Matteo Zandi

Theoretical Physics

On group theory:

groups, representations and all that

October 22, 2023

Study notes taken during the master degree

Contents

Page

Part I

Lie groups and representations

Chapter 1

Groups

The justification of a group can be found in the structure of a physics transformation: given two transformation, the composition of them should be defined together with the unit one, i.e. nothing happens, and the inverse one, i.e. if you want to return back to the initial system.

Definition 1.1 (Group)
A group is a set of elements $G = \{g_i\}$ associated with a composition map

$$\cdot \colon G \times G \to G$$

satisfying the following properties

1. *closure, i.e.*
$$g_1 g_2 \in G \quad \forall g_1, g_2 \in G$$

2. *associativity, i.e.*
$$(g_1 g_2) g_3 = g_1 (g_2 g_3) = g_1 g_2 g_3 \quad \forall g_1, g_2, g_3 \in G$$

3. *unit element, i.e.*
$$\exists! g_0 \in G \colon g_0 g = g g_0 = g \quad \forall g \in G$$

4. *inverse element, i.e.*
$$\exists! g^{-1} \in G \colon g^{-1} g = g g^{-1} = g_0 \quad \forall g \in G$$

Definition 1.2 (Abelian group)
A group is said to abelian if

5. *commutativity, i.e.*
$$g_1 g_2 = g_2 g_1 \quad \forall g_1, g_2 \in G$$

Definition 1.3 (Subgroup)
A subgroup is a subset $H \subset G$ of a group which is also a group itself with closed restricted composition map.

Example 1.1 (Groups)**.** Examples of groups are

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ with composition map $+$ and unit element 0,
2. $\mathbb{Q} \setminus \{0\}$, $\mathbb{Q} \setminus \{0\}$ with composition map $\times$ and unit element 1,
3. $\mathbb{Z}_n = \{z \in [0,\ n-1] \ : \ a+n = a\}$ with composition map $+$ and unit element 0.

Example 1.2 (Matrix groups)**.** Matrices are a non-abelian group with matrix multiplication as composition map:

1. $GL(n) = \{M \in Mat_{n\times n}(\mathbb{R}) \ : \ \det M \neq 0\}$

Given a fixed invertible $n \times n$ matrix B, a subgroup of $GL(n)$ is the set of matrices M which preserve this matrix, i.e. $M^t BM = B$:

2. $O(n) = \{R \in Mat_{n\times n}(\mathbb{R}) \ : \ R^t \mathbb{1} R = \mathbb{1}\}$ with Euclidean metric $B = \mathbb{1}$,
3. $O(1,\ n-1) = \{\Lambda \in Mat_{n\times n}(\mathbb{R}) \ : \ \Lambda^t \eta \Lambda = \eta\}$ with Minkovskian metric $B = \eta$.

Over the complex field

4. $U(n) = \{U \in Mat_{n\times n}(\mathbb{C}) \ : \ U^\dagger U = \mathbb{1}\}$

Imposing $\det M = 1$, we find the special groups

5. $SL(n) = \{M \in GL(n) \ : \ \det M = 1\}$,
6. $SO(n) = \{R \in O(n) \ : \ \det R = 1\}$,
7. $SO(1,\ n-1) = \{\Lambda \in O(1,\ n-1) \ : \ \det \Lambda = 1\}$,
8. $SU(n) = \{M \in U(n) \ : \ \det U = 1\}$.

Chapter 2

Lie groups

Definition 2.1 (Lie group)
A Lie group is a group endowed with a manifold structure such that composition and inverse are smooth maps, i.e.

$$\begin{aligned} \mu\colon G \times G &\to G \\ (x, y) &\mapsto x^{-1}y \end{aligned}$$

In Lie groups, we can introduce the notions of closeness and power series. The tangent space at $g_0 \in G$, i.e. elements of the group infinitesimally away from the unit element, gives rise of the Lie algebra.

