Conformal Field Theory

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Chapter 1

Conformal Symmetry

Conformal symmetry is defined as an angle-preserving symmetry in the complex plane. However, this definition is not wrong but isn't very useful either. It requires us to define what constitutes an angle to fully comprehend it. There has been discussion in the literature where physicists have argued that such a definition only makes sense when we are only concerned with spatial directions. In special relativity and beyond, one must incorporate time as well, and the notion of an angle with time is not a very well-defined object. Therefore, an alternative that also holds for angle-preserving transformations is given by:

$$x \to \lambda x$$

Note that, this is **not** a general coordinate transformation which relabels the coordinates but it is actually changing the underlying geometry. Under this transformation, we observe:

$$\begin{aligned}\cos\theta &= \frac{\vec{x}\cdot\vec{y}}{|x||y|} \\ &= \frac{g_{\mu\nu}x^{\mu}y^{\nu}}{\sqrt{g_{\mu\nu}x^{\mu}x^{\nu}}\sqrt{g_{\mu\nu}y^{\mu}y^{\nu}}}\end{aligned}$$

under conformal transformation $g_{\mu\nu} \to \Omega(x)g_{\mu\nu}$

$$\begin{aligned}&= \frac{\Omega(x)g_{\mu\nu}x^{\mu}y^{\nu}}{\sqrt{\Omega(x)g_{\mu\nu}x^{\mu}x^{\nu}}\sqrt{\Omega(x)g_{\mu\nu}y^{\mu}y^{\nu}}} \\ &= \cos\theta\end{aligned}$$

Therefore, conformal transformation could be defined as one which scales the metric and as such preserves the angles.

$$g'_{\mu\nu}(x') = \Omega(x)g_{\mu\nu}(x) \qquad \text{(with } \Omega(x) > 0\text{)}$$

The approach to studying a Conformal Field Theory (CFT) is very different from that of Quantum Field Theory (QFT). In QFT, we typically write the Lagrangian first and then study the equations of motion. However, this is rarely the case in CFT. Instead, we focus little on writing the Lagrangian but rather utilize the conformal symmetry directly to guess the functional form of the correlation function. This approach is known as the conformal bootstrap.

1.1 Infinitesimal Conformal Transformation

The fundamental essence of conformal transformations resides in their infinitesimal form, which serves as a crucial tool for investigating how fields transform under these symmetries. It plays a pivotal role in defining the generator of the conformal group and, subsequently, constraining the set of possible correlators that are compatible with conformal symmetry. Any infinitesimal transformation can be expressed as:

$$x'^{\mu} = x^{\mu} + \underbrace{\epsilon^{\mu}(x)}_{\text{infinitesimal}}$$

and subsequently,

$$x^{\mu} = x'^{\mu} - \epsilon^{\mu}(x)$$

therefore, the metric transforms like:

$$
\begin{aligned}
g'_{\mu\nu} &= \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}}_{\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha(x)} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\
&= \left[\delta^\alpha_\mu - \frac{\partial \epsilon^\alpha(x)}{\partial x'^\mu}\right]\left[\delta^\beta_\nu - \frac{\partial \epsilon^\beta(x)}{\partial x'^\nu}\right] g_{\alpha\beta} \\
&= \delta^\alpha_\mu \delta^\beta_\nu g_{\alpha\beta} - \delta^\beta_\nu \partial_\mu \epsilon^\alpha(x) g_{\alpha\beta} - \delta^\alpha_\mu \partial_\nu \epsilon^\beta(x) g_{\alpha\beta} + \mathcal{O}(\epsilon^2) \\
\Omega(x) g_{\mu\nu} &= g_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu
\end{aligned}
$$

In the third step, we used chain rule on $\epsilon^\alpha(x)$ and ignored $\mathcal{O}((\partial\epsilon)^2)$ terms. From the last line, it is reasonable to expect that:

$$
\begin{aligned}
\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &\propto g_{\mu\nu} \\
\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) &= f(x) g_{\mu\nu}
\end{aligned} \tag{1.1}
$$

Contracting Indicies

$$
\begin{aligned}
\partial^\mu \epsilon_\mu(x) + \partial^\mu \epsilon_\mu(x) &= f(x)\delta^\mu_\mu \\
2(\partial \cdot \epsilon) &= \underset{\uparrow\text{dimension of spacetime}}{d}\, f(x) \\
f(x) &= \frac{2}{d}\frac{\partial \epsilon_\mu(x)}{\partial x_\mu}
\end{aligned}
$$

Substituing back in (1.1)

$$
\boxed{\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \frac{2}{d}(\partial \cdot \epsilon)\ g_{\mu\nu}} \tag{1.2}
$$

Now, we operate by ∂^ν

$$
\begin{aligned}
\frac{\partial}{\partial x'_\nu}\left[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)\right] &= \frac{\partial}{\partial x'_\nu}\left(\frac{2}{d}\partial \cdot \epsilon(x)\ \overset{\text{assuming flat metric}\downarrow}{g_{\mu\nu}}\right) \\
\partial_\mu \underbrace{\partial^\nu \epsilon_\nu}_{\partial\cdot\epsilon} + \underbrace{\partial^\nu \partial_\nu}_{\Box} \epsilon_\mu &= \frac{2}{d} g_{\mu\nu} \partial^\nu \partial \cdot \epsilon \\
\partial_\mu(\partial \cdot \epsilon) + \Box\epsilon &= \frac{2}{d}\partial_\mu(\partial \cdot \epsilon)
\end{aligned}
$$

Operating by ∂_ν

$$
\partial_\nu[\partial_\mu(\partial \cdot \epsilon) + \Box\epsilon] = \partial_\nu\left[\frac{2}{d}\partial_\mu(\partial \cdot \epsilon)\right]
$$

$$
\left(1 - \frac{2}{d}\right)\partial_\nu \partial_\mu(\partial \cdot \epsilon) + \Box(\partial_\nu \epsilon_\mu) = 0 \tag{1.3}
$$

under relabeling $\mu \leftrightarrow \nu$

$$
\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box(\partial_\mu \epsilon_\nu) = 0 \tag{1.4}
$$

adding (1.3) and (1.4)

$$
\begin{aligned}
2\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box\underbrace{[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)]}_{\frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu}} &= 0 \\
\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \frac{1}{d}\Box(\partial \cdot \epsilon) g_{\mu\nu} &= 0 \\
\left[g_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu\right](\partial \cdot \epsilon) &= 0
\end{aligned} \tag{1.5}
$$

Contracting the indicies

$$
[d\Box + (d-2)\Box](\partial \cdot \epsilon) = 0
$$

$$2(d-1)\Box(\partial\cdot\epsilon)=0$$

hence,

$$\boxed{(d-1)\Box(\partial\cdot\epsilon)=0} \tag{1.6}$$

if $d=1 \implies$ any $\epsilon^\mu(x)$ satisfies (1.6), therefore, is conformal transformation. It is interesting to note that any 1D QFT is conformal field theory, but for our purpose it's not very useful. We will be concerned with $d\neq 1$ for the rest of this notes unless stated otherwise. Consider the action of ∂_α on (1.2) and then cyclic relabeling of incidies as $\alpha\to\mu\to\nu$

$$\partial_\alpha[\partial_\mu\epsilon_\nu(x)+\partial_\nu\epsilon_\mu(x)]=\frac{2}{d}\partial_\alpha g_{\mu\nu}(\partial\cdot\epsilon) \tag{1.7}$$

$$\partial_\mu[\partial_\nu\epsilon_\alpha(x)+\partial_\alpha\epsilon_\nu(x)]=\frac{2}{d}\partial_\mu g_{\nu\alpha}(\partial\cdot\epsilon) \tag{1.8}$$

$$\partial_\nu[\partial_\alpha\epsilon_\mu(x)+\partial_\mu\epsilon_\alpha(x)]=\frac{2}{d}\partial_\nu g_{\alpha\mu}(\partial\cdot\epsilon) \tag{1.9}$$

Adding the first two equation and subtracting from the last, we get [(1.7) + (1.8) − (1.9)]:

$$\cancel{2}\partial_\alpha\partial_\mu\epsilon_\nu=\frac{\cancel{2}}{d}[g_{\mu\nu}\partial_\alpha+g_{\nu\alpha}\partial_\mu-g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon)$$

$$\partial_\alpha\partial_\mu\epsilon_\nu=\frac{1}{d}[g_{\mu\nu}\partial_\alpha+g_{\nu\alpha}\partial_\mu-g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon) \tag{1.10}$$

Referring to eqn 1.7.11 of "Ideas and Methods of Supersymmetry and Supergravity" by Sergio M. Kuzenko, we find that the 3rd order derivation of $\epsilon^\mu(x)$ vanishes. Therefore, the most general conformal transformation is of the type:

$$x'^\mu=x^\mu+\underbrace{a^\mu+b^\mu{}_\nu x^\nu+c^\mu{}_{\nu\alpha}x^\nu x^\alpha}_{\epsilon^\mu}$$

Where, $a^\mu, b^\mu{}_\nu$ and $c^\mu{}_{\nu\alpha}$ are parameters relevant to their transformation. The goal here is simple:

- First find the relevant transformations
- Then based on the transformation rule, find the generators.

For $\epsilon^\mu=a^\mu$:

$$\begin{aligned}x'^\mu&=x^\mu+a^\mu\\&=x^\mu+\delta^\mu_\nu a^\nu\\&=x^\mu+(\partial_\nu x^\mu)a^\nu\\&=[1+ia^\nu(-i\partial_\nu)]x^\mu\end{aligned}$$

Thus, the generator of translation is $P_\mu-i\partial_\mu$[1]. For $\epsilon^\mu=b^\mu{}_\alpha x^\alpha$, we refer to (1.2)

$$\begin{aligned}\partial_\mu\epsilon_\nu(x)+\partial_\nu\epsilon_\mu(x)&=\frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu}\\ \partial_\mu(b_{\nu\alpha}x^\alpha)+\partial_\nu(b_{\mu\alpha}x^\alpha)&=\frac{2}{d}(\partial^\mu b_{\mu\alpha}x^\alpha)g_{\mu\nu}\\ b_{\nu\alpha}\delta^\alpha_\mu+b_{\mu\alpha}\delta^\alpha_\nu&=\frac{2}{d}(b_{\mu\alpha}g^{\alpha\mu})g_{\mu\nu}\\ b_{\nu\mu}+b_{\mu\nu}&=\frac{2}{d}b^\alpha{}_\alpha g_{\mu\nu}\\ \frac{b_{\nu\mu}+b_{\mu\nu}}{2}&=\frac{1}{d}b^\alpha{}_\alpha g_{\mu\nu}\end{aligned}$$

now,

$$\begin{aligned}b_{\mu\nu}&=\frac{b_{\mu\nu}-b_{\nu\mu}}{2}+\frac{b_{\mu\nu}+b_{\nu\mu}}{2}\\&=M_{\mu\nu}+\lambda g_{\mu\nu}\end{aligned}$$

If $b_{\mu\nu}=\lambda g_{\mu\nu}(M_{\mu\nu}=0)$

$$x'^\mu=x_\mu+b^\mu{}_\nu x^\nu$$

[1] if we use $[1-a^\nu(\partial_\nu)]x^\mu$ as the definition, then $P_\mu=i\partial_\mu$ would be the generator

$$
\begin{aligned}
&= x^\mu + \lambda g^{\mu\alpha} \underbrace{g_{\alpha\nu} x^\nu}_{x_\alpha} \\
&= x^\mu + \lambda x^\mu \\
&= x^\mu + \lambda x^\nu \delta^\mu_\nu \\
&= x^\mu + \lambda x^\nu (\partial_\nu x^\mu) \\
&= x^\mu + i\lambda x^\nu (-i\partial_\nu x^\mu) \\
&= (1 + i\lambda(-ix^\nu \partial_\nu)) x^\mu
\end{aligned}
$$

Thus, the generator of dilatation is $D = -ix^\mu \partial_\mu$. For $b_{\mu\nu} = M_{\mu\nu}(\lambda = 0)$.

$$
\begin{aligned}
x'^\mu &= x^\mu + M^\mu{}_\nu x^\nu \\
&= x^\mu + M^\alpha{}_\nu \delta^\mu_\alpha x^\nu \\
&= x^\mu + M^\alpha{}_\nu (\partial_\alpha x^\mu) x^\nu \\
&= x^\mu + M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu \\
&= x^\mu + \frac{M_{\alpha\nu} - M_{\nu\alpha}}{2} (\partial^\alpha x^\mu) x^\nu \\
&= x^\mu + \frac{1}{2} M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu - \overset{\text{relabeling } \nu \leftrightarrow \alpha}{\boxed{\frac{1}{2} M_{\nu\alpha} (\partial^\alpha x^\mu) x^\nu}} \\
&= x^\mu + \frac{1}{2} M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu - \frac{1}{2} M_{\alpha\nu} (\partial^\nu x^\mu) x^\alpha \\
&= x^\mu + \frac{1}{2} M_{\alpha\nu} (x^\nu \partial^\alpha - x^\alpha \partial^\nu) x^\mu \\
&= x^\mu + \frac{i}{2} M_{\alpha\nu} \left\{ -i(x^\nu \partial^\alpha - x^\alpha \partial^\nu) \right\} x^\mu \\
&= x^\mu + \frac{i}{2} M_{\alpha\nu} \{ \underbrace{i(x^\alpha \partial^\nu - x^\nu \partial^\alpha)}_{L^{\alpha\nu}} \} x^\mu
\end{aligned}
$$

Thus, the generator of rotation is $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Now, the last part $\epsilon^\mu = c^\mu{}_{\nu\alpha} x^\nu x^\alpha = c^\mu{}_{\alpha\nu} x^\nu x^\alpha$, we refer to (1.10):

$$
\begin{aligned}
\partial_\alpha \partial_\mu \epsilon_\nu &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] (\partial \cdot \epsilon) \\
\partial_\alpha \partial_\mu (c_{\nu\sigma\beta} x^\sigma x^\beta) &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] \partial^\mu (c_{\mu\sigma\beta} x^\sigma x^\beta) \\
c_{\nu\sigma\beta} \partial_\alpha (\delta^\sigma_\mu x^\beta + x^\sigma \delta^\beta_\mu) &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] c_{\mu\sigma\beta} (g^{\sigma\mu} x^\beta + x^\sigma g^{\beta\mu}) \\
c_{\nu\sigma\beta} (\delta^\sigma_\mu \delta^\beta_\alpha + \delta^\sigma_\alpha \delta^\beta_\mu) &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] (c^\sigma{}_{\sigma\beta} x^\beta + c^\beta{}_{\sigma\beta} x^\sigma) \\
2 c_{\nu\mu\alpha} &= \frac{2}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] c^\sigma{}_{\sigma\beta} x^\beta \\
c_{\nu\mu\alpha} &= \underbrace{\frac{1}{d} c^\sigma{}_{\sigma\beta}}_{b_\beta} [g_{\mu\nu} \delta^\beta_\alpha + g_{\nu\alpha} \delta^\beta_\mu - g_{\alpha\mu} \delta^\beta_\nu] \\
&= g_{\nu\mu} b_\alpha + g_{\nu\alpha} b_\mu - g_{\mu\alpha} b_\nu
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\epsilon_\mu &= c_{\mu\alpha\beta} x^{\alpha\beta} \\
&= (g_{\mu\alpha} b_\beta + g_{\mu\beta} b_\alpha - g_{\alpha\beta} b_\mu) x^\alpha x^\beta \\
&= x_\mu (b \cdot x) + x_\mu (b \cdot x) - b_\mu (x \cdot x) \\
&= 2x_\mu (b \cdot x) - x^2 b_\mu
\end{aligned}
$$

Hence, the Special Conformal Transformation looks like:

$$
\begin{aligned}
x'^\mu &= x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu \\
&= x^\mu + 2(b \cdot x) x^\nu \delta^\mu_\nu - x^2 b^\nu \delta^\mu_\nu \\
&= x^\mu + 2(b \cdot x) x^\nu \partial_\nu x^\mu - x^2 b^\nu \partial_\nu x^\mu
\end{aligned}
$$

$$
\begin{aligned}
&= [1 + 2(b \cdot x)x^\nu \partial_\nu - x^2 b^\nu \partial_\nu] x^\mu \\
&= [1 + \{2b^\alpha x_\alpha x^\nu \partial_\nu - x^2 b^\alpha \partial_\alpha\}] x^\mu \\
&= [1 + ib^\alpha \{\underbrace{-i(2x_\alpha x^\nu \partial_\nu - x^2 \partial_\alpha)}_{K_\alpha}\}] x^\mu
\end{aligned}
$$

Hence, the generator for Special Conformal Transformations (SCT) takes the form $K^\mu = -i(2x^\mu x \cdot \partial - x^2 \partial^\mu)$. We will now list all the **infinitesimal** transformations and their generators we found in this section.

1. Translation

$$x'^\mu = x^\mu + a^\mu \qquad\qquad P_\mu = -i\partial_\mu \tag{1.11}$$

2. Rotation

$$x'^\mu = x^\mu + M^\mu{}_\nu x^\nu \qquad\qquad L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{1.12}$$

3. Dilatation

$$x'^\mu = (1 + \lambda) x^\mu \qquad\qquad D = -ix^\mu \partial_\mu \tag{1.13}$$

4. Special Conformal Transformation

$$x'^\mu = x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu \qquad\qquad K^\mu = -i(2x^\mu x \cdot \partial - x^2 \partial^\mu) \tag{1.14}$$

In the above listed transfomations, the parameters $a^\mu, M^\mu{}_\nu, \lambda$ and b^μ are all infinitesimal.

1.2 Finite Conformal Transformation

In the previous section, we considered the infinitesimal conformal transformation, however in this section we will consider the finite conformal transformation.

1. Translation

$$x'^\mu = x^\mu + \underset{\uparrow \text{finite vector}}{a^\mu} = e^{ia^\nu P_\nu} x^\mu$$

2. Dilatation

$$x'^\mu = \left(1 + \frac{\lambda}{N}\right) x^\mu$$

   In order to achieve the finite dilatation, we use the infinitesimal transformation recursively by dividing the finite λ into infinitely many λ/N pieces and then transforming

$$
\begin{aligned}
x'^\mu &= \left(1 + \frac{\lambda''}{N}\right) \underbrace{\left(1 + \frac{\lambda'}{N}\right)\left(1 + \frac{\lambda}{N}\right) x^\mu}_{x''^\mu} \\
&= \lim_{N \to \infty} \left(1 + \frac{\lambda}{N}\right)^N x^\mu \\
&= e^\lambda x^\mu = e^{i\lambda D} x^\mu
\end{aligned}
$$

3. Rotation

$$
\begin{aligned}
x'^\mu &= \left[1 + \frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta}\right]^\mu{}_\nu x^\nu \\
&= \left[e^{\frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta}}\right]^\mu{}_\nu x^\nu = \Lambda^\mu{}_\nu x^\nu
\end{aligned}
$$

4. The special conformal transformation

$$x'^\mu = x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu$$

nitesimal parameter, i.e. t is small.

let $b^\mu = t\, e^\mu$

$$x'^\mu(t) \equiv x^\mu(t) = x^\mu + 2t(e\cdot x)x^\mu - x^2 t e^\mu$$

To find the finite form of the transformation we have to recursively apply the above equation multiple times (Lie Algebra sence). The usual way is to integrate the infinitesimal form. The other way, and since we know that the transformations satisfy the conformal Killing equation, is to find the integral curve of the corresponding conformal Killing vector field as they are equivalent (Differential Geometry sense). Consider the $t-$derivative of the above[2].

