) + \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{0}\exp(i(-\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y}))\Big) \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^3}\frac{1}{2\sqrt{\omega_{\mathbf{p}}}\omega_{\mathbf{q}}}\Big(\underbrace{\delta^3(\mathbf{p}-\mathbf{q})\exp(i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \\
&\qquad - \underbrace{\delta^3(\mathbf{q}-\mathbf{p})\exp(i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\Big(\underbrace{\exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} - \underbrace{\exp(i\mathbf{p}\cdot(-\mathbf{x}+\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})}\Big) = 0 \ .
\end{aligned}$$

For the conjugate-conjugate commutator, using (6.6), (6.5) and (5.1)

$$
\begin{aligned}
[\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] &= [\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^{\dagger}_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big), \\
&\quad \int \frac{d^3q}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\Big)\Big(\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{y}) - \hat{a}^{\dagger}_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{y})\Big)] \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6}\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)[\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^{\dagger}_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x}), \\
&\quad \hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{y}) - \hat{a}^{\dagger}_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{y})] \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6}\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(\underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{0}\exp(i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})) - \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}\exp(i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y} \\
&\quad - \underbrace{[\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p})}\exp(i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})) + \underbrace{[\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}]}_{0}\exp(i(-\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y}))\Big) \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^3}\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(-\underbrace{\delta^3(\mathbf{p}-\mathbf{q})\exp(i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \\
&\quad + \underbrace{\delta^3(\mathbf{q}-\mathbf{p})\exp(i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\Big(-\frac{\omega_{\mathbf{p}}}{2}\Big)\Big(-\underbrace{\exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} + \underbrace{\exp(i\mathbf{p}\cdot(-\mathbf{x}+\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})}\Big) = 0\ .
\end{aligned}
$$

For the field-conjugate commutator, using (6.6), (6.4), (6.5) and (5.1)

$$
\begin{aligned}
[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] &= [\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x}) \Big), \\
&\quad \int \frac{d^3q}{(2\pi)^3} \Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}} \Big) \Big(\hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{y}) - \hat{a}^\dagger_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{y}) \Big)] \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6} \Big(-\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}} \Big) [\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x}), \\
&\quad \hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{y}) - \hat{a}^\dagger_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{y})] \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6} \Big(-\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}} \Big) \Big(\underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{0} \exp(i(\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})) - \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \exp(i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})) \\
&\quad + \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p})} \exp(i(-\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})) - \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{0} \exp(i(-\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})) \Big) \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^3} \Big(-\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}} \Big) \Big(-\underbrace{\delta^3(\mathbf{p}-\mathbf{q}) \exp(i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \\
&\quad - \underbrace{\delta^3(\mathbf{q}-\mathbf{p}) \exp(i(-\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \Big(\frac{i}{2}\Big) \Big(\underbrace{\exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} + \underbrace{\exp(i\mathbf{p}\cdot(-\mathbf{x}+\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} \Big) \\
&= \frac{i}{2} 2\delta^3(\mathbf{x}-\mathbf{y}) = i\delta^3(\mathbf{x}-\mathbf{y}) \ .
\end{aligned}
$$

q.e.d.

The hamiltonian is

$$H = \frac{1}{2}\int d^3x\ (\pi^2 + (\boldsymbol{\nabla}\varphi)^2 + m^2\varphi^2) \ .$$

If we make a function study of the classical hamiltonian, we notice that it has quadratic terms and a minimum at $\varphi_0(t, \mathbf{x}) = const$ which we could consider as the ground state with $\varphi_0 = 0$. Quantising the theory means that we consider quantum (small) fluctuations $\delta\varphi$ around this ground state such that

$$\varphi(t, \mathbf{x}) = \underbrace{\varphi(t, \mathbf{x})_0}_{0} + \delta\varphi(t, \mathbf{x}) \ .$$

The hamiltonian operator in quantum field theory becomes

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{1}{2}\int d^3p\ \omega_{\mathbf{p}} \delta^3(0) \ . \tag{6.7}$$

Proof. Infact, the conjugate field is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi} = \partial_0 \varphi = \dot{\varphi} \tag{6.8}$$

and using (3.1) and (**??**)

$$\begin{aligned}
H &= \int d^3x \; T^{00} \\
&= \int d^3x \; (\pi \underbrace{\dot{\varphi}}_{\pi} - \mathcal{L}) \\
&= \int d^3x \; (\pi^2 - \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} m^2 \varphi^2) \\
&= \int d^3x \; (\pi^2 - \frac{1}{2}\partial_0 \varphi \partial^0 \varphi - \frac{1}{2}\partial_i \varphi \partial^i \varphi + \frac{1}{2} m^2 \varphi^2) \\
&= \int d^3x \; (\pi^2 - \frac{1}{2}\underbrace{\partial_0 \varphi \partial^0 \varphi}_{\pi^2} - \frac{1}{2}\underbrace{\partial_i \varphi \partial^i \varphi}_{-\nabla^2 \varphi} + \frac{1}{2} m^2 \varphi^2) \\
&= \frac{1}{2} \int d^3x \; (\pi^2 + (\boldsymbol{\nabla} \varphi)^2 + m^2 \varphi^2) \; .
\end{aligned}$$

Furthermore, using (6.6), (6.4), (6.5) and (5.1)

$$\begin{aligned}
\hat{H} &= \frac{1}{2} \int d^3x \; \Big(\hat{\pi}^2 + (\boldsymbol{\nabla} \hat{\varphi})^2 + m^2 \hat{\varphi}^2\Big) \\
&= \frac{1}{2} \int d^3x \; \Big(\int \frac{d^3p}{(2\pi)^3} \Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big) \Big(\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&\quad \Big(\int \frac{d^3q}{(2\pi)^3} \Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\Big) \Big(\hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big) \\
&\quad + \nabla \Big(\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&\quad \nabla \Big(\int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} \Big(\hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big) \\
&\quad + m^2 \Big(\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&\quad \Big(\int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} \Big(\hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\ \Big(\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) - \hat{a}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}) \\
&\quad - \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) + \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})\Big) \\
&\quad + \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(i\mathbf{p}\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - i\mathbf{p}\hat{a}^{\dagger}_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\cdot \\
&\quad \Big(i\mathbf{q}\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) - i\mathbf{q}\hat{a}^{\dagger}_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big) \\
&\quad + m^2\frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) + \hat{a}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}) \\
&\quad + \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) + \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\Big(\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})} - \hat{a}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} \\
&\quad - \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} + \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})}\Big) \\
&\quad + \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(-\mathbf{p}\cdot\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})} + \mathbf{p}\cdot\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} \\
&\quad + \mathbf{p}\cdot\mathbf{q}\hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} - \mathbf{p}\cdot\mathbf{q}\hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})}\Big) \\
&\quad + \frac{m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})} + \hat{a}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} \\
&\quad + \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} + \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^3p\ d^3q}{(2\pi)^3}\Big(\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \\
&\quad - \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}}\Big) \\
&\quad + \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(-\mathbf{p}\cdot\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}} + \mathbf{p}\cdot\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \\
&\quad + \mathbf{p}\cdot\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} - \mathbf{p}\cdot\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}}\Big) \\
&\quad + \frac{m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \\
&\quad + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}}\Big)\Big) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\Big(\Big(-\frac{\omega_{\mathbf{p}}}{2}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} - \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\Big) \\
&\quad + \Big(\frac{|\mathbf{p}|^2}{2\omega_{\mathbf{p}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\Big) \\
&\quad + \Big(\frac{m^2}{2\omega_{\mathbf{p}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\Big)\Big) \\
&= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\mathbf{p}}}\Big((\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}})\underbrace{(-\omega_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2)}_{0} \\
&\quad + (\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}})\underbrace{(\omega_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2)}_{2\omega_{\mathbf{p}}^2}\Big) \\
&= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{2\omega_{\mathbf{p}}^{\cancel{2}}}{\cancel{\omega_{\mathbf{p}}}}(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\underbrace{\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}}_{[\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{p}}]+\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\underbrace{[\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p})} + 2\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \\
&= \frac{1}{2}\int d^3p\ \omega_{\mathbf{p}}\delta^3(0) + \int \frac{d^3p}{(2\pi)^3}\ \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\ ,
\end{aligned}
$$

where we have used the fact that $\omega_{-\mathbf{p}} = \sqrt{|-\mathbf{p}|^2 + m^2} = \sqrt{|\mathbf{p}|^2 + m^2} = \omega_{\mathbf{p}}$.

q.e.d.

The first term of (6.7) counts simply how what is the relativistic energy of each particle $\omega_{\mathbf{p}}$ and through the number operator $\hat{N}_{\mathbf{p}} = \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}$ and the integral, we sum all over the possible value of $\mathbf{p}$. However, most of them may be zero and we do not have to worry about divergences.

6.2 Vacuum energy

Things are different if we look at the second term of (6.7), beacuse, in analogy with the energy of the single harmonic oscillator, we interpret it as the energy of the vacuum and it diverges for two reasons

1. infrared divergence, i.e.
$$\delta^3(0) \to \infty \ ,$$
2. ultraviolet divergence, i.e. for $|\mathbf{p}| \to \infty$
$$\int d^3p \ \omega_{\mathbf{p}} \to \infty \ ,$$
since for $|\mathbf{p}| \to \infty$
$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2} \simeq |\mathbf{p}| \ .$$

This can be better understood by applying the hamiltonian operator to the vacuum state $|0\rangle$, i.e. the state such that it is annihilated by all the annihilation operators is for all $\mathbf{p}$
$$\hat{a}_{\mathbf{p}}|0\rangle = 0 \quad \forall \mathbf{p} \ .$$
Therefore
$$\hat{H}|0\rangle = E_0|0\rangle = \infty|0\rangle$$
and the vaccum energy is infinite.

Proof. Infact, using (6.7)
$$\hat{H}|0\rangle = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\underbrace{\hat{a}_{\mathbf{p}}|0\rangle}_{0} + \underbrace{\left(\frac{1}{2}\int d^3p \ \omega_{\mathbf{p}}\delta^3(0)\right)}_{\infty}|0\rangle = \infty|0\rangle = E_0|0\rangle \ .$$

q.e.d.

IR divergence

The infrared divergence is due to the fact that space is infinitely large. This means that in every point of spacetime there is an harmonic oscilators. To prove this, consider a box of sides L and periodic boundary conditions for the fields. The volume of the box is just the Dirac delta inside the integrand of the energy vacuum. Infact

$$(2\pi)^3\delta^3(0) = \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x\ \exp(-i\mathbf{p}\cdot\mathbf{x})\Big|_{\mathbf{p}=0} = \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x = L^3 = V\ .$$

This divergence can be removed by studying energy densities instead of pure energies.

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{2}\ .$$

UV divergence

However, still the energy density is infinite because of the ultraviolet divergence, since for $|\mathbf{p}| \to \infty$

$$\mathcal{E}_0 \to \infty\ .$$

The reason is the following: we made a strong assumption considering the theory valid for any large value of energy and now we have found where the theory breaks, since this divergence arises indeed from the fact that our theory is not valid for arbitrarily high energies. What we need to do id to introduce a cut-off, i.e. a maximum energy after which the theory is not anymore valid. Since gravity cannot be neglected and becomes strongly coupled at Planck mass $M_P \simeq 10^{19}GeV$, we therefore set the cut-off at this energy.

Computationally, we measure only energy differences between excited estates, which are particles, and the vacuum energy, which becomes irrelevant and it can be set to zero. This procedure is called *normal ordering*. We define a new hamiltonian operator

$$:\hat{H}: \ = \hat{H} - E_0 = \hat{H} - \langle 0|\hat{H}|0\rangle\ ,$$

such that

$$:\hat{H}:|0\rangle = \underbrace{\hat{H}|0\rangle}_{E_0|0\rangle} - E_0|0\rangle = 0\ .$$

The difference between $\hat{H}$ and $:\hat{H}:$ is due to an ambiguity in going from classical to quantum theory. Infact, normal ordering means to set a rule to order annihilation and creation operators: all annihilation operators are pleced to the right and, consequently, creation operatore to the left (dagger always first). We emphasise that in the interaction theory, vaccume energy cannot be anymore set to zero.

As we said, different ordering in the classical hamiltonians bring different hamiltonian operators. Infact, if we rewrite the hamiltonian of the classical harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}\omega^2 q^2 = \frac{1}{2}(\omega q - ip)(\omega q + ip) \ ,$$

we notice that the first one leads us to

$$\hat{H} = \omega(\hat{a}^\dagger \hat{a} + \frac{\mathbb{I}}{2}) \ ,$$

while the second one to

$$\hat{H} = \omega a^\dagger \hat{a} \ .$$

Proof. For the first hamiltonian

$$\begin{aligned}
\hat{H} &= \frac{1}{2}(-i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger))^2 + \frac{1}{2}\omega^2(\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger))^2 \\
&= -\frac{\omega}{4}(\cancel{\hat{a}^2} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \cancel{(\hat{a}^\dagger)^2}) + \frac{\omega}{4}(\cancel{\hat{a}^2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \cancel{(\hat{a}^\dagger)^2}) \\
&= \frac{\omega}{4}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\
&= \frac{\omega}{2}(\underbrace{\hat{a}\hat{a}^\dagger}_{[\hat{a},\hat{a}^\dagger]+\hat{a}^\dagger\hat{a}} + \hat{a}^\dagger\hat{a}) \\
&= \frac{\omega}{2}(\underbrace{[\hat{a},\hat{a}^\dagger]}_{\mathbb{I}} + 2\hat{a}^\dagger\hat{a}) \\
&= \omega(\frac{\mathbb{I}}{2} + \hat{a}^\dagger\hat{a}) \ ,
\end{aligned}$$

while for the second hamiltonian

$$\begin{aligned}
\hat{H} &= \frac{1}{2}\Big(\omega\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger) - i(-i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger))\Big)\Big(\omega\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger) + i(-i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger))\Big) \\
&= \frac{\omega}{4}(\cancel{\hat{a}} + \hat{a}^\dagger - \cancel{\hat{a}} + \hat{a}^\dagger)(\hat{a} + \cancel{\hat{a}^\dagger} + \hat{a} - \cancel{\hat{a}^\dagger}) \\
&= \omega\hat{a}^\dagger\hat{a} \ .
\end{aligned}$$

q.e.d.

Finally, the normal ordered hamiltonian of the Klein-Gordon theory is

$$:\hat{H}: \ = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ . \tag{6.9}$$

Proof. Infact, since

$$\hat{H} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \ ,$$

we have

$$:\hat{H}: \ = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ .$$

q.e.d.

Furthermore, by analogy of the harmonic oscillator, the hamiltonian (6.7) and the annihilation and creation operators satisfies the commutation relations

$$[\hat{H}, \hat{a}_{\mathbf{p}}] = -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ , \quad [\hat{H}, \hat{a}^\dagger_{\mathbf{p}}] = \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} \ .$$

Proof. For the first commutator

$$\begin{aligned}
[\hat{H}, \hat{a}_{\mathbf{p}}] &= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}[\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{p}}] \\
&= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}(\hat{a}^\dagger_{\mathbf{q}}\underbrace{[\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{p}}]}_{0} + \underbrace{[\hat{a}^\dagger_{\mathbf{q}}, \hat{a}_{\mathbf{p}}]}_{-(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}\hat{a}_{\mathbf{q}}) \\
&= -\int \frac{d^3q}{\cancel{(2\pi)^3}}\omega_{\mathbf{q}}\cancel{(2\pi)^3}\delta^3(\mathbf{p}-\mathbf{q})\hat{a}_{\mathbf{q}} \\
&= -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ .
\end{aligned}$$

For the second commutator

$$\begin{aligned}
[\hat{H}, \hat{a}^\dagger_{\mathbf{p}}] &= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}[\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{p}}] \\
&= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}(\hat{a}^\dagger_{\mathbf{q}}\underbrace{[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} + \underbrace{[\hat{a}^\dagger_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{p}}]}_{0}\hat{a}_{\mathbf{q}}) \\
&= \int \frac{d^3q}{\cancel{(2\pi)^3}}\omega_{\mathbf{q}}\cancel{(2\pi)^3}\delta^3(\mathbf{p}-\mathbf{q})\hat{a}^\dagger_{\mathbf{q}} \\
&= \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} \ .
\end{aligned}$$

q.e.d.