Definition 2.2 (Lie algebra)
A Lie algebra is a linear space equipped with an anti-symmetric product, called Lie brackets

$$[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the following properties

1. *linearity, i.e.*

$$[\alpha X + \beta Y,\ Z] = \alpha[X,\ Z] + \beta[Y,\ Z] \quad \forall X,\ Y,\ Z \in \mathfrak{g} \quad \forall \alpha,\ \beta \in \mathbb{R}$$

2. *anti-symmetry, i.e.*

$$[X,\ Y] = -[Y,\ X] \quad \forall X,\ Y \in \mathfrak{g}$$

3. *Jacobi identity, i.e.*

$$[X,\ [Y,\ Z]] + [Y,\ [Z,\ X]] + [Z,\ [X,\ Y]] = 0 \quad \forall X,\ Y,\ Z \in \mathfrak{g}$$

A Lie algebra can be encoded with the structure constants f_{ijk}

$$[T_i,\ T_j] = f_{ijk} T_k$$

where $\{T_i\}$ is a basis of $\mathfrak{g}$.

The exponential map helps to construct a group element that is finitely away as $g = \exp(X)$ from $X \in \mathfrak{g}$, tied to the existence of a unique path $\gamma \colon \mathbb{R} \to G$ such that $\gamma(0) = g_0$ and $\gamma(1) = g$ which is a one-parameter subgroup $\{\gamma(s) \ : \ s \in \mathbb{R}\}$ whose tangent vector at g_0 is X.
A useful formula is the Baker-Campbell-Hausdorff one, which connects the group composition with the Lie brackets

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X,\ Y] + \dots) \tag{2.1}$$

Summarizing, from structure constants we define a Lie algebra, from the exponential map and the BCH formula we define Lie group elements in terms of generators of the Lie algebra $\{T_i\}$.

Chapter 3

Representations

In physics, it is useful to study how groups act on objects, in particular matrix groups act on vectors belonging to linear spaces.

Definition 3.1 (Representation)
A linear representation of a group G is a group homomorphism

$$\rho\colon G \to Aut(V)$$

satisfying the following property

1. *composition map, i.e.*

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) \quad \forall g_1, g_2 \in G$$

It is possible to derive other properties

2. unit element, i.e.

$$\rho(g_0) = \mathbb{1}_V$$

3. inverse element, i.e.

$$\rho(g^{-1}) = \rho^{-1}(g) \quad \forall g \in G$$

For finite-dimensional linear spaces, i.e. $\dim V = n$, and $Aut(V) \simeq GL(n)$ after picking a basis. Furthermore, $\dim(\rho,\ V) = \dim V$. A representation $(\rho,\ V)$ acts as a linear transformation on a vector $v \in V$ as $\rho(g)v$.

Definition 3.2 (Reducible, irreducible representation)
A representation $(\rho,\ V)$ is reducible if

$$\nexists U \subset V \ :\ \rho(g)u \in U \quad \forall u \in U$$

otherwise, it is irreducible.

A reducible representation can be always put in block triangle form, choosing a suitable basis

$$\rho(g) = \begin{bmatrix} \rho_1(g) & B(g) \\ 0 & \rho_2(g) \end{bmatrix}$$

with the invariant subspace is $U = \{(u,\ 0) \in V\}$. If $B(g) = 0$, the representation is completely reducible and decomposes into the direct sum of $\rho = \rho_1 \oplus \rho_2$.

Definition 3.3 (Equivalent representations)
Two representations ρ_1 and ρ_2 of the same dimension are equivalent if

$$\exists S invertible \ : \ \rho_2(g) = S^{-1}\rho_1(g)S \quad \forall g \in G$$

which means that there exists a basis change that relate the representations.

Definition 3.4 (Faithful representation)
A representation ρ is faithful if

$$g_1 \neq g_2 \ \Rightarrow \ \rho(g_1) \neq \rho(g_1)$$

For non-faithful representations, there exists $H \subset G$ such that $\rho(h) = \mathbb{1} \quad \forall h \in H$.

