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Theoretical Physics

On differential geometry:

manifolds and all that

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Study notes taken during the master degree

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Manifolds and tensors

1.1 Differentiable Manifolds

A differential manifold $\mathcal{M}$ is a topological space which looks locally like $\mathbb{R}^N$.
In a topological space, the notions of contiguity and continuity are well defined. A topological space $(\mathcal{M},\ \{A_i\})$ is a set of points $\mathcal{M}$ in which is defined a family of open sets $\{A_i\}$ such that $\{\emptyset,\ \mathcal{M},\ \cup_i A_i,\ \cap_{i<\infty} A_i\} \in \{A_i\}$. In particular, an Haussdorf space has the property that $\forall P,\ Q \in \mathcal{M} \quad \exists U \in P,\ V \in Q \quad : \quad U \cap V = \emptyset$. Two points are contiguous if they belong to the same open subset, called neighbourhood.
A map is an application $\phi\colon D \subset \mathcal{M} \to \mathbb{R}^n$. In a topological space, a map is continuous if maps open sets into open sets.
A chart is a pair $A,\ \phi$, where $A \subset \mathcal{M}$ and $\phi\colon A \to \mathbb{R}^n$ invertible continuous, which associates a set of n real coordinates $x^i = \phi$ for the open set A. An atlas is a colection of charts that covers entirely the manifold $\mathcal{A} = \{\{(A_i\ \phi_i)\}\colon \cup_i A_i \supseteq \mathcal{M}\}$.
A consistency map between two charts ϕ_1 and ϕ_2, over a point $P \in A_1 \cap A_2$, is $\phi\colon \phi(A_2) \subseteq \mathbb{R}^n \to \psi(A_2) \subseteq \mathbb{R}^n$ invertible such that $\psi(\phi_1(P)) = \phi_2(P)$ or $(\phi_2^{-1} \circ \psi \circ \phi_1) = \mathbb{I}$ or, equivalently, $\psi^{-1}(\phi_2(P)) = \phi_1(P)$ or $(\phi_1^{-1} \circ \psi \circ \phi_2) = \mathbb{I}$. ψ is a change of coordinates in $\mathbb{R}^n$. It follows that the dimension n must be the same for all charts, hence it is the dimension of the manifold. If $\psi \in C^p(\mathbb{R}^n)$, the manifold is a p-differentiable manifold.
A manifold is an equivalence class of atlases, where two atlases are equivalent if there exists a bijective correspondence between them.

1.2 Curves

A curve is a continuous map $\gamma\colon I \subseteq \mathbb{R} \to \mathcal{M}$. Introducing a chart $\phi\circ\gamma\colon I \subseteq \mathbb{R} \to \mathbb{R}^n$, or $x^i = x^i(\lambda)$, where λ is a real parameter. If $x^i(\lambda) \in C^p(\mathbb{R})$, then *gamma* is p-differentiable. A reparameterization $\gamma' = \gamma'(\gamma)$ defines a different curve, although the images of the curves coincide.

1.3 Scalars

A function is a map $f\colon \mathcal{M} \to \mathbb{R}$. Introducting a chart $f \circ \phi^{-1}\colon \mathbb{R}^n \to \mathbb{R}$, or $f = f(x^i)$. If ϕ' is another chart, then $f'(x'(P)) = f(x(P))$, showing that it is indeed a scalar.

1.4 Vectors

A vector at a point $P \in \mathcal{M}$ is a map that associates to the derivative to a function defined in a neighbourhood of P $v_\gamma\colon f \to v_\gamma(f) = \frac{df}{d\lambda}\Big|_{\lambda_P} \in \mathbb{R}$, where $\gamma(\lambda_P) = P$. Introducing a chart

$$\begin{aligned} v_{\gamma,\ P}(f) &= \frac{d(f\circ\gamma)}{d\lambda}\Big|_{\lambda_P} \\ &= \frac{d}{d\lambda}(f\circ\mathbb{I}\circ\gamma)\Big|_{lambda_P} \\ &= \frac{d}{d\lambda}(f\circ\phi^{-1}\circ\phi\circ\gamma)\Big|_{lambda_P} \\ &= \frac{d}{d\lambda}(f(x^i)\circ x^i(\lambda))\Big|_{\lambda_P} \\ &= \frac{d}{d\lambda}f(x^i(\lambda))\Big|_{\lambda_P} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda} \end{aligned}$$

and since it is true $\forall f$

$$v_\gamma = dv\lambda = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} \tag{1.1}$$

which means that a vector is the tangent to a curve γ at a point P.
By definition a vector is a linear functional

$$v_\gamma(af+bg) = \frac{d}{d\lambda}(af+bg) = a\frac{df}{d\lambda} + b\frac{dg}{d\lambda}$$

From (1.1)

$$v = \underbrace{\frac{dx^i}{d\lambda}}_{v_i}\underbrace{\frac{\partial}{\partial x^i}}_{e_i} = v^i e_i$$

where v^i are the components and e_i are the coordinate basis vectors, whose vectors tangent to the coordinate line defined by constant x^j for $i \neq j$.
Under a change of coordinates

$$y^i = y^i(x^j)$$

components transform by

$$v^i = \frac{dx^i}{d\lambda} = \frac{\partial x^i}{\partial y^{j'}}\frac{dy^{j'}}{d\lambda} = \frac{\partial x^i}{\partial y^{j'}}v^{j'}$$

and basis vectors transform by

$$e_i = \frac{\partial}{\partial x^i} = \frac{\partial y^{j'}}{\partial x^i}\frac{\partial}{\partial y^{j'}} = \frac{\partial y^{j'}}{\partial x^i} e_{j'}$$

Remember that components transform but basis change too, inversely, then the vector remains the same.
A vector field in an open set $U \subseteq \mathcal{M}$ is map from each point $P \in U$ into a vector $v(P)$. Introducing a chart, $v(x^i) = v \circ \phi^{-1}$.
The coordinate vectors $e_i = \frac{\partial}{\partial x^i}$ form a basis of a linear space composed by all the vectors tangent to a point P, called the tangent space T_P.

