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Theoretical Physics

On differential geometry:

differential manifolds, Lie derivatives, differential
forms and Riemannian manifolds

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Study notes taken during the master degree

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Part I

Manifolds

Chapter 1

Manifolds and tensors

1.1 Differentiable Manifolds

A differential manifold \mathcal{M} is a topological space which looks locally like \mathbb{R}^N .

In a topological space, the notions of contiguity and continuity are well defined. A topological space $(\mathcal{M}, \{A_i\})$ is a set of points \mathcal{M} in which is defined a family of open sets $\{A_i\}$ such that $\{\emptyset, \mathcal{M}, \cup_i A_i, \cap_{i<\infty} A_i\} \in \{A_i\}$. In particular, an Hausssdorf space has the property that $\forall P, Q \in \mathcal{M} \quad \exists U \in P, V \in Q \quad : \quad U \cap V = \emptyset$. Two points are contiguous if they belong to the same open subset, called neighbourhood. A map is an application $\phi: D \subset \mathcal{M} \rightarrow \mathbb{R}^n$. In a topological space, a map is continuous if maps open sets into open sets.

A chart is a pair A, ϕ , where $A \subset \mathcal{M}$ and $\phi: A \rightarrow \mathbb{R}^n$ invertible continuous, which associates a set of n real coordinates $x^i = \phi$ for the open set A . An atlas is a collection of charts that covers entirely the manifold $\mathcal{A} = \{(A_i, \phi_i)\} : \cup_i A_i \supseteq \mathcal{M}$. A consistency map between two charts ϕ_1 and ϕ_2 , over a point $P \in A_1 \cap A_2$, is $\phi: \phi(A_2) \subseteq \mathbb{R}^n \rightarrow \psi(A_2) \subseteq \mathbb{R}^n$ invertible such that $\psi(\phi_1(P)) = \phi_2(P)$ or $(\phi_2^{-1} \circ \psi \circ \phi_1) = \mathbb{I}$ or, equivalently, $\psi^{-1}(\phi_2(P)) = \phi_1(P)$ or $(\phi_1^{-1} \circ \psi \circ \phi_2) = \mathbb{I}$. ψ is a change of coordinates in \mathbb{R}^n . It follows that the dimension n must be the same for all charts, hence it is the dimension of the manifold. If $\psi \in C^p(\mathbb{R}^n)$, the manifold is a p -differentiable manifold.

A manifold is an equivalence class of atlases, where two atlases are equivalent if there exists a bijective correspondence between them.

1.2 Curves

A curve is a continuous map $\gamma: I \subseteq \mathbb{R} \rightarrow \mathcal{M}$. Introducing a chart $\phi \circ \gamma: I \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$, or $x^i = x^i(\lambda)$, where λ is a real parameter. If $x^i(\lambda) \in C^p(\mathbb{R})$, then *gamma* is p -differentiable. A reparameterization $\gamma' = \gamma'(\gamma)$ defines a different curve, although the images of the curves coincide.

1.3 Scalars

A function is a map $f: \mathcal{M} \rightarrow \mathbb{R}$. Introducing a chart $f \circ \phi^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}$, or $f = f(x^i)$. If ϕ' is another chart, then $f'(x'(P)) = f(x(P))$, showing that it is indeed a scalar.

1.4 Vectors

A vector at a point $P \in \mathcal{M}$ is a map that associates to the derivative to a function defined in a neighbourhood of P $v_\gamma: f \rightarrow v_\gamma(f) = \left. \frac{df}{d\lambda} \right|_{\lambda_P} \in \mathbb{R}$, where $\gamma(\lambda_P) = P$. Introducing a chart