The momentum operator is defined as

$$\hat{\mathbf{P}} = -\int d^3x \ \hat{\pi}\boldsymbol{\nabla}\hat{\varphi} = \int \frac{d^3p}{(2\pi)^3}\mathbf{p}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ . \tag{6.10}$$

Proof. Infact, using (3.2)

$$\begin{aligned}\hat{\mathbf{P}} &= \int d^3x\ T^{0i} \\ &= \int d^3x\ \hat{\pi}\boldsymbol{\nabla}\hat{\varphi}\ .\end{aligned}$$

Furthermore, using (6.6), (6.4), (6.5) and (5.1)

$$\begin{aligned}\hat{\mathbf{P}} = &-\int d^3x\ \Big(\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\ &\nabla\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big) \\ = &-\int d^3x\ \Big(\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\ &\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\Big(i\mathbf{q}\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) - i\mathbf{q}\hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big) \\ = &-\int\frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\Big(-\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\Big)(i\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) - i\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}) \\ &- i\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) + i\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})) \\ = &-\int\frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\Big(\frac{\mathbf{q}}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\Big)(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} \\ &- \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})}) \\ = &-\int\frac{d^3p\ d^3q}{(2\pi)^3}\Big(\frac{\mathbf{q}}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\Big)(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{q}=-\mathbf{p}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} \\ &- \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{q}=-\mathbf{p}}) \\ = &-\int\frac{d^3p}{(2\pi)^3}\Big(\frac{\mathbf{p}}{2}(-\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}) - \frac{\mathbf{p}}{2}(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}})\Big) \\ = &\int\frac{d^3p}{(2\pi)^3}\Big(\frac{\mathbf{p}}{2}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}) + \underbrace{\frac{\mathbf{p}}{2}(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}})}_{0}\Big)\ ,\end{aligned}$$

where in the last row, the second term vanishes because it is an odd function integrated all over $\mathbb{R}^3$. Finally, in normal ordering

$$\hat{\mathbf{P}} = \int\frac{d^3p}{(2\pi)^3}\Big(\frac{\mathbf{p}}{2}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}})\Big) = \int\frac{d^3p}{(2\pi)^3}\mathbf{p}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\ .$$

q.e.d.

6.3 1-particle states

Now, we build the energy eigenstates of a 1-particle state. In analogy with the harmonic oscillator, we require the following properties:

1. the vacuum state is annihilated by all the annihilation operators for all $\mathbf{p}$
$$\hat{a}_{\mathbf{p}}|0\rangle = 0 \quad \forall \mathbf{p} \ ,$$
2. a generic state can be defined by the creation operators actiong on the vacuum
$$|\mathbf{p}\rangle = \hat{a}^\dagger_{\mathbf{p}}|0\rangle \ .$$

The state $|\mathbf{p}\rangle$ is the momentum eigentstate of a single scalar (spinless) particle with mass m. Infact, it is the momentum eigenstate

$$\hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \ ,$$

Proof. Infact, using (6.10)

$$\begin{aligned}
\hat{\mathbf{P}}|\mathbf{p}\rangle &= \hat{\mathbf{P}}\hat{a}^\dagger_{\mathbf{p}}|0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\mathbf{q}\hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}}_{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]+\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}} |0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\mathbf{q}\hat{a}^\dagger_{\mathbf{q}}(\underbrace{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} +\hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{p}})|0\rangle}_{0} \\
&= \int \frac{d^3q}{\cancel{(2\pi)^3}}\mathbf{q}\hat{a}^\dagger_{\mathbf{q}}\cancel{(2\pi)^3}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}}|0\rangle \\
&= \mathbf{p}\hat{a}^\dagger_{\mathbf{p}}|0\rangle \\
&= \mathbf{p}|\mathbf{p}\rangle \ .
\end{aligned}$$

q.e.d.

Furthermore, this states is also the energy eigenstate, since it is a function of the momentum

$$\hat{H}|\mathbf{p}\rangle = E_{\mathbf{p}}|\mathbf{p}\rangle = \omega_{\mathbf{p}}|\mathbf{p}\rangle \ .$$

Proof. Infact, using (6.9)

$$
\begin{aligned}
\hat{H}|\mathbf{p}\rangle &= \hat{H}\hat{a}^\dagger_{\mathbf{p}}|0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}}_{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]+\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}} |0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}}(\underbrace{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} +\hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{p}})|0\rangle}_{0} \\
&= \int \frac{d^3q}{\cancel{(2\pi)^3}}\omega_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}}\cancel{(2\pi)^3} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} |0\rangle \\
&= \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}|0\rangle \\
&= \omega_{\mathbf{p}}|\mathbf{p}\rangle \ .
\end{aligned}
$$

q.e.d.

6.4 n-particle states

We can generalise for a system composed by n particles. The state becomes

$$|\mathbf{p}_1,\ldots \mathbf{p}_n\rangle = \hat{a}^\dagger_{\mathbf{p}_1}\ldots\hat{a}^\dagger_{\mathbf{p}_n}|0\rangle \ .$$

Notice that the state is symmetric under exchange of any two particles, since

$$[\hat{a}^\dagger_{\mathbf{p}_i},\hat{a}^\dagger_{\mathbf{p}_j}] = 0 \ .$$

Proof. For instance, given two particles of momenta $\mathbf{p}$ and $\mathbf{q}$, we have

$$|\mathbf{p},\mathbf{q}\rangle = \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}|0\rangle = \hat{a}^\dagger_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = |\mathbf{q},\mathbf{p}\rangle \ .$$

q.e.d.

This means that the Klein-Gordon theory describes bosons. It is indeed spin-statistics relation and it is a consequence of quantum field theory and the commutation relations imposed to quantise (not quantum mechanics).

A basis of the Fock space is built by all the possible combination of creation operators acting on the vacuum state

$$\{|0\rangle, \hat{a}^\dagger_{\mathbf{p}_1}|0\rangle, \hat{a}^\dagger_{\mathbf{p}_1}\hat{a}^\dagger_{\mathbf{p}_2}|0\rangle,\ldots\}$$

, where $|0\rangle$ is the vacuum state, $\hat{a}^\dagger_{\mathbf{p}_1}|0\rangle$ is the 1-particle state, $\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}^\dagger_{\mathbf{p}_2}|0\rangle$ is the 2-particles state, etc. The total Fock space is

$$\mathcal{F} = \bigoplus_n \mathcal{H}_n$$

where $\mathcal{H}_n$ is the Hilbert space for n particles.

We can define the number operator wich counts the number of particle in a given state

$$\hat{N} = \int \frac{d^3p}{(2\pi)^3} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \ , \tag{6.11}$$

such that

$$\hat{N}|\mathbf{p}_1, \ldots \mathbf{p}_n\rangle = \hat{N}\hat{a}^\dagger_{\mathbf{p}_1} \ldots \hat{a}^\dagger_{\mathbf{p}_n}|0\rangle = n|\mathbf{p}_1, \ldots \mathbf{p}_n\rangle \ .$$

Notice that the particle number is conserved, since

$$[\hat{H}, \hat{N}] = 0 \ .$$

This means that if the system has initially n particles, this number will remain the same. This happens only in a free theory, because interactions move the system between different sectors of the Fock space.

Proof. Infact, using (6.9) and (6.11)

$$\begin{aligned}
[\hat{H}, \hat{N}] = {} & [\int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}, \int \frac{d^3q}{(2\pi)^3} \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{q}}] \\
= {} & \int \frac{d^3p \ d^3q}{(2\pi)^6} \omega_{\mathbf{p}} [\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{q}}] \\
= {} & \int \frac{d^3p \ d^3q}{(2\pi)^6} \omega_{\mathbf{p}} (\hat{a}^\dagger_{\mathbf{p}} [\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{q}}] + [\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{q}}] \hat{a}_{\mathbf{p}}) \\
= {} & \int \frac{d^3p \ d^3q}{(2\pi)^6} \omega_{\mathbf{p}} (\hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}} \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{0} + \hat{a}^\dagger_{\mathbf{p}} \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \hat{a}_{\mathbf{q}} + \hat{a}^\dagger_{\mathbf{q}} \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \hat{a}_{\mathbf{p}} + \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{0} \hat{a}_{\mathbf{q}} \hat{a}_{\mathbf{p}}) \\
= {} & \int \frac{d^3p \ d^3q}{(2\pi)^3} \omega_{\mathbf{p}} (\hat{a}^\dagger_{\mathbf{p}} \underbrace{\delta^3(\mathbf{p} - \mathbf{q})}_{\mathbf{p}=\mathbf{q}} \hat{a}_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}} \underbrace{\delta^3(\mathbf{p} - \mathbf{q})}_{\mathbf{p}=\mathbf{q}} \hat{a}_{\mathbf{p}}) \\
= {} & \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} (\cancel{\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}} - \cancel{\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}}) = 0 \ .
\end{aligned}$$

q.e.d.

Chapter 7

Two real (or complex) Klein-Gordon field

Consider two real Klein-Gordon fields φ_1 and φ_2 with different masses $m_1 \neq m_2$. Their lagrangian is $\forall i = 1, 2$

$$\mathcal{L} = \sum_{i=1}^{2} \left(\frac{1}{2} \partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{1}{2} m_i^2 \varphi_i^2 \right) ,$$

such that the equations of motion are 2 independent Klein-Gordon equations for each field

$$(\Box + m_i^2)\varphi_i(x) = 0 .$$

Since the fields are not interacting, in normal ordering the total hamiltonian operator is

$$\hat{H} = \hat{H}_1 + \hat{H}_2 ,$$

where

$$\hat{H}_i = \int \frac{d^3p}{(2\pi)^3} \omega_{i,\mathbf{p}} \hat{a}^\dagger_{i,\mathbf{p}} \hat{a}_{i,\mathbf{p}}$$

and

$$\omega_{i,\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m_i^2} ,$$

the total momentum operator is

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}_1 + \hat{\mathbf{P}}_2 ,$$

where

$$\hat{\mathbf{P}}_i = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \hat{a}^\dagger_{i,\mathbf{p}} \hat{a}_{i,\mathbf{p}}$$

and the total number operator is

$$\hat{N} = \hat{N}_1 + \hat{N}_2 ,$$

where

$$\hat{N}_i = \int \frac{d^3p}{(2\pi)^3} \hat{a}^\dagger_{i,\mathbf{p}} \hat{a}_{i,\mathbf{p}} \ .$$

For particle states, the vacuum state is

$$\hat{a}_{i,\mathbf{p}}|0\rangle = 0$$

and the action of $\hat{a}^\dagger_{i,\mathbf{p}}$ on it creates a relativistic particle with mass m_i

$$|\mathbf{p}_i\rangle = \hat{a}^\dagger_{i,\mathbf{p}}|0\rangle \ .$$

Furthermore, they are eigenstates of the total operators

$$\hat{H}|\mathbf{p}_i\rangle = \omega_{i,\mathbf{p}}|\mathbf{p}_i\rangle \ , \quad \hat{\mathbf{P}}|\mathbf{p}_i\rangle = \mathbf{p}|\mathbf{p}_i\rangle \ , \quad \hat{N}|\mathbf{p}_i\rangle = 1|\mathbf{p}_i\rangle \ .$$

Notice that they seem degenerate in $\mathbf{p}$ since they have the same momentum, but they can always be distinguished by a measurement of their energy since it is different

$$\omega_{1,\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m_1^2} \neq \omega_{2,\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m_2^2} \ .$$

7.1 Electrical charge via $O(2)$ symmetry

The more interesting case is the equal-mass one $m_1 = m_2$, because it arises a new symmetry of the action. Rewriting the lagrangian in term of a vector and its transpose one

$$\boldsymbol{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \ , \boldsymbol{\varphi}^T = \begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix} \ ,$$

we obtain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\varphi}^T)(\partial^\mu \boldsymbol{\varphi}) - \frac{1}{2} m^2 \boldsymbol{\varphi}^T \boldsymbol{\varphi} \ .$$

Since this lagrangian is invariant by an $O(2)$ rotation in the field space, the Noether's theorem allows us to define a charge operator

$$\hat{Q} = -i \int \frac{d^3p}{(2\pi)^3} (\hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{p}} - \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{p}})$$

where we have not used normal ordering.

Proof. The lagrangian is invariant under an $O(2)$ rotation in the (φ_1, φ_2) plane. Infact, for a rotation

$$\boldsymbol{\varphi}' = R\boldsymbol{\varphi} \ ,$$

we have

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu(\boldsymbol{\varphi}')^T)(\partial^\mu \boldsymbol{\varphi}') - \frac{1}{2}m^2(\boldsymbol{\varphi}')^T\boldsymbol{\varphi}'$$

$$\begin{aligned} &= \frac{1}{2}(\partial_\mu(R\boldsymbol{\varphi})^T)(\partial^\mu R\boldsymbol{\varphi}) - \frac{1}{2}m^2(R\boldsymbol{\varphi})^T R\boldsymbol{\varphi} \\ &= \frac{1}{2}(\partial_\mu\boldsymbol{\varphi}^T)\underbrace{R^T R}_{\mathbb{I}}(\partial^\mu\boldsymbol{\varphi}) - \frac{1}{2}m^2\boldsymbol{\varphi}^T\underbrace{R^T R}_{\mathbb{I}}\boldsymbol{\varphi} \\ &= \frac{1}{2}(\partial_\mu\boldsymbol{\varphi}^T)(\partial^\mu\boldsymbol{\varphi}) - \frac{1}{2}m^2\boldsymbol{\varphi}^T\boldsymbol{\varphi} = \mathcal{L}\ , \end{aligned}$$

since the lagrangian depends only on the lenght of $|\varphi|^2 = \boldsymbol{\varphi}^T\boldsymbol{\varphi}$. Now, we compute the conserved current by considering an infinitesimal transformation matrix

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \simeq \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix}\ ,$$

and for the fields

$$\begin{bmatrix} {\varphi'}_1 \\ {\varphi'}_2 \end{bmatrix} = \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

which implies an infinitesimal transformation of the fields

$$\delta\varphi_1 = {\varphi'}_1 - \varphi_1 = \theta\varphi_2\ , \quad \delta\varphi_2 = {\varphi'}_2 - \varphi_2 = -\theta\varphi_1\ .$$

By the Noether's theorem, the conserved current (2.1) is

$$J^\mu = \underbrace{\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_i}}_{\partial^\mu\varphi_i}\delta\varphi_i = \partial^\mu\varphi_1\delta\varphi_1 + \partial^\mu\varphi_2\delta\varphi_2 = \theta((\partial^\mu\varphi_1)\varphi_2 - (\partial^\mu\varphi_2)\varphi_1)\ ,$$

where $K^\mu = 0$, and conserved charge is

$$Q = \int d^3x\ J^0 = \int d^3x\ ((\partial^0\varphi_1)\varphi_2 - (\partial^0\varphi_2)\varphi_1) = \int d^3x\ (\dot{\varphi}_1\varphi_2 - \dot{\varphi}_2\varphi_1)$$

where we have omitted a constant θ.

