Matteo Zandi

On Lie groups, Lie algebras and representations:

$SO(3)$, $SO^{+}(1,3)$ and $ISO(1,3)$

November 5, 2023

Theoretical Physics

Contents

Page

Part I

Lie groups and representations

Chapter 1

Lie groups

Groups are the mathematical language of physics transformations or symmetries. This can be justified by noticing that the set of transformations must include the identity one, i.e. nothing happens, the inverse one, i.e. if you want to return back to the initial system, and the composition of two of them, i.e. two consecutive transformations (from the first to the second system and then from the second to the third one) are equivalent as if you go from the first to the third system with the composition of the two of them. This is indeed the definition of a group and it will be the topic of this chapter. In particular, we are interested in infinitesimal continuous transformations, which are related to Lie groups, and in what happens at the identity, which is related to Lie algebras.

1.1 Groups

Definition 1.1 (Group)
A group is a set of elements $G = \{g_i\}$ *associated with a composition map*

$$\begin{aligned} &G \times G \to G \\ &(g_1, g_2) \mapsto g_1 g_2 \ , \end{aligned}$$

such that it satisfies the following properties $\forall g, g_1, g_2, g_3 \in G$

1. *closure, i.e.*
$$g_1 g_2 \in G \ ,$$
2. *associativity, i.e.*
$$(g_1 g_2) g_3 = g_1 (g_2 g_3) = g_1 g_2 g_3 \ ,$$
3. *identity element, i.e.*
$$\exists! g_0 \in G \colon g_0 g = g g_0 = g \ , \tag{1.1}$$

4. *inverse element, i.e.*

$$\exists! g^{-1} \in G\colon\, g^{-1}g = gg^{-1} = g_0\ . \tag{1.2}$$

Definition 1.2 (Abelian group)
A group is said to be abelian if it satisfies the additional property

5. *commutativity, i.e.*
$$g_1 g_2 = g_2 g_1 \quad \forall g_1, g_2 \in G\ .$$

Definition 1.3 (Subgroup)
A subgroup is a subset $H \subset G$ of a group which is also a group itself with closed restricted composition map.

The trivial subgroup $\{g_0\}$ is always a subgroup.

Examples of groups

Example 1.1 (Set groups)**.** Examples of set groups are

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ with composition map $+$ and identity element 0,
2. $\mathbb{Q}\setminus\{0\}$, $\mathbb{R}\setminus\{0\}$ with composition map $\times$ and identity element 1,
3. $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{z \in \mathbb{N}, z \in [0, n-1]\colon\, a + n = a\}$ with composition map $+$ and identity element 0.

Example 1.2 (Subgroups)**.** Examples of sugroups are

1. $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$,
2. $\mathbb{Z}_n \subset \mathbb{Z}$.

Example 1.3 (Matrix groups)**.** Matrices are a non-abelian group with matrix multiplication as composition map, even though in order to contain the inverse matrix they must have non-zero determinant:

1. general linear group, i.e.

$$GL(n) = \{M \in Mat_{n\times n}(\mathbb{R})\colon\, \det M \neq 0\}\ . \tag{1.3}$$

In particular, given a fixed invertible matrix $B \in Mat_{n\times n}$, a subgroup of $GL(n)$ is the set of matrices M which preserve this matrix, i.e. $M^T B M = B$:

2. orthogonal group, i.e.

$$O(n) = \{R \in Mat_{n\times n}(\mathbb{R})\colon\, R^T \mathbb{I} R = \mathbb{I}\}\ ,$$

   where B is the Euclidean metric $\mathbb{I}$,

3. Lorentz group, i.e.

$$O(1,3) = \{\Lambda \in Mat_{4\times 4}(\mathbb{R})\colon\ \Lambda^T\eta\Lambda = \eta\}\ ,$$

   where B is the Minkovskian metric η,

4. symplectic group, i.e.

$$Sp(n) = \{M \in Mat_{2n\times 2n}(\mathbb{R})\colon\ M^T J M = J\}\ ,$$

   where B is the symplectic matrix $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Over the complex field:

4. unitary group, i.e.

$$U(n) = \{U \in Mat_{n\times n}(\mathbb{C})\colon\ U^\dagger U = \mathbb{1}\}\ .$$

It is useful to impose the additional condition of unit determinant $\det M = 1$, which gives rise to subgroups called special groups:

5. special linear group, i.e.

$$SL(n) = \{M \in GL(n)\colon\ \det M = 1\}\ ,$$

6. special orthogonal group, i.e.

$$SO(n) = \{R \in O(n)\colon\ \det R = 1\}\ , \tag{1.4}$$

7. special Lorentz group, i.e.

$$SO(1,3) = \{\Lambda \in O(1,3)\colon\ \det \Lambda = 1\}\ , \tag{1.5}$$

8. special unitary group, i.e.

$$SU(n) = \{U \in U(n)\colon\ \det U = 1\}\ .$$

1.2 Lie groups and Lie algebras

Definition 1.4 (Lie group)
A Lie group is a group endowed with a manifold structure such that the composition and the inverse maps are smooth, i.e.

$$\begin{aligned} \mu\colon G \times G &\to G \\ (x,y) &\mapsto x^{-1}y\ . \end{aligned}$$

In Lie groups, we can introduce the notions of closeness and power series. Infact, an infinitesimal transformation M can be Taylor expand into $M = \mathbb{I} + X$ such that its action on a vector is $Mv = v + \delta v$, where $\delta v = Xv$. Therefore, the set of infinitesimal trasformations X is the tangent space at the identity element $g_0 \in G$. It is a linear space, such that its elements could be different from those of the Lie group. Furthermore, it has also a composition map, which in general is not invertible.

Definition 1.5 (Lie algebra)
A Lie algebra is a linear space equipped with an anti-symmetric product, called Lie brackets

$$[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\ ,$$

such that it satisfies the following properties $\forall X, Y, Z \in \mathfrak{g}$, $\forall \alpha, \beta \in \mathbb{R}$

1. *linearity, i.e.*
$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]\ ,$$

2. *anti-symmetry, i.e.*
$$[X, Y] = -[Y, X]\ , \tag{1.6}$$

3. *Jacobi identity, i.e.*
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\ . \tag{1.7}$$

Definition 1.6 (Structure constants)
Given a basis $\{T_i\}$ *of* $\mathfrak{g}$*, the Lie algebra can be completely determines by the structure constants* $f_{ijk} \in \mathbb{R}$ *in the following way*

$$[T_i, T_j] = f_{ijk} T_k\ . \tag{1.8}$$

Since they must obey the anti-symmetry condition, they satisfy the property

$$f_{ijk} = -f_{jik}\ .$$

Proof. We put (**??**) into (1.6)

$$f_{ijk} T_k = [T_i, T_j] = -[T_j, T_i] = -f_{jik} T_k\ .$$

Hence

$$f_{ijk} = -f_{jik}\ .$$

q.e.d.

Moreover, since they must obey the Jacobi identity, they satisfy the property

$$f_{ilm} f_{jkl} + f_{jlm} f_{kil} + f_{klm} f_{ijl} = 0\ .$$

Proof. We put (**??**) into (1.7)

$$
\begin{aligned}
0 &= [T_i,[T_j,T_k]] + [T_j,[T_k,T_i]] + [T_k,[T_i,T_j]] \\
&= [T_i, f_{jkl}T_l] + [T_j, f_{kil}T_l] + [T_k, f_{ijl}T_l] \\
&= f_{jkl}[T_i,T_l] + f_{kil}[T_j,T_l] + f_{ijl}[T_k,T_l] \\
&= f_{jkl}f_{ilm}T_m + f_{kil}f_{jlm}T_m + f_{ijl}f_{klm}T_m \\
&= (f_{jkl}f_{ilm} + f_{kil}f_{jlm} + f_{ijl}f_{klm})T_m \\
&= (f_{ilm}f_{jkl} + f_{jlm}f_{kil} + f_{klm}f_{ijl})T_m \ .
\end{aligned}
$$

Since it is true for any T_m

$$f_{ilm}f_{jkl} + f_{jlm}f_{kil} + f_{klm}f_{ijl} = 0 \ .$$

q.e.d.

Exponential map and Baker-Campbell-Hausdorff formula

Definition 1.7 (Exponential map)
Given the existence of a unique path $\gamma\colon \mathbb{R} \to G$ such that $\gamma(0) = g_0$ and $\gamma(1) = g$, which is a one-parameter subgroup $\{\gamma(s)\colon s \in \mathbb{R}\}$ such that the tangent vector at g_0 is X, the exponential map is the map that gives a group element $G \ni g = \exp(X)$ which is finitely away from $X \in \mathfrak{g}$.

Notice that we can exponentiate only elements which are connected to the identity element.

Theorem 1.1 (Baker-Campbell-Hausdorff formula)
Let X, Y be infinitesimal matrices. Then the matrix Z, which is solution of the equation

$$\exp(X)\exp(Y) = \exp(Z) \ , \tag{1.9}$$

is

$$Z = X + Y + \frac{1}{2}[X,\ Y] + O(X^3) + O(Y^3) \ . \tag{1.10}$$

Proof. Given a matrix X, its exponential $\exp(X)$ can be calculated with the Taylor expansion

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \ . \tag{1.11}$$

Another useful Taylor expansion is the logarithmic one

$$\log(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(X-1)^n \ . \tag{1.12}$$

We isolate Z in the equation (1.9) and Taylor expand (1.12) with $X = \exp(X)\exp(Y)$

$$\begin{aligned} Z &= \log(\exp X \exp Y) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\exp(X)\exp(Y) - 1)^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\sum_{p=0}^{\infty} \frac{X^p}{p!} \sum_{q=0}^{\infty} \frac{X^q}{q!} - 1)^n \ . \end{aligned}$$

We neglect terms of third or higher order in order to compute the second order expansion

$$\begin{aligned} Z &= \sum_{n=1}^{2} \frac{(-1)^{n+1}}{n} (\sum_{p=0}^{\infty} \frac{X^p}{p!} \sum_{q=0}^{\infty} \frac{X^q}{q!} - 1)^n \\ &= \sum_{p=0}^{\infty} \frac{X^p}{p!} \sum_{q=0}^{\infty} \frac{X^q}{q!} - 1 - \frac{1}{2} \Big(\sum_{p=0}^{\infty} \frac{X^p}{p!} \sum_{q=0}^{\infty} \frac{X^q}{q!} - 1 \Big)^2 \\ &= (1 + X + \frac{1}{2}X^2 + O(X^3))(1 + Y + \frac{1}{2}Y^2 + O(Y^3)) - 1 \\ &\quad - \frac{1}{2}\Big((1 + X + O(X^2))(1 + Y + O(Y^2)) - 1 \Big)^2 \\ &= X + \frac{1}{2}X^2 + Y + \frac{1}{2}Y^2 + XY - \frac{1}{2}(X^2 + Y^2 + XY + YX) + O(X^3) + O(Y^3) \\ &= X + Y + \frac{1}{2}(XY - YX) + +O(X^3) + O(Y^3) \\ &= X + Y + \frac{1}{2}[X,\ Y] + O(X^3) + O(Y^3) \ . \end{aligned}$$

q.e.d.

The procedure to go from the Lie group to the Lie algebra is the following one: from the identity element of the Lie group, we move infinitesimally around it and find the generators of the Lie algebra. Conversely, to go from the Lie algebra to an element of the Lie group, we start from the generators of the Lie algebra, and by using the exponential map and the BCH formula, we find an element of the Lie group.