$$\begin{aligned} v_{\gamma, P}(f) &= \left. \frac{d(f \circ \gamma)}{d\lambda} \right|_{\lambda_P} \\ &= \left. \frac{d}{d\lambda} (f \circ \mathbb{I} \circ \gamma) \right|_{\lambda_P} \\ &= \left. \frac{d}{d\lambda} (f \circ \phi^{-1} \circ \phi \circ \gamma) \right|_{\lambda_P} \\ &= \left. \frac{d}{d\lambda} (f(x^i) \circ x^i(\lambda)) \right|_{\lambda_P} \\ &= \left. \frac{d}{d\lambda} f(x^i(\lambda)) \right|_{\lambda_P} = \frac{\partial f}{\partial x^i} \frac{dx^i}{d\lambda} \end{aligned}$$

and since it is true $\forall f$

$$v_\gamma = dv\lambda = \frac{dx^i}{d\lambda} \frac{\partial}{\partial x^i} \quad (1.1)$$

which means that a vector is the tangent to a curve γ at a point P .

By definition a vector is a linear functional

$$v_\gamma(af + bg) = \frac{d}{d\lambda}(af + bg) = a \frac{df}{d\lambda} + b \frac{dg}{d\lambda}$$

From (1.1)

$$v = \underbrace{\frac{dx^i}{d\lambda}}_{v_i} \underbrace{\frac{\partial}{\partial x^i}}_{e_i} = v^i e_i$$

where v^i are the components and e_i are the coordinate basis vectors, whose vectors tangent to the coordinate line defined by constant x^j for $i \neq j$.

Under a change of coordinates

$$y^i = y^i(x^j)$$

components transform by

$$v^i = \frac{dx^i}{d\lambda} = \frac{\partial x^i}{\partial y^{j'}} \frac{dy^{j'}}{d\lambda} = \frac{\partial x^i}{\partial y^{j'}} v^{j'}$$

and basis vectors transform by

$$e_i = \frac{\partial}{\partial x^i} = \frac{\partial y^{j'}}{\partial x^i} \frac{\partial}{\partial y^{j'}} = \frac{\partial y^{j'}}{\partial x^i} e_{j'}$$

Remember that components transform but basis change too, inversely, then the vector remains the same.

A vector field in an open set $U \subseteq \mathcal{M}$ is map from each point $P \in U$ into a vector $v(P)$. Introducing a chart, $v(x^i) = v \circ \phi^{-1}$.

The coordinate vectors $e_i = \frac{\partial}{\partial x^i}$ form a basis of a linear space composed by all the vectors tangent to a point P , called the tangent space T_P .

Proof. First, every tangent vector can be expressed as linear combination.

Consider two curves, parametrized by λ and σ , across a point P which generate two vectors $v = \frac{\partial}{\partial \lambda}$ and $w = \frac{d}{d\sigma}$. Hence, a generic linear combination of them

$$av + bw = a \frac{d}{d\lambda} + b \frac{d}{d\sigma} = a \frac{\partial x^i}{\partial \lambda} \frac{\partial}{\partial x^i} + b \frac{dx^i}{d\sigma} \frac{\partial}{\partial x^i} = \left(a \frac{\partial x^i}{\partial \lambda} + b \frac{dx^i}{d\sigma} \right) \frac{\partial}{\partial x^i} = \left(a \frac{\partial x^i}{\partial \lambda} + b \frac{dx^i}{d\sigma} \right) e_i$$

Since there are n coordinates x^i , we have n independent curves.

Second, the coordinate basis vectors are linearly independent.

The determinant of the jacobian matrix of $y^i = y^i(x^j)$ must not vanish

$$\det J = \det \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

hence, there are n columns (or rows) which are linearly independent and also n basis vector

$$e_i = \frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

q.e.d.

It is important to remark that coordinate basis vectors at different points belong to different tangent space and cannot be linearly combined.

1.5 Fiber bundles

A tangent bundle is the set of all tangent space at each point together with the manifold itself $\mathcal{TM} = \{\mathcal{M}, \{T_P : \forall P \in \mathcal{M}\}\}$. It can be shown that \mathcal{TM} is a manifold too.