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = 2(e\cdot x)x^\mu - x^2 e^\mu \tag{1.15}$$

defining $y^\mu(t) = \frac{x^\mu(t)}{x^2(t)}$

quotient rule

$$\begin{aligned}
\dot{y}^\mu(t) &= \frac{x^2\dot{x}^\mu - 2(\dot{x}\cdot x)x^\mu}{(x^2)^2} \\
&= \frac{x^2[2(e\cdot x)x^\mu - x^2e^\mu] - 2[2(e\cdot x)x^\nu - x^2e^\nu]x_\nu x^\mu}{x^4} \\
&= \frac{x^2[2(e\cdot x)x^\mu - x^2e^\mu] - 2[2(e\cdot x)x^2 - x^2(e\cdot x)]x^\mu}{x^4} \\
&= \frac{x^2[\cancel{2(e\cdot x)x^\mu} - x^2e^\mu] - \cancel{2(e\cdot x)x^2x^\mu}}{x^4} \\
\dot{y}^\mu(t) &= -e^\mu
\end{aligned}$$

Solving the above differential equation

$$\begin{aligned}
y^\mu(t) &= y^\mu(0) - te^\mu \\
\frac{x^\mu(t)}{x^2(t)} &= \frac{x^\mu(0)}{x^2(0)} - te^\mu
\end{aligned}$$

going back to the old notation $x'^\mu \equiv x^\mu(t)$

$$\begin{aligned}
\frac{x'^\mu}{x'^2} &= \frac{x^\mu}{x^2} - te^\mu \\
&= \frac{x^\mu}{x^2} - b^\mu
\end{aligned} \tag{1.16}$$

Squaring both sides

$$\begin{aligned}
\left(\frac{x'^\mu}{x'^2}\right)^2 \equiv \frac{x'^\mu}{x'^2}\frac{x'_\mu}{x'^2} &= \left(\frac{x^\mu}{x^2} - b^\mu\right)^2 \\
\frac{x'^2}{x'^4} &= \left(\frac{x^\mu}{x^2}\right)^2 + b^2 - \frac{2(x\cdot b)}{x^2} \\
\frac{1}{x'^2} &= \frac{1 + b^2x^2 - 2(x\cdot b)}{x^2} \\
x'^2 &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned} \tag{1.17}$$

referring to (1.16)

$$x'^\mu = x'^2\left[\frac{x^\mu}{x^2} - b^\mu\right]$$

and substituting (1.17)

$$\begin{aligned}
x'^\mu &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}\left[\frac{x^\mu}{x^2} - b^\mu\right] \\
&= \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned}$$

[2]when we consider the differential equation, we are no longer thinking of it as transformation but rather flow along a trajectory parameterized by t. This part was taken from pg 16 of "Four point function in momentum spaces and topological terms in gravity"

Above procedure also suggests that, finite SCT could be described as a sequence of inversion→translation→inversion. Where inversion is defined as:

$$I(x^\mu) = \frac{x^\mu}{x^2}$$

First we note that the inversion is a global conformal transformation and since it is undefined at origin, it does not have an infinitesimal part i.e. we can not expect inversion to be obtained by exponentiating an element from the conformal Lie algebra. Another interesting point to note is that there is no parameter associated with the transformation here such as λ for dilatation or b^μ for SCT. Lastly, it is also closely related to the stereographic projection. To show this let us study the stereographic projection of sphere onto a plane.

Let $x \in \mathbb{R}^n$, and define stereographic projection from the **north pole** to the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as:

$$X^i = \frac{2x^i}{1+|x|^2}, \quad X^{n+1} = \frac{|x|^2-1}{1+|x|^2}$$

where x^i are coordinates on the projected plane and X^i are the coordiantes on the sphere in embedding space. Now project this point on the sphere back to $\mathbb{R}^n$ via stereographic projection from the **south pole**:

$$x'^i = \frac{X^i}{1+X^{n+1}}$$

Substituting:

$$x'^i = \frac{\frac{2x^i}{1+|x|^2}}{1+\frac{|x|^2-1}{1+|x|^2}} = \frac{2x^i}{(1+|x|^2)+(|x|^2-1)} = \frac{2x^i}{2|x|^2} = \frac{x^i}{|x|^2}$$

Hence, the composition gives:

$$x^i \mapsto \frac{x^i}{|x|^2}$$

which is the **inversion** in the unit sphere.

Therefore, it is reasonable to find the killing vector for streographic projection. In general, we note that these two transformation would have the following form:

$$x'^\mu = \Omega(x)x^\mu$$

If $\partial_\mu\partial_\nu(\frac{1}{\Omega}) \propto g_{\mu\nu}$. The killing vector associated with it will have the form:

$$K^A{}_\mu = \frac{1}{\Omega^2}\frac{\partial x'^A}{\partial x^\mu}$$

Coming back to special conformal transformation which was the topic at hand, we now look at how they scale the metric tensor.

$$\frac{\partial x'^\mu}{\partial x^\nu} = \left\{\frac{\delta^\mu_\nu - 2b^\mu x_\nu}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\nu + 2b^2 x_\nu)}{\Lambda^2}\right\}$$

$$\begin{aligned}
g_{\alpha\beta}(x) = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g'_{\mu\nu}(x')\Big|_{x'=x'(x)} &= \left\{\frac{\delta^\mu_\alpha - 2b^\mu x_\alpha}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^2}\right\} \\
&\times \left\{\frac{\delta^\nu_\beta - 2b^\nu x_\beta}{\Lambda} - \frac{(x^\nu - b^\nu x^2)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^2}\right\} g'_{\mu\nu}(x') \\
&= \left\{\frac{\delta^\mu_\alpha - 2b^\mu x_\alpha}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\alpha + 2b^2 x'_\alpha)}{\Lambda^2}\right\} \\
&\times \left\{\frac{g'_{\mu\beta} - 2b_\mu x_\beta}{\Lambda} - \frac{(x_\mu - b_\mu x^2)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^2}\right\} \\
&= \frac{g'_{\alpha\beta} - 2b_\beta x_\alpha - 2b_\alpha x_\beta + 4b^2 x_\alpha x_\beta}{\Lambda^2} - \frac{(x_\beta - b_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&+ \frac{(2(b\cdot x)x_\beta - 2b^2 x_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&- \frac{(x_\alpha - 2b_\alpha x^2)(-2b_\alpha + 2b^2 x_\beta)}{\Lambda^3}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(2(b\cdot x)x_\alpha - 2b^2x^2x_\alpha)(-2b_\beta + 2b^2x_\beta)}{\Lambda^3}\\
&+\frac{(x^\mu - b^\mu x^2)(x_\mu - b_\mu x^2)(-2b_\alpha + 2b^2x_\alpha)(-2b_\beta + 2b^2x_\beta)}{\Lambda^4}\\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{(-2b_\beta x_\alpha - 2_\alpha x_\beta + 4b^2x_\alpha x_\beta)(1 - 2b\cdot x + b^2x^2)}{\Lambda^3}\\
&+\frac{1}{\Lambda^3}\left\{2b_\alpha x_\mu - 2b_\alpha b_\beta x^2 - 2b^2x_\alpha x_\beta + 2b^2x^2x_\alpha b_\beta\right.\\
&\left.-4(b\cdot x)b_\alpha x_\beta + 4b^2x^2b_\alpha x_\beta + 4b^2(b\cdot x)x_\alpha x_\beta - 4b^\mu x^2 x_\alpha x_\beta\right\} + (\alpha \leftrightarrow \beta)\\
&+\frac{\left\{x^2 - 2(b\cdot x)x^2 + b^2x^4\right\}\left\{4b_\alpha b_\beta - 4b^2b_\beta x_\alpha - 4b^2b_\alpha x_\beta + 4b^4x_\alpha x_\beta\right\}}{\Lambda^4}\\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3}\\
&\times\left(-\cancel{2b_\beta x_\alpha} - \cancel{2b_\alpha x_\beta} + \cancel{4b^2x_\alpha x_\beta} + \cancel{4(b\cdot x)b_\alpha x_\beta} - \cancel{8b^2(b\cdot x)x_\alpha x_\beta}\right.\\
&-\cancel{2b^2x^2b_\beta x_\alpha} - \cancel{2b^2x^2b_\alpha x_\beta} + 4b^\mu x^2x_\alpha x_\beta + \cancel{2b_\alpha x_\beta} + \cancel{2b_\beta x_\alpha}\\
&-4b_\alpha b_\beta x^2 - \cancel{4b^2x_\alpha x_\beta} + \cancel{2b^2x^2x_\alpha b_\beta} + \cancel{2b^2x^2x_\beta x_\alpha} - \cancel{4bxb_\alpha x_\beta}\\
&\left.-\cancel{4bxb_\beta x_\alpha} + 4b^2x^2b_\alpha x_\beta + 4b^2x^2b_\beta x_\alpha + \cancel{8b^2b^2x_\alpha x_\beta} - 8b^4x^2x_\alpha x_\beta\right)\\
&+x^2\frac{\Lambda}{\Lambda^4}\left\{4b_\alpha b_\beta - 4b^2b_\beta x_\alpha - 4b^2b_\alpha x_\beta + 4b^4x_\alpha x_\beta\right\}\\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3}\left\{-4b_\alpha b_\beta x^2 + 4b^2x^2b_\alpha x_\beta + 4b^2x^2b_\beta x_\alpha - 4b^4x^2x_\alpha x_\beta\right\}\\
&\quad+x^2\frac{1}{\Lambda^3}\left\{4b_\alpha b_\beta - 4b^2b_\beta x_\alpha - 4b^2b_\alpha x_\beta + 4b^4x_\alpha x_\beta\right\}
\end{aligned}
$$

$$g'_{\alpha\beta}(x') = \Lambda^2 g_{\alpha\beta}(x)$$

Jacobian of the Transformation

The following part is taken from "Conformal Field Theory Primer in $D \geq 3$" by Andrew Evans, pg 36:

$$
\begin{aligned}
\text{Translation:}\quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1\\
\text{Rotation:}\quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1\\
\text{Dilataion:}\quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \lambda^{-d}\\
\text{Inversion:}\quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \left(\frac{1}{\tilde{x}^2}\right)^d
\end{aligned}
$$

Since the rest are easier to show, we will only focus on showing the last part:

$$
\begin{aligned}
\frac{\partial x^\mu}{\partial \tilde{x}^\nu} &= \frac{1}{\tilde{x}^2}\left[\delta^\mu_\nu - 2\frac{\tilde{x}^\mu\tilde{x}_\nu}{\tilde{x}^2}\right]\\
\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) &= \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}}\frac{\partial x^{\mu_2}}{\partial \tilde{x}^{\nu}_2}\cdots\frac{\partial x^{\mu_d}}{\partial \tilde{x}^{\nu}_d}\\
&= \left(\frac{1}{\tilde{x}^2}\right)^d\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\left[\delta^{\mu_1}_{\nu_1} - 2\frac{\tilde{x}^{\mu_1}\tilde{x}_{\nu_1}}{\tilde{x}^2}\right]\left[\delta^{\mu_2}_{\nu_2} - 2\frac{\tilde{x}^{\mu_2}\tilde{x}_{\nu_2}}{\tilde{x}^2}\right]\cdots\left[\delta^{\mu_d}_{\nu_d} - 2\frac{\tilde{x}^{\mu_d}\tilde{x}_{\nu_d}}{\tilde{x}^2}\right]\\
&= \left(\frac{1}{\tilde{x}^2}\right)^d\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{i=1}^{d}\delta^{\mu_i}_{\nu_i} - 2\sum_{j=1}^{d}\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\tilde{x}^{\mu_j}\tilde{x}_{\nu_j}}{\tilde{x}^2}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} + 0
\end{aligned}
$$

Now we use the identity

$$\epsilon_{\color{red}{i_1\ldots i_k}\,i_{k+1}\ldots i_n}\epsilon^{\color{red}{i_1\ldots i_k}\,j_{k+1}\ldots j_n} = \delta^{i_1\ldots i_k\,j_{k+1}\ldots j_n}_{i_1\ldots i_k\,i_{k+1}\ldots i_n} = k!\,\delta^{j_{k+1}\ldots j_n}_{i_{k+1}\ldots i_n}$$

which in our case becomes

$$\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} = (d-1)!\delta^{\mu_j}_{\nu_j}$$

Hence

$$
\begin{aligned}
\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) &= \left(\frac{1}{\tilde{x}^2}\right)^d \left(\frac{d! - 2\sum_{j=1}^{d}(d-1)!}{d!}\right) \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d \left(1 - \frac{2d(d-1)!}{d!}\right) = -\left(\frac{1}{\tilde{x}^2}\right)^d
\end{aligned}
$$

How distances transform

Under translation

$$x'^\mu = x^\mu + a^\mu$$

So,

$$
\begin{aligned}
x_a'^\mu - x_b'^\mu &= x_a^\mu + a^\mu - x_b^\mu - a^\mu \\
&= x_a^\mu - x_b^\mu
\end{aligned}
$$

Thus, the distances are invariant under translation:

$$|x_a' - x_b'| = |x_a^\mu - x_b^\mu|$$

Under dilatation

$$x'^\mu = (1+\lambda)x^\mu$$

So,

$$
\begin{aligned}
x_a'^\mu - x_b'^\mu &= (1+\lambda)x_a^\mu - (1+\lambda)x_b^\mu \\
&= (1+\lambda)(x_a^\mu - x_b^\mu)
\end{aligned}
$$

We find that the distances between two point scales under dilatation, therefore the natural quantity which is invariant under both translation and dilatation is

$$
\begin{aligned}
\frac{\left|x_a'^\mu - x_b'^\mu\right|}{\left|x_c'^\mu - x_d'^\mu\right|} &= \frac{\cancel{1+\lambda}\,|x_a^\mu - x_b^\mu|}{\cancel{1+\lambda}\,|x_c^\mu - x_d^\mu|} \\
&= \frac{|x_a^\mu - x_b^\mu|}{|x_c^\mu - x_d^\mu|}
\end{aligned}
$$

Under special conformal transformation

$$
\begin{aligned}
x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2x^2} \\
&= \frac{x^\mu - b^\mu x^2}{\Lambda^2(x)}
\end{aligned}
$$

So,

$$
\begin{aligned}
x_a'^\mu &= \frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} \\
x_b'^\mu &= \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}
\end{aligned}
$$

and,

$$x_a'^\mu - x_b'^\mu = \frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}$$

squaring both sides

$$(x_a'^\mu - x_b'^\mu)^2 = \left(\frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}\right)^2$$

$$
\begin{aligned}
&= \frac{x_a^2 + b^2(x_a^2)^2 - 2x_a^2(x_a\cdot b)}{\Lambda^4(x_a)} + \frac{x_b^2 + b^2(x_b^2)^2 - 2x_b^2(x_b\cdot b)}{\Lambda^4(x_b)} \\
&\quad - \frac{2}{\Lambda^2(x_a)\Lambda^2(x_b)}\left[x_a\cdot x_b - x_b^2(x_a\cdot b) - x_a^2(b\cdot x_b) + b^2x_a^2x_b^2\right] \\
&= x_a^2\left[\frac{1-2(x_a\cdot b)}{\Lambda^4(x_a)} + \frac{\overbrace{2(b\cdot x_b) - b^2x_b^2}^{1-\Lambda^2(x_b)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1-2(x_b\cdot b)}{\Lambda^4(x_b)} + \frac{\overbrace{2(b\cdot x_a) - b^2x_a^2}^{1-\Lambda^2(x_a)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2\left[\frac{1-2(x_a\cdot b) - \Lambda^2(x_a)}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1-2(x_b\cdot b)-\Lambda^2(x_b)}{\Lambda^4(x_b)}\right. \\
&\quad \left. + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2\left[\frac{-b^2x_a^2}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{-b^2x_b^2}{\Lambda^4(x_b)} + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= \frac{(x_a - x_b)^2}{\Lambda^2(x_a)\Lambda^2(x_b)}
\end{aligned}
$$

Thus, we find that the ratio of distances are not invariant under SCT.

$$
\frac{\left|x_a'^{\mu} - x_b'^{\mu}\right|}{|x_a - x_b|} = \frac{1}{\Lambda(x_a)\Lambda(x_b)}
$$

where $\Lambda(x_a) = \sqrt{1 - 2(x_a\cdot b) + b^2x_a^2}$ We can however, construct another quantity which is invariant under SCT.

$$
\begin{aligned}
\frac{|x_a' - x_b'|}{|x_b' - x_d'|}\frac{|x_d' - x_c'|}{|x_c' - x_a'|} &= \frac{\frac{|x_a - x_b|}{\Lambda(x_a)\Lambda(x_b)}}{\frac{|x_b - x_d|}{\Lambda(x_b)\Lambda(x_d)}}\frac{\frac{|x_d - x_c|}{\Lambda(x_d)\Lambda(x_c)}}{\frac{|x_c - x_a|}{\Lambda(x_c)\Lambda(x_a)}} \\
&= \frac{|x_a - x_b|}{|x_b - x_d|}\frac{|x_d - x_c|}{|x_c - x_a|}
\end{aligned}
$$

Such expressions are called, anharmonic ratios or cross-ratios.

1.3 Lie Algebra of Generators

$$
\begin{aligned}
[P_\mu, P_\nu] &= [-i\partial_\mu, -i\partial_\nu] \\
&= -[\partial_\mu, \partial_\nu] = 0
\end{aligned}
$$

Some useful identities

$$
\begin{aligned}
[x_\alpha, \partial_\beta]f &= x_\alpha\partial_\beta f - \underbrace{\partial_\beta(x_\alpha f)}_{(\partial_\beta x_\alpha)f + x_\alpha\partial_\beta f} \\
&= x_\alpha\partial_\beta f - x_\alpha\partial_\beta f - (\partial_\beta x_\alpha)f \\
&= -(\partial_\beta x_\alpha)f
\end{aligned}
$$

$$
\begin{aligned}
[x_\alpha, \partial_\beta] &= -\partial_\beta x_\alpha = -g_{\beta\alpha}\partial^\mu x_\alpha \\
&= g_{\beta\alpha}
\end{aligned} \tag{1.18}
$$

next is,

$$
[x^2, \partial_\beta] = [x^\alpha x_\alpha, \partial_\beta]
$$

$$
\begin{aligned}
&= x^\alpha[x_\alpha, \partial_\beta] + [x^\alpha, \partial_\beta]x_\alpha \\
&= -x^\alpha g_{\beta\alpha} - \delta^\alpha_\beta x_\alpha \\
&= -x_\beta - x_\beta \\
&= -2x_\beta
\end{aligned} \tag{1.19}
$$

and the last one is,

$$
\begin{aligned}
[x_\mu x^\nu, \partial_\beta] &= x_\mu[x^\nu, \partial_\beta] + [x_\mu, \partial_\beta]x^\nu \\
&= -x_\nu \delta^\nu_\beta - x^\nu g_{\beta\alpha}
\end{aligned} \tag{1.20}
$$

We will now consider, the lie algebra of different operators one by one.