Proof. First, every tangent vector can be expressed as linear combination.
Consider two curves, parametrized by λ and σ, across a point P which generate two vectors $v = \frac{\partial}{\partial \lambda}$ and $w = \frac{d}{d\sigma}$. Hence, a generic linear combination of them

$$av+bw = a\frac{d}{d\lambda}+b\frac{d}{d\sigma} = a\frac{\partial x^i}{\partial \lambda}\frac{\partial}{\partial x^i}+b\frac{dx^i}{d\sigma}\frac{\partial}{\partial x^i} = \left(a\frac{\partial x^i}{\partial \lambda}+b\frac{dx^i}{d\sigma}\right)\frac{\partial}{\partial x^i} = \left(a\frac{\partial x^i}{\partial \lambda}+b\frac{dx^i}{d\sigma}\right)e_i$$

Since there are n coordinates x^i, we have n indipendent curves.
Second, the coordinate basis vectors are linearly independent.
The determinant of the jacobian matrix of $y^i = y^i(x^j)$ must not vanish

$$\det J = \det \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

hence, there are n columns (or rows) which are linearly independent and also n basis vector

$$e_i = \frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

q.e.d.

It is important to remark that coordinate basis vectors at different points belong to different tangent space and cannot be linearly combined.

1.5 Fiber bundles

A tangent bundle is the set of all tangent space at each point together with the manifold itself $\mathcal{TM} = \{\mathcal{M}, \ \{T_P \colon \forall P \in \mathcal{M}\}\}$. It can be shown that $\mathcal{TM}$ is a manifold too.

1.6 Exponential map

An integral curve $\gamma = \gamma(\lambda)$ of a vector field V is the curve which as tangent vector $\frac{d}{d\lambda}$ has the element of V in $P \in \gamma$, i.e.

$$V = \frac{d}{d\lambda}$$

Introducing a point P_0 and a chart x^i

$$\begin{aligned} V^i(\lambda) &= \frac{dx^i(\lambda)}{d\lambda} \\ x^i(P_0) &= x^i(\lambda_0) \end{aligned} \tag{1.2}$$

which are a system of n Cauchy problems and the components of V at an arbitrary point $P = \phi^{-1}(x^i(\lambda))$ are $V(P) = V^i(x^j(\lambda))\frac{\partial}{\partial x^i} = V^i(\lambda)\frac{\partial}{\partial x^i}$.
Theorems of calculus in $\mathbb{R}^n$ ensure that locally tha solution of (1.2) always exists, which is indeed the integral curve $\gamma(\lambda)$.
Formally, the solution of (1.2) is the exponential map

$$x^i(\lambda) = \exp((\lambda - \lambda_0)V)x^i\Big|_{\lambda_0}$$

which describes the flow of V in a neighbourhood of P.

Proof. Let $V = \frac{d}{d\lambda}$ be a vector fields with integral curve $\gamma = \gamma(\lambda)$. Introducing a chart x^i and Taylor expanding around P_0 along γ

$$\begin{aligned} x^i(\lambda_0 + \epsilon) &= x^i(\lambda_0) + \epsilon\frac{dx^i}{d\lambda}\Big|_{\lambda_0} + \frac{\epsilon^2}{2}\frac{d^2x^i}{d\lambda^2}\Big|_{\lambda_0} + \dots \\ &= \left(1 + \epsilon\frac{d}{d\lambda}\Big|_{\lambda_0} + \frac{\epsilon^2}{2}\frac{d^2}{d\lambda^2}\Big|_{\lambda_0} + \dots\right) x^i(\lambda_0) \\ &= \exp(\epsilon\frac{d}{d\lambda})\ x^i\Big|_{\lambda_0} \\ &= \exp(\epsilon V)\ x^i\Big|_{\lambda_0} \end{aligned}$$

q.e.d.

For an arbitrary function f in a neighbourhood of P

$$f(\lambda_0 + \epsilon) = \exp\left(\epsilon\frac{d}{d\lambda}\right) f\Big|_{\lambda_0} = \exp(\epsilon V)\ f\Big|_{\lambda_0}$$

1.7 Lie brackets

Introducing a chart x^i, the Lie brackets of two vector fields $V = \frac{d}{d\lambda} = v^i \frac{\partial}{\partial x^i}$ and $W = \frac{d}{d\mu} = w^i \frac{\partial}{\partial x^i}$ are

$$\begin{aligned}
[V,\ W] &= \frac{d}{d\lambda}\frac{d}{d\mu} - \frac{d}{d\mu}\frac{d}{d\lambda} \\
&= v^i \frac{\partial}{\partial x^i}\Big(w^j \frac{\partial}{\partial x^j}\Big) - w^i \frac{\partial}{\partial x^i}\Big(v^j \frac{\partial}{\partial x^j}\Big) \\
&= \cancel{v^i w^j \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}} + v^i \frac{\partial w^j}{\partial x^i}\frac{\partial}{\partial x^j} - \cancel{v^j w^i \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}} - w^i \frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial x^j} \\
&= \Big(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}\Big)\frac{\partial}{\partial x^j}
\end{aligned}$$

where it is used the facf that the partial derivatives commute. Hence the commutator of two vectors is still a vector.
The geometrical meaning of the commutator is the following: consider two vector fields $V = \frac{d}{d\lambda}$ and $W = \frac{d}{d\mu}$. Using the exponential map, the coordinates of A, moving before along V and then along W, are

$$x^i(A) = \exp\Big(\epsilon_2 \frac{d}{d\mu}\Big) \exp\Big(\epsilon_1 \frac{d}{d\lambda}\Big)\ x^i\Big|_P$$

whereas the coordinates of B, moving before along W and then along Y, are

$$x^i(B) = \exp\Big(\epsilon_1 \frac{d}{d\lambda}\Big) \exp\Big(\epsilon_2 \frac{d}{d\mu}\Big)\ x^i\Big|_P$$

Computing the difference

$$x^i(B) - x^i(A) = \epsilon_1 \epsilon_2 \Big[\frac{d}{d\lambda},\ \frac{d}{d\mu}\Big]\ x^i\Big|_P + O(\epsilon^3)$$

Hence, if the commutator does not vanish, the final points are different $A \neq B$ and the path $PA \cup PB$ does not close.
A sufficient and necessary condition for a set of fields to be coordinate vectors is that their commutator vanishes.

