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On quantum field theory I:

how to secondly quantise?

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Theoretical Physics

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Part I

Classical field theory

Chapter 1

Action

A field is a physical quantity $\phi(t, \mathbf{x})$ which is defined ad every point in spacetime. The dynamics of a field is governed by an action, which is a functional that associates a real number to each field configuration for a fixed time interval $[t_1, t_2]$

$$S[\phi_i(x), \partial_\mu \phi_i(x)] = \int_{t_1}^{t_2} dt\ L = \int_{t_1}^{t_2} dt\ \int d^3x\ \mathcal{L} = \int d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i)\ , \tag{1.1}$$

where $\mathcal{L}$ is the lagrangian density, defined by

$$L = \int d^3x\ \mathcal{L}\ .$$

In natural units, the dimensional analysis is

$$[S] = 0\ [d^4x] = -4\ [\mathcal{L}] = 4\ .$$

1.1 The principle of stationary action

The dynamics of the system can be determined by the principle of stationary action.

Principle 1.1
The system evolve from an initial configuration at time t_1 to a final configuration at time t_2 along a path in configuration space which extremises the action (1.1), i.e.

$$\delta S = 0\ . \tag{1.2}$$

with the additional conditions

1. *fields vanish at spatial infinity*

$$\phi_i(t, \mathbf{x}) \to 0 \quad |\mathbf{x}| \to \infty\ ,$$

hence

$$\delta\phi_i(t, \infty) = 0\ , \tag{1.3}$$

2. *fields vanish at time extremes*

$$\delta\phi_i(t_1, \mathbf{x}) = \delta\phi_i(t_2, \mathbf{x}) = 0 \ . \tag{1.4}$$

The equation of motion of the system are the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} = 0 \ . \tag{1.5}$$

Proof. The variation of the action is

$$\delta S = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\partial_\mu\phi_i\right) = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\partial_\mu\delta\phi_i\right) ,$$

where

$$\delta\phi_i = \phi'_i(x) - \phi_i(x) \ ,$$

and

$$\delta\partial_\mu\phi_i(x) = \partial_\mu\phi'_i - \partial_\mu\phi(x) = \partial_\mu(\phi'_i(x) - \phi_i(x)) = \partial_\mu\delta\phi(x) \ .$$

By integration by parts, we obtain

$$\delta S = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\partial_\mu\delta\phi_i\right) = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right) + \int d^4x \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right) .$$

Notice that the last term is a total derivative and it vanishes at the boundary by the condition (1.4) and (1.3)

$$\int d^4x \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right) = 0 \ .$$

Hence, we find

$$\delta S = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right) ,$$

and, by the principle of stationary action (1.2)

$$\int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right) = 0 \ .$$

Finally, since $\delta_i\phi$ is arbitrary, we obtain (1.5)

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} = 0 \ .$$

q.e.d.

In order to quantise the theory, we need the hamiltonian formalism.

Definition 1.1
The conjugate field $\phi^i(x)$ associated to the field ϕ_i is

$$\phi^i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(x)}$$

The hamiltonian density is given by the Legendre transformation

$$\mathcal{H} = \Phi^i \dot{\phi}_i - \mathcal{L}$$

where the hamiltonian is

$$H = \int d^3x \ \mathcal{H}$$

Chapter 2

Noether's theorem

Symmetries are fundamental in quantum field theory and they can be classified into

1. spacetime
   a) global
      i. continuous (Poincarè)
      ii. discrete (Parity, time reversal)
   b) local
      i. continuous (General relativity)
      ii. discrete (Parity coordinate dependent)
2. internal
   a) global
      i. continuous (Flavour)
      ii. discrete ($\mathbb{Z}_2$)
   b) local
      i. continuous ($SU(3) \times SU(2) \times U(1)$)
      ii. discrete ($\mathbb{Z}_2(x)$)

Through the Noether's theorem, we can associate conserved quantities to continuous symmetries.

Theorem 2.1 (Noether's)
Every continuous symmetry $\delta\phi_i$ of the action (1.1) give rise to a conserved current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi - K^\mu \tag{2.1}$$

such that it satisfies a continuity equation

$$\partial_\mu J^\mu = 0 \tag{2.2}$$

Proof. We consider an infinitesimal transformation for a continuous symmetry of the system

$$\phi'_i = \phi_i + \delta\phi_i$$

which induces a transformation of the lagrangian

$$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$$

In order to be a symmetry of the system, we require that the action is not invariant, but we allow to be up to a boundary term $K^\mu(\phi_i)$, because the dynamics of the system, i.e. the equations of tmotion, do not change with a boundary term. Hence

$$S' = S + \int \partial_\mu K^\mu(\phi_i)$$

but

$$\delta S = \int \partial_\mu K^\mu(\phi_i) \tag{2.3}$$

Explicitly, we obtain

$$\begin{aligned}
\delta S &= \delta \int d^4x \ \mathcal{L} \\
&= \int d^4x \ \Big(\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\partial_\mu\phi_i\Big) \\
&= \int d^4x \ \Big(\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\partial_\mu\delta\phi_i\Big) \\
&= \int d^4x \ \Big(\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i - \partial_\mu\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\phi_i\Big) + \int d^4x \ \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i\Big) \\
&= \int d^4x \ \delta\phi_i \underbrace{\Big(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\Big)}_{0} + \int d^4x \ \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i\Big) \\
&= \int d^4x \ \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i\Big)
\end{aligned}$$

where we used the fact that partial derivatives and symmetries commute, the equation of motions (1.5) and we integrated by parts. Hence, by requiring that it is a symmetry

$$\delta S = \int d^4x \ \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi_i\Big) = \int d^4x \partial_\mu K^\mu$$

or equivalently

$$\int d^4x \; \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi_i - K^\mu \Big) = 0$$

Since it is for arbitrary integration, the integrand vanishes and

$$\partial_\mu J^\mu = 0$$

with

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta \phi_i - K^\mu$$

q.e.d.

Notice that every conserved current can be related to a conserved quantity Q by

$$Q = \int_{\mathbb{R}^3} d^3x \; J^0$$

This means that Q is conserved locally, i.e. any charge carrier leaving a finite volume V is associated to a flow of current $\mathbf{J}$ out of the volume.

