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On path integral:

an alternative way to quantise?

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Theoretical Physics

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Chapter 1

Introduction

In quantum mechanics, there are two equivalent method to quantise: operatorial formalism and path integrals. The latter has been introduced by Feynman and it is useful for two reasons: quantisation of (non-abelian) gauge theories in the standard model and relation between quantum field theory and statistical mechanics.

1.1 Two slit experiment

Consider an electron, created by a source, that passes through a barrier with two slits and reach a detector. The standard way to study such system is to highlight the wave behaviour of the electron and calculate the interference pattern with the Huygens principle. However, Feynman proposes an alternative way. The electron is a particle which accomplish all the possible path with an associated amplitude. Therefore, for each path c_i, there is an amplitude $A(c_i)$ and the total amplitude $A_{tot} = \sum_i A(c_i)$ is related to the probability to find the particle in a point of the detector screen $p = |A_{tot}|^2$. To each amplitude, we associated a unit norm and a phase equal to $\frac{S}{\hbar}$, where S is the action of the particle,

$$A(c_i) = \exp(\frac{i}{\hbar})S(c_i)$$

and the total amplitude is

$$A_{tot} = \sum_i \exp(\frac{i}{\hbar})S(c_i) \ .$$

Heuristical free particle

Consider a free particle with associated action

$$S[q] = \int_0^T dt \ \frac{m}{2}\dot{q}^2 \ .$$

We suppose that the difference between the two path is $d \ll D$ and we evaluate the action fot the path c_1

$$S(c_1) = \frac{m}{2}\frac{D^2}{T^2}T = \frac{m}{2}\frac{D^2}{T}$$

and for the path c_2

$$S(c_2) = \frac{m}{2}\frac{(D+d)^2}{T^2}T = \frac{m}{2T}(D^2+2Dd+O(d^2)) = \frac{m}{2}\frac{D^2}{T}+\frac{mDd}{T}O(d^2) = \frac{m}{2}\frac{D^2}{T}+pd+O(d^2) = S(c_1)+p$$

where we roughly estimate $p \sim \frac{mD}{T}$. Therefore, the total amplitude is

$$A_{tot} = A(c_1)+A(c_2) = \exp(\frac{i}{\hbar}S(c_1))+\exp(\frac{i}{\hbar}S(c_2)) = \exp(\frac{i}{\hbar}S(c_1))\Big(1+\exp(\frac{i}{\hbar}pd))+O(d^2)\ .$$

We notice that the maximum probability is given when

$$\exp(\frac{i}{\hbar}pd) = 1 \quad \Rightarrow \quad \frac{pd}{\hbar} = 2\pi n\ ,$$

where $n \in \mathbb{Z}$. We recover quantum mechanics, since we recognise the Compton wavelength of the electron

$$\lambda = \frac{\hbar}{p}$$

and we find the relation

$$\frac{d}{\lambda} = n\ .$$

We can generalise by increasing the number of slits and intermediary screens to have all the possible paths between the initial point (source) and the final point (detector). The action becomes

$$S[q] = \int t_i{}^{t_f} dt\ L(q, \dot{q})$$

and the transition amplitude in the continuum limit

$$A = \sum_i \exp(\frac{i}{\hbar}S(c_i)) = \int \mathcal{D}q\ \exp(\frac{i}{\hbar}S[q])\ ,$$

where $S[q]$ is a functional, A is an integral functional and $\mathcal{D}q$ is the measure in the path space.

We can recover the classical limit by noticing that, for macroscopic systems, the rate $\frac{S}{\hbar}$ is very big and small variations $\frac{\delta S}{\hbar}$ are bigger than $i\pi$. Therefore, the amplitudes of nearby paths cancel by destructive interference, unless it is the real classical path, since $\delta S = 0$.

1.2 Schroedinger equation

Consider a 1-dimensional particle. The lagrangian action in the configuration space

$$S[x(t)] = \int dt \Big(\frac{m}{2} \dot{x}^2 - V(x) \Big) \ .$$

The momentum is defined as

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \ .$$

By a Legendre transformation, we can introduce the hamiltonian

$$H = p\dot{x} - L = p \frac{p}{m} + \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} \ .$$

The hamiltonian action in the phase space

$$S[x(t)] = \int dt \Big(p\dot{x} - \frac{p^2}{2m} - V(x) \Big) \ .$$

Given 2 function in phase space f and g, the Poisson brackets are

$$\{x, x\} = \{p, p\} = 0 \ , \quad \{x, p\} = 1 \ .$$

The canonical quantisation consists in promoting position and momentum to operators $\hat{x}$ and $\hat{p}$ such that they satisfy the canonical commutation relations

$$[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0 \ , \quad [\hat{x}, \hat{p}] = i\hbar \ .$$

Therefore, also the hamiltonian is promoted to an operator acting on the Hilbert space $\mathcal{H}$

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + V(\hat{x}) \ .$$

The Schroedinger equation reads as

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle \ ,$$

where $|\psi\rangle \in \mathcal{H}$ is a state in the Hilbert space.

In the position representation, the eigenstates of the position are

$$\hat{x} |x\rangle = x |x\rangle \ ,$$

such that they satisfy

$$\langle x | x' \rangle = \delta(x - x') \ , \quad \mathbb{I} = \int dx \ |x\rangle\langle x| \ .$$

The momentum operator in this representation is

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \ .$$

Hence, the Schroedinger equation is

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = \Big(\frac{1}{2m}\hat{p}^2 + V(\hat{x})\Big)\psi(t,x) = \Big(- \frac{\hbar^2}{2m}\frac{\partial^2 x}{\partial +^2} V(x)\Big)\psi(t,x) \ .$$

However, for any time-independent hamiltonian, we can solve the Schroedinger equation introducing an evolution operator

$$|\psi(t)\rangle = \exp(-\frac{i}{\hbar}\hat{H}t)|\psi_i\rangle \ .$$

Proof. In fact,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\Big(- \frac{i}{\hbar}\hat{H}\Big)|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \ .$$

q.e.d.

The transition amplitude to find the system from the intial state ψ_i to the final state ψ_f is

$$A(x_i, x_f, T) = \langle\psi_f|\psi(t)\rangle = \langle\psi_f|\exp(-\frac{i}{\hbar}\hat{H}T)|\psi_i\rangle \ .$$

Proof. In fact,

$$\begin{aligned}
\langle\psi_f|\psi(t)\rangle &= \langle\psi_f|\exp(-\frac{i}{\hbar}\hat{H}T)|\psi_i\rangle \\
&= \langle\psi_f|\mathbb{I}\exp(-\frac{i}{\hbar}\hat{H}T)\mathbb{I}|\psi_i\rangle \\
&= \int dx_f \ \underbrace{\langle\psi_f|x_i\rangle}_{\psi_f(x_f)}\langle x_i|\exp(-\frac{i}{\hbar}\hat{H}T)\int dx_i \ |x_i\rangle \underbrace{\langle x_i|\psi_i\rangle}_{\psi_i(x_i)} \\
&= \int dx_i \int dx_f \ \psi_f^*(x_f)\psi_i(x_i)\langle x_f|\exp(-\frac{i}{\hbar}\hat{H}T)|x_i\rangle \ ,
\end{aligned}$$

which shows that it is a matrix element between position eigenstates of the evolution operator. q.e.d.

Chapter 2

Path integrals in phase space

The path integral in phase space is

$$A = \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\Big) \exp(\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\Big(p_k\frac{x_k - x_{k-1}}{\epsilon} - H(x_{k-1}, p_k)))$$

$$= \int \mathcal{D}x\ \mathcal{D}p\ \exp(\frac{i}{\hbar}S[x,p])\ ,$$

where $\epsilon = \frac{T}{N}$.

Proof. In fact

$$\begin{aligned}
A(x_i, x_f, T) &= \langle x_f|\exp(-\frac{i}{\hbar}\hat{H}T)|x_i\rangle \\
&= \langle x_f|\exp(-\frac{i}{\hbar}\hat{H}\underbrace{\frac{T}{N}}_{\epsilon}N)|x_i\rangle \\
&= \langle x_f|\exp(-\frac{i}{\hbar}\hat{H}\epsilon)^N|x_i\rangle \\
&= \langle x_f|\mathbb{I}\exp(-\frac{i}{\hbar}\hat{H}\epsilon)\mathbb{I}\ldots\mathbb{I}\exp(-\frac{i}{\hbar}\hat{H}\epsilon)\mathbb{I}|x_i\rangle \\
&= \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N}\langle x_k|\exp(-\frac{i}{\hbar}\hat{H}\epsilon)|x_{k-1}\rangle\Big) \\
&= \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N}\langle x_k|\mathbb{I}\exp(-\frac{i}{\hbar}\hat{H}\epsilon)|x_{k-1}\rangle\Big) \\
&= \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N}\frac{dp_k}{2\pi\hbar}\Big)\prod_{k=1}^{N}\underbrace{\langle x_k|p_k\rangle}_{\exp(\frac{i}{\hbar}p_k x_k)}\langle p_k|\exp(-\frac{i}{\hbar}\hat{H}\epsilon)|x_{k-1}\rangle \\
&= \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N}\frac{dp_k}{2\pi\hbar}\Big)\prod_{k=1}^{N}\exp(\frac{i}{\hbar}p_k x_k)\langle p_k|\exp(-\frac{i}{\hbar}\hat{H}\epsilon)|x_{k-1}\rangle\ .
\end{aligned}$$