Proof. First, the sufficient condition. Consider two coordinate vector fields $V = \frac{\partial}{\partial x^1}$ and $W = \frac{\partial}{\partial x^2}$. Then $v^i = \delta^i{}_1$, $w^j = \delta^j{}_2$ and the commutator vanishes, since the derivative of a constant is so.
Second, the necessary condition. Consider two commuting vector fields $V = \frac{\partial}{\partial x^1}$ and $W = \frac{\partial}{\partial x^2}$. Suppose also that they are linearly independent

$$aV(P) + bW(P) = 0 \qquad \Longleftrightarrow \qquad a = b = 0. \tag{1.3}$$

Introducing a chart x^i, moving from P along V by $\Delta\lambda = \alpha$ to a point R

$$x^(R) = \exp\left(\alpha \frac{d}{d\lambda}\right) x^i\Big|_P$$

and then along W by $\Delta\mu = \beta$ to a point Q

$$x^(Q) = \exp\left(\beta \frac{d}{d\mu}\right) \exp\left(\alpha \frac{d}{d\lambda}\right) x^i\Big|_P \tag{1.4}$$

If α and β are coordinates, the corresponding basis vectors are $\frac{\partial}{\partial\alpha} = \frac{\partial x^i}{\partial\alpha}$ and $\frac{\partial}{\partial\beta} = \frac{\partial x^i}{\partial\beta}$. Hence, using (1.4)

$$\begin{aligned}
\frac{\partial x^i}{\partial\alpha} &= \frac{\partial}{\partial\alpha}\left(\exp\left(\beta\frac{d}{d\mu}\right)\exp\left(\alpha\frac{d}{d\lambda}\right) x^i\Big|_P\right) \\
&= \exp\left(\beta\frac{d}{d\mu}\right)\frac{\partial}{\partial\alpha}\left(\exp\left(\alpha\frac{d}{d\lambda}\right) x^i\Big|_P\right) \\
&= \exp\left(\beta\frac{d}{d\mu}\right)\exp\left(\alpha\frac{d}{d\lambda}\right) \frac{dx^i}{d\lambda}\Big|_P
\end{aligned}$$

and, similarly,

$$\frac{\partial x^i}{\partial\beta} = \exp\left(\beta\frac{d}{d\mu}\right)\exp\left(\alpha\frac{d}{d\lambda}\right) \frac{dx^i}{d\mu}\Big|_P$$

This shows that $\frac{\partial}{\partial\alpha}$ and $\frac{\partial}{\partial\beta}$ are respectively the vector fields $\frac{d}{d\lambda}$ and $\frac{\partial}{\partial\mu}$ evaluated in Q, using the fact that the commutator is vanishing. Then the determinant of the Jacobians is not zero

$$\det J = \det\begin{bmatrix} \frac{\partial x^1}{\partial\alpha} & \frac{\partial x^2}{\partial\alpha} \\ \frac{\partial x^1}{\partial\beta} & \frac{\partial x^2}{\partial\beta} \end{bmatrix} \neq 0$$

since we assumed the linearly independence of $\frac{d}{d\lambda}$ and $\frac{d}{d\mu}$. q.e.d.

1.8 1-forms

A 1-form is a linear functional w acting on a vector $w\colon T_P \to \mathbb{R}$ such that $w(\alpha v + \beta u) = \alpha w(v) + \beta w(u)$ and $(\alpha w + \alpha z)(v) = \alpha w(v) + \beta z(v)$. Linearity implies that the action of a 1-form is completely determined by the action on a basis of T_P. 1-forms acting on the same T_P form a linear space T_P^*, called the cotangent space. A cotangent bundle is the set of all cotangent space at each point together with the manifold itself $\mathcal{T}^*\mathcal{M} = \{\mathcal{M},\ \{T_P^*\colon \forall P \in \mathcal{M}\}\}$. A 1-form field is a map associates a 1-form of T^*P to each point $P \in \mathcal{M}$.

One way of thinking 1-forms is in the following way: given an arbitrary function, a vector field is defined by $V(f) = \frac{df}{d\lambda}$ whereas given an arbitrary vector field, a 1-form is defined by $V(f) = \frac{df}{d\lambda} = df(\frac{d}{d\lambda})$. The difference is the in the former V is fixed and f is arbitrary, whereas in the latter f is fixed and V is arbitrary. Introducing a chart x^i

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda} = \nabla_i f \frac{dx^i}{d\lambda} = df_i \frac{dx^i}{d\lambda}$$

where df_i are the components of the 1-form df, called the gradient of f.
Geometrically, the gradient of a function can be seen as the number of contour lines that a vector V crosses in a neighbourhood of P. Generalizing for 1-forms, they can be seen as a set of level surfaces whose action on a vector is the number of surface the vector crosses.
Let $\{e_i\}$ be a basis of T_P. A basis of T^*_P is not related to it, however it is convenient to choose the dual basis, which completely defined a basis of T^*P by a basis in T_P in the following way

$$e^i(e_j) = \delta^i{}_j \tag{1.5}$$

or, equivalently, applying it to a vector v

$$e^i(v) = e^i(v^j e_j) = v^j e^i(e_j) = v^j \delta^i{}_j = v^i$$

Consequently, $\mathcal{M}$, T_P and T^*_P have the same dimension n. $\{e^i\}$ are actually a basis of T^*_P, since given an arbitrary 1-form q

$$q(v) = q(v^i e_i) = v^i q(e_i) = v^i q_i$$

and

$$v^i q_i = q_i e^i(v)$$

Remember that under a change of coordinates, vectors or 1-forms do not change, only their components and basis change, the latter inversely.
Differentials are the basis 1-form dual to the coordinate basis vectors

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda} = \frac{\partial f}{\partial x^i} dx^i\left(\frac{dx^j}{d\lambda}\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i}\frac{dx^j}{d\lambda} dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i}\frac{\partial x^j}{\partial \lambda}\delta^i{}_j = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \lambda}$$

where it has been used the dual basis.