Proof. Infact, by using (2.2)

$$\begin{aligned} \frac{dQ}{dt} &= \frac{d}{dt} \int_{\mathbb{R}^3} d^3x \; J^0 \\ &= \int_{\mathbb{R}^3} d^3x \; \frac{\partial J^0}{\partial t} \\ &= - \int_{\mathbb{R}^3} d^3x \; \nabla \cdot \mathbf{J} = 0 = - \int_{\partial \mathbb{R}^3} d\mathbf{S} \cdot \mathbf{J} = 0 \end{aligned}$$

where we used the Stoke's theorem and the fact that $\mathbf{J} \to 0$ for $|\mathbf{x}| \to 0$. q.e.d.

Chapter 3

Energy-momentum tensor

Spacetime translations give rise to 4 conserved currents, which corresponds to the conservation of energy and momentum. Infact, we consider an infinitesimal spacetime translation

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu}$$

such that fields change by

$$\phi'_i = \phi_i(x + \epsilon) = \phi(x) + \epsilon^{\mu}\partial_{\mu}\phi_i(x)$$

We considered an active transformation, where there is not a change of frame but fields themselves are indeed translated into new fields such that

$$\phi'_i(x') = \phi(x) = \phi(x' + \epsilon)$$

A passive transformation would have acted as

$$\phi'_i = \phi_i(x - \epsilon)$$

Since the lagrangian is a function of the coordinates via fields, we have the following transformation

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \epsilon^{\mu}\partial_{\mu}\mathcal{L} = \epsilon^{\mu}\partial_{\nu}(\delta^{\nu}{}_{\mu}\mathcal{L})$$

Hence, the boundary term is

$$K^{\mu} = \delta^{\mu}{}_{\nu}\mathcal{L}$$

We apply the Noether's theorem (2.1) and find 4 different conserved currents labelled by ν

$$(J^{\mu})_{\nu} = \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_i}\partial_{\nu}\phi_i - \delta^{\mu}{}_{\nu}\mathcal{L}$$

and we define the energy-momentum tensor, or stress-energy tensor,

$$T^{\mu}{}_{\nu} = (J^{\mu})_{\nu}$$

such that

$$\partial_\mu T^\mu{}_\nu = 0$$

In natural units, the dimensional analysis is

$$T^\mu{}_\nu = [\mathcal{L}] = 4$$

The 4 conserved charges are

$$Q_\nu = \int_{\mathbb{R}^3} d^3x\ (J^0)_\nu = \int_{\mathbb{R}^3} d^3x\ T^0{}_\nu$$

which correspond to the 4-momentum

$$P^\mu = \int_{\mathbb{R}^3} d^3x\ T^{0\mu}$$

In particular, the 0-th component is the energy

$$\begin{aligned} P^0 &= \int d^3x\ T^{00} \\ &= \int d^3x\ \Big(\frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial^0 \phi_i - \delta^{00} \mathcal{L}\Big) \\ &= \int d^3x\ \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}}_{\pi^i} \dot{\phi}_i - \mathcal{L}\Big) = \int d^3x\ (\underbrace{\pi^i \dot{\phi}_i - \mathcal{L}}_{\mathcal{L}}) = \int d^3x\ \mathcal{H} = H \end{aligned} \tag{3.1}$$

such that

$$\frac{dH}{dt} = 0$$

and the j-th components are the momentum

$$\begin{aligned} P^j &= \int d^3x\ T^{0j} \\ &= \int d^3x\ \Big(\frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \underbrace{\partial^j \phi_i}_{-\partial_j \phi_i} - \underbrace{\delta^{0j}}_{0} \mathcal{L}\Big) \\ &= \int d^3x\ \Big(- \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}}_{\pi^i} \partial_j \phi_i\Big) \\ &= -\int d^3x\ \pi^i \partial_j \phi_i \end{aligned} \tag{3.2}$$

such that

$$\frac{dP^i}{dt} = 0$$

Chapter 4

An example: electrodynamics

Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot E = \rho \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \tag{4.1}$$

can be written in covariant form

$$\partial_\mu F^{\mu\nu} = J^\nu \quad \partial_\mu F^{*\mu\nu} = 0$$

where $F^{\mu\nu}$ is the electromagnetic tensor and $F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}$ is its dual.

Furthermore, they can be written in terms of the scalar ϕ and the vector potentials $\mathbf{A}$, defined by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Maxwell's equations do not change under this transformation.

Proof. Maybe in the future. q.e.d.

In covariant form, we can write the electromagnetic tensor as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Maxwell's equations can be seen as the equations of motion of the electromagnetic lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J_\mu A^\mu$$

or, equivalenty written in terms of the 4-potential,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 - A_\mu J^\mu$$

Proof. First, we prove that they are equivalent. Maybe in the future.

Second, we prove that it leads to the Maxwell's equations. Maybe in the future.

q.e.d.

In natural units, the dimensional analysis is

$$[F^{\mu\nu}] = 2 \quad [A_\mu] = 1 \quad [J^\mu] = 3$$

The minus sign garanties that the kinetic energy has a positive one

$$-\frac{1}{2}\partial_0 A_i \underbrace{\partial^0}_{\partial_0} \underbrace{A^i}_{-A_i} = \frac{1}{2}\dot{A}_i^2$$

The fourth field A_0 is not a dynamical quantity, since there is no kinetic energy in terms of $\dot{A}_0^2$, because the first $-\frac{1}{2}\partial_0 A_0 \partial^0 A^0$ cancels out with $\frac{1}{2}(\partial_0 A_0)^2$. Therefore, there are only 3 degrees of freedom. However, since electrodynamics is a gauge theory, it is possible to restrict to only 2 degrees of freedom, which correspond to the 2 transversal polarisations direction of an electromagnetic wave.

The energy-momentum tensor is

$$T^{\mu\nu} = \partial^\nu A^\mu \partial_\rho A^\rho - \partial^\mu A^\rho \partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{4.2}$$

Proof. Maybe in the future. q.e.d.

However, the first term in (4.2) is not symmetric under change $\mu \leftrightarrow \nu$, but in order to take into account general relativity, this tensor must be symmetric, since $R_{\mu\nu}$ and $g_{\mu\nu}$ are so in

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

To do it, we defined a new energy-momentum tensor starting from the old one with the addition of an extra term: the partial derivative of a 3 indices anti-symmetric in the first 2 indices tensor $K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}$

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

This garanties that it is conserved as well

$$\partial_\mu \tilde{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \underbrace{\partial_\mu \partial_\lambda}_{symm} \underbrace{K^{\lambda\mu\nu}}_{anti} = \partial_\mu T^{\mu\nu} = 0$$

In the electromagnetic case, we choose K to be

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$$

and the symmetric energy-momentum tensor becomes

$$\tilde{T}^{\mu\nu} = F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4}\eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

which is called the Belifante-Rosenfeld tensor.