Finally, we promote it to charge operator

$$
\begin{aligned}
\hat{Q} &= \int d^3x\ (\hat{\pi}_1\hat{\varphi}_2 - \hat{\pi}_2\hat{\varphi}_1) \\
&= \int d^3x\ \Big(\int \frac{d^3q}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\Big)\Big(\hat{a}_{1,\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) - \hat{a}^\dagger_{1,\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big) \\
&\qquad \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{2,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{2,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\qquad - \int \frac{d^3q}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\Big)\Big(\hat{a}_{2,\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) - \hat{a}^\dagger_{2,\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big) \\
&\qquad \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{1,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{1,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&= -\frac{i}{2}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\Big(\hat{a}_{1,\mathbf{q}}\hat{a}_{2,\mathbf{p}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}+\mathbf{p})} + \hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{p}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}-\mathbf{p})} \\
&\qquad - \hat{a}^\dagger_{1,\mathbf{q}}\hat{a}_{2,\mathbf{p}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}-\mathbf{p})} - \hat{a}^\dagger_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{p}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}+\mathbf{p})} \\
&\qquad - \hat{a}_{2,\mathbf{q}}\hat{a}_{1,\mathbf{p}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}+\mathbf{p})} - \hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{p}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}-\mathbf{p})} \\
&\qquad + \hat{a}^\dagger_{2,\mathbf{q}}\hat{a}_{1,\mathbf{p}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}-\mathbf{p})} + \hat{a}^\dagger_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{p}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}+\mathbf{p})}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{i}{2}\int \frac{d^3p\, d^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\Big(\hat{a}_{1,\mathbf{q}}\hat{a}_{2,\mathbf{p}}\underbrace{\delta^3(\mathbf{q}+\mathbf{p})}_{\mathbf{q}=-\mathbf{p}} + \hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{p}}\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{q}=\mathbf{p}} \\
&\quad - \hat{a}^\dagger_{1,\mathbf{q}}\hat{a}_{2,\mathbf{p}}\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{q}=\mathbf{p}} - \hat{a}^\dagger_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{p}}\underbrace{\delta^3(\mathbf{q}+\mathbf{p})}_{\mathbf{q}=-\mathbf{p}} \\
&\quad - \hat{a}_{2,\mathbf{q}}\hat{a}_{1,\mathbf{p}}\underbrace{\delta^3(\mathbf{q}+\mathbf{p})}_{\mathbf{q}=-\mathbf{p}} - \hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{p}}\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{q}=\mathbf{p}} \\
&\quad + \hat{a}^\dagger_{2,\mathbf{q}}\hat{a}_{1,\mathbf{p}}\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{q}=\mathbf{p}} + \hat{a}^\dagger_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{p}}\underbrace{\delta^3(\mathbf{q}+\mathbf{p})}_{\mathbf{q}=-\mathbf{p}}\Big) \\
&= -\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{1,-\mathbf{p}}\hat{a}_{2,\mathbf{p}} + \hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}} - \hat{a}^\dagger_{1,\mathbf{p}}\hat{a}_{2,\mathbf{p}} - \hat{a}^\dagger_{1,-\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}} \\
&\quad - \hat{a}_{2,-\mathbf{p}}\hat{a}_{1,\mathbf{p}} - \hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}} + \hat{a}^\dagger_{2,\mathbf{p}}\hat{a}_{1,\mathbf{p}} + \hat{a}^\dagger_{2,-\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}}) \\
&= -\frac{i}{2}\Big(\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{1,-\mathbf{p}}\hat{a}_{2,\mathbf{p}} - \hat{a}_{2,-\mathbf{p}}\hat{a}_{1,\mathbf{p}}) + \int \frac{d^3p}{(2\pi)^3}(\hat{a}^\dagger_{2,-\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}} - \hat{a}^\dagger_{1,-\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}}) \\
&\quad + \int \frac{d^3p}{(2\pi)^3}(\hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}} - \hat{a}^\dagger_{1,\mathbf{p}}\hat{a}_{2,\mathbf{p}} - \hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}} + \hat{a}^\dagger_{2,\mathbf{p}}\hat{a}_{1,\mathbf{p}})\Big) \\
&= -\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}} - \underbrace{\hat{a}^\dagger_{1,\mathbf{p}}\hat{a}_{2,\mathbf{p}}}_{\hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}}} - \hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}} + \underbrace{\hat{a}^\dagger_{2,\mathbf{p}}\hat{a}_{1,\mathbf{p}}}_{\hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}}}) \\
&= -\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}} - \hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}} - \hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}} + \hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}}) \\
&= -i\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}} - \hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}})
\end{aligned}
$$

where in the fourth row from end, the first two integrals vanish because they are odd functions since they commute. q.e.d.

It is hermitian

$$\hat{Q}^\dagger = \hat{Q}\ .$$

Proof. Infact,

$$
\begin{aligned}
\hat{Q}^\dagger &= i\int \frac{d^3p}{(2\pi)^3}((\hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}})^\dagger - (\hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}})^\dagger) \\
&= i\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}} - \hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}}) \\
&= -i\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{1,\mathbf{p}}\hat{a}^\dagger_{2,\mathbf{p}} - \hat{a}_{2,\mathbf{p}}\hat{a}^\dagger_{1,\mathbf{p}}) = \hat{Q}\ .
\end{aligned}
$$

q.e.d.

It is conserved by the hamiltonian

$$[\hat{Q}, \hat{H}] = 0 \ .$$

Proof. Infact,

$$\begin{aligned}
[\hat{Q}, \hat{H}] &= [-i \int \frac{d^3p}{(2\pi)^3} (\hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{p}} - \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{p}}), \int \frac{d^3q}{(2\pi)^3} (\omega_{1,\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{q}} \hat{a}_{1,\mathbf{q}} + \omega_{2,\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{q}} \hat{a}_{2,\mathbf{q}})] \\
&= -i \int \frac{d^3p \ d^3q}{(2\pi)^6} [\hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{p}} - \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{p}}, \omega_{1,\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{q}} \hat{a}_{1,\mathbf{q}} + \omega_{2,\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{q}} \hat{a}_{2,\mathbf{q}}] \\
&= -i \int \frac{d^3p \ d^3q}{(2\pi)^6} (\omega_{1,\mathbf{q}} [\hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}} \hat{a}_{1,\mathbf{q}}] - \omega_{1,\mathbf{q}} [\hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}} \hat{a}_{1,\mathbf{q}}] \\
&\quad + \omega_{2,\mathbf{q}} [\hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}} \hat{a}_{2,\mathbf{q}}] - \omega_{2,\mathbf{q}} [\hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}} \hat{a}_{2,\mathbf{q}}]) \\
&= -i \int \frac{d^3p \ d^3q}{(2\pi)^6} (\omega_{1,\mathbf{q}} \hat{a}_{1,\mathbf{p}} [\hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}} \hat{a}_{1,\mathbf{q}}] + \omega_{1,\mathbf{q}} [\hat{a}_{1,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}} \hat{a}_{1,\mathbf{q}}] \hat{a}^\dagger_{2,\mathbf{p}} \\
&\quad - \omega_{1,\mathbf{q}} \hat{a}_{2,\mathbf{p}} [\hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}} \hat{a}_{1,\mathbf{q}}] - \omega_{1,\mathbf{q}} [\hat{a}_{2,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}} \hat{a}_{1,\mathbf{q}}] \hat{a}^\dagger_{1,\mathbf{p}} \\
&\quad + \omega_{2,\mathbf{q}} \hat{a}_{1,\mathbf{p}} [\hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}} \hat{a}_{2,\mathbf{q}}] + \omega_{2,\mathbf{q}} [\hat{a}_{1,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}} \hat{a}_{2,\mathbf{q}}] \hat{a}^\dagger_{2,\mathbf{p}} \\
&\quad - \omega_{2,\mathbf{q}} \hat{a}_{2,\mathbf{p}} [\hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}} \hat{a}_{2,\mathbf{q}}] - \omega_{2,\mathbf{q}} [\hat{a}_{2,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}} \hat{a}_{2,\mathbf{q}}] \hat{a}^\dagger_{1,\mathbf{p}})
\end{aligned}$$

$$
\begin{aligned}
&= -i \int \frac{d^3p\ d^3q}{(2\pi)^6} (\omega_{1,\mathbf{q}} \hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}_{1,\mathbf{q}}]}_{0} + \omega_{1,\mathbf{q}} \hat{a}_{1,\mathbf{p}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}}]}_{0} \hat{a}_{1,\mathbf{q}} \\
&\quad + \omega_{1,\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{q}} \underbrace{[\hat{a}_{1,\mathbf{p}}, \hat{a}_{1,\mathbf{q}}]}_{0} \hat{a}^\dagger_{2,\mathbf{p}} + \omega_{1,\mathbf{q}} \underbrace{[\hat{a}_{1,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \hat{a}_{1,\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{p}} \\
&\quad - \omega_{1,\mathbf{q}} \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}_{1,\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} - \omega_{1,\mathbf{q}} \hat{a}_{2,\mathbf{p}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}}]}_{0} \hat{a}_{1,\mathbf{q}} \\
&\quad - \omega_{1,\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{q}} \underbrace{[\hat{a}_{2,\mathbf{p}}, \hat{a}_{1,\mathbf{q}}]}_{0} \hat{a}^\dagger_{1,\mathbf{p}} - \omega_{1,\mathbf{q}} \underbrace{[\hat{a}_{2,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}}]}_{0} \hat{a}_{1,\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{p}} \\
&\quad + \omega_{2,\mathbf{q}} \hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}_{2,\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} + \omega_{2,\mathbf{q}} \hat{a}_{1,\mathbf{p}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}}]}_{0} \hat{a}_{2,\mathbf{q}} \\
&\quad + \omega_{2,\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{q}} \underbrace{[\hat{a}_{1,\mathbf{p}}, \hat{a}_{2,\mathbf{q}}]}_{0} \hat{a}^\dagger_{2,\mathbf{p}} + \omega_{2,\mathbf{q}} \underbrace{[\hat{a}_{1,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}}]}_{0} \hat{a}_{2,\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{p}} \\
&\quad - \omega_{2,\mathbf{q}} \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}_{2,\mathbf{q}}]}_{0} - \omega_{2,\mathbf{q}} \hat{a}_{2,\mathbf{p}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}}]}_{0} \hat{a}_{2,\mathbf{q}} \\
&\quad - \omega_{2,\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{q}} \underbrace{[\hat{a}_{2,\mathbf{p}}, \hat{a}_{2,\mathbf{q}}]}_{0} \hat{a}^\dagger_{1,\mathbf{p}} - \omega_{2,\mathbf{q}} \underbrace{[\hat{a}_{2,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \hat{a}_{2,\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{p}}) \\
&= -i \int \frac{d^3p\ d^3q}{(2\pi)^6} (\omega_{1,\mathbf{q}} \underbrace{[\hat{a}_{1,\mathbf{p}}, \hat{a}^\dagger_{1,\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \hat{a}_{1,\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{p}} - \omega_{1,\mathbf{q}} \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{p}}, \hat{a}_{1,\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \\
&\quad + \omega_{2,\mathbf{q}} \hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{p}}, \hat{a}_{2,\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} - \omega_{2,\mathbf{q}} \underbrace{[\hat{a}_{2,\mathbf{p}}, \hat{a}^\dagger_{2,\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \hat{a}_{2,\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{p}}) \\
&= -i \int \frac{d^3p\ d^3q}{(2\pi)^3} (\omega_{1,\mathbf{q}} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} \hat{a}_{1,\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{p}} + \omega_{1,\mathbf{q}} \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{q}} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} \\
&\quad - \omega_{2,\mathbf{q}} \hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{q}} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} - \omega_{2,\mathbf{q}} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} \hat{a}_{2,\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{p}}) \\
&= -i \int \frac{d^3p}{(2\pi)^3} (\omega_{1,\mathbf{p}} \hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{p}} + \omega_{1,\mathbf{p}} \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{p}} - \omega_{2,\mathbf{p}} \hat{a}_{1,\mathbf{p}} \hat{a}^\dagger_{2,\mathbf{p}} - \omega_{2,\mathbf{p}} \hat{a}_{2,\mathbf{p}} \hat{a}^\dagger_{1,\mathbf{p}})
\end{aligned}
$$

q.e.d.

Notice that the definition of the charge operator is ambiguous, since we may define a new charge operator

$$\hat{Q}' = c_1 \hat{Q} + c_2 \ ,$$

where $c_1, c_2 \in \mathbb{R}$ and it still satisfy the conservation law (in the Heisenberg picture)

$$\frac{d\hat{Q}'}{dt} = 0 \ .$$

Proof. Infact

$$\frac{d\hat{Q}'}{dt} = \frac{dc_1 Q + c_2}{dt} = c_1 \frac{dQ}{dt} = 0 \ .$$

q.e.d.

The ambiguity of c_1 can be used to set the units, while the one of c_2 can be exploit to go in normal ordering. Infact, setting $c_1 = 1$ and using

$$\hat{Q}|0\rangle = 0 \ ,$$

we obtain

$$\hat{Q}' = c_2 \ .$$

Proof. Infact

$$\hat{Q}'|0\rangle = \hat{Q}|0\rangle + c_2|0\rangle = c_2|0\rangle \ , \tag{7.1}$$

$$\langle 0|\hat{Q}'|0\rangle = c_2\langle 0|0\rangle = c_2$$

and

$$:\hat{Q}': \ = \hat{Q}' - \langle 0|\hat{Q}'|0\rangle = \hat{Q} + c_2 - c_2 = \hat{Q} \ .$$

q.e.d.

We define new ladder operators

$$\hat{a}_{\pm,\mathbf{p}} = \frac{\hat{a}_{1,\mathbf{p}} \pm i\hat{a}_{2,\mathbf{p}}}{\sqrt{2}} \ , \quad \hat{a}^\dagger_{\pm,\mathbf{p}} = \frac{\hat{a}^\dagger_{1,\mathbf{p}} \mp i\hat{a}^\dagger_{2,\mathbf{p}}}{\sqrt{2}} \ ,$$

such that they satisfy

$$[\hat{Q}, \hat{a}_{\pm,\mathbf{p}}] = \mp\hat{a}_{\pm,\mathbf{p}} \ , \quad [\hat{Q}, \hat{a}^\dagger_{\pm,\mathbf{p}}] = \pm\hat{a}^\dagger_{\pm,\mathbf{p}} \ .$$

Proof. For the annihilation commutator

$$
\begin{aligned}
[\hat{Q}, \hat{a}_{\pm,\mathbf{p}}] &= \frac{[\hat{Q}, \hat{a}_{1,\mathbf{p}}] \pm i[\hat{Q}, \hat{a}_{2,\mathbf{p}}]}{\sqrt{2}} \\
&= -\frac{i}{\sqrt{2}} \int \frac{d^3q}{(2\pi)^3} ([\hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{q}} - \hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}_{1,\mathbf{p}}] \pm i[\hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{q}} - \hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}]) \\
&= -\frac{i}{\sqrt{2}} \int \frac{d^3q}{(2\pi)^3} ([\hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{q}}, \hat{a}_{1,\mathbf{p}}] - [\hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}_{1,\mathbf{p}}] \pm i[\hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}] \mp i[\hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}]) \\
&= -\frac{i}{\sqrt{2}} \int \frac{d^3q}{(2\pi)^3} (\hat{a}_{1,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{q}}, \hat{a}_{1,\mathbf{p}}]}_{0} + \underbrace{[\hat{a}_{1,\mathbf{q}}, \hat{a}_{1,\mathbf{p}}]}_{0} \hat{a}^\dagger_{2,\mathbf{q}} - \hat{a}_{2,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}_{1,\mathbf{p}}]}_{-(2\pi)\delta^3(\mathbf{p}-\mathbf{q})} - \underbrace{[\hat{a}_{2,\mathbf{q}}, \hat{a}_{1,\mathbf{p}}]}_{0} \hat{a}^\dagger_{1,\mathbf{q}} \\
&\qquad \pm i\hat{a}_{1,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}]}_{-(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \pm i \underbrace{[\hat{a}_{1,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}]}_{0} \hat{a}^\dagger_{2,\mathbf{q}} \mp i\hat{a}_{2,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}]}_{0} \mp i \underbrace{[\hat{a}_{2,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}]}_{0} \hat{a}^\dagger_{1,\mathbf{q}}) \\
&= -\frac{i}{\sqrt{2}} \int d^3q \; (\hat{a}_{2,\mathbf{q}} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \mp i\hat{a}_{1,\mathbf{q}} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}}) \\
&= -\frac{i}{\sqrt{2}}\hat{a}_{2,\mathbf{p}} - \frac{i}{\sqrt{2}}(\mp i\hat{a}_{1,\mathbf{p}}) \\
&= \frac{\mp\hat{a}_{1,\mathbf{p}} - i\hat{a}_{2,\mathbf{p}}}{\sqrt{2}} \\
&= \mp\frac{\hat{a}_{1,\mathbf{p}} \pm i\hat{a}_{2,\mathbf{p}}}{\sqrt{2}} = \mp\hat{a}_{\pm,\mathbf{p}} \; .
\end{aligned}
$$