Symmetries in quantum mechanics

To clarify how symmetries are generated by infinitesimal transformations, recall what happens in quantum mechanics. For example, infinitesimal spatial translations generat the momentum operator

$$\langle x|\hat{P}|\psi\rangle = -i\frac{\partial}{\partial x}\psi(x) \ ,$$

which leads to a finite spatial translation

$$\langle x|\exp(-\frac{i}{\hbar}y\hat{P})|\psi\rangle = \psi(x+y) \ ,$$

or infinitesimal time translations generate the energy operator via the Schroedinger equation

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle \ ,$$

which leads to a finite time evolution

$$|\psi(t+\tau)\rangle = \exp(-\frac{i}{\hbar}\hat{H}\tau)|\psi(t)\rangle \ .$$

Chapter 2

Representations

In physics, it is useful to study how groups act on objects, in particular matrix groups act on vectors belonging to linear spaces of different dimensions. In this chapter, we will study how a group, a Lie group or a Lie algebra acts on objects living in a linear space, which is called a representation.

2.1 Representations

Definition 2.1 (Automorphism)
The automorphism $Aut(V)$ of a linear space V is the set of invertible linear maps into the the linear space itself.

Definition 2.2 (Representation)
A linear representation (ρ, V) of a group G is a group homomorphism into the set of automorphisms on V

$$\rho \colon G \to Aut(V)$$

such that it satisfies the following property $\forall g_1, g_2 \in G$

1. *composition map, i.e.*

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) \ . \tag{2.1}$$

It is possible to derive other properties $\forall g \in G$

2. identity element, i.e.

$$\rho(g_0) = \mathbb{I}_V \ , \tag{2.2}$$

3. inverse element, i.e.

$$\rho(g^{-1}) = \rho^{-1}(g) \ .$$

Proof. For the identity element, we choose $g_1 = g_0$ and $g_2 = g$ in (2.1)

$$\rho(g_0 g) = \rho(g_0)\rho(g) \ ,$$

and by (1.1) and the property of the identity $\mathbb{I}_V$

$$\rho(g_0 g) = \rho(g) = \mathbb{I}_V \rho(g) \ .$$

Hence

$$\rho(g_0) = \mathbb{I}_V \ .$$

For the inverse element, we choose $g_1 = g^{-1}$ and $g_2 = g$ in (2.1)

$$\rho(g^{-1}g) = \rho(g^{-1})\rho(g) \ ,$$

and by (1.2) and (2.2)

$$\rho(g^{-1}g) = \rho(g_0) = \mathbb{I}_V \ .$$

Hence

$$\rho(g^{-1})\rho(g) = \mathbb{I}_V \ ,$$

or, equivalently,

$$\rho(g^{-1}) = \rho^{-1}(g) \ ,$$

q.e.d.

If we restrict to finite-dimensional linear spaces, i.e. $\dim V = n$, given a basis of V, we have $Aut(V) \simeq GL(n)$, where $GL(n)$ is (1.3). Furthermore, $\dim(\rho, V) = \dim V$. A representation (ρ, V) acts as a linear transformation on a vector $v \in V$ as $\rho(g)v$.

Definition 2.3 (Reducible, irreducible representation)
A representation $(\rho,\ V)$ is reducible if

$$\nexists \emptyset \neq U \subset V \ ,$$

such that $\forall u \in U$

$$\rho(g)u \in U \ .$$

Otherwise, it is irreducible.

A reducible representation can be always put in block triangle form, by choosing a suitable basis

$$\rho(g) = \begin{bmatrix} \rho_1(g) & B(g) \\ 0 & \rho_2(g) \end{bmatrix} \ ,$$

where the invariant subspace is $U = \{\begin{bmatrix} u & 0 \end{bmatrix} \in V\}$. In this way, a smaller dimensional representation can be construct. If $B(g) = 0$, the representation is completely reducible and decomposes into the direct sum of $\rho = \rho_1 \oplus \rho_2$.

Definition 2.4 (Equivalent representations)
Two representations ρ_1 and ρ_2 of the same dimension are equivalent if $\exists S$ invertible such that $\forall g \in G$

$$\rho_2(g) = S^{-1}\rho_1(g)S \ ,$$

which means that there exists a basis change that relates the two representations.

Definition 2.5 (Faithful representation)
A representation ρ is faithful if

$$g_1 \neq g_2 \ \Rightarrow \ \rho(g_1) \neq \rho(g_1) \ .$$

For non-faithful representations, there exists $H \subset G$ such that $\rho(h) = \mathbb{I} \quad \forall h \in H$.

Definition 2.6
Over the complex field, i.e. $\rho\colon G \to GL(n,\ \mathbb{C})$, a representation is unitary if

$$\rho(g^{-1}) = \rho^{-1}(g) = \rho(g)^\dagger \ , \tag{2.3}$$

where $\rho^\dagger = (\rho^*)^T$.

The physical meaning of unitary representations is that they preserve probabilities in quantum mechanics.

Proof. Infact through (2.3)

$$||\rho(g)|\psi\rangle||^2 = \langle\psi|\rho^\dagger(g)\rho(g)|psi\rangle = \langle\psi|\rho^{-1}(g)\rho(g)|\psi\rangle = \langle\psi|\psi\rangle = |||\psi\rangle||^2 \ .$$

q.e.d.

For any group, there exists a trivial 1-dimensional representation, $\rho(g) = \mathbb{I}$. For any matrix group, there exists a non-trivial 1-dimensional representation, $\rho(g) = \det(g)$. The defining representation is $\rho(g) = g$. For a $n \times n$ matrix group, the defining representation has the dimension n.

2.2 Representations of Lie groups and Lie algebras

Definition 2.7 (Endomorphism)
The endomorphism $End(V)$ of a linear space V is the set of (not in general invertible) linear maps into the the linear space itself.

Definition 2.8 (Representation of a Lie algebra)
A representation of a Lie algebra is a Lie algebra homomorphism into the set of endomorphisms on V

$$\rho_{\mathfrak{g}} \colon \mathfrak{g} \to End(V) \ ,$$

such that it satifies the following compatibility condition with the Lie brackets $\forall X, Y \in \mathfrak{g}$

$$\rho_{\mathfrak{g}}[X,Y] = \rho_{\mathfrak{g}}X\rho_{\mathfrak{g}}Y - \rho_{\mathfrak{g}}Y\rho_{\mathfrak{g}}X \ , \tag{2.4}$$

or, given a set of generators $\{T_i\}$,

$$\rho_{\mathfrak{g}}[T_i, T_j] = f_{ijk}\rho_{\mathfrak{g}}T_k \ .$$

From Lie group representations to Lie algebra representations

Any representation of a Lie group (ρ, V) induces a representation of its Lie algebra.

Proof. Infact, an element of the group $g = \exp(tX)$ gives a path of transformations on V and we can define a representation on V with

$$\rho_{\mathfrak{g}}(X)(v) = \frac{d}{dt}\rho(\exp(tX))\Big|_{t=0} \ . \tag{2.5}$$

Hence, $\rho_{\mathfrak{g}}X$ is the same size of $\rho(g)$.

To show that it respects the Lie brackets (2.4), we use (1.10), (2.1), (1.6) and the linearity property of the representation

$$\begin{aligned}
\rho_{\mathfrak{g}}X\rho_{\mathfrak{g}}Y - \rho_{\mathfrak{g}}Y\rho_{\mathfrak{g}}X &= \frac{d}{dt}(\rho(\exp(tX))\rho(\exp(tY)) - \rho(\exp(tY))\rho(\exp(tX)))\Big|_{t=0} \\
&= \frac{d}{dt}\Big(\rho(\exp(t(X+Y+\frac{1}{2}[X,Y]))) - \rho(\exp(t(Y+X+\frac{1}{2}[Y,X])))\Big)\Big|_{t=0} \\
&= \frac{d}{dt}\Big(\rho(\exp(t(X+Y+\frac{1}{2}[X,Y] - Y - X - \frac{1}{2}[Y,X])))\Big)\Big|_{t=0} \\
&= \frac{d}{dt}\Big(\rho(\exp(t(\frac{1}{2}[Y,X] - \frac{1}{2}[Y,X])))\Big)\Big|_{t=0} \\
&= \frac{d}{dt}\Big(\rho(\exp(t(\frac{1}{2}[Y,X] + \frac{1}{2}[X,Y])))\Big)\Big|_{t=0} \\
&= \frac{d}{dt}\Big(\rho(\exp(t[Y,X]))\Big)\Big|_{t=0} \\
&= \rho_{\mathfrak{g}}[X,Y] \ .
\end{aligned}$$

Hence $(\rho_{\mathfrak{g}}, V)$ is representation of the Lie algebra. q.e.d.

For a unitary representation, the Lie algebra representations are anti-Hermitian matrices,

$$\rho_{\mathfrak{g}}^{\dagger}(X) = -\rho_{\mathfrak{g}}(X) \ .$$

Proof. Because of unitarity (2.3),

$$\rho^{\dagger}(\exp(tX))\rho(\exp(tX)) = 1 \ ,$$

we have, by using (2.5)

$$\begin{aligned}
0 &= \frac{d}{dt}\rho(\exp(tX))^{\dagger}\rho(\exp(tX))\Big|_{t=0} \\
&= \rho_{\mathfrak{g}}^{\dagger}(X)\rho^{\dagger}(\exp(tX))\rho(\exp(tX))\Big|_{t=0} + \rho_{\mathfrak{g}}(X)\rho(\exp(tX))^{\dagger}\rho(\exp(tX))\Big|_{t=0} \\
&= \left(\rho_{\mathfrak{g}}^{\dagger}(X) + \rho_{\mathfrak{g}}(X)\right) \underbrace{\rho^{\dagger}(\exp(tX))\rho(\exp(tX))\Big|_{t=0}}_{1} \\
&= \rho_{\mathfrak{g}}^{\dagger}(X) + \rho_{\mathfrak{g}}(X) \ .
\end{aligned}$$

Hence

$$\rho_{\mathfrak{g}}^{\dagger}(X) = -\rho_{\mathfrak{g}}(X) \ .$$

q.e.d.

In physics, it is more convenient working with Hermitian matrices, then, by introducing the related represention

$$\tilde{\rho}_{\mathfrak{g}}(X) = i\rho_{\mathfrak{g}} \ ,$$

the commutator becomes

$$[\tilde{\rho}_{\mathfrak{g}}(T_i), \tilde{\rho}_{\mathfrak{g}}(T_j)] = -[\rho_{\mathfrak{g}}(T_i), \rho_{\mathfrak{g}}(T_j)] = -f_{ijk}\rho_{\mathfrak{g}}(T_k) = if_{ijk}\rho_{\mathfrak{g}}(T_k) \ .$$

From Lie algebra representations to Lie group representations

Not all the representations of a Lie algebra extend to a representation of the Lie groups, because the Lie algebra gives information only locally, around the identity, while the Lie group could have different global topology.

However, if the group is simply connected, i.e. all closed paths are contractible (deformable to a point), the Lie algebra representation is also a Lie group representation. Furthermore, if the Lie group G is not simply-connected, there is always another Lie group $\tilde{G}$, the universal cover, which has the same representation as the Lie algebra. Universal cover means that there is a surjective projection homomorphism $\phi\colon \tilde{G} \to G$. An important property between the two groups is that they have the same Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}$.

Proof. Geometrically, there is always an open neighbourhood such that the projection map is a diffeomorphism. This means that the tangent space at every point is isomorphic to the tangent space at the identity. q.e.d.

Representations in quantum mechanics

In quantum mechanics, a state is not uniquely defined. Infact it is characterized by a Hilbert space vector up to a phase factor, which is called a ray in the Hilbert space. Two states are indeed the same physical state if

$$|\psi\rangle = \exp(i\lambda)|\phi\rangle$$

where λ is the phase factor.

Hence, we can relax the group structure and define another kind of representation, different from the linear one, called a projective representation.

Definition 2.9 (Projective representation)
A projective representation satisfies the property

$$\rho(g)\rho(h) = \exp(i\phi(g,h))\rho(gh)$$

where $\phi(g,h) \in \mathbb{R}$.

All finite-dimensional unitary projective representations come from a unitary linear representation of the universal covering group.