$$
\begin{aligned}
[P_\mu, D] &= [-i\partial_\mu, -ix^\alpha\partial_\alpha] \\
&= -[\partial_\mu, x^\alpha\partial_\alpha] \\
&= -x^\alpha[\partial_\mu, \partial_\alpha] - [\partial_\mu, x^\alpha]\partial_\alpha \\
&= -\delta^\alpha_\mu\partial_\alpha = -\partial_\mu = -i(-i\partial_\mu) \\
&= -iP_\mu
\end{aligned}
$$

$$
\begin{aligned}
[P_\mu, L_{\alpha\beta}] &= [-i\partial_\mu, -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\
&= -[\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha] \\
&= -[\partial_\mu, x_\alpha]\partial_\beta + [\partial_\mu, x_\beta]\partial_\alpha \\
&= g_{\alpha\mu}\partial_\beta - g_{\beta\mu}\partial_\alpha \\
&= i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
[P_\mu, K_\nu] &= [-i\partial_\mu, -i(2x_\mu x^\alpha\partial_\alpha - x^2\partial_\nu)] \\
&= -[\partial_\mu, 2x_\nu x^\alpha\partial_\alpha - x^2\partial_\nu] \\
&= -2x_\nu x^\alpha[\partial_\mu, \partial_\alpha] - 2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + x^2[\partial_\mu, \partial_\nu] + [\partial_\mu, x^2]\partial_\nu \\
&= -2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + [\partial_\mu, x^2]\partial_\nu \\
&= -2(g_{\mu\nu}x^\alpha + \delta^\alpha_\mu x_\nu)\partial_\alpha + 2x_\mu\partial_\nu \\
&= -2g_{\mu\nu}x^\alpha\partial_\alpha - 2(x_\nu\partial_\mu - x_\mu\partial_\nu) \\
&= -2ig_{\mu\nu}D - 2iL_{\mu\nu} \\
&= -2i(g_{\mu\nu}D - L_{\mu\nu})
\end{aligned}
$$

$$
\begin{aligned}
[D, K_\mu] &= -[x^\alpha\partial_\alpha, 2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu] \\
&= -2[x^\alpha\partial_\alpha, x_\mu x^\beta\partial_\beta] + [x^\alpha\partial_\alpha, x^2\partial_\mu] \\
&= -2\left\{x^\alpha[\partial_\alpha, x_\mu x^\beta]\partial_\beta + x_\mu x^\beta[x^\alpha, \partial_\beta]\partial_\alpha\right\} \\
&\quad + x^\alpha[\partial_\alpha, x^2]\partial_\mu + x^2[x^\alpha, \partial_\mu]\partial_\alpha \\
&= -2\left\{x^\alpha\left(g_{\alpha\mu}x^\beta + \delta^\beta_\alpha x_\mu\right)\partial_\beta + x_\mu x^\beta(-\delta^\alpha_\beta)\partial_\alpha\right\} \\
&\quad + 2x^2\partial_\mu - \cancel{x^\alpha x^2\partial_\alpha\partial_\mu} + \cancel{x^2 x^\alpha\partial_\alpha\partial_\mu} - x^2\partial_\mu \\
&= -\cancel{2x_\mu x^\beta\partial_\beta} - 2x^\beta x_\mu\partial_\beta + \cancel{2x_\mu x^\beta\partial_\beta} + x^2\partial_\mu \\
&= -(2x^\beta x_\mu\partial_\beta - x^2\partial_\mu) \\
&= -iK_\mu
\end{aligned}
$$

$$
\begin{aligned}
[K_\mu, L_{\alpha\beta}] &= [-i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu), i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\
&= [2x_\mu x^\nu\partial_\nu - x^2\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha] \\
&= 2[x_\mu x^\nu\partial_\nu, x_\alpha\partial_\beta] - [x^2\partial_\mu, x_\alpha\partial_\beta] + \underbrace{2[x_\mu x^\nu\partial_\nu, x_\beta\partial_\alpha] - [x^2\partial_\mu, x_\beta\partial_\alpha]}_{\alpha\leftrightarrow\beta} \\
&= 2\left\{x_\mu x^\nu[\partial_\nu, x_\alpha]\partial_\beta + x_\alpha[x_\mu x^\nu, \partial_\beta]\partial_\nu\right\} - x^2[\partial_\mu, x_\alpha]\partial_\beta - x_\alpha[x^2, \partial_\beta]\partial_\mu - (\alpha\leftrightarrow\beta) \\
&= \cancel{2x_\mu x^\nu(g_{\nu\alpha})\partial_\beta} - 2x_\alpha\left(g_{\mu\beta}x^\nu + \cancel{\delta^\nu_\beta x_\mu}\right)\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + 2x_\alpha x_\beta\partial_\mu - (\alpha\leftrightarrow\beta)
\end{aligned}
$$

$$
\begin{aligned}
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + \cancel{2x_\alpha x_\beta\partial_\mu} + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2 g_{\mu\beta}\partial_\alpha - \cancel{2x_\beta x_\alpha\partial_\mu}\\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2 g_{\mu\beta}\partial_\alpha\\
&= -g_{\mu\beta}(2x_\alpha x^\nu\partial_\nu - x^2\partial_\alpha) + g_{\mu\alpha}(2x_\beta x^\nu\partial_\nu - x^2\partial_\beta)\\
&= ig_{\mu\alpha}K_\beta - ig_{\mu\beta}K_\alpha = i(g_{\mu\alpha}K_\beta - g_{\mu\beta}K_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
[K_\mu, K_\nu] &= -[2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu, 2x_\nu x^\beta\partial_\beta - x^2\partial_\nu]\\
&= -4[x_\mu x^\alpha\partial_\alpha, x_\nu x^\beta\partial_\beta] + 2[x_\mu x^\alpha\partial_\alpha, x^2\partial_\nu] + 2[x^2\partial_\mu, x_\nu x^\beta\partial_\beta] - [x^2\partial_\mu, x^2\partial_\nu]\\
&= -4x_\nu x^\beta\left[x_\mu x^\alpha, \partial_\beta\right]\partial_\alpha - 4x_\mu x^\alpha\left[\partial_\alpha, x_\nu x^\beta\right]\partial_\beta + 2x_\mu x^\alpha\left[\partial_\alpha, x^2\right]\partial_\nu + 2x^2\left[x_\mu x^\alpha, \partial_\nu\right]\partial_\alpha\\
&\quad + 2x^2\left[\partial_\mu, x_\nu x^\beta\right]\partial_\beta + 2x_\nu x^\beta\left[x^2, \partial_\beta\right]\partial_\mu - x^2[\partial_\mu, x^2]\partial_\nu - x^2[x^2, \partial_\nu]\partial_\mu\\
&= \cancel{4x_\nu x^\beta(g_{\mu\beta}x^\alpha + \delta^\alpha_\beta x_\mu)\partial_\alpha} - \cancel{4x_\mu x^\alpha(g_{\alpha\nu}x^\beta + \delta^\beta_\alpha x_\nu)\partial_\beta} + 4x_\mu x^2\partial_\nu - 2x^2(\cancel{g_{\mu\nu}x^\alpha} + \delta^\alpha_\nu x_\mu)\partial_\alpha\\
&\quad + 2x^2(\cancel{g_{\mu\nu}x^\beta} + \delta^\beta_\mu x_\nu)\partial_\beta - 4x_\nu x^2\partial_\mu - 2x^2 x_\mu\partial_\nu + 2x^2 x_\nu\partial_\mu\\
&= \cancel{4x_\mu x^2\partial_\nu - 2x_\mu x^2\partial_\nu} + \cancel{2x_\nu x^2\partial_\mu - 4x_\nu x^2\partial_\mu} - \cancel{2x^2 x_\mu\partial_\nu} + \cancel{2x^2 x_\nu\partial_\mu}\\
&= 0
\end{aligned}
$$

Next, we will see that Conformal Algebra in d dimensions is isomorphic to the Lie algebra of the Lorentz group in $d+2$ dimensions, any conformal covariant correlator in d dimensions should be obtainable from Lorentz covariant expressions in $d+2$ dimensions via some kind of dimensional reduction procedure. This is essentially the idea behind **Embedding Formalism**. We define the following set of new operators:

$$
\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu}\\
J_{0,\mu} &= \frac{1}{2}\left(P_\mu + K_\mu\right)\\
J_{-1,\mu} &= \frac{1}{2}\left(P_\mu - K_\mu\right)\\
J_{-1,0} &= D
\end{aligned}
$$

with the property that

$$J_{ab} = -J_{ba}$$

where

$$a, b \in \{-1, 0, 1, \cdots,\ \overset{\text{d is dimension of spacetime}}{d}\ \}$$

These new generators, obey $SO(d+1,1)$ lie algebra:

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{bc}J_{ad}\right) \tag{1.21}$$

In this section, we will explicitly assume the form of flat metric as being euclidean, and given as:

$$g_{\mu\nu} = \eta_{\mu\nu} = (\underbrace{1, 1, \cdots, 1}_{d})$$

Our metric in (1.21) would be given as:

$$\eta_{ab} = (\ \underset{\eta_{-1-1}=-1}{-1}\ ,\ \underset{\eta_{00}=1}{1}\ , \underbrace{1, \cdots, 1}_{\mu,\nu}) \tag{1.22}$$

If our original metric was Minkowski, we would have had:

$$\eta_{ab} = (-1, 1, \underbrace{-1, \cdots, 1}_{d})$$

We will now check, if (1.21) holds true:

$$
\begin{aligned}
[J_{\mu\nu}, J_{0,\alpha}] &= \left[L_{\mu\nu}, \frac{1}{2}(P_\alpha + K_\alpha)\right]\\
&= \frac{1}{2}[L_{\mu\nu}, P_\alpha] + \frac{1}{2}[L_{\mu\nu}, K_\alpha]\\
&= -\frac{1}{2}[P_\alpha, L_{\mu\nu}] - \frac{1}{2}[K_\alpha, L_{\mu\nu}]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\left(\eta_{\alpha\mu}P_\mu - \eta_{\alpha\nu}P_\mu\right) - \frac{1}{2}(\eta_{\alpha\mu}K_\nu - \eta_{\alpha\nu}K_\nu) \\
&= -\eta_{\alpha\mu}\left[\frac{1}{2}(P_\nu + K_\nu)\right] + \eta_{\alpha,\nu}\left[\frac{1}{2}(P_\mu + K_\mu)\right] \\
&= -i\eta_{\alpha\mu}J_{0,\nu} + i\eta_{\alpha\nu}J_{0,\mu}
\end{aligned}
$$

$$
\begin{aligned}
[J_{0,\mu}, J_{-1,0}] &= \left[\frac{1}{2}(P_\mu + K_\mu), D\right] \\
&= \frac{1}{2}[P_\mu, D] + \frac{1}{2}[K_\mu, D] \\
&= -\frac{1}{2}iP_\mu - \frac{1}{2}(-iK_\mu) = \frac{-i}{2}(P_\mu - K_\mu) = -iJ_{-1,\mu}
\end{aligned}
$$

If we assume that the metric in (1.21) is indeed given by (1.22). Then, the algebra (1.21) holds true. This shows the isomorphism between the conformal group in $d-$dimensions and the group $SO(d+1,1)$ with $1/2(d+1)(d+2)$ parameters.

Conformal Generators on the Field

Finite form of conformal transformation $(x' = \Lambda x)$[3]

$$
\begin{aligned}
\Phi'_a(x') &= U(\Lambda)\Phi_a(x)U^{-1}(\Lambda) \\
\Phi'_a(\Lambda x) &= \sum_b \pi_{ab}(\Lambda)\Phi_b(x) \\
&= \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x') \\
&= \pi_{ab}(e^{i\omega_g c_g})\Phi_b(e^{-i\omega_g c_g}x')
\end{aligned}
\tag{1.23}
$$

We have dropped the $\sum$ sign and summation over repeated indices are implied. Infinitesimal form of (1.23):

$$
\begin{aligned}
\Phi'_a(x') &= (1 - i\omega_g \overset{\text{generator only acting on field}}{T_g})_{ab}\Phi_b(\Lambda^{-1}x') \\
&= (1 - i\omega_g T_g)_{ab}\underbrace{\Phi_b[(1 - i\omega_g \overset{\text{generator which only acts on } x'^\mu}{c_g})x'^\mu]}_{\Phi_b(x') + \{(1-i\omega_g c_g)x'^\mu - x'^\mu\}\partial_\mu\Phi_b(x')} \\
&= (1 - i\omega_g T_g)_{ab}[\Phi_b(x') - i\omega_g c_g x'^\mu\partial_\mu\Phi_b(x')] \\
\Phi'(x') &= \Phi(x') - i\omega_g\left[T_g + \underbrace{c_g x'^\mu\frac{\partial}{\partial x'^\mu}}_{\text{accounts for the change in argument of field}}\right]\Phi(x') + \mathcal{O}(\omega_g^2)
\end{aligned}
$$

However, we will not use this approach but rather we will consider the transformations at origin and then translate it to every other point. This approach is based on studying the stabilizer subgroup of the Conformal Symmetry.[4] So, if we study the same at origin:

$$
\begin{aligned}
\Phi'(0) &= \Phi(x') - i\omega_g\left[T_g + c_g x'^\mu\frac{\partial}{\partial x'^\mu}\right]\Phi(x')\bigg|_{x'=0} \\
&= \Phi(0) - i\omega_g T_g\Phi(0)
\end{aligned}
$$

using translation operator

$$
\begin{aligned}
e^{ix^\lambda P_\lambda}\Phi'(0)e^{-ix^\alpha P_\alpha} &= e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} - e^{ix^\lambda P_\lambda}i\omega_g T_g\Phi(0)e^{-ix^\alpha P_\alpha} \\
\Phi'(x) &= \Phi(x) - e^{ix^\lambda P_\lambda}i\omega_g T_g e^{-ix^\sigma P_\sigma}e^{ix^\beta P_\beta}\Phi(0)e^{-ix^\alpha P_\alpha} \\
&= \Phi(x) - i\omega_g \underbrace{e^{ix^\lambda P_\lambda}T_g e^{-ix^\sigma P_\sigma}}_{\text{we will find these “translated operators” later}}\Phi(x)
\end{aligned}
$$

[3] tobias osborne's lecture notes pg 18

[4] pg 7 of "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications"

For translation

$$\Phi'(x+a) = e^{ia^\lambda P_\lambda}\Phi(x)e^{-ia^\alpha P_\alpha}$$
$$= e^{ia^\lambda [P_\lambda,\ ]}\Phi(x)$$

using (1.25)

$$= e^{a\cdot\partial}\Phi(x)$$

For rotation, at $x'^\mu = 0. \implies x^\mu = 0$

$$\Phi'_a(0) = \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}0) = \pi_{ab}(\Lambda)\Phi_b(0)$$

Now, assuming the generator of rotation $T_g = L_{\mu\nu}$ acts like[5]

$$L_{\mu\nu}\Phi_a(0) = S_{\mu\nu}\Phi_a(0) \tag{1.24}$$

at origin. At any other point, it will behave as:

$$\begin{aligned} L_{\mu\nu}\Phi_a(x) &= e^{ix^\beta P_\beta}L_{\mu\nu}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\ &= \underbrace{e^{ix^\beta P_\beta}L_{\mu\nu}e^{-ix^\sigma P_\sigma}}_{?}\underbrace{e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}}_{\Phi_a(x)} \end{aligned}$$

by taking the derivative of second term, we obtain the following commutator

$$\begin{aligned} \Phi_a(x) &= e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\ \partial_\mu\Phi_a(x) &= (\partial_\mu e^{ix^\lambda P_\lambda})\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)(\partial_\mu e^{-ix^\alpha P_\alpha}) \\ &= iP^\mu e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}(-iP^\mu) \\ &= iP^\mu\Phi_a(x) - i\Phi_a(x)P^\mu \\ &= i[P^\mu, \Phi_a(x)] \end{aligned} \tag{1.25}$$

We will now derive the form of $e^{ix^\beta P_\beta}L_{\mu\nu}e^{-ix^\sigma P_\sigma}$[6]:

$$\begin{aligned} e^{ix^\beta P_\beta}L_{\mu\nu}e^{-ix^\sigma P_\sigma} &= L_{\mu\nu} + [L_{\mu\nu}, -ix^\alpha P_\alpha] + \frac{1}{2!}[[L_{\mu\nu}, -ix^\alpha P_\alpha], -ix^\alpha P_\alpha] + \dots \\ &= L_{\mu\nu} + ix^\alpha \underbrace{[P_\alpha, L_{\mu\nu}]}_{i(g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu)} + \dots \\ &= L_{\mu\nu} + i^2x^\alpha(g_{\alpha\mu}P_\nu - g_{\alpha\nu}uP_\mu) \\ &= L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \\ &= L_{\mu\nu} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{we found in section 1.1}} \end{aligned}$$

we know, at $x' = 0$ we have $L_{\mu\nu} = S_{\mu\nu}$, so for the sake of consistency we get

$$e^{ix^\beta P_\beta}L_{\mu\nu}e^{-ix^\sigma P_\sigma} = \underbrace{S_{\mu\nu}}_{\text{Spin Operator}} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{transforms the argument of field}}$$

The exponential map of above can be found in any textbook on QFT which describes rotation or Lorentz transformation.[7] If we ignore $S_{\mu\nu}$, then we can see how the last part acts on field:

$$\begin{aligned} x'^\mu &= \left(\delta^\mu_\nu + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right)x^\mu \\ &= x^\mu + \frac{i}{2}\omega_{\alpha\beta}i\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right)x^\nu \\ \Phi'(x) &= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\Phi(x) \end{aligned}$$

[5] pg 10, paragraph 2 of "Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions"

[6] using BCH lemma $e^ABe^{-A} = e^{[A,\ ]}B$

[7] check eqn 1.141 and 1.150 of "QFT in curved spacetime" by Leonard Parker

$$
\begin{aligned}
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta} i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha g^{\beta\sigma}\partial_\sigma - x^\beta g^{\alpha\sigma}\partial_\sigma)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)x^\sigma\partial_\sigma\Phi(x) \\
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}x\cdot\partial\Phi(x) \\
&\approx \Phi\left(x^\mu - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}x^\nu\right) \\
\Phi'(x') &= \Phi(x)
\end{aligned}
$$

For dilatation, at $x'^\mu = 0$, $x'^\mu = (1+\lambda)x^\mu = 0 \implies x^\mu = 0$. We have $\omega_g = \lambda$ and $T_g = D$:

$$D\Phi_a(0) = \tilde{\Delta}\Phi_a(0) \tag{1.26}$$

corresponding commutator (by operating it on eigenstate of dilatation)

$$
\begin{aligned}
D\left|\Delta\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle + \Phi_\Delta(0)D\left|0\right\rangle \\
\tilde{\Delta}\left|\Delta\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle + 0 \\
\tilde{\Delta}\Phi_\Delta(0)\left|0\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle
\end{aligned}
$$

Applying the same procedure, we consider:

$$
\begin{aligned}
e^{ix^\beta P_\beta} D e^{-ix^\sigma P_\sigma} &= D + [D, -ix^\beta P_\beta] + \frac{1}{2!}[[D, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= D - ix^\alpha(iP_\alpha) \\
&= D + x^\alpha P_\alpha \\
&= D - ix^\alpha\partial_\alpha
\end{aligned}
\tag{1.27}
$$

for the sake of consistency at $x' = 0$

$$= \tilde{\Delta} - ix^\alpha\partial_\alpha$$

Now, we consider

$$D\Phi_a(x) = (\tilde{\Delta} - ix^\alpha\partial_\alpha)\Phi_a(x)$$

redefining $\tilde{\Delta} \equiv -i\Delta$, we get

$$D\Phi_a(x) = -i(\Delta + x^\alpha\partial_\alpha)\Phi_a(x)$$

Similarly,[8]

$$
\begin{aligned}
[D, \Phi_a(x)] &= De^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma}D \\
&= e^{ix^\lambda P_\lambda}\underbrace{e^{ix^\alpha P_\alpha}De^{-ix^\beta P_\beta}}_{=D+x^\alpha P_\alpha}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)\underbrace{e^{-ix^\sigma P_\sigma}De^{-ix^\alpha P_\alpha}}_{=D+x^\alpha P_\alpha}e^{-ix^\lambda P_\lambda} \\
&= e^{ix^\beta P_\beta}[D + x^\alpha P_\alpha, \Phi_a(0)]e^{-ix^\sigma P_\sigma} \\
&= e^{ix^\beta P_\beta}\underbrace{[D, \Phi_a(0)]}_{\tilde{\Delta}\Phi_a(0)}e^{-ix^\sigma P_\sigma} + e^{ix^\beta P_\beta}\underbrace{[x^\alpha P_\alpha, \Phi_a(0)]}_{=x^\alpha[P_\alpha,\Phi_a(0)]}e^{-ix^\sigma P_\sigma} \\
&= \tilde{\Delta}\Phi_a(x) - ix\cdot\partial\Phi_a(x) \\
&= -i(\Delta + x\cdot\partial)\Phi_a(x)
\end{aligned}
$$

Finite Dilatation[9], we consider

$$x' = e^\lambda x = e^{i\lambda D}x = \left(1 + i\frac{\lambda}{N}\overset{Dx^\mu = -ix\cdot\partial x^\mu}{D}\right)\dots\left(1 + i\frac{\lambda}{N}D\right)x$$

[8]from pg 31 of 2309.10107, and x is not an operator here but a number

[9]look up *Lectures Notes For An Introduction to Conformal Field Theory A Course Given By Dr. Tobias Osborne*, pg 19

then at origin, the field transforms (active transformation) as:

$$\begin{aligned}
\Phi'_a(0) &= \left(1+i\frac{\lambda}{N}D\right)\dots\left(1+i\frac{\lambda}{N}D\right)\Phi_a(0) \\
&= \left(1+i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1+i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) && \text{(using } D\Phi(0)=\tilde{\Delta}\Phi) \\
&= e^{i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{-\lambda\Delta_a}\Phi_a(0)
\end{aligned}$$

In passive transformation

$$\begin{aligned}
\Phi'_a(0) &= \left(1-i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1-i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) \\
&= e^{-i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{\lambda\Delta}\Phi_a(0)
\end{aligned}$$