For the creation commutator

$$
\begin{aligned}
[\hat{Q}, \hat{a}^\dagger_{\pm,\mathbf{p}}] &= \frac{[\hat{Q}, \hat{a}^\dagger_{1,\mathbf{p}}] \mp i[\hat{Q}, \hat{a}^\dagger_{2,\mathbf{p}}]}{\sqrt{2}} \\
&= -\frac{i}{\sqrt{2}} \int \frac{d^3q}{(2\pi)^3} ([\hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{q}} - \hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}^\dagger_{1,\mathbf{p}}] \mp i[\hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{q}} - \hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}^\dagger_{2,\mathbf{p}}]) \\
&= -\frac{i}{\sqrt{2}} \int \frac{d^3q}{(2\pi)^3} ([\hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{q}}, \hat{a}^\dagger_{1,\mathbf{p}}] - [\hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}^\dagger_{1,\mathbf{p}}] \mp i[\hat{a}_{1,\mathbf{q}}\hat{a}^\dagger_{2,\mathbf{q}}, \hat{a}^\dagger_{2,\mathbf{p}}] \pm i[\hat{a}_{2,\mathbf{q}}\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}^\dagger_{2,\mathbf{p}}]) \\
&= -\frac{i}{\sqrt{2}} \int \frac{d^3q}{(2\pi)^3} (\hat{a}_{1,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{q}}, \hat{a}^\dagger_{1,\mathbf{p}}]}_{0} + \underbrace{[\hat{a}_{1,\mathbf{q}}, \hat{a}^\dagger_{1,\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p})} \hat{a}^\dagger_{2,\mathbf{q}} - \hat{a}_{2,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}^\dagger_{1,\mathbf{p}}]}_{0} - \underbrace{[\hat{a}_{2,\mathbf{q}}, \hat{a}_{1,\mathbf{p}}]}_{0} \hat{a}^\dagger_{1,\mathbf{q}} \\
&\qquad \mp i\hat{a}_{1,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{2,\mathbf{q}}, \hat{a}^\dagger_{2,\mathbf{p}}]}_{0} \mp i \underbrace{[\hat{a}_{1,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}]}_{0} \hat{a}^\dagger_{2,\mathbf{q}} \pm i\hat{a}_{2,\mathbf{q}} \underbrace{[\hat{a}^\dagger_{1,\mathbf{q}}, \hat{a}_{2,\mathbf{p}}]}_{0} \pm i \underbrace{[\hat{a}_{2,\mathbf{q}}, \hat{a}^\dagger_{2,\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \hat{a}^\dagger_{1,\mathbf{q}}) \\
&= -\frac{i}{\sqrt{2}} \int d^3q \; (\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{p}=\mathbf{q}} \hat{a}^\dagger_{2,\mathbf{q}} \pm i \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \hat{a}^\dagger_{1,\mathbf{q}}) \\
&= -\frac{i}{\sqrt{2}} \hat{a}^\dagger_{2,\mathbf{p}} \pm \frac{1}{\sqrt{2}} \hat{a}^\dagger_{1,\mathbf{p}} \\
&= \pm \frac{\hat{a}^\dagger_{1,\mathbf{p}} \mp i\hat{a}^\dagger_{2,\mathbf{p}}}{\sqrt{2}} = \pm \hat{a}^\dagger_{\pm,\mathbf{p}} \; .
\end{aligned}
$$

q.e.d.

Now, we study the spectrum of $\hat{Q}$. We define the eigenstate of the charge operators as

$$\hat{Q}|S\rangle = q|S\rangle \; ,$$

where q is the charge. The action of the ladder operators (**??**) is to add or subtract a unit of charge in the system

$$\hat{Q}\hat{a}^\dagger_{\pm,\mathbf{p}}|S\rangle = (q \pm 1)\hat{a}^\dagger_{\pm,\mathbf{p}} \; .$$

Proof. Infact,

$$\underbrace{\hat{Q}\hat{a}^\dagger_{\pm,\mathbf{p}}}_{[\hat{Q}, a^\dagger_{\pm,\mathbf{p}}] + a^\dagger_{\pm,\mathbf{p}}\hat{Q}} \quad |S\rangle = \underbrace{[\hat{Q}, a^\dagger_{\pm,\mathbf{p}}]}_{\pm a^\dagger_{\pm,\mathbf{p}}} |S\rangle + a^\dagger_{\pm,\mathbf{p}}\hat{Q}|S\rangle = \pm a^\dagger_{\pm,\mathbf{p}}|S\rangle + a^\dagger_{\pm,\mathbf{p}} \underbrace{\hat{Q}|S\rangle}_{q} |S\rangle = (q \pm 1)\hat{a}^\dagger_{\pm,\mathbf{p}} \; .$$

q.e.d.

Since $\hat{Q}$ commute with $\hat{H}$ and $\hat{\mathbf{P}}$ and the three operators are linear combinations of the ladder operators, $|S\rangle$ is a common eigenstate of them. Consider a system of

n particles such that they have charge $\pm q$. Therefore the common eigenstates $|S_n^\pm\rangle$ are defined by

$$\hat{Q}|0\rangle = 0 \ , \quad |S_n^\pm\rangle = \prod_{i=1}^n \hat{a}^\dagger_{\pm,\mathbf{p}}|0\rangle \ ,$$

where $|S_n^+\rangle$ corresponds to n positively-charge particles and $|S_n^-\rangle$ corresponds to n negatively-charge particles, such that they satisfy the properties

$$\hat{H}|S_n^\pm\rangle = \Big(\sum_{i=1}^n \omega_{\mathbf{p}_i}\Big)|S_n^\pm\rangle \ , \quad \hat{\mathbf{P}}|S_n^\pm\rangle = \Big(\sum_{i=1}^n \mathbf{p}_i\Big)|S_n^\pm\rangle \ , \quad \hat{N}|S_n^\pm\rangle = n|S_n^\pm\rangle$$

and

$$\hat{Q}|S_n^\pm\rangle = \pm n|S_n^\pm\rangle \ .$$

This means that a particle states is characterised by its energy, momentum and charge eigestate. Furthermore, notice that the last espression give us the physical interpretation of the charge operator: q is indeed the electric charge such that positively-charged states are particles and negatively-charged states are antiparticles. To be more precise, we could allow all also all the linear combination between them, e.g. the first is q, the second is $-q$, etc.

However, a single Klein-Gordon field can describe only chargeless particles, since for $\varphi_1 = \varphi_2$ we have

$$\hat{Q} = 0 \ .$$

Hence, you need at least two degrees of freedom to descrive particles and antiparticles with non-zero electric charge.

Proof. Infact for $\varphi_1 = \varphi_2 = \varphi$

$$Q = \int d^3x \ (\dot{\varphi}_1\varphi_2 - \dot{\varphi}_2\varphi_1) = \int d^3x \ (\dot{\varphi}^2 - \dot{\varphi}^2) = 0 \ .$$

q.e.d.

7.2 Complex Klein-Gordon field

The description of two real Klein-Gordon fields is equivalent to a complex Klein-Gordon field, since the degrees of freedom are still two. For a the latter, they are

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \ , \quad \varphi^* = \frac{\varphi_1 - i\varphi_2}{\sqrt{2}}$$

and the corresponding lagrangian is

$$\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi \ .$$

Proof. Infact

$$\begin{aligned}\mathcal{L} &= \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi \\ &= \partial_\mu\frac{\varphi_1 - i\varphi_2}{\sqrt{2}}\partial^\mu\frac{\varphi_1 + i\varphi_2}{\sqrt{2}} - m^2\frac{\varphi_1 - i\varphi_2}{\sqrt{2}}\frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\ &= \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_2 - \frac{1}{2}m^2\varphi_1\varphi_2 \ .\end{aligned}$$

q.e.d.

7.3 Electric charge via $U(1)$ symmetry

This lagrangian is invariant with an $U(1)$ rotation (which is equivalent to an $O(2)$ rotation) and the Noether's theorem allows us to define a charge operator

$$\hat{Q} = \int\frac{d^3p}{(2\pi)^3}(\hat{a}^\dagger_{+,\mathbf{p}}\hat{a}_{+,\mathbf{p}} - \hat{a}^\dagger_{-,\mathbf{p}}\hat{a}_{-,\mathbf{p}}) = \hat{N}_+ - \hat{N}_- \ ,$$

where we have used normal ordering and the number operators are

$$\hat{N}_\pm = \int\frac{d^3p}{(2\pi)^3}\hat{a}^\dagger_{\pm,\mathbf{p}}\hat{a}_{\pm,\mathbf{p}} \ .$$

This can be seen by the definition of the field operator

$$\hat{\varphi}(\mathbf{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\Big(\hat{a}_{+,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{-,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \tag{7.2}$$

and the conjugate field operator

$$\hat{\varphi}^*(\mathbf{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\Big(\hat{a}_{-,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{+,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \ , \tag{7.3}$$

where $\hat{a}^\dagger_{+,\mathbf{p}}$ create a particle with momentum $\mathbf{p}$ and energy $\omega_\mathbf{p}$, $\hat{a}^\dagger_{-,\mathbf{p}}$ create an antiparticle with momentum $\mathbf{p}$ and energy $\omega_\mathbf{p}$, $\hat{a}_{+,\mathbf{p}}$ destroys a particle with momentum $\mathbf{p}$ and energy $\omega_\mathbf{p}$ and $\hat{a}_{-,\mathbf{p}}$ destroys an antiparticle with momentum $\mathbf{p}$ and energy $\omega_\mathbf{p}$.

Proof. The lagrangian is invariant under a global $U(1)$ rotation. Infact, for a rotation

$$\varphi' = \exp(i\theta)\varphi \ , \quad \varphi'^* = \exp(-i\theta)\varphi^* \ ,$$

we have

$$\begin{aligned}\mathcal{L}' &= (\partial_\mu(\varphi')^*)(\partial^\mu\varphi') - m^2(\varphi')^*\varphi' \\ &= (\partial_\mu\varphi^*)\cancel{\exp(-i\theta)}\cancel{\exp(i\theta)}(\partial^\mu\varphi) - m^2\varphi^*\cancel{\exp(-i\theta)}\cancel{\exp(i\theta)}\varphi \\ &= \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi = \mathcal{L} \ .\end{aligned}$$

Now, we compute the conserved current by considering an infinitesimal transformation for the fields

$$\varphi' = \exp(i\theta)\varphi \simeq \varphi + i\theta\varphi \ , \quad \varphi'^{*} = \exp(-i\theta)\varphi^{*} \simeq \varphi^{*} - i\theta\varphi^{*} \ ,$$

which implies an infinitesimal transformation of the fields

$$\delta\varphi = \varphi' - \varphi = i\theta\varphi \ , \quad \delta\varphi^{*} = \varphi'^{*} - \varphi^{*} = -\theta\varphi^{*} \ .$$

By the Noether's theorem, the conserved current (2.1) is

$$J^{\mu} = \underbrace{\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi_i}}_{\partial^{\mu}\varphi_i}\delta\varphi_i = \partial^{\mu}\varphi\delta\varphi + \partial^{\mu}\varphi^{*}\delta\varphi^{*} = i\theta((\partial^{\mu}\varphi^{*})\varphi - (\partial^{\mu}\varphi)\varphi^{*}) \ ,$$

where $K^{\mu} = 0$, and conserved charge is

$$Q = \int d^3x \ J^0 = i\int d^3x \ ((\partial^0\varphi^{*})\varphi - (\partial^0\varphi)\varphi^{*}) = i\int d^3x \ (\dot{\varphi}^{*}\varphi - \dot{\varphi}\varphi^{*})$$

where we have omitted a constant θ.

Hence, the charge is

$$Q = i\int d^3x \ (\varphi\dot{\varphi}^{*} - \varphi^{*}\dot{\varphi}) \ .$$

Now, we find the field operator, using (6.4)

$$\begin{aligned}
\varphi &= \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\Big(\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{1,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^{\dagger}_{1,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\qquad + i\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{2,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^{\dagger}_{2,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\underbrace{\frac{\hat{a}_{1,\mathbf{p}} + i\hat{a}_{2,\mathbf{p}}}{\sqrt{2}}}_{\hat{a}_{+,\mathbf{p}}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \underbrace{\frac{\hat{a}^{\dagger}_{1,\mathbf{p}} + i\hat{a}^{\dagger}_{2,\mathbf{p}}}{\sqrt{2}}}_{\hat{a}^{\dagger}_{-,\mathbf{p}}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{+,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^{\dagger}_{-,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)
\end{aligned}$$

and the complex conjugate field operator is

$$\begin{aligned}
\varphi^* &= \frac{\varphi_1 - i\varphi_2}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\Big(\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{1,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{1,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\qquad - i\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{2,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{2,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\underbrace{\frac{\hat{a}_{1,\mathbf{p}} - i\hat{a}_{2,\mathbf{p}}}{\sqrt{2}}}_{\hat{a}_{-,\mathbf{p}}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \underbrace{\frac{\hat{a}^\dagger_{1,\mathbf{p}} - i\hat{a}^\dagger_{2,\mathbf{p}}}{\sqrt{2}}}_{\hat{a}^\dagger_{+,\mathbf{p}}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{-,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{+,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\ .
\end{aligned}$$

Furthermore, the conjugate field is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi}^*$$

and the complex conjugate of the conjugate field is

$$\pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}^*} = \dot{\varphi}\ .$$

Hence

$$\begin{aligned}
\pi^* &= \frac{\pi_1 + i\pi_2}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\Big(\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{1,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{1,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\qquad + i\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{2,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{2,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\underbrace{\frac{\hat{a}_{1,\mathbf{p}} + i\hat{a}_{2,\mathbf{p}}}{\sqrt{2}}}_{\hat{a}_{+,\mathbf{p}}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \underbrace{\frac{\hat{a}^\dagger_{1,\mathbf{p}} + i\hat{a}^\dagger_{2,\mathbf{p}}}{\sqrt{2}}}_{\hat{a}^\dagger_{+,\mathbf{p}}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{+,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{-,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)
\end{aligned}$$

and

$$
\begin{aligned}
\pi &= \frac{\pi_1 - i\pi_2}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\Big(\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{1,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{1,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\qquad - i\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{2,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{2,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\underbrace{\frac{\hat{a}_{1,\mathbf{p}} - i\hat{a}_{2,\mathbf{p}}}{\sqrt{2}}}_{\hat{a}_{-,\mathbf{p}}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \underbrace{\frac{\hat{a}^\dagger_{1,\mathbf{p}} - i\hat{a}^\dagger_{2,\mathbf{p}}}{\sqrt{2}}}_{\hat{a}^\dagger_{+,\mathbf{p}}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{-,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{+,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\ .
\end{aligned}
$$

