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On group theory:

groups, representations and all that

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Theoretical Physics

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Part I

Lie groups and representations

Chapter 1

Groups

The justification of a group can be found in the structure of a physics transformation: given two transformation, the composition of them should be defined together with the unit one, i.e. nothing happens, and the inverse one, i.e. if you want to return back to the initial system.

Definition 1.1 (Group)
A group is a set of elements $G = \{g_i\}$ associated with a composition map

$$\cdot\colon G \times G \to G$$

satisfying the following properties

1. *closure, i.e.*
$$g_1 g_2 \in G \quad \forall g_1, g_2 \in G$$
2. *associativity, i.e.*
$$(g_1 g_2) g_3 = g_1 (g_2 g_3) = g_1 g_2 g_3 \quad \forall g_1, g_2, g_3 \in G$$
3. *unit element, i.e.*
$$\exists! g_0 \in G\colon g_0 g = g g_0 = g \quad \forall g \in G$$
4. *inverse element, i.e.*
$$\exists! g^{-1} \in G\colon g^{-1} g = g g^{-1} = g_0 \quad \forall g \in G$$

Definition 1.2 (Abelian group)
A group is said to abelian if

5. *commutativity, i.e.*
$$g_1 g_2 = g_2 g_1 \quad \forall g_1, g_2 \in G$$

Definition 1.3 (Subgroup)
A subgroup is a subset $H \subset G$ of a group which is also a group itself with closed restricted composition map.

Example 1.1 (Groups)**.** Examples of groups are

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ with composition map $+$ and unit element 0,
2. $\mathbb{Q} \setminus \{0\}$, $\mathbb{Q} \setminus \{0\}$ with composition map $\times$ and unit element 1,
3. $\mathbb{Z}_n = \{z \in [0,\ n-1] \ :\ a + n = a\}$ with composition map $+$ and unit element 0.

Example 1.2 (Matrix groups)**.** Matrices are a non-abelian group with matrix multiplication as composition map:

1. $GL(n) = \{M \in Mat_{n\times n}(\mathbb{R}) \ :\ \det M \neq 0\}$

Given a fixed invertible $n \times n$ matrix B, a subgroup of $GL(n)$ is the set of matrices M which preserve this matrix, i.e. $M^t B M = B$:

2. $O(n) = \{R \in Mat_{n\times n}(\mathbb{R}) \ :\ R^t \mathbb{1} R = \mathbb{1}\}$ with Euclidean metric $B = \mathbb{1}$,
3. $O(1,\ n-1) = \{\Lambda \in Mat_{n\times n}(\mathbb{R}) \ :\ \Lambda^t \eta \Lambda = \eta\}$ with Minkovskian metric $B = \eta$.

Over the complex field

4. $U(n) = \{U \in Mat_{n\times n}(\mathbb{C}) \ :\ U^\dagger U = \mathbb{1}\}$

Imposing $\det M = 1$, we find the special groups

5. $SL(n) = \{M \in GL(n) \ :\ \det M = 1\}$,
6. $SO(n) = \{R \in O(n) \ :\ \det R = 1\}$,
7. $SO(1,\ n-1) = \{\Lambda \in O(1,\ n-1) \ :\ \det \Lambda = 1\}$,
8. $SU(n) = \{M \in U(n) \ :\ \det U = 1\}$.

Chapter 2

Lie groups

Definition 2.1 (Lie group)
A Lie group is a group endowed with a manifold structure such that composition and inverse are smooth maps, i.e.

$$\begin{aligned}\mu\colon G \times G &\to G \\ (x, y) &\mapsto x^{-1}y\end{aligned}$$

In Lie groups, we can introduce the notions of closeness and power series. The tangent space at $g_0 \in G$, i.e. elements of the group infinitesimally away from the unit element, gives rise of the Lie algebra.

Definition 2.2 (Lie algebra)
A Lie algebra is a linear space equipped with an anti-symmetric product, called Lie brackets

$$[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the following properties

1. *linearity, i.e.*

$$[\alpha X + \beta Y,\ Z] = \alpha[X,\ Z] + \beta[Y,\ Z] \quad \forall X,\ Y,\ Z \in \mathfrak{g} \quad \forall \alpha,\ \beta \in \mathbb{R}$$

2. *anti-symmetry, i.e.*

$$[X,\ Y] = -[Y,\ X] \quad \forall X,\ Y \in \mathfrak{g}$$

3. *Jacobi identity, i.e.*

$$[X,\ [Y,\ Z]] + [Y,\ [Z,\ X]] + [Z,\ [X,\ Y]] = 0 \quad \forall X,\ Y,\ Z \in \mathfrak{g}$$

A Lie algebra can be encoded with the structure constants f_{ijk}

$$[T_i,\ T_j] = f_{ijk} T_k$$

where $\{T_i\}$ is a basis of $\mathfrak{g}$.

The exponential map helps to construct a group element that is finitely away as $g = \exp(X)$ from $X \in \mathfrak{g}$, tied to the existence of a unique path $\gamma\colon \mathbb{R} \to G$ such that $\gamma(0) = g_0$ and $\gamma(1) = g$ which is a one-parameter subgroup $\{\gamma(s) \ : \ s \in \mathbb{R}\}$ whose tangent vector at g_0 is X.

A useful formula is the Baker-Campbell-Hausdorff one, which connects the group composition with the Lie brackets

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X,\ Y] + \ldots) \tag{2.1}$$

Summarizing, from structure constants we define a Lie algebra, from the exponential map and the BCH formula we define Lie group elements in terms of generators of the Lie algebra $\{T_i\}$.

Chapter 3

Representations

In physics, it is useful to study how groups act on objects, in particular matrix groups act on vectors belonging to linear spaces.

Definition 3.1 (Representation)
A linear representation of a group G is a group homomorphism

$$\rho \colon G \to Aut(V)$$

satisfying the following property

1. *composition map, i.e.*

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) \quad \forall g_1, g_2 \in G$$

It is possible to derive other properties

2. unit element, i.e.
$$\rho(g_0) = \mathbb{1}_V$$

3. inverse element, i.e.
$$\rho(g^{-1}) = \rho^{-1}(g) \quad \forall g \in G$$

For finite-dimensional linear spaces, i.e. $\dim V = n$, and $Aut(V) \simeq GL(n)$ after picking a basis. Furthermore, $\dim(\rho,\ V) = \dim V$. A representation $(\rho,\ V)$ acts as a linear transformation on a vector $v \in V$ as $\rho(g)v$.

Definition 3.2 (Reducible, irreducible representation)
A representation $(\rho,\ V)$ is reducible if

$$\nexists U \subset V \ : \ \rho(g)u \in U \quad \forall u \in U$$

otherwise, it is irreducible.

A reducible representation can be always put in block triangle form, choosing a suitable basis

$$\rho(g) = \begin{bmatrix} \rho_1(g) & B(g) \\ 0 & \rho_2(g) \end{bmatrix}$$

with the invariant subspace is $U = \{(u,\ 0) \in V\}$. If $B(g) = 0$, the representation is completely reducible and decomposes into the direct sum of $\rho = \rho_1 \oplus \rho_2$.