For arbitrary point (ignoring the change in argument of field and thus generator c_g):

$$\begin{aligned}
\Phi'_a(x') &= \pi_{ab}(e^{i\lambda D})\Phi_b(x) \\
&= [e^{i\lambda\tilde{\Delta}}]_{ab}\Phi_b(x) \\
\Phi'_a(e^{\lambda}x) &= [e^{-\lambda\Delta}]_{ab}\Phi_b(x) = e^{-\lambda\Delta}\Phi_a(x)
\end{aligned}$$

For SCT, $x'^\mu = 0 \implies x^\mu = 0$. Hence, we will consider the same equations, but in this context:

$$K_\mu\Phi_a(0) = \kappa_\mu\Phi_a(0)$$

Again, applying the same procedure,

$$\begin{aligned}
e^{ix^\beta P_\beta}K_\mu e^{-ix^\sigma P_\sigma} &= K_\mu + [K_\mu, -ix^\beta P_\beta] + \frac{1}{2!}[[K_\mu, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu - ix^\beta[K_\mu, P_\beta] + \frac{1}{2}[-ix^\beta[K_\mu, P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu + 2x^\beta(g_{\mu\beta}D - L_{\mu\beta}) + \frac{1}{2}[2x^\beta(g_{\mu\beta}D - L_{\mu\beta}), -ix^\alpha P_\alpha] \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} - ix_\mu x^\alpha[D, P_\alpha] + ix^\beta x^\alpha \underbrace{[L_{\mu\beta}, P_\alpha]}_{-i(g_{\alpha\mu}P_\beta - g_{\alpha\beta}P_\mu)} \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + x_\mu x^\alpha P_\alpha + x_\mu x^\beta P_\beta - x_\alpha x^\alpha P_\mu \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + 2x_\mu x^\alpha P_\alpha - x_\alpha x^\alpha P_\mu
\end{aligned}$$

From the generator of dilatation and SCT, we have[10]

$$[D, K_\mu] = -iK_\mu \qquad \text{at } x'^\mu = 0 \implies \qquad [\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu$$

and

$$[D, L_{\mu\nu}] = 0 \qquad \text{at } x'^\mu = 0 \implies \qquad [\tilde{\Delta}, S_{\mu\nu}] = 0$$

For primary fields:

$$K_\mu\Phi_a(0) = 0$$

Since, for primary field $\tilde{\Delta}$ commutes with all other operators which belong to the stability subgroup. By Schur's lemma $\tilde{\Delta} \propto I$, where I is an identity operator. The SCT and momentum generator acts as ladder operator for Dilatation.

$$\begin{aligned}
[D, [P_\mu, \Phi(0)]] = [P_\mu, [D, \Phi(0)]] + [[D, P_\mu], \Phi(0)] = -i(\Delta + 1)[P_\mu, \Phi(0)] \\
[D, [K_\mu, \Phi(0)]] = [K_\mu, [D, \Phi(0)]] + [[D, K_\mu], \Phi(0)] = -i(\Delta - 1)[K_\mu, \Phi(0)]
\end{aligned}$$

[10] same notes, look at eqn 65 to 70 (pg 18-19), all these commutators are for T_g

Finite Conformal Transformation of Fields

We begin by noting that *translation* and *rotation* do not introduce any new thing that we hadn't encountered in QFT, it is only the dilatation which does. Upon exponentiating the infinitesimal dilatation:

$$\begin{aligned}\Phi'(x') &= e^{-i\omega_g\left[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}\right]}\Phi(x') \\ &= e^{-i\omega_g T_g} e^{-i\omega_g c_g x' \cdot \frac{\partial}{\partial x'}}\Phi(x') \\ &= e^{-i\omega_g T_g}\Phi(e^{-i\omega_g c_g} x')\end{aligned}$$

This section is taken from "advanced mathematical methods - conformal field theory" by David Duffins.[11] (verify the statement, footnote can be verified from weinbegs' QFT pg 191)

$$\begin{aligned}\Phi'_a(x') = U(\Lambda)\Phi_a(x')U^{-1}(\Lambda) &= e^{-i\omega_g T_g}\Phi_a e^{i\omega_g T_g} \\ &= e^{-i\omega_g [T_g, \ ]}\Phi_a(x')\end{aligned}$$

For translation

$$\Phi(x) = e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} = e^{x\partial}\Phi(0)$$

Or,

$$\begin{aligned}\Phi'(x') &= \Phi(x) \\ &= \Phi(x' - a) \\ &= e^{-a\frac{\partial}{\partial x'}}\Phi(x') \\ &= e^{-iaP}\Phi(x')\end{aligned}$$

For dilatation ($x' = e^\lambda x$)

$$\begin{aligned}\Phi'_a(x') &= e^{-i\lambda D}\Phi_a(x') \\ &= e^{-\lambda(\Delta + x'\cdot\partial)}\Phi_a(x') \\ &= e^{-\lambda\Delta}\underbrace{e^{-\lambda x\cdot\partial}\Phi_a(x')}_{\Phi_a[e^{-\lambda}x']} \\ &= e^{-\lambda\Delta}\Phi_a(x)\end{aligned}$$

The last part could be understood as:

$$\Phi_a\left[\left(1 - \frac{\lambda}{N}\right)x\right] = e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a(x)$$
$$\Phi_a\left[\underbrace{\left(1 - \frac{\lambda}{N}\right)\dots\left(1 - \frac{\lambda}{N}\right)}_{N \text{ terms}}x\right] = e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a\left[\underbrace{\left(1 - \frac{\lambda}{N}\right)\dots\left(1 - \frac{\lambda}{N}\right)}_{N-1 \text{ terms}}x\right]$$
$$\Phi_a(e^{-\lambda}x) = e^{-\lambda x\cdot\partial}\Phi_a(x)$$

or, alternatively

$$\begin{aligned}e^{-\lambda x\cdot\partial}\Phi_a(x) &= \left(1 - \frac{\lambda}{N}x\cdot\partial\right)^N\Phi_a(x) \\ &= \left(1 - \frac{\lambda}{N}x\cdot\partial\right)\dots\underbrace{\left(1 - \frac{\lambda}{N}x\cdot\partial\right)\Phi_a(x)}_{\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right]} \\ &= \Phi_a\left[\left(1 - \frac{\lambda}{N}\right)^N x\right] \\ &= \Phi_a(e^{-\lambda}x)\end{aligned}$$

[11] Active coordinate transformation is given as: $\Phi(x') = U(\Lambda)\Phi(x)U^{-1}(\Lambda)$ whereas passive transformation is given as $\Phi(x') = U^{-1}(\Lambda)\Phi(x)U(\Lambda)$

Chapter 2

Embedding coordinates for Euclidean Space

Consider the embedding space coordinates

$$X^0, \underbrace{X^1, X^2, \ldots, X^d}_{X^\mu}, X^{d+1}$$

we introduce the following null coordinates, $X^M = (X^+, X^-, X^\mu)$, where

$$\left.\begin{aligned} X^+ &= X^0 + X^{d+1} \\ X^- &= X^0 - X^{d+1} \end{aligned}\right\} X^0 = \frac{X^+ + X^-}{2}; \quad X^{d+1} = \frac{X^+ - X^-}{2}$$

with the mostly plus metric in $\mathbb{R}^{d+1,1}$, reads as

$$ds^2 = \sum_{\mu=1}^{d} (dX^\mu)^2 - dX^+ dX^-$$

with the metric given as:

$$\eta_{MN} = \begin{pmatrix} 0 & -1/2 & 0 & 0 & \cdots \\ -1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 0 & 1 & \\ & & & & \ddots \end{pmatrix}$$

We can easily show that the generators in $d+2$ dimensional space reduces to d dimensional conformal generators in Euclidean space. We go from (X_{-1}, X_0, X_μ) coordinates to (ρ, η, x_μ) by following coordinate transformation.

$$\begin{aligned} X_{-1} &= \frac{\rho(1 - \eta^2 + \vec{x}^2)}{2} \\ X_0 &= \frac{\rho(1 + \eta^2 - \vec{x}^2)}{2} \\ X_\mu &= \rho x_\mu \end{aligned}$$

inverting above,

$$\begin{aligned} \rho &= X_{-1} + X_0 \\ \eta &= \frac{\sqrt{\eta_{MN} X^M X^N}}{X_{-1} + X_0} \leftarrow \rho\eta = \sqrt{\eta_{MN} X^M X^N} \\ x_\mu &= \frac{X_\mu}{X_{-1} + X_0} \end{aligned}$$

We can write the change of basis as:

$$\frac{\partial}{\partial X_{-1}} = \frac{\partial}{\partial \rho} - \frac{1 + \eta^2 + \vec{x}^2}{2\rho\eta} \frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho} \frac{\partial}{\partial x_\mu}$$

$$
\begin{aligned}
\frac{\partial}{\partial X_0} &= \frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\
\frac{\partial}{\partial X_\mu} &= \frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}
\end{aligned}
$$

Then, the generators for Lorentz transformation in $d+2$ dimensional space transforms as:

$$
\begin{aligned}
P_\mu = J_{-1,\mu} + J_{0,\mu} &= (X_{-1}+X_0)\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} - \frac{\partial}{\partial X_{-1}}\right) \\
&= \rho\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - \rho x_\mu\left(\frac{1}{\rho\eta}\frac{\partial}{\partial \eta}\right) \\
&= \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x^\mu}
\end{aligned}
$$

$$
\begin{aligned}
K_\mu = J_{0,\mu} - J_{-1,\mu} &= (X_0 - X_{-1})\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_{-1}}\right) \\
&= \rho(\eta^2-\vec{x}^2)\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - 2\rho x_\mu\left[\frac{\partial}{\partial \rho} - \left(\frac{\eta^2+\vec{x}^2}{2\rho\eta}\right) - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&= \frac{\eta^2-\vec{x}^2}{\eta}x_\mu\frac{\partial}{\partial \eta} + (\eta^2-\vec{x}^2)\frac{\partial}{\partial x_\mu} - 2\rho x_\mu\frac{\partial}{\partial \rho} + x_\mu\frac{\eta^2+\vec{x}^2}{\eta}\frac{\partial}{\partial \eta} + 2x_\mu(x\cdot\partial) \\
&= 2x_\mu(x\cdot\partial) - \vec{x}^2\partial_\mu + \eta^2\partial_\mu - 2\rho x_\mu\frac{\partial}{\partial \rho} + 2x_\mu\eta\frac{\partial}{\partial \eta}
\end{aligned}
$$

$$
\begin{aligned}
D = J_{-10} = X_{-1}\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X^{-1}} &= X_{-1}\frac{\partial}{\partial X_0} + X_0\frac{\partial}{\partial X_{-1}} \\
&= \frac{\rho(1-\eta^2+\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&\quad + \frac{\rho(1+\eta^2-\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&= \rho\frac{\partial}{\partial \rho} - x^\mu\frac{\partial}{\partial x^\mu} - \eta\frac{\partial}{\partial \eta}
\end{aligned}
$$

For $\rho = 1$ and $\eta = 0$

$$
\begin{aligned}
P_\mu &= \partial_\mu \\
K_\mu &= 2x_\mu(x\cdot\partial) - x^\nu x_\nu\partial_\mu \\
D &= -x^\mu\partial_\mu
\end{aligned}
$$

Note that conformal algebra is satisfied by both $\pm P_\mu$ and $\pm K_\mu$. Needless to say it is extremely simpler to construct Lorentz covariant expressions than conformal covariant ones. Therefore the problem is: once we have constructed Lorentz covariant expressions in $d+2$ dimensions **how do we descend to d dimensions without breaking the covariance?** This can be done as follows:

We first note that null light cone $X^2 = 0$ is Lorentz Invariant in embedding space. Therefore, we will consider a null cone in the Minkowski space $\mathbb{R}^{d+1,1}$, i.e. the space of null rays passing through origin defined via:

$$
\begin{aligned}
X^2 &= -(X^0)^2 + (X^1)^2 + \ldots + (X^{d+1})^2 \\
&= -X^+X^- + \underbrace{(X^1)^2 + \ldots + (X^d)^2}_{\sum_{\mu=1}^{d}(X^\mu)^2} \\
&= 0
\end{aligned}
$$

We can use this constraint to remove one of the coordinates, say X^- out of $d+2$ coordinates in embedding space. Next, we will think of the embedding space as a fiber bundle over the $d-$dimensional base space (the physical space where the CFT is defined). The fibers in this fiber bundle are the null lines in the $(d+2)-$dimensional space that project down to points in the $d-$dimensional space. At each point in the base space, section (a map) is defined, which is a specific choice of null vector from each fiber. Points in the original $d-$dimensional space are represented by null vectors X^A subject to the identification $X^A \sim \lambda X^A$ for any non-zero λ. All null vectors isomorphic to each other up to dilation belong to the same fiber. This has an important consequence: Lorentz transformations map a point to another point outside the Euclidean section (since they are rotation about a plane). To return to the Euclidean section, we need to use dilation, which is taken care of due to this identification. Therefore, in order to eliminate another one of the coordinate, say X^+, we consider an Euclidean section of the embedding space defined by:

$$X^+ = f(X^\mu) \equiv f(x^\mu)$$

This will help us identifying X^μ with the Euclidean space coordinates x^μ.

$$X^\mu \equiv x^\mu$$

This leads to definition of X^- based on null condition as:

$$X^- = \frac{\sum_{\mu=1}^{d}(X^\mu)^2}{X^+} = \frac{x^2}{f(X^\mu)}$$

The spacetime interval on this section is given as:

$$ds^2 = dx^2 - dX^+ dX^- \Big|_{X^+=f(X^i), X^-=\frac{x^2}{X^+}}$$

This section satisfies two of the following conditions:

- section intersects each of the light rays at some point
- maps each point in d dimensional Euclidean space to a point on the null cone in Embedding space.

Let us now analyze how Lorentz Transformation acts on a generic section. The Lorentz transformation acting as rotation on the point X^A in the null-cone will move it to another point on the null cone outside the the section $X^B = \Lambda^B{}_A X^A$. However, suppose via some conformal transformation (dilatation) in d dimensional Euclidean Space, we can move X^B to X^C back into the section. In all these steps, the only thing that changes the metric is the rescaling to get back into the section.

$$\begin{aligned}
ds_B^2 &= dX^M dX_M \\
&= d(\lambda(X) X^M) d(\lambda(X) X_M) \\
&= [\lambda dX^M + X^M(\boldsymbol{\nabla}\lambda \cdot dX)][\lambda dX_M + X_M(\boldsymbol{\nabla}\lambda \cdot dX)] \\
&= \lambda^2 dX^M dX_M + 2\lambda \underbrace{dX^M X_M}_{=0}(\boldsymbol{\nabla}\lambda \cdot dX) + \underbrace{X^M X_M}_{=0}(\boldsymbol{\nabla}\lambda \cdot dX)^2 \\
&= \lambda^2 dX^M dX_M = \lambda^2 ds_C^2
\end{aligned}$$

where we used, $X^2 = 0$ and $X^\mu dX_\mu = 0$ for restricting it to null cone. Assuming the three conditions we used for simplification applies, the Lorentz Transformation in $d+2$-dimensional spacetime is equivalent to conformal transformation in $d-$dimensional spacetime iff metric in $d-$dimensional space is Euclidean thus, dX_+ in ds^2 has to vanish. It gives us the condition for defining the Euclidean section as $X^+ = \textit{constant}$ and thus, for the sake of simplificity, we take it as 1. Thus, we have two conditions which we can use to eliminate two extra degree of freedom.

In the embedding space formalism, choosing an Euclidean section corresponds to picking a specific way to embed the $d-$dimensional space in the $(d+2)-$dimensional space. We define the following map between d dimensional Euclidean Space with conformal symmetry to null cone in $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$

$$(X^+, X^-, X^\mu) \equiv \left(1, x^2, x^\mu\right)$$

Here, we note that choosing a constant value for X^+ would give us a section on the cone on which the induced metric is Euclidean.

2.1 Tensors in Embedding Space

In this section, we will only concern ourselves with traceless and symmetric fields in $\mathbb{R}^d$ and leave the anti-symmetric tensors for future. Consider a symmetric and traceless tensor $O_{M_1 \ldots M_S}$ defined on the cone $X^2 = 0$ in $\mathbb{R}^{d+1,1}$. Under the rescaling $X \to \lambda X$, the tensor transforms as

$$O_{M_1 \ldots M_S}(\lambda X) = \lambda^{-\Delta} O_{M_1 \ldots M_S}(X)$$

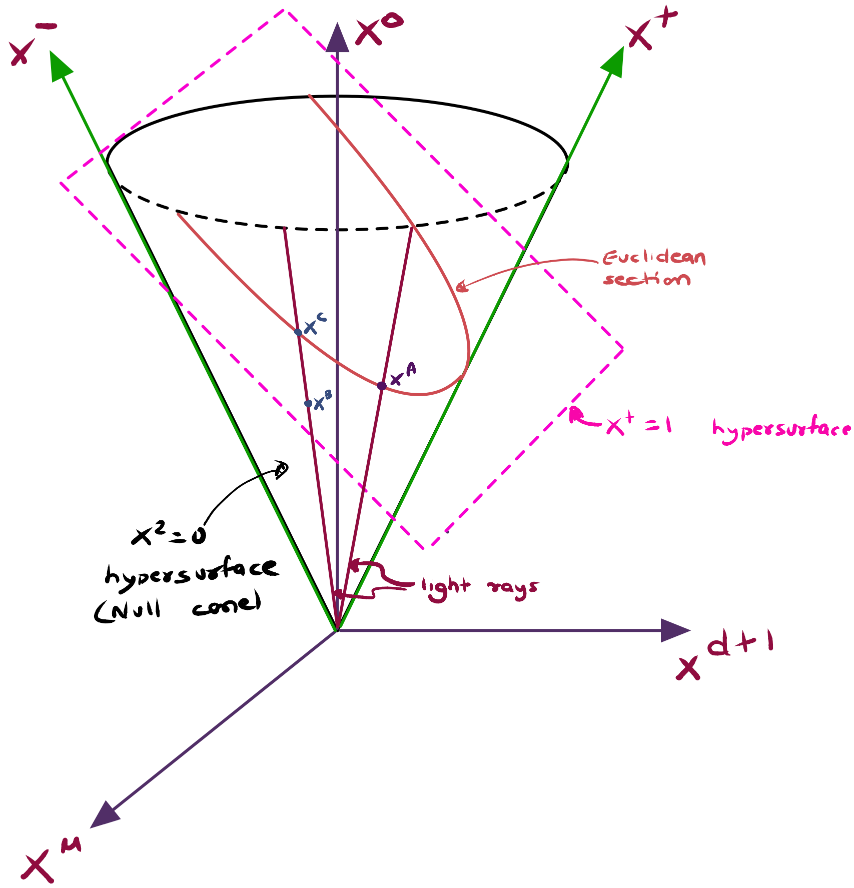

Figure 2.1: The hypersurface perpendicular to X^+ axis cutting at $X^+ = 1$ is shown as a plane and the null hypersurface is shown as the cone. The intersection of these two hypersurfaces describes the Euclidean Section. Dilatations are rotation in the X^0X^{d+1} plane and SCT or momentum generators are rotations in $X^\mu X^{d+1}$ with X^0X^{d+1} plane.