So far, we compute the exaxt development. Now, we go infinitesimally with $N \to \infty$

$$
\begin{aligned}
A &= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k \Big) \Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} \Big) \prod_{k=1}^{N} \exp(\frac{i}{\hbar} p_k x_k) \langle p_k | \exp(-\frac{i}{\hbar} \hat{H} \epsilon) | x_{k-1} \rangle \\
&= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k \Big) \Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} \Big) \prod_{k=1}^{N} \exp(\frac{i}{\hbar} (p_k (x_k - x_{k-1}) - H(x_{k-1}, p_k) epsilon)) \\
&= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k \Big) \Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} \Big) \prod_{k=1}^{N} \exp(\frac{i}{\hbar} \underbrace{\epsilon}_{dt} (\underbrace{p_k}_{p} \underbrace{\frac{x_k - x_{k-1}}{\epsilon}}_{\dot{x}} - \underbrace{H(x_{k-1}, p_k)}_{H})) \\
&= \int \mathcal{D}x \; \mathcal{D}p \; \exp(\frac{i}{\hbar} S[x,p]) \; ,
\end{aligned}
$$

where we have used

$$
\langle p_k | \exp(-\frac{i}{\hbar} \hat{H} \epsilon) | x_{k-1} \rangle = \langle p_k | x_{k-1} \rangle \exp(-\frac{i}{\hbar} H(x_{k-1}, p_k) \epsilon) \; .
$$

q.e.d.

Chapter 3

Path integrals in configuration space

The path integral in configuration space is

$$A = \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\frac{m}{2\pi i\hbar\epsilon}\Big)^{\frac{N}{2}} \exp\Big(\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\Big(\frac{m}{2}\frac{(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1})\Big)\Big) = \int \mathcal{D}x \ \exp(\frac{i}{\hbar}S[x]) \ .$$

Proof. With the use of the gaussian integral,

$$\int_{-\infty}^{\infty} dp \ \exp(-\frac{\alpha}{2}p^2 + \beta p) = \sqrt{\frac{2\pi}{\alpha}} \exp(\frac{\beta^2}{2\alpha}) \ ,$$

where in our case they are

$$\alpha = \frac{i\epsilon}{\hbar m} \ , \quad \beta = \frac{i}{\hbar}(x_k - x_{k-1}) \ ,$$

we have

$$\begin{aligned} A &= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\frac{1}{2\pi\hbar}\Big(\frac{2\pi\hbar m}{i\epsilon}\Big)^{\frac{N}{2}} \exp\Big(\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\epsilon\Big(\frac{m(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1})\Big)\Big)\Big) \\ &= \lim_{N\to\infty} \int \prod_{k=1}^{N-1} dx_k \Big(\frac{m}{2\pi\hbar i\epsilon}\Big)^{\frac{N}{2}} \exp\Big(\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\epsilon\Big(\frac{m(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1})\Big)\Big) \\ &= \int \mathcal{D}x \ \exp(\frac{i}{\hbar}S[x]) \ . \end{aligned}$$

q.e.d.

Chapter 4

Free particle

Consider a particle with $N = 1$ and $T = \epsilon$, the path integral is

$$A = \sqrt{\frac{m}{2\pi\hbar i T}} \exp(\frac{i}{\hbar}\frac{m(x_f - x_i)^2}{2T}) \ .$$

Proof. In fact, heuristically for $x_{cl}(t) = x_i + \frac{x_f - x_i}{T} t$

$$S[x_{cl}] = \int_0^T \frac{m}{2}\frac{(x_f - x_i)^2}{T^2} = T\frac{m}{2}\frac{(x_f - x_i)^2}{T^2} = \frac{m(x_f - x_i)^2}{2T} \ .$$

Moreover, considering $x(t) = x_{cl}(t) + q(t)$

$$\begin{aligned}
A &= \int \mathcal{D}x \exp(\frac{i}{\hbar}S[x(t)]) \\
&= \int \mathcal{D}x \exp(\frac{i}{\hbar}S[x_{cl}(t) + q(t)]) \\
&= \int \mathcal{D}x \exp(\frac{i}{\hbar}S[x_{cl}(t)] + S[q(t)]) \\
&= \int \mathcal{D}x \exp(\frac{i}{\hbar}S[x_{cl}(t)]) \underbrace{\int \mathcal{D}x \exp(\frac{i}{\hbar}S[q(t)])}_{N} \\
&= N \exp(\frac{i}{\hbar}S[x_{cl}(t)]) \ ,
\end{aligned}$$

where we have used for the mixed term

$$S[x] \propto \int dt \frac{d}{dt}(x_{cl} + q)\frac{d}{dt}(x_{cl} + q) = \int dt 2\dot{x}\dot{q} = \int dt \underbrace{\ddot{x}}_{0} q = 0$$

q.e.d.

It is a semiclassical approximation, since it solves the Schroedinger equation

$$i\hbar\frac{\partial}{\partial t}A(x_i, x_f, T) = -\frac{\hbar^2}{2m}A(x_i, x_f, T) \ .$$

Chapter 5

Wick rotation

Quantum mechanics can be related to statistical mechanics by an analitic continuation $T \to i\beta$, called a Wick rotation.

The Schroedinger equation becomes a diffusion equation

$$\frac{\partial}{\partial\beta}A = \frac{1}{2m}\frac{\partial^2}{\partial x_f^2}A \ .$$

Proof. In fact

$$-\frac{1}{2m}\frac{\partial^2}{\partial x_f^2}A = -\frac{1}{2m}\frac{1}{i}\frac{\partial}{\partial T} = -\frac{1}{2m}\frac{1}{i}\frac{\partial^2}{\partial x_f^2}A = \frac{1}{2m}\frac{\partial^2}{\partial x_f^2}A \ .$$

q.e.d.

The fundamental solution of the heat equation is

$$\begin{cases} A(x_i, x_f, \beta) = \sqrt{\frac{m}{2\pi\beta}}\exp\Big(-\frac{m(x_f-x_i)^2}{2\beta}\Big) \\ A(x_i, x_f, \beta = 0) = \delta(x_i - x_f) \end{cases} \ .$$

Through a Wick rotation, we can obtain an euclidean metric

$$ds^2 = -dt^2 + dx^2 \to ds^2 dt^2 + dx^2 \ .$$

Therefore, we have an euclidean action

$$iS[x] = i\int_0^T dt\Big(\frac{m}{2}\dot{x}^2 - V(x)\Big) = \int_0^\beta da\Big(-\frac{m}{2}\Big(\frac{dx}{da}\Big)^2 - V(x)\Big) = -\int_0^\beta da\Big(\frac{m}{2}\Big(\frac{dx}{da}\Big)^2 + V(x)\Big) = -S$$

with an euclidean path integral

$$\int \mathcal{D}x \exp(-S_E[x]) \ .$$

To summarise

$$A(x_i, x_f, T) = \langle x_f | \exp(-it\hat{H}) | x_i \rangle = \int \mathcal{D}x \exp(itS[x])$$
$$\to A(x_i, x_f, T) = \langle x_f | \exp(-\beta\hat{H}) | x_i \rangle = \int \mathcal{D}x \exp(-S_E[x])$$

and

$$i\frac{\partial}{\partial t}A = \hat{H}A \to \frac{\partial}{\partial\beta}A = -\hat{H}A \ .$$

The connection with statistical mechanics can be made by computing the canonical partition function, in quantum mechanics

$$Z = \operatorname{tr}\exp(-i\hat{H}t) = \sum_n \exp(-iE_n t) = \int dx \langle x | \exp(-i\hat{H}t) | x \rangle = \int_{PBC} \mathcal{D}x \exp(iS[x]) \ ,$$

where we have used a discrete energy spectrum (otherwise there would have been an integral), we have used position eigenstates and the path integral is on a circle with periodic boundary condition $x(T) = x(0)$; and in statistical mehcanics

$$Z_E = \operatorname{tr}\exp(-\beta\hat{H}) = \sum_n \exp(-\beta E_n) = \int dx \langle x | \exp(-\beta\hat{H}) | x \rangle = \int_{PBC} \mathcal{D}x \exp(-S_E[x]) \ ,$$

where the path integral is on a circle with periodic boundary condition $x(\beta) = x(0)$.