Definition 3.3 (Equivalent representations)
Two representations ρ_1 and ρ_2 of the same dimension are equivalent if

$$\exists S invertible\ :\ \ \rho_2(g) = S^{-1}\rho_1(g)S \quad \forall g \in G$$

which means that there exists a basis change that relate the representations.

Definition 3.4 (Faithful representation)
A representation ρ is faithful if

$$g_1 \neq g_2 \ \Rightarrow\ \rho(g_1) \neq \rho(g_1)$$

For non-faithful representations, there exists $H \subset G$ such that $\rho(h) = \mathbb{1} \quad \forall h \in H$.

Definition 3.5
Over the complex field, i.e. $\rho\colon G \to GL(n,\ \mathbb{C})$, a representation is unitary if

$$\rho(g^{-1}) = \rho^{-1}(g) = \rho(g)^{\dagger}$$

For any group, there exists a trivial 1-dimensional representation, $\rho(g) = 1$. For any matrix group, there exists a non-trivial 1-dimensional representation, $\rho(g) = \det(g)$.

3.1 Representations of Lie Groups and Lie Algebras

The defining representation is $\rho(g) = g$. For a $n \times n$ matrix group, the defining representation has the dimension n.

A representation of a Lie algebra is a set of endomorphisms on a vector space V

$$\rho_{\mathfrak{g}}\colon \mathfrak{g} \to End(V)$$

where End means the set of linear maps $V \to V$. Given a basis, it becomes the set of matrices. Furthermore, the compatibility with the Lie algebra require the further condition

$$\rho_{\mathfrak{g}}[X, Y] = \rho_{\mathfrak{g}}X\rho_{\mathfrak{g}}Y - \rho_{\mathfrak{g}}Y\rho_{\mathfrak{g}}X$$

or, given a set of generators $\{T_i\}$,

$$\rho_{\mathfrak{g}}[T_i, T_j] = f_{ijk}\rho_{\mathfrak{g}}T_k$$

3.2 From Lie group rep to Lie algebra rep

Any representation of a Lie group (ρ, V) induces a representation of its Lie algebra. Infact, an element of the group $g = \exp(tX)$ gives a path of transformations on V and we can define a representation on V with

$$\rho_{\mathfrak{g}}(X)(v) = \frac{d}{dt}\rho(\exp(tX))\Big|_{t=0}$$

Hence, $\rho_{\mathfrak{g}}X$ is the same size of $\rho(g)$ and respects the Lie brackets. $(\rho_{\mathfrak{g}}, V)$ is representation of the Lie algebra.

For a unitary representation, the Lie algebra representations are anti-Hermitian matrices, i.e. $\rho_{\mathfrak{g}}^{\dagger}(X) = -\rho_{\mathfrak{g}}(X)$.

Proof. Because of unitarity,

$$\rho(\exp(tX))^{\dagger}\rho(\exp(tX)) = 1$$

and deriving it

$$0 = \frac{d}{dt}\rho(\exp(tX))^{\dagger}\rho(\exp(tX))\Big|_{t=0} = \frac{d}{dt}(\rho_{\mathfrak{g}}^{\dagger}(X)+\rho_{\mathfrak{g}}(X))\rho(\exp(tX))^{\dagger}\rho(\exp(tX))\Big|_{t=0} = \rho_{\mathfrak{g}}^{\dagger}(X)+\rho_{\mathfrak{g}}(X)$$

Hence

$$\rho_{\mathfrak{g}}^{\dagger}(X) = i\rho_{\mathfrak{g}}(X)$$

q.e.d.

In Physics, it is more convenient working with Hermitian matrices, then introducing the related represention

$$\tilde{\rho}_{\mathfrak{g}}(X) = i\rho_{\mathfrak{g}}$$

and the commutator becomes

$$[\tilde{\rho}_{\mathfrak{g}}(T_i), \tilde{\rho}_{\mathfrak{g}}(T_j)] = -[\rho_{\mathfrak{g}}(T_i), \rho_{\mathfrak{g}}(T_j)] = -f_{ijk}\rho_{\mathfrak{g}}(T_k) = if_{ijk}\rho_{\mathfrak{g}}(T_k)$$

3.3 From Lie algebra rep to Lie group rep

Not all the representations of a Lie algebra extend to a rep of the Lie groups, because the Lie algebra gives information only locally, around the identity, while the Lie group could have different global topology.

However, if the group is simply connected, i.e. all closed paths are contractible (deformable to a point), the Lie algebra rep is also a Lie group rep. If the Lie group G is not simply-connected, there is always another Lie group $\tilde{G}$, the universal cover, which has the same rep as the Lie algebra. Cover because there is a surjective projection homomorphism $\phi\colon \tilde{G} \to G$. Furthermore, they have the same Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}$.

Proof. Geometrically, there is always an open neighbourhood such that the projection map is a diffeomorphism. This means that the tangent space at evry point is isomorphic to the tangent space at the identity. q.e.d.

3.4 Reps in QM

In quantum mechanics, a state is not uniquely defined. Infact it is characterized by a Hilbert space vector up to a phase factor. Two states are the same physical state if

$$|\psi\rangle = \exp(i\lambda)|\phi\rangle$$

Hence, we can relax the definition of the group structure, allowing also a projective representation

$$\rho(g)\rho(h) = \exp(i\phi(g,h))\rho(gh)$$

where $\phi(g,h) \in \mathbb{R}$.

All unitary projective representations come from a linear representation of the covering group.

Part II

SO(3) and SO(1, 3)

Chapter 4

SO(3)

4.1 SO(3) as a Lie group

In this chapter, we will study the three-dimensional rotations group $O(3)$. Computing the determinant, we can decomposed the Lie group into two parts according to the sign of it:

$$\Rightarrow O(3) = \underbrace{\{\det R = +1\}}_{SO(3)} \cup \{\det R = -1\} = SO(3) \cup \det R = -1$$

Since there is no continuos path that connect the two parts and only $SO(3)$ contains the identity, we are going to study $SO(3)$ and recover the other one with a reflexion along an axis.