1.6 Exponential map

An integral curve $\gamma = \gamma(\lambda)$ of a vector field V is the curve which as tangent vector $\frac{d}{d\lambda}$ has the element of V in $P \in \gamma$, i.e.

$$V = \frac{d}{d\lambda}$$

Introducing a point P_0 and a chart x^i

$$\begin{aligned} V^i(\lambda) &= \frac{dx^i(\lambda)}{d\lambda} \\ x^i(P_0) &= x^i(\lambda_0) \end{aligned} \tag{1.2}$$

which are a system of n Cauchy problems and the components of V at an arbitrary point $P = \phi^{-1}(x^i(\lambda))$ are $V(P) = V^i(x^j(\lambda)) \frac{\partial}{\partial x^i} = V^i(\lambda) \frac{\partial}{\partial x^i}$.

Theorems of calculus in \mathbb{R}^n ensure that locally the solution of (1.2) always exists, which is indeed the integral curve $\gamma(\lambda)$.

Formally, the solution of (1.2) is the exponential map

$$x^i(\lambda) = \exp((\lambda - \lambda_0)V)x^i \Big|_{\lambda_0}$$

which describes the flow of V in a neighbourhood of P .

Proof. Let $V = \frac{d}{d\lambda}$ be a vector field with integral curve $\gamma = \gamma(\lambda)$. Introducing a chart x^i and Taylor expanding around P_0 along γ

$$\begin{aligned} x^i(\lambda_0 + \epsilon) &= x^i(\lambda_0) + \epsilon \frac{dx^i}{d\lambda} \Big|_{\lambda_0} + \frac{\epsilon^2}{2} \frac{d^2 x^i}{d\lambda^2} \Big|_{\lambda_0} + \dots \\ &= \left(1 + \epsilon \frac{d}{d\lambda} \Big|_{\lambda_0} + \frac{\epsilon^2}{2} \frac{d^2}{d\lambda^2} \Big|_{\lambda_0} + \dots \right) x^i(\lambda_0) \\ &= \exp\left(\epsilon \frac{d}{d\lambda}\right) x^i \Big|_{\lambda_0} \\ &= \exp(\epsilon V) x^i \Big|_{\lambda_0} \end{aligned}$$

q.e.d.

For an arbitrary function f in a neighbourhood of P

$$f(\lambda_0 + \epsilon) = \exp\left(\epsilon \frac{d}{d\lambda}\right) f \Big|_{\lambda_0} = \exp(\epsilon V) f \Big|_{\lambda_0}$$

1.7 Lie brackets

Introducing a chart x^i , the Lie brackets of two vector fields $V = \frac{d}{d\lambda} = v^i \frac{\partial}{\partial x^i}$ and $W = \frac{d}{d\mu} = w^i \frac{\partial}{\partial x^i}$ are

$$\begin{aligned} [V, W] &= \frac{d}{d\lambda} \frac{d}{d\mu} - \frac{d}{d\mu} \frac{d}{d\lambda} \\ &= v^i \frac{\partial}{\partial x^i} \left(w^j \frac{\partial}{\partial x^j} \right) - w^i \frac{\partial}{\partial x^i} \left(v^j \frac{\partial}{\partial x^j} \right) \\ &= \cancel{v^i w^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}} + v^i \frac{\partial w^j}{\partial x^i} \frac{\partial}{\partial x^j} - \cancel{v^j w^i \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}} - w^i \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial x^j} \\ &= \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \end{aligned}$$

where it is used the fact that the partial derivatives commute. Hence the commutator of two vectors is still a vector.