Putting together

$$
\begin{aligned}
\hat{Q} &= i\int d^3x\ (\hat{\varphi}\hat{\pi} - \hat{\varphi}^*\hat{\pi}^*) \\
&= i\int d^3x\ \Big(\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{+,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{-,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\qquad \int \frac{d^3q}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\Big)\Big(\hat{a}_{-,\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) - \hat{a}^\dagger_{+,\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big) \\
&\qquad - \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{-,\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{+,\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\qquad \int \frac{d^3q}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\Big)\Big(\hat{a}_{+,\mathbf{q}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{-,\mathbf{q}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&= \frac{1}{2}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\Big(\hat{a}_{+,\mathbf{p}}\hat{a}_{-,\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}+\mathbf{p})} - \hat{a}_{+,\mathbf{p}}\hat{a}^\dagger_{+,\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}-\mathbf{p})} \\
&\qquad + \hat{a}^\dagger_{-,\mathbf{p}}\hat{a}_{-,\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}-\mathbf{p})} - \hat{a}^\dagger_{-,\mathbf{p}}\hat{a}^\dagger_{+,\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}+\mathbf{p})} \\
&\qquad - \hat{a}_{-,\mathbf{p}}\hat{a}_{+,\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}+\mathbf{p})} + \hat{a}_{-,\mathbf{p}}\hat{a}^\dagger_{-,\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}-\mathbf{p})} \\
&\qquad - \hat{a}^\dagger_{+,\mathbf{p}}\hat{a}_{+,\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}-\mathbf{p})} + \hat{a}^\dagger_{+,\mathbf{p}}\hat{a}^\dagger_{-,\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{q}+\mathbf{p})}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^3p\ d^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\Big(\hat{a}_{+,\mathbf{p}}\hat{a}_{-,\mathbf{q}}\underbrace{\delta^3(\mathbf{q}+\mathbf{p})}_{\mathbf{q}=-\mathbf{p}} - \hat{a}_{+,\mathbf{p}}\hat{a}^\dagger_{+,\mathbf{q}}\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{q}=\mathbf{p}} \\
&\quad + \hat{a}^\dagger_{-,\mathbf{p}}\hat{a}_{-,\mathbf{q}}\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{q}=\mathbf{p}} - \hat{a}^\dagger_{-,\mathbf{p}}\hat{a}^\dagger_{+,\mathbf{q}}\underbrace{\delta^3(\mathbf{q}+\mathbf{p})}_{\mathbf{q}=-\mathbf{p}} \\
&\quad - \hat{a}_{-,\mathbf{p}}\hat{a}_{+,\mathbf{q}}\underbrace{\delta^3(\mathbf{q}+\mathbf{p})}_{\mathbf{q}=-\mathbf{p}} + \hat{a}_{-,\mathbf{p}}\hat{a}^\dagger_{-,\mathbf{q}}\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{q}=\mathbf{p}} \\
&\quad - \hat{a}^\dagger_{+,\mathbf{p}}\hat{a}_{+,\mathbf{q}}\underbrace{\delta^3(\mathbf{q}-\mathbf{p})}_{\mathbf{q}=\mathbf{p}} + \hat{a}^\dagger_{+,\mathbf{p}}\hat{a}^\dagger_{-,\mathbf{q}}\underbrace{\delta^3(\mathbf{q}+\mathbf{p})}_{\mathbf{q}=-\mathbf{p}}\Big) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{+,\mathbf{p}}\hat{a}_{-,-\mathbf{p}} - \hat{a}_{+,\mathbf{p}}\hat{a}^\dagger_{+,\mathbf{p}} + \hat{a}^\dagger_{-,\mathbf{p}}\hat{a}_{-,\mathbf{p}} - \hat{a}^\dagger_{-,\mathbf{p}}\hat{a}^\dagger_{+,-\mathbf{p}} \\
&\quad - \hat{a}_{-,\mathbf{p}}\hat{a}_{+,-\mathbf{p}} + \hat{a}_{-,\mathbf{p}}\hat{a}^\dagger_{-,\mathbf{p}} - \hat{a}^\dagger_{+,\mathbf{p}}\hat{a}_{+,\mathbf{p}} + \hat{a}^\dagger_{+,\mathbf{p}}\hat{a}^\dagger_{-,-\mathbf{p}}) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}(\hat{a}_{+,\mathbf{p}}\hat{a}_{-,-\mathbf{p}} - \hat{a}_{-,\mathbf{p}}\hat{a}_{+,-\mathbf{p}}) + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}(\hat{a}^\dagger_{+,\mathbf{p}}\hat{a}^\dagger_{-,-\mathbf{p}} - \hat{a}^\dagger_{-,\mathbf{p}}\hat{a}^\dagger_{+,-\mathbf{p}}) \\
&\quad + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}(\hat{a}^\dagger_{-,\mathbf{p}}\hat{a}_{-,\mathbf{p}} + \hat{a}_{-,\mathbf{p}}\hat{a}^\dagger_{-,\mathbf{p}} - \hat{a}_{+,\mathbf{p}}\hat{a}^\dagger_{+,\mathbf{p}} - \hat{a}^\dagger_{+,\mathbf{p}}\hat{a}_{+,\mathbf{p}})\ ,
\end{aligned}
$$

where in the last row, the first two integrals vanish because they are odd functions since they commute. Finally, in normal ordering, we obtain

$$
\hat{Q} = \int \frac{d^3p}{(2\pi)^3}(\hat{a}^\dagger_{-,\mathbf{p}}\hat{a}_{-,\mathbf{p}} - \hat{a}^\dagger_{+,\mathbf{p}}\hat{a}_{+,\mathbf{p}})\ .
$$

q.e.d.

We remark that this is possible because the theory is free. If there are interactions, the number operators are not anymore conserved but the charge operator still is: interactions create and destroy particles and antiparticles under the constrain of conserved total charge.

Chapter 8

Manifestly Lorentz covariance

8.1 Lorentz covariance

The vacuum state is normalised

$$\langle 0|0\rangle = 1 \ ,$$

while 1-particle states satisfiy the orthogonality relation

$$\langle \mathbf{p}|\mathbf{q}\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$$

and the completeness relation

$$\mathbb{I} = \int \frac{d^3p}{(2\pi)^3}|\mathbf{p}\rangle\langle\mathbf{p}| \ ,$$

where $\mathbb{I}$ is the identity operator.

Proof. Maybe in the future. q.e.d.

However, we want Lorentz covariance, since the identity operator is so but the right side of the completeness relation is not, given that the measure $\int d^3p$ and the projector $|\mathbf{p}\rangle\langle\mathbf{p}|$ are not separately so. We know that $\in d^4p$ is Lorentz covariant, because

$$d^4p' = \underbrace{|\det\Lambda|}_{1} d^4p = d^4p \ .$$

Therefore, we change the orthogonality relation into

$$\langle p|q\rangle = (2\pi)^3 2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}\delta^3(\mathbf{p}-\mathbf{q})$$

and the completeness relation into

$$\mathbb{I} = \int \frac{d^4p}{(2\pi)^3}\delta(p_0^2-|\mathbf{p}|^2-m^2)\theta(p_0)|\mathbf{p}\rangle\langle\mathbf{p}| \ ,$$

where $p_0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ and the manifestly invariant states are

$$|p\rangle = \sqrt{2E_{\mathbf{p}}}|\mathbf{p}\rangle \ . \tag{8.1}$$

Proof. Maybe in the future. q.e.d.

8.2 Heisenberg picture

Classical theory is Lorentz-invariant since the lagrangian $\mathcal{L}$ is manifestly so. However, so far in the Schroedinger picture, we worked at a preferred time in which field operators are $\hat{\varphi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$. The time evolution is governed by the Schroedinger equation

$$i\frac{d}{dt}|\mathbf{p}(t)\rangle = \hat{H}|\mathbf{p}(t)\rangle \ ,$$

in which a state evolves as

$$|\mathbf{p}(t)\rangle = \exp(-iE_{\mathbf{p}}t)|\mathbf{p}\rangle \ .$$

In Heisenberg's picture, an operator is related to the Schroedinger's one as

$$\hat{O}_H = \exp(i\hat{H}t)\hat{O}_S \exp(-i\hat{H}t) \ ,$$

where its time evolution is governed by the Heisenberg equation

$$\frac{d}{d\hat{O}_H} = i[\hat{H}, \hat{O}_H] \ .$$

Proof. In fact

$$\begin{aligned}
\frac{d}{dt}\hat{O}_H &= \frac{d}{dt}\Big(\exp(i\hat{H}t)\hat{O}_S \exp(-i\hat{H}t)\Big) \\
&= \frac{d}{dt}\Big(\exp(i\hat{H}t)\Big)\hat{O}_S \exp(-i\hat{H}t) + \exp(i\hat{H}t)\cancel{\frac{d}{dt}\Big(\hat{O}_S\Big)} \exp(-i\hat{H}t) + \exp(i\hat{H}t)\hat{O}_S\frac{d}{dt}\Big(\exp(-i\hat{H}t)\Big) \\
&= i\hat{H}\underbrace{\exp(i\hat{H}t)\hat{O}_S \exp(-i\hat{H})}_{\hat{O}_H} - i\underbrace{\exp(i\hat{H}t)\hat{O}_S \exp(-i\hat{H})}_{\hat{O}_H}\hat{H} \\
&= i\hat{H}\hat{O}_H - i\hat{O}_H\hat{H} \\
&= i[\hat{H}, \hat{O}_H] \ .
\end{aligned}$$

q.e.d.

Therefore, in Schoredinger picture we have $\hat{\varphi}(\mathbf{x})$ while in Heisenberg picture we have $\hat{\varphi}(x)$, where they agree at $t = 0$. The commutation relation at equal time t becomes

$$[\hat{\varphi}(t,\mathbf{x}),\hat{\varphi}(t,\mathbf{y})] = [\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{y})] = 0 \ , \quad [\hat{\varphi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}) \ .$$

The time evolution of $\hat{\varphi}(x)$ is

$$\frac{\partial}{\partial t}\hat{\varphi}(x) = \hat{\pi}(x) \ .$$

Proof. In fact

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\varphi}(x) &= i[\hat{H},\hat{\varphi}(x)] \\
&= i[\frac{1}{2}\int d^3y(\hat{\pi}^2(t,\mathbf{y}) + \nabla^2\hat{\varphi}(t,\mathbf{y}) + m^2\hat{\varphi}^2(t,\mathbf{y})),\hat{\varphi}(t,\mathbf{x})] \\
&= \frac{i}{2}\int d^3y\Big(\underbrace{[\hat{\pi}^2(t,\mathbf{y}),\hat{\varphi}(t,\mathbf{x})]}_{\pi(t,\mathbf{y})[\hat{\pi}(t,\mathbf{y}),\hat{\varphi}(t,\mathbf{x})]+[\hat{\pi}(t,\mathbf{y}),\hat{\varphi}(t,\mathbf{x})]\pi(t,\mathbf{y})} + \underbrace{[\nabla^2\hat{\varphi}(t,\mathbf{y}),\hat{\varphi}(t,\mathbf{x})]}_{0} + m^2\underbrace{[\hat{\varphi}^2(t,\mathbf{y}),\hat{\varphi}(t,\mathbf{x})]}_{0}\Big) \\
&= \frac{i}{2}\int d^3y\Big(\pi(t,\mathbf{y})\underbrace{[\hat{\pi}(t,\mathbf{y}),\hat{\varphi}(t,\mathbf{x})]}_{-i\delta^3(\mathbf{x}-\mathbf{y})} + \underbrace{[\hat{\pi}(t,\mathbf{y}),\hat{\varphi}(t,\mathbf{x})]}_{-i\delta^3(\mathbf{x}-\mathbf{y})}\pi(t,\mathbf{y})\Big) \\
&= \frac{1}{2}\int d^3y\Big(\pi(t,\mathbf{y})\underbrace{\delta^3(\mathbf{x}-\mathbf{y})}_{\mathbf{x}=\mathbf{y}} + \underbrace{\delta^3(\mathbf{x}-\mathbf{y})}_{\mathbf{x}=\mathbf{y}}\pi(t,\mathbf{y})\Big) \\
&= \frac{1}{2}(\pi(t,\mathbf{x}) + \pi(t,\mathbf{x})) \\
&= \pi(t,\mathbf{x}) \ ,
\end{aligned}$$

where we have used $[\boldsymbol{\nabla}_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}),\hat{\varphi}(t,\mathbf{x})] = 0$ since the right-handed side depends on $\mathbf{y}$ and the left-handed side depends on $\mathbf{x}$. q.e.d.

The time evolution of $\hat{\pi}(x)$ is

$$\frac{\partial}{\partial t}\hat{\pi}(x) = (\nabla^2 - m^2)\hat{\varphi}(x) \ .$$

Proof. In fact

$$
\begin{aligned}
\frac{\partial}{\partial t}\hat{\pi}(x) &= i[\hat{H},\hat{\pi}(x)] \\
&= i[\frac{1}{2}\int d^3y(\hat{\pi}^2(t,\mathbf{y})+\nabla^2\hat{\varphi}(t,\mathbf{y})+m^2\hat{\varphi}^2(t,\mathbf{y})),\hat{\pi}(t,\mathbf{x})] \\
&= \frac{i}{2}\int d^3y\Big(\underbrace{[\hat{\pi}^2(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{0}+\underbrace{[\nabla^2\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{[\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}+m^2\underbrace{[\hat{\varphi}^2(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{\hat{\varphi}(t,\mathbf{y})[\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]+[\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]\hat{\varphi}(t,\mathbf{y})}\Big) \\
&= \frac{i}{2}\int d^3y\Big(\underbrace{[\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\cdot[\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]+[\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})} \\
&\quad+m^2\hat{\varphi}(t,\mathbf{y})\underbrace{[\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{i\delta^3(\mathbf{x}-\mathbf{y})}+m^2\underbrace{[\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{i\delta^3(\mathbf{x}-\mathbf{y})}\hat{\varphi}(t,\mathbf{y})\Big) \\
&= \frac{i}{2}\int d^3y\Big(\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\cdot\underbrace{[\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{\nabla_{\mathbf{y}}[\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}+\underbrace{[\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{\nabla_{\mathbf{y}}[\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}) \\
&\quad+im^2\hat{\varphi}(t,\mathbf{y})i\delta^3(\mathbf{x}-\mathbf{y})+im^2\delta^3(\mathbf{x}-\mathbf{y})\hat{\varphi}(t,\mathbf{y})\Big) \\
&= \frac{i}{2}\int d^3y\Big(\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\cdot\nabla_{\mathbf{y}}\underbrace{[\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{i\delta^3(\mathbf{x}-\mathbf{y})}+\nabla_{\mathbf{y}}\underbrace{[\hat{\varphi}(t,\mathbf{y}),\hat{\pi}(t,\mathbf{x})]}_{i\delta^3(\mathbf{x}-\mathbf{y})}\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}) \\
&\quad+im^2\hat{\varphi}(t,\mathbf{y})i\delta^3(\mathbf{x}-\mathbf{y})+im^2\delta^3(\mathbf{x}-\mathbf{y})\hat{\varphi}(t,\mathbf{y})\Big) \\
&= \frac{i}{2}\int d^3y\Big(i\underbrace{\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\cdot\nabla_{\mathbf{y}}\delta^3(\mathbf{x}-\mathbf{y})}_{-\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\delta^3(\mathbf{x}-\mathbf{y})+\text{boundary term}}+i\underbrace{\nabla_{\mathbf{y}}\delta^3(\mathbf{x}-\mathbf{y})\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})}_{-\delta^3(\mathbf{x}-\mathbf{y})\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})+\text{boundary term}} \\
&\quad+im^2\hat{\varphi}(t,\mathbf{y})i\delta^3(\mathbf{x}-\mathbf{y})+im^2\delta^3(\mathbf{x}-\mathbf{y})\hat{\varphi}(t,\mathbf{y})\Big) \\
&= \frac{i}{2}\int d^3y\Big(-i\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\underbrace{\delta^3(\mathbf{x}-\mathbf{y})}_{\mathbf{x}=\mathbf{y}}-i\underbrace{\delta^3(\mathbf{x}-\mathbf{y})}_{\mathbf{x}=\mathbf{y}}\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{y}}\hat{\varphi}(t,\mathbf{y}) \\
&\quad+im^2\hat{\varphi}(t,\mathbf{y})i\underbrace{\delta^3(\mathbf{x}-\mathbf{y})}_{\mathbf{x}=\mathbf{y}}+im^2\underbrace{\delta^3(\mathbf{x}-\mathbf{y})}_{\mathbf{x}=\mathbf{y}}\hat{\varphi}(t,\mathbf{y})\Big) \\
&= \nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{x}}\hat{\varphi}(t,\mathbf{x})-m^2\hat{\varphi}(t,\mathbf{x})=(\nabla^2-m^2)\hat{\varphi}(t,\mathbf{x})\ .
\end{aligned}
$$

q.e.d.

Combining the two time evolutions, we obtain the Klein-Gordon equation (6.1)

$$(\Box+m^2)\hat{\varphi}(x)\ .$$

Proof. In fact,

$$\ddot{\hat{\varphi}}(x) = \dot{\hat{\pi}}(x) = (\nabla^2 - m^2)\hat{\psi}(x) \ ,$$

hence

$$0 = (\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\hat{\varphi}(x) = (\Box + m^2)\hat{\varphi}(x) \ .$$

q.e.d.