Any $SO(3)$ rotation can be parametrized by a unit vector, perpendicular to the rotation plane, and a rotation angle θ:

$$R(\theta,\ \mathbf{n})_{ij} = \cos\theta\ \delta_{ij} + (1-\cos\theta)\ n_i n_j - \sin\theta\ \epsilon_{ijk} n_k$$

Hence, an infinitesimal rotation $\delta\theta$ near the identity $R(\theta=0,\ \mathbf{n}) = \delta_{ij}$ is

$$R(\delta\theta,\ \mathbf{n})_{ij} = \underbrace{\cos\delta\theta}_{1}\ \delta_{ij} + (1-\underbrace{\cos\delta\theta}_{1})\ n_i n_j - \underbrace{\sin\delta\theta}_{\delta\theta}\ \epsilon_{ijk} n_k = \delta_{ij} - \delta\theta\ \epsilon_{ijk} n_k$$

and its action on an arbitrary vector $\mathbf{v}$ is

$$R(\delta\theta,\ \mathbf{n})_{ij} v_i = \delta_{ij} v_i - \delta\theta\ \epsilon_{ijk} v_i n_k = \delta_{ij} v_i + \delta\theta\ \epsilon_{jik} v_i n_k$$

or

$$R(\delta\theta,\ \mathbf{n})\mathbf{v} = \mathbf{v} + \delta\theta\ \epsilon jik v_i n_k$$

4.2 SO(3) representations

In this chapter,

Chapter 5

SO(1, 3)

The Lorentz group is defined by the matrices Λ such that preserve the Minkovski metric

$$\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}\ \ \nu\eta_{\alpha\beta} = \eta_{\mu\nu} \tag{5.1}$$

First, the Lorentz group can be decomposed into two parts according to their determinant

$$\det(\Lambda^T\eta\Lambda) = \det\Lambda^T\det\eta\det\Lambda = \overset{2}{\det}\Lambda = \det\eta = 1$$

Hence

$$\det\Lambda = \pm 1$$

and the Lorentz group can be written as

$$O(1,3) = \underbrace{\{\det\Lambda = +1\}}_{SO(1,3)}\cup\{\det\Lambda = -1\}$$

where $SO(1,3)$ is called the proper Lorentz group.

Second, the proper Lorentz group can be decomposed into two parts according to their (0, 0) component

$$\begin{aligned}\eta_{00} &= \Lambda^{\alpha}{}_{0}\Lambda^{\beta}{}_{0}\eta_{\alpha\beta}\\ -1 &= -(\Lambda^{0}{}_{0})^2 + (\Lambda^{i}{}_{i})^2\\ (\Lambda^{0}{}_{0})^2 &= 1 + (\Lambda^{i}{}_{i})^2 \geq 1\end{aligned}$$

Hence

$$\Lambda^{0}{}_{0} \in]\infty, -1] \cup [1, \infty[$$

and the proper Lorentz group can be written as

$$SO(1,3) = \underbrace{\{\Lambda^{0}{}_{0} \in]\infty, -1]\}}_{SO(1,3)^+}\cup\{\Lambda^{0}{}_{0} \in [1, \infty[\}$$

where $SO(1,3)^+$ is called the proper orthochronous Lorentz group.

From now on, only the proper orthochronous Lorentz group will be studied because is the only group containing the identity.

5.1 Lie algebra: generators of $SO(1,3)^+$

Consider an infinitesimal Lorentz transformation around the identity

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{5.2}$$

where $\omega^\mu{}_\nu \ll 1$ is an infinitesimal matrix.

Using (5.1),

$$(\delta^\alpha{}_\mu + \omega^\alpha{}_\mu)(\delta^\beta{}_\nu + \omega^\beta{}_\nu)\eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\delta^\alpha{}_\mu\delta^\beta{}_\nu\eta_{\alpha\beta} + \delta^\alpha{}_\mu\omega^\beta{}_\nu\eta_{\alpha\beta} + \omega^\alpha{}_\mu\delta^\beta{}_\nu\eta_{\alpha\beta} + \omega^\alpha{}_\mu\omega^\beta{}_\nu\eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\cancel{\eta_{\mu\nu}} + \omega_{\mu\nu} + \omega_{\nu\mu} + O(\omega^2) = \cancel{\eta_{\mu\nu}}$$

Hence, the matrices $\omega_{\mu\nu}$ are anti-symmetric

$$\omega_{\mu\nu} = \omega_{\nu\mu}$$

Using the exponential map, a generic $SO(1,3)^+$ transformation can be written as

$$\Lambda^\mu{}_\nu = \exp(-\frac{i}{2}\omega^{\alpha\beta}M_{\alpha\beta})^\mu{}_\nu$$

where $M_{\alpha\beta}$ are the generators of the Lie algebra $\mathfrak{so}(1,3)$. Since they must be antisymmetric, otherwise they would vanish, there are six independent generators of $\mathfrak{so}(1,3)$.

5.2 Lie algebra: commutators of $SO(1,3)^+$

To find the explicit expression of the commutator of two generators, first it will be computed the following expression using (5.2)

$$\begin{aligned}
(\tilde\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\alpha{}_\beta\tilde\Lambda^\beta{}_\gamma\Lambda^\gamma{}_\nu &= (\delta^\mu{}_\alpha - \tilde\omega^\mu{}_\alpha)(\delta^\alpha{}_\beta - \omega^\alpha{}_\beta)(\delta^\beta{}_\gamma + \tilde\omega^\beta{}_\gamma)(\delta^\gamma{}_\nu + \omega^\gamma{}_\nu) + O(\tilde\omega^2) + O(\omega^2)\\
&= \delta^\mu{}_\alpha\delta^\alpha{}_\beta\delta^\beta{}_\gamma\delta^\gamma{}_\nu - \tilde\omega^\mu{}_\alpha\delta^\alpha{}_\beta\delta^\beta{}_\gamma\delta^\gamma{}_\nu - \delta^\mu{}_\alpha\omega^\alpha{}_\beta\delta^\beta{}_\gamma\delta^\gamma{}_\nu + \delta^\mu{}_\alpha\delta^\alpha{}_\beta\tilde\omega^\beta{}_\gamma\delta^\gamma{}_\nu\\
&\quad + \delta^\mu{}_\alpha\delta^\alpha{}_\beta\delta^\beta{}_\gamma\tilde\omega^\beta{}_\gamma + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\beta\delta^\beta{}_\gamma\delta^\gamma{}_\nu - \tilde\omega^\mu{}_\alpha\delta^\alpha{}_\beta\delta^\beta{}_\gamma\omega^\gamma{}_\nu - \delta^\mu{}_\alpha\omega^\alpha{}_\beta\tilde\omega^\beta{}_\gamma\delta^\gamma{}_\nu\\
&\quad + \delta^\mu{}_\alpha\tilde\omega^\alpha{}_\beta\delta^\beta{}_\gamma\omega^\gamma{}_\nu + O(\tilde\omega^2) + O(\omega^2)\\
&= \delta^\mu{}_\nu - \cancel{\tilde\omega^\mu{}_\nu} - \cancel{\omega^\mu{}_\nu} + \cancel{\tilde\omega^\mu{}_\nu} + \cancel{\omega^\mu{}_\nu} + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu - \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu - \omega^\mu{}_\gamma\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu\\
&\quad + O(\omega^2) + O(\tilde\omega^2)\\
&= \delta^\mu{}_\nu + \cancel{\tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu} - \cancel{\tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu} - \omega^\mu{}_\gamma\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu + O(\omega^2) + O(\tilde\omega^2)\\
&= \delta^\mu{}_\nu - \omega^\mu{}_\gamma\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu + O(\omega^2) + O(\tilde\omega^2)
\end{aligned}$$

and second using the BCH formula (2.1)

$$
\begin{aligned}
(\tilde\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\alpha{}_\beta\tilde\Lambda^\beta{}_\gamma\Lambda^\gamma{}_\nu &= \exp(\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta})\exp(\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho})\exp(-\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta})\exp(-\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho})\\
&= \exp(\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta}+\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}+[\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta},\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}])\\
&\quad \exp(-\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta}-\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}+[\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta},\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}])+O(\omega^2)+O(\tilde\omega^2)\\
&= \exp(\cancel{\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta}}+\cancel{\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}}-\cancel{\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta}}-\cancel{\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}}\\
&\quad +[\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta},\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}])+O(\omega^2)+O(\tilde\omega^2)\\
&= \exp([\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta},\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}])+O(\omega^2)+O(\tilde\omega^2)
\end{aligned}
$$