Proof. Maybe in the future. q.e.d.

The energy density is

$$\mathcal{E} = \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2)$$

Proof. Maybe in the future. q.e.d.

The momentum density is

$$\mathcal{P}^i = (\mathbf{E} \times \mathbf{B})^i$$

Proof. Maybe in the future. q.e.d.

Part II

Klein-Gordon theory

Chapter 5

Canonical or second quantisation

In Schoedinger picture, where states evolve in time while operators do not, recall that standard quantisation from classical mechanics to quantum mechanics works in this way:

1. hamiltonian formalism $H \mapsto$ hamiltonian operator $\hat{H}$,
2. generalised coordinates and conjugate momenta $(q_i, p^i = \frac{\partial L}{\partial \dot{q}_i}) \mapsto$ operators on a Hilbert space $\hat{q}_i$ and $\hat{p}^i$,
3. Poissons brackets $\{q_i, p^j\} = \delta_i^{\ j}$ and $\{p^i, p^j\} = \{q_i, q_j\} = 0 \mapsto$ commutators $[q_i, p^j] = i\delta_i^{\ j}$ and $[p^i, p^j] = [q_i, q_j] = 0$.

Similarly, the second quantisation from classical field theory to quantum field theory works in this way:

1. fields and conjugate fields $(\varphi_i(t, \mathbf{x}), \pi^i(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i}) \mapsto$ operators on a Fock space $\hat{\varphi}_i(t, \mathbf{x})$ and $\hat{\pi}^i(t, \mathbf{x})$,
2. canonical commutation relations $[\hat{\varphi}_i(t, \mathbf{x}), \hat{\pi}^j(t, \mathbf{y})] = i\delta_i^{\ j}\delta^3(\mathbf{x}-\mathbf{y})$ and $[\hat{\varphi}_i(t, \mathbf{x}), \hat{\varphi}_j(t, \mathbf{y})] = [\hat{\pi}^i(t, \mathbf{x}), \hat{\pi}^j(t, \mathbf{y})] = 0$.

States which live in the Fock state $|\psi\rangle$ evolve in time via the Schoedinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

where $|\psi\rangle$ is a wave functional such that its modulus square gives the density probability to find the field in a certain configuration and $\hat{H}(\varphi_i(t, \mathbf{x}), \pi^i(t, \mathbf{x}))$ is an operator, since $\varphi_i(t, \mathbf{x})$ and $\pi^i(t, \mathbf{x})$ are.

In order to solve the theory, we need to find the eigenstates of $\hat{H}$, but it is too difficult expect in the case of a free theory, which the lagrangian is quadratic and the equations od motion are linear and solvable.

5.1 Harmonic oscillator

Recall some feature of the harmonic oscillator.

5.2 Dirac delta

Recall that the integral representation of the Dirac delta is

$$\delta^3(\mathbf{x}-\mathbf{y}) = \int \frac{d^3p}{(2\pi)^3} \exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})) = \int \frac{d^3p}{(2\pi)^3} \exp(-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})) \ . \tag{5.1}$$

Chapter 6

Single real Klein-Gordon field

6.1 Hamiltonian

The simplest relativistic field theory is the Klein-Gordon theory of a single real scalar field chargeless and spinless. Its lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

and its equations of motion are

$$(\Box + m^2)\varphi(x) = 0 \ . \tag{6.1}$$

Proof. Infact, using (1.5)

$$\begin{aligned}
0 &= \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} \\
&= \frac{\partial}{\partial\varphi}\Big(\cancel{\frac{1}{2}\partial_\mu\varphi\partial^\mu}\varphi - \underbrace{\frac{1}{2}m^2\varphi^2}_{m^2\varphi}\Big) + \partial_\mu\frac{\partial}{\partial\partial_\mu\varphi}\Big(\underbrace{\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi}_{\partial_\mu\partial^\mu\varphi} - \cancel{\frac{1}{2}m^2\varphi^2}\Big) \\
&= \underbrace{\partial_\mu\partial^\mu}_{\Box}\varphi + m^2\varphi \\
&= (\Box + m^2)\varphi \ .
\end{aligned}$$

q.e.d.

It is a system of infinitely many degrees of freedom and to decouple them we need to perform a Fourier transform

$$\varphi(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t, \mathbf{p}) \ , \tag{6.2}$$

which in momentum space becomes

$$\Big(\frac{\partial^2}{\partial t^2} + |\mathbf{p}|^2 + m^2\Big)\tilde{\varphi}(t, \mathbf{x}) = 0$$

and its solution is an harmonic oscillator for each $\mathbf{p}$ of frequency

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}\ . \tag{6.3}$$

Hence, the most general solution of the Klein-Gordon equation (6.1) is a superposition of simple harmonic oscillators, each vibrating with different frequency and amplitude. To quantise the theory and φ, we need to quantise this set of infinitely decoupled harmonic oscillators.

Proof. We decompose (6.1) into time and space components

$$0 = (\Box + m^2)\varphi = (\underbrace{\partial_0}_{\partial^0}\partial^0 + \underbrace{\partial_i}_{-\partial^i}\partial^i + m^2)\varphi = ((\partial^0)^2 - (\partial^i)^2 + m^2)\varphi = (\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\varphi\ ,$$

and we substitute (6.2)

$$\begin{aligned} 0 &= (\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\int \frac{d^3p}{(2\pi)^3}\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t,\mathbf{p}) \\ &= \int \frac{d^3p}{(2\pi)^3}(\frac{\partial^2}{\partial t^2} - \underbrace{\nabla^2}_{-i^2|\mathbf{p}|^2} + m^2)(\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t,\mathbf{p})) \\ &= \int \frac{d^3p}{(2\pi)^3}(\frac{\partial^2}{\partial t^2} - i^2|\mathbf{p}|^2 + m^2)\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t,\mathbf{p}) \\ &= \int \frac{d^3p}{(2\pi)^3}(\frac{\partial^2}{\partial t^2} + |\mathbf{p}|^2 + m^2)\exp(i\mathbf{p}\cdot\mathbf{x})\tilde{\varphi}(t,\mathbf{p})\ , \end{aligned}$$

where the integrand vanishes with the exponential. Finally, we define the energy (6.3) and we obtain

$$(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{p}})^2\tilde{\varphi}(t,\mathbf{p}) = 0\ ,$$

which is indeed the equation of an harmonic oscillator in the form $\ddot{x} + \omega^2 x = 0$. q.e.d.