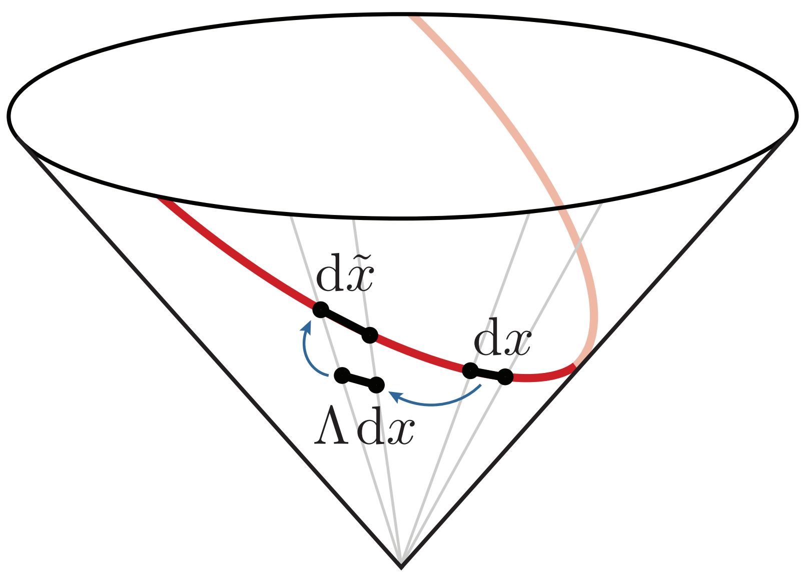

Figure 2.2: Upon Lorentz transformation, the points get mapped to different section however by utilizing the dilatation, we bring it back inside the original Euclidean section.

i.e. it is a homogeneous function of degree Δ. We expect $O_{M_1 \ldots M_S}$ to get mapped to traceless and symmetric primary field in $\mathbb{R}^d$. Since, each index go from 0 to $d+1$, in $\mathbb{R}^{d+1,1}$ we find that, for $d+2-$dimensional fields other than scalar have 2 more degree of freedom per index than $d-$dimensional fields. In order to remove the extra degree of freedom, we consider the transversality condition.

$$X^{M_1} O_{M_1 \ldots M_S} = 0$$

We define the physical field to be:

$$\phi_{\mu\nu\lambda\ldots}(x) = \frac{\partial X^{M_1}}{\partial x^\mu} \frac{\partial X^{M_2}}{\partial x^\nu} \frac{\partial X^{M_3}}{\partial x^\lambda} \ldots O_{M_1 \ldots M_S}(X) \Bigg|_{X=X(x)}$$

Note that, this definition implies a redundancy. Indeed, anything proportional to X^M gives zero since

$$X^2 = 0 \implies X_M \frac{\partial X^M}{\partial x^\mu} = 0$$

Therefore, $O_{M_1 \ldots M_S}(X) \to O_{M_1 \ldots M_S}(X) + X_{M_1} F_{M_2 \ldots M_S}(X)$ gets mapped to the same physical field. This $SO(d+1,1)$ tensor is sometimes referred to as pure gauge in the literature. It is this gauge redundancy that reduces another degree of freedom per index by making it unphysical.

2.2 Examples: Two point and Three point correlator

In the last section we showed how the embedding space formalism put in place could be used to deduce the conformally invariant correlator. In this section we will utilize the formalism and explicitly construct two point and three point function using the formalism developed thus far. From 1.2, we know that the ratios are only invariant under dilatation and translation. Therefore, we seek to construct an invariant out of these ratios and metric tensor which is also invariant under SCT and the exchange of indices $\mu \leftrightarrow \nu$.. The ansatz is[1]

$$\langle J_\mu(x_1) J_\nu(x_2) \rangle = C \underbrace{\frac{1}{|x_1 - x_2|^{2\Delta}}}_{\text{same as scalar case}} \left[g_{\mu\nu} + \delta \frac{(x_1 - x_2)_\mu (x_1 - x_2)_\nu}{(x_1 - x_2)^2} \right]$$

The correlation function is invariant under translation, therefore we will consider following redefinition:

$$\begin{aligned} \langle J_\mu(x_1) J_\nu(x_2) \rangle &= \langle J_\mu(x_1 - x_2) J_\nu(0) \rangle \\ &= \langle J_\mu(x_{12}) J_\nu(0) \rangle = \langle J_\mu(x) J_\nu(0) \rangle \end{aligned}$$

Since SCT is just inversion→translation→inversion, we can this property to our advantage. As the correlation function is already invariant under translations, it suffices to verify its invariance under inversions. If this property holds, then by extension, the correlation function will also be invariant under SCT. The inversion transformation is given as[2]:

$$x'_\mu = \frac{x_\mu}{x^2} \qquad\qquad |x'|^2 = \frac{1}{|x|^2}$$

and

$$\frac{\partial x'_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \frac{x_\nu}{x^2} = \frac{1}{x^2} \left[g_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2} \right] = x'^2 \underbrace{\left[g_{\mu\nu} - 2 \frac{x'_\mu x'_\nu}{x'^2} \right]}_{I_{\mu\nu}}$$

The vector field would transform as

$$\langle J_\mu(x'_1) J_\nu(x'_2) \rangle = \underbrace{\left| \frac{\partial x_\alpha}{\partial x'^\mu} \right|^{\Delta/d} \left| \frac{\partial x_\beta}{\partial x'^\nu} \right|^{\Delta/d}}_{\substack{\text{this was used to derive} \\ \text{the correlation function for scalar case}}} \overbrace{\frac{\partial x'_\alpha}{\partial x^\mu} \frac{\partial x'_\beta}{\partial x^\nu}}^{\text{without conformal factor}} \langle J^\alpha(x_1) J^\beta(x_2) \rangle$$

[1]pg 24 of "CFT with boundary and defects" by Herzog

[2]pg 17-18 of "Quantum Gravity and Cosmology based on Conformal Field Theory" and section 4.5 of "A conformal field theory primer in $D \geq 3$" by Andrew Evans

we see that

$$\underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|_{x=x_1}^{\Delta/d}}_{|x_1'|^{-2\Delta}} \left|\frac{\partial x_\beta}{\partial x'^\nu}\right|_{x=x_2}^{\Delta/d} \frac{1}{|x_{12}|^{2\Delta}} = |x_1'|^{-2\Delta}|x_2'|^{-2\Delta}\frac{1}{|x_{12}|^{2\Delta}} = \frac{1}{|x_{12}'|^{2\Delta}}$$

where we used

$$\begin{aligned}
|x_{12}'|^2 &= \left(\frac{x_1^\mu}{x_1^2} - \frac{x_2^\mu}{x_2^2}\right)^2 \\
&= \frac{|x_1|^2}{x_1^4} + \frac{|x_2|^2}{x_2^4} - 2\frac{x_1^\mu}{x_1^2}\frac{x_{2\mu}}{x_2^2} \\
&= \frac{1}{x_1^2} - 2\frac{x_1}{x_1^2}\cdot\frac{x_2}{x_2^2} + \frac{1}{x_2^2} \\
&= \frac{(x_1-x_2)^2}{x_1^2x_2^2} = \frac{|x_{12}|^2}{|x_1|^2|x_2|^2} = \frac{|x_{12}|^2}{|x_1'|^{-2}|x_2'|^{-2}}
\end{aligned}$$

Then, we only have to ensure that $g_{\mu\nu} + \delta\frac{x'_\mu x'_\nu}{x'^2}$ is invariant under inversion. We consider the transformation of $g_{\mu\nu} + \delta x'_\mu x'_\nu / x'^2$ directly.[3]

$$\begin{aligned}
g^{\mu\nu} + \delta\frac{(x_{12}')^\mu(x_{12}')^\nu}{(x_{12}')^2} &= \left(\delta_\alpha^\mu - 2\frac{x_1^\mu x_{1\alpha}}{x_1^2}\right)\left(\delta_\beta^\nu - 2\frac{x_2^\nu x_{2\beta}}{x_2^2}\right)\left[g^{\alpha\beta} + \delta\frac{x_{12}^\alpha x_{12}^\beta}{x_{12}^2}\right] \\
&= \left(\delta_\alpha^\mu - 2\frac{x_1'^\mu x_{1\alpha}'}{|x_1|'^2}\right)\left(\delta_\beta^\nu - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right)\left[g^{\alpha\beta} + \delta\frac{(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\alpha(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2}\right] \\
&= \left[g^{\mu\beta} + \delta\frac{(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\mu(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\beta}{|x_1|'^2}\right. \\
&\quad \left. - 2\delta\frac{x_1'^\mu(\cancel{|x_1'|^2}|x_2'|^2 - x_1'\cdot x_2'\cancel{|x_1'|^2})(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2\cancel{|x_1'|^2}|x_2'|^2}\right]\left(\delta_\beta^\nu - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right) \\
&= \left[g^{\mu\beta} - \delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\beta}{|x_1|'^2}\right]\left(\delta_\beta^\nu - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right) \\
&= \left[g^{\mu\nu} - \delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2}\right] \\
&\quad - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 2\delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu(x_2'\cdot x_1'\cancel{|x_2'|^2} - \cancel{|x_2'|^2}|x_1'|^2)x_2'^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2\cancel{|x_2'|^2}} + 4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad - \delta\frac{\{x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2')\}^\mu\{x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2(x_1'\cdot x_2')x_2'\}^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad - \delta\frac{\{x_2'|x_1'|^2 + |x_1' - x_2'|^2x_1' - |x_1'|^2x_1'\}^\mu\{x_1'|x_2'|^2 + |x_1' - x_2'|^2x_2' - |x_2'|^2x_2'\}^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2}
\end{aligned}$$

[3] the conformal factor is there following eqn 55 of TASI Lectures on the Conformal Bootstrap. The tensor operator under inversion transforms as mentioned in eqn 3.18 of Conformal Field Theory with Boundaries and Defects or eqn 1.55 and 1.60 of EPFL Lectures on Conformal Field Theory in D¿= 3 Dimensions

$$O'^\mu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\frac{\Delta+1}{d}}\frac{\partial x'^\mu}{\partial x^\nu}O^\nu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\Delta/d} I_\nu^\mu(x')O^\nu(x')$$

$$O'_\mu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\frac{\Delta-1}{d}}\frac{\partial x^\nu}{\partial x'^\mu}O_\nu(x')$$

$$
\begin{aligned}
&+\delta\frac{x_{12}'^{\mu}x_{12}'^{\nu}}{x_{12}'^{2}}-\delta\frac{x_1'^{\mu}}{|x_1'|^2}x_{12}'^{\nu}+\delta\frac{x_2'^{\nu}}{|x_2'|^2}x_{12}'^{\mu}-\delta|x_{12}'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2}\\
&=g^{\mu\nu}-(\delta+2)\frac{x_1'^{\mu}x_1'^{\nu}}{|x_1|'^2}-(\delta+2)\frac{x_2'^{\nu}x_2'^{\mu}}{|x_2'|^2}+2(\delta+2)\frac{x_1'^{\mu}x_2'^{\nu}(x_1'\cdot x_2')}{|x_1|'^2}+\cancel{\delta\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2}}+\cancel{\delta\frac{x_1'^{\mu}x_2'^{\nu}}{|x_2'|^2}}\\
&-\cancel{\delta|x_1'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2}}-\cancel{\delta|x_2'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2}}+\delta\frac{x_{12}'^{\mu}x_{12}'^{\nu}}{x_{12}'^{2}}
\end{aligned}
$$

which implies $\delta=-2$. Hence, the two point function is given as

$$\langle J_\mu(x)J_\nu(0)\rangle=\frac{C}{|x|^{2\Delta}}\left[g_{\mu\nu}-2\frac{x_\mu x_\nu}{x^2}\right]$$

The embedding space formalism gives the same answer[4]: Considering a tensor field of $SO(d+1,1)$ denoted as $O_{A_1\ldots A_n}(X)$, with the properties

- defined on the null-cone $X^2=0$,
- traceless and symmetric,
- homogeneous of degree $-\Delta$ in X, i.e., $O_{A_1\ldots A_n}(\lambda X)=\lambda^{-\Delta}O_{A_1\ldots A_n}(X)$,
- transverse $X^{A_i}O_{A_1\ldots A_n}(X)=0$, with $i=1,\ldots,n$

It is clear that those are conditions rendering $O_{A_1\ldots A_n}(X)$ manifestly invariant under $SO(d+1,1)$. In order to find the corresponding tensor in $\mathbb{R}^d$, one has to restrict $O_{A_1\ldots.A_n}(X)$ to the Poincaré section and project the indices as

$$\langle O^\mu(x_1)O^\nu(x_2)\rangle=\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_1^B}{\partial x_2^\nu}\langle O_A(X_1)O_B(X_2)\rangle$$

For example, the most general form of the two-point function of two operators with spin-1 and dimension Δ can be derived as:[5]

$$\left\langle O^A(X_1)O^B(X_2)\right\rangle=\frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta^{AB}+\alpha\frac{X_2^AX_1^B}{X_1\cdot X_2}+\beta\frac{X_1^AX_2^B}{X_1\cdot X_2}\right]$$

We will drop the last term as it projects to zero anyways.

$$\left\langle O^A(X_1)O^B(X_2)\right\rangle=\frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta^{AB}+\alpha\frac{X_2^AX_1^B}{X_1\cdot X_2}\right]$$

According to the transverse condition

$$X_{A1}\left\langle O^A(X_1)O^B(X_2)\right\rangle=\frac{C_{12}}{(X_1\cdot X_2)^\Delta}[X_1^B+\alpha X_1^B]=0\implies\alpha=-1$$

we now use the projection to find the correlation function in $\mathbb{R}^d$:

$$
\begin{aligned}
\langle O_\mu(x_1)O_\nu(x_2)\rangle&=\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\langle O_A(X_1)O_B(X_2)\rangle\\
&=\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta_{AB}-\frac{X_{A2}X_{B1}}{X_1\cdot X_2}\right]\\
&=\frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\eta_{AB}-\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\frac{X_{A2}X_{B1}}{X_1\cdot X_2}\right]\\
&=\frac{C_{12}}{(x_1-x_2)^{2\Delta}}\left[g_{\mu\nu}-2\frac{(x_1-x_2)_\mu(x_1-x_2)_\nu}{(x_1-x_2)^2}\right]
\end{aligned}
$$

[4]section 2.4 of "Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions" or section 5.2.2 of "Conformal Field Theory" by Loriano Bonora

[5]here the terms in bracket is chosen such that they are invariant under the replacement $x\to\lambda x$. We are not using the transformation law for any of them. Under which, even the metric will change to $\eta_{AB}\to\lambda^{-2}\eta_{AB}$.

where we used $X^A = (X^a, X^+, X^-) = (x^a, 1, x^2)$, $X_B = \left(x_a, -\frac{1}{2}x^2, -\frac{1}{2}\right)$ and $\eta_{ab} = I_{d\times d}$ with $\eta_{+-} = \eta_{-+} = -1/2$

$$
\begin{aligned}
&\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\eta_{AB} = g_{\mu\nu} \\
&\qquad \frac{\partial X_1^A}{\partial x_1^\mu}X_{A2} = \eta_{ab}\frac{\partial x_1^a}{\partial x_1^\mu}x_2^b - \frac{1}{2}\cancelto{0}{\frac{\partial 1}{\partial x_1^\mu}}x_2^2 - \frac{1}{2}\frac{\partial x_1^2}{\partial x_1^\mu}1 \\
&\qquad\qquad = \eta_{ab}\delta_\mu^a x_2^b - x_{1\mu} = (x_2 - x_1)_\mu = -(x_1 - x_2)_\mu \\
&\qquad \frac{\partial X_2^B}{\partial x_2^\nu}X_{B1} = \eta_{ab}\frac{\partial x_2^a}{\partial x_2^\mu}x_1^b - \frac{1}{2}\cancelto{0}{\frac{\partial 1}{\partial x_2^\mu}}x_1^2 - \frac{1}{2}\frac{\partial x_2^2}{\partial x_2^\mu}1 = (x_1 - x_2)_\nu \\
&(X_1 - X_2)^A(X_1 - X_2)_A = (x_1 - x_2)^a(x_1 - x_2)_a - \frac{1}{2}(1-1)(x_1^2 - x_2^2) - (x_1^2 - x_2^2)(\tfrac{1}{2} - \tfrac{1}{2}) \\
&\qquad \implies X_1 \cdot X_2 = -\frac{1}{2}(x_1 - x_2)^2
\end{aligned}
$$

Next, we boostrap three point correlator. [6] On the null cone we will have

$$
\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle = \frac{W_M}{(-2X_1\cdot X_2)^{\alpha_{123}}(-2X_1\cdot X_3)^{\alpha_{132}}(-2X_2\cdot X_3)^{\alpha_{231}}}
$$

where the powers α_{ijk} of the scalar factor are determined by the dilatation as in case of scalar operators and
r the tensor structure W_M equals to

$$
W_M = \frac{(-2X_2\cdot X_3)X_{1M} - (-2X_1\cdot X_3)X_{2M} - (-2X_1\cdot X_2)X_{3M}}{(-2X_1\cdot X_2)^{\frac{1}{2}}(-2X_1\cdot X_3)^{\frac{1}{2}}(-2X_2\cdot X_3)^{\frac{1}{2}}}.
$$

Let us comment a few things on the tensor structure. The relative sign is, as before, fixed by transversality. We drop the term proportional to X_{3M}, since would project to zero anyway. The scaling behavior of correlation function under dilatation is completely determined in the scalar part so the tensor structure have scaling 0 in all variables ($X \to \lambda X \implies W_\mu \to \lambda^0 W_\mu$). Finally, it is immediate to check that the tensor structure is transverse, i.e. $(X_3)_M W_M = 0$. Projecting to physical space as:

$$
\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle = \frac{\partial X_3^M}{\partial x_3^\mu}\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle
$$

we find, as explicitly computed before,

$$
\begin{aligned}
\frac{\partial X_3^M}{\partial x_3^\mu}X_{iM} &= (x_i - x_3)_\mu, \quad i = 1,2 \\
-2X_i\cdot X_j &= (x_i - x_j)^2, \quad i = 1,2,3\ (i<j),
\end{aligned}
$$

so that we end up with the tensor structure

$$
W_\mu = \frac{|x_2 - x_3|^2(x_1 - x_3)_\mu - |x_1 - x_3|^2(x_2 - x_3)_\mu}{|x_1 - x_2||x_1 - x_3||x_2 - x_3|} = \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}
$$

Therefore, the three-point function of two scalars and one spin 1 operators in physical space is given by

$$
\begin{aligned}
\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle &= \frac{\frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \\
&= \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2+1}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1+1}}
\end{aligned}
$$

The three-point function of higher-spin operators $J_{\mu_1\ldots\mu_\ell}$ is constructed from the above, analogously as what we did for the two-point functions, since it turns out that W_μ is the only indexed object for three points that is conformal invariant.

[6] pg 30 of Masters Thesis on "Spinning Correlators at Finite Temperature" of Oscar Arandes Tejerina

2.3 Fermions in Embedding Space

Following is taken from section 3.2 of Lectures on Conformal Field Theories by Hugh Osborn. To discuss spinor fields in the embedding formalism requires extending the usual $d-$dimensional gamma matrices to $d+2$ dimensions. For $d = 2n$, we define[7]

$$\begin{aligned}
a_0^{\pm} &= \frac{1}{2}(\pm\gamma^0 + \gamma^1) \\
a_1^{\pm} &= \frac{1}{2}(\gamma^2 \pm i\gamma^3) \\
a_2^{\pm} &= \frac{1}{2}(\gamma^4 \pm i\gamma^5) \\
&\vdots \\
a_{\frac{d-2}{2}}^{\pm} &= \frac{1}{2}(\gamma^{d-2} \pm i\gamma^{d-1})
\end{aligned}$$

where gamma matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

One can show that:

$$\begin{aligned}
&\{a_i^-, a_j^-\} = \{a_i^+, a_j^+\} = 0 \\
&\{a_i^-, a_j^+\} = \delta_{ij} \qquad i,j = 0,1,2\ldots {}^{d-2}\!/\!_2.
\end{aligned} \tag{2.1}$$

In the literature ${}^{d-2}\!/\!_2$ is defined as another variable labeled by k, but for the sake for clarity we will keep it explicit. This is the algebra of raising and lowering operators for ${}^{d}\!/\!_2$ independent two-level systems. We ask how many basis vectors are there (including lowest weight state) which could be formed by operating ${}^{d}\!/\!_2$ raising a_i^+ on lowest weight state:[8]

$$\sum_{r=0}^{d/2} {}^{d/2}C_r = 2^{d/2}$$

It implies that in $d-$dimensions, we have $2^{d/2} \times 2^{d/2}$ dimensional matrix representation for $\gamma-$matrices. We will use the highest weight representation to determine $a_j^{\pm}$ and then use them to construct γ_μ. From (2.1), we quickly observe that

$$(a_i^-)^2 = 0 = (a_i^+)^2$$

It implies that we can only act a_i or $a_i^\dagger$ once on a state, the second time it acts the state is annihilation. We will build off our intuition from harmonic oscillator (fermionic) and assume that there is a lowest weight state $|\xi\rangle$ such that

$$a_i^- |\xi\rangle = 0 \quad \text{for all } i$$

Similarly, acting on it once by each $a_i^\dagger$ for all i, we can construct states in the representation. The states can be labeled $s = (s_0, s_1, ..., s_{\frac{d-2}{2}})$, where each of the $s_a = \pm\frac{1}{2}$:

$$|\xi^{(s)}\rangle = (a_{\frac{d-2}{2}}^+)^{s_{\frac{d-2}{2}}+\frac{1}{2}} \ldots (a_0^+)^{s_0+\frac{1}{2}} |\xi\rangle \tag{2.2}$$

The lowest weight state $|\xi\rangle$ corresponds to all $s_a = -\frac{1}{2}$. Taking the $|\xi^s\rangle$ as a basis, we derive the matrix elements of γ_μ from the definitions and the anti-commutation relation. Starting with $d = 2$, we have a single two-level system:

$$|\xi^{(\frac{1}{2})}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad\qquad |\xi^{(-\frac{1}{2})}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

we can construct the raising and lowering operator connecting these two matrices as:

$$a_0^- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \qquad\qquad a_0^+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

[7] we have abused notation for the sake of avoiding cluttering of indices and $\pm$

[8] we would like to remind ourselves that number of linearly independent basis is defined as the dimension of space.

we find:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $d = 4$, we have 2 independent fermionic oscillator:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we construct, the following a_i^+ and a_i^- operators for $i = 0, 1$.

$$a_0^+ = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_0^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$a_1^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_1^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

From (2.2) we see that[9]

$$a_0^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \qquad a_1^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \qquad a_0^+ a_1^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, we conclude that the gamma matrices are gives as:

$$\gamma^0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

The above choice of gamma matrices satisfy the clifford algebra, however the chosen basis is not familiar from QFT textbooks. Given a representation γ^μ in d dimensions, we can construct a representation Γ^μ in $d+2$ dimensions using the prescription,

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\gamma^\mu \otimes \sigma^3, \quad \mu = 0, \ldots, d-3,$$

$$\Gamma^{d-2} = \mathbb{I} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^1, \quad \Gamma^{d-1} = \mathbb{I} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^2$$

where the σ^i obey

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

The 2×2 matrices that we add act on the index $s_{d-2/2}$, which newly appears in going from $d = 2k$ to $2k+2$ dimensions. In odd dimensions the first $d-1$ gamma matrices can be constructed as above, and $\Gamma_d = \pm\Gamma_1\Gamma_2 \ldots \Gamma_{d-1}$ completes the gamma matrix algebra. There are two independent representations of the gamma matrix algebra in odd dimensions, differing in the sign of Γ_d. These representations are exchanged by parity, and both representations appear in a parity-conserving theory.