Hence, putting together

$$
\exp([\frac{i}{2}\tilde\omega^{\alpha\beta}M_{\alpha\beta},\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}]) = \delta^\mu{}_\nu - \omega^\mu{}_\gamma\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu = \exp(-\frac{i}{2}(-\omega^\mu{}_\gamma\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu)M_\mu{}^\nu)
$$

$$
-\frac{1}{4}\tilde\omega^{\alpha\beta}\omega^{\sigma\rho}[M_{\alpha\beta},M_{\sigma\rho}] = -\frac{i}{2}(-\omega^\mu{}_\alpha\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu)M_\mu{}^\nu = -\frac{i}{2}\eta_{\alpha\gamma}(-\omega^{\mu\alpha}\tilde\omega^{\gamma\nu}+\tilde\omega^{\mu\alpha}\omega^{\gamma\nu})M_{\mu\nu}
$$

Hence

$$
-\frac{1}{4}\tilde\omega^{\alpha\beta}\omega^{\sigma\rho}[M_{\alpha\beta},M_{\sigma\rho}] = -\frac{i}{2}\eta_{\alpha\gamma}(-\omega^{\mu\alpha}\tilde\omega^{\gamma\nu}+\tilde\omega^{\mu\alpha}\omega^{\gamma\nu})M_{\mu\nu}
$$

Consider an ansatz

$$
[M_{\alpha\beta},M_{\sigma\rho}] = T^{(1)}_{\alpha\beta}M_{\sigma\rho} + T^{(2)}_{\alpha\sigma}M_{\beta\rho} + T^{(3)}_{\alpha\rho}M_{\beta\sigma} + T^{(4)}_{\beta\sigma}M_{\alpha\rho} + T^{(5)}_{\beta\rho}M_{\alpha\sigma} + T^{(6)}_{\sigma\rho}M_{\alpha\beta}
$$

and inserting into the previous, there are no matching term with $T^{(1)} = T^{(2)} = 0$.

Furthermore, using $M_{\mu\nu} = -M_{\nu\mu}$

$$
0 = [M_{\alpha\beta},M_{\sigma\rho}] + [M_{\beta\alpha}.M_{\sigma\rho}]
$$

5.3 Rep of Lorentz algebra

$SO^+(1,3)$ is not compact, since the values of the boosts do not have an upper buondary. For non-compact non abelian Lie groups, any non-trivial unitary representation must be infinite-dimensional.

Now, let us focus in finite-dimensional representations. They are not unitary, but for fields representations do not matter since there is no scalar product between operators.

The defining rep of the Lie algebra is

$$
(M_{\alpha\beta})^\mu{}_\nu = i(\delta^\mu{}_\alpha\eta_{\beta\nu} - \delta^\mu{}_\beta\eta_{\alpha\nu})
$$

The generators of the rotations are hermitian

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$$

while the generators of the boosts are anti-hermitian, beacuse they do not have a finite range,

$$K_i = M_{0i}$$

The procedure to construct Lorentz algebra irrep is complexification: consider a complex linear combination $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$.

Defining new generators

$$A_i = \frac{1}{2}(J_i + iK_i) \quad B_i = \frac{1}{2}(J_i - iK_i)$$

their new commutation relations are

$$[A_i, A_j] = i\epsilon_{ijk}A_K \quad [B_i, B_j] = i\epsilon_{ijk}B_K \quad [A_i, B_j] = 0$$

Hence $\mathfrak{so}(1,3) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In this way, we can use representations of $\mathfrak{su}(2)$ labelled by a pair (j_1, j_2).

Formally, an irreducible rep of a direct sum $\mathfrak{g} \oplus \mathfrak{h}$ can be built by the tensor product of $(\rho_{\mathfrak{g}}, V_{\mathfrak{g}})$ and $(\rho_{\mathfrak{h}}, V_{\mathfrak{h}})$

$$\rho\colon \mathfrak{g} \oplus \mathfrak{h} \to End(V) = End(V_{\mathfrak{g}} \otimes V_{\mathfrak{h}})$$

such that

$$\rho(X+Y)(v \otimes w) = \rho_{\mathfrak{g}}(v) \otimes w + v \otimes \rho_{\mathfrak{h}}(w)$$

For the Lorentz algebra

$$\rho_{j_1,j_2}\Big(\sum_m \lambda_m A_m + \sum_n k_n B_n\Big) = \sum_m \lambda_m \rho_{j_1}(A_m) \otimes id_{V_{j_2}} + \sum_n k_n id_{V_{j_1}} \otimes \rho_{j_2}(B_n)$$

with dimension $\dim(V_{j_1} \otimes V_{j_2}) = \dim V_{j_1} \dim V_{j_2} = (2j_1+1)(2j_2+1)$.

For the rotation algebra $J_i = A_i + B_i$, a similar closed algebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, using the Clebsh-Gordon decomposition, this reduces into a sum of irreducible reps of spin j such that $|j| \leq j_1 + j_2$.

Notice that if they are both integer of half-integer, the sum representation is integer, like a bosonic one. On the other hand, if one is integer and the other one half-integer, the sum representation is half-integer, like a fermionic one.

$SO^+(1,3)$ is not simply connected, so its universal cover is $SL(2,\mathbb{C})$. In particular $SO^+(1,3) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$. Hence the spinor representations of $SO^+(1,3)$ is $SL(2,\mathbb{C})$.

5.4 Spinor rep

There are two different spinor reps: the left-handed Weyl spinor ρ_L with $(\frac{1}{2},0)$ and the right-handed Weyl spinor ρ_R with $(0,\frac{1}{2})$. They are called spinors because the only irrep of the rotations restriction hae $j=\frac{1}{2}$.