Part II

$SO(3)$ and $SO^+(1,3)$

Chapter 3

$SO(3)$

In this chapter, we will apply all the notions we learnt in the previous ones to (1.4), the unit-determinant 3-dimensional rotation group $SO(3)$.

3.1 $SO(3)$ as a Lie group

The determinant of a $O(3)$ rotation has only two possible values, i.e. $\det R = \pm 1$.

Proof. By using (1.4) and the product property of the determinant

$$1 = \det \mathbb{I} = \det R^T R = \det R^T \det R = \det^2 R \ .$$

Hence

$$\det R = \pm 1 \ .$$

q.e.d.

We can decomposed $O(3)$ into two parts according to the sign of the determinant

$$\Rightarrow O(3) = \underbrace{\{\det R = +1\}}_{SO(3)} \cup \{\det R = -1\} = SO(3) \cup \{\det R = -1\} \ .$$

Since there is no continuous path that connect the two parts and only $SO(3)$ contains the identity, we are going to study $SO(3)$ and recover the other one with a reflexion along an axis.

Any $SO(3)$ rotation can be parametrized by a unit vector, perpendicular to the rotation plane, and a rotation angle θ by the formula

$$R(\theta, \mathbf{n})_{ij} = \cos\theta \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta \epsilon_{ijk} n_k$$

Proof. Consider a unit vector n and decompose another vector $v$into it

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \ ,$$

where

$$v_{\parallel} = (\mathbf{v} \cdot \mathbf{n})\mathbf{n} \ ,$$

and

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n} \ .$$

We exploit the triple vector product formula, with the unit norm of the unit vector

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{v}) = (\mathbf{n} \cdot \mathbf{v})\mathbf{n} - \underbrace{(\mathbf{n} \cdot \mathbf{n})}_{1} \mathbf{v} = (\mathbf{n} \cdot \mathbf{v})\mathbf{n} - \mathbf{v}$$

and obtain

$$\mathbf{v} - (\mathbf{n} \cdot \mathbf{v})\mathbf{n} = -\mathbf{n} \times (\mathbf{n} \times \mathbf{v})$$

We put it in the previous equation and find

$$\mathbf{v}_{\perp} = -\mathbf{n} \times (\mathbf{n} \times \mathbf{v})$$

Furthermore, we have the identity

$$\mathbf{n} \times \mathbf{v} = \mathbf{n} \times (\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}) = \underbrace{\mathbf{n} \times \mathbf{v}_{\parallel}}_{0} + \mathbf{n} \times \mathbf{v}_{\perp} = \mathbf{n} \times \mathbf{v}_{\perp}$$

Now, we rotate around the unit vector $\mathbf{n}$ and go into polar coordinates

$$\mathbf{v}'_{\perp} = \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{n} \times \mathbf{v}_{\perp} = \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{n} \times \mathbf{v}$$

and

$$\mathbf{v}'_{\parallel} = \mathbf{v}_{\parallel}$$

Hence

$$\begin{aligned}
\mathbf{v}' &= R(\theta, \mathbf{n})\mathbf{v} \\
&= \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp} \\
&= \mathbf{v}_{\parallel} + \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{n} \times \mathbf{v} \\
&= \mathbf{v}_{\parallel} + \cos\theta(\mathbf{v} - \mathbf{v}_{\parallel}) + \sin\theta \mathbf{n} \times \mathbf{v} \\
&= \cos\theta \mathbf{v} + (1 - \cos\theta)\mathbf{v}_{\parallel} + \sin\theta \mathbf{n} \times \mathbf{v} \\
&= \cos\theta \mathbf{v} + (1 - \cos\theta)\mathbf{v}_{\parallel} + \sin\theta \mathbf{n} \times \mathbf{v} \\
&= \cos\theta \mathbf{v} + (1 - \cos\theta)(\mathbf{v} \cdot \mathbf{n})\mathbf{n} + \sin\theta \mathbf{n} \times \mathbf{v}
\end{aligned}$$

Since it is true for any vector $\mathbf{v}$, in index notation it becomes

$$R(\theta, \mathbf{n})_{ij} = \cos\theta \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta \epsilon_{ijk} n_k$$

q.e.d.

An infinitesimal rotation $\delta\theta$ near the identity $R(\theta = 0, \mathbf{n}) = \delta_{ij}$ is

$$R(\delta\theta, \mathbf{n})_{ij} = \underbrace{\cos\delta\theta}_{1}\,\delta_{ij} + (1 - \underbrace{\cos\delta\theta}_{1})n_i n_j - \underbrace{\sin\delta\theta}_{\delta\theta}\,\epsilon_{ijk}n_k = \delta_{ij} - \delta\theta\ \epsilon_{ijk}n_k$$

and its action on an arbitrary vector $\mathbf{v}$ is

$$R(\delta\theta, \mathbf{n})_{ij}v_i = \delta_{ij}v_i - \delta\theta\epsilon_{ijk}v_i n_k = \delta_{ij}v_i + \delta\theta\epsilon_{jik}v_i n_k$$

or

$$R(\delta\theta, \mathbf{n})\mathbf{v} = \mathbf{v} + \delta\theta\epsilon_{jik}v_i n_k$$

The generators L_i of the Lie algebra, which are a basis of $\mathfrak{so}(3)$, are

$$(L_i)_{jk} = -\epsilon_{ijk}$$

Proof. If we substitute the unit vector $\mathbf{n}$ with the Euclidean unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, we find

$$R(\delta\theta, \begin{bmatrix}1\\0\\0\end{bmatrix}) = \mathbf{v}\delta\theta\begin{bmatrix}0\\-v_3\\v_2\end{bmatrix} = \mathbf{v} + \delta\theta\underbrace{\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}}_{L_1}\mathbf{v}$$

$$R(\delta\theta, \begin{bmatrix}0\\1\\0\end{bmatrix}) = \mathbf{v}\delta\theta\begin{bmatrix}v_3\\0\\-v_1\end{bmatrix} = \mathbf{v} + \delta\theta\underbrace{\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}}_{L_2}\mathbf{v}$$

$$R(\delta\theta, \begin{bmatrix}0\\0\\1\end{bmatrix}) = \mathbf{v}\delta\theta\begin{bmatrix}-v_2\\v_1\\0\end{bmatrix} = \mathbf{v} + \delta\theta\underbrace{\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}}_{L_1}\mathbf{v}$$

q.e.d.

Any rotation belonging ot $SO(3)$ can be written with the exponential map

$$R(\theta, \mathbf{n}) = \exp(\theta n_i L_i) = \exp(\theta\mathbf{n}\cdot\mathbf{L})$$

The Lie algebra is $\mathfrak{so}(3) = \{\text{anti-symmetric traceless} 3\times 3 \text{matrices}\}$, a 3-dimensional real vector space.

Proof. Orthogonality require

$$\mathbb{I} = R(\theta, \mathbf{n})^T R(\theta, \mathbf{n}) = \exp(\theta n_i(L_i^T + L_i) + \ldots)$$

then the generators are antisymmetric

$$L_i^T = -L^i$$

and the determinant condition $\det R = 1$ means

$$trL_i = 0$$

q.e.d.

The commutation relations are

$$[L_i, L_j] = \epsilon_{ijk} L_k$$

In physics, it is useful define generators as $J_i = iL_i$. Thus

$$R(\theta, \mathbf{n}) = \exp(-i\theta\mathbf{n} \cdot \mathbf{J})$$

and the commutation relations become

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

3.2 $\mathfrak{so}(3)$ as a Lie algebra

Now we study the finite dimensional representation of $SO(3)$. In the previous section, we used the Cartesian basis: Therefore, by a basis change of the Lie algebra, we introduce the ladders operators

$$J_\pm = J_1 \pm iJ_2$$

such that they satisfy the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_3$$

Since J_i are Hermitian, so $J_\pm$ are.

Proof. First, the commutation relations between J_3 and $J_\pm$ are

$$[J_3, J_\pm] = [J_3, J_1 \pm iJ_2] = [J_3, J_1] \pm i[J_3, J_2] = iJ_2 \pm i(-iJ_1) = J_1 \pm iJ_2 = \pm J_\pm \ .$$

Secondly, the commutation relations between J_+ and J_- are

$$[J_+, J_-] = [J_1+iJ_2, J_1-iJ_2] = \underbrace{[J_1, J_1]}_{0} + i[J_2, J_1] - i[J_1, J_2] + \underbrace{[J_2, J_2]}_{0} = i(-iJ_3) - i(iJ_3) = 2J_3$$

q.e.d.

We define the Hermitian square modulus operator

$$J^2 = J_1^2 + J_2^2 + J_3^2$$

It is a Casimir operator, i.e. it commutes with all the Lie algebra generators.

Proof. First, the commutation relations between J^2 and $J_\pm$ are

$$\begin{aligned}
[J^2, J_\pm] &= [J^2, J_1 \pm iJ_2] \\
&= [J_1^2 + J_2^2 + J_3^2, J_1 \pm iJ_2] \\
&= \underbrace{[J_1^2, J_1]}_{0} + [J_2^2, J_1] + [J_3^2, J_1] \pm i[J_1^2, J_2] \pm i \underbrace{[J_2^2, J_2]}_{0} \pm i[J_3^2, J_2] \\
&= J_2[J_2, J_1] + [J_2, J_1]J_2 + J_3[J_3, J_1] + [J_3, J_1]J_3 \pm iJ_1[J_1, J_2] \pm i[J_1, J_2]J_1 \pm iJ_3[J_3, J_2] \pm i[J_3, \\
&= -i \underbrace{J_2J_3}_{0} + -i \underbrace{J_3J_2}_{0} + i \underbrace{J_3J_2}_{0} + i \underbrace{J_2J_3}_{0} \mp \underbrace{J_1J_3}_{0} \mp \underbrace{J_3J_1}_{0} \pm \underbrace{J_3J_1}_{0} \pm \underbrace{J_1J_3}_{0} = 0
\end{aligned}$$

Second, the commutation relations between J^2 and J_i are

$$\begin{aligned}
[J^2, J_i] &= [J_1^2 + J_2^2 + J_3^2, J_i] \\
&= [J_1^2, J_i] + [J_2^2, J_i] + [J_3^2, J_i] \\
&= J_1[J_1, J_i] + [J_1, J_i]J_1 + J_2[J_2, J_i] + [J_2, J_i]J_2 + J_3[J_3, J_i] + [J_3, J_i]J_3
\end{aligned}$$

If $i = 1$

$$\begin{aligned}
[J^2, J_1] &= J_1 \underbrace{[J_1, J_1]}_{0} + \underbrace{[J_1, J_1]}_{0} J_1 + J_2[J_2, J_1] + [J_2, J_1]J_2 + J_3[J_3, J_1] + [J_3, J_1]J_3 \\
&= -i \underbrace{J_2J_3}_{0} - i \underbrace{J_3J_2}_{0} + i \underbrace{J_3J_2}_{0} + i \underbrace{J_2J_3}_{0} = 0
\end{aligned}$$

and similarly for $i = 2, 3$. q.e.d.

Eigenvalues

Since J_3 and J^2 are commuting Hermitian matrices on a Hilbert space, the spectral theorem garanties us that there is a basis $\{|\xi, m\rangle\}$ such that it is orthonormal $\langle\xi, m|\xi', m'\rangle = \delta_{mm'}\delta_{\xi\xi'}$ and it is a simultaneously eigenvector of the two operators, i.e. $J^2|\xi, m\rangle = \xi|\xi, m\rangle$ and $J_3|\xi, m\rangle = m|\xi, m\rangle$. There could be degeneracies, i.e. same eigenvalue could have more than one eigenket, but the ladder operators prevent that because we can restrict to 1-dimensional subspaces which correspond to same labels.