[9] $a_0^{\pm}$ acts like raising and lowering operator in the same oscillator while $a_1^{\pm}$ changes the oscillator.

We now move onto calculating the correlation function involving spinors in embedding space formalism. To define spinor fields on null cone in embedding space requires that the number of component in $\mathbb{R}^d$ is half the number of components in $\mathbb{R}^{d+1,1}$.

$$\psi(x) \to \Psi(X), \qquad \bar{\psi}(x) \to \bar{\Psi}(X)$$

which satisfies the following homogeneity condition:

$$\Psi(\lambda X) = \lambda^{-\Delta+\frac{1}{2}}\Psi(X), \qquad \bar{\Psi}(\lambda X) = \lambda^{-\Delta+\frac{1}{2}}\bar{\Psi}(X)$$

The degrees of freedom of $\Psi; \bar{\Psi}$ are reduced to those for $\psi; \bar{\psi}$ by imposing the transversality condition like before:

$$\bar{\Gamma}_A X^A \Psi(X) = 0 \qquad \bar{\Psi}(X)\Gamma_A X^A = 0$$

It introduces the gauge invariance and thus the degrees of freedom are now halved by imposing the equivalence relations

$$\Psi(X) \sim \Psi + \bar{\Gamma}_A X^A \zeta(X) \qquad \bar{\Psi}(X) \sim \bar{\Psi} + \bar{\zeta}(X)\Gamma_A X^A \tag{2.3}$$

for arbitrary spinor $\zeta(X); \bar{\zeta}(X)$ of appropriate homogeneity. From standard QFT, we are familiar that

$$V_A = \bar{\Psi}\Gamma_A\Psi'$$

transforms like a vector and from (2.3), we have

$$\bar{\Psi}\Gamma_A\Psi' \sim -\bar{\Psi}'\bar{\Gamma}_A\Psi$$

whereas,

$$\bar{\Psi}\Psi$$

transforms like a scalar. However, the above is only under (2.3) in odd dimensions. So it does not correspond to a scalar on the projective null cone in even dimensions.

Chapter 3

de Sitter

We can write the metric for de Sitter space in conformal coordinates as:

$$ds^2 = \frac{1}{(H\eta)^2}(-d\eta^2 + d\vec{x}^2) = \frac{1}{(H\eta)^2}\eta_{\mu\nu}dx^\mu dx^\nu$$

Since the metric is independent of x^i, it has 3 killing vector associated with translation. Since Then $\xi^\mu = \delta^\mu_i$ and

$$\xi_\mu = g_{\mu\nu}\xi^\nu = g_{\mu\nu}\delta^\nu_i = g_{\mu i}$$

From killing equation:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = \xi_{\mu,\nu} + \xi_{\nu,\mu} - 2\Gamma^\rho_{\mu\nu}\xi_\rho$$

Substituting $\xi_\rho = g_{\rho i}$, we get:

$$g_{\mu i,\nu} + g_{\nu i,\mu} - g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})g_{\rho i}$$

Now, using $g_{\rho i}g^{\rho\sigma} = \delta^\sigma_i$, the Killing condition simplifies to:

$$g_{\mu i,\nu} + g_{\nu i,\mu} - (g_{i\mu,\nu} + g_{i\nu,\mu} - g_{\mu\nu,i}) = 0$$

Hence,

$$\xi^x = (0,1,0,0) \qquad \xi^y = (0,0,1,0) \qquad \xi^z = (0,0,0,1)$$

or more compactly

$$\boxed{\xi = \xi^i\partial_i = \partial_i}$$

To generate translation from them, we just exponentiate the generators:

$$e^{a\partial_x}x = x + a\partial_x x + 0 = x + a$$

We can find the remaining by solving the killing equation. The non vanishing connection terms are given as:

$$\Gamma^x_{\eta x} = -\frac{1}{\eta} \qquad \Gamma^y_{\eta y} = -\frac{1}{\eta}$$

$$\Gamma^z_{\eta z} = -\frac{1}{\eta} \qquad \Gamma^\eta_{xx} = -\frac{1}{\eta}$$

$$\Gamma^\eta_{yy} = -\frac{1}{\eta} \qquad \Gamma^\eta_{zz} = -\frac{1}{\eta}$$

$$\Gamma^\eta_{\eta\eta} = -\frac{1}{\eta}$$

We can write the killing equation as

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$$
$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - 2\Gamma^\lambda_{\mu\nu}\xi_\lambda = 0$$

for $\mu = \nu$, we have following four expressions:

$$\partial_x\xi_x = -\frac{1}{\eta}\xi_\eta \qquad \partial_y\xi_y = -\frac{1}{\eta}\xi_\eta$$

$$\partial_z \xi_z = -\frac{1}{\eta}\xi_\eta \qquad\qquad \partial_\eta \xi_\eta = -\frac{1}{\eta}\xi_\eta$$

We can solve the last equation as:

$$\frac{d\xi_\eta}{\xi_\eta} = -\frac{d\eta}{\eta} \quad \Rightarrow \quad \ln \xi_\eta = -\ln \eta + \ln f(x, y, z)$$

which leads to

$$\boxed{\xi_\eta(\eta, x, y, z) = \frac{f(x, y, z)}{\eta}}$$

Now we determine $f(x, y, z)$:

$$\xi_\eta = \frac{f(x, y, z)}{\eta} \Rightarrow \partial_\eta \xi_\eta = -\frac{1}{\eta}\xi_\eta = -\frac{f(x, y, z)}{\eta^2}$$

So,

$$\partial_x \xi_x = -\frac{f(x, y, z)}{\eta^2}; \qquad \partial_y \xi_y = -\frac{f(x, y, z)}{\eta^2}; \qquad \partial_z \xi_z = -\frac{f(x, y, z)}{\eta^2}$$

For now let us focus on the specific case where,

$$f(x, y, z) = \text{const} = A \Rightarrow \xi_\eta = \frac{A}{\eta}$$

We will determine the specific form of $f(x, y, z)$ later. Then:

$$\partial_x \xi_x = -\frac{A}{\eta^2} \Rightarrow \xi_x = -\frac{A}{\eta^2}x + \phi_1(y, z, \eta)$$
$$\partial_y \xi_y = -\frac{A}{\eta^2} \Rightarrow \xi_y = -\frac{A}{\eta^2}y + \phi_2(x, z, \eta)$$
$$\partial_z \xi_z = -\frac{A}{\eta^2} \Rightarrow \xi_z = -\frac{A}{\eta^2}z + \phi_3(x, y, \eta)$$

For now lets set $\phi_i = 0$ which describes rotation and fix coefficient A such that in the component form:

$$\boxed{\xi^\mu = g^{\mu\nu}\xi_\nu = (\eta, x, y, z)}$$

is the killing vector associated with dilatation. With the basis, it looks like:

$$D = \xi^\mu \partial_\mu = x^\mu \partial_\mu$$

Next, we solve for $\mu \neq \nu$ case where $\Gamma^\lambda_{\mu\nu} = 0$:

$$\partial_x \xi_y + \partial_y \xi_x = 0$$
$$\partial_x \xi_z + \partial_z \xi_x = 0$$
$$\partial_y \xi_z + \partial_z \xi_y = 0$$

Since $\partial_i g^{\mu\nu} = 0$, we can rewrite them as

$$\partial_x \xi^y + \partial_y \xi^x = 0$$
$$\partial_x \xi^z + \partial_z \xi^x = 0$$
$$\partial_y \xi^z + \partial_z \xi^y = 0$$

These are same as killing vector for flat space rotation.

$$\xi = \epsilon^{(i)jk} x_j \partial_k$$

The remaining ones which lead to SCT are as follows:

$$\partial_i \xi_\eta + \partial_\eta \xi_i + \frac{2}{\eta}\xi_i = 0$$

Going back we found the solution to

$$\partial_\eta\, \xi_\eta \;=\; -\frac{1}{\eta}\,\xi_\eta,$$

as

$$\xi_\eta(\eta, x, y, z) \;=\; \frac{f(x,y,z)}{\eta} \quad \text{for some function } f(x,y,z).$$

This time we won't assume it to just be a constant. Therefore the killing equation to be solved becomes:

$$\frac{1}{\eta}\,\partial_i f(x,y,z) \;+\; \partial_\eta\,\xi_i \;+\; \frac{2}{\eta}\,\xi_i = 0,$$

which is a first-order linear ordinary differential equation in η for each spatial component ξ_i. The homogeneous part $\partial_\eta \xi_i + (2/\eta)\xi_i = 0$ integrates immediately to $\xi_i^{(\mathrm{hom})}(\eta) = C_i(x,y,z)/\eta^2$. A particular solution of the inhomogeneous equation can be found by considering power series in η, defined as $\xi_i = A(x,y,z)\eta^\alpha$

$$\begin{aligned} A\alpha\eta^{\alpha-1} + 2A\eta^{\alpha-1} &= -\eta^{-1}\partial_i f(x,y,z) \\ (\alpha+2)A\eta^{\alpha-1} &= -\eta^{-1}\partial_i f(x,y,z) \implies \alpha = 0, A = -\frac{1}{2}\partial_i f(x,y,z) \end{aligned}$$

Therefore, $\xi_i^{(\mathrm{part})} = -\frac{1}{2}\,\partial_i f(x,y,z)$. Hence the full solution is

$$\begin{aligned} \xi_\eta &= \frac{f(x,y,z)}{\eta} \\ \xi_i(\eta,x,y,z) &= \frac{C_i(x,y,z)}{\eta^2} \;-\; \frac{1}{2}\,\partial_i f(x,y,z) \end{aligned}$$

Meanwhile, from the $i \neq j$ Killing equations:

$$\partial_i \xi_j + \partial_j \xi_i = \frac{1}{\eta^2}(\partial_i C_j + \partial_j C_i) - \partial_i \partial_j f = 0$$

we infer $\partial_i\partial_j f = 0$ and $\partial_i C_j + \partial_j C_i = 0$. From $\partial_i\partial_j f = 0$ one concludes f is at most linear in (x,y,z), so

$$f(x,y,z) = a_x\,x + a_y\,y + a_z\,z + A$$

with constant coefficients a_i, A. There are further constraints on C_i as:

$$\begin{aligned} \partial_i \xi_i &= \frac{1}{\eta^2}\partial_i C_i(x,y,z) - \frac{1}{2}\partial_i\partial_i f(x,y,z) = -\frac{f(x,y,z)}{\eta^2} \\ \partial_i C_i(x,y,z) &= -f(x,y,z) \end{aligned}$$

Since we have already studied dilatation, where we studied the effect of $A \neq 0$, therefore let us set A to zero,

$$C_i = -\int f(x,y,z)dx_i = -(a\cdot x)x_i + \frac{1}{2}a_i(x^i)^2 + g(x_j \forall j \neq i)$$

Then

$$\partial_i C_j + \partial_j C_i = 0 \implies C_i = -\left(\sum_{j=1}^{3} a_j x^j\right)x_i + \frac{1}{2}a_i\left(\sum_{j=1}^{3} x^j x^j\right)$$

Let choose $(a_x, a_y, a_z) = -\frac{2}{H^2}(b_x, b_y, b_z)$ so ξ_i appears without fraction. Then

$$\xi_\eta(\eta,x,y,z) = -\frac{2b_i x^i}{H^2\eta} \qquad \xi_i(\eta,x,y,z) \begin{aligned}[t] &= \frac{2(b\cdot x)x_i - b_i\vec{x}^2}{H^2\eta^2} - \frac{1}{2H^2}\partial_i\big(-2b_j x^j\big) \\ &= \frac{2(b\cdot x)x_i - b_i\vec{x}^2}{H^2\eta^2} + \frac{b_i}{H^2} \end{aligned}$$

Raising indices gives[1]

$$\xi^\eta = 2\,\eta\,\big(b_i x^i\big), \qquad \xi^i = (\eta^2 - \vec{x}^2)b^i + 2(b\cdot x)x^i$$

Since $x^\mu = (\eta, x^i)$ and $b^\mu = (0, b^i)$, this coincides exactly with

$$\xi^\mu = 2(b\cdot x)x^\mu - b^\mu\big(-\eta^2 + \vec{x}^2\big), \quad b^0 = 0$$

If $b^0 = 0$ condition was not imposed then the above generator would be generating SCT however, we can quickly note that on $\eta = 0$ hypersurface, the killing vector written above does generate special conformal transformation over $\mathbb{R}^3$. Hence, we note the killing vectors as following which matches with 1.11

[1] indices of b_i and x_i are raised and lowered by $\eta_{\mu\nu}$

- Translation
$$P_i = \partial_i$$
- Rotation
$$L_{ij} = x_i\partial_j - x_j\partial_i$$
- Dilatation
$$D = x^\mu \partial_\mu$$
- SCT
$$K^i = [2(b^j c_j)x^i - b^i(\eta_{kl}x^k x^l)]\partial_i$$

3.1 Ambient space

The generators of $D+1$ dimensional lorentz transformations are given as:

$$J_{MN} = X_M\partial_N - X_N\partial_M$$

where $M, N = 0, 1, 2, \dots D$. In the flat slicing, the coordinates are given as:

$$\left.\begin{aligned} X^0 &= \tfrac{\rho}{2(-\eta)}(1-s) \\ X^D &= \tfrac{\rho}{2(-\eta)}(1+s) \end{aligned}\right\} \implies \rho = -\eta(X^D + X^0)$$
$$X^i = \frac{\rho}{-\eta}x^i \implies x^i = \frac{-\eta}{\rho}X^i = \frac{X^i}{X^0 + X^D}$$

where

$$s = \eta^2 - \delta_{ij}x^i x^j$$

and $i = 1, 2, \dots D-1$. We can note from below that $\rho =$constant hypersurface are hyperbolic in nature.

$$g_{AB}X^A X^B = -(X^0)^2 + (X^D)^2 + \delta_{ij}X^i X^j = \frac{\rho^2 s^2}{\eta^2} + \frac{\rho^2}{\eta^2}\delta_{ij}x^i x^j = \rho^2$$

Inverting above relation,

$$\begin{aligned}
\rho &= \sqrt{g_{AB}X^A X^B} \implies \frac{\partial\rho}{\partial X^0} = -\frac{X^0}{\rho} = \frac{s-1}{2(-\eta)} \\
\eta &= -\frac{\rho}{X^0 + X^D} \implies \frac{\partial\eta}{\partial X^0} = \frac{X_0X_D + X_D^2 + X^iX_i}{(X^D + X^0)^2\sqrt{g_{AB}X^AX^B}} = \frac{\frac{1+s}{2} + \delta_{ij}x^ix^j}{\rho} \\
x^i &= \frac{X^i}{X^0 + X^D}
\end{aligned}$$

we also have

$$\begin{aligned}
\frac{\partial}{\partial X^0} &= \frac{1}{2(-\eta)}(-1 + \eta^2 - \delta_{ij}x^ix^j)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(1 + \eta^2 + \delta_{ij}x^ix^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i \\
\frac{\partial}{\partial X^D} &= \frac{1}{2(-\eta)}(1 + \eta^2 - \delta_{ij}x^ix^j)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(-1 + \eta^2 + \delta_{ij}x^ix^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i \\
\frac{\partial}{\partial X^i} &= \frac{x_i}{-\eta}\frac{\partial}{\partial\rho} - \frac{x_i}{\rho}\frac{\partial}{\partial\eta} - \frac{\eta}{\rho}\partial_i
\end{aligned}$$

In this coordinate

$$ds^2 = -(dX^0)^2 + (dX^D)^2 + \delta_{ij}dX^idX^j = d\rho^2 + \rho^2\frac{(-d\eta^2 + \delta_{ij}dx^idx^j)}{\eta^2}$$

Then

$$\begin{aligned}
P_i = J_{Di} - J_{0i} &= X_D\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^D} - X_0\frac{\partial}{\partial X^i} + X_i\frac{\partial}{\partial X^0} \\
&= (X_D - X_0)\frac{\partial}{\partial X^i} - X_i\left(\frac{\partial}{\partial X^D} - \frac{\partial}{\partial X^0}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (X^D + X^0)\frac{\partial}{\partial X^i} - X^i\left(\frac{\partial}{\partial X^D} - \frac{\partial}{\partial X^0}\right) \\
&= \frac{\rho}{-\eta}\left(\frac{x^i}{-\eta}\frac{\partial}{\partial\rho} - \frac{x^i}{\rho}\frac{\partial}{\partial\eta} - \frac{\eta}{\rho}\partial_i\right) - \frac{\rho}{-\eta}x^i\left(\frac{1}{-\eta}\frac{\partial}{\partial\rho} - \frac{1}{\rho}\frac{\partial}{\partial\eta}\right) \\
&= \partial_i
\end{aligned}
$$

since $i = 1, 2, \ldots D-1$, there are $D-1$ momenta and SCT generator.

$$
\begin{aligned}
K_i = J_{Di} + J_{0i} &= X_D\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^D} + X_0\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^0} \\
&= (X_D + X_0)\frac{\partial}{\partial X^i} - X_i\left(\frac{\partial}{\partial X^D} + \frac{\partial}{\partial X^0}\right) \\
&= (X^D - X^0)\frac{\partial}{\partial X^i} - X^i\left(\frac{\partial}{\partial X^D} + \frac{\partial}{\partial X^0}\right) \\
&= \frac{\rho s}{-\eta}\left(\cancel{\frac{x^i}{-\eta}\frac{\partial}{\partial\rho}} - \frac{x^i}{\rho}\frac{\partial}{\partial\eta} - \frac{\eta}{\rho}\partial_i\right) - \frac{\rho}{-\eta}x^i\left(\cancel{\frac{s}{-\eta}\frac{\partial}{\partial\rho}} + \frac{\eta^2 + \delta_{ij}x^ix^j}{\rho}\frac{\partial}{\partial\eta} + \frac{2\eta}{\rho}x^j\partial_j\right) \\
&= \cancel{\frac{x^is}{\eta}\frac{\partial}{\partial\eta}} + s\partial_i + \frac{2\eta^2 - \cancel{s}}{\eta}x_i\frac{\partial}{\partial\eta} + 2x_i(x^j\partial_j) \\
&= (\eta^2 - \delta_{jk}x^jx^k)\partial_i + 2x_i\eta\partial\eta + 2x_i(x^j\partial_j) \\
&= 2x_i(x\cdot\partial) - (-\eta^2 + \vec{x}^2)\partial_i
\end{aligned}
$$

$$
\begin{aligned}
D = J_{D0} &= X_D\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X^D} = X_D\frac{\partial}{\partial X^0} + X^0\frac{\partial}{\partial X^D} \\
&= \frac{\rho}{2(-\eta)}(1+s)\left[\frac{1}{2(-\eta)}(-1+s)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(1+\eta^2+\delta_{ij}x^ix^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i\right] \\
&\quad + \frac{\rho}{2(-\eta)}(1-s)\left[\frac{1}{2(-\eta)}(1+s)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(-1+\eta^2+\delta_{ij}x^ix^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i\right] \\
&= -\eta\frac{\partial}{\partial\eta} - x^i\partial_i
\end{aligned}
$$