Since the rep $j=\frac{1}{2}$ is generated by the Pauli matrices and $j=0$ is the trivial one, the generators of left spinors are

$$\rho_L(A_i) = \frac{1}{2}\sigma_i \otimes \mathbb{I}_1 = \frac{1}{2}\sigma_i \quad \rho_R(B_i) = \mathbb{I}_2 \otimes 0 = 0$$

and the generators of right spinors are

$$\rho_R(A_i) = 0 \otimes \mathbb{I}_1 = 0 \quad \rho_R(B_i) = \mathbb{I}_1 \otimes \frac{1}{2}\sigma_i = \frac{1}{2}\sigma_i$$

Hence, the original generators for left spinors are

$$\rho_L(J_i) = \rho_L(A_i) + \rho_L(B_i) = \frac{1}{2}\sigma_i \quad \rho_L(K_i) = -i(\rho_L(A_i) - \rho_L(B_i)) = -\frac{i}{2}\sigma_i$$

and the original generators for right spinors are

$$\rho_R(J_i) = \rho_R(A_i) + \rho_R(B_i) = \frac{1}{2}\sigma_i \quad \rho_R(K_i) = -i(\rho_R(A_i) - \rho_R(B_i)) = \frac{i}{2}\sigma_i$$

The associated rep is generated by a real linear combination

$$V_{(\frac{1}{2},0)} \ni \phi \mapsto \exp(-i\theta\vec{n}\cdot\rho_L(\vec{J}) - i\vec{\cdot}\rho_L(\vec{K}))\phi$$

and

$$V_{(0,\frac{1}{2})} \ni \phi \mapsto \exp(-i\theta\vec{n}\cdot\rho_R(\vec{J}) - i\vec{\cdot}\rho_R(\vec{K}))\phi = \exp(-\frac{1}{2}(-i\theta\vec{n}+\vec{v})\cdot\vec{\sigma})\phi$$

Notice that they are the complex conjugates of each other. Using the identity

$$(i\sigma_2)\sigma_i(-i\sigma_1) = (i\sigma_2)\overline{\sigma}_i(i\sigma_2)^{-1} = -\sigma_i$$

then defining $\chi = i\sigma_2\phi$

$$\begin{aligned}
\chi \mapsto i\sigma_2\phi = i\sigma_2\overline{\exp(-i\theta\vec{n}\cdot\rho_L(\vec{J}) - i\vec{\cdot}\rho_L(\vec{K}))\phi}\\
i\sigma_2\overline{\exp(-\frac{1}{2}(-i\theta\vec{n}+\vec{v})\cdot\vec{\sigma})\phi}\\
&= i\sigma_2\exp(-\frac{1}{2}(-i\theta\vec{n}+\vec{v})\cdot\overline{\vec{\sigma}})\overline{\phi}\\
&= i\sigma_2(i\sigma_2)^{-1}\exp(-\frac{1}{2}(-i\theta\vec{n}+\vec{v})\cdot\vec{\sigma})(i\sigma_2)\overline{\phi}\\
&= \exp(-i\theta\vec{n}\cdot\rho_R(\vec{J}) - i\vec{v}\cdot\rho_R(\vec{K}))\chi
\end{aligned}$$

Hence, up to a basis change, the complex conjugate of ϕ transforms like χ.

Another way to introduce spinors is through the Dirac matrices: 4×4 matrices which satisfy the anticommutator relations

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$$

They allow to construct a rep of the Lorentz algebra

$$M_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta] = \frac{i}{4}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$$

such that

$$[M_{\alpha\beta}, M_{\sigma\rho}] = -i(\eta_{\alpha\sigma}M_{\beta\rho} - \eta_{\alpha\rho}M_{\beta\sigma} - \eta_{\beta\sigma}M_{\alpha\rho} + \eta_{\beta\rho}M_{\alpha\sigma})$$

which is a 4-dimensional rep that leads to the Dirac equation.

It is called the bi-spinor rep and it is related to the Weyl one via

$$V_D = V_{frac12,0} \oplus V_{0,\frac{1}{2}}$$

but with basis

$$\psi_1 = \frac{\phi_1 + \chi_1}{\sqrt{2}} \quad \psi_2 = \frac{\phi_2 + \chi_2}{\sqrt{2}}$$
$$\psi_3 = \frac{\phi_1 - \chi_1}{\sqrt{2}} \quad \psi_4 = \frac{\phi_2 - \chi_2}{\sqrt{2}}$$

where (ϕ_1, ϕ_2) is a basis of $V_{frac12,0}$ and (χ_1, χ_2) of $V_{0,\frac{1}{2}}$.

5.5 Vector rep

Consider a rep $(\frac{1}{2}, \frac{1}{2})$ which corresponds to a bosonic field, since its restrictions are $j = \frac{1}{2} + \frac{1}{2} = 1$ and $j = \frac{1}{2} - \frac{1}{2} = 0$. It is equivalent to the defining rep of $\mathfrak{so}(1, 3)$. Picking a basis of $V_{\frac{1}{2}}$ that is $\{|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle\} = \{|+\rangle, |-\rangle\}$, the natural basis becomes

$$|+\rangle \oplus |+\rangle = |++\rangle \quad |+\rangle \oplus |-\rangle = |+-\rangle \quad |-\rangle \oplus |+\rangle = |-+\rangle \quad |-\rangle \oplus |-\rangle = |--\rangle$$

or more useful

$$|e_1\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \quad |e_2\rangle = \frac{|++\rangle - |--\rangle}{\sqrt{2}}$$
$$|e_3\rangle = \frac{-|++\rangle - i|--\rangle}{\sqrt{2}} \quad |e_4\rangle = \frac{-|+-\rangle - |-+\rangle}{\sqrt{2}}$$

The rep of the generators becomes

$$\rho(A_i) = \rho_{\frac{1}{2}}(A_i) \otimes \mathbb{I}_2 = \mathcal{I}_i^{(\frac{1}{2})} \otimes \mathbb{I}_2$$

$$\rho(B_i) = \mathbb{I}_2 \otimes \rho_{\frac{1}{2}}(B_i) = \mathbb{I}_2 \otimes \mathcal{I}_i^{(\frac{1}{2})}$$

which act on basis

$$\rho(A_i)|+-\rangle = \mathcal{I}_i^{(\frac{1}{2})}|+\rangle \otimes |-\rangle \quad \rho(B_i)|+-\rangle = |+\rangle \otimes \mathcal{I}_i^{(\frac{1}{2})}|-\rangle$$

The original Lorentz generators becomes

$$\rho(J_i) = \rho(A_i) + \rho(B_i) = \mathcal{I}_i^{(\frac{1}{2})} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathcal{I}_i^{(\frac{1}{2})}$$

and

$$\rho(K_i) = -i(\rho(A_i) - \rho(B_i)) = -i\mathcal{I}_i^{(\frac{1}{2})} \otimes \mathbb{I}_2 + i\mathbb{I}_2 \otimes \mathcal{I}_i^{(\frac{1}{2})}$$

For example, studing the generators J_3

$$\rho(J_3)|++\rangle = |++\rangle \quad \rho(J_3)|--\rangle = -|--\rangle \quad \rho(J_3)|-+\rangle = \rho(J_3)|+-\rangle = 0$$

or using the other basis

$$\rho(J_3)|e_1\rangle = \rho(J_3)|e_4\rangle = 0 \quad \rho(J_3)|e_2\rangle = i|e_3\rangle \quad \rho(J_3)|e_2\rangle = -i|e_2\rangle$$

Hence the matrix is

$$\rho(J_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For example, studing the generators K_3

$$\rho(K_3)|++\rangle = \rho(K_3)|--\rangle = 0 \quad \rho(K_3)|+-\rangle = -i|+-\rangle \quad \rho(K_3)|-+\rangle = i|-+\rangle$$

or using the other basis

$$\rho(K_3)|e_1\rangle = i|e_4\rangle \quad \rho(K_3)|e_2\rangle = \quad \rho(K_3)|e_3\rangle = 0 \quad \rho(K_3)|e_4\rangle = i|e_1\rangle$$

Hence the matrix is

$$\rho(K_3) = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

As we expected, it is indeed the defining rep.