Definition 3.5
Over the complex field, i.e. $\rho\colon G \to GL(n,\ \mathbb{C})$, a representation is unitary if

$$\rho(g^{-1}) = \rho^{-1}(g) = \rho(g)^\dagger$$

3.1 Representations of Lie Groups and Lie Algebras

Part II

SO(3) and SO(1, 3)

Chapter 4

SO(3)

4.1 SO(3) as a Lie group

In this chapter, we will study the three-dimensional rotations group $O(3)$. Computing the determinant, we can decomposed the Lie group into two parts according to the sign of it:

$$\Rightarrow O(3) = \underbrace{\{\det R = +1\}}_{SO(3)} \ \cup \ \{\det R = -1\} = SO(3) \ \cup \ \det R = -1$$

Since there is no continuos path that connect the two parts and only $SO(3)$ contains the identity, we are going to study $SO(3)$ and recover the other one with a reflexion along an axis.
Any $SO(3)$ rotation can be parametrized by a unit vector, perpendicular to the rotation plane, and a rotation angle θ:

$$R(\theta,\ \mathbf{n})_{ij} = \cos\theta\ \delta_{ij} + (1-\cos\theta)\ n_i n_j - \sin\theta\ \epsilon_{ijk} n_k$$

Hence, an infinitesimal rotation $\delta\theta$ near the identity $R(\theta = 0,\ \mathbf{n}) = \delta_{ij}$ is

$$R(\delta\theta,\ \mathbf{n})_{ij} = \underbrace{\cos\delta\theta}_{1}\ \delta_{ij} + (1 - \underbrace{\cos\delta\theta}_{1})\ n_i n_j - \underbrace{\sin\delta\theta}_{\delta\theta}\ \epsilon_{ijk} n_k = \delta_{ij} - \delta\theta\ \epsilon_{ijk} n_k$$

and its action on an arbitrary vector $\mathbf{v}$ is

$$R(\delta\theta,\ \mathbf{n})_{ij} v_i = \delta_{ij} v_i - \delta\theta\ \epsilon_{ijk} v_i n_k = \delta_{ij} v_i + \delta\theta\ \epsilon_{jik} v_i n_k$$

or

$$R(\delta\theta,\ \mathbf{n})\mathbf{v} = \mathbf{v} + \delta\theta\ \epsilon jik v_i n_k$$

4.2 SO(3) representations

In this chapter,

Chapter 5

SO(1, 3)

The Lorentz group is defined by the matrices Λ such that preserve the Minkovski metric

$$\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}\ \ \nu\eta_{\alpha\beta} = \eta_{\mu\nu} \tag{5.1}$$

First, the Lorentz group can be decomposed into two parts according to their determinant

$$\det(\Lambda^T\eta\Lambda) = \det\Lambda^T \det\eta \det\Lambda = \overset{2}{\det}\Lambda = \det\eta = 1$$

Hence

$$\det\Lambda = \pm 1$$

and the Lorentz group can be written as

$$O(1,3) = \underbrace{\{\det\Lambda = +1\}}_{SO(1,3)} \cup \{\det\Lambda = -1\}$$

where $SO(1,3)$ is called the proper Lorentz group.
Second, the proper Lorentz group can be decomposed into two parts according to their (0, 0) component

$$\eta_{00} = \Lambda^{\alpha}{}_{0}\Lambda^{\beta}{}_{0}\eta_{\alpha\beta}$$
$$-1 = -(\Lambda^{0}{}_{0})^2 + (\Lambda^{i}{}_{i})^2$$
$$(\Lambda^{0}{}_{0})^2 = 1 + (\Lambda^{i}{}_{i})^2 \geq 1$$

Hence

$$\Lambda^{0}{}_{0} \in]\infty, -1] \cup [1, \infty[$$

and the proper Lorentz group can be written as

$$SO(1,3) = \underbrace{\{\Lambda^{0}{}_{0} \in]\infty, -1]\}}_{SO(1,3)^+} \cup \{\Lambda^{0}{}_{0} \in [1, \infty[\}$$

where $SO(1,3)^+$ is called the proper orthochronous Lorentz group.
From now on, only the proper orthochronous Lorentz group will be studied because is the only group containing the identity.