Since field operators evolve in time and they depend on the ladder operators, the latters evolve in time as well

$$(\hat{a}_{\mathbf{p}})_H = \exp(-iE_{\mathbf{p}}t)(\hat{a}_{\mathbf{p}})_S \ , \quad (\hat{a}^\dagger_{\mathbf{p}})_H = \exp(iE_{\mathbf{p}}t)(\hat{a}^\dagger_{\mathbf{p}})_S \ .$$

Proof. Using the formula $\hat{H}^n\hat{a}_{\mathbf{p}} = \hat{a}_{\mathbf{p}}(\hat{H} - E_{\mathbf{p}})^n$ which can be proved

$$[\hat{H}, \hat{a}_{\mathbf{p}}] = \hat{H}\hat{a}_{\mathbf{p}} - \hat{a}_{\mathbf{p}}\hat{H} = -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}} = -E_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ ,$$

$$\hat{H}\hat{a}_{\mathbf{p}} = \hat{a}_{\mathbf{p}}(\hat{H} - E_{\mathbf{p}}) \ ,$$

hence

$$\hat{H}^2\hat{a}_{\mathbf{p}} = \hat{H}\hat{a}_{\mathbf{p}}(\hat{H} - E_{\mathbf{p}}) = \hat{a}_{\mathbf{p}}(\hat{H} - E_{\mathbf{p}})^2 \ , \tag{8.2}$$

by induction

$$\hat{H}^n\hat{a}_{\mathbf{p}} = \hat{a}_{\mathbf{p}}(\hat{H} - E_{\mathbf{p}})^n \ .$$

For the annihilation operator

$$\begin{aligned}
(\hat{a}_{\mathbf{p}})_H &= \exp(i\hat{H}t)(\hat{a}_{\mathbf{p}})_S \exp(-i\hat{H}t) \\
&= \underbrace{\exp(i\hat{H}t)}_{\sum_n \frac{(it)^n}{n!}\hat{H}^n} \hat{a}_{\mathbf{p}} \exp(-i\hat{H}t) \\
&= \sum_n \frac{(it)^n}{n!} \underbrace{\hat{H}^n\hat{a}_{\mathbf{p}}}_{\hat{a}_{\mathbf{p}}(\hat{H}-E_{\mathbf{p}})^n} \exp(-i\hat{H}t) \\
&= \underbrace{\sum_n \frac{1}{n!}(it(\hat{H} - E_{\mathbf{p}}))^n}_{\exp(it(\hat{H}-E_{\mathbf{p}}))} \hat{a}_{\mathbf{p}} \exp(-i\hat{H}t) \\
&= \exp(it(\cancel{\hat{H}} - E_{\mathbf{p}}))\hat{a}_{\mathbf{p}}\cancel{\exp(-i\hat{H}t)} \\
&= \exp(-iE_{\mathbf{p}}t)\hat{a}_{\mathbf{p}} \ .
\end{aligned}$$

Using the formula $\hat{H}^n\hat{a}_{\mathbf{p}} = \hat{a}_{\mathbf{p}}(\hat{H} - E_{\mathbf{p}})^n$ which can be proved

$$[\hat{H}, \hat{a}^\dagger_{\mathbf{p}}] = \hat{H}\hat{a}^\dagger_{\mathbf{p}} - \hat{a}^\dagger_{\mathbf{p}}\hat{H} = \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} = E_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} \ ,$$

$$\hat{H}\hat{a}^\dagger_{\mathbf{p}} = \hat{a}^\dagger_{\mathbf{p}}(\hat{H} + E_{\mathbf{p}}) \ ,$$

hence

$$\hat{H}^2\hat{a}^\dagger_{\mathbf{p}} = \hat{H}\hat{a}^\dagger_{\mathbf{p}}(\hat{H} + E_{\mathbf{p}}) = \hat{a}^\dagger_{\mathbf{p}}(\hat{H} + E_{\mathbf{p}})^2 \ , \tag{8.3}$$

by induction

$$\hat{H}^n\hat{a}^\dagger_{\mathbf{p}} = \hat{a}^\dagger_{\mathbf{p}}(\hat{H} + E_{\mathbf{p}})^n \ .$$

For the annihilation operator

$$\begin{aligned}
(\hat{a}^\dagger_{\mathbf{p}})_H &= \exp(i\hat{H}t)(\hat{a}^\dagger_{\mathbf{p}})_S \exp(-i\hat{H}t) \\
&= \underbrace{\exp(i\hat{H}t)}_{\sum_n \frac{(it)^n}{n!}\hat{H}^n} \hat{a}^\dagger_{\mathbf{p}} \exp(-i\hat{H}t) \\
&= \sum_n \frac{(it)^n}{n!} \underbrace{\hat{H}^n\hat{a}^\dagger_{\mathbf{p}}}_{\hat{a}^\dagger_{\mathbf{p}}(\hat{H}+E_{\mathbf{p}})^n} \exp(-i\hat{H}t) \\
&= \underbrace{\sum_n \frac{1}{n!}(it(\hat{H} + E_{\mathbf{p}}))^n}_{\exp(it(\hat{H}+E_{\mathbf{p}}))} \hat{a}^\dagger_{\mathbf{p}} \exp(-i\hat{H}t) \\
&= \exp(it(\cancel{\hat{H}} + E_{\mathbf{p}}))\hat{a}^\dagger_{\mathbf{p}}\cancel{\exp(-i\hat{H}t)} \\
&= \exp(iE_{\mathbf{p}}t)\hat{a}^\dagger_{\mathbf{p}} \ .
\end{aligned}$$

q.e.d.

Finally, the field operators are

$$\hat{\varphi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(\hat{a}^\dagger_{\mathbf{p}} \exp(ipx) + \hat{a}_{\mathbf{p}} \exp(-ipx))$$

and

$$\hat{\pi}(x) = \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{E_{\mathbf{p}}}{2}}\Big)(\hat{a}_{\mathbf{p}} \exp(-ipx) - \hat{a}^\dagger_{\mathbf{p}} \exp(ipx)) \ ,$$

where $px = p^\mu x_\mu = E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x}$.

Proof. For the field operator

$$\begin{aligned}
\hat{\varphi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}((\hat{a}^\dagger_{\mathbf{p}})_H \exp(-i\mathbf{p}\cdot\mathbf{x}) + (\hat{a}_{\mathbf{p}})_H \exp(i\mathbf{p}\cdot\mathbf{x})) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(\hat{a}^\dagger_{\mathbf{p}} \exp(i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})) + \hat{a}_{\mathbf{p}} \exp(-i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x}))) \qquad (8.4) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(\hat{a}^\dagger_{\mathbf{p}} \exp(ipx) + \hat{a}_{\mathbf{p}} \exp(-ipx)) \ .
\end{aligned}$$

For the conjugate field operator

$$\begin{aligned}
\hat{\pi}(x) &= \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{E_{\mathbf{p}}}{2}}\Big)(-(\hat{a}^\dagger_{\mathbf{p}})_H \exp(-i\mathbf{p}\cdot\mathbf{x}) + (\hat{a}_{\mathbf{p}})_H \exp(i\mathbf{p}\cdot\mathbf{x})) \\
&= \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{E_{\mathbf{p}}}{2}}\Big)(-\hat{a}^\dagger_{\mathbf{p}} \exp(i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})) + \hat{a}_{\mathbf{p}} \exp(-i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x}))) \\
&= \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{E_{\mathbf{p}}}{2}}\Big)(-\hat{a}^\dagger_{\mathbf{p}} \exp(ipx) + \hat{a}_{\mathbf{p}} \exp(-ipx)) \ .
\end{aligned} \tag{8.5}$$

q.e.d.

8.3 Casuality

So far, almost everything is manifestly Lorentz invariant except the commutation relations, because they privileged a time since they must be evaluated at equal time. We need to work out commutation relations at arbitrary times

$$[\hat{O}_1(t, \mathbf{x}), \hat{O}_2(t', \mathbf{y})] = [\hat{O}_1(x), \hat{O}_2(y)] \ .$$

Its physical meaning is the effects of one onto the other one. This means that in order to preserve causality, it must be zero outside the light cone since the signal cannot travel faster than light. Obviously, if it is inside the light cone, it can be non-zero. This means that

$$[\hat{O}_1(x), \hat{O}_2(y)] = 0 \quad \forall (x-y)^2 = (x^\mu - y^\mu)(x_\mu - y_\mu) < 0 \ .$$

We define the quantity

$$\Delta(x-y) = [\hat{\varphi}(x), \hat{\varphi}(y)] \ .$$

It is Lorentz invariant

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big(\exp(-ip(x-y)) - \exp(ip(x-y))\Big) \ .$$

Proof. In fact,

$$
\begin{aligned}
\Delta(x-y) &= [\hat{\varphi}(x), \hat{\varphi}(y)] \\
&= [\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(\hat{a}_{\mathbf{p}} \exp(-ipx) + \hat{a}^\dagger_{\mathbf{p}} \exp(ipx) \Big), \\
&\quad \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(\hat{a}_{\mathbf{q}} \exp(-iqy) + \hat{a}^\dagger_{\mathbf{q}} \exp(iqy) \Big)] \\
&= \int \frac{d^3p\; d^3q}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} \Big(\underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{0} exp(-ipx-iqy) + \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{-(2\pi)^3\delta(p-q)} exp(ipx-iqy) \\
&\quad + \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{(2\pi)^3\delta(p-q)} exp(-ipx+iqy) + \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{p}}]}_{0} exp(ipx+iqy) \Big) \\
&= \int \frac{d^3p\; d^3q}{(2\pi)^3} \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} \Big(-\underbrace{\delta(p-q)}_{p=q} exp(ipx-iqy) + \underbrace{\delta(p-q)}_{p=q} exp(-ipx+iqy) \Big) \\
&== \int \frac{d^3p\; d^3q}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \Big(exp(-ip(x-y)) - exp(ip(x-y)) \Big) \\
&=
\end{aligned}
$$

q.e.d.

It satisfies the causality constrains

1. inside the light cone, for timelike separations

$$\Delta(x-y) \neq 0 \quad \forall (x-y)^2 > 0 \ ,$$

2. outside the light cone, for spacelike separations

$$\Delta(x-y) = 0 \quad \forall (x-y)^2 < 0 \ .$$

Proof. For inside the light cone, we choose one particular case with $x^\mu = (t, 0, 0, 0)$ and $y^\mu = (0, 0, 0, 00)$, where we are static in space. Hence

$$
\begin{aligned}
\Delta(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Big(\exp(-i\underbrace{p(x-y)}_{E_{\mathbf{p}}t}) - \exp(i\underbrace{p(x-y)}_{E_{\mathbf{p}}t}) \Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Big(\exp(-iE_{\mathbf{p}}t) - \exp(iE_{\mathbf{p}}t) \Big) \ .
\end{aligned}
$$

In polar coordinates $(|p|, \theta, \varphi)$

$$\Delta(x-y) = \int_0^\infty \frac{d|p|}{(2\pi)^3} \frac{|p|^2}{2E_{\mathbf{p}}} \Big(\exp(-iE_{\mathbf{p}}t) - \exp(iE_{\mathbf{p}}t) \Big) \underbrace{\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta}_{4\pi}$$

$$= 4\pi \int_0^\infty \frac{d|p|}{(2\pi)^3} \frac{|p|^2}{2\sqrt{|p|^2+m^2}} \Big(\exp(-i\sqrt{|p|^2+m^2}t) - \exp(i\sqrt{|p|^2+m^2}t) \Big) .$$

By a change of variable $|p| = E_{\mathbf{p}}$ with differential $|p|^2 d|p| = E_{\mathbf{p}} dE_{\mathbf{p}} \sqrt{E_{\mathbf{p}}^2 - m^2}$

$$\Delta(x-y) = \frac{1}{4\pi^2} \int_m^\infty dE_{\mathbf{p}} \sqrt{E_{\mathbf{p}} - m^2} (\exp(-iE_{\mathbf{p}}t) - \exp(iE_{\mathbf{p}}t))$$
$$= \frac{m}{8\pi t} (Y_1(mt) + iJ_1(mt) - Y_1(-mt) - iJ_1(-mt)) ,$$

where have used the Bessel function of first order. notice that their behaviour at infinity is

$$J_1(x) \xrightarrow{x\to\infty} \sqrt{\frac{2\pi}{x}} \cos x , \quad Y_1(x) \xrightarrow{x\to\infty} \sqrt{\frac{2}{\pi x}} \sin x ,$$

hence

$$Y_1(mt) + mJ_2(mt) \xrightarrow{t\to\infty} \sqrt{\frac{2}{\pi m t}} (\sin(mt) + i\cos(mt)) = i\sqrt{\frac{2}{\pi m t}} \exp(-imt)$$

and

$$\Delta(x-y) \xrightarrow{t\to\infty} \propto \exp(-imt) - \exp(imt) \neq 0 .$$

For outside the light cone, because of the Lorentz invariance, we need to prove only a particular spacelike separations and it becomes true for all spacelike separations. We choose the one at the same t

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Big(\exp(-i \underbrace{p(x-y)}_{-\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}) - \exp(i \underbrace{p(x-y)}_{-\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}) \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Big(\exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})) - \exp(-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})) \Big) = 0 ,$$

where we echanged $\mathbf{p}$ in $-\mathbf{p}$ in the second term of the integrand. q.e.d.

We have just proved that the Klein-Gordon theory preserves causality.

8.4 Correlators

Another way to study is via the propagators. Consider a particle at a spacetime point y. What is the probability to find it st x? We define the propagator

$$D(x-y) = \langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle .$$

which is

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \exp(-ip(x-y)) \ . \tag{8.6}$$

Proof. In fact

$$\begin{aligned}
D(x-y) &= \langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle \\
&= \langle 0| \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (\hat{a}^\dagger_{\mathbf{p}} \exp(ipx) + \hat{a}_{\mathbf{p}} \exp(-ipx)) \\
&\quad \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} (\hat{a}^\dagger_{\mathbf{q}} \exp(iqy) + \hat{a}_{\mathbf{q}} \exp(-iqy))|0\rangle \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}}}} \langle 0| \Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}} \exp(-ipx - iqy) + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}} \exp(ipx - iqy) \\
&\quad + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}} \exp(-ipx + iqy) + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} \exp(ipx + iqy) \Big) |0\rangle \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}}}} \Big(\langle 0|\hat{a}_{\mathbf{p}} \underbrace{\hat{a}_{\mathbf{q}}|0\rangle}_{0} \exp(-ipx - iqy) + \langle 0|\hat{a}^\dagger_{\mathbf{p}} \underbrace{\hat{a}_{\mathbf{q}}|0\rangle}_{0} \exp(ipx - iqy) \\
&\quad + \langle 0| \underbrace{\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}}_{[\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{q}}]+\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}} |0\rangle \exp(-ipx + iqy) + \underbrace{\langle 0|\hat{a}^\dagger_{\mathbf{p}}}_{0} \hat{a}^\dagger_{\mathbf{p}}|0\rangle \exp(ipx + iqy) \Big) \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}}}} \Big(\langle 0| \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{(2\pi)^3\delta(\mathbf{p}-\mathbf{q})} |0\rangle \exp(-ipx + iqy) + \langle 0|\hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{p}}|0\rangle}_{0} \exp(-ipx + iqy) \Big) \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}}}} \underbrace{\delta(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \underbrace{\langle 0|0\rangle}_{1} \exp(-ipx + iqy) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \exp(-ip(x-y)) \ .
\end{aligned}$$

q.e.d.

Outside the light cone, it does not vanish.

Proof. For spacelike separation, like $(x-y) = (0, \mathbf{r})$, we have

$$D(x-y) = \frac{1}{(2\pi)^2 r} \int_m^\infty dy \ \frac{y \exp(-yr)}{\sqrt{y^2 - m^1}} \ ,$$

or with asymptotic behaviour at infinity

$$D(x-y) \xrightarrow{r\to\infty} \propto \exp(-mr) \ ,$$

which means that the propagators decays exponentially quickly outside the light cone but it does not vanish. q.e.d.