1.9 Tensors

A tensor (n, m) at P is a linear functional that maps n 1-forms and m vectors into a real number

$$T\colon \underbrace{T^*_P \otimes \cdots \otimes T^*_P}_{n\ times} \otimes \underbrace{T_P \otimes \cdots \otimes T_P}_{m\ times} \to \mathbb{R}$$

A tensor can be also seen as the outer product of 1-forms and vectors. A tensor (1, 0) is a vector and a tensor (0, 1) is a 1-form. A tensor (n, m) can be written in terms of the dual basis

$$T = T^{i_1 \cdots i_n}_{j_1 \cdots j_m} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e^{j_1} \otimes \cdots \otimes e^{j_m}$$

where the components are

$$T^{i_1\cdots i_n}_{j_1\cdots j_m} = T(e^{i_1},\ \cdots,\ e^{i_n},\ e_{j_1},\ \cdots,\ e^{j_m})$$

A change of basis is determined by a 4×4 non-degenerate matrix $\Lambda \in GL(n)$. On a vector basis, it acts as

$$e'_j = \Lambda^i{}_j e_i \tag{1.6}$$

This transformation has no effects on the dual space, however, in order to keep the duality of the basis, it must induce a transformation with the inverse matrix

$$e'^j = \Lambda^j{}_i e^i$$

Proof. Recalling (1.5), to preserve the duality, also the transformed dual basis must obey

$$e'^i(e'_j) = \delta^i_j \tag{1.7}$$

Hence, given an arbitrary transformation matrix,

$$e'^i = M^i{}_k e^k$$

and putting into (1.7), using(1.6)

$$\delta^i{}_j = e'^i(e'_j) = M^i{}_k e^k(\Lambda^l{}_j e_l) = M^i{}_k \Lambda^l{}_j e^k(e_l) = M^i{}_k \Lambda^l{}_j \delta^k{}_l = M^i{}_k \Lambda^k{}_j$$

then, M must satisfy

$$M^i{}_k \Lambda^k{}_j = \delta^i$$

and it is indeed the inverse matrix. q.e.d.

It is possible to perform several operations on tensors at P:

1. scalar multiplication, i.e.
$$S^{(n,m)} = aT^{(n,m)} \quad \forall a \in \mathbb{R}$$
2. addition, i.e.
$$S^{(n,m)} = T^{(n,m)} + Q^{(n,m)}$$
3. outer product, i.e.
$$S^{(n+p,m+q)} = T^{(n,m)} \otimes Q^{(p,q)}$$
4. saturation with 1-forms, i.e.
$$T^{(n-1,m)} = T^{(n,m)}(\dots,\ w,\ \dots)$$

5. saturation with vector, i.e.

$$T^{(n,m-1)} = T^{(n,m)}(\dots,\ v,\ \dots)$$

The last two can be generalised to an arbitrary saturation of a (n,m) tensor with a $(p<n, q<m)$ tensor.
For a change of basis in the tangent space to correspond a change of coordinates on the manifold, the transformation matrix must obey the condition

$$\frac{\partial \Lambda^j{}_i}{\partial x^k} = \frac{\partial \Lambda^j{}_k}{\partial x^i} \tag{1.8}$$

Proof. Consider two charts x^i and y^i that overlap at P. The transformation matrix between basis is

$$\Lambda^i{}_j = \frac{\partial x^i}{\partial y^j}$$

and the inverse is

$$\Lambda^j{}_i = \frac{\partial y^j}{\partial x^i}$$

If we move continuously to another point Q insider the charts, the matrix transformation will become a field $\Lambda(Q) = \Lambda(x^i(Q)) = \Lambda(y^i(Q))$ and, since the partial derivatives commute

$$\frac{\partial \Lambda^j{}_i}{\partial x^k} = \frac{\partial}{\partial x^k} pdvy^j x^i = \frac{\partial}{\partial x^i} pdvy^j x^k = \frac{\partial \Lambda^j{}_k}{\partial x^i}$$

q.e.d.

1.10 Metric tensor

The notions of lenght and angles on a manifold can be introduced with the metric tensor.
A metric tensor g is a $(2,0)$ tensor which maps two vectors into a real number, satisfying the following properties

1. symmetry, i.e.

$$g(v,w) = g(w,v) = g(v^i e_i, w^j e_j) = g(e_i, e_j)v^i w^j = g_{ij} v^i v^j \quad \forall v,\ w \in T_P$$

2. non-degeneracy, i.i

$$g(v,w) = 0 \quad \forall w \in T_P \qquad \iff \qquad v = 0$$

or, equivalenlty, if $\det g_{ij} \neq 0$

A metric tensor defines a scalar product

$$g(v, w) = v \cdot w$$

and introduces the notions of norm of a vector

$$v^2 = g(v, v) = v \cdot v = g_{ij} v^i v^j$$

and angle between two vectors

$$g(v, w) = vw \cos \theta$$

Although, the latter only with Riemannian metrics.
The metric tensor, under a change of basis Λ, change

$$g' = \Lambda^T g \Lambda$$

where ${g'}_{ij} = g({e'}_i, {e'}_j)$. Since it is symmetric, it can be always possible to find two matrices $O^{-1} = O^T$ and $D = D^T = diag(\frac{1}{\sqrt{|g_{ii}^{(diag)}|}})$ such that

$$g' = D^T O^T g O D = D g^{(diag)} D$$

and put in canonical form

$${g'}_{ij} = \pm \delta_{ij}$$

which defines an orthonormal basis at P, i.e. $g(e_i, e_j) = \pm \delta_{ij}$.
The $\pm$ cannot be eliminated and the sum of the diagonal element is called the signature. A sign inversion does not affect the signature. The diagonal elements can classify the metric in the following way:

1. Riemannian metric, i.e. all of the same sign
2. pseduo-Riemannian metric, i.e. both signs appear (Lorentzian metric if one is of one kind and all the others of the other kind)

Metric tensors define a map between T_P and T_P^*, to lower indices and the inverse to raise them. Infact, a vector $v \in T_P$ can be mapped into a 1-form

$$v_i = v(e_i) = g(v^j e_j, e_i) = v^i g(e_j, e_i) = v^i g_{ij}$$

and a 1-form $w \in T_P^*$ can be mapped into a vector

$$w^i = e^i(w) = g(e^i, w_j e^j) = w_j g(e^i, e^j) = w_j g^{ij}$$

Consequently, at P a vector and a 1-form are equivalent.
The inverse metric tensor in defined by

$$g_{ij}^{-1} = g^{ij} \quad g_{ij} g^{jk} = {\delta^k}_i$$

If the metric is in canonical form, the dual basis will be orthonormal.
A metric tensor field is a map that associates each point of $\mathcal{M}$ into a metric tensor. The manifold becomes a metric manifold $(\mathcal{M},\ g)$. The metric tensor field in terms of coordinate vectors and dual basis is

$$g(x) = g_{ij}(x) dx^i \otimes dx^j$$

which is written as line element

$$ds^2 = g_{ij}(x) dx^i dx^j$$

Consider the integral curve γ of a vector field $v = \frac{d}{d\lambda}$. The scalar infinitesimal displacement along v is

$$ds^2 = dx \cdot dx = g(dx, dx) = g(v d\lambda, v d\lambda) = g(v, v) d\lambda^2$$

Integrating along γ, the length of the path between λ_1 and λ_2 is

$$s(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} d\lambda\ \sqrt{g(v,v)} = \int_{\lambda_1}^{\lambda_2} d\lambda\ \sqrt{g_{ij}(\lambda) v^i(\lambda) v^j(\lambda)}$$

Introducing a chart x^i,

$$s(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} d\lambda\ \sqrt{g_{ij}(\lambda) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$$

It is always possible to find a change of coordinate that put the metric tensor field in the locally canonical form

$$g_{ij}(x) = \pm\delta_{ij} + \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}\Big|_{x_P} \delta x^k \delta x^l$$

which means to find a locally orthogonal coordinates x^i such that $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \pm\delta_{ij}$. However, this holds only locally, not on the entire manifold.