By quantum mechanics similarity, we have the following eigenvalue relations

$$J^2|j, m\rangle = j(j+1)|j, m\rangle \qquad J_3|j, m\rangle = m|j, m\rangle \tag{3.1}$$

where $j \in \frac{\mathbb{N}_0}{2}$ and $|m| \leq j$. Furthermore, the ladder operators act on these kets by

$$J_\pm|j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

Notice that different values of j correspons to different representations, which have dimension $2j+1$. The irreducible representations are labelled by the eigenvalues of Casimir operator.

For $j = 1$, we have a 3-dimensional rep. For $j > 1$, we have a tensor product of irreps $V_1^{\otimes j} = V_1 \otimes \ldots \otimes V_1$. However half-integers are not a rep of $SO(3)$.

Proof. The entries of the representation matrices $M = \rho_j(X)$ for any $X \in \mathfrak{so}(3)$ are

$$M_{ab} = \langle j, a|M|j, b\rangle$$

and its action on kets is

$$M|j,m\rangle = \sum_k |j,k\rangle M_{km}$$

By using the exponential map, we can find a representation of an element which could be in the universal cover. Infact, if we could take for istance the second unit vector n_2, we have

$$\rho_j(R(\theta))_{ab} = \langle j, a| \exp(-i\theta J_2^{(j)})|j, b\rangle = \langle j, a| \exp(-\frac{1}{2}\theta(J_+^{(j)} - J_-^{(j)}))|j, b\rangle$$

where $J_k^{(j)} = \rho_j(L_k)$.

Explicitly, for $\theta = \pi$ we have

$$\rho(R(\pi))_{ab} = (-1)^{j-b}\delta_{a,-b}$$

Hence, a full rotation 2π is

$$\begin{aligned}
\rho_j(R(\pi)R(\pi))_{ab} &= (\rho_j(R(\pi))\rho_j(R(\pi)))_{ab} \\
&= \sum_c \rho(R(\pi))_{ac}\rho_j(R(\pi))_{cb} \\
&= \sum_c (-1)^{j-c}\delta_{a,-c}(-1)^{j-b}\delta_{c,-b} \\
&= \sum_c (-1)^{2j-c-b}\delta_{a,-c}\delta_{c,-b} \\
&= (-1)^{2j}\delta_{ab}
\end{aligned}$$

Therefore $R(\pi)R(\pi) \neq \mathbb{I}$ and a full rotation is not the unit matrix for half-integer representations. q.e.d.

Observe that, since the difference is just a phase factor, it is indeed a projective rep.

3.3 $SO(3)$ and $SU(2)$

The spinor representation of $SU(2)$ with $j = \frac{1}{2}$ is a 2-dimensional representation with basis

$$|e_1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle \qquad |e_2\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$$

Hence, the matrix representation of the J_3 operator is

$$J_3^{(\frac{1}{2})} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

of the ladder operators are

$$J_-^{(\frac{1}{2})} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad J_-^{(\frac{1}{2})} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and of the J_i operators are

$$J_1^{(\frac{1}{2})} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad J_2^{(\frac{1}{2})} = \begin{bmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}$$

Proof. First, the J_3 operator with (3.1)

$$\langle e_1|J_3^{(\frac{1}{2})}|e_1\rangle = -\langle e_2|J_3^{(\frac{1}{2})}|e_2\rangle = -\frac{1}{2} \quad \langle e_1|J_3^{(\frac{1}{2})}|e_2\rangle = \langle e_2|J_3^{(\frac{1}{2})}|e_1\rangle = 0$$

Second, the ladder operators

$$J_-^{(\frac{1}{2})}|e_1\rangle = |e_2\rangle \quad J_+^{(\frac{1}{2})}|e_1\rangle = J_-^{(\frac{1}{2})}|e_2\rangle == 0 \quad J_+^{(\frac{1}{2})}|e_2\rangle = |e_1\rangle$$

Third, the J_i operators, with the inverse relations of (**??**)

$$J_1^{(\frac{1}{2})} = \frac{1}{2}(J_+^{(\frac{1}{2})} + J_-^{(\frac{1}{2})}) \quad J_2^{(\frac{1}{2})} = \frac{1}{2i}(J_+^{(\frac{1}{2})} - J_-^{(\frac{1}{2})}) \ .$$

q.e.d.

Notice that

$$J_i^{(\frac{1}{2})} = \frac{1}{2}\sigma_i$$

where σ_i are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

such that thet satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

This three matrices are linearly independent and they generate the $SU(2)$ group, i.e. the set of Hermitian traceless matrices.

Proof. This can be seen by writing down a complex 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and by requiring the condition to be a $SU(2)$ matrix, i.e. $b = c^*$ because of hermiticy and $a = -d$ because of tracelessness. Hence

$$\begin{pmatrix} a & b \\ b^* & -a \end{pmatrix}$$

which leaves three degrees of freedom, because a is real. Now we decompose into the real and the complex part of b

$$\begin{pmatrix} a & b_r - ib_i \\ b_r + ib_i & -a \end{pmatrix} = b_r \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b_i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = b_r\sigma_1 + b_i\sigma_2 + a\sigma_3$$

which proves that they are indeed a basis for $\mathfrak{su}(2)$ q.e.d.

.

Thus a matrix $U \in SU(2)$ can be written in terms of the exponential map

$$U(\theta, \mathbf{n}) = \exp(-i\theta\mathbf{n} \cdot \mathbf{J}^{(\frac{1}{2})}) = \exp(-\frac{i}{2}\theta\mathbf{n} \cdot \sigma) \ ,$$

or explicitly

$$U(\theta, \mathbf{n}) = \begin{bmatrix} \cos\frac{\theta}{2} - in_3 \sin\frac{\theta}{2} & -i(n_1 - in_2)\sin\frac{\theta}{2} \\ -i(n_1 + in_2)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_3 \sin\frac{\theta}{2} \end{bmatrix} .$$

Alternatively, we can show that $SU(2) \simeq S^3$, by keeping complex parameters a and b

$$U(a, b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{3.2}$$

such that $\det U = |a|^2 + |b|^2 = 1$.

$\mathfrak{so}(3) = \mathfrak{su}(2)$

In order to see that the Lie algebras are the same, we introduce a map between the two linear spaces

$$X = \sum_i x_i\sigma_i \mapsto \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

with the Euclidean scalar product

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}tr(XY)$$

Thus, given an element $X \in \mathfrak{su}(2)$ we have for any $u \in SU(2)$ both the tracelessness and the hermiticy properties

$$tr(UXU^{-1}) = tr(U^{-1}UX) = 0 \quad (UXU^{-1})^\dagger = (U^{-1})^\dagger X^\dagger U^\dagger = UXU^{-1}$$

and an action $\mathfrak{su}(2) \simeq \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$X \mapsto UXU^{-1}$$

Furthermore, it is orthogonal

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2} tr(XY) \mapsto \frac{1}{2} tr(UXU^{-1}UYU^{-1}) = \frac{1}{2}(XY) = \mathbf{x} \cdot \mathbf{y}$$

and, since $SU(2)$ is connected, the identities are mapped and this defines a 3-dimensional rep

$$\rho_{ad} \colon SU(2) \to SO(3) \subset Aut(V) \quad V \ni X \mapsto \rho_{ad}(U)(X) = UXU^{-1}$$

called the adjoint representation. In this case, it coincides with the spin $j = 1$ representation of $SO(3)$.

Explicitly, by using (3.2)

$$\begin{aligned} U(a,b)\sigma_1 U(a,b)^{-1} &= Re(a^2 - b^2)\sigma_1 - Im(a^2 - b^2)\sigma_2 + 2Re(ab^*)\sigma_3 \\ U(a,b)\sigma_2 U(a,b)^{-1} &= Im(a^2 + b^2)\sigma_1 + Re(a^2 + b^2)\sigma_2 + 2Im(ab^*)\sigma_3 \\ U(a,b)\sigma_3 U(a,b)^{-1} &= -Re(ab)\sigma_1 + 2Im(ab)\sigma_2 + (|a|^2 - |b|^2)\sigma_3 \end{aligned}$$

or, in matrix notation,

$$\rho_{ad}(U(a,b)) = \begin{bmatrix} Re(a^2 - b^2) & Im(a^2 + b^2) & -Re(ab) \\ -Im(a^2 - b^2) & Re(a^2 + b^2) & 2Im(ab) \\ 2Re(ab^*) & 2Im(ab^*) & |a|^2 - |b|^2 \end{bmatrix}$$

We can recover rotation matrices around each Cartesian axis, by computing

$$R(\delta\theta, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \rho_{ad}(U(\cos\frac{\theta}{2}, -i\sin\frac{\theta}{2}))$$

$$R(\delta\theta, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \rho_{ad}(U(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}))$$

$$R(\delta\theta, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \rho_{ad}(U(\exp(-\frac{i}{2}\theta), 0))$$

Therefore, ρ_{ad} defines a group homomorphism. The kernel is $\rho_{ad}(\mathbb{I}_2) = \rho_{ad}(-\mathbb{I}_2) = \mathbb{I}_3$. Hence, the universal cover $SU(2)$ is a two-fold cover of $SO(3)$, i.e. $SO(3) \simeq SU(2)/\ker(\rho_{ad}) = SU(2)/\mathbb{Z}_2$.

Proof. If $U(a,b)$ is mapped into the identity, then $Re(ab^*) = 0$, $|a|^2 - |b|^2 = 1$ and $Re(a^2 - b^2) = 1$. Hence $b = 0$ and $a = \pm 1$. q.e.d.

For a general $SO(n)$ rotation group, since it is not connected, we have a two-fold universal cover $Spin(n)$, which is for $n \leq 6$

$$Spin(3) \simeq SU(2) \quad Spin(4) \simeq SU(2) \times SU(2) Spin(5) \simeq Sp(2) \quad Spin(6) \simeq SU(6)$$

However, for $n > 6$, there is not a matrix group. Each group has a spinor representation. For $n = 2m$, it has dimension 2^{m-1} and for $n = 2m + 1$, it has dimension 2^m.s

Chapter 4

$SO^+(1,3)$

In this chapter, we will apply all the notions we learnt in the previous ones to (1.5), the proper Lorentz group $SO(1,3)$. In particular, we are interested in the proper orthochronous Lorentz group $SO^+(1,3)$.

4.1 $SO^+(1,3)$ as a Lie group

The Lorentz group $O(1,3)$ is defined by the matrices Λ such that they preserve the Minkovski metric

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu} \tag{4.1}$$

First, the Lorentz group can be decomposed into two parts according to their determinant

$$\det(\Lambda^T \eta \Lambda) = \det \Lambda^T \det \eta \det \Lambda = \overset{2}{\det} \Lambda = \det \eta = 1$$

Hence

$$\det \Lambda = \pm 1$$

and the Lorentz group can be written as

$$O(1,3) = \underbrace{\{\det \Lambda = +1\}}_{SO(1,3)} \cup \{\det \Lambda = -1\} = SO(1,3) \cup \{\det \Lambda = -1\}$$

where $SO(1,3)$ is called the proper Lorentz group.

Second, the proper Lorentz group can be decomposed into two parts according to their $(0,0)$ component

$$\eta_{00} = \Lambda^\alpha{}_0 \Lambda^\beta{}_0 \eta_{\alpha\beta}$$

$$-1 = -(\Lambda^0{}_0)^2 + (\Lambda^i{}_i)^2$$

$$(\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_i)^2 \geq 1$$

Hence

$$\Lambda^0{}_0 \in]\infty, -1] \cup [1, \infty[$$

and the proper Lorentz group can be written as

$$SO(1,3) = \underbrace{\{\Lambda^0{}_0 \in]\infty, -1]\}}_{SO^+(1,3)} \cup \{\Lambda^0{}_0 \in [1, \infty[\} = SO^+(1,3) \cup \{\Lambda^0{}_0 \in [1, \infty[\}$$

where $SO(1,3)^+$ is called the proper orthochronous Lorentz group.