However, causality is not violated because even though the propagator is non-zero, the commutator $\Delta(x-y)$ is so and it can be written as

$$\Delta(x-y) = D(x-y) - D(y-x) \ ,$$

which means that there is a decostructive interference, with the Feynman interpretation as

$$\Delta(x-y) = \langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle - \langle 0|\hat{\varphi}(y)\hat{\varphi}(x)|0\rangle \ ,$$

where the first term is the probability amplitude to go from **y** to **x** at a speed larger than that of light and the latter term is the probability amplitude to go from **x** to **y** at a speed larger than that of light. The physical intuition, for a complex Klein-Gordon field, is a particle and an antiparticle that travels backward in time

$$\Delta(x-y) = \langle 0|\hat{\varphi}(x)\hat{\varphi}^*(y)|0\rangle - \langle 0|\hat{\varphi}^*(y)\hat{\varphi}(x)|0\rangle \ .$$

Part III

Dirac theory

Chapter 9

Dirac action

9.1 Spinor representation of the Lorentz group

The reducible representation of a Dirac spinor is

$$\psi'_D = \exp(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\psi_D \ ,$$

where ψ_D is a four-components complex vector, $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and the gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_4 \ .$$

In the Weyl basis, the Dirac matrices become

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix} \ .$$

It is useful to redefine the matrix

$$S^{\mu\nu} = -i\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \ ,$$

such that

$$\psi'^{\alpha}_D(x) = \exp(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu})^\alpha{}_\beta\psi^\beta_D(x) = S^\alpha_\beta\psi^\beta_D(x) \ , \tag{9.1}$$

where $\alpha, \beta = 1, 2, 3, 4$.

Notice that

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \ . \tag{9.2}$$

Proof. Maybe in the future. q.e.d.

9.2 Invariant quantities

In order to have a Lorentz invariant action, we need to built Lorentz invariant quantities in function of ψ. Observe that the quantity $\psi\psi^\dagger$ is not a scalar.

Proof. In fact,

$$\psi'^\dagger\psi' = \psi^\dagger S^\dagger S\psi \neq \psi^\dagger\psi \ ,$$

since $S^\dagger \neq S^{-1}$. q.e.d.

However, S satisfies $S^\dagger = \gamma^0 S^{-1}\gamma^0$.

Proof. In fact, consider the spinor representation

$$S = \exp(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}) \ ,$$

its hermitian

$$S = \exp(\frac{1}{2}\omega_{\mu\nu}(S^\dagger)^{\mu\nu})$$

and its inverse

$$S = \exp(-\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}) \ .$$

q.e.d.

We define the adjoint Dirac spinor

$$\overline{\psi}(x) = \psi^\dagger(x)\gamma^0 \ .$$

With this, we can construct a Lorentz invariant bilinear spinor

$$\overline{\psi}\psi \ .$$

Proof. In fact,

$$\overline{\psi}'\psi' = \psi'^\dagger\gamma^0\psi' = \psi^\dagger \underbrace{S^\dagger}_{\gamma^0 S^{-1}\gamma^0} \gamma^0 S\psi == \psi^\dagger\gamma^0 S^{-1}\underbrace{\gamma^0\gamma^0}_{1} S\psi = \psi^\dagger\gamma^0 \underbrace{S^{-1}S}_{1}\psi = \psi^\dagger\gamma^0\psi = \overline{\psi}\psi \ .$$

q.e.d.

Now, we want to build a 4-vector $\overline{\psi}\gamma^\mu\psi$ such that

$$\overline{p}si'\gamma^\mu\psi' = \overline{\psi}\Lambda^\mu{}_\nu\gamma^\nu\psi \ ,$$

or equivalently

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu \ .$$

Proof. Maybe in the future. q.e.d.

We obtained a Lorentz invariant scalar by contracting γ^μ with the first order derivative ∂_μ.

Furthermore, $\Sigma^{\mu\nu}$ is a 2-tensor

$$\overline{p}si'\Sigma^{\mu\nu}\psi' = \overline{\psi}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Sigma^{\alpha\beta}\psi \ .$$

Proof. Maybe in the future. q.e.d.

9.3 Dirac action

Now, we have all the tools to build a Lorentz invariant lagrangian

$$\mathcal{L} = \overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\overline{\psi}(x)\psi(x) = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \ .$$

We have added an i factor to ensure that $\mathcal{L} \in \mathbb{R}$.

Proof. Maybe in the future. q.e.d.

The dymensional analysis is

$$[S] = 0 \ , [d^4x] = - \ , [\mathcal{L}] = 4 \ , [\psi] = \frac{3}{2} \ , [\partial_\mu] = 1 \ , [m] = 1 \ .$$

Notice that in the Klein Gordon theory, we had $[\varphi] = 1$. However, in a renormalisable theory, the coupling between operators must be of dimension 4. This means that only terms like $\varphi\overline{\psi}\psi$ are allowed. Another difference in the Dirac theory is that the lagrangian in at first order whereas in the Klein-Gordon theory is at second order. This is possible only because the gamma matrices exists only in the Dirac theory, while in the Klein-Gordon we have to constract to partial derivatives to get a scalar.

The equations of motion can be obtained by the Euler-Lagrange equations: the Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

and the conjugate Dirac equation is

$$\overline{\psi}(x)(i\gamma^\mu\overleftarrow{\partial_\mu} + m) = 0 \ .$$

Proof. Maybe in the future. q.e.d.

9.4 Dirac and Klein-Gordon equations

The four-components of the Dirac spinor satisfy the Dirac equation, but each components separately satisfy the Klein-Gordon equation, because it means that particles ensures the mass-shell condition.

Proof. Maybe in the future. q.e.d.

Chapter 10

Chiral spinors

Recall that the Dirac representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ is reducible and it can be decomposed into 2 irreducible Weyl representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

We introduce the γ^5 matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

such that it satisfies

$$\{\gamma^\mu,\gamma^5\}=0\ ,\quad (\gamma^5)^2=\mathbb{I}\ ,\quad (\gamma^5)^\dagger=\gamma^5\ .$$

In the Weyl basis it becomes

$$\gamma^5=\begin{bmatrix}-\mathbb{I}_2 & 0\\ 0 & \mathbb{I}_2\end{bmatrix}\ .$$

With γ^5, we can define the projection operators

$$P_L=\frac{\mathbb{I}-\gamma^5}{2}\ ,\quad P_R=\frac{\mathbb{I}+\gamma^5}{2}\ .$$

such that they satisfy

$$P_L^2=P_L\ ,\quad P_R^2=P_R\quad P_L^\dagger=P_L\ ,\quad P_R^\dagger=P_R\ ,\quad P_LP_R=P_RP_L=0\ ,\quad P_L+P_R=\mathbb{I}\ .$$

and they decompose the Dirac spinor into a left-handed Weyl spinor $\psi_L^{(W)}$ and a right-handed Weyl spinor $\psi_R^{(W)}$

$$\psi_L=\begin{bmatrix}\psi_L^{(W)}\\ 0\end{bmatrix}=P_L\psi=\frac{\mathbb{I}-\gamma^5}{2}\psi\ ,\quad \psi_R=\begin{bmatrix}0\\ \psi_R^{(W)}\end{bmatrix}=P_R\psi=\frac{\mathbb{I}+\gamma^5}{2}\psi\ .$$

Furthermore, their eigenvalues are

$$\gamma^5\psi_L=(-1)\psi_L\ ,\quad \gamma^5\psi_R=(+1)\psi_R\ .$$

The Dirac lagrangian in terms of the Weyl spinors is

$$\mathcal{L} = \overline{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \overline{\psi}_R i\gamma^\mu \partial_\mu \psi_R - m(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L) \ .$$

Notice that for a massive fermions, we do not know if it is right-handed or left-handed because of the last mixed term. Instead for massless fermions, we know.

Proof. Maybe in the future. q.e.d.

In terms of the Weyl spinors, the Dirac equation becomes

$$\begin{cases} i\frac{\partial}{\partial t}\psi_R^{(W)}(x) + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi_R^{(W)} - m\psi_L^{(W)} = 0 \\ i\frac{\partial}{\partial t}\psi_L^{(W)}(x) + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi_L^{(W)} - m\psi_R^{(W)} = 0 \end{cases} .$$

Proof. Maybe in the future. q.e.d.

For massless fermions, which have a hamiltonian $\hat{H} = |\hat{p}|$, the Weyl equations become

$$\begin{cases} (\hat{\mathbf{S}}\cdot\mathbf{p})\psi_R^{(W)}(x) = (+1)\psi_R^{(W)}(x) \\ (\hat{\mathbf{S}}\cdot\mathbf{p})\psi_L^{(W)}(x) = (-1)\psi_L^{(W)}(x) \end{cases}$$

where $\mathbf{p}$ is the direction of motion and $\hat{S}$ is the spin operator. The quantity $\hat{\mathbf{S}}\cdot\mathbf{p}$ is called helicity and it is the projection of the spin along the direction of motion.

Proof. Maybe in the future. q.e.d.

10.1 Parity

The parity operator transforms a right-handed Weyl spinor into a left-handed Weyl spinor and viceversa

$$\begin{cases} {\psi'}_L^{(W)} = \psi_R^{(W)} \\ {\psi'}_R^{(W)} = \psi_L^{(W)} \end{cases} .$$

Proof. Maybe in the future. q.e.d.

Chapter 11

Solutions of the Dirac equation

Since each components of the Dirac spinor $\psi(x)$ satisfies the Klein-Gordon equation, the plane waves are solutions

$$\psi_\alpha(x) = u_\alpha(\mathbf{p}) \exp(-ipx) \ ,$$

where $u_\alpha(\mathbf{p})$ is the polarisation vector with 4 components $\alpha = 1,2,3,4$ and $p_0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$. Furthermore, in order to satisfy the Dirac equation, $u_\alpha(\mathbf{p})$ satisfies

$$\begin{bmatrix} -m & p^\mu \sigma_\mu \\ p^\mu \overline{\sigma}_\mu & -m \end{bmatrix} u(\mathbf{p}) = 0 \ , \tag{11.1}$$

where $\sigma^\mu = (\mathbb{I}_2, \sigma^i)$ and $\overline{\sigma}^\mu = (\mathbb{I}_2, -\sigma^i)$.

Proof. In fact,

$$0 = (i\gamma^\mu \partial_\mu - m)\psi(x) = (i\gamma^\mu(-ip_\mu) - m)u(\mathbf{p}) \exp(-ipx) \ .$$

Hence

$$0 = (\gamma^\mu p_\mu - m)u(\mathbf{p}) = \Big(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} p_0 + \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} p_i - m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Big) u(\mathbf{p}) = \begin{bmatrix} -m & p^\mu \sigma_\mu \\ p^\mu \overline{\sigma}_\mu & -m \end{bmatrix} u(\mathbf{p}) = 0 \ .$$

q.e.d.

Moreover, we can split the polarisation vector $u(\mathbf{p})$ into the right and the left-handed part

$$u(\mathbf{p}) = \begin{bmatrix} u_L(\mathbf{p}) \\ u_R(\mathbf{p}) \end{bmatrix} \ , \tag{11.2}$$

which can be intepret as the positive frequency solution.

Proof. In fact, putting (11.2) into (11.1)

$$\begin{cases} (p^\mu \overline{\sigma}_\mu) u_L = m u_R \\ (p^\mu \sigma_\mu) u_R = m u_L \end{cases} \ . \tag{11.3}$$

Notice that

$$
\begin{aligned}
(p_\mu \sigma^\mu)(p_\nu \overline{\sigma}^\nu) &= (p_0 + p_i \sigma^i)(p_0 + p_j \overline{\sigma}^j) \\
&= (p_0 + p_i \sigma^i)(p_0 - p_j \sigma^j) \\
&= p_0^2 - p_i p_j \underbrace{\sigma^i \sigma^j}_{\delta^{ij} + i\epsilon^{ijk}\sigma_k} \\
&= p_0^2 - p_i p_j \underbrace{\delta^{ij}}_{i=j} + i\, \cancel{\underbrace{p_i p_j}_{symm} \underbrace{\epsilon^{ijk}}_{anti}} \sigma_k \\
&= p_0^2 - |\mathbf{p}|^2 = m^2 \ .
\end{aligned}
$$

We choose the form of u_L such that

$$
u_L(\mathbf{p}) = A p^\mu \sigma_\mu \chi \ ,
$$

where A is a constant and χ is 2-components spinor. Hence, the first equation of (11.3)

$$
m u_R = (p^\mu \overline{\sigma}_\mu) u_L = A \underbrace{(p^\mu \overline{\sigma}_\mu)(p^\nu \sigma_\nu)}_{m^2} \chi = A m^2 \chi
$$

and

$$
u_R(\mathbf{p}) = m A \chi.
$$

In this way, the second equation of (11.3) is automatically satisfied

$$
m u_L = p^\mu \sigma_\mu \underbrace{m A \chi}_{u_R} = p^\mu \sigma_\mu u_R \ .
$$

Therefore

$$
u(\mathbf{p}) = A \begin{bmatrix} (p^\mu \sigma_\mu)\chi \\ m\chi \end{bmatrix} \ .
$$

We choose $A = \frac{1}{m}$ and $\chi = \sqrt{p^\mu \overline{\sigma}_\mu}\xi$, where ξ is a constant 2-components spinor normalised such that $\xi^\dagger \xi = 1$. Hence

$$
u(\mathbf{p}) = \frac{1}{m} \begin{bmatrix} (p^\mu \sigma_\mu)\sqrt{p^\nu \overline{\sigma}_\nu}\xi \\ m\sqrt{p^\mu \overline{\sigma}_\mu}\xi \end{bmatrix} = \frac{1}{m} \begin{bmatrix} \sqrt{p^\mu \sigma_\mu}\, \underbrace{\sqrt{p^\alpha \sigma_\alpha}\sqrt{p^\nu \overline{\sigma}_\nu}}_{m}\, \xi \\ m\sqrt{p^\mu \overline{\sigma}_\mu}\xi \end{bmatrix} = \begin{bmatrix} \sqrt{p^\mu \sigma_\mu}\xi \\ \sqrt{p^\mu \overline{\sigma}_\mu}\xi \end{bmatrix} \ .
$$

q.e.d.

Actually, there is another class of plane waves solutions, the negative frequency solutions

$$
\psi(x) = v(\mathbf{p}) \exp(ipx) \ ,
$$

where $v(\mathbf{p})$ is the polarisation vector

$$
v(\mathbf{p}) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\eta \\ -\sqrt{p_\mu \overline{\sigma}^\mu}\eta \end{bmatrix} \ ,
$$

where η is a constant 2-components spinor normalised such that $\eta^\dagger\eta = 1$.