Proof. Around P, the metric tensor field g_{ij} can be Taylor expanded in $x = x_P + \delta x$

$$g_{ij} = g_{ij}(x_P) + \frac{\partial g_{ij}}{\partial x^k}\Big|_{x_P} \delta x^k + \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}\Big|_{x_P} \delta x^k \delta x^l + \dots \tag{1.9}$$

as well as the transformation matrix

$$\frac{\partial x^i}{\partial y^j}(x) = \frac{\partial x^i}{\partial y^j}(x_P) + \frac{\partial}{\partial x^k} \frac{\partial x^i}{\partial y^j}\Big|_{x_P} \delta x^k + \frac{1}{2} \frac{\partial^2}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial y^j}\Big|_{x_P} \delta x^k \delta x^l + \dots \tag{1.10}$$

and the metric in the new coordinates

$$g'_{ij} = g'_{ij}(y_P) + \frac{\partial g'_{ij}}{\partial y^k}\Big|_{y_P} \delta y^k + \frac{1}{2} \frac{\partial^2 g'_{ij}}{\partial y^k \partial y^l}\Big|_{y_P} \delta y^k \delta y^l + \dots \tag{1.11}$$

Using

$$g'_{ij} = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl}$$

then the left-handed side is

$$\begin{aligned}
\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl} = &\Big(\frac{\partial x^k}{\partial y^i} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\delta x^a + \frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^e}\frac{\partial x^k}{\partial y^i}\delta x^a \delta x^e + \dots\Big)\\
&\Big(\frac{\partial x^l}{\partial y^j} + \frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}\delta x^b + \frac{1}{2}\frac{\partial^2}{\partial x^b \partial x^f}\frac{\partial x^l}{\partial y^j}\delta x^b \delta x^f + \dots\Big)\\
&\Big(g_{kl} + \frac{\partial g_{kl}}{\partial x^c}\delta x^c + \frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^c \partial x^d}\delta x^c \delta x^d + \dots\Big)\\
= &\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}\delta x^b g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}\delta x^b \frac{\partial g_{kl}}{\partial x^c}\delta x^c + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^c}\delta x^c\\
&+ \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\delta x^a \frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\delta x^a \frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}\delta x^b g_{kl} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\delta x^a \frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^c}\delta x^c\\
&+ \frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^e}\frac{\partial x^k}{\partial y^i}\delta x^a \delta x^e \frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{1}{2}\frac{\partial^2}{\partial x^b \partial x^f}\frac{\partial x^l}{\partial y^j}\delta x^b \delta x^f g_{kl}\\
&+ \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^c \partial x^d}\delta x^c \delta x^d + \dots\\
= &\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl} + \delta x^a \Big(\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^a}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^a} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl}\Big)\\
&+ \delta x^a \delta x^b \Big(\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^a}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^b} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^b}\\
&\qquad + \frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^b}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^b}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^a \partial x^b}\Big)
\end{aligned}$$

Comparing infinitesimal of the same order

$$\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl} = g'_{ij}$$

$$\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^a}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^a} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl} = \frac{\partial g'_{ij}}{\partial y^k}$$

$$\begin{aligned}
&\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^a}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^b} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^b}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial}{\partial x^a}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{\partial g_{kl}}{\partial x^b}\\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^b}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^b}\frac{\partial x^l}{\partial y^j}g_{kl} + \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\frac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^a \partial x^b} = \frac{1}{2}\frac{\partial^2 g'_{ij}}{\partial y^k \partial y^l}
\end{aligned}$$

Looking at this system of equations, we find 1 degree of freedom for the first one, n for the second one and n^2 for the third one. Hence, since Λ has $n^2 - 1$ degrees of freedom with -1 coming from (1.8), we only have enough degree of freedom to put

$$g'_{ij}(y_P) = \pm\delta_{ij}$$

and

$$\left.\frac{\partial g_{ij}}{\partial y^k}\right|_{y_P} = 0$$

but not enough to put

$$\left.\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}\right|_{y_P} = 0$$

q.e.d.

Chapter 2

Lie derivatives

2.1 Active and passive transformation

In the passive interpretation of a diffeomorphism on $\mathcal{M}$, the points remain the same but their coordinates changes. The diffeomorphism does not act on $\mathcal{M}$, but on the coordinates in $\mathbb{R}^n$. It can be sees as a change of coordinates $x' = x'(x, \epsilon)$ where ϵ is a parameter such that $x'(x, 0) = x$. For instance, a function changes in such a way that $\Phi'(x') = \Phi(x)$ where $\Phi(x) = (f \o \phi^{-1})(x)$.
In the active interpretation of a diffeomorphism, the points are actually moved (along the flow of an integral curve). The diffeomorphism do act on $\mathcal{M}$.
The Lie dragged or push forward of a function f from a point P to a point P' is a new function such that $f^*(P') = f(P)$.