Few comments are to be made. First, we need some (dimensional) regularisation, for instance we need an imaginary time to preserve gauge symmetry. Second, since so far we only have compute the 1-dimensional case, we need to generalise for the finite or infinite degrees of freedom. For instance, in the 3-dimensional particle

$$\int_{\mathbf{R}^3} \mathcal{D}x \ \exp(iS[x]) = \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} d^3 x_k \Big) \Big(\frac{m}{2\pi i\hbar\epsilon} \Big)^{\frac{3N}{2}} \exp\Big(\frac{i}{\hbar} \sum_{k=1}^{N} \epsilon \Big(\frac{m}{2} \frac{(\mathbf{x}_k - \mathbf{x}_{k-1})^2}{\epsilon^2} - V(\mathbf{x}_{k-1}) \Big) \Big) \ .$$

Third, there is an ambiguity in choosing the argument of the potential: the prepoint discretisation chooses $V(x_{k-1})$, the midpoint discretisation chooses $V(\frac{x_k - x_{k-1}}{2})$ and the postpoint discretisation chooses $V(x_k)$. For gauge theoru, the midpoint prescription is chosen. However all the quantisation methods (among ordering) are equivalent, since there are the same ambiguities in all methods.

Chapter 6

Correlation functions

We define the normalised n-point correlation function as

$$\begin{aligned}\langle x(t_1), \ldots x(t_n)\rangle &= \frac{\int \mathcal{D}x \; x(t_1)\ldots x(t_n)\exp(\frac{i}{\hbar}S[x])}{\int \mathcal{D}x \;\; \exp(\frac{i}{\hbar}S[x])} \\ &= \frac{1}{Z}\int \mathcal{D}x \; x(t_1)\ldots x(t_n)\exp(\frac{i}{\hbar}S[x]) \ ,\end{aligned}$$

where $x(t)$ is a dynamical variable.

The generating functional of correlation functions is

$$Z[J] = \int \mathcal{D}x \;\; \exp\left(\frac{i}{\hbar}S[x] + \frac{i}{\hbar}\int_0^T dt \;\, J(t)x(t)\right) \ ,$$

where the functional $J(t)$ is called the source.

We can write the normalised n-point correlation function in terms of the generating functional in the following way

$$\langle x(t_1), \ldots x(t_n)\rangle = \frac{1}{Z[0]}\left(\frac{\hbar}{i}\right)^n \frac{\delta^n Z[J]}{\delta J(t_1)\ldots \delta J(t_n)}\Big|_{J=0} \ .$$

The 0-point function is

$$\langle 1\rangle = \frac{Z}{Z} = 1 \ .$$

If you know $Z[J]$, you know all the correlation functions.

In the Schroedinger picture, the correlation function is

$$\langle x(t_1)\ldots x(t_n)\rangle = \frac{1}{Z}\langle x_f|\exp(-\frac{i}{\hbar}\hat{H}(t_f - T_n)\hat{x}\ldots\hat{x}\exp(-\frac{i}{\hbar}\hat{H}(T_1 - t_i)))|x_i\rangle \ ,$$

where we have ordered to start from the earliest T_1 to the latest T_n. In the Heisenberg picture, the correlation function is

$$\langle x(t_1)\ldots x(t_n)\rangle = \frac{1}{Z}\langle x_f, t_f|T\hat{h}_H(t_1)\ldots\hat{h}_H(t_n)|x_i, t_i\rangle \ ,$$

where T is the time-ordered prescription. For example, a 2-point correlation function is

$$\langle 0|T\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle \ .$$

Chapter 7

Gaussian integrals

We make a list of useful gaussian integrals

$$\int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \exp(-\frac{1}{2}K\phi^2) = \frac{1}{\sqrt{K}} \ ,$$

with $K \geq 0$.

$$\int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \exp(-\frac{1}{2}K\phi^2 + J\phi) = \frac{1}{\sqrt{K}} \exp(\frac{1}{2K}J^2) \quad K \geq 0 \ .$$

$$\int_{\mathbb{R}^n} \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}} \exp(-\frac{1}{2}\phi^i K_{ij}\phi^j) = \frac{1}{\sqrt{\det k_{ij}}} \ ,$$

where K_{ij} is symmetric, real, positive defined and $\det K_{ij}$ is the product of the eigenvalues.

$$\int_{\mathbb{R}^n} \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}} \exp(-\frac{1}{2}\phi^i K_{ij}\phi^j + J_i\phi^i) = \frac{1}{\sqrt{\det k_{ij}}} \exp(\frac{1}{2}J_i G^{ij} J_j) \ ,$$

where G^{ij} is the inverse of K_{ij} such that $K_{ij}G^{jl} = \delta_i{}^l$.

By analitic continuation, we have

$$\int_{\mathbb{R}^n} \frac{d^n\phi}{(-i2\pi)^{\frac{n}{2}}} \exp(-\frac{i}{2}\phi^i K_{ij}\phi^j + iJ_i\phi^i) = \frac{1}{\sqrt{\det k_{ij}}} \exp(\frac{i}{2}J_i G^{ij} J_j) \ ,$$

but in order to converge, we must have $K_{ij} - i\epsilon\delta_{ij}$, so that a damping exponential compares.

Chapter 8

Hypercondensed notation

In the hypercondensed notation, we can write

$$x(t) \to \phi^i \ , \quad x \to \phi \ , t \to i \ ,$$

$$A_\mu(x^\nu) \to \phi^i \ , \quad A \to \phi \ , (\ _\mu, x^\nu) \to i \ ,$$

Notice that now i is a continuous index. Furthermore

$$\phi^i \phi_i = \int dt x(t) x(t) = \int dt \int dt' x(t) \delta(t-t') x(t') \ .$$

or

$$\phi^i \phi_i = \int d^4x A_\mu(x) A^\mu(y) = \int d^4x \int d^4y A_\mu(x) \eta^{\mu\nu} \delta^4(x-y) A_\nu(y) \ .$$

We can introduce the generating functional of connected correlation functional

$$W[J] = \frac{\hbar}{i} \ln ZJ \ ,$$

which implies that

$$ZJ = \exp(\frac{i}{\hbar} W[J]) \ .$$

We can write the normalised n-point correlation function in terms of the generating functional of connected correlation functions in the following way

$$\langle \phi^{i_1} \dots \phi^{i_n} \rangle_c = \left(\frac{\hbar}{i} \right)^{n-1} \frac{\delta^n W[J]}{\delta J_{i_1} \dots \delta J_{i_n}} \Big|_{J=0} \ .$$

The effective action-generating functional of 1-particle irreducible is

$$\Gamma(\varphi) = \min_J (W[J] - J^i \varphi_i) \ ,$$

where the procedure to calculate φ is

$$\varphi^i = \frac{\delta W[J]}{\delta J_i} \ .$$

Proof. The condition of minimum implies

$$\frac{\delta W[J]}{\delta J_i} - \varphi^i = 0 \ .$$

Therefore

$$\varphi^i = \frac{\delta W[J]}{\delta J_i} \ ,$$

and we can invert the relation from $\varphi^i = \varphi^i(J)$ to $T_i = T_i(\varphi)$. q.e.d.

Notice that it is a Legendre transform.

Part II

Free theory

Chapter 9

Free theory

Consider a quadratic action

$$S[\phi] = \frac{1}{2}\phi^i K_{ij}\phi^j \ ,$$

where K_{ij} is an invertible matrix and the equations of motion are

$$K_{ij}\phi^j = 0 \ .$$

The generating functional is

$$Z[J] = \int \mathcal{D}x \ \exp(iS[\phi] + iJ_i\phi^i) = \overset{-1/2}{\det} K_{ij} \exp(\frac{1}{2}J_i G^{ij} J_j) \ ,$$

through which we can obtain all normalised correlation functions. The generating functional of connected correlation functions is

$$W[J] = \frac{1}{2}J_i G^{ij} J_i - \Lambda \ ,$$

where $\Lambda = \frac{i}{2}\ln\det K_{ij}$.

Proof. In fact

$$W[J] = -i\ln Z[J] = -\underbrace{\frac{i}{2}\ln\det K_{ij}}_{\Lambda} + \frac{1}{2}J_i G^{ij} J_i = \frac{1}{2}J_i G^{ij} J_i - \Lambda \ .$$

q.e.d.

The effective action is

$$\Gamma[\varphi] = -\frac{1}{2}\varphi^i K_{ij}\varphi^j - \Lambda \ .$$

Proof. In fact

$$\varphi^i = \frac{\delta W}{\delta J_i} = G^{ij} J_j$$

and inverting it

$$J_i = K_{ij} \varphi^i \ .$$

Therefore

$$\Gamma[\varphi] = \min_J (W[J] - J_i \varphi^i) = \frac{1}{2} J_i G^{ij} J_j - \Lambda - \underbrace{\varphi^i K_{ij} \varphi^j}_{J_i G^{ij} J_j} = -\frac{1}{2} \varphi^i K_{ij} \varphi^j - \Lambda \ .$$

q.e.d.