They can be distinguished since

$$(\gamma^\mu p_\mu - m)u(\mathbf{p}) = 0 \ , \quad (\gamma^\mu p_\mu + m)v(\mathbf{p}) = 0 \ ,$$

and

$$\hat{H}\psi(x) = i\frac{\partial}{\partial t}(u(\mathbf{p})\exp(-ipx)) = E_{\mathbf{p}}\psi(x) \ , \quad \hat{H}\psi(x) = i\frac{\partial}{\partial t}(u(\mathbf{p})\exp(ipx)) = -E_{\mathbf{p}}\psi(x) \ .$$

Consider a massive particle in the rest frame $p^\mu = (m, 0, 0, 0)$. The positive frequency solutions look like

$$\psi(x) = \sqrt{m}\exp(-iE_{\mathbf{p}}t)\begin{bmatrix}\xi \\ \xi\end{bmatrix} \ .$$

Using (9.1), we restrict to spatial rotations in which the generators are

$$S^{ij} = -\frac{1}{2}\epsilon^{ijk}\begin{bmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{bmatrix} \ ,$$

where $i \neq j$ and the parameters are

$$\omega_{ij} = -\epsilon_{ijk}\theta^k \ .$$

Therefore, the matrix rotation is

$$\exp(\frac{1}{2}\omega_{ij}S^{ij}) = \begin{bmatrix}\exp(\frac{i}{2}\theta^i\sigma_i) & 0 \\ 0 & \exp(\frac{i}{2}\theta^i\sigma_i)\end{bmatrix}$$

and the Dirac spinor transforms as

$$\psi'(x) = \begin{bmatrix}\exp(\frac{i}{2}\theta^i\sigma_i) & 0 \\ 0 & \exp(\frac{i}{2}\theta^i\sigma_i)\end{bmatrix}\psi(x) \ ,$$

which induces a transformation on ξ such that

$$\xi' = \exp(\frac{i}{2}\theta^i\sigma_i)\xi \ .$$

This is indeed an $SU(2)$ transformation, where we can recognise tha spin operator $\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\sigma}$ and ξ is a 2-components spinot which describes particle with spin $\frac{1}{2}$. Since $\xi^\dagger\xi = 1$, we choose, for the spin up

$$\xi = \begin{bmatrix}1 \\ 0\end{bmatrix} : \sigma_3\xi = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = (+1)\xi \ ,$$

for the spin down

$$\xi = \begin{bmatrix}0 \\ 1\end{bmatrix} : \sigma_3\xi = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix} = (-1)\xi \ ,$$

Chapter 12

Useful formulas

12.1 Inner product

We introduce a basis of the 2-components spinors

$$\xi^r \ , \eta^s \ ,$$

where $r, s = 1, 2$ such that they satisfy

$$(\xi^\dagger)^r \xi^s = \delta^{rs} \ , \quad (\eta^\dagger)^r \eta^s = \delta^{rs} \ .$$

For example,

$$\xi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \quad \xi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \quad \eta^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \quad \eta^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ .$$

We define the following inner products

1. $$(u^\dagger)^r(\mathbf{p}) u^s(\mathbf{p}) = 2p^0 \delta^{rs} \ ,$$

2. $$\overline{u}^r(\mathbf{p}) u^s(\mathbf{p}) = 2m \delta^{rs} \ ,$$

3. $$(v^\dagger)^r(\mathbf{p}) v^s(\mathbf{p}) = 2p^0 \delta^{rs} \ ,$$

4. $$\overline{v}^r(\mathbf{p}) v^s(\mathbf{p}) = -2m \delta^{rs} \ ,$$

5. $$\overline{u}^r(\mathbf{p}) v^s(\mathbf{p}) = \overline{v}^r(\mathbf{p}) u^s(\mathbf{p}) = 0 \ ,$$

6.

$$(u^\dagger)^r(\mathbf{p})v^s(-\mathbf{p}) = (v^\dagger)^r(\mathbf{p})u^s(-\mathbf{p}) = 0 \ .$$

Proof. For the first one,

$$\begin{aligned}
(u^\dagger)^r(\mathbf{p})u^s(\mathbf{p}) &= \begin{bmatrix}\sqrt{p^\mu\sigma_\mu}(\xi^\dagger)^r & \sqrt{p^\mu\overline{\sigma}_\mu}(\xi^\dagger)^r\end{bmatrix}\begin{bmatrix}\sqrt{p^\mu\sigma_\mu}\xi^s \\ \sqrt{p^\mu\overline{\sigma}_\mu}\xi^s\end{bmatrix} \\
&= (\xi^\dagger)^r p^\mu\sigma_\mu\xi^s + (\xi^\dagger)^r p^\mu\overline{\sigma}_\mu\xi^s \\
&= (\xi^\dagger)^r p^0 \underbrace{\sigma_0}_{\mathbb{I}_2} \xi^s + (\xi^\dagger)^r p^0 \underbrace{\overline{\sigma}_0}_{\mathbb{I}_2} \xi^s + (\xi^\dagger)^r p^i\sigma_i\xi^s + (\xi^\dagger)^r p^i \underbrace{\overline{\sigma}_i}_{-\sigma_i} \xi^s \\
&= (\xi^\dagger)^r p^0\xi^s + (\xi^\dagger)^r p^0\xi^s + \cancel{(\xi^\dagger)^r p^i\sigma_i\xi^s} - \cancel{(\xi^\dagger)^r p^i\sigma_i\xi^s} \\
&= 2p_0 \underbrace{(\xi^\dagger)^r\xi^s}_{\delta^{rs}} \\
&= 2p_0\delta^{rs} \ .
\end{aligned}$$

For the second one,

$$\begin{aligned}
\overline{u}^r(\mathbf{p})u^s(\mathbf{p}) &= \begin{bmatrix}\sqrt{p^\mu\sigma_\mu}(\xi^\dagger)^r & \sqrt{p^\mu\overline{\sigma}_\mu}(\xi^\dagger)^r\end{bmatrix}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{p^\mu\sigma_\mu}\xi^s \\ \sqrt{p^\mu\overline{\sigma}_\mu}\xi^s\end{bmatrix} \\
&= (\xi^\dagger)^r p^\mu \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\nu\overline{\sigma}_\nu}}_{m} \xi^s + (\xi^\dagger)^r \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\nu\overline{\sigma}_\nu}}_{m} \xi^s \\
&= 2m \underbrace{(\xi^\dagger)^r\xi^s}_{\delta^{rs}} \\
2m &= \delta^{rs} \ .
\end{aligned}$$

For the third one,

$$\begin{aligned}
(v^\dagger)^r(\mathbf{p})v^s(\mathbf{p}) &= \begin{bmatrix}\sqrt{p^\mu\sigma_\mu}(\eta^\dagger)^r & -\sqrt{p^\mu\overline{\sigma}_\mu}(\eta^\dagger)^r\end{bmatrix}\begin{bmatrix}\sqrt{p^\mu\sigma_\mu}\eta^s \\ -\sqrt{p^\mu\overline{\sigma}_\mu}\eta^s\end{bmatrix} \\
&= (\eta^\dagger)^r p^\mu\sigma_\mu\eta^s + (\eta^\dagger)^r p^\mu\overline{\sigma}_\mu\eta^s \\
&= (\eta^\dagger)^r p^0 \underbrace{\sigma_0}_{\mathbb{I}_2} \eta^s + (\eta^\dagger)^r p^0 \underbrace{\overline{\sigma}_0}_{\mathbb{I}_2} \eta^s + (\eta^\dagger)^r p^i\sigma_i\eta^s + (\eta^\dagger)^r p^i \underbrace{\overline{\sigma}_i}_{-\sigma_i} \eta^s \\
&= (\eta^\dagger)^r p^0\eta^s + (\eta^\dagger)^r p^0\eta^s + \cancel{(\eta^\dagger)^r p^i\sigma_i\eta^s} - \cancel{(\eta^\dagger)^r p^i\sigma_i\eta^s} \\
&= 2p_0 \underbrace{(\eta^\dagger)^r\eta^s}_{\delta^{rs}} \\
&= 2p_0\delta^{rs} \ .
\end{aligned}$$

For the fourth one,

$$
\begin{aligned}
\overline{v}^r(\mathbf{p})v^s(\mathbf{p}) &= \left[\sqrt{p^\mu\sigma_\mu}(\eta^\dagger)^r \quad -\sqrt{p^\mu\overline{\sigma}_\mu}(\eta^\dagger)^r\right]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{p^\mu\sigma_\mu}\eta^s\\ -\sqrt{p^\mu\overline{\sigma}_\mu}\eta^s\end{bmatrix}\\
&= -(\eta^\dagger)^r p^\mu \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\nu\overline{\sigma}_\nu}}_{m}\eta^s - (\eta^\dagger)^r \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\nu\overline{\sigma}_\nu}}_{m}\eta^s\\
&= -2m\underbrace{(\eta^\dagger)^r\eta^s}_{\delta^{rs}}\\
&- 2m = \delta^{rs} \ .
\end{aligned}
$$

For the fifth one,

$$
\begin{aligned}
\overline{u}^r(\mathbf{p})v^s(\mathbf{p}) &= \left[\sqrt{p^\mu\sigma_\mu}(\xi^\dagger)^r \quad \sqrt{p^\mu\overline{\sigma}_\mu}(\xi^\dagger)^r\right]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{p^\mu\sigma_\mu}\eta^s\\ -\sqrt{p^\mu\overline{\sigma}_\mu}\eta^s\end{bmatrix}\\
&= -(\xi^\dagger)^r \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\nu\overline{\sigma}_\nu}}_{m}\eta^s + (\xi^\dagger)^r \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\nu\overline{\sigma}_\nu}}_{m}\eta^s\\
&= m(\cancel{(-\xi^\dagger)^r\eta^s} + \cancel{(\xi^\dagger)^r\eta^2}) = 0 \ .
\end{aligned}
$$

For the second in the fifth one,

$$
\begin{aligned}
\overline{v}^r(\mathbf{p})u^s(\mathbf{p}) &= \left[\sqrt{p^\mu\sigma_\mu}(\eta^\dagger)^r \quad \sqrt{p^\mu\overline{\sigma}_\mu}(\eta^\dagger)^r\right]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{p^\mu\sigma_\mu}\xi^s\\ -\sqrt{p^\mu\overline{\sigma}_\mu}\xi^s\end{bmatrix}\\
&= -(\eta^\dagger)^r \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\nu\overline{\sigma}_\nu}}_{m}\xi^s + (\eta^\dagger)^r \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\nu\overline{\sigma}_\nu}}_{m}\xi^s\\
&= m(\cancel{(-\eta^\dagger)^r\xi^s} + \cancel{(\eta^\dagger)^r\xi^2}) = 0 \ .
\end{aligned}
$$

For the sixth one,

$$
\begin{aligned}
(u^\dagger)^r(\mathbf{p})v^s(-\mathbf{p}) &= \left[\sqrt{p^\mu\sigma_\mu}(\xi^\dagger)^r \quad \sqrt{p^\mu\overline{\sigma}_\mu}(\xi^\dagger)^r\right]\begin{bmatrix}\sqrt{\overline{p}^\mu\sigma_\mu}\eta^s\\ -\sqrt{\overline{p}^\mu\overline{\sigma}_\mu}\eta^s\end{bmatrix}\\
&= (\xi^\dagger)^r \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{\overline{p}^\mu\sigma_\mu}}_{m}\eta^s - (\xi^\dagger)^r \underbrace{\sqrt{p^\mu\overline{\sigma}_\mu}\sqrt{\overline{p}^\mu\overline{\sigma}_\mu}}_{m}\eta^s\\
&= m((\xi^\dagger)^r\eta^s - (\xi^\dagger)^r\eta^s) = 0 \ .
\end{aligned}
$$

For the second in the sixth one,

$$
\begin{aligned}
(v^\dagger)^r(\mathbf{p})u^s(-\mathbf{p}) &= \left[\sqrt{p^\mu\sigma_\mu}(\eta^\dagger)^r \quad -\sqrt{p^\mu\overline{\sigma}_\mu}(\eta^\dagger)^r\right]\begin{bmatrix}\sqrt{\overline{p}^\mu\sigma_\mu}\xi^s\\ \sqrt{\overline{p}^\mu\overline{\sigma}_\mu}\xi^s\end{bmatrix}\\
&= (\eta^\dagger)^r \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{\overline{p}^\mu\sigma_\mu}}_{m}\xi^s - (\eta^\dagger)^r \underbrace{\sqrt{p^\mu\overline{\sigma}_\mu}\sqrt{\overline{p}^\mu\overline{\sigma}_\mu}}_{m}\xi^s\\
&= m((\eta^\dagger)^r\xi^s - (\eta^\dagger)^r\xi^s) = 0 \ .
\end{aligned}
$$

q.e.d.

12.2 Outer product

We define the following outer products

1.
$$\sum_{s=1}^{2} u^s(\mathbf{p})\overline{u}^s(\mathbf{p}) = \gamma^\mu p_\mu + m\mathbb{I}_4 \ ,$$

2.
$$\sum_{s=1}^{2} v^s(\mathbf{p})\overline{v}^s(\mathbf{p}) = \gamma^\mu p_\mu - m\mathbb{I}_4 \ .$$

Proof. For the first one

$$\begin{aligned}
\sum_{s=1}^{2} u^s(\mathbf{p})\overline{u}^s(\mathbf{p}) &= \sum_{s=1}^{2} \begin{bmatrix} \sqrt{p^\mu\sigma_\mu}\xi^s \\ \sqrt{p^\mu\overline{\sigma}_\mu}\xi^s \end{bmatrix} \begin{bmatrix} \sqrt{p^\mu\sigma_\mu}(\xi^\dagger)^r & \sqrt{p^\mu\overline{\sigma}_\mu}(\xi^\dagger)^r \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \sum_{s=1}^{2} \begin{bmatrix} \sqrt{p^\mu\sigma_\mu}\xi^s(\xi^\dagger)^s\sqrt{p^\mu\overline{\sigma}_\mu} & \sqrt{p^\mu\sigma_\mu}\xi^s(\xi^\dagger)^s\sqrt{p^\mu\sigma_\mu} \\ \sqrt{p^\mu\overline{\sigma}_\mu}\xi^s(\xi^\dagger)^s\sqrt{p^\mu\overline{\sigma}_\mu} & \sqrt{p^\mu\overline{\sigma}_\mu}\xi^s(\xi^\dagger)^s\sqrt{p^\mu\sigma_\mu} \end{bmatrix} \\
&= \begin{bmatrix} \underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\mu\overline{\sigma}_\mu}}_{m} & \sqrt{p^\mu\sigma_\mu}\sqrt{p^\mu\sigma_\mu} \\ \sqrt{p^\mu\overline{\sigma}_\mu}\sqrt{p^\mu\overline{\sigma}_\mu} & \underbrace{\sqrt{p^\mu\overline{\sigma}_\mu}\sqrt{p^\mu\sigma_\mu}}_{m} \end{bmatrix} \\
&= \begin{bmatrix} m & p^\mu\sigma_\mu \\ p^\mu\overline{\sigma}_\mu & m \end{bmatrix} \\
&= \gamma^\mu p_\mu + m \ ,
\end{aligned}$$

where we have used

$$\sum_{s=1}^{2} \xi^s(\xi^\dagger)^s = \mathbb{I}_2 \ .$$

For the first one

$$\begin{aligned}
\sum_{s=1}^{2} v^s(\mathbf{p})\overline{v}^s(\mathbf{p}) &= \sum_{s=1}^{2} \begin{bmatrix} \sqrt{p^\mu\sigma_\mu}\eta^s \\ -\sqrt{p^\mu\overline{\sigma}_\mu}\eta^s \end{bmatrix} \begin{bmatrix} \sqrt{p^\mu\sigma_\mu}(\eta^\dagger)^r & -\sqrt{p^\mu\overline{\sigma}_\mu}(\eta^\dagger)^r \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \sum_{s=1}^{2} \begin{bmatrix} -\sqrt{p^\mu\sigma_\mu}\eta^s(\eta^\dagger)^s\sqrt{p^\mu\overline{\sigma}_\mu} & \sqrt{p^\mu\sigma_\mu}\eta^s(\eta^\dagger)^s\sqrt{p^\mu\sigma_\mu} \\ \sqrt{p^\mu\overline{\sigma}_\mu}\eta^s(\eta^\dagger)^s\sqrt{p^\mu\overline{\sigma}_\mu} & -\sqrt{p^\mu\overline{\sigma}_\mu}\eta^s(\eta^\dagger)^s\sqrt{p^\mu\sigma_\mu} \end{bmatrix} \\
&= \begin{bmatrix} -\underbrace{\sqrt{p^\mu\sigma_\mu}\sqrt{p^\mu\overline{\sigma}_\mu}}_{m} & \sqrt{p^\mu\sigma_\mu}\sqrt{p^\mu\sigma_\mu} \\ \sqrt{p^\mu\overline{\sigma}_\mu}\sqrt{p^\mu\overline{\sigma}_\mu} & -\underbrace{\sqrt{p^\mu\overline{\sigma}_\mu}\sqrt{p^\mu\sigma_\mu}}_{m} \end{bmatrix} \\
&= \begin{bmatrix} -m & p^\mu\sigma_\mu \\ p^\mu\overline{\sigma}_\mu & -m \end{bmatrix} \\
&= \gamma^\mu p_\mu - m \ ,
\end{aligned}$$

where we have used

$$\sum_{s=1}^{2} \eta^s(\eta^\dagger)^s = \mathbb{I}_2 \ .$$

q.e.d.

Chapter 13

How to not quantise the Dirac theory

List of Theorems