2.2 Congruence

A congruence of a vector field V is a set of integral curve which start from a curve Σ_0, that is an hypersurface of dimension $n-1$ and uniquely cover a portion of $\mathcal{M}$.
A Lie dragging or a push-forward $\phi_{\Delta\lambda} \colon \mathcal{M} \to \mathcal{M}$ is the motion of a point $P(\lambda_0)$ in $P(\lambda_0 + \Delta\lambda)$ such that $\phi_{\Delta\lambda}$ is continuous and invertible. If $V \in C^\infty$, the push-forward becomes a diffeomorphism and form a group. Infact, $\phi_{\lambda_1} \o \phi_{\lambda_2} = \phi_{\lambda_1+\lambda_2}$, $\phi_\lambda^{-1} = \phi_{-\lambda}$ and $\phi_{\lambda=0} = \mathbb{I}$.
The push-forward of a function f along a congruence of $V = \frac{d}{d\lambda}$ is a map f^* from a point P to a point $Q = \phi_{\Delta\lambda}(P)$ such that

$$f^*_{\Delta\lambda}(Q) = f(P)$$

If it is true $\forall Q$ along the integral curve of V, $f^*_{\Delta\lambda}$ is constant and $\frac{df}{d\lambda} = 0$.
The push-forward of a vector field $W = \frac{d}{d\mu}$ along a congruence of $V = \frac{d}{d\lambda}$ is a map f^* from a point P to a point $Q = \phi_{\Delta\lambda}(P)$ such that

$$W^*_{\Delta\lambda}(f^*_{\Delta\lambda})|_Q = W(f)|_P$$

where f is an arbitrary function. It can be also written as

$$\frac{df^*}{d\mu}\Big|_{\lambda_0+\Delta\lambda} = \frac{df}{d\mu}$$

where $\lambda(P) = \lambda_0$ and $\lambda(Q) = \lambda_0 + \Delta\lambda$.
Furthermore, the commutator between V and W^* vanishes

$$[V, W^*] = [\frac{d}{d\lambda}, \frac{d}{d\mu^*}] = 0 \tag{2.1}$$

Proof. Fixing f and varying $\Delta\lambda$, $\frac{df^*}{d\mu^*}$ is constant along the congruences of V. Mapping the initial curve Σ_0 into a new curve $\Sigma_{\Delta\lambda}$ and since λ is constant, it can be used as a coordinate.
Since W^* is tangent to $\Sigma_{\Delta\lambda}$ its parameter μ^* is constant along the congruences of V. Hence there are two coordinates (λ, μ^*) are coordinates and their coordinate vectors commute. q.e.d.

2.3 Lie derivatives

The Lie derivative of a function f along a vector field $V = \frac{d}{d\lambda}$ is

$$\pounds_V f|_{\lambda_0} = \lim_{\Delta\lambda\to 0} \frac{f^*_{-\Delta\lambda}(\lambda_0) - f(\lambda_0)}{\Delta\lambda} = \lim_{\Delta\lambda\to 0} \frac{f(\lambda_0+\Delta\lambda) - f(\lambda_0)}{\Delta\lambda} = \frac{df}{d\lambda}\Big|_{\lambda_0} = V(f)$$

where it has been used the push-back $\phi_{-\Delta\lambda}(P(\lambda_0+\Delta\lambda)) = P(\lambda_0)$ and $f^*_{-\Delta\lambda}(\lambda_0) = f(\lambda_0+\Delta\lambda)$. If f is constant along the congruences, then $\pounds_V f = 0$.
The Lie derivative of a vector field $W = \frac{d}{d\mu}$ along a vector field $V = \frac{d}{d\lambda}$ is

$$\pounds_V W(f)|_{\lambda_0} = \lim_{\Delta\lambda\to 0} \frac{W^*_{-\Delta\lambda} - W}{\Delta\lambda}(f)\Big|_{\lambda_0}$$

or it can be written as

$$\pounds_V W = [V, W]$$

in components

Proof. Taylor expanding around $\lambda_0 + \Delta\lambda$

$$W^*_{-\Delta\lambda}(f)\Big|_{\lambda_0} = \frac{df}{d\mu^*}\Big|_{\lambda_0} = \frac{df}{d\mu^*}\Big|_{\lambda_0+\Delta\lambda} - \Delta\lambda\Big(\frac{d}{d\lambda}\frac{d}{d\mu^*}f\Big)\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2)$$

Using (2.1) and at first order $\frac{d}{d\mu} = \frac{d}{d\mu^*}$,

$$\begin{aligned}
\frac{df}{d\mu^*}\Big|_{\lambda_0} &= \frac{df}{d\mu}\Big|_{\lambda_0+\Delta\lambda} - \Delta\lambda\Big(\frac{d}{d\lambda}\frac{d}{d\mu^*}f\Big)\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2) \\
&= \frac{df}{d\mu}\Big|_{\lambda_0} + \Delta\lambda\frac{d}{d\lambda}\frac{d}{d\mu^*}f\Big|_{\lambda_0} - \Delta\lambda\Big(\frac{d}{d\lambda}\frac{d}{d\mu^*}f\Big)\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2) \\
&= \frac{df}{d\mu}\Big|_{\lambda_0} + \Delta\lambda\Big(\frac{d}{d\lambda}\frac{d}{d\mu^*}f - \frac{d}{d\mu^*}\frac{d}{d\lambda}f\Big)\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2) \\
&= \frac{df}{d\mu}\Big|_{\lambda_0} + \Delta\lambda\Big[\frac{d}{d\lambda},\frac{d}{d\mu}\Big]f\Big|_{\lambda_0+\Delta\lambda} + O(\Delta\lambda^2)
\end{aligned}$$

Hence

$$W^*_{-\Delta\lambda}(f)\Big|_{\lambda_0} = W(f)\Big|_{\lambda_0} + \Delta\lambda\Big[\frac{d}{d\lambda},\frac{d}{d\mu}\Big]f\Big|_{\lambda_0} + O(\Delta\lambda^2)$$

and

$$\pounds_V W(f)|_{\lambda_0} = \lim_{\Delta\lambda\to 0}\frac{\Delta\lambda\Big[\frac{d}{d\lambda},\frac{d}{d\mu}\Big]f\Big|_{\lambda_0}(f)\Big|_{\lambda_0} + O(\Delta\lambda^2)}{\Delta\lambda} = \Big[\frac{d}{d\lambda},\frac{d}{d\mu}\Big]f\Big|_{\lambda_0}$$

q.e.d.