The correlation function is

$$\langle \phi^{i_1} \dots \phi^{i_n} \rangle = \frac{1}{Z} (\frac{1}{i})^n \frac{\delta^n Z[J]}{\delta J_{i_1} \dots \delta J_{i_n}} \Big|_{J=0} \ .$$

Some examples are

$$\langle 1 \rangle = \frac{Z}{Z} = 1 \ ,$$
$$\langle \phi^i \rangle = \frac{1}{Z} \frac{1}{i} \frac{\delta Z}{\delta J_i} \Big|_{J=0} = \frac{1}{iZ} G^{ij} J_j \exp(\frac{1}{2} J_i G^{ij} J_j) \Big|_{J=0} = 0 \ ,$$
$$\langle \phi^i \phi^j \rangle = \frac{1}{Z} (\frac{1}{i})^2 \frac{\delta^2 Z[J]}{\delta J_i \delta J_j} \Big|_{J=0} = -iG^{ij} \ .$$

Notice that 1 point correlation function vanishes since the action is invariant under $\phi' = -\phi$ and it can be generalise for all odd point functions. On the other hand, even point functions can be always decomposed into 2 point ones. For example

$$\langle \phi^1 \phi^2 \phi^3 \phi^3 \rangle = (\frac{1}{i})^4 \frac{\delta^4 \exp(\frac{1}{2} J_i G^{ij} J_j)}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} = -G^{12} G^{34} - G^{13} G^{24} - G^{14} G^{23} = \langle \phi^1 \phi^2 \rangle \langle \phi^3 \phi^4 \rangle + \langle \phi^1 \phi^3 \rangle \langle \phi^2 \phi^4 \rangle + \langle \phi$$

This result is called the Wick's theorem, ehich states that in a quadratic theory, odd point functions vanish and even point functions can be expressed by 2 point functions and connect in all possible ways all the points. There are $(n-1)!!$ Wick contractions.

Reintroducing $\hbar$, we obtain

$$S[\phi] = -\frac{1}{2} \phi^i K_{ij} \phi^j \ ,$$
$$Z[J] = \overset{-1/2}{\det} K_{ij} \exp(\frac{1}{2\hbar} J_i G^{ij} J_j) \ ,$$
$$W[J] = \frac{1}{2} J_i G^{ij} J_j - \hbar \Lambda \ ,$$
$$\Lambda[\varphi] = -\frac{1}{2} \phi^i K_{ij} \phi^j - \hbar \Lambda = S[\varphi] + \hbar \text{corrections} \ ,$$
$$\langle \phi^i \phi^j \rangle = -i\hbar G^{ij} \ .$$

Chapter 10

Harmonic oscillator

Consider an harmonic oscillator with mass $m = 1$ and frequency ω. Its action is

$$\begin{aligned} S[x] &= \int_{-\infty}^{\infty} dt \, \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2\right) \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} dt \, \left(\frac{d^2}{dt^2} + \omega^2\right) x^2 \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' x(t)\left(\frac{d^2}{dt^2} + \omega^2\right)\delta(t-t')x(t') \\ &= -\frac{1}{2}\phi^i K_{ij}\phi^j \ , \end{aligned}$$

where we have intergated by parts and we define

$$K_{ij} = \left(\frac{d^2}{dt^2} + \omega^2\right)\delta(t-t') \ , \quad \phi^i = x(t) \ .$$

The Green function representation in Fourier space is

$$G(t,t') = \int \frac{dp}{2\pi}\frac{\exp(-ip(t-t'))}{-p^2+\omega^2} \ .$$

Proof. From the definition

$$K(t,t')G(t',t'') = \delta(t-t'') \ ,$$

we have

$$G(t,t') = \int \frac{dp}{2\pi}\frac{\exp(-ip(t-t'))}{-p^2+\omega^2} \ .$$

q.e.d.

Using the Feynman-Stueckelberg prescription, we obtain

$$G(t,t') = \frac{i}{2\pi}\exp(-i\omega|t-t'|) \ ,$$

which means adding an $\epsilon \to 0^+$ factor in order to have a damping factor in the path integrals. Therefore, the path integral is

$$Z[J] = \mathcal{N}\exp(\frac{i}{2\hbar}\int dt \int dt' J(t)G(t,t'0J(t'))) \ .$$

Proof. In fact

$$\begin{aligned}
Z[J] = &\int \mathcal{D}x \ \exp(\frac{i}{\hbar}S[x] + \frac{i}{\hbar}\int dt \ J(t)x(t)) \\
&= \int \mathcal{D}x \ \exp(\int dt \int dt' \ (\frac{1}{2}x(t)K(t,t')x(t') - J(t)\delta(t-t')x(t'))) \\
&= \int \mathcal{D}x \ \exp(\int dt \int dt' \ (\frac{1}{2}x(t)K(t,t')x(t') - J(t)\delta(t-t')x(t')) + \frac{1}{2}J(t)G(t,t')J(t') - \frac{1}{2}J(t) \\
&= \exp\left(\frac{i}{2\hbar}\int dt \int dt' \ J(t)G(t,t')J(t')\right)\int \mathcal{D}x \ \exp(-\frac{i}{\hbar}\int dt \int dt' \ \tilde{x}(t)K(t,t')\tilde{x}(t')) \\
&= \exp\left(\frac{i}{2\hbar}\int dt \int dt' \ J(t)G(t,t')J(t')\right)\underbrace{\int \mathcal{D}\tilde{x} \ \exp(-\frac{i}{\hbar}\int dt \int dt' \ \tilde{x}(t)K(t,t')\tilde{x}(t'))}_{\mathcal{N}} \\
&= \mathcal{N}\exp(\frac{i}{2\hbar}\int dt \int dt' J(t)G(t,t'0J(t'))) \ ,
\end{aligned}$$

where we have used the translation invariance of the measure and

$$\tilde{x}(t) = x(t) \quad \int dt'' \ G(t,t'')J(t'') \ .$$

q.e.d.

The 2 point function is

$$\langle x(t)x(t')\rangle = \frac{\int \mathcal{D}x \ x(t)x(t)\exp(\frac{i}{\hbar}S[x])}{\int \mathcal{D}x \ \exp(\frac{i}{\hbar}S[x])} = \frac{1}{Z[0]}(\frac{\hbar}{i})^2\frac{\delta^2 Z[J]}{\delta J(t)\delta J(t')} = -i\hbar G(t,t') = \frac{\hbar}{2\omega}\exp(-i\omega|t-t') \ .$$

10.1 Harmonic oscillator in Euclidean time

The path integral in Euclidean time, using a discrete diagonalised energy eigenbasis,

$$Z_E = \mathrm{tr}\exp(-\beta\hat{H}) = \sum_n \exp(-\beta E_n) \xrightarrow{\beta\to\infty} \exp(-\beta E_0) + \text{subleading terms} = \langle 0|\exp(-\beta\hat{H})|0\rangle \ ,$$

which is the energy of the vacuum state. Therefore

$$Z_E[x] = \int_{PBC} \mathcal{D}x \ \exp(-S_E[x] + \int d\tau \ J(\tau)x(\tau)) \xrightarrow{\beta\to\infty} \ ,$$

where

$$S_E[x] = \int_{-\beta/2}^{\beta/2} d\tau \ (\frac{1}{2}m\dot{x}^2 + \frac{\omega^2}{2}x^2) \ .$$

The Green function is

$$G_E(\tau,\tau') = \int \frac{dp}{2\pi} \frac{\exp(-ip(t-t'))}{p_E^2+\omega^2} = \frac{1}{2\omega}\exp(-\omega|\tau-\tau'|) \ .$$

The 2 point function is

$$\langle x(\tau)x(\tau')\rangle = G_E(\tau,\tau') = \frac{1}{2\omega}\exp(-\omega|\tau-\tau'|) \ .$$

Notice that in order to preserve the Fourier transform, we need to inverse Wick rotate the momentum, i.e. $t \to -i\tau$ and $p_M \to iP_E$. Hence, after a Wick rotation

$$\begin{aligned}\langle x(\tau)x(\tau')\rangle = G_E(\tau,\tau') &= \int \frac{dp_E}{2\pi} \frac{\exp(-ip_E(t-t'))}{p_E^2+\omega^2} \\ &\xrightarrow{\text{Wick}} -i\int \frac{dp_M}{2\pi} \frac{\exp(-ip_M(t-t'))}{p_E^2+\omega^2} = -iG(t,t') = \langle x(t)x(t')\rangle\end{aligned}$$

and

$$\frac{1}{2\omega}\exp(-\omega|\tau-\tau'|) \xrightarrow{\text{Wick}} \frac{1}{2\omega}\exp(-i\omega|t-t'|) \ .$$

Chapter 11

Klein-Gordon theory

Consider a Klein-Gordon field. Its action is

$$
\begin{aligned}
S[x] &= \int d^4x \,\Big(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\Big) \\
&= \int d^4x \,(\frac{1}{2}\phi\Box\phi - \frac{1}{2}\phi m^2\phi) \\
&= -\frac{1}{2}\int d^4x \int d^4y \phi(x)(-\Box_x + m^2)\delta^4(x-y)\phi(y) \\
&= -\frac{1}{2}\phi^i K_{ij}\phi^j \ ,
\end{aligned}
$$

where we have intergated by parts and we define

$$
K_{ij} = \Big(- \Box_x + m^2\Big)\delta^4(x-y) \ , \quad \phi^i = \phi(x) \ .
$$

The Green function representation in Fourier space is

$$
G(x,y) = \int \frac{d^4p}{(2\pi)^4}\frac{\exp(-ip_\mu(x^\mu - y^\mu))}{p^2+\omega^2} \ .
$$

We can recover the harmonic oscillator from the Klein-Gordon theory by rescricting ourselves to the situation in which we have only the time 0-dimension. In fact, the Klein-Gordon action becomes

$$
\int dt(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2)
$$

where we can identify $\omega = m$.