By analogy with the simple quantum harmonic oscillator, we define the field operator

$$\hat{\varphi}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \tag{6.4}$$

and the conjugate operator

$$\hat{\pi}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\ , \tag{6.5}$$

such that they satisfies the commutation relations for annihilation and creation operators

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}] = [\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}] = 0 \ , \quad [\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \ . \tag{6.6}$$

Therefore, the canonical commutation relations become

$$[\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{y})] = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = 0$$

and

$$[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}) \ .$$

Proof. For the field-field commutator, using (6.6), (6.4) and (5.1)

$$\begin{aligned}
[\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{y})] = {} & [\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^{\dagger}_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x}) \Big), \\
& \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} \Big(\hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{y}) + \hat{a}^{\dagger}_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{y}) \Big)] \\
= {} & \int \frac{d^3p\ d^3q}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} [\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^{\dagger}_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x}), \\
& \hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{y}) + \hat{a}^{\dagger}_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{y})] \\
= {} & \int \frac{d^3p\ d^3q}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} \Big(\underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{0} \exp(i(\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})) + \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \exp(i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})) \\
& + \underbrace{[\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p})} \exp(i(-\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})) + \underbrace{[\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}]}_{0} \exp(i(-\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})) \Big) \\
= {} & \int \frac{d^3p\ d^3q}{(2\pi)^3} \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} \Big(\underbrace{\delta^3(\mathbf{p} - \mathbf{q}) \exp(i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \\
& - \underbrace{\delta^3(\mathbf{q} - \mathbf{p}) \exp(i(-\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \Big) \\
= {} & \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \Big(\underbrace{\exp(i\mathbf{p}\cdot(\mathbf{x} - \mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} - \underbrace{\exp(i\mathbf{p}\cdot(-\mathbf{x} + \mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} \Big) = 0 \ .
\end{aligned}$$

For the conjugate-conjugate commutator, using (6.6), (6.5) and (5.1)

$$
\begin{aligned}
[\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = {} & [\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big), \\
& \int \frac{d^3q}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\Big)\Big(\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{y}) - \hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{y})\Big)] \\
= {} & \int \frac{d^3p\; d^3q}{(2\pi)^6}\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)[\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x}), \\
& \hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{y}) - \hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{y})] \\
= {} & \int \frac{d^3p\; d^3q}{(2\pi)^6}\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(\underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{0}\exp(i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})) - \underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}\exp(i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y} \\
& - \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]}_{-(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p})}\exp(i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})) + \underbrace{[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{0}\exp(i(-\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y}))\Big) \\
= {} & \int \frac{d^3p\; d^3q}{(2\pi)^3}\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(-\underbrace{\delta^3(\mathbf{p}-\mathbf{q})\exp(i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \\
& + \underbrace{\delta^3(\mathbf{q}-\mathbf{p})\exp(i(-\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}}\Big) \\
= {} & \int \frac{d^3p}{(2\pi)^3}\Big(-\frac{\omega_{\mathbf{p}}}{2}\Big)\Big(-\underbrace{\exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} + \underbrace{\exp(i\mathbf{p}\cdot(-\mathbf{x}+\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})}\Big) = 0\ .
\end{aligned}
$$

For the field-conjugate commutator, using (6.6), (6.4), (6.5) and (5.1)

$$
\begin{aligned}
[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = [&\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \Big(\hat{a}_\mathbf{p} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_\mathbf{p} \exp(-i\mathbf{p}\cdot\mathbf{x}) \Big), \\
&\int \frac{d^3q}{(2\pi)^3} \Big(-i\sqrt{\frac{\omega_\mathbf{q}}{2}} \Big) \Big(\hat{a}_\mathbf{q} \exp(i\mathbf{q}\cdot\mathbf{y}) - \hat{a}^\dagger_\mathbf{q} \exp(-i\mathbf{q}\cdot\mathbf{y}) \Big)] \\
= &\int \frac{d^3p\ d^3q}{(2\pi)^6} \Big(-\frac{i}{2}\sqrt{\frac{\omega_\mathbf{q}}{\omega_\mathbf{p}}} \Big) [\hat{a}_\mathbf{p} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_\mathbf{p} \exp(-i\mathbf{p}\cdot\mathbf{x}), \\
&\hat{a}_\mathbf{q} \exp(i\mathbf{q}\cdot\mathbf{y}) - \hat{a}^\dagger_\mathbf{q} \exp(-i\mathbf{q}\cdot\mathbf{y})] \\
= &\int \frac{d^3p\ d^3q}{(2\pi)^6} \Big(-\frac{i}{2}\sqrt{\frac{\omega_\mathbf{q}}{\omega_\mathbf{p}}} \Big) \Big(\underbrace{[\hat{a}_\mathbf{p}, \hat{a}_\mathbf{q}]}_{0} \exp(i(\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})) - \underbrace{[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{q}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} \exp(i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})) \\
&+ \underbrace{[\hat{a}^\dagger_\mathbf{p}, \hat{a}_\mathbf{q}]}_{-(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p})} \exp(i(-\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})) - \underbrace{[\hat{a}^\dagger_\mathbf{p}, \hat{a}^\dagger_\mathbf{q}]}_{0} \exp(i(-\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})) \Big) \\
= &\int \frac{d^3p\ d^3q}{(2\pi)^3} \Big(-\frac{i}{2}\sqrt{\frac{\omega_\mathbf{q}}{\omega_\mathbf{p}}} \Big) \Big(-\underbrace{\delta^3(\mathbf{p}-\mathbf{q}) \exp(i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \\
&- \underbrace{\delta^3(\mathbf{q}-\mathbf{p}) \exp(i(-\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y}))}_{\mathbf{p}=\mathbf{q}} \Big) \\
= &\int \frac{d^3p}{(2\pi)^3} \Big(\frac{i}{2} \Big) \Big(\underbrace{\exp(i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} + \underbrace{\exp(i\mathbf{p}\cdot(-\mathbf{x}+\mathbf{y}))}_{\delta^3(\mathbf{x}-\mathbf{y})} \Big) \\
= &\frac{i}{2} 2\delta^3(\mathbf{x}-\mathbf{y}) = i\delta^3(\mathbf{x}-\mathbf{y})\ .
\end{aligned}
$$

q.e.d.