5.6 Other rep

1. $(0,0)$ is the trivial rep. This fields are called scalar fields. They are bosonic.
2. $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ or $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ are the spinor rep. They correspond to all the matter particles.
3. $(\frac{1}{2},\frac{1}{2})$. They correspond to all the force bosons.
4. $(1,0)$ or $(0,1)$. The first one correspond to self-dual 2 form fields and the second one is the anti self-dual 2 form fields. In Standard Model there are no particle such that, but compare in string theory.
5. $(1,0)\oplus(0,1)$. They correspond to parity invariant 2-forms, like the electromagnetic tensor.
6. $(1,\frac{1}{2})\otimes(\frac{1}{2},1)$. They correspond to Rarita-Schwinger fields, like the gravitino.
7. $(1,1)$. They correspond to traceless symmetric tensor fields, like the graviton.

Part III

QFT

Chapter 6

Lorentz group

6.1

The Lorentz group is the group of isometries of the Minkovski spacetime, i.e. which leaves the metric unchanged

$$\Lambda^T \eta \Lambda = \eta$$

In particular, the proper orthochronous Lorentz group, the one which leaves discrete spacial P and time T inversion.

It has 6 parameters

1. 3-dimensional compact space rotations, i.e.

$$\theta = (\theta_1, \theta_2, \theta_3)$$

2. 3-dimensional non-compact boosts, i.e.

$$\beta = (\beta_1, \beta_2, \beta_3)$$

hence, it is a 6-dimensional Lie group and its continuous parameters can be gathered into an antisymmetric matrix ω

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

in the following way

$$\omega_{ij} = \epsilon_{ijk}\theta_k \quad \omega_{0i} = \beta_i$$

The basis of the corresponding Lie algebra sare the 6 generators

$$M^{\mu\nu} = -M^{\nu\mu}$$

which satisfy the commutator relation

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\rho\sigma} + M^{\rho\sigma}\eta^{\mu\sigma} - M^{\mu\rho}\eta_{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho})$$

A generic element of the Lorentz group can be written as

$$\Lambda = \exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}) \simeq 1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}$$

In its finite-dimensional representations, each element is represented by an operator (matrix) which acts in a finite-dimensional space. In its infinite-dimensional representations, each element is represented by an operator (field) which acts in a infinite-dimensional space (Hilbert space).

The generators can be decomposed in hermitian generators of rotations

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$$

and in anti-hermitian generators of boosts

$$K_i = M_{0i}$$

satisfing the commutation relation

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad [K_i, K_j] = -i\epsilon_{ijk}J_k \quad [J_i, K_j] = i\epsilon_{ijk}K_k$$

Notice that the algebra of J is $SU(2)$, but the other algebra is not closed. However, if we take a complex linear combination,

$$A_i = \frac{1}{2}(J_i + iK_i) \quad B_i = \frac{1}{2}(J_i - iK_i)$$

we can construct closed algebras of hermitian generators

$$[A_i, A_j] = i\epsilon_{ijk}A_k \quad [B_i, B_j] = i\epsilon_{ijk}B_k \quad [A_i, B_j] = 0$$

Hence, $\mathfrak{so}^+(1,3) \simeq \mathfrak{su}(2) \times \mathfrak{su}(2)$. Under parity,

$$x^0 \to x^0 \quad x^i \to -x^i$$

we can switch to the other generator

$$J_i \to J_i \quad K_i \to -K_i \quad \Rightarrow \quad A_i \rightleftarrows B_i$$

The physical spin $J = A + B$ can be then decomposed into two spinors belonging to $SU(2)$: left-handed spinors labelled by J_A and right-handed spinors labelled by J_B. In Physics, particles can be labelled by their representations

1. scalar $(0,0)$ with total spin $s = 0$;

2. vectors $(\frac{1}{2}, \frac{1}{2})$ with total spin $s = 0$;
3. Weyl spinors $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ with total spin $s = \frac{1}{2}$;
4. Dirac spinors $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ with total spin $s = \frac{1}{2}$;
5. Rarita-Schwinger $(0, 0)$ with total spin $s = \frac{3}{2}$;
6. graviton $(0, 0)$ with total spin $s = 2$;

6.2 Finite-dimensional representations

The trivial representation $(0, 0)$ is

$$M^{\mu\nu} = 0$$

and each element is

$$\Lambda = \exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}) = 1$$

It is associated to scalars

$$\phi' = \Lambda\phi = 1\phi = \phi$$

The vector representation $(\frac{1}{2}, \frac{1}{2})$ is

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = -i(\eta^{\mu\sigma}\delta^{\rho}{}_{\nu} - \eta^{\rho\mu}\delta^{\sigma}{}_{\nu})$$

and each element is

$$\Lambda = \exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$$

It is associated to 4-vectors

$$(V')^{\rho} = \Lambda^{\rho}{}_{\sigma}V^{\sigma} = (\exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}))^{\rho}{}_{\sigma}V^{\sigma}$$

It is the fundamental representation.

The spinorial representation is not of $SO^{+}(1, 3)$ but of its double cover $SL(2, \mathbb{C})$. Infact there is an isomorphism between $SO^{+}(1, 3) \simeq SL(2, \mathbb{C})/\mathbb{Z}_2$. This means that for each element of $SO^{+}(1, 3)$ there are two corresponding element of $SL(2, \mathbb{C})$. Infact, using the Pauli matrices

$$\sigma^{\mu} = \mathbb{I}, \vec{\sigma}$$

we can correspond a 2×2 matrix X with a 4-vector in the following way

$$X = x_{\mu}\sigma^{\mu} = \begin{matrix} x_0 + ix_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{matrix}$$

Hence if Λ preserve the metric $ds^2 = x_0^2 - |\vec{x}|^2$, we can correspond with a matrix $N \in SL(2,\mathbb{C})$

$$X' = NXZ$$

such that

$$\det X' = \det X = x_0^2 - |\vec{x}|^2$$

Hence there is a $2-1$ map between them, e.g. for $N = \pm\mathbb{I}_2$ there is $\Lambda = 1$.