Chapter 12

Perturbation expansion

The model that we have in mind is the anharmonic oscillator, which action is

$$S[x] = \int dt(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \frac{g}{3!}x^3 - \frac{\lambda}{4!}x^4 + \ldots) = S_0 + S_{int}$$

where g and λ are small perturbative quantity, called coupling constants.

The path integral is

$$\begin{aligned}
Z[J] &= \int \mathcal{D}x \ \exp(\frac{i}{\hbar}S[x] + \frac{i}{\hbar}\int dt \ J(t)x(t)) \\
&= \int \mathcal{D}x \ \exp(\frac{i}{\hbar}S_{int}[x]) \underbrace{\exp(\frac{i}{\hbar}S_0[x] + \frac{i}{\hbar}\int dt \ J(t)x(t))}_{Z_0[J]} \\
&= \int \mathcal{D}x \ \Big(1 + (\frac{i}{\hbar}S_{int}[x]) + \frac{1}{2}(\frac{i}{\hbar}S_{int}[x])^2 + \ldots\Big) Z_0[J] \\
&= \langle \exp(\frac{i}{\hbar}S_{int}[x])\rangle_{0,U} \\
&= \langle 1\rangle_{0,U} + \langle \frac{i}{\hbar}S_{int}[x]\rangle_{0,U} + \frac{1}{2}\langle(\frac{i}{\hbar}S_{int}[x])^2\rangle_{0,U} \ .
\end{aligned}$$

where 0 means free theory and U not normalised. We can intepret this expression as a differential operator

$$\int \mathcal{D}x \ \exp(\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}])Z_0[J] = \int \mathcal{D}x \ \Big(1+(\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}])+\frac{1}{2}(\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}])^2+\ldots\Big)Z_0[J] \, .$$

12.1 Vacuum diagrams

In this section, we will work in Euclidean time. The path integral is

$$Z_E[J] = \int \mathcal{D}x \ \exp(-S_E[x] + \int d\tau \ x(\tau J(\tau))) \ ,$$

where the Euclidean action is

$$S_E = \lim_{\beta\to\infty} \int_{-\beta/2}^{\beta/2} d\tau (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{g}{3!}x^3 + \frac{\lambda}{4!}x^4) \ .$$

Consider the correction to the vacuum energy

$$\begin{aligned} Z_E[0] &= \int \mathcal{D}x \ \exp(-S_E[x]) \\ &= \langle 1 \rangle_U \\ &= \lim_{\beta\to\infty} \langle 0| \exp(-\beta\hat{H})|0\rangle = \lim_{\beta\to\infty} \exp(-\beta E_0) \\ &= \langle \exp(-S_{E,int}[x])\rangle_{U,0} \\ &= \lim_{\beta\to\infty} \exp(-\beta(E_0^{(0)} + \Delta E)) \ , \end{aligned}$$

where

$$E_0 = E_0^{(0)} + \Delta E \ .$$

Now we compute the energy corrections ΔE.

First case

Consider the case in which $g = 0$ and $\lambda \neq 0$. The path integral is

$$\begin{aligned} Z[0] &= \langle 1 - S_{E,int}[x] + \ldots \rangle \\ &= \langle 1 \rangle_{U,0} - \int_{-\beta/2}^{\beta/2} d\tau \ \frac{\lambda}{4!} \langle x^4(\tau)\rangle_{U,0} \frac{\langle 1 \rangle}{\langle 1 \rangle} + \ldots \\ &= \langle 1 \rangle (1 - \int_{-\beta/2}^{\beta/2} d\tau \ \frac{\lambda}{4!} \langle x(\tau)x(\tau)x(\tau)x(\tau)\rangle_{U,0}) \ . \end{aligned}$$

Now, we have to compute the 4 point function (which has $(n-1)!! = 3$ terms)

$$\langle x(\tau)x(\tau)x(\tau)x(\tau)\rangle = 3\langle x(\tau)x(\tau)\rangle^2 \ ,$$

where we have used the Wick theorem. For the harmonic oscillator is

$$\langle x(\tau)x(\tau)x(\tau)x(\tau)\rangle = \frac{3}{4\omega^2} \ .$$

Therefore

$$\begin{aligned} Z[0] &= \langle 1 \rangle (1 - \int_{-\beta/2}^{\beta/2} d\tau \ \frac{\lambda}{4!} \frac{3}{4\omega^2} + \ldots) \\ &= \langle 1 \rangle (1 - \frac{\lambda\beta}{32\omega^2} + \ldots) \\ &= \langle 1 \rangle \exp(-\frac{\beta\lambda}{32\omega^2}) \ . \end{aligned}$$

Finally, the energy correction is

$$\Delta E = \frac{\lambda}{32\omega^2} \ .$$

Second case

Consider the case in which $g \neq 0$ and $\lambda = 0$. The path integral is

$$\begin{aligned} Z[0] &= \langle 1 - S_{E,int}[x] + \frac{1}{2} S_{E,int}^2[x] \dots \rangle \\ &= \langle 1 \rangle (1 - \int_{-\beta/2}^{\beta/2} d\tau \ \frac{g}{3!} \cancel{\langle x^3(\tau) \rangle} + \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' \ (\frac{g}{3!})^2 \langle x^3(\tau) x^3(\tau') \rangle) \ . \end{aligned}$$

Now, we have to compute the 6 point function (which has $(n-1)!! = 15$ terms)

$$\langle x^3(\tau) x^3(\tau') \rangle = 6 G_E^3(\tau - \tau') + 9 G_E(\tau - \tau) G_E(\tau - \tau') G_E(\tau' - \tau') \ ,$$

where we have used the Wick theorem. For the harmonic oscillator is

$$\langle x^3(\tau) x^3(\tau') \rangle = \frac{6}{8\omega^3} \exp(-3\omega|\tau - \tau'|) + \frac{9}{8\omega^3} \exp(-\omega|\tau - \tau'|) \ .$$

Therefore

$$\begin{aligned} Z[0] &= \langle 1 \rangle (1 + \frac{1}{2} (\frac{g}{3!})^2 \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' \ (\frac{6}{8\omega^3} \exp(-3\omega|\tau - \tau'|) + \frac{9}{8\omega^3} \exp(-\omega|\tau - \tau'|)) + \dots) \\ &= \langle 1 \rangle (1 + \frac{1}{2} (\frac{g}{3!})^2 \frac{1}{8\omega^3} \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' \ (6 \exp(-3\omega|\tau - \tau'|) + 9 \exp(-\omega|\tau - \tau'|)) + \dots) \ . \end{aligned}$$

Now, we have to evaluate the integral

$$\int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' \ (6 \exp(-3\omega|\tau - \tau'|) + 9 \exp(-\omega|\tau - \tau'|)) \ .$$

We make a change of variable $|\sigma| = |\tau - \tau'|$ and compute the limit $\beta \to \infty$

$$\begin{aligned} & \underbrace{\int_{-\beta/2}^{\beta/2} d\tau}_{\beta} \int_{-\infty}^{\infty} d\sigma \ (6 \exp(-3\omega|\sigma|) + 9 \exp(-\omega|\sigma|)) \\ &= \beta(\int_0^{\infty} d\sigma \ (6 \exp(-3\omega\sigma) + 9 \exp(-\omega\sigma)) + \int_{-\infty}^{0} d\sigma \ (6 \exp(3\omega\sigma) + 9 \exp(\omega\sigma))) \ . \end{aligned}$$

We make a change of variable $\sigma = -\sigma$ in the second integrand

$$\begin{aligned} & \beta(\int_0^{\infty} d\sigma \ (6 \exp(-3\omega\sigma) + 9 \exp(-\omega\sigma)) - \int_{-\infty}^{0} d\sigma \ (-6 \exp(3\omega\sigma) + 9 \exp(-\omega\sigma))) \\ &= 2\beta \int_0^{\infty} d\sigma \ (6 \exp(-3\omega\sigma) + 9 \exp(-\omega\sigma)) \\ &= 2\beta (\frac{6}{3\omega} + \frac{9}{\omega}) \ . \end{aligned}$$

Therefore,

$$\begin{aligned}
Z[0] &= \langle 1\rangle(1+\frac{1}{2}(\frac{g}{3!})^2\frac{1}{8\omega^3}2\beta(\frac{6}{3\omega}+\frac{9}{\omega})+\ldots) \\
&= \langle 1\rangle(1+(\frac{g}{3!})^2\frac{1}{8\omega^3}\beta\frac{11}{\omega}+\ldots) \\
&= \langle 1\rangle\exp(\frac{11\beta g^4}{288\omega^4})\ .
\end{aligned}$$