The Lie derivative satifies the properties

1. vanishes if the components of W are constant along V
2. Leibniz rule, i.e.
$$\pounds_V(fW) = f\pounds_V(W) + \pounds_V(f)W$$
3. linearity, i.e.
$$\pounds_V + \pounds_W = \pounds_{V+W}$$
4. commutator, i.e.
$$[\pounds_V,\pounds_W] = \pounds_{[V,W]}$$
5. Jacobi identity, i.e.
$$[[\pounds_V,\pounds_W],\pounds_Z] + [[\pounds_W,\pounds_Z],\pounds_X] + [[\pounds_Z,\pounds_V],\pounds_W] = 0$$

The Lie derivative of a 1-form ω along a vector field $V = \frac{d}{d\lambda}$ is

$$(\pounds_V\omega)(W) = \pounds_V(\omega(W)) - \omega(\pounds_V W)$$

or introducing a chart x^i and the sual basis

$$(\pounds_V\omega)_i = V^k\frac{\partial\omega_i}{\partial x^k} + \omega_k\frac{\partial V^k}{\partial x^i}$$

Proof. Using the Leibniz rule,

$$\pounds_V(\omega(W)) = (\pounds_V\omega)(W) + \omega(\pounds_V W)$$

$$(\pounds_V\omega)(W) = \pounds_V(\omega(W)) - \omega(\pounds_V W)$$

Introducing a coordinate basis, the Lie derivative of a scalar and the components of a commutator

$$\begin{aligned}(\pounds_V\omega)_i &= (\pounds_V\omega)(e_i)\\ &= \pounds_V(\omega(e_i)) - \omega(\pounds_V e_i) = \frac{d\omega(e_i)}{d\lambda} - \omega([V, e_i])\\ &= V^k\frac{\partial\omega^i}{\partial x^k} + \omega^k\frac{\partial V^k}{\partial x^i}\end{aligned}$$

q.e.d.

The Lie derivative of a tensor (n, m)

$$T(\omega_1, \ldots, \omega_n, W^1, \ldots, W^m) \colon \mathcal{M} \to \mathbb{R}$$

along a vector field $V = \frac{d}{d\lambda}$ is

$$\begin{aligned}\pounds_V T(\omega_1, \ldots, \omega_n, W^1, \ldots, W^n) = (\pounds_V T)&(\omega_1, \ldots, \omega_n, W^1, \ldots, W^n)\\ &+ T(\pounds_V\omega_1, \ldots, \omega_n, W^1, \ldots, W^n) + \ldots\\ &+ T(\pounds_V\omega_1, \ldots, \pounds_V\omega_n, W^1, \ldots, W^n)\\ &+ T(\omega_1, \ldots, \omega_n, \pounds_V W^1, \ldots, W^n) + \ldots\\ &+ T(\omega_1, \ldots, \omega_n, W^1, \ldots, \pounds_V W^n)\end{aligned}$$

2.4 Symmetries

Symmetries are no longer referred to the manifold, but to tensor defined in it. Furthermore, their geometrical meaning is a local feature.
A submanifold is a subset $\mathcal{S} \subset \mathcal{M}$ of dimension $\dim \mathcal{S} \leq \dim \mathcal{M}$ sucin which there exist a chart x^i such that $U \cap \mathcal{S} \subseteq \mathcal{M}$ and $x^{n-m+1} = \ldots = x^n = 0$ for all $P \in mathcalS$.
The tangent space in a point $P \in mathcalS$ has dimension

$$\dim T_P^{(\mathcal{M})} = n \geq \dim T_P^{(\mathcal{M})} = m$$

Curves and vectors in $\mathcal{S}$ maps to $\mathcal{M}$

$$\gamma_{\mathcal{S}} = (x^1(\lambda), \ldots, x^m(\lambda)) \rightleftarrows \gamma_{\mathcal{M}} = (x^1(\lambda), \ldots, x^m(\lambda), 0, \ldots, 0)$$

and

$$V_{\mathcal{S}} = (V^1, \ldots, V^m) \rightleftarrows V_{\mathcal{M}} = (V^1, \ldots, V^m, 0, \ldots, 0)$$

but the inverse is not unique, infact there are infinitely many curves or vectors created putting a different number from 0 in the places with index greater than m.
A 1-form in the submanifold is defined as

$$\omega_{\mathcal{S}}(V) = \omega_{\mathcal{M}}(V, 0, \ldots, 0)$$

where $V \in T_P^{(\mathcal{S})}$. Also here, the inverse is not unique, infact there are infinitely many 1-forms created putting a different number from 0 in the places with index greater than m.
A set of vector fields $V^{(k)}$ with $k = 1, \ldots, p$ is linearly independent if there exist a_k constants such that

$$\sum_{k=1}^{p} a_k V^k(P) = 0 \quad \forall P \in \mathcal{M}$$

This does not mean that at a given P, they are linearly independent, because the coefficients could depend on point $a_k = a_k(P)$.

Theorem 2.1 (Frobenius)
Let $V^{(k)}$ be a set of linearly independent vector fields with $k = 1, \ldots, p$ such that forms a Lie algebra

$$[V^{(i)}, V^{(j)}] = C^{ij}{}_k V^{(k)}$$

where $C^{ij}{}_k \in \mathbb{R}$. Then the integral curves of $V^{(k)}$ form a family of submanifolds or foliations of $\mathcal{M}$ of dimension $m \leq p$.

A vector field V is a symmetry of a tensor field T if

$$\pounds_V T = 0$$

Theorem 2.2
Let $V^{(i)}$ be a set of linearly independent vector fields with $i = 1, \ldots, p$ and $T^{(k)}$ be a set of linearly independent vector fields with $k = 1, \ldots, q$ such that

$$\pounds_{\sum_i a_i V^{(i)}} \sum_k b_k T^{(k)} = 0$$

Then $V^{(i)}$ form a Lie algebra

Proof. Given a two symmetries $V^{(1)}$ and $V^{(2)}$, using a property of the Lie derivative

$$[\pounds_{V^{(1)}}, \pounds_{V^{(2)}}] = \pounds_{[V^{(1)}, V^{(2)}]} = 0$$

Hence, $[V^{(1)}, V^{(2)}]$ is a symmetry as well. Generalizing for a linear combination $aV^{(1)} + bV^{(2)}$, the only condition to satift the hypothesis is that a and b are independent of P and the structure constant as well. q.e.d.

Corollary 2.1
$V^{(i)}$ *define a submanifold of dimension* $m \leq p$.

An isometries is a symmetry of the metric tensor

$$\pounds_V g = 0$$

where V is called the Killing vector. Hence, congruences along a Killing vector preserves lengths and angles.
In special relativity, inertial observers can be seen as coordinate frames along Killing vectors, using the Minkovski metric $g = \eta$.