From now on, only the proper orthochronous Lorentz group will be studied because is the only group containing the identity. The rest of the group can be obtained by composing a proper orthochronous Lorentz transformation with time or spatial reversals.

4.2 $\mathfrak{so}(1,3)$ as a Lie algebra

To find the Lie algebra, we consider an infinitesimal Lorentz transformation around the identity

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{4.2}$$

where $\omega^\mu{}_\nu \ll 1$ is an infinitesimal matrix.

In order to preserve the metric $\omega^\mu{}_\nu$ is antisymmetric.

Proof. By using (4.1),

$$(\delta^\alpha{}_\mu + \omega^\alpha{}_\mu)(\delta^\beta{}_\nu + \omega^\beta{}_\nu)\eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\delta^\alpha{}_\mu \delta^\beta{}_\nu \eta_{\alpha\beta} + \delta^\alpha{}_\mu \omega^\beta{}_\nu \eta_{\alpha\beta} + \omega^\alpha{}_\mu \delta^\beta{}_\nu \eta_{\alpha\beta} + \omega^\alpha{}_\mu \omega^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\cancel{\eta_{\mu\nu}} + \omega_{\mu\nu} + \omega_{\nu\mu} + O(\omega^2) = \cancel{\eta_{\mu\nu}}$$

Hence, the matrices $\omega_{\mu\nu}$ are anti-symmetric

$$\omega_{\mu\nu} = \omega_{\nu\mu}$$

q.e.d.

By means of the exponential map, a generic $SO(1,3)^+$ transformation can be written as

$$\Lambda^\mu{}_\nu = \exp(-\frac{i}{2}\omega^{\alpha\beta}M_{\alpha\beta})^\mu{}_\nu$$

where $M_{\alpha\beta}$ are the generators of the Lie algebra $\mathfrak{so}(1,3)$. Since they must be antisymmetric, otherwise they would vanish because ω are so as well, there are six independent generators of $\mathfrak{so}(1,3)$.

The commutation relations between the generators are

$$[M_{\alpha\beta}, M_{\sigma\rho}] = -i(\eta_{\alpha\sigma}M_{\beta\rho} - \eta_{\alpha\rho}M_{\beta\sigma} - \eta_{\beta\sigma}M_{\alpha\rho} + \eta_{\beta\rho}M_{\alpha\sigma})$$

or in matrix indices notation

$$[M_{\alpha\beta}, M_{\sigma\rho}]^{\mu}{}_{\nu} = (M_{\alpha\beta})^{\mu}{}_{\gamma}(M_{\sigma\rho})^{\gamma}\ \ \nu - (M_{\sigma\rho})^{\mu}{}_{\gamma}(M_{\alpha\beta})^{\gamma}\ \ \nu$$

Proof. To find the explicit expression of the commutator of two generators, first it will be computed the following expression using (4.2)

$$\begin{aligned}
(\tilde{\Lambda}^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\alpha}{}_{\beta}\tilde{\Lambda}^{\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\nu} &= (\delta^{\mu}{}_{\alpha} - \tilde{\omega}^{\mu}{}_{\alpha})(\delta^{\alpha}{}_{\beta} - \omega^{\alpha}{}_{\beta})(\delta^{\beta}{}_{\gamma} + \tilde{\omega}^{\beta}{}_{\gamma})(\delta^{\gamma}{}_{\nu} + \omega^{\gamma}{}_{\nu}) + O(\tilde{\omega}^2) + O(\omega^2)\\
&= \delta^{\mu}{}_{\alpha}\delta^{\alpha}{}_{\beta}\delta^{\beta}{}_{\gamma}\delta^{\gamma}{}_{\nu} - \tilde{\omega}^{\mu}{}_{\alpha}\delta^{\alpha}{}_{\beta}\delta^{\beta}{}_{\gamma}\delta^{\gamma}{}_{\nu} - \delta^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\beta}\delta^{\beta}{}_{\gamma}\delta^{\gamma}{}_{\nu} + \delta^{\mu}{}_{\alpha}\delta^{\alpha}{}_{\beta}\tilde{\omega}^{\beta}{}_{\gamma}\delta^{\gamma}{}_{\nu}\\
&\quad + \delta^{\mu}{}_{\alpha}\delta^{\alpha}{}_{\beta}\delta^{\beta}{}_{\gamma}\tilde{\omega}^{\beta}{}_{\gamma} + \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\beta}\delta^{\beta}{}_{\gamma}\delta^{\gamma}{}_{\nu} - \tilde{\omega}^{\mu}{}_{\alpha}\delta^{\alpha}{}_{\beta}\delta^{\beta}{}_{\gamma}\omega^{\gamma}{}_{\nu} - \delta^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\beta}\tilde{\omega}^{\beta}{}_{\gamma}\delta^{\gamma}{}_{\nu}\\
&\quad + \delta^{\mu}{}_{\alpha}\tilde{\omega}^{\alpha}{}_{\beta}\delta^{\beta}{}_{\gamma}\omega^{\gamma}{}_{\nu} + O(\tilde{\omega}^2) + O(\omega^2)\\
&= \delta^{\mu}{}_{\nu} - \cancel{\tilde{\omega}^{\mu}{}_{\nu}} - \cancel{\omega^{\mu}{}_{\nu}} + \cancel{\tilde{\omega}^{\mu}{}_{\nu}} + \cancel{\omega^{\mu}{}_{\nu}} + \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu} - \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu} - \omega^{\mu}{}_{\gamma}\tilde{\omega}^{\alpha}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu}\\
&\quad + O(\omega^2) + O(\tilde{\omega}^2)\\
&= \delta^{\mu}{}_{\nu} + \cancel{\tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu}} - \cancel{\tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu}} - \omega^{\mu}{}_{\gamma}\tilde{\omega}^{\alpha}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu} + O(\omega^2) + O(\tilde{\omega}^2)\\
&= \delta^{\mu}{}_{\nu} - \omega^{\mu}{}_{\gamma}\tilde{\omega}^{\alpha}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu} + O(\omega^2) + O(\tilde{\omega}^2)
\end{aligned}$$

and second using the BCH formula (**??**)

$$\begin{aligned}
(\tilde{\Lambda}^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\alpha}{}_{\beta}\tilde{\Lambda}^{\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\nu} &= \exp(\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta})\exp(\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho})\exp(-\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta})\exp(-\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho})\\
&= \exp(\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta} + \frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho} + [\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta}, \frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}])\\
&\quad \exp(-\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta} - \frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho} + [\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta}, \frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}]) + O(\omega^2) + O(\tilde{\omega}^2)\\
&= \exp(\cancel{\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta}} + \cancel{\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}} - \cancel{\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta}} - \cancel{\frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}}\\
&\quad + [\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta}, \frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}]) + O(\omega^2) + O(\tilde{\omega}^2)\\
&= \exp([\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta}, \frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}]) + O(\omega^2) + O(\tilde{\omega}^2)
\end{aligned}$$

Hence, putting together

$$\exp([\frac{i}{2}\tilde{\omega}^{\alpha\beta}M_{\alpha\beta}, \frac{i}{2}\omega^{\sigma\rho}M_{\sigma\rho}]) = \delta^{\mu}{}_{\nu} - \omega^{\mu}{}_{\gamma}\tilde{\omega}^{\alpha}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu} = \exp(-\frac{i}{2}(-\omega^{\mu}{}_{\gamma}\tilde{\omega}^{\alpha}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu})M_{\mu}{}^{\nu})$$

$$-\frac{1}{4}\tilde{\omega}^{\alpha\beta}\omega^{\sigma\rho}[M_{\alpha\beta}, M_{\sigma\rho}] = -\frac{i}{2}(-\omega^{\mu}{}_{\alpha}\tilde{\omega}^{\alpha}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu})M_{\mu}{}^{\nu} = -\frac{i}{2}\eta_{\alpha\gamma}(-\omega^{\mu\alpha}\tilde{\omega}^{\gamma\nu} + \tilde{\omega}^{\mu\alpha}\omega^{\gamma\nu})M_{\mu\nu}$$

Hence

$$-\frac{1}{4}\tilde{\omega}^{\alpha\beta}\omega^{\sigma\rho}[M_{\alpha\beta}, M_{\sigma\rho}] = -\frac{i}{2}\eta_{\alpha\gamma}(-\omega^{\mu\alpha}\tilde{\omega}^{\gamma\nu} + \tilde{\omega}^{\mu\alpha}\omega^{\gamma\nu})M_{\mu\nu}$$

Consider an ansatz

$$[M_{\alpha\beta}, M_{\sigma\rho}] = T^{(1)}_{\alpha\beta}M_{\sigma\rho} + T^{(2)}_{\alpha\sigma}M_{\beta\rho} + T^{(3)}_{\alpha\rho}M_{\beta\sigma} + T^{(4)}_{\beta\sigma}M_{\alpha\rho} + T^{(5)}_{\beta\rho}M_{\alpha\sigma} + T^{(6)}_{\sigma\rho}M_{\alpha\beta}$$

and inserting into the previous, there are no matching term with $T^{(1)} = T^{(2)} = 0$.

Furthermore, using $M_{\mu\nu} = -M_{\nu\mu}$

$$0 = [M_{\alpha\beta}, M_{\sigma\rho}] + [M_{\beta\alpha}.M_{\sigma\rho}]$$

q.e.d.

It is also possbile to derive the 4×4 matrix representation, which is

$$(M_{\alpha\beta})^{\mu}{}_{\nu} = i(\delta^{\mu}{}_{\alpha}\eta_{\beta\nu} - \delta^{\mu}{}_{\beta}\eta_{\alpha\nu})$$

By resticting to spatial indices

$$[M_{ij}, M_{kl}] = -i(\delta_{ik}M_{jl} - \delta_{il}M_{jk} - \delta_{jk}M_{il} + \delta_{ji}M_{ij})$$

which gives rise to three independent generators $J_m = -\frac{1}{2}\epsilon_{mij}M_{ij}$ or equivalently $M_{ij} = -\epsilon_{ijk}J_k$, with commuting relations $[J_m, J_n] = i\epsilon mnkJ_k$. Hence, the proper orthochronous Lorentz group contains the rotation group as subgroup.

There are other generators $K_i = M_{0i}$ associated to the temporal index, which are the generators of boosts in the i-th direction.. Hence

$$[K_i, K_j] = -i(\eta_{00}M_{ij} + \delta_{ij}M_{00}) = -iM_{ij} = -i\epsilon_{ijk}J_k$$

and

$$[J_i, K_j] = -\frac{1}{2}\epsilon_{imn}[M_{mn}, M_{0j}] = -\frac{i}{2}(-\delta_{mj}M_{n0} + \delta_{nj}M_{m0}) = i\epsilon_{ijn}K_n$$

If the velocity is $\mathbf{v} = c\tanh(r)\mathbf{u}$ where r is the rapidity, then a boost transformation is

$$\Lambda = \begin{pmatrix} \cosh(r) & \sinh(r)\mathbf{u}^T \\ \sinh(r)\mathbf{u} & \mathbb{I}_3 + (\cosh(r) - 1)\mathbf{u}\mathbf{u}^T \end{pmatrix}$$

Hence, a general proper orthochronous Lorentz transformation is

$$\Lambda(\theta, \mathbf{n}, r, \mathbf{u}) = \exp(-i\theta\mathbf{n}\cdot\mathbf{J} - i\mathbf{v}\cdot\mathbf{K}) = \exp(-i\theta\mathbf{n}\cdot\mathbf{J} - ic\tanh(r)\mathbf{u}\cdot\mathbf{K})$$

4.3 Representations of the Lorentz algebra

$SO^{+}(1,3)$ is not compact, since the values of the boosts do not have an upper-bound limit. For non-compact non-abelian Lie groups, any non-trivial unitary representation must be infinite-dimensional. However, let us focus in finite-dimensional representations, even though they are not unitary. We can justified this choice by noticing that for field representations, which become quantum operator, there is no scalar product between them.