The hamiltonian is

$$H = \frac{1}{2} \int d^3x\ (\pi^2 + (\boldsymbol{\nabla}\varphi)^2 + m^2\varphi^2)\ .$$

If we make a function study of the classical hamiltonian, we notice that it has quadratic terms and a minimum at $\varphi_0(t, \mathbf{x}) = const$ which we could consider as the ground state with $\varphi_0 = 0$. Quantising the theory means that we consider quantum (small) fluctuations $\delta\varphi$ around this ground state such that

$$\varphi(t, \mathbf{x}) = \underbrace{\varphi(t, \mathbf{x})_0}_{0} + \delta\varphi(t, \mathbf{x})\ .$$

The hamiltonian operator in quantum field theory becomes

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \omega_\mathbf{p} \hat{a}^\dagger_\mathbf{p} \hat{a}_\mathbf{p} + \frac{1}{2} \int d^3p\ \omega_\mathbf{p} \delta^3(0)\ . \tag{6.7}$$

Proof. Infact, the conjugate field is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi} = \partial_0 \varphi = \dot{\varphi} \tag{6.8}$$

and using (3.1) and (**??**)

$$\begin{aligned}
H &= \int d^3x \; T^{00} \\
&= \int d^3x \; (\pi \underbrace{\dot{\varphi}}_{\pi} - \mathcal{L}) \\
&= \int d^3x \; (\pi^2 - \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} m^2 \varphi^2) \\
&= \int d^3x \; (\pi^2 - \frac{1}{2}\partial_0 \varphi \partial^0 \varphi - \frac{1}{2}\partial_i \varphi \partial^i \varphi + \frac{1}{2} m^2 \varphi^2) \\
&= \int d^3x \; (\pi^2 - \frac{1}{2}\underbrace{\partial_0 \varphi \partial^0 \varphi}_{\pi^2} - \frac{1}{2}\underbrace{\partial_i \varphi \partial^i \varphi}_{-\nabla^2 \varphi} + \frac{1}{2} m^2 \varphi^2) \\
&= \frac{1}{2} \int d^3x \; (\pi^2 + (\boldsymbol{\nabla} \varphi)^2 + m^2 \varphi^2) \; .
\end{aligned}$$

Furthermore, using (6.6), (6.4), (6.5) and (5.1)

$$\begin{aligned}
\hat{H} &= \frac{1}{2} \int d^3x \; \Big(\hat{\pi}^2 + (\boldsymbol{\nabla} \hat{\varphi})^2 + m^2 \hat{\varphi}^2\Big) \\
&= \frac{1}{2} \int d^3x \; (\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&\quad \Big(\int \frac{d^3q}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\Big)\Big(\hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big) \\
&\quad + \nabla\Big(\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&\quad \nabla\Big(\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big) \\
&\quad + m^2\Big(\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\hat{a}_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}} \exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\Big) \\
&\quad \Big(\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{q}} \exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\ \Bigg(\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})\\
&\quad - \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})\Big)\\
&\quad + \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(i\mathbf{p}\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - i\mathbf{p}\hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big)\cdot\\
&\quad \Big(i\mathbf{q}\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) - i\mathbf{q}\hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\\
&\quad + m^2\frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})\\
&\quad + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\Bigg(\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})}\\
&\quad - \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})}\Big)\\
&\quad + \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(-\mathbf{p}\cdot\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})} + \mathbf{p}\cdot\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})}\\
&\quad + \mathbf{p}\cdot\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} - \mathbf{p}\cdot\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})}\Big)\\
&\quad + \frac{m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})}\\
&\quad + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})}\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^3p\ d^3q}{(2\pi)^3}\Bigg(\Big(-\frac{1}{2}\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \\
&\quad - \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}}\Big) \\
&\quad + \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(-\mathbf{p}\cdot\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}} + \mathbf{p}\cdot\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \\
&\quad + \mathbf{p}\cdot\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} - \mathbf{p}\cdot\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}}\Big) \\
&\quad + \frac{m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} \\
&\quad + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{p}=\mathbf{q}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{p}=-\mathbf{q}}\Big)\Bigg) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\Bigg(\Big(-\frac{\omega_{\mathbf{p}}}{2}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} - \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\Big) \\
&\quad + \Big(\frac{|\mathbf{p}|^2}{2\omega_{\mathbf{p}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\Big) \\
&\quad + \Big(\frac{m^2}{2\omega_{\mathbf{p}}}\Big)\Big(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\Big)\Bigg) \\
&= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\mathbf{p}}}\Big((\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}})\underbrace{(-\omega^2_{\mathbf{p}} + |\mathbf{p}|^2 + m^2)}_{0} \\
&\quad + (\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}})\underbrace{(\omega^2_{\mathbf{p}} + |\mathbf{p}|^2 + m^2)}_{2\omega^2_{\mathbf{p}}}\Big) \\
&= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{2\omega_{\mathbf{p}}^{\cancel{2}}}{\cancel{\omega_{\mathbf{p}}}}(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\underbrace{\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}}_{[\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{p}}]+\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\underbrace{[\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p})} + 2\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \\
&= \frac{1}{2}\int d^3p\ \omega_{\mathbf{p}}\delta^3(0) + \int \frac{d^3p}{(2\pi)^3}\ \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\ ,
\end{aligned}
$$

where we have used the fact that $\omega_{-\mathbf{p}} = \sqrt{|-\mathbf{p}|^2 + m^2} = \sqrt{|\mathbf{p}|^2 + m^2} = \omega_{\mathbf{p}}$.

q.e.d.

The first term of (6.7) counts simply how what is the relativistic energy of each particle $\omega_{\mathbf{p}}$ and through the number operator $\hat{N}_{\mathbf{p}} = \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}$ and the integral, we sum all over the possible value of $\mathbf{p}$. However, most of them may be zero and we do not have to worry about divergences.