These generators do not have any dependence on de Sitter radius l. Therefore, we could work in sterographic projection of de sitter coordinates onto minkowski space (sterographic projection as conformal transformation). We begin by recalling that de Sitter space can be realized as a hyperboloid embedded in $d+1$ dimensions:

$$
-X_0^2 + X_1^2 + \cdots + X_{d-1}^2 + X_d^2 = l^2.
$$

The normal to this surface could be given as:

$$
n_A = \frac{\partial}{\partial x^A}\left(X^BX_B - l^2\right) = 2X_A
$$

Therefore, any tensor field K which is tangent to this hypersurface satisfies the following transversality condition.

$$
X^AK_A = 0
$$

We define the streographic projection by the following coordinate transformation:

$$
\begin{aligned}
X^\mu \equiv r^\mu &= \Omega(x)x^\mu \\
X^4 \equiv r^4 &= l\Omega(x)\left(1 - \frac{x^2}{4l^2}\right) \\
\Omega(x) &= \frac{1}{1 + \frac{x^2}{4l^2}}
\end{aligned}
$$

Here, r^a is the cartesian coordinate on de Sitter space and x^a is the coordinate on projected minkowski space. We can note that $r_Ar^A = l^2$ is imposed on de Sitter coordinates but no such condition is imposed on $x^2 = \eta_{\mu\nu}x^\mu x^\nu$.Then we define the following object:[2]

$$
K^\mu_A = \frac{1}{\Omega(x)^2}\frac{\partial r_A}{\partial x_\mu} = \frac{\delta^\mu_A}{\Omega} - r_A\frac{\partial}{\partial x_\mu}\left(\frac{1}{\Omega(x)}\right)
$$

[2]this insight was provided in A unified construction of Skyrme-type non-linear sigma models via the higher dimensional Landau models

$$= \left(1 + \frac{x^2}{4l^2}\right)^2 \frac{\partial r_A}{\partial x_\mu}$$

and we can derive their explicit form by using:

$$\frac{\partial r_\nu}{\partial x_\mu} = \Omega(x)\delta^\mu_\nu - x_\nu \Omega(x)^2 \frac{x^\mu}{2l^2} \qquad \frac{\partial r_5}{\partial x^\mu} = -\Omega(x)^2 \frac{x_\mu}{l}$$

which satisfies the transversality condition and as such can be used as projection tensor. However, we will derive their explicit form in somewhat lengthy but illuminating way which will reveal their true nature. These K^μ_A satisfy the conformal killing equation

$$\begin{aligned} \frac{\partial}{\partial x^\mu} K^A{}_\nu + \frac{\partial}{\partial x^\nu} K^A{}_\mu &= \delta^A_\mu \partial_\nu \left(\frac{1}{\Omega}\right) - \delta^A_\nu \partial_\mu \left(\frac{1}{\Omega}\right) - x^A \partial_\mu \partial_\nu \left(\frac{1}{\Omega}\right) \\ &\quad + \delta^A_\nu \partial_\mu \left(\frac{1}{\Omega}\right) - \delta^A_\mu \partial_\nu \left(\frac{1}{\Omega}\right) - x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) \\ &= -2x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) = 2f(x) g_{\mu\nu} \end{aligned}$$

and

$$\begin{aligned} f(x) g_{\mu\nu} &= -x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) \\ f(x) g^{\mu\nu} g_{\mu\nu} &= -x^A \partial_\mu \partial^\mu \left(\frac{1}{\Omega}\right) = \partial_\mu K^{A\mu} \\ f(x) &= \frac{\partial_\mu K^{A\mu}}{d} \end{aligned}$$

So, $K^A{}_\mu$ could be interpreted as killing vector as long as $\partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) \propto g_{\mu\nu}$. There is additional property that $K^A{}_\mu$ satisfies, namely transversality condition:[3]

$$r^A K_A{}^\mu \Big|_{r_A r^A = l^2} = 0 \implies x^\nu K_\nu{}^\mu + l\left(1 - \frac{x^2}{4l^2}\right) K_4{}^\mu = 0$$

using the general form of conformal killing vectors:

$$K^\mu_a = t^\mu_a + \epsilon_a x^\mu + \omega^{\mu\nu}_a x_\nu + \lambda^\mu_a x^2 - 2\lambda^\sigma_a x_\sigma x^\mu$$

$$\begin{aligned} &x^\nu (t^\mu_\nu + \epsilon_\nu x^\mu + \omega^{\mu\sigma}_\nu x_\sigma + \lambda^\mu_\nu x^2 - 2\lambda^\sigma_\nu x_\sigma x^\mu) \\ &\qquad + l\left(1 - \frac{x^\nu x_\nu}{4l^2}\right)(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) = 0 \end{aligned}$$

Or,

$$\begin{aligned} &x^\nu t^\mu_\nu + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + \lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu) \\ &\qquad + l(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) \\ &\qquad - \frac{x^\nu x_\nu}{4l}(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) = 0 \end{aligned}$$

It can be conveniently written as:

$$\begin{aligned} &l t^\mu_4 + (x^\nu t^\mu_\nu + l\epsilon_4 x^\mu + l\omega^{\mu\nu}_4 x_\nu) + \left[-\frac{x^2}{4l} t^\mu_4 + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + l\lambda^\mu_4 x^2 - 2l\lambda^\sigma_4 x_\sigma x^\mu\right] \\ &\qquad + \left[\lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu - \frac{1}{4l}\epsilon_4 x^2 x^\mu - \frac{1}{4l}\omega^{\mu\nu}_4 x^2 x_\nu\right] + \left[-\frac{x^4}{4l}\lambda^\mu_4 + \frac{2x^2 x_\sigma x^\mu}{4l}\lambda^\sigma_4\right] = 0 \end{aligned}$$

[3] following the procedure outlined in section 2 of Gauge Theories on de Sitter space and Killing Vectors

Let us set coefficient of x^ν to zero, order by order.

$$\begin{aligned}
t_4^\mu &= 0 \\
x^\nu t_\nu^\mu + l\epsilon_4 x^\mu + l\omega_4^{\mu\nu} x_\nu &= 0 \\
-\frac{x^2}{4l} t_4^\mu + \epsilon_\nu x^\nu x^\mu + \omega_\nu^{\mu\sigma} x^\nu x_\sigma + l\lambda_4^\mu x^2 - 2l\lambda_4^\sigma x_\sigma x^\mu &= 0 \\
\lambda_\nu^\mu x^\nu x^2 - 2\lambda_\nu^\sigma x^\nu x_\sigma x^\mu - \frac{1}{4l}\epsilon_4 x^2 x^\mu - \frac{1}{4l}\omega_4^{\mu\nu} x^2 x_\nu &= 0 \\
-\frac{x^4}{4l}\lambda_4^\mu + \frac{2x^2 x_\sigma x^\mu}{4l}\lambda_4^\sigma &= 0
\end{aligned}$$

contracting the last four equations with x_μ:

$$\begin{aligned}
t_\nu^\mu x_\mu x^\nu + l\epsilon_4 x^2 &= 0 \\
\epsilon_\nu x^\nu x^2 + l\lambda_4^\mu x_\mu x^2 - 2l\lambda_4^\sigma x_\sigma x^2 &= 0 \\
\lambda_\nu^\mu x_\mu x^\nu x^2 - 2\lambda_\nu^\sigma x^\nu x_\sigma x^2 - \frac{1}{4l}\epsilon_4 x^4 &= 0 \\
x^4 \lambda_4^\mu &= 0
\end{aligned}$$

Hence,

$$\begin{aligned}
t_\nu^\mu x_\mu x^\nu + l\epsilon_4 x^2 &= 0 \\
\epsilon_\nu x^\nu x^2 &= 0 \\
-\lambda_\nu^\sigma x^\nu x_\sigma x^2 - \frac{1}{4l}\epsilon_4 x^4 &= 0
\end{aligned}$$

Since t generates translation, we have $t_\nu^\mu = \delta_\nu^\mu$:

$$\begin{aligned}
\epsilon_4 &= -\frac{1}{l} \\
\lambda_{\mu\nu} &= \frac{1}{4l^2} g_{\mu\nu}
\end{aligned}$$

Substituting them in Killing vector, we get:

$$\begin{aligned}
K_\nu^\mu &= \left(1 + \frac{x^2}{4l^2}\right)\delta_\nu^\mu - \frac{x_\nu x^\mu}{2l^2} \\
&= \delta_\nu^\mu + \frac{1}{4l^2}(x^2\delta_\nu^\mu - 2x_\nu x^\mu) \\
K_4^\mu &= -\frac{x^\mu}{l}
\end{aligned}$$

These satisfy few properties:

$$\begin{aligned}
K_A^\mu K^{A\nu} &= \left(1 + \frac{x^2}{4l^2}\right)^2 g^{\mu\nu} = \frac{1}{\Omega(x)^2} g^{\mu\nu} \\
K_B^\mu K_{C\mu} &= \left(1 + \frac{x^2}{4l^2}\right)^2 (\delta_{BC} - r_B r_C)
\end{aligned}$$

verify above part, there could be missing scaling on the last $r_B r_C$ term. It could be used to make projection operator like[4]

$$\begin{aligned}
A_\mu &= \frac{\partial r^B}{\partial x^\mu}\hat{A}_B = \Omega(x)^2 K^{B\mu}\hat{A}_B \\
K_C^\mu A_\mu &= \Omega(x)^2 K^{B\mu} K_C^\mu \hat{A}_B = (\delta_C^B - r^B r_C)\hat{A}_B \\
\hat{A}_B &= K_B^\mu A_\mu \qquad \text{(where we used } r^B\hat{A}_B = 0\text{)}
\end{aligned}$$

[4]hat tensors are on de Sitter and without hat ones are on Minkowski

We can define the derivative on de Sitter space as:

$$\begin{aligned}\hat{\partial}_A &= \frac{\partial}{\partial r^A} - \frac{r_A r_B}{l^2}\frac{\partial}{\partial r_B} \implies r^A\hat{\partial}_A = 0\\ &\equiv K^\mu_A \frac{\partial}{\partial x^\mu}\end{aligned}$$

using $r_A r^A = l^2$ and $\eta_{AB} = (-, +, + \ldots)$

$$\begin{aligned}J_{AB} &= r_A\frac{\partial}{\partial r^B} - r_B\frac{\partial}{\partial r^A}\\ &= r_A\left(\hat{\partial}_B + \frac{r_B r_C}{l^2}\frac{\partial}{\partial r_C}\right) - r_B\left(\hat{\partial}_A + \frac{r_A r_C}{l^2}\frac{\partial}{\partial r_C}\right)\\ &= (r_A\hat{\partial}_B - r_B\hat{\partial}_A)\\ &= (r_A K^\mu_B - r_B K^\mu_A)\partial_\mu\end{aligned}$$

which could be put in more illuminating form:

$$r^A J_{AB} = l^2 K^\mu_B \partial_\mu$$

Then, for $A, B \neq 4$

$$\begin{aligned}L_{ab} &= \Omega(x)\left[x_a\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_b - \frac{x_a x_b x^\mu}{4l^2} - x_b\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_a + \frac{x_a x_b x^\mu}{4l^2}\right]\partial_\mu\\ &= (x_a\delta^\mu_b - x_b\delta^\mu_a)\partial_\mu\end{aligned}$$

These are spatial rotation. Boosts in the ambient space could be given as

$$K_i = J_{0i}$$

Now we calculate the de Sitter Momenta

$$\begin{aligned}l\mathbf{P}_\nu = J_{4\nu} &= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left\{\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu - \frac{x_\nu x^\mu}{2l^2}\right\} + \frac{x_\nu x^\mu}{l}\right]\partial_\mu\\ &= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu - \frac{x_\nu x^\mu}{2l} + \frac{x^2}{4l^2}\frac{x_\nu x^\mu}{2l} + \frac{x_\nu x^\mu}{l}\right]\partial_\mu\\ &= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu + \frac{x^2}{4l^2}\frac{x_\nu x^\mu}{2l} + \frac{x_\nu x^\mu}{2l}\right]\partial_\mu\\ &= \left[l\left(1-\frac{x^2}{4l^2}\right)\delta^\mu_\nu + \frac{x_\nu x^\mu}{2l}\right]\partial_\mu\\ &= lP_\nu + \frac{1}{4l}K_\nu\end{aligned}$$

The zeroth component of this momenta (which used to be dilatation) becomes our new Hamiltonian:

$$H = J_{40}$$

Since under Wigner Innou contraction the rotations about these planes will appear as translation and they are generated by Momentum operator. We will often refer to these as de Sitter momenta and in the limit $l \to \infty$, they begin to commute and we recover Lorentz algebra.

$$\begin{aligned}[J_{\mu\nu}, J_{\lambda\rho}] &= \eta_{\mu\lambda}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\lambda} + \eta_{\nu\rho}J_{\mu\lambda} - \eta_{\nu\lambda}J_{\mu\rho}\\ [J_{\mu\nu}, P_\lambda] &= \frac{1}{l}[J_{\mu\nu}, J_{4\lambda}] = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu\\ [P_\mu, P_\nu] &= \frac{1}{c^2l^2}[J_{4\mu}, J_{4\nu}] = \frac{1}{c^2l^2}J_{\mu\nu}\end{aligned}$$

The scalar curvature of de Sitter space is given by:

$$R = \frac{d(d-1)}{l^2} = \frac{2d}{d-2}\Lambda$$

3.2 Wigner Innou Contraction

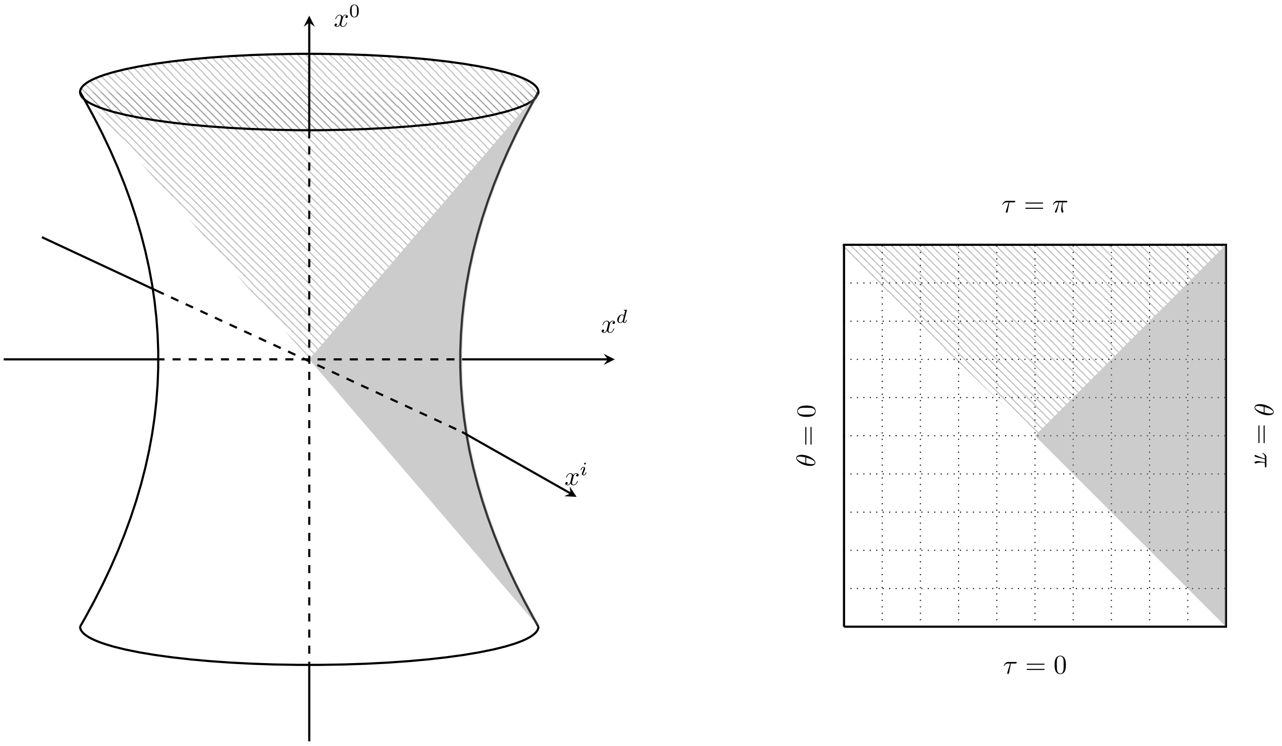

Figure 3.1: To take the $l \to \infty$ limit, imagine placing a plane at $x^d = l$. The plane will be coordinatized by x^μ and the rotation in x^0x^d plane as well as x^ix^d plane will appear as translation. That's why we split the generators of the ambient space into 6 lorentz generator + 4 translation.