The defining representation of the Lie algebra $\mathfrak{so}^+(1,3)$ is

$$(M_{\alpha\beta})^\mu{}_\nu = i(\delta^\mu{}_\alpha \eta_{\beta\nu} - \delta^\mu{}_\beta \eta_{\alpha\nu})$$

The generators of rotations are

$$J_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$$

which are hermitian, i.e. $J_i^\dagger = J_i$, while the generators of boosts are

$$K_i = M_{0i}$$

which are anti-hermitian, i.e. $K_i^\dagger = -K_i$, because they do not have a finite range. Explicitly, they are

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \quad J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.3}$$

and

$$K_1 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

Proof. For J_1 and similarly for the others

$$(J_1)^\mu{}_\nu = -(M_{23})^\mu{}_\nu = -i(\underbrace{\delta^\mu{}_2}_{\mu=2}\underbrace{\eta_{3\nu}}_{\nu=3} + \underbrace{\delta^\mu{}_3}_{\mu=3}\underbrace{\eta_{2\nu}}_{\nu=2}) = -i\delta^2{}_3 + i\delta^3{}_2$$

For K_1 and similarly for the others

$$(K_1)^\mu{}_\nu = -(M_{01})^\mu{}_\nu = -i(\underbrace{\delta^\mu{}_0}_{\mu=0}\underbrace{\eta_{1\nu}}_{\nu=1} + \underbrace{\delta^\mu{}_1}_{\mu=1}\underbrace{\eta_{0\nu}}_{\nu=0}) = i\delta^0{}_1 + i\delta^1{}_0$$

q.e.d.

The commutation relations between thes generators are

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad [J_i, K_j] = i\epsilon_{ijk} K_k \quad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

Proof. Maybe in the future. q.e.d.

The procedure to construct Lorentz algebra irreducible representations is called complexification and it consists of a complex linear combination of the generators $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. In particular, for the Lorentz algebra, we define new generators

$$A_i = \frac{1}{2}(J_i + iK_i) \quad B_i = \frac{1}{2}(J_i - iK_i)$$

such that the new commutation relations are

$$[A_i, A_j] = i\epsilon_{ijk} A_K \quad [B_i, B_j] = i\epsilon_{ijk} B_K \quad [A_i, B_j] = 0$$

Proof. Maybe in the future. q.e.d.

Hence $\mathfrak{so}(1,3) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In this way, we can use representations of $\mathfrak{su}(2)$ labelled by a pair (j_1, j_2) to construct a representation of the whole Lorentz algebra. Formally, an irreducible representation of a direct sum $\mathfrak{g} \oplus \mathfrak{h}$ can be built by the tensor product of $(\rho_{\mathfrak{g}}, V_{\mathfrak{g}})$ and $(\rho_{\mathfrak{h}}, V_{\mathfrak{h}})$ by

$$\rho \colon \mathfrak{g} \oplus \mathfrak{h} \to End(V) = End(V_{\mathfrak{g}} \otimes V_{\mathfrak{h}})$$

such that

$$\rho(X + Y)(v \otimes w) = \rho_{\mathfrak{g}}(v) \otimes w + v \otimes \rho_{\mathfrak{h}}(w)$$

In particular, for the Lorentz algebra

$$\rho_{j_1,j_2}\Big(\sum_m \lambda_m A_m + \sum_n k_n B_n\Big) = \sum_m \lambda_m \rho_{j_1}(A_m) \otimes \mathbb{I}_{V_{j_2}} + \sum_n k_n \mathbb{I}_{V_{j_1}} \otimes \rho_{j_2}(B_n)$$

where A_m and B_n are independent generators of $\mathfrak{su}(2)$ and the dimension is $\dim(V_{j_1} \otimes V_{j_2}) = \dim V_{j_1} \dim V_{j_2} = (2j_1 + 1)(2j_2 + 1)$.

If we restrict to the rotation algebra $J_i = A_i + B_i$, a similar closed algebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ can be studied inside $\mathfrak{so}(1,3)$. By means of the Clebsch-Gordan decomposition, this reduces into a sum of irreducible representation of spin j such that $|j| \leq j_1 + j_2$, with total dimension $(2j_1 + 1)(2j_2 + 1)$.

Notice that if they are both integers or half-integers, the sum representation is integer, like a bosonic one. On the other hand, if one is integer and the other one half-integer, the sum representation is half-integer, like a fermionic one.

$SO^+(1,3)$ is not simply connected. Therefore, we need its universal cover is $SL(2,\mathbb{C})$. In particular, we have also here a two-fold cover $SO^+(1,3) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$. Hence the spinor representations of $SO^+(1,3)$ is in $SL(2,\mathbb{C})$.

Spinor representations

Given the fact that we have two labels j_1 and j_2 for the representations, there are two different 2-dimensional spinor representations: the left-handed Weyl spinor $\rho_L = \rho_{(\frac{1}{2},0)}$ with $(\frac{1}{2},0)$ and the right-handed Weyl spinor $\rho_R = \rho_{(0,\frac{1}{2})}$ with $(0,\frac{1}{2})$. They are called spinors because the only irreducible representation of rotations restriction are the one corresponding to $j = \frac{1}{2}$.

Since the rep $j = \frac{1}{2}$ is generated by the Pauli matrices and $j = 0$ is the trivial one, the generators of left spinors are

$$\rho_L(A_i) = \frac{1}{2}\sigma_i \otimes \mathbb{I}_1 = \frac{1}{2}\sigma_i \quad \rho_R(B_i) = \mathbb{I}_2 \otimes 0 = 0$$

and the generators of right spinors are

$$\rho_R(A_i) = 0 \otimes \mathbb{I}_1 = 0 \quad \rho_R(B_i) = \mathbb{I}_1 \otimes \frac{1}{2}\sigma_i = \frac{1}{2}\sigma_i$$

Hence, by linearity, the original generators for left spinors are

$$\rho_L(J_i) = \rho_L(A_i) + \rho_L(B_i) = \frac{1}{2}\sigma_i \quad \rho_L(K_i) = -i(\rho_L(A_i) - \rho_L(B_i)) = -\frac{i}{2}\sigma_i$$

and the original generators for right spinors are

$$\rho_R(J_i) = \rho_R(A_i) + \rho_R(B_i) = \frac{1}{2}\sigma_i \quad \rho_R(K_i) = -i(\rho_R(A_i) - \rho_R(B_i)) = \frac{i}{2}\sigma_i$$

The associated representation is generated by real linear combinations: for left spinors is

$$V_{(\frac{1}{2},0)} \ni \phi \mapsto \exp(-i\theta\vec{n}\cdot\rho_L(\vec{J}) - i\vec{\cdot}\rho_L(\vec{K}))\phi$$

and for right spinors is

$$V_{(0,\frac{1}{2})} \ni \phi \mapsto \exp(-i\theta\vec{n}\cdot\rho_R(\vec{J}) - i\vec{\cdot}\rho_R(\vec{K}))\phi = \exp(-\frac{1}{2}(-i\theta\vec{n}+\vec{v})\cdot\vec{\sigma})\phi$$

They are the complex conjugates of each other.

Proof. By using the identity

$$(i\sigma_2)\sigma_i(-i\sigma_1) = (i\sigma_2)\overline{\sigma}_i(i\sigma_2)^{-1} = -\sigma_i$$

we define a new vector $\chi = i\sigma_2\phi$ which maps as

$$\begin{aligned}
\chi \mapsto i\sigma_2\phi &= i\sigma_2\overline{\exp(-i\theta\vec{n}\cdot\rho_L(\vec{J}) - i\vec{\cdot}\rho_L(\vec{K}))\phi} \\
&= i\sigma_2\overline{\exp(-\frac{1}{2}(-i\theta\vec{n}+\vec{v})\cdot\vec{\sigma})\phi} \\
&= i\sigma_2\exp(-\frac{1}{2}(-i\theta\vec{n}+\vec{v})\cdot\overline{\vec{\sigma}})\overline{\phi} \\
&= i\sigma_2(i\sigma_2)^{-1}\exp(-\frac{1}{2}(-i\theta\vec{n}+\vec{v})\cdot\vec{\sigma})(i\sigma_2)\overline{\phi} \\
&= \exp(-i\theta\vec{n}\cdot\rho_R(\vec{J}) - i\vec{v}\cdot\rho_R(\vec{K}))\chi
\end{aligned}$$

Hence, up to a basis change, the complex conjugate of ϕ transforms like χ. q.e.d.

Dirac representation

The direct sum of the two Weyl spinors is a reducible representation, called the bispinor representation. It is related to the Weyl one via

$$V_D = V_{\frac{1}{2},0} \oplus V_{0,\frac{1}{2}}$$

but with different basis

$$\psi_1 = \frac{\phi_1 + \chi_1}{\sqrt{2}} \quad \psi_2 = \frac{\phi_2 + \chi_2}{\sqrt{2}} \quad \psi_3 = \frac{\phi_1 - \chi_1}{\sqrt{2}} \quad \psi_4 = \frac{\phi_2 - \chi_2}{\sqrt{2}}$$

where (ϕ_1, ϕ_2) is a basis of $V_{frac12,0}$ and (χ_1, χ_2) of $V_{0,\frac{1}{2}}$.

Another way to introduce bispinors is through the Dirac matrices, i.e. 4×4 matrices such that they satisfy the anticommutator relations

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$$

They allow to construct a 4-dimensional representation of the Lorentz algebra

$$M_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta] = \frac{i}{4}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$$

such that the commutator relations are

$$[M_{\alpha\beta}, M_{\sigma\rho}] = -i(\eta_{\alpha\sigma}M_{\beta\rho} - \eta_{\alpha\rho}M_{\beta\sigma} - \eta_{\beta\sigma}M_{\alpha\rho} + \eta_{\beta\rho}M_{\alpha\sigma})$$

which leads to the Dirac equation. Further insights can be found in the RQM or in the QFT course.

Vector representation

Now, we study the vector representation, which is a representation where the spin label is $(\frac{1}{2}, \frac{1}{2})$. Since its restrictions to rotations are $j = \frac{1}{2} + \frac{1}{2} = 1$ and $j = \frac{1}{2} - \frac{1}{2} = 0$, they both have integer spin and it corresponds to a bosonic field.

We pick a basis of $V_{\frac{1}{2}}$, which is

$$\{|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle\} = \{|+\rangle, |-\rangle\}$$

and it induces a natural basis for $V_{(\frac{1}{2},\frac{1}{2})}$

$$|+\rangle \oplus |+\rangle = |++\rangle \quad |+\rangle \oplus |-\rangle = |+-\rangle \quad |-\rangle \oplus |+\rangle = |-+\rangle \quad |-\rangle \oplus |-\rangle = |--\rangle$$

However, it can be more useful to introduce another basis for the same linear space

$$|e_1\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \quad |e_2\rangle = \frac{|++\rangle - |--\rangle}{\sqrt{2}}$$
$$|e_3\rangle = \frac{-|++\rangle - i|--\rangle}{\sqrt{2}} \quad |e_4\rangle = \frac{-|+-\rangle - |-+\rangle}{\sqrt{2}}$$

It is equivalent to the defining rep of $\mathfrak{so}(1,3)$.