6.3 Finite-dimensional representations of $SL(2,\mathbb{C})$

The fundamental representation is the left-handed Weyl spinor $(\frac{1}{2},0)$

$$(\psi)_\alpha = N_\alpha{}^\beta \phi_\beta \quad \alpha,\beta = 1,2 \quad N \in SL(2,\mathbb{C})$$

The complex conjugate representation is the right-handed Weyl spinor $(0,\frac{1}{2})$

$$\overline{\chi}_{\dot{\alpha}} = (N^*)_{\dot{\alpha}}{}^{\dot{\beta}} \overline{\chi}_{\dot{\beta}} \quad \dot{\alpha},\dot{\beta} = 1,2 \quad N^* \in SL(2,\mathbb{C})$$

Its direct product $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ gives rise to the reducible representation of the Dirac spinors

$$\psi_D = \begin{bmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{bmatrix}$$

The invariant tensor to raise and lower indises is

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

since

$$(\epsilon')^{\alpha\beta} = \epsilon^{\sigma\rho} N_\rho{}^\alpha N_\sigma{}^\beta = \epsilon^{\alpha\beta} \underbrace{\det N}_{1} = \epsilon^{\alpha\beta}$$

Hence

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta$$

and

$$\overline{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \overline{\chi}_{\dot{\beta}} \tag{6.1}$$

The generators of $SL_(2,\mathbb{C})$ are

1. left-handed Weyl spinor

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu)_\alpha{}^\beta$$

hence

$$(\psi')_\alpha = (\exp(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}))_\alpha{}^\beta \psi_\beta$$

2. right-handed Weyl spinor

$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}(\overline{\sigma}^{\mu}\sigma^{\nu} - \overline{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}{}_{\dot{\beta}}$$

hence

$$(\chi')^{\dot{\alpha}} = (\exp(-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}))^{\dot{\alpha}}{}_{\dot{\beta}}\chi^{\dot{\beta}}$$

3. Dirac spinor

$$\Sigma^{\mu\nu} = \frac{i}{4}\gamma^{\mu\nu} = \begin{bmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{bmatrix}$$

hence

$$\psi'_D = (\exp(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}))\psi_D$$

where $\sigma^{\mu} = (\mathbb{I}, \vec{\sigma})$, $\overline{\sigma}^{\mu} = (\mathbb{I}, -\vec{\sigma})$ and $\gamma^{\mu\nu} = [\gamma^{\mu}, \gamma^{\nu}]$.

Introducing the chirality operator

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix}$$

its action on a Dirac spinor

$$\gamma^5\psi_D = \begin{bmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \psi_{\alpha} \\ \overline{\chi}^{\dot{\alpha}} \end{bmatrix} = \begin{bmatrix} -\psi_{\alpha} \\ +\overline{\chi}^{\dot{\alpha}} \end{bmatrix}$$

Hence, a left-handed Weyl spinor has eigenvalue equals to $+1$ and a right-handed Weyl spinor has eigenvalue equals to -1.

The projectors operators are

1. left-handed Weyl spinor

$$P_L = \frac{1}{2}(\mathbb{I} - \gamma^5)$$

and its action is

$$P_L\psi_D = \begin{pmatrix} \psi_{\alpha} \\ 0 \end{pmatrix}$$

2. right-handed Weyl spinor

$$P_R = \frac{1}{2}(\mathbb{I} + \gamma^5)$$

and its action is

$$P_R\psi_D = \begin{pmatrix} 0 \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}$$

The Dirac conjugate is

$$\overline{\psi}_D = \psi_D^\dagger \gamma^0 = \begin{pmatrix} \chi^\alpha & \overline{\psi}_{\dot\alpha} \end{pmatrix}$$

The charge conjugate is

$$\psi_D^C = C\overline{\psi}_D^T = \begin{pmatrix} \chi_\alpha \\ \overline{\psi}_{\dot\alpha} \end{pmatrix}$$

where the charge conjugation matrix exchanges particles with antiparticles

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$$

A Majorana spinor is such that $\psi_\alpha = \chi_\alpha$

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix} = \psi_M^C$$

hence

$$\psi_D = \psi_{M_1} + i\psi_{M_2} \quad \psi_D = \psi_{M_1}^C - i\psi_{M_2}$$

6.4 Infinite-dimensional representations of $SO^*(1,3)$

Each field representation element is represented by an operator acting on non-constant objects like fields, recalling that not only the field changes but also the coordinates

$$\psi_a(x) \to \underbrace{(\exp(-\frac{i}{2}\omega_\mu\nu S^{\mu\nu}))_{ab}}_{\text{internal finite-dimensional}} \quad \underbrace{\exp(-\frac{i}{2}\omega_\mu\nu L^{\mu\nu})}_{\text{external infinite-dimensional}} \quad \phi_b(x) = (\exp(-\frac{i}{2}\omega_\mu\nu J^{\mu\nu}))_{ab}\phi_b(x)$$

where $J^\mu = S^{\mu\nu} + L^{\mu\nu}$, in particular

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

and

$$S^{\mu\nu} = \begin{cases} 0 & \text{scalar} \\ (M^{\mu\nu})^\rho{}_\sigma & \text{vector} \\ \sigma^{\mu\nu}, \sigma^{\mu\nu}, \Sigma^{\mu\nu} & \text{spinors} \end{cases}$$

If you quantise the theory, fields become operators on a Foch space and can have infinite-dimensional representations: each element is represented by a unitary operator acting on quantum states of a 1-particle Hilbert space. By Wigner's theorem, unitary operators have hermitian generators. Infact

$$\Lambda \to U(\Lambda)$$

and

$$|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}} a^\dagger_{\vec{p}} |0\rangle$$

and

$$U(\Lambda)|\vec{p}, s\rangle = |\Lambda\vec{p}, s\rangle = \sqrt{2E_{\Lambda\vec{p}}} a^\dagger_{\Lambda\vec{p}} |0\rangle$$

Chapter 7

Poincaré group

In the Poincaré group, we add the spacetime translations to $SO^+(1,3)$, and the most general transformation becomes

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

where a^μ is a 4-vector. This group is also called the inhomogeneous proper orthochronous Lorentz group and it is a 10-dimensional non-compact Lie group. Hence, we need to add further 4 generators P^μ such that the commutation relations become

$$[M^{\mu\nu} P^\sigma] = i(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma})$$

and

$$[P^\mu, P^\nu] = 0 \tag{7.1}$$

This means that it is an abelian subgroup.

In the field representation, $P^\mu = i\partial^\mu$.

7.1 1-particle Hilbert space representation of the Poincaré group

In quantum mechanics, given an operator A and a generator of a transformation T, if they commute $[A, T] = 0$ then the observable/aignevalue associated to A is invariant under the tranformation generated by T. Infact

$$A|\phi_a\rangle = a|\phi_A\rangle$$

and

$$|\phi'_A\rangle = \exp(i\alpha T)|\phi_a\rangle = |\phi_a\rangle + i\alpha T|\phi_a\rangle + O(\alpha^2)$$

and

$$A|\phi'_A\rangle = A|\phi_a\rangle + i\alpha AT|\phi_a\rangle = a|\phi_a\rangle + i\alpha TA|\phi_a\rangle = a(\mathbb{1} + i\alpha T)|\phi_a\rangle = a|\phi'_a\rangle$$

Hence, even though $|\phi_a\rangle = |\phi'_a\rangle$, they have the same eigenvalue.