Finally, the energy correction is

$$\Delta E = -\frac{11g^4}{288\omega^4}\ .$$

12.2 Feynman diagrams

Part III

Fermionic path integral

Chapter 13

Grassmann or anticommuting numbers

An n-dimensional Grassman algebra $\mathfrak{G}_n$ is generate by a set of generators θ_i, called Grassmann variables or anticommuting numbers, such that they satisfy

$$\{\theta_i, \theta_j\} = 0 \ , \quad \theta_i \theta_i = 0 \ .$$

13.1 Operation with Grassmann variables

Functions

A function of the Grassmann variables can be defined by its Taylor expansion. For $n = 1$, we have only θ such that $\theta^2 = 0$ and the function f can be expressed as

$$f(\theta) = f_0 + f_1 \theta \ ,$$

in which we cannot have more terms since $\theta^2 = 0$ cancels them out. For $n = 2$, we have θ_1, θ_2 such that $\theta_1\theta_2 = -\theta_2\theta_1$, $\theta_1^2 = \theta_2^2 = 0$ and the function f can be expressed as

$$f(\theta_1, \theta_2) = f_0 + f_1\theta_1 + f_2\theta_2 + f_3\theta_1\theta_2 \ .$$

where the term $\theta_2\theta_1$ is not written since it is not independent by the anticommutation relation. We should have written something like

$$a\theta_1\theta_2 + b\theta_2\theta_1 = a\theta_1\theta_2 + b\theta_2\theta_1 = \underbrace{(a - b)}_{f_3}\theta_1\theta_2 = f_3\theta_1\theta_2 \ .$$

An example of function is the exponential

$$\exp(\theta) = 1 + \theta \ ,$$

where $f_0 = 1$ and $f_1 = 1$. Terms with an even number of Grassmann variables are called Grassmann even and terms with an odd number of Grassmann variables are called Grassmann odd.

Derivatives

Derivatives are simple since all the functions are linear in Grassmann variables. However, we can define 2 kind of derivatives: left and right. The only difference, by the anticommutativity relation, is only a minus sign. For example in $n = 2$, we have

$$\frac{\partial_l}{\partial\theta_1} f(\theta_1, \theta_2) = \frac{\partial_l}{\partial\theta_1}(f_0 + f_1\theta_1 + f_2\theta_2 + f_3\theta_1\theta_2) = f_1 + f_3\theta_2 \ ,$$

$$\frac{\partial_r}{\partial\theta_1} f(\theta_1, \theta_2) = \frac{\partial_r}{\partial\theta_1}(f_0+f_1\theta_1+f_2\theta_2+f_3\theta_1\theta_2) = \frac{\partial_r}{\partial\theta_1}(f_0+f_1\theta_1+f_2\theta_2-f_3\theta_2\theta_1) = f_1-f_3\theta_2 \ .$$

In these notes, we will use always left derivatives.

Integrals

Berezin integrals are defined to be identical with derivatives

$$\int d\theta = \frac{\partial}{\partial\theta} \ .$$

It is translational invariant

$$\int d\theta f(\theta+\eta) = \int d\theta f(\theta) \ ,$$

where $\theta, \eta \in \mathfrak{G}$.

Proof. In fact

$$\int d\theta f(\theta+\eta) = \frac{\partial}{\partial\theta} f(\theta+\eta) = \frac{\partial}{\partial\theta}(f_0 + f_1(\theta+\eta)) = f_1 = \int d\theta f(\theta) \ .$$

q.e.d.

As a consequence, we have

$$\int d\theta f(\theta+\eta) = \int d(\theta+\eta) f(\theta+\eta) = \int d\theta f(\theta) \ .$$

Reality properties

A Grassmann variable can be either real, i.e. $\theta = \theta^*$, or complex, i.e. $\theta \neq \theta^*$. The product of two conjugate Grassmann variables is

$$(\theta_1\theta_2)^* = \theta_2^*\theta_1^* \ .$$

The complex conjugate of the product of 2 real Grassmann variables $\theta_1^* = \theta_1$ and $\theta_2^* = \theta_2$ is purely imaginary

$$(\theta_2\theta_1)^* = \theta_1^*\theta_2^* = \theta_1\theta_2 = -\theta_2\theta_1 \ ,$$

whereas if we multiply by i we obtain a real Grassmann variable

$$(i\theta_2\theta_1)^* = -i\theta_1^*\theta_2^* = -i\theta_1\theta_2 = i\theta_2\theta_1 \ .$$

We can always express complex Grassmann variables η in terms of real ones θ_1 and θ_2

$$\eta = \frac{\theta_1 + i\theta_2}{\sqrt{2}} \ , \quad \eta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}} \ .$$

Gaussian integration

For a single Grassmann variable θ, the Gaussian integral is trivial, since $\theta^2 = 0$ and

$$\int d\theta \exp(a\theta^2) = \int d\theta 1 = \frac{\partial}{\partial\theta} 1 = 0 \ .$$

We need at least 2 Grassmann variables θ_1 and θ_2 to have a non trivial Gaussian integral

$$\int d\theta_1 d\theta_2 \exp(-a\theta_1\theta_2) = \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}(1 - a\theta_1\theta_2) = \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}(1 + a\theta_2\theta_1) = \frac{\partial}{\partial\theta_1}(a\theta_1) = a \ .$$

We can rewritten this integral in terms of a skew-symmetric matrix A^{ij} defined as

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \ ,$$

so that it becomes

$$\int d\theta_1 d\theta_2 \exp(-\frac{1}{2}\theta_i A^{ij}\theta_j) = \overset{1/2}{\det} A^{ij} \ ,$$

where in our case $\det A = a^2 \geq 0$. Notice that in the bosonic case, we had $\det^{-1/2}$. Generalising for an even number of Grassmann variables $n = 2$, we have

$$\int d^n\theta \exp(-\frac{1}{2}\theta_i A^{ij}\theta_j) = \overset{1/2}{\det} A^{ij} \ ,$$

where $d^n\theta = d\theta_1 \ldots d\theta_n$ and A is a skew-symmetric block-diagonal matrix

$$A = \begin{bmatrix} 0 & -a_1 & 0 & \ldots & 0 \\ -a_1 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & 0 & 0 & a_n \\ 0 & \ldots & 0 & -a_n & 0 \end{bmatrix} \ .$$

whose determinant is $\det^{1/2} A = a_1 \ldots a_n$. The skew-diagonalisation can be made by an orthogonal matrix. For complex variables η_i, we have

$$\int d^n\eta^* \int d^n\eta \exp(-\frac{1}{2}\eta_i^* A^{ij} \eta_j) = \det A^{ij} \ ,$$

where $\int d^n\eta^* \int d^n\eta = d\eta_1^* d\eta_1 \ldots d\eta_n^* d\eta_n$.

Chapter 14

Canonical quantisation

In order to find physical application, we consider an infinite-dimensional Grassman algebra $\mathfrak{G}_\infty$, in hypercondensed notation $\theta_i \to \theta(t)$.

14.1 Fermionic harmonic oscillator

Consider an harmonic oscillator with mass $m = 1$ and frequency ω. Its action is

$$S[x,p] = \int dt (p\dot{x} - \frac{1}{2}(p^2 + m^2 x^2)) \ .$$

We can make a change of basis, using the ladder operators

$$a = \frac{\omega x + ip}{\sqrt{2\omega}} \ , \quad a^* = \frac{\omega x - ip}{\sqrt{2\omega}} \ ,$$

hence

$$S[a, a^*] = \int dt (i a^* \dot{a} - \omega a^* a) + \text{total derivatives} \ .$$

We promote it to operators $\hat{a}$ and $\hat{a}^\dagger$ via the commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$ living in a Fock space.

Now, we consider a fermionic harmonic oscillator, described by complex Grassmann variables $\psi(t)$ and $\psi^*(t)$, with a similar action

$$S[\psi, \psi^*] = \int dt (i\psi^* \dot{\psi} - \omega \psi^* \psi) \ .$$

The equation of motion is

$$i\dot{\psi} - \omega\psi = 0 \ .$$

Notice that it is similar to the time components of the Dirac equation, with $m = \omega$ and $\gamma^0 = -i$.

Now, we use the canonical quantisation method in phase space. The conjugate momentum is

$$\pi = \frac{\partial L}{\partial \psi} = -i\psi^* \ .$$

The Poisson brackets are

$$\{\pi, \psi\}_{PB} = -1 \ , \quad \{\pi, \psi\}_{PB} = -i \ .$$

In the language of second quantisation, we have

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1 \ , \quad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0 \ .$$

By the Fock construction, we can build the space in which the states live, which is the irreducible representation for the Grassmann algebra. The vacuum $|0\rangle$ is the state such that

$$\hat{\psi}|0\rangle = \langle 0|\hat{\psi}^\dagger = 0 \ , \quad \langle 0|0\rangle = 1 \ .$$

The first excited state $|1\rangle$ is the state such that

$$|1\rangle = \hat{\psi}^\dagger|0\rangle \ , \quad \langle 1| = \langle 0|\hat{\psi} \ , \quad \langle 0|1\rangle = \langle 0|\hat{\psi}^\dagger|0\rangle = 0 \ , \quad \langle 1|1\rangle = \langle 0|\hat{\psi}\hat{\psi}^\dagger|0\rangle = \langle 0|\underbrace{\{\hat{\psi}, \hat{\psi}^\dagger\}}_{1}|0\rangle + \langle 0|\hat{\psi}^\dagger \underbrace{\hat{\psi}|0\rangle}_{0} =$$