Proof. The complex generators of the representation are

$$\rho(A_i) = \rho_{\frac{1}{2}}(A_i) \otimes \mathbb{J}_2 = \mathcal{J}_i^{(\frac{1}{2})} \otimes \mathbb{J}_2$$

and

$$\rho(B_i) = \mathbb{J}_2 \otimes \rho_{\frac{1}{2}}(B_i) = \mathbb{J}_2 \otimes \mathcal{J}_i^{(\frac{1}{2})}$$

where their actions on the natural basis are

$$\rho(A_i)|+-\rangle = \mathcal{J}_i^{(\frac{1}{2})}|+\rangle \otimes |-\rangle \quad \rho(B_i)|+-\rangle = |+\rangle \otimes \mathcal{J}_i^{(\frac{1}{2})}|-\rangle$$

Moreover, the original generators become

$$\rho(J_i) = \rho(A_i) + \rho(B_i) = \mathcal{J}_i^{(\frac{1}{2})} \otimes \mathbb{J}_2 + \mathbb{J}_2 \otimes \mathcal{J}_i^{(\frac{1}{2})}$$

and

$$\rho(K_i) = -i(\rho(A_i) - \rho(B_i)) = -i\mathcal{J}_i^{(\frac{1}{2})} \otimes \mathbb{J}_2 + i\mathbb{J}_2 \otimes \mathcal{J}_i^{(\frac{1}{2})}$$

where $\mathcal{J}^{(\frac{1}{2})}$ is the single $\mathfrak{su}(2)$ representation.

For example, we take the generator J_3, whose action on the natural basis is

$$\rho(J_3)|++\rangle = |++\rangle \quad \rho(J_3)|--\rangle = -|--\rangle \quad \rho(J_3)|-+\rangle = \rho(J_3)|+-\rangle = 0$$

or on the other basis is

$$\rho(J_3)|e_1\rangle = \rho(J_3)|e_4\rangle = 0 \quad \rho(J_3)|e_2\rangle = i|e_3\rangle \quad \rho(J_3)|e_2\rangle = -i|e_2\rangle$$

Hence the corresponding matrix becomes

$$\rho(J_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is indeed the same matrix as (4.3).

For example, we take the generator K_3, whose action on the natural basis is

$$\rho(K_3)|++\rangle = \rho(K_3)|--\rangle = 0 \quad \rho(K_3)|+-\rangle = -i|+-\rangle \quad \rho(K_3)|-+\rangle = i|-+\rangle$$

or on the other basis is

$$\rho(K_3)|e_1\rangle = i|e_4\rangle \quad \rho(K_3)|e_2\rangle = \quad \rho(K_3)|e_3\rangle = 0 \quad \rho(K_3)|e_4\rangle = i|e_1\rangle$$

Hence the corresponding matrix becomes

$$\rho(K_3) = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

which is indeed the same matrix as (4.4). q.e.d.

Other representations

We conclude this chapter with a list of some relevant representations in physics. See Table 4.1.

Representation	Field	Physics
$(0,0)$	scalar	Higgs
$(\frac{1}{2},0)$, $(0,\frac{1}{2})$ or $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	fermionic	matter
$(\frac{1}{2},\frac{1}{2})$	bosonic	forces
$(1,0)$ (or $(0,1)$)	(anti) self-dual 2-form	only string theory
$(1,0)\oplus(0,1)$	parity invariant 2-form	EM tensor
$(1,\frac{1}{2})\oplus(\frac{1}{2},1)$	Rarita-Schwinger	gravitino
$(1,1)$	traceless symmetric tensor	graviton

Table 4.1: Other representations of the Lorentz group

Part III

QFT

Chapter 5

Lorentz group

5.1

The Lorentz group is the group of isometries of the Minkovski spacetime, i.e. which leaves the metric unchanged

$$\Lambda^T \eta \Lambda = \eta$$

In particular, the proper orthochronous Lorentz group, the one which leaves discrete spacial P and time T inversion.

It has 6 parameters

1. 3-dimensional compact space rotations, i.e.

$$\theta = (\theta_1, \theta_2, \theta_3)$$

2. 3-dimensional non-compact boosts, i.e.

$$\beta = (\beta_1, \beta_2, \beta_3)$$

hence, it is a 6-dimensional Lie group and its continuous parameters can be gathered into an antisymmetric matrix ω

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

in the following way

$$\omega_{ij} = \epsilon_{ijk}\theta_k \quad \omega_{0i} = \beta_i$$

The basis of the corresponding Lie algebra sare the 6 generators

$$M^{\mu\nu} = -M^{\nu\mu}$$

which satisfy the commutator relation

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\rho\sigma} + M^{\rho\sigma}\eta^{\mu\sigma} - M^{\mu\rho}\eta_{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho})$$

A generic element of the Lorentz group can be written as

$$\Lambda = \exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}) \simeq 1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}$$

In its finite-dimensional representations, each element is represented by an operator (matrix) which acts in a finite-dimensional space. In its infinite-dimensional representations, each element is represented by an operator (field) which acts in a infinite-dimensional space (Hilbert space).

The generators can be decomposed in hermitian generators of rotations

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$$

and in anti-hermitian generators of boosts

$$K_i = M_{0i}$$

satisfing the commutation relation

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad [K_i, K_j] = -i\epsilon_{ijk}J_k \quad [J_i, K_j] = i\epsilon_{ijk}K_k$$

Notice that the algebra of J is $SU(2)$, but the other algebra is not closed. However, if we take a complex linear combination,

$$A_i = \frac{1}{2}(J_i + iK_i) \quad B_i = \frac{1}{2}(J_i - iK_i)$$

we can construct closed algebras of hermitian generators

$$[A_i, A_j] = i\epsilon_{ijk}A_k \quad [B_i, B_j] = i\epsilon_{ijk}B_k \quad [A_i, B_j] = 0$$

Hence, $\mathfrak{so}^+(1,3) \simeq \mathfrak{su}(2) \times \mathfrak{su}(2)$. Under parity,

$$x^0 \to x^0 \quad x^i \to -x^i$$

we can switch to the other generator

$$J_i \to J_i \quad K_i \to -K_i \quad \Rightarrow \quad A_i \rightleftarrows B_i$$

The physical spin $J = A + B$ can be then decomposed into two spinors belonging to $SU(2)$: left-handed spinors labelled by J_A and right-handed spinors labelled by J_B. In Physics, particles can be labelled by their representations

1. scalar $(0,0)$ with total spin $s = 0$;

2. vectors $(\frac{1}{2},\frac{1}{2})$ with total spin $s=0$;
3. Weyl spinors $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ with total spin $s=\frac{1}{2}$;
4. Dirac spinors $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ with total spin $s=\frac{1}{2}$;
5. Rarita-Schwinger $(0,0)$ with total spin $s=\frac{3}{2}$;
6. graviton $(0,0)$ with total spin $s=2$;

5.2 Finite-dimensional representations

The trivial representation $(0,0)$ is

$$M^{\mu\nu}=0$$

and each element is

$$\Lambda=\exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})=1$$

It is associated to scalars

$$\phi'=\Lambda\phi=1\phi=\phi$$

The vector representation $(\frac{1}{2},\frac{1}{2})$ is

$$(M^{\rho\sigma})^{\mu}{}_{\nu}=-i(\eta^{\mu\sigma}\delta^{\rho}{}_{\nu}-\eta^{\rho\mu}\delta^{\sigma}{}_{\nu})$$

and each element is

$$\Lambda=\exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$$

It is associated to 4-vectors

$$(V')^{\rho}=\Lambda^{\rho}{}_{\sigma}V^{\sigma}=(\exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}))^{\rho}{}_{\sigma}V^{\sigma}$$

It is the fundamental representation.

The spinorial representation is not of $SO^{+}(1,3)$ but of its double cover $SL(2,\mathbb{C})$. Infact there is an isomorphism between $SO^{+}(1,3)\simeq SL(2,\mathbb{C})/\mathbb{Z}_2$. This means that for each element of $SO^{+}(1,3)$ there are two corresponding element of $SL(2,\mathbb{C})$. Infact, using the Pauli matrices

$$\sigma^{\mu}=\mathbb{I},\vec{\sigma}$$

we can correspond a 2×2 matrix X with a 4-vector in the following way

$$X=x_{\mu}\sigma^{\mu}=\begin{matrix} x_0+ix_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{matrix}$$

Hence if Λ preserve the metric $ds^2 = x_0^2 - |\vec{x}|^2$, we can correspond with a matrix $N \in SL(2,\mathbb{C})$

$$X' = NXZ$$

such that

$$\det X' = \det X = x_0^2 - |\vec{x}|^2$$

Hence there is a $2-1$ map between them, e.g. for $N = \pm\mathbb{I}_2$ there is $\Lambda = 1$.

5.3 Finite-dimensional representations of $SL(2,\mathbb{C})$

The fundamental representation is the left-handed Weyl spinor $(\frac{1}{2},0)$

$$(\psi)_\alpha = N_\alpha{}^\beta \phi_\beta \quad \alpha,\beta = 1,2 \quad N \in SL(2,\mathbb{C})$$

The complex conjugate representation is the right-handed Weyl spinor $(0,\frac{1}{2})$

$$\overline{\chi}_{\dot{\alpha}} = (N^*)_{\dot{\alpha}}{}^{\dot{\beta}}\overline{\chi}_{\dot{\beta}} \quad \dot{\alpha},\dot{\beta} = 1,2 \quad N^* \in SL(2,\mathbb{C})$$

Its direct product $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ gives rise to the reducible representation of the Dirac spinors

$$\psi_D = \begin{bmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{bmatrix}$$

The invariant tensor to raise and lower indises is

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

since

$$(\epsilon')^{\alpha\beta} = \epsilon^{\sigma\rho} N_\rho{}^\alpha N_\sigma{}^\beta = \epsilon^{\alpha\beta}\underbrace{\det N}_{1} = \epsilon^{\alpha\beta}$$

Hence

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$$

and

$$\overline{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\chi}_{\dot{\beta}} \tag{5.1}$$

The generators of $SL_(2,\mathbb{C})$ are

1. left-handed Weyl spinor

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)_\alpha{}^\beta$$

hence

$$(\psi')_\alpha = (\exp(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}))_\alpha{}^\beta\psi_\beta$$

2. right-handed Weyl spinor

$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}(\overline{\sigma}^{\mu}\sigma^{\nu} - \overline{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}{}_{\dot{\beta}}$$

hence

$$(\chi')^{\dot{\alpha}} = (\exp(-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}))^{\dot{\alpha}}{}_{\dot{\beta}}\chi^{\dot{\beta}}$$

3. Dirac spinor

$$\Sigma^{\mu\nu} = \frac{i}{4}\gamma^{\mu\nu} = \begin{bmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{bmatrix}$$

hence

$$\psi'_D = (\exp(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}))\psi_D$$

where $\sigma^{\mu} = (\mathbb{I}, \vec{\sigma})$, $\overline{\sigma}^{\mu} = (\mathbb{I}, -\vec{\sigma})$ and $\gamma^{\mu\nu} = [\gamma^{\mu}, \gamma^{\nu}]$.

Introducing the chirality operator

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix}$$

its action on a Dirac spinor

$$\gamma^5\psi_D = \begin{bmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \psi_{\alpha} \\ \overline{\chi}^{\dot{\alpha}} \end{bmatrix} = \begin{bmatrix} -\psi_{\alpha} \\ +\overline{\chi}^{\dot{\alpha}} \end{bmatrix}$$

Hence, a left-handed Weyl spinor has eigenvalue equals to $+1$ and a right-handed Weyl spinor has eigenvalue equals to -1.