6.2 Vacuum energy

Things are different if we look at the second term of (6.7), beacuse, in analogy with the energy of the single harmonic oscillator, we interpret it as the energy of the vacuum and it diverges for two reasons

1. infrared divergence, i.e.
$$\delta^3(0) \to \infty \ ,$$
2. ultraviolet divergence, i.e. for $|\mathbf{p}| \to \infty$
$$\int d^3p \ \omega_{\mathbf{p}} \to \infty \ ,$$
since for $|\mathbf{p}| \to \infty$
$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2} \simeq |\mathbf{p}| \ .$$

This can be better understood by applying the hamiltonian operator to the vacuum state $|0\rangle$, i.e. the state such that it is annihilated by all the annihilation operators is for all $\mathbf{p}$
$$\hat{a}_{\mathbf{p}}|0\rangle = 0 \quad \forall\mathbf{p} \ .$$
Therefore
$$\hat{H}|0\rangle = E_0|0\rangle = \infty|0\rangle$$
and the vaccum energy is infinite.

Proof. Infact, using (6.7)
$$\hat{H}|0\rangle = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\underbrace{\hat{a}_{\mathbf{p}}|0\rangle}_{0} + \underbrace{\left(\frac{1}{2}\int d^3p \ \omega_{\mathbf{p}}\delta^3(0)\right)}_{\infty}|0\rangle = \infty|0\rangle = E_0|0\rangle \ .$$

q.e.d.

IR divergence

The infrared divergence is due to the fact that space is infinitely large. This means that in every point of spacetime there is an harmonic oscilators. To prove this, consider a box of sides L and periodic boundary conditions for the fields. The volume of the box is just the Dirac delta inside the integrand of the energy vacuum. Infact

$$(2\pi)^3\delta^3(0) = \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x \, \exp(-i\mathbf{p}\cdot\mathbf{x})\Big|_{\mathbf{p}=0} = \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x = L^3 = V \, .$$

This divergence can be removed by studying energy densities instead of pure energies.

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{2} \, .$$

UV divergence

However, still the energy density is infinite because of the ultraviolet divergence, since for $|\mathbf{p}| \to \infty$

$$\mathcal{E}_0 \to \infty \, .$$

The reason is the following: we made a strong assumption considering the theory valid for any large value of energy and now we have found where the theory breaks, since this divergence arises indeed from the fact that our theory is not valid for arbitrarily high energies. What we need to do id to introduce a cut-off, i.e. a maximum energy after which the theory is not anymore valid. Since gravity cannot be neglected and becomes strongly coupled at Planck mass $M_P \simeq 10^{19} GeV$, we therefore set the cut-off at this energy.

Computationally, we measure only energy differences between excited estates, which are particles, and the vacuum energy, which becomes irrelevant and it can be set to zero. This procedure is called *normal ordering*. We define a new hamiltonian operator

$$:\hat{H}: \; = \hat{H} - E_0 = \hat{H} - \langle 0|\hat{H}|0\rangle \, ,$$

such that

$$:\hat{H}:|0\rangle = \underbrace{\hat{H}|0\rangle}_{E_0|0\rangle} - E_0|0\rangle = 0 \, .$$

The difference between $\hat{H}$ and $:\hat{H}:$ is due to an ambiguity in going from classical to quantum theory. Infact, normal ordering means to set a rule to order annihilation and creation operators: all annihilation operators are pleced to the right and, consequently, creation operatore to the left (dagger always first). We emphasise that in the interaction theory, vaccume energy cannot be anymore set to zero.

As we said, different ordering in the classical hamiltonians bring different hamiltonian operators. Infact, if we rewrite the hamiltonian of the classical harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}\omega^2 q^2 = \frac{1}{2}(\omega q - ip)(\omega q + ip) \ ,$$

we notice that the first one leads us to

$$\hat{H} = \omega(\hat{a}^\dagger \hat{a} + \frac{\mathbb{I}}{2}) \ ,$$

while the second one to

$$\hat{H} = \omega a^\dagger \hat{a} \ .$$

Proof. For the first hamiltonian

$$\begin{aligned}
\hat{H} &= \frac{1}{2}(-i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger))^2 + \frac{1}{2}\omega^2(\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger))^2 \\
&= -\frac{\omega}{4}(\cancel{\hat{a}^2} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \cancel{(\hat{a}^\dagger)^2}) + \frac{\omega}{4}(\cancel{\hat{a}^2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \cancel{(\hat{a}^\dagger)^2}) \\
&= \frac{\omega}{4}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\
&= \frac{\omega}{2}(\underbrace{\hat{a}\hat{a}^\dagger}_{[\hat{a},\hat{a}^\dagger]+\hat{a}^\dagger\hat{a}} + \hat{a}^\dagger\hat{a}) \\
&= \frac{\omega}{2}(\underbrace{[\hat{a}, \hat{a}^\dagger]}_{\mathbb{I}} + 2\hat{a}^\dagger\hat{a}) \\
&= \omega(\frac{\mathbb{I}}{2} + \hat{a}^\dagger\hat{a}) \ ,
\end{aligned}$$

while for the second hamiltonian

$$\begin{aligned}
\hat{H} &= \frac{1}{2}\Big(\omega\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger) - i(-i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger))\Big)\Big(\omega\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger) + i(-i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger))\Big) \\
&= \frac{\omega}{4}(\cancel{\hat{a}} + \hat{a}^\dagger - \cancel{\hat{a}} + \hat{a}^\dagger)(\hat{a} + \cancel{\hat{a}^\dagger} + \hat{a} - \cancel{\hat{a}^\dagger}) \\
&= \omega\hat{a}^\dagger\hat{a} \ .
\end{aligned}$$

q.e.d.

Finally, the normal ordered hamiltonian of the Klein-Gordon theory is

$$: \hat{H} : \ = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ . \tag{6.9}$$

Proof. Infact, since

$$\hat{H} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \ ,$$

we have

$$:\hat{H}: \ = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ .$$

q.e.d.