Notice that we cannot build anymore kets since we would have

$$|2\rangle = \underbrace{(\hat{\psi}^\dagger)^2}_{0}|0\rangle = 0 \ .$$

Therefore, the Fock space is

$$\mathcal{F}_2 = \mathrm{span}\{|0\rangle, |1\rangle\} \ .$$

The basis is orthonormal $braketmn = \delta_{mn}$, with $m, n = 0, 1$. We can write the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ as matrices

$$\hat{\psi} = \begin{bmatrix} \langle 0|\underbrace{\hat{\psi}|0\rangle}_{0} & \langle 0|\underbrace{\hat{\psi}|1\rangle}_{|0\rangle} \\ \langle 1|\underbrace{\hat{\psi}|0\rangle}_{0} & \langle 1|\underbrace{\hat{\psi}|1\rangle}_{|0\rangle} \end{bmatrix} = \begin{bmatrix} 0 & \underbrace{\langle 0|0\rangle}_{1} \\ 0 & \underbrace{\langle 1|0\rangle}_{0} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ , \quad \hat{\psi}^\dagger = \begin{bmatrix} \langle 0|\underbrace{\hat{\psi}^\dagger|0\rangle}_{|1\rangle} & \langle 0|\underbrace{\hat{\psi}^\dagger|1\rangle}_{0} \\ \langle 1|\underbrace{\hat{\psi}^\dagger|0\rangle}_{|1\rangle} & \langle 1|\underbrace{\hat{\psi}^\dagger|1\rangle}_{0} \end{bmatrix} = \begin{bmatrix} \underbrace{\langle 0|1\rangle}_{0} & \\ \underbrace{\langle 1|1\rangle}_{1} & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, they satisfy the anticommutation relations.

Proof. For the first,

$$\{\hat{\psi}, \hat{\psi}\} = \hat{\psi}\hat{\psi} + \hat{\psi}\hat{\psi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 + 0 = 0 \ .$$

For the second,

$$\{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = \hat{\psi}^\dagger\hat{\psi}^\dagger + \hat{\psi}^\dagger\hat{\psi}^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

For the third,

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = \hat{\psi}\hat{\psi}^\dagger + \hat{\psi}^\dagger\hat{\psi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

q.e.d.

14.2 Hamiltonian structure of bosonic phase space

The hamiltonian action is written as

$$S[x,p] = \int dt(p\dot{x} - H(x,p)) ,$$

where the first term is the sympletic one, which gives rise to the Poisson brackets. In fact, we can generally rewritten the action as

$$S[z] = \int dt(\frac{1}{2}z^a(\Omega^{-1})_{ab}\dot{z}^b - H(z)) ,$$

where it is in first order in time since the Hamilton equations are and Ω_{ab} is an invertible antisymmetric constant-valued matrix. The latter in canonical coordinates is

$$\Omega_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad (\Omega^{-1})_{ab} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Given 2 functions $f(z)$ and $g(z)$, the Poisson brackets are defined as

$$\{f(z), g(z)\}_{PB} = \frac{\partial f}{\partial z^a}\Omega^{ab}\frac{\partial g}{\partial z^b} .$$

It satisfies the following properties

1. antysymmetry, i.e.
$$\{f,g\}_{PB} = -\{g,f\}_{PB} ,$$
2. Leibniz rule, i.e.
$$\{f,gh\}_{PB} = \{f,g\}_{PB}h + g\{f,h\}_{PB} ,$$
3. Jacobi identity, i.e.
$$\{f,\{g,h\}_{PB}\}_{PB} + \{g,\{h,f\}_{PB}\}_{PB} + \{h,\{f,g\}_{PB}\}_{PB} .$$

Notice that $\{z^a, z^b\}_{PB} = \Omega^{ab}$.

In canonical quantisation, we promote the Poisson brackets to commutators of operators

$$[\hat{z}^a, \hat{z}^b] = i\hbar\Omega^{ab} \ .$$

and then you need to find the irreducible representation of this algebra to see the space in which states live.

14.3 Hamiltonian structure of fermionic phase space

Generalising for Grassmann variables, we have

$$Z^A = (z^a, \theta^\alpha) \ ,$$

where z^a are bosonic coordinates, θ^α are fermionic ones. The hamiltonian action is

$$S[z] = \int dt(\frac{1}{2}Z^A(\Omega^{-1})_{AB}\dot{Z}^B - H(Z)) \ ,$$

where all terms must be Grassmann evens. The graded-sympletic matrix Ω is

$$\Omega^{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ ,$$

where the bosonic block is antisymmetric and the fermionic block is symmetric.

Proof. If the bosonic block were symmetric, we would have a total derivative, which in the action it is a boundary term

$$\frac{1}{2}\begin{bmatrix} x & p \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = p\dot{x} + \dot{x}p = \frac{d(px)}{dt} \ .$$

Instead, we have

$$\frac{1}{2}\begin{bmatrix} x & p \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = p\dot{x} - \dot{x}p \neq \frac{d(px)}{dt}$$

If the fermionic block were antisymmetric, we would have a total derivative, which in the action it is a boundary term

$$\frac{1}{2}\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1 \end{bmatrix} = \theta_1\dot{\theta}_2 - \dot{\theta}_2\theta_1 = \theta_1\dot{\theta}_2 + \theta_1\dot{\theta}_2 = \frac{d(\theta_1\theta_2)}{dt} \ .$$

Instead, we have

$$\frac{1}{2}\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1 \end{bmatrix} = \theta_1\dot{\theta}_2 + \dot{\theta}_2\theta_1 = \theta_1\dot{\theta}_2 - \theta_1\dot{\theta}_2 \neq \frac{d(\theta_1\theta_2)}{dt} \ .$$

q.e.d.

Given 2 functions $f(Z)$ and $g(Z)$, the Poisson brackets are defined as

$$\{f(Z), g(Z)\}_{PB} = \frac{\partial_r f}{\partial Z^A}\Omega^{AB}\frac{\partial_l g}{\partial Z^B} \ .$$

It satisfies the following properties

1. graded antysymmetry, i.e.

$$\{f, g\}_{PB} = (-1)^{\epsilon_f \epsilon_g + 1}\{g, f\}_{PB} \ ,$$

where

$$\epsilon_f = \begin{cases} 0 & \text{if f is bosonic} \\ 1 & \text{if f is fermionic} \end{cases}$$

2. Leibniz rule, i.e.

$$\{f, gh\}_{PB} = \{f, g\}_{PB} h + (-1)^{\epsilon_f \epsilon_g} g\{f, h\}_{PB} \ ,$$

3. Jacobi identity, i.e.

$$\{f, \{g, h\}_{PB}\}_{PB} + (-1)^{\epsilon_f(\epsilon_g+\epsilon_h)}\{g, \{h, f\}_{PB}\}_{PB} + (-1)^{\epsilon_h(\epsilon_f+\epsilon_g)}\{h, \{f, g\}_{PB}\}_{PB} \ .$$

Notice that $\{Z^A, Z^B\}_{PB} = \Omega^{AB}$.

In canonical quantisation, we promote the Poisson brackets to graded commutators of operators

$$[\hat{Z}^A, \hat{Z}^B\} = i\hbar\Omega^{AB} \ .$$

where the graded commutator is defined as

$$[\hat{A}, \hat{B}\} = \begin{cases} [\hat{A}, \hat{B}] & \text{if at least there is one bosonic operator} \\ \{\hat{A}, \hat{B}\} & \text{if both are fermionic operators} \end{cases}$$

and then you need to find the irreducible representation of this algebra to see the space in which states live.

Examples

Consider a single real Grassmann variable ψ, called also a Maiorana fermion in $0+1$ dimension. The action is

$$S[\psi] = \int dt \frac{i}{2}\psi\dot{\psi} \ ,$$

where we have put i in order to have a real action and no hamiltonian because the only most simple quadratic term vanishes $\psi^2 = 0$. Therefore, the graded sympletic action is $\Omega^{-1} = i$ or $\Omega = -i$ and the canonical quantised anticommutator is

$$\{\hat{\psi}, \hat{\psi}\} = i\hbar(-i) = \hbar \ .$$

Finally, the only possibility is that $\hat{\psi}$ is a number

$$\hat{\psi} = \sqrt{\frac{\hbar}{2}} \ .$$

The Fock space is consistuted by only the vacuum state $|0\rangle$.

Now, consider n real Grassmann variables ψ^i. The action is

$$S[\psi] = \int dt \frac{i}{2} \psi^i \delta_{ij} \dot{\psi}^j - H(\psi^i) \ .$$

Therefore, the graded sympletic action is $(\Omega^{-1})_{ij} = i\delta_{ij}$ or $\Omega^{ij} = -i\delta^{ij}$ and the canonical quantised anticommutator is

$$\{\hat{\psi}^i, \hat{\psi}^j\} = i\hbar(-i\delta^{ij}) = \hbar\delta^{ij} \ .$$

Notice that, if we define $\hat{\psi} = \sqrt{\frac{\hbar}{2}}$, we recover the Clifford algebra $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. We need to find the irreducible representation of this algebra to see the space in which states live. The dimension is

$$\dim \mathcal{F} = 2^{[n/2]} \times 2^{[n/2]} \ ,$$

where $[n/2]$ is the integer part. The Fock space has dimension 2 if $n = 2, 3$, dimension $4, 5$ if $n = 4, 5$, etc.