The geometrical meaning of the commutator is the following: consider two vector fields $V = \frac{d}{d\lambda}$ and $W = \frac{d}{d\mu}$. Using the exponential map, the coordinates of A, moving before along V and then along W , are

$$x^i(A) = \exp \left(\epsilon_2 \frac{d}{d\mu} \right) \exp \left(\epsilon_1 \frac{d}{d\lambda} \right) x^i \Big|_P$$

whereas the coordinates of B, moving before along W and then along Y , are

$$x^i(B) = \exp \left(\epsilon_1 \frac{d}{d\lambda} \right) \exp \left(\epsilon_2 \frac{d}{d\mu} \right) x^i \Big|_P$$

Computing the difference

$$x^i(B) - x^i(A) = \epsilon_1 \epsilon_2 \left[\frac{d}{d\lambda}, \frac{d}{d\mu} \right] x^i \Big|_P + O(\epsilon^3)$$

Hence, if the commutator does not vanish, the final points are different $A \neq B$ and the path $PA \cup PB$ does not close.

A sufficient and necessary condition for a set of fields to be coordinate vectors is that their commutator vanishes.

Proof. First, the sufficient condition. Consider two coordinate vector fields $V = \frac{\partial}{\partial x^1}$ and $W = \frac{\partial}{\partial x^2}$. Then $v^i = \delta^i_1$, $w^j = \delta^j_2$ and the commutator vanishes, since the derivative of a constant is so.

Second, the necessary condition. Consider two commuting vector fields $V = \frac{\partial}{\partial x^1}$ and $W = \frac{\partial}{\partial x^2}$. Suppose also that they are linearly independent

$$aV(P) + bW(P) = 0 \quad \Longleftrightarrow \quad a = b = 0. \quad (1.3)$$

Introducing a chart x^i , moving from P along V by $\Delta\lambda = \alpha$ to a point R

$$x(R) = \exp\left(\alpha \frac{d}{d\lambda}\right) x^i \Big|_P$$

and then along W by $\Delta\mu = \beta$ to a point Q

$$x(Q) = \exp\left(\beta \frac{d}{d\mu}\right) \exp\left(\alpha \frac{d}{d\lambda}\right) x^i \Big|_P \quad (1.4)$$

If α and β are coordinates, the corresponding basis vectors are $\frac{\partial}{\partial\alpha} = \frac{\partial x^i}{\partial\alpha}$ and $\frac{\partial}{\partial\beta} = \frac{\partial x^i}{\partial\beta}$. Hence, using (1.4)

$$\begin{aligned} \frac{\partial x^i}{\partial\alpha} &= \frac{\partial}{\partial\alpha} \left(\exp\left(\beta \frac{d}{d\mu}\right) \exp\left(\alpha \frac{d}{d\lambda}\right) x^i \Big|_P \right) \\ &= \exp\left(\beta \frac{d}{d\mu}\right) \frac{\partial}{\partial\alpha} \left(\exp\left(\alpha \frac{d}{d\lambda}\right) x^i \Big|_P \right) \\ &= \exp\left(\beta \frac{d}{d\mu}\right) \exp\left(\alpha \frac{d}{d\lambda}\right) \frac{dx^i}{d\lambda} \Big|_P \end{aligned}$$

and, similarly,

$$\frac{\partial x^i}{\partial\beta} = \exp\left(\beta \frac{d}{d\mu}\right) \exp\left(\alpha \frac{d}{d\lambda}\right) \frac{dx^i}{d\mu} \Big|_P$$

This shows that $\frac{\partial}{\partial\alpha}$ and $\frac{\partial}{\partial\beta}$ are respectively the vector fields $\frac{d}{d\lambda}$ and $\frac{d}{d\mu}$ evaluated in Q , using the fact that the commutator is vanishing. Then the determinant of the Jacobians is not zero

$$\det J = \det \begin{bmatrix} \frac{\partial x^1}{\partial\alpha} & \frac{\partial x^2}{\partial\alpha} \\ \frac{\partial x^1}{\partial\beta} & \frac{\partial x^2}{\partial\beta} \end{bmatrix} \neq 0$$

since we assumed the linearly independence of $\frac{d}{d\lambda}$ and $\frac{d}{d\mu}$.

q.e.d.

1.8 1-forms

1.9 Tensors

List of Theorems