Furthermore, by analogy of the harmonic oscillator, the hamiltonian (6.7) and the annihilation and creation operators satisfies the commutation relations

$$[\hat{H}, \hat{a}_{\mathbf{p}}] = -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ , \quad [\hat{H}, \hat{a}^\dagger_{\mathbf{p}}] = \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} \ .$$

Proof. For the first commutator

$$\begin{aligned}
[\hat{H}, \hat{a}_{\mathbf{p}}] &= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}[\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{p}}] \\
&= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}(\hat{a}^\dagger_{\mathbf{q}}\underbrace{[\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{p}}]}_{0} + \underbrace{[\hat{a}^\dagger_{\mathbf{q}}, \hat{a}_{\mathbf{p}}]}_{-(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}\hat{a}_{\mathbf{q}}) \\
&= -\int \frac{d^3q}{\cancel{(2\pi)^3}}\omega_{\mathbf{q}}\cancel{(2\pi)^3}\delta^3(\mathbf{p}-\mathbf{q})\hat{a}_{\mathbf{q}} \\
&= -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ .
\end{aligned}$$

For the second commutator

$$\begin{aligned}
[\hat{H}, \hat{a}^\dagger_{\mathbf{p}}] &= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}[\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{p}}] \\
&= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}(\hat{a}^\dagger_{\mathbf{q}}\underbrace{[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} + \underbrace{[\hat{a}^\dagger_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{p}}]}_{0}\hat{a}_{\mathbf{q}}) \\
&= \int \frac{d^3q}{\cancel{(2\pi)^3}}\omega_{\mathbf{q}}\cancel{(2\pi)^3}\delta^3(\mathbf{p}-\mathbf{q})\hat{a}^\dagger_{\mathbf{q}} \\
&= \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} \ .
\end{aligned}$$

q.e.d.

The momentum operator is defined as

$$\hat{\mathbf{P}} = -\int d^3x \ \hat{\pi}\boldsymbol{\nabla}\hat{\varphi} = \int \frac{d^3p}{(2\pi)^3}\mathbf{p}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ . \tag{6.10}$$

Proof. Infact, using (3.2)

$$
\begin{aligned}
\hat{\mathbf{P}} &= \int d^3x\ T^{0i} \\
&= \int d^3x\ \hat{\pi}\boldsymbol{\nabla}\hat{\varphi}\ .
\end{aligned}
$$

Furthermore, using (6.6), (6.4), (6.5) and (5.1)

$$
\begin{aligned}
\hat{\mathbf{P}} = &-\int d^3x\ \Big(\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\nabla\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\Big(\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) + \hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big) \\
= &-\int d^3x\ \Big(\int \frac{d^3p}{(2\pi)^3}\Big(-i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big)\Big(\hat{a}_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{x}) - \hat{a}^\dagger_{\mathbf{p}}\exp(-i\mathbf{p}\cdot\mathbf{x})\Big) \\
&\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\Big(i\mathbf{q}\hat{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}) - i\mathbf{q}\hat{a}^\dagger_{\mathbf{q}}\exp(-i\mathbf{q}\cdot\mathbf{x})\Big)\Big) \\
= &-\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\Big(-\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\Big)(i\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) - i\mathbf{q}\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}) \\
&- i\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x}) + i\mathbf{q}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})) \\
= &-\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\Big(\frac{\mathbf{q}}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\Big)(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} \\
&- \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}-\mathbf{q})} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\exp(i(-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})}_{\delta^3(\mathbf{p}+\mathbf{q})}) \\
= &-\int \frac{d^3p\ d^3q}{(2\pi)^3}\Big(\frac{\mathbf{q}}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\Big)(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{q}=-\mathbf{p}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} \\
&- \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\underbrace{\delta^3(\mathbf{p}+\mathbf{q})}_{\mathbf{q}=-\mathbf{p}}) \\
= &-\int \frac{d^3p}{(2\pi)^3}\Big(\frac{\mathbf{p}}{2}(-\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}) - \frac{\mathbf{p}}{2}(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}})\Big) \\
= &\int \frac{d^3p}{(2\pi)^3}\Big(\frac{\mathbf{p}}{2}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}) + \frac{\mathbf{p}}{2}(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}})\Big)\ ,
\end{aligned}
$$

which in normal ordering becomes

$$
\hat{\mathbf{P}} = \int \frac{d^3p}{(2\pi)^3}\Big(\frac{\mathbf{p}}{2}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}})\Big) = \int \frac{d^3p}{(2\pi)^3}\mathbf{p}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\ .
$$

q.e.d.

6.3 1-particle states

Now, we build the energy eigenstates of a 1-particle state. In analogy with the harmonic oscillator, we require the following properties:

1. the vacuum state is annihilated by all the annihilation operators for all $\mathbf{p}$

$$\hat{a}_{\mathbf{p}}|0\rangle = 0 \quad \forall \mathbf{p} \ ,$$

2. a generic state can be defined by the creation operators actiong on the vacuum

$$|\mathbf{p}\rangle = \hat{a}^\dagger_{\mathbf{p}}|0\rangle \ .$$

The state $|\mathbf{p}\rangle$ is the momentum eigentstate of a single scalar (spinless) particle with mass m. Infact, it is the momentum eigenstate

$$\hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \ ,$$

Proof. Infact, using (6.10)

$$\begin{aligned}
\hat{\mathbf{P}}|\mathbf{p}\rangle &= \hat{\mathbf{P}}\hat{a}^\dagger_{\mathbf{p}}|0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\mathbf{q}\hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}}_{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]+\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}} |0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\mathbf{q}\hat{a}^\dagger_{\mathbf{q}}(\underbrace{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} +\hat{a}^\dagger_{\mathbf{q}}\underbrace{\hat{a}_{\mathbf{p}})|0\rangle}_{0} \\
&= \int \frac{d^3q}{\cancel{(2\pi)^3}}\mathbf{q}\hat{a}^\dagger_{\mathbf{q}}\cancel{(2\pi)^3}\underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}}|0\rangle \\
&= \mathbf{p}\hat{a}^\dagger_{\mathbf{p}}|0\rangle \\
&= \mathbf{p}|\mathbf{p}\rangle \ .
\end{aligned}$$

q.e.d.

Furthermore, this states is also the energy eigenstate, since it is a function of the momentum

$$\hat{H}|\mathbf{p}\rangle = E_{\mathbf{p}}|\mathbf{p}\rangle = \omega_{\mathbf{p}}|\mathbf{p}\rangle \ .$$

Proof. Infact, using (6.9)

$$
\begin{aligned}
\hat{H}|\mathbf{p}\rangle &= \hat{H}\hat{a}^\dagger_{\mathbf{p}}|0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}}_{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]+\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}} |0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}}(\underbrace{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} +\hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{p}})|0\rangle}_{0} \\
&= \int \frac{d^3q}{\cancel{(2\pi)^3}}\omega_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}}\cancel{(2\pi)^3} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} |0\rangle \\
&= \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}|0\rangle \\
&= \omega_{\mathbf{p}}|\mathbf{p}\rangle \ .
\end{aligned}
$$

q.e.d.