Now, consider a complex Grassmann variables ψ and ψ^*. The action is

$$S[\psi, \psi^*] = \int dt (i\psi^* \dot{\psi} - H(\psi, \psi^*)) \ .$$

Therefore, the graded sympletic action is $\Omega^{ij} = -i$ and the canonical quantised anticommutator is

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = i\hbar(-i) = \hbar \ .$$

Hence, we have recovered the fermionic harmonic oscillator and wit hthe same procedure of the Fock construction, we can build the Fock space.

Chapter 15

Coherent states

It is useful to introduce a change of basis for the Hilbert space, using the coherent states.

15.1 Bosonic coherent states

Recall that the bosonic harmonic oscillator has ladder operators

$$[\hat{a}, \hat{a}^\dagger] = 1 \ , \quad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \ .$$

The infinite-dimensional Fock space can be expressed by an orthonormal diagonal basis $|0\rangle, |1\rangle, \ldots |n\rangle, \ldots$ such that

$$\hat{a}|0\rangle = 0 \ , \quad \hat{a}^\dagger|0\rangle = |1\rangle \ , \quad \ldots \ , \quad \frac{(\hat{a}^\dagger)^n}{n!}|0\rangle = |n\rangle \ , \quad \ldots \ .$$

The coherent states are defined as eigenstates of the annihilation operator

$$\hat{a}|a\rangle = a|a\rangle \ , \quad \langle a|\hat{a}^\dagger = \langle a|a \ ,$$

where $a \in \mathbb{C}$. An explicit realisation can be

$$|a\rangle = \exp(a\hat{a}^\dagger)|0\rangle \ , \quad \langle a^*| = \langle 0|\exp(a^*\hat{a})$$

Proof. In fact

$$\begin{aligned}|a\rangle = |a\rangle &= \exp(a\hat{a}^\dagger)|0\rangle \\ &= (1 + a\hat{a}^\dagger + \ldots + \frac{(a\hat{a}^\dagger)^n}{n!} + \ldots)|0\rangle \\ &= (|0\rangle + a|1\rangle + \ldots + a^n|n\rangle + \ldots) \ ,\end{aligned}$$

hence

$$\hat{a}|a\rangle = \hat{a}(|0\rangle + a|1\rangle + \ldots + a^n|n\rangle + \ldots) = 0 + a|0\rangle + \ldots + \frac{a^n}{\sqrt{(n-1)!}}|n-1\rangle + \ldots = a(|0\rangle + \ldots +$$

q.e.d.

They satisfy the following properties

1. scalar product
$$\langle a^*|a\rangle = \exp(a^* a) \ ,$$
2. resolution of the identity, i.e.
$$\int \frac{da^* da}{2\pi i} \exp(-a^* a)|a\rangle\langle a^*| \ ,$$
3. trace, i.e.
$$\mathrm{tr}\, \hat{A} = \int \frac{da^* da}{2\pi i} \langle a^*|\hat{A}|a\rangle \ .$$

Proof. Maybe in the future. q.e.d.

15.2 Fermionic coherent states

The fermionic harmonic oscillator has ladder operators

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1 \ , \quad \{\hat{a}, \hat{a}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0 \ .$$

The Fock space is only spanned by $|0\rangle$ and $|1\rangle$.

The coherent states are defined as eigenstates of the annihilation operator

$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle \ , \quad \langle\psi^*|\hat{\psi}^\dagger = \langle\psi^*|\psi^* \ ,$$

where ψ is a complex Grassmann variable. An explicit realisation can be

$$|\psi\rangle = \exp(\hat{\psi}^\dagger\psi)|0\rangle \ , \quad \langle\psi^*| = \langle 0| \exp(\psi^*\hat{\psi})$$

Proof. In fact

$$|\psi\rangle = \exp(\hat{\psi}^\dagger\psi)|0\rangle = (1 + \hat{\psi}^\dagger\psi)|0\rangle = |0\rangle - \psi|1\rangle \ ,$$

hence

$$\hat{\psi}|\psi\rangle = \hat{\psi}(|0\rangle - \psi|1\rangle) = \psi\hat{\psi}|1\rangle = \psi|0\rangle = \psi(|0\rangle - \psi|1\rangle) = \psi|\psi\rangle \ ,$$

where we have used the fact that $\psi^2 = 0$. q.e.d.

They satisfy the following properties

1. scalar product
$$\langle \psi^* | \psi \rangle = \exp(\psi^* \psi) \ ,$$

2. resolution of the identity, i.e.
$$\int d\psi^* d\psi \exp(-\psi^* \psi) |\psi\rangle\langle\psi^*| \ ,$$

3. trace, i.e.
$$\mathrm{tr}\, \hat{A} = \int d\psi^* d\psi \langle -\psi^* | \hat{A} | \psi \rangle \ .$$

4. supertrace, i.e.
$$\mathrm{str}\, \hat{A} = \int d\psi^* d\psi \langle \psi^* | \hat{A} | \psi \rangle \ .$$

Proof. Maybe in the future. q.e.d.

Chapter 16

Fermionic path integral

Consider a normal ordered hamiltonian $\hat{H}(\hat{\psi}, \hat{\psi}^\dagger)$, which can be $\hat{H} = \omega_0 + \omega\hat{\psi}^\dagger\hat{\psi}$. The transition amplitude between the initial state $|\psi_i\rangle$ and the final state $\langle\psi_f^*|$ is

$$\begin{aligned}
\langle\psi_f^*|\exp(-i\hat{H}T)|\psi_i\rangle &= \langle\psi_f^*|\exp(-i\hat{H}\frac{T}{N}N)|\psi_i\rangle \\
&= \langle\psi_f^*|\exp(-i\hat{H}\epsilon)\ldots\exp(-i\hat{H}\epsilon)|\psi_i\rangle \\
&= \langle\psi_f^*|\exp(-i\hat{H}\epsilon)\mathbb{I}\ldots\mathbb{I}\exp(-i\hat{H}\epsilon)|\psi_i\rangle \\
&= \int(\prod_{k=1}^{N-1} d\psi_k^* d\psi\exp(-\psi_k^*\psi_k))\prod_{k=1}^{N}\langle\psi_k^*|\exp(-i\hat{H}\epsilon)|\psi_{k-1}\rangle \\
&= \int(\prod_{k=1}^{N-1} d\psi_k^* d\psi\exp(-\psi_k^*\psi_k))\prod_{k=1}^{N}\langle\psi_k^*|(1-i\epsilon\hat{H}+\ldots)|\psi_{k-1}\rangle \\
&= \int(\prod_{k=1}^{N-1} d\psi_k^* d\psi\exp(-\psi_k^*\psi_k))\prod_{k=1}^{N}\langle\psi_k^*|\psi_{k-1}\rangle(1-i\epsilon H(\psi_k^*,\psi_{k-1})+\ldots) \\
&= \int(\prod_{k=1}^{N-1} d\psi_k^* d\psi\exp(-\psi_k^*\psi_k))\prod_{k=1}^{N}\exp(\psi_k^*\psi_{k-1})\exp(-i\epsilon H(\psi_k^*,\psi_{k-1}))\ ,
\end{aligned}$$

hence

$$
\begin{aligned}
&\lim_{N\to\infty}\int(\prod_{k=1}^{N-1} d\psi_k^* d\psi \exp(-\psi_k^*\psi_k))\prod_{k=1}^{N}\exp(\psi_k^*\psi_{k-1})\exp(-i\epsilon H(\psi_k^*,\psi_{k-1}))\\
&=\lim_{N\to\infty}\int(\prod_{k=1}^{N-1} d\psi_k^* d\psi \exp(-\psi_k^*\psi_k))\exp(\sum_{k=1}^{N} psi_k^*\psi_{k-1} - i\epsilon H(\psi_k^*,\psi_{k-1}) + \psi_k^*\psi_k - \psi_k^*\psi_k)\\
&=\lim_{N\to\infty}\int(\prod_{k=1}^{N-1} d\psi_k^* d\psi \exp(-\psi_k^*\psi_k))\exp(\sum_{k=1}^{N} -psi_k^*\frac{\psi_k-\psi_{k-1}}{\epsilon} - i\epsilon H(\psi_k^*,\psi_{k-1}) + \psi_N^*\psi_N)\\
&=\int \mathcal{D}\psi^*\mathcal{D}\psi \exp(i\int dt(i\psi^*\dot{\psi} - H(\psi^*,\psi)) + \psi^*(T)\psi(T))\\
&=\int \mathcal{D}\psi^*\mathcal{D}\psi \exp(iS[\psi^*,\psi])\ ,
\end{aligned}
$$

where the action is

$$
S[\psi,\psi^*] = \int_0^T dt(i\psi^*\dot{\psi} - H(\psi,\psi^*)) - i\psi^*(T)\psi(T)\ .
$$

The last term is there to ensure the only allowed boundary conditions $\psi(0) = \psi_i$ and $\psi^*(T) = \psi_f$.