If an operator commutes with all the generators of the Poincaré group, it is called a Casimir operator, its associated quantity are invariant over a Poincaré transformation and it defines a class of states, called multiplets, which are labelled by different eigenvalues of the Casimir operators.

In the Poincaré group, there are 2 Casimir operators

$$C_1 = P^\mu P_\mu$$

and

$$C_2 = W^\mu W_\mu$$

where $W^\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ is the Pauli-Lubanski vector.

7.2 Massive representations

In this case $P^\mu P_\mu = P^2$ has corresponding eigenvalues $p^2 = E_{\vec{p}}^2 - |\vec{p}|^2 = m^2 \neq 0$ and there is a set of infinitely many momenta $\{p^\mu\}$, generated by starting with a fixed p^μ and applying a Poincaré transformation. The multiplet is labelled by $|m; p^\mu\rangle$ and by the eigenvalues of W^μ.

The little group is the subgroup of the Poincaré group such that p^μ is fixed and built from all the generators which commute with P^μ. Wigner's theorem shows that the structure of it does not depend on the way we choose p^μ or $|m; p^\mu\rangle$. Hence, we choose the case as simple as possible: the rest frame

$$p^\mu = (m, 0, 0, 0)$$

It is invariant under space rotations and its little group is $SO(3) \simeq SU(2)/\mathbb{Z}_2$. Given that

$$[P^\mu, W_\mu] = 0$$

p^μ is invariant under tranformations generated by W^μ.

The components of the Pauli-Lubanski vector are

$$W_0 = \frac{1}{2}\epsilon_{0\nu\rho\sigma}P^\nu M^{\rho\sigma} = \frac{1}{2}\underbrace{\epsilon_{00\rho\sigma}}_{0} mM^{\rho\sigma} = 0$$

and

$$W_i = \frac{1}{2}\epsilon_{i\nu\rho\sigma}P^\nu M^{\rho\sigma} = \frac{1}{2}\underbrace{\epsilon_{i0jk}}_{-\epsilon_{ijk}} mM^{ij} = -\frac{m}{2}\epsilon_{ijk}M^{jk} = -mJ_i$$

Hence, the second casimir operator is

$$C_2 = W^\mu W_\mu = W^i W_i = -m^2 J^i J_i = -m^2 J^2$$

and it is associated with the spin eigenvalues j and j_3 such that $|j_3| \leq j$ and there are $2j+1$ values.

The multiplet of a massive representations is

$$|m, j; p^\mu, j_3\rangle$$

where p^μ is a continuous variable and j_3 is a discrete variable.

7.3 Massless representations

In this case $P^\mu P_\mu = P^2$ has corresponding eigenvalues $p^2 = E_{\vec{p}}^2 - |\vec{p}|^2 = m^2 = 0$ and $E_{\vec{p}} = |\vec{p}|$. For the little group, we choose $\vec{p}$ along the z-axis

$$p^\mu = (E, 0, 0, E)$$

It is invariant under space rotations in the (x, y) plane, i.e. $SO(2) \simeq U(1)$, but the little group is bigger.

The components of the Pauli-Lubanski vector are

$$\begin{aligned}
W_0 &= \frac{1}{2}\epsilon_{0\nu\rho\sigma}P^\nu M^{\rho\sigma} \\
&= \frac{1}{2}\underbrace{\epsilon_{00\rho\sigma}}_{0}EM^{\rho\sigma} + \frac{1}{2}\epsilon_{03\rho\sigma}EM^{\rho\sigma} \\
&= \frac{1}{2}\underbrace{\epsilon_{0312}}_{1}EM^{12} + \frac{1}{2}\underbrace{\epsilon_{0321}}_{-1}EM^{21} \\
&= E\underbrace{\frac{1}{2}(M^{12} - M^{21})}_{J_3} \\
&= EJ_3
\end{aligned}$$

and similarly

$$W_1 = -E(J_1 + K_2) \quad W_2 = E(-J_2 + K_1) \quad W_3 = -EJ_3 = -W_0$$

Hence, there are only 3 independent generators which satisfy the algebra

$$[W_1, W_2] = 0 \quad [W_3, W_1] = -iEW_2 \quad [W_3, W_2] = iEW_1$$

which is the algebra of the 2-dimensional Euclidean group $E(2)$, the group consistued by the isometries of the 2-dimensional metric. Its dimension is 3: 2 translations generated by W_1 and W_2 and 1 rotations in a plane by W_3.

The eigenvalues associated to W_3 are discrete while the ones associated to W_1 and W_2 are continuous, but continuous spin representations are not seen in Nature. Hence, we set by experimentally evidence $W_1 = W_2 = 0$.

Since

$$W^\mu = J_3(E, 0, 0E) = J_3 P^\mu \tag{7.2}$$

the eigenvalues of the second Casimir operator are

$$W^\mu W_\mu \propto P^\mu P_\mu = 0$$

but we can introduce the helicity, i.e. the projection of the spin along the direction of the motion, which an be left and right, and its eigenvalues are $\lambda = 0, \frac{1}{2}, 1, \ldots$. We want also the negative values to incorporate the parity invariance in our theory.

The multiplet of a massless representations is

$$|0, 0; p^\mu, \pm\lambda\rangle$$

The helicity of the particles are

1. Higgs boson has $\lambda = 0$,
2. quarks and leptons have $\lambda = \pm\frac{1}{2}$,
3. photons, $W^\pm$, Z^0, gluons have $\lambda = \pm 1$,
4. graviton has $\lambda = \pm 2$.

Notice that the photons has only two degrees of freedom but $W^\pm$ and Z^0, after the Higgs mechanism, acquire the thirs one which correspond to $\lambda = 0$.

Putting $E = 1$, the generators become

$$W_1 = -(J_1 + K_2) \quad W_2 = K_1 - J_2 \quad W_3 = -J_3$$

Physically, W_3 generates rotations in the (p_x, p_y) plane but W_1 and W_2 makes a more complicated transformation. Infact, the generator K_2 is

$$K_2 = M^{20} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the generator J_1 is

$$J_1 = M^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

and the generator W_1 is

$$W_1 = -(J_1 + K_2) = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

An infinitesimal trasformation of W_1 is

$$(p')^{\mu} = (\exp(-i\lambda W_1))^{\mu}{}_{\nu} p^{\nu} = p^{\mu} - i\lambda (W_1)^{\mu}{}_{\nu} p^{\nu}$$

Explicitly

$$\begin{cases} E' = E - \lambda p_y \\ p'_x = p_x \\ p'_x = p_x + \lambda(E - p_z) \\ p'_z = p_z + \lambda p_y \end{cases}$$

and for

$$p^{\mu} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

remains invariant

$$\begin{cases} E' = 1 - \lambda 0 = 1 \\ p'_x = 0 \\ p'_x = 0 + \lambda(1 - 1) = 0 \\ p'_z = 1 + 0 = 1 \end{cases}$$

List of Theorems