Chapter 3

Integrals and forms

3.1 p-forms

A p-form is an antysymmetric tensor $(0, p)$ in the tangent T_P. p-forms form a linear space.
A 2-form ω is

$$\omega_{[ij]} = \frac{1}{2!}(\omega_{ij} - \omega_{ji})$$

A 3-form ω is

$$\omega_{[ijk]} = \frac{1}{3!}(\omega_{ijk} + \omega_{jki} + \omega_{kij} - \omega_{ikj} - \omega_{kji} - \omega_{jki})$$

A general p-form ω is

$$\omega_{[i_1 \ldots i_p]} = \frac{1}{p!}(\omega_{i_1 \ldots i_p} + \text{permutations})$$

The number of independent components of a p-form is the binomial coefficient

$$\binom{n}{p}$$

with the condition $\sum_p \binom{n}{p} = n^2$.
Introducing the wedge product

$$\omega = \omega_{i_1 \ldots i_p} e^{i_1} \wedge \ldots \wedge e^{i_p} = \omega_{i_1 \ldots i_p} \frac{1}{p!}(e^{i_1} \otimes \ldots \otimes e^{i_p} + \text{permutations})$$

Moreover, the wedge product can be used to compose a p-form and a q-form into a (p+q)-form

$$\text{p-form} \wedge \text{q-form} = \text{(p+q)-form}$$

and to contract a p-form with a vector to obtain a (p-1)-form

$$\begin{aligned} p(V, \ldots) &= (\omega_{i_1 \ldots i_p} e^{i_1} \wedge \ldots \wedge e^{i_p})(V^k e_k) \\ &= \frac{1}{p!}(\omega_{i_1 \ldots i_p} e^{i_1}(e_k) \otimes \ldots \otimes e^{i_p} + \text{permutations}) \\ &= V^k \omega_{i_1 \ldots i_p} e^{i_2} \wedge \ldots \wedge e^{i_p} \end{aligned}$$

3.2 Volume

A polyhedron in $\mathcal{M}$ is defined by n linearly independent vectors and its volume is a number. Therefore it is natural to associate an n-form, given the additional antysymmetric property, i.e. to vanish if two vectors are linearly dependent. In a coordinate basis, the n vectors are

$$\Delta x_k = dx^i_{(k)} \frac{\partial}{\partial x^i}$$

and the n-form is

$$\omega = f e^1 \wedge \ldots \wedge e^n$$

Putting together, the volume of an infinitesimal polyhedron is

$$\omega(\Delta x_{(1)}, \ldots, \Delta x_{(n)}) = f e^i(\Delta x_{(1)}) \ldots e^n(\Delta x_{(n)}) + \text{permutations}$$

Choosing coordinate basis,

$$\begin{aligned} \omega(\Delta x_{(1)}, \ldots, \Delta x_{(n)}) &= f dx^1_{(1)} \ldots dx^n_{(n)} + 0 + \ldots + 0 \\ &= f dx^1_{(1)} \ldots dx^n_{(n)} \\ &= dV \end{aligned}$$

Introducing a lattice of charts, the volume of a region $U \subseteq \mathcal{M}$ is the integral

$$V(U) = \int_{\phi(U)} f dx^1 \ldots dx^n = \int_U \omega$$

It is a scalar, since under a change of coordinates $y^i(x^i)$

$$V = \int_U \omega = \int_{\phi'(U)} f(y) J(y) d^n y$$

where J is the determinant of the jacobian, which shows that the volume is coordinate independent.

3.3 Area

In a submanifold of dimension $n-1$, the infinitesimal area of an hypersurface uses a (n-1)-form, taken by contracting the n-form of the volume with a vector $v \in T_P^{(\mathcal{M})} \notin T_P^{(\mathcal{S})}$ which is $A = \omega(v, \ldots)$.
The infinitesimal area, choosing coordinate basis, is

$$\begin{aligned}\omega(v, \omega_{(1)}, \ldots, \omega_{(n-1)}) &= A(\omega_{(1)}, \ldots, \omega_{(n-1)}) \\ &= vfe^1(\omega_{(1)}) \wedge \ldots \wedge e^{n-1}(\omega_{(n-1)}) \\ &= vfdx^1 \ldots dx^{n-1} \\ &= dA\end{aligned}$$

and the area of a portion $\Sigma \subseteq \mathcal{S}$ is

$$A(\Sigma) = \int_{\phi(\Sigma)} fvdx^1 \ldots dx^{n-1} = \int_\Sigma A$$

It is a scalar, since under a change of coordinates $y^i(x^i)$

$$A' = J^{(n-1)}A$$

where $J^{(n-1)}$ is the determinant of the jacobian restricted to the image of Σ.

3.4 Integrating with the metric

In a point P, the metric can be put in canonical form

$$g_{ij}(P) = \pm\delta_{ij}$$

The natural volume n-form is

$$\omega_g = e^1 \wedge \ldots \wedge e^n$$

Under a local change of coordinates $y^i(x^i)$

$$\omega_g = J\omega'_g = Jdy^1 \wedge \ldots \wedge dy^n$$

Using $g' = \Lambda^T g \Lambda$

$$\det g' = \det(\Lambda^T g \Lambda) = \det g \overset{2}{\det} \Lambda = \det g J^2 = \pm J^2$$

where $J = \sqrt{|\det g'|}$.
Hence, the volume of U becomes

$$V(U) = \int_U \omega_g = \int_{\phi(U)} \sqrt{|\det g'|} dy^1 \ldots dy^n$$

Similarly, the natural area (n-1)-form is

$$A_g = \omega_g(\ldots, e_n) = e^1 \wedge \ldots \wedge e^{n-1}$$

and the area of a portion Σ is

$$A = \int_\Sigma A_g = \int_{\phi(\Sigma)} \sqrt{|\det g^{(n-1)}|} dx^1 \ldots dx^{n-1}$$

where the metric locally is

$$g_{ij} = \begin{bmatrix} g_{ij}^{(n-1)} & 0 \\ 0 & \pm 1 \end{bmatrix}$$

3.5 Differential forms

The exterior derivative of a p-form

$$\omega = \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

is

$$d\omega = (\partial_k \omega_{i_1 \ldots i_p}) dx^k \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

It satisfies the following properties

1. addition, i.e.
$$d(\omega + \sigma) = d\omega + d\sigma$$

2. wedge product, i.e.
$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^p \omega \wedge d\sigma$$

3. vanishing boundary, i.e.
$$d(d\omega) = 0$$

For a function, this is the differential

$$df = \partial_i f dx^i = df$$

such that

$$d(df) = \partial_i \partial_j f dx^i \wedge dy^j = 0$$

List of Theorems