The projectors operators are

1. left-handed Weyl spinor

$$P_L = \frac{1}{2}(\mathbb{I} - \gamma^5)$$

and its action is

$$P_L\psi_D = \begin{pmatrix} \psi_{\alpha} \\ 0 \end{pmatrix}$$

2. right-handed Weyl spinor

$$P_R = \frac{1}{2}(\mathbb{I} + \gamma^5)$$

and its action is

$$P_R\psi_D = \begin{pmatrix} 0 \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}$$

The Dirac conjugate is

$$\overline{\psi}_D = \psi_D^\dagger \gamma^0 = \begin{pmatrix} \chi^\alpha & \overline{\psi}_{\dot{\alpha}} \end{pmatrix}$$

The charge conjugate is

$$\psi_D^C = C\overline{\psi}_D^T = \begin{pmatrix} \chi_\alpha \\ \overline{\psi}_{\dot{\alpha}} \end{pmatrix}$$

where the charge conjugation matrix exchanges particles with antiparticles

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$

A Majorana spinor is such that $\psi_\alpha = \chi_\alpha$

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix} = \psi_M^C$$

hence

$$\psi_D = \psi_{M_1} + i\psi_{M_2} \quad \psi_D = \psi_{M_1}^C - i\psi_{M_2}$$

5.4 Infinite-dimensional representations of $SO^*(1,3)$

Each field representation element is represented by an operator acting on non-constant objects like fields, recalling that not only the field changes but also the coordinates

$$\psi_a(x) \to \underbrace{(\exp(-\frac{i}{2}\omega_\mu\nu S^{\mu\nu}))_{ab}}_{\text{internal finite-dimensional}} \quad \underbrace{\exp(-\frac{i}{2}\omega_\mu\nu L^{\mu\nu})}_{\text{external infinite-dimensional}} \quad \phi_b(x) = (\exp(-\frac{i}{2}\omega_\mu\nu J^{\mu\nu}))_{ab}\phi_b(x)$$

where $J^\mu = S^{\mu\nu} + L^{\mu\nu}$, in particular

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

and

$$S^{\mu\nu} = \begin{cases} 0 & \text{scalar} \\ (M^{\mu\nu})^\rho{}_\sigma & \text{vector} \\ \sigma^{\mu\nu}, \sigma^{\mu\nu}, \Sigma^{\mu\nu} & \text{spinors} \end{cases}$$

If you quantise the theory, fields become operators on a Foch space and can have infinite-dimensional representations: each element is represented by a unitary operator acting on quantum states of a 1-particle Hilbert space. By Wigner's theorem, unitary operators have hermitian generators. Infact

$$\Lambda \to U(\Lambda)$$

and

$$|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}} a^\dagger_{\vec{p}} |0\rangle$$

and

$$U(\Lambda)|\vec{p}, s\rangle = |\Lambda\vec{p}, s\rangle = \sqrt{2E_{\Lambda\vec{p}}} a^\dagger_{\Lambda\vec{p}} |0\rangle$$

Chapter 6

Poincaré group

In the Poincaré group, we add the spacetime translations to $SO^+(1,3)$ and the most general transformation becomes

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

where a^μ is a 4-vector. This group is also called the inhomogeneous proper orthochronous Lorentz group $ISO(1,3)$ and it is a 10-dimensional non-compact Lie group. Hence, we need to add 4 extra generators P^μ such that the commutation relations become

$$[M^{\mu\nu}, P^\sigma] = i(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}) \tag{6.1}$$

and

$$[P^\mu, P^\nu] = 0 \tag{6.2}$$

This means that it is an abelian subgroup.

In the field representation, $P^\mu = i\partial^\mu$.

6.1 1-particle Hilbert space representation of the Poincaré group

In quantum mechanics, given an operator A and a generator of a transformation T, if they commute $[A,T] = 0$ then the eigenvalue (observable) associated to A is invariant under the tranformation generated by T, i.e.

$$A|\phi_a\rangle = a|\phi_a\rangle \quad \Rightarrow \quad A|\phi'{}_a\rangle = a|\phi'{}_a\rangle$$

where $|\phi'{}_a\rangle = \exp(i\alpha T)|\phi_a\rangle$.

Proof. We Taylor expand to the first order in α

$$|\phi'{}_a\rangle = \exp(i\alpha T)|\phi_a\rangle = |\phi_a\rangle + i\alpha T|\phi_a\rangle + O(\alpha^2)$$

and we find

$$A|\phi'_A\rangle = A|\phi_a\rangle + i\alpha AT|\phi_a\rangle = a|\phi_a\rangle + i\alpha TA|\phi_a\rangle = a(\mathbb{I} + i\alpha T)|\phi_a\rangle = a|\phi'_a\rangle$$

Hence, even though $|\phi_a\rangle \neq |\phi'_a\rangle$, they have the same eigenvalue $a' = a$. q.e.d.

6.2 Casimir operators

If an operator commutes with all the generators of the group, it is called a Casimir operator. Its associated quantity are invariant over a Poincaré transformation and it defines a class of states, called multiplets, which are labelled by different eigenvalues of the Casimir operators.

In the Poincaré group, there are 2 Casimir operators

$$C_1 = P^\mu P_\mu \tag{6.3}$$

and

$$C_2 = W^\mu W_\mu \tag{6.4}$$

where

$$W^\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} \tag{6.5}$$

is the Pauli-Lubanski vector.

Proof. Firstly, we check C_1 by computing its commutators with P_ν

$$[C_1, P_\nu] = [P^\mu P_\mu, P_\nu] = P^\mu[P_\mu, P_\nu] + [P^\mu, P_\nu]P_\mu = P^\mu \underbrace{[P_\mu, P_\nu]}_{0} + \eta^{\mu\sigma} \underbrace{[P_\sigma, P_\nu]}_{0} P_\mu = 0$$

and its commutator with $M_{\mu\nu}$

$$[C_1, M_{\mu\nu}] = [P^\sigma P_\sigma, M_{\mu\nu}] = P^\sigma \underbrace{[P_\sigma, M_{\mu\nu}]}_{0} + [P^\sigma, M_{\mu\nu}]P_\sigma = \eta^{\mu\sigma} \underbrace{[P_\sigma, P_\nu]}_{0} P_\mu$$

q.e.d.

6.3 Massive representations

In this case $P^\mu P_\mu = P^2$ has corresponding eigenvalues $p^2 = E_{\vec{p}}^2 - |\vec{p}|^2 = m^2 \neq 0$ and there is a set of infinitely many momenta $\{p^\mu\}$, generated by starting with a fixed p^μ and applying a Poincaré transformation. The multiplet is labelled by $|m; p^\mu\rangle$ and by the eigenvalues of W^μ.

The little group is the subgroup of the Poincaré group such that p^μ is fixed and built from all the generators which commute with P^μ. Wigner's theorem shows that the structure of it does not depend on the way we choose p^μ or $|m; p^\mu\rangle$. Hence, we choose the case as simple as possible: the rest frame

$$p^\mu = (m, 0, 0, 0)$$

It is invariant under space rotations and its little group is $SO(3) \simeq SU(2)/\mathbb{Z}_2$.
Given that

$$[P^\mu, W_\mu] = 0$$

p^μ is invariant under tranformations generated by W^μ.

The components of the Pauli-Lubanski vector are

$$W_0 = \frac{1}{2}\epsilon_{0\nu\rho\sigma}P^\nu M^{\rho\sigma} = \frac{1}{2}\underbrace{\epsilon_{00\rho\sigma}}_{0} mM^{\rho\sigma} = 0$$

and

$$W_i = \frac{1}{2}\epsilon_{i\nu\rho\sigma}P^\nu M^{\rho\sigma} = \frac{1}{2}\underbrace{\epsilon_{i0jk}}_{-\epsilon_{ijk}} mM^{ij} = -\frac{m}{2}\epsilon_{ijk}M^{jk} = -mJ_i$$

Hence, the second casimir operator is

$$C_2 = W^\mu W_\mu = W^i W_i = -m^2 J^i J_i = -m^2 J^2$$

and it is associated with the spin eigenvalues j and j_3 such that $|j_3| \leq j$ and there are $2j + 1$ values.

The multiplet of a massive representations is

$$|m, j; p^\mu, j_3\rangle$$

where p^μ is a continuous variable and j_3 is a discrete variable.

6.4 Massless representations

In this case $P^\mu P_\mu = P^2$ has corresponding eigenvalues $p^2 = E_{\vec{p}}^2 - |\vec{p}|^2 = m^2 = 0$ and $E_{\vec{p}} = |\vec{p}|$. For the little group, we choose $\vec{p}$ along the z-axis

$$p^\mu = (E, 0, 0, E)$$

It is invariant under space rotations in the (x, y) plane, i.e. $SO(2) \simeq U(1)$, but the little group is bigger.

The components of the Pauli-Lubanski vector are

$$\begin{aligned}
W_0 &= \frac{1}{2}\epsilon_{0\nu\rho\sigma}P^\nu M^{\rho\sigma} \\
&= \frac{1}{2}\underbrace{\epsilon_{00\rho\sigma}}_{0}EM^{\rho\sigma} + \frac{1}{2}\epsilon_{03\rho\sigma}EM^{\rho\sigma} \\
&= \frac{1}{2}\underbrace{\epsilon_{0312}}_{1}EM^{12} + \frac{1}{2}\underbrace{\epsilon_{0321}}_{-1}EM^{21} \\
&= E\underbrace{\frac{1}{2}(M^{12} - M^{21})}_{J_3} \\
&= EJ_3
\end{aligned}$$

and similarly

$$W_1 = -E(J_1 + K_2) \quad W_2 = E(-J_2 + K_1) \quad W_3 = -EJ_3 = -W_0$$

Hence, there are only 3 independent generators which satisfy the algebra

$$[W_1, W_2] = 0 \quad [W_3, W_1] = -iEW_2 \quad [W_3, W_2] = iEW_1$$

which is the algebra of the 2-dimensional Euclidean group $E(2)$, the group consistued by the isometries of the 2-dimensional metric. Its dimension is 3: 2 translations generated by W_1 and W_2 and 1 rotations in a plane by W_3.

The eigenvalues associated to W_3 are discrete while the ones associated to W_1 and W_2 are continuous, but continuous spin representations are not seen in Nature. Hence, we set by experimentally evidence $W_1 = W_2 = 0$.

Since

$$W^\mu = J_3(E, 0, 0E) = J_3P^\mu \tag{6.6}$$

the eigenvalues of the second Casimir operator are

$$W^\mu W_\mu \propto P^\mu P_\mu = 0$$

but we can introduce the helicity, i.e. the projection of the spin along the direction of the motion, which an be left and right, and its eigenvalues are $\lambda = 0, \frac{1}{2}, 1, \ldots$. We want also the negative values to incorporate the parity invariance in our theory.

The multiplet of a massless representations is

$$|0, 0; p^\mu, \pm\lambda\rangle$$

The helicity of the particles are

1. Higgs boson has $\lambda = 0$,

2. quarks and leptons have $\lambda = \pm\frac{1}{2}$,
3. photons, $W^{\pm}$, Z^0, gluons have $\lambda = \pm 1$,
4. graviton has $\lambda = \pm 2$.

Notice that the photons has only two degrees of freedom but $W^{\pm}$ and Z^0, after the Higgs mechanism, acquire the thirs one which correspond to $\lambda = 0$.

Putting $E = 1$, the generators become

$$W_1 = -(J_1 + K_2) \quad W_2 = K_1 - J_2 \quad W_3 = -J_3$$

Physically, W_3 generates rotations in the (p_x, p_y) plane but W_1 and W_2 makes a more complicated transformation. Infact, the generator K_2 is

$$K_2 = M^{20} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the generator J_1 is

$$J_1 = M^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

and the generator W_1 is

$$W_1 = -(J_1 + K_2) = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

An infinitesimal trasformation of W_1 is

$$(p')^\mu = (\exp(-i\lambda W_1))^\mu{}_\nu p^\nu = p^\mu - i\lambda (W_1)^\mu{}_\nu p^\nu$$

Explicitly

$$\begin{cases} E' = E - \lambda p_y \\ p'_x = p_x \\ p'_x = p_x + \lambda(E - p_z) \\ p'_z = p_z + \lambda p_y \end{cases}$$

and for

$$p^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

remains invariant

$$\begin{cases} E' = 1 - \lambda 0 = 1 \\ p'_x = 0 \\ p'_x = 0 + \lambda(1 - 1) = 0 \\ p'_z = 1 + 0 = 1 \end{cases}$$

List of Theorems