6.4 n-particle states

We can generalise for a system composed by n particles. The state becomes

$$|\mathbf{p}_1,\dots\mathbf{p}_n\rangle = \hat{a}^\dagger_{\mathbf{p}_1}\dots\hat{a}^\dagger_{\mathbf{p}_n}|0\rangle \ .$$

Notice that the state is symmetric under exchange of any two particles, since

$$[\hat{a}^\dagger_{\mathbf{p}_i},\hat{a}^\dagger_{\mathbf{p}_j}] = 0 \ .$$

Proof. For instance, given two particles of momenta $\mathbf{p}$ and $\mathbf{q}$, we have

$$|\mathbf{p},\mathbf{q}\rangle = \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}|0\rangle = \hat{a}^\dagger_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = |\mathbf{q},\mathbf{p}\rangle \ .$$

q.e.d.

This means that the Klein-Gordon theory describes bosons. It is indeed spin-statistics relation and it is a consequence of quantum field theory and the commutation relations imposed to quantise (not quantum mechanics).

A basis of the Fock space is built by all the possible combination of creation operators acting on the vacuum state

$$\{|0\rangle, \hat{a}^\dagger_{\mathbf{p}_1}|0\rangle, \hat{a}^\dagger_{\mathbf{p}_1}\hat{a}^\dagger_{\mathbf{p}_2}|0\rangle, \dots\}$$

, where $|0\rangle$ is the vacuum state, $\hat{a}^\dagger_{\mathbf{p}_1}|0\rangle$ is the 1-particle state, $\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}^\dagger_{\mathbf{p}_2}|0\rangle$ is the 2-particles state, etc. The total Fock space is

$$\mathcal{F} = \bigoplus_n \mathcal{H}_n$$

where $\mathcal{H}_n$ is the Hilbert space for n particles.

We can define the number operator wich counts the number of particle in a given state

$$\hat{N} = \int \frac{d^3p}{(2\pi)^3}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ ,$$

such that

$$\hat{N}|\mathbf{p}_1, \dots \mathbf{p}_n\rangle = \hat{N}\hat{a}^\dagger_{\mathbf{p}_1}\dots\hat{a}^\dagger_{\mathbf{p}_n}|0\rangle = n|\mathbf{p}_1, \dots \mathbf{p}_n\rangle \ .$$

Notice that the particle number is conserved, since

$$[\hat{H}, \hat{N}] = 0 \ .$$

This means that if the system has initially n particles, this number will remain the same. This happens only in a free theory, because interactions move the system between different sectors of the Fock space.

Proof. Infact

$$\begin{aligned}
[\hat{H}, \hat{N}] &= \hat{H}\hat{N} - \hat{N}\hat{H} \\
&= \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \int \frac{d^3q}{(2\pi)^3}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}} - \int \frac{d^3q}{(2\pi)^3}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}} \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6}\omega_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}} \underbrace{\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}}_{[\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{q}}]+\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}} \hat{a}_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}} \underbrace{\hat{a}_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}}_{[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}]+\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}} \hat{a}_{\mathbf{p}}) \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6}\omega_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}}(\underbrace{[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}})\hat{a}_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}}(\underbrace{[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{p}}]}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})} + \hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}})\hat{a}_{\mathbf{p}}) \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^6}\omega_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}}((2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}})\hat{a}_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}}((2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) + \hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}})\hat{a}_{\mathbf{p}}) \\
&= \int \frac{d^3p\ d^3q}{(2\pi)^{\cancel{6}}}\omega_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}}\cancel{(2\pi)^3} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} \hat{a}_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}}\cancel{(2\pi)^3} \underbrace{\delta^3(\mathbf{p}-\mathbf{q})}_{\mathbf{q}=\mathbf{p}} \hat{a}_{\mathbf{p}}) \\
&\qquad + \int \frac{d^3p\ d^3q}{(2\pi)^6}\omega_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\hat{a}_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \\
&= \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(\cancel{\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}} - \cancel{\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}}) + \int \frac{d^3p\ d^3q}{(2\pi)^6}\omega_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\hat{a}_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{p}})
\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{d^3p\, d^3q}{(2\pi)^6} \omega_{\mathbf{p}} (\hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{q}} \hat{a}_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{p}}) \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^6} \omega_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{q}} \hat{a}_{\mathbf{q}} - \int \frac{d^3p\, d^3q}{(2\pi)^6} \omega_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{p}} \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^6} \omega_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{q}} \hat{a}_{\mathbf{q}} - \int \frac{d^3p\, d^3q}{(2\pi)^6} \omega_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{q}} \hat{a}_{\mathbf{q}} \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^6} (\omega_{\mathbf{p}} - \omega_{\mathbf{q}}) \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{q}} \hat{a}_{\mathbf{q}} \\
&=
\end{aligned}
$$

where we have exchanged $\mathbf{p} \leftrightarrow \mathbf{q}$, since they are integral variables.

TO COMPLETE!

q.e.d.

6.5 Lorentz covariance

The vacuum state is normalised

$$\langle 0|0\rangle = 1 \ ,$$

while 1-particle states satisfiy the orthogonality relation

$$\langle \mathbf{p}|\mathbf{q}\rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q})$$

and the completeness relation

$$\mathbb{I} = \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}\rangle\langle\mathbf{p}| \ ,$$

where $\mathbb{I}$ is the identity operator.

Proof. Maybe in the future. q.e.d.

However, we want Lorentz covariance, since the identity operator is so but the right side of the completeness relation is not, given that the measure $\int d^3p$ and the projector $|\mathbf{p}\rangle\langle\mathbf{p}|$ are not separately so. We know that $\in d^4p$ is Lorentz covariant, because

$$d^4p' = \underbrace{|\det \Lambda|}_{1} d^4p = d^4p \ .$$

Therefore, we change the orthogonality relation into

$$\langle p|q\rangle = (2\pi)^3 2\sqrt{E_{\mathbf{p}} E_{\mathbf{q}}} \delta^3(\mathbf{p} - \mathbf{q})$$

and the completeness relation into

$$\mathbb{I} = \int \frac{d^4p}{(2\pi)^3}\delta(p_0^2 - |\mathbf{p}|^2 - m^2)\theta(p_0)|\mathbf{p}\rangle\langle\mathbf{p}| \ ,$$

where $p_0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ and the manifestly invariant states are

$$|p\rangle = \sqrt{2E_{\mathbf{p}}}|\mathbf{p}\rangle \ . \tag{6.11}$$

Proof. Maybe in the future. q.e.d.

Chapter 7

Two real (or complex) Klein-Gordon field

List of Theorems