5.1 Lie algebra: generators of $SO(1,3)^+$

Consider an infinitesimal Lorentz transformation around the identity

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{5.2}$$

where $\omega^\mu{}_\nu \ll 1$ is an infinitesimal matrix.
Using (5.1),

$$(\delta^\alpha{}_\mu + \omega^\alpha{}_\mu)(\delta^\beta{}_\nu + \omega^\beta{}_\nu)\eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\delta^\alpha{}_\mu\delta^\beta{}_\nu\eta_{\alpha\beta} + \delta^\alpha{}_\mu\omega^\beta{}_\nu\eta_{\alpha\beta} + \omega^\alpha{}_\mu\delta^\beta{}_\nu\eta_{\alpha\beta} + \omega^\alpha{}_\mu\omega^\beta{}_\nu\eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\cancel{\eta_{\mu\nu}} + \omega_{\mu\nu} + \omega_{\nu\mu} + O(\omega^2) = \cancel{\eta_{\mu\nu}}$$

Hence, the matrices $\omega_{\mu\nu}$ are anti-symmetric

$$\omega_{\mu\nu} = \omega_{\nu\mu}$$

Using the exponential map, a generic $SO(1,3)^+$ transformation can be written as

$$\Lambda^\mu{}_\nu = \exp(-\frac{i}{2}\omega^{\alpha\beta}M_{\alpha\beta})^\mu{}_\nu$$

where $M_{\alpha\beta}$ are the generators of the Lie algebra $\mathfrak{so}(1,3)$. Since they must be antisymmetric, otherwise they would vanish, there are six independent generators of $\mathfrak{so}(1,3)$.

5.2 Lie algebra: commutators of $SO(1,3)^+$

To find the explicit expression of the commutator of two generators, first it will be computed the following expression using (5.2)

$$\begin{aligned}
(\tilde\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\alpha{}_\beta\tilde\Lambda^\beta{}_\gamma\Lambda^\gamma{}_\nu &= (\delta^\mu{}_\alpha - \tilde\omega^\mu{}_\alpha)(\delta^\alpha{}_\beta - \omega^\alpha{}_\beta)(\delta^\beta{}_\gamma + \tilde\omega^\beta{}_\gamma)(\delta^\gamma{}_\nu + \omega^\gamma{}_\nu) + O(\tilde\omega^2) + O(\omega^2) \\
&= \delta^\mu{}_\alpha\delta^\alpha{}_\beta\delta^\beta{}_\gamma\delta^\gamma{}_\nu - \tilde\omega^\mu{}_\alpha\delta^\alpha{}_\beta\delta^\beta{}_\gamma\delta^\gamma{}_\nu - \delta^\mu{}_\alpha\omega^\alpha{}_\beta\delta^\beta{}_\gamma\delta^\gamma{}_\nu + \delta^\mu{}_\alpha\delta^\alpha{}_\beta\tilde\omega^\beta{}_\gamma\delta^\gamma{}_\nu \\
&\quad + \delta^\mu{}_\alpha\delta^\alpha{}_\beta\delta^\beta{}_\gamma\tilde\omega^\beta{}_\gamma + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\beta\delta^\beta{}_\gamma\delta^\gamma{}_\nu - \tilde\omega^\mu{}_\alpha\delta^\alpha{}_\beta\delta^\beta{}_\gamma\omega^\gamma{}_\nu - \delta^\mu{}_\alpha\omega^\alpha{}_\beta\tilde\omega^\beta{}_\gamma\delta^\gamma{}_\nu \\
&\quad + \delta^\mu{}_\alpha\tilde\omega^\alpha{}_\beta\delta^\beta{}_\gamma\omega^\gamma{}_\nu + O(\tilde\omega^2) + O(\omega^2) \\
&= \delta^\mu{}_\nu - \cancel{\tilde\omega^\mu{}_\nu} - \cancel{\omega^\mu{}_\nu} + \cancel{\tilde\omega^\mu{}_\nu} + \cancel{\omega^\mu{}_\nu} + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu - \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu - \omega^\mu{}_\gamma\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu \\
&\quad + O(\omega^2) + O(\tilde\omega^2) \\
&= \delta^\mu{}_\nu + \cancel{\tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu} - \cancel{\tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu} - \omega^\mu{}_\gamma\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu + O(\omega^2) + O(\tilde\omega^2) \\
&= \delta^\mu{}_\nu - \omega^\mu{}_\gamma\tilde\omega^\alpha{}_\nu + \tilde\omega^\mu{}_\alpha\omega^\alpha{}_\nu + O(\omega^2) + O(\tilde\omega^2)
\end{aligned}$$

and second using the BCH formula (2.1)

$$(\tilde\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\alpha{}_\beta\tilde\Lambda^\beta{}_\gamma\Lambda^\gamma{}_\nu =$$

List of Theorems