We will first use the stereographic projection to see how the Newton Hook limit could be achieved.

$$\begin{aligned} P_\mu &= \frac{J_{4\mu}}{l} \\ L_{ab} &= J_{ab} = x_a\partial_b - x_b\partial_a \\ -K_a = L_{a0} &= \frac{J_{a0}}{c} \end{aligned}$$

where $a = 1, 2, 3$ and $\mu = 0, 1, 2, 3$ with mostly plus metric signature:

$$[J_{AB}, J_{CD}] = \eta_{AC}J_{BD} - \eta_{AD}J_{BC} + \eta_{BD}J_{AC} - \eta_{BC}J_{AD}$$

which leads to

$$\begin{aligned} [J_{\mu\nu}, J_{\lambda\rho}] &= \eta_{\mu\lambda}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\lambda} + \eta_{\nu\rho}J_{\mu\lambda} - \eta_{\nu\lambda}J_{\mu\rho} \\ [J_{\mu\nu}, P_\lambda] &= \frac{1}{l}[J_{\mu\nu}, J_{4\lambda}] = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu \\ [P_\mu, P_\nu] &= \frac{1}{c^2l^2}[J_{4\mu}, J_{4\nu}] = \frac{1}{c^2l^2}J_{\mu\nu} \end{aligned}$$

Then, the algebra becomes:[5]

$$\begin{aligned} [L_{ab}, L_{de}] &= \delta_{ad}L_{be} + \delta_{be}L_{ad} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd} \\ [L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0} \\ [L_{0b}, L_{0e}] &= \frac{1}{c^2}[J_{0b}, J_{0e}] = \eta_{00}\frac{1}{c^2}J_{be}, \\ [L_{ab}, P_d] &= \frac{1}{l}[J_{ab}, J_{4d}] = \delta_{bd}\frac{J_{a4}}{l} - \delta_{ad}\frac{J_{b4}}{l} = \delta_{ad}P_b - \delta_{bd}P_a \end{aligned}$$

[5]This form of the analysis was suggested in section 2.4.2 of de Sitter Relativity: Foundation and some physical implications

$$
\begin{aligned}
[L_{a0}, P_b] &= \frac{1}{cl}[J_{a0}, J_{4b}] = -\frac{1}{cl}\delta_{ab}J_{04} = \frac{1}{c}\delta_{ab}P_0,\\
[L_{a0}, P_0] &= \frac{1}{cl}[J_{a0}, J_{40}] = \eta_{00}\frac{1}{cl}J_{a4} = \frac{1}{c}P_a,\\
[L_{ab}, P_0] &= \frac{1}{l}[J_{ab}, J_{40}] = 0,\\
[P_a, P_b] &= \frac{1}{l^2}[J_{4a}, J_{4b}] = \frac{\eta_{44}}{l^2}J_{ab} = \frac{1}{l^2}L_{ab},\\
[P_a, P_0] &= \frac{1}{l^2}[J_{4a}, J_{40}] = \frac{\eta_{44}}{l^2}L_{a0} = -\frac{c}{l^2}K_a\\
[P_0, P_0] &= \frac{1}{l^2}[J_{40}, J_{40}] = 0.
\end{aligned}
$$

We find that unless we scale P_0 by c. The resulting algebra will be ill defined as $[L_{a0}, P_0] \to 0$ and it will not match with Galilean group where $[L_{a0}, P_0] = P_a$.

$$
\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{be}L_{ad} - \delta_{ad}L_{be} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd}\\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0}\\
[L_{0b}, L_{0e}] &= -\frac{1}{c^2}L_{be},\\
[L_{ab}, P_d] &= \delta_{ad}P_b - \delta_{bd}P_a\\
[L_{a0}, P_b] &= \frac{1}{c^2}\delta_{ab}P_0,\\
[L_{a0}, P_0] &= P_a,\\
[L_{ab}, P_0] &= 0,\\
[P_a, P_b] &= \frac{1}{\tau^2 c^2}L_{ab},\\
[P_a, P_0] &= \frac{1}{\tau^2}L_{a0},\\
[P_0, P_0] &= 0.
\end{aligned}
$$

where $\tau = \frac{l}{c}$ is kept constant during the process of taking the limit $l \to \infty$ and $c \to \infty$:

$$
\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{be}L_{ad} - \delta_{ad}L_{be} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd}\\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0}\\
[L_{0b}, L_{0e}] &= 0\\
[L_{ab}, P_d] &= \delta_{ad}P_b - \delta_{bd}P_a\\
[L_{a0}, P_b] &= 0,\\
[L_{a0}, P_0] &= P_a,\\
[L_{ab}, P_0] &= 0,\\
[P_a, P_b] &= 0\\
[P_a, P_0] &= -\frac{1}{\tau^2}K_a,\\
[P_0, P_0] &= 0.
\end{aligned}
$$

which is same as

$$
\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk}J_k, && [J_i, P_j] = \epsilon_{ijk}P_k, && [J_i, K_j] = \epsilon_{ijk}K_k,\\
&[P_i, P_j] = 0, && [K_i, K_j] = 0, && [K_i, P_j] = 0,\\
&[H, J_i] = 0, && [H, P_i] = \frac{c^2\Lambda}{3}K_i, && [H, K_i] = P_i,
\end{aligned}
$$

for

$$
J_i = \frac{1}{2}\epsilon_{ijk}J_{jk} \qquad\qquad P_0 = H
$$

Alternatively, we can derive it by considering the lie algebra of the killing vector fields in the ambient space in the following form:

$$
\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, P_j] = \epsilon_{ijk} P_k, && [J_i, K_j] = \epsilon_{ijk} K_k,\\
&[P_i, P_j] = \epsilon_{ijk} J_k, && [K_i, K_j] = -\epsilon_{ijk} J_k, && [K_i, P_j] = -\delta_{ij} H,\\
&[H, J_i] = 0, && [H, P_i] = K_i, && [H, K_i] = P_i,
\end{aligned}
$$

Under the Wigner-Innou contraction to be considered here, we only have to examine Lie brackets which doesn't involve J_i. Therefore, we only have to worry about the following 5 commutators:

$$
\begin{aligned}
&[P_i, P_j] = \epsilon_{ijk} J_k, && [K_i, K_j] = -\epsilon_{ijk} J_k, && [K_i, P_j] = -\delta_{ij} H,\\
&[H, P_i] = K_i, && [H, K_i] = P_i,
\end{aligned}
$$

We will consider the following rescaling as suggested in section 2 of covariant formulation of newton-hooke particle and its canonical analysis[6]. The ambient space boost is scaled down by c for taking non relativistic limit, H is scaled up by c for consistency and P_i is scaled down by l for small Λ limit:

$$
\tilde{P}_i = \frac{P_i}{cl} = \frac{J_{4i}}{cl} \qquad \tilde{P}_0 \equiv \tilde{H} = \frac{H}{l} = \frac{J_{40}}{l} \qquad \tilde{K}_i = \frac{K_i}{c} = \frac{J_{0i}}{c}
$$

Note that Hamiltonian is nothing but the Dilatation operator.

$$
\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, P_j] = \epsilon_{ijk} P_k, && [J_i, K_j] = \epsilon_{ijk} K_k,\\
&[\tilde{P}_i, \tilde{P}_j] = \frac{1}{c^2 l^2} \epsilon_{ijk} J_k && [\tilde{K}_i, \tilde{K}_j] = -\frac{1}{c^2} \epsilon_{ijk} J_k && [\tilde{K}_i, \tilde{P}_j] = -\frac{1}{c^2 l} \delta_{ij} \tilde{H}\\
&[\tilde{H}, J_i] = 0, && [\tilde{H}, \tilde{P}_i] = \frac{1}{l^2} K_i, && [\tilde{H}, \tilde{K}_i] = \tilde{P}_i,
\end{aligned}
$$

Here the parameter l is defined as $l = \frac{c^2\Lambda}{3}$. In the limit $c \to \infty$ and $\Lambda \to 0$ while keeping $c^2\Lambda$.

$$
\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, P_j] = \epsilon_{ijk} P_k, && [J_i, K_j] = \epsilon_{ijk} K_k,\\
&[P_i, P_j] = 0, && [K_i, K_j] = 0, && [K_i, P_j] = 0,\\
&[H, J_i] = 0, && [H, P_i] = \frac{c^2\Lambda}{3} K_i, && [H, K_i] = P_i,
\end{aligned}
$$

3.3 Representation

Representation of the generator associated with Newton-Hooke algebra could be given as:[7]

$$
\begin{aligned}
H &= \partial_\eta\\
P_i &= -\cos\left(\frac{\eta}{l}\right)\partial_i\\
K_i &= -l\sin\left(\frac{\eta}{l}\right)\partial_i\\
J_i &= \epsilon_{ijk} x^k \partial_j
\end{aligned}
$$

In the limit $l \to \infty$

$$
\begin{aligned}
H &= \partial_\eta\\
P_i &= -\partial_i\\
K_i &= -\eta\partial_i\\
J_i &= \epsilon_{ijk} x^k \partial_j
\end{aligned}
$$

In the limit $\eta \to 0$:

$$
\begin{aligned}
H &= 0\\
P_i &= -\partial_i\\
K_i &= 0\\
J_i &= \epsilon_{ijk} x^k \partial_j
\end{aligned}
$$

[6]Boosts are scaled down by c for Galilean contraction.

[7]borrowed from equation 16 of Covariant Formulation of the Newton-Hooke Particle and its Canonical Analysis

Chapter 4

Ward Identities in de Sitter space

The infinitesimal conformal transformation which respects the de Sitter isometry will look like

$$\begin{aligned} \text{dilation:} \quad & \eta \to \eta(1+\lambda), && \mathbf{x} \to \mathbf{x}(1+\lambda), \\ \text{SCT:} \quad & \eta \to \eta(1-2\mathbf{b}\cdot\mathbf{x}), && \mathbf{x} \to \mathbf{x} - 2(\mathbf{b}\cdot\mathbf{x})\mathbf{x} + (\mathbf{x}^2-\eta^2)\mathbf{b}, \end{aligned}$$

We consider the following quantity:

$$\begin{aligned} e^{i\lambda D}\langle\ldots\rangle = \langle\ldots\rangle &\implies D\langle\ldots\rangle = 0 \\ e^{i\vec{b}\cdot\vec{K}}\langle\ldots\rangle = \langle\ldots\rangle &\implies \vec{b}\cdot\vec{K}\langle\ldots\rangle \equiv \underbrace{\mathbf{b}\cdot\mathbf{K}}_{\text{function, not an operator}}\langle\ldots\rangle = 0 \end{aligned}$$

Since, at late time ($\eta \to 0$) we will be decomposing our fields like:

$$\Phi = \sum_{\{\Delta\}} \eta^{\Delta} O_{\Delta}(\vec{x})$$

$\eta = \lim_{t\to\infty}$

where Δ is the scaling dimension of O_Δ. We deduce how the generators act on the boundary operator O_Δ by using the fact that scaling dimension of Φ is zero in de sitter space:

$$S =$$

Let us now come back to dilatation and study how it acts on O:

$$\begin{aligned} D\Phi &= -x\cdot\partial\Phi \\ &= -x\cdot\partial\sum_{\{\Delta\}}\eta^{\Delta}O_{\Delta} \\ &= \sum_{\{\Delta\}}(-\eta\partial_\eta - \vec{x}\partial_{\vec{x}})\eta^{\Delta}O_{\Delta} \\ &= \sum_{\{\Delta\}}\eta^{\Delta}\underbrace{(-\Delta-\vec{x}\partial_{\vec{x}})}_{D}O_{\Delta} \\ &= D\sum_{\{\Delta\}}\eta^{\Delta}O_{\Delta} \\ b\cdot K\Phi &= b^{\mu}[-2x_\mu\eta\partial_\eta - 2x_\mu\vec{x}\cdot\partial_{\vec{x}} - (\eta^2-|\vec{x}|^2)\partial_\mu]\Phi \\ &= [-2(\vec{b}\cdot\vec{x})\eta\partial_\eta - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} - (\eta^2-|\vec{x}|^2)\vec{b}\cdot\partial_{\vec{x}}]\Phi \\ &= \sum_{\{\Delta\}}\underbrace{[-2(\vec{b}\cdot\vec{x})\Delta - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} - (\eta^2-|\vec{x}|^2)\vec{b}\cdot\partial_{\vec{x}}]}_{b\cdot K}\eta^{\Delta}O_{\Delta} \end{aligned}$$

in the limit $\eta \to 0$

$$= [-2(\vec{b}\cdot\vec{x})\Delta - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} + |\vec{x}|^2\vec{b}\cdot\partial_{\vec{x}}]\Phi$$

In momentum space, the above operators take the following form:

$$D : (3-\eta\partial_\eta) + k^i\partial_{k_i},$$

$$\mathbf{b}\cdot\mathbf{K} : (3-\eta\partial_\eta)\, 2b^i\partial_{k_i} - \mathbf{b}\cdot\mathbf{k}\partial_{k_i}\partial_{k_i} + 2k^i\partial_{k_i}b^j\partial_{k_j}.$$

or, simply

$$\begin{aligned} D &: (3-\Delta) + k^i\partial_{k_i}, \\ \mathbf{b}\cdot\mathbf{K} &: (3-\Delta)\, 2b^i\partial_{k_i} - \mathbf{b}\cdot\mathbf{k}\partial_{k_i}\partial_{k_i} + 2k^i\partial_{k_i}b^j\partial_{k_j}. \end{aligned}$$

There are two ways to derive this expression, we will discuss both of them. The first is based on using Fourier Transform:

$$f(\vec{x}) = \int d^3x e^{i\vec{x}\cdot\vec{k}} f(\vec{k})$$

In our case

$$\begin{aligned} D\underbrace{\int d^3k e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k})}_{O_\Delta(\vec{x})} &= -\left(\Delta + x^j\frac{\partial}{\partial x^j}\right)\int d^3k e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k}) \\ &= \int d^3k\left[-\Delta - x^j\frac{\partial}{\partial x^j}\right] e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k}) \\ &= \int d^3k[-\Delta e^{i\vec{x}\cdot\vec{k}} - x^j e^{i\vec{x}\cdot\vec{k}}(ik_j)] O_\Delta(\vec{k}) \end{aligned}$$

we have to get rid of x^j, so we consider:

$$\begin{aligned} &= \int d^3k\left[-\Delta e^{i\vec{x}\cdot\vec{k}} - \left(-i\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right) ik_j\right] O_\Delta(\vec{k}) \\ &= \int d^3k\left[-\Delta e^{i\vec{x}\cdot\vec{k}} - \left(\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right) k_j\right] O_\Delta(\vec{k}) \end{aligned}$$

integrating by parts the second term

$$\begin{aligned} &= \int d^3k e^{i\vec{x}\cdot\vec{k}}\left[-\Delta + \left(\frac{\partial}{\partial k_j}\right) k_j\right] O_\Delta(\vec{p}) \\ &= \int d^3k e^{i\vec{x}\cdot\vec{k}}\left(-\Delta + \cancelto{3}{\frac{\partial k_j}{\partial k_j}} + k_j\frac{\partial}{\partial k_j}\right) O_\Delta(k)). \end{aligned}$$

Thus the action of the dilatation generator in momentum space is

$$DO_\Delta(\vec{k}) = \left(3 - \Delta + k^j\frac{\partial}{\partial k^j}\right) O_\Delta(\vec{k})$$

The second way is to replace the following in the expression in coordinate space (it can be seen operating the corresponding derivative operator on $e^{ix^\mu k_\mu}$ and the corresponding integration by parts).

$$\begin{aligned} x_\mu &\to -i\frac{\partial}{\partial k^\mu} \\ \frac{\partial}{\partial x_\mu} &\to -ik^\mu \end{aligned}$$

Substituting in

$$DO_\Delta(\vec{x}) = -\left(\Delta + x^j\frac{\partial}{\partial x^j}\right) O_\Delta(\vec{x})$$

we get

$$DO_\Delta(\vec{k}) = -\left[\Delta - i\frac{\partial}{\partial k^j}\left(-ik^j\right)\right] O_\Delta(\vec{k}) = -\left(\Delta - 3 - k^j\frac{\partial}{\partial k^j}\right) O_\Delta(\vec{k})$$

4.1 Conformal Ward Identity in momentum space

The following is taken from "Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex". We start by assuming that the generator of conformal transformation annihilates the correlation function.

$$\begin{aligned}
D\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\right\rangle &= 0\\
e^{ix_n\cdot P}D\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\right\rangle e^{-ix_n\cdot P} &= 0\\
\underbrace{e^{ix_n\cdot P}De^{-ix_n\cdot P}}_{D+\vec{x}_n\cdot\partial_{\vec{x}_n}}\underbrace{e^{ix_n\cdot P}\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\right\rangle e^{-ix_n\cdot P}}_{\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\rangle} &= 0
\end{aligned}$$

now, in Fourier space:

$$\begin{aligned}
\left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\right\rangle &= \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n-1} ik_j\cdot(x_j-x_n)+0}\\
&\quad\times\delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle\\
&= \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n-1} ik_j\cdot x_j - ix_n\cdot(\sum_{j=1}^{n-1}k_j)}\\
&\quad\times\delta^d\left(k_n+\sum_j^{n-1} k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle\\
&\equiv \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n} ik_j\cdot x_j}\\
&\quad\times\delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle
\end{aligned}$$

From first and third equality, we observe that the replacement we need to make are:

$$\begin{aligned}
(x_j-x_n)_\mu &\to -i\frac{\partial}{\partial k_j^\mu}\\
\frac{\partial}{\partial x_\mu} &\to -ik^\mu
\end{aligned}$$

Thus,

$$\begin{aligned}
(\underbrace{D+\vec{x}_n\cdot\partial_{\vec{x}_n}}_{-(\sum_{j=1}^n\Delta_j+x_j\cdot\partial_{x_j})+x_n\partial_{x_n}})\left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\right\rangle &= 0\\
-\left[\sum_j^n\Delta_j+\sum_j^{n-1}(x_j-x_n)\cdot\partial_{x_j}\right]\left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\right\rangle &= 0\\
\underbrace{-\left[\sum_j^n\Delta_j+\sum_j^{n-1}\left(-i\frac{\partial}{\partial k_j^\mu}\right)\cdot(-ik_j^\mu)\right]}_{-[\Delta-(n-1)d-\sum_{j=1}^{n-1}k_j\cdot\partial_{k_j}]}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle\delta^d\left(\sum_j^n k_j\right) &= 0
\end{aligned}$$

The alternate way to do the same is by explicitly doing it. We then will have to use

$$\int dx f(x)\partial_x\delta(x-a) = -\int dx\partial_x f(x)\delta(x-a) = -\partial_x f(a)$$

Consider the following integral:

$$\begin{aligned}
I_{\alpha\beta} &= \int d^dk\left[\frac{\partial}{\partial x^\alpha}\delta^d(k^\mu)\right]k_\beta\\
&= -\int d^dk\delta^d(k^\mu)\underbrace{\frac{\partial k_\beta}{\partial x^\alpha}}_{g_{\alpha\beta}} = -g_{\alpha\beta}
\end{aligned} \tag{4.1}$$

However, we also know that:

$$
\begin{aligned}
\int d^d k \delta^d(k^\mu) &= 1 \\
\int d^d k \delta^d(k^\mu) \frac{k^2}{k^2} &= 1 \\
\int d^d k \delta^d(k^\mu) \frac{g^{\alpha\beta} k_\alpha k_\beta}{k^2} &= 1 \\
\int d^d k \frac{\delta^d(k^\mu)}{k^2} k_\alpha k_\beta &= \frac{1}{d} g_{\alpha\beta}
\end{aligned} \tag{4.2}
$$

from (4.1) and (4.2), we get

$$
\frac{\partial}{\partial k^\alpha} \delta^d(k^\mu) = -\frac{d}{k^2} k_\alpha \delta^d(k^\mu)
$$

using the above, we derive:

$$
k^\alpha \frac{\partial}{\partial k^\alpha} \delta^3(\vec{k}) = -\frac{3}{k^2} k^\alpha k_\alpha \delta^3(\vec{k}) = -3\delta^3(\vec{k})
$$

Then, we have:

$$
\begin{aligned}
-\sum_{j=1}^{n} &\left(\Delta_j + x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right) \langle O_1(\vec{x}_1) O_2(\vec{x}_2) \dots O_n(\vec{x}_n) \rangle \\
&= -\sum_{j=1}^{3} \left(\Delta_j + x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right) \\
&\quad \int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{i k_l x_l} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle \\
&= -\int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{i k_l x_l} \\
&\quad \left(\sum_{j=1}^{n} \Delta_j - \underbrace{\sum_{j=1}^{n} d}_{nd} - \sum_{j=1}^{n} k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \right) \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle + \delta'_{\text{term}}
\end{aligned}
$$

where

$$
\delta'_{\text{term}} = \sum_{j=1}^{n} \int \prod_{l=1}^{n} d^d k_l \left[k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) \right] e^{i k_l \cdot x_l} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle
$$

defining $P = \sum_{i=1}^n \vec{k}_i$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \int \prod_{l=1}^{n} d^d k_l \left[k_j^\alpha \frac{\partial}{\partial P^\alpha} \delta^d(P) \right] e^{i k_l \cdot x_l} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle \\
&= \int \prod_{l=1}^{n} d^d k_l \left[P^\alpha \frac{\partial}{\partial P^\alpha} \delta^d(P) \right] e^{i k_l \cdot x_l} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle \\
&= -d \int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{i k_l x_l} \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle
\end{aligned}
$$

Thus, finally

$$
\begin{aligned}
-\sum_{j=1}^{n} \left(x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + \Delta_j \right) \langle O_1(\vec{x}_1) O_2(\vec{x}_2) \dots O_n(\vec{x}_n) \rangle &= -\int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n k_i) e^{i k_l x_l} \\
&\left(\sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n} k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \right) \left\langle O_1(\vec{k}_1) O_2(\vec{k}_2) \dots O_n(\vec{k}_n) \right\rangle
\end{aligned} \tag{4.3}
$$

Before we proceed, there is another identity we'd like to derive which would be very helpful.

$$
\begin{aligned}
\partial_\alpha \partial_\beta \delta^d(k^\mu) &= \partial_\alpha \left[-\frac{d}{k^2} \delta^d(k^\mu) k_\beta \right] \\
&= -d \left(\frac{\partial}{\partial k^\alpha} k^{-2} \right) \delta^d(k^\mu) k_\beta - \frac{d}{k^2} [\partial_\alpha \delta^d(k)] k_\beta - \frac{d}{k^2} \delta^d(k^\mu) \partial_\alpha k_\beta \\
&= \frac{2d}{k^3} \frac{k_\alpha}{k} \delta^d(k^\mu) k_\beta + \frac{d^2}{k^4} \delta^d(k^\mu) k_\alpha k_\beta - \frac{d}{k^2} \delta^d(k^\mu) g_{\alpha\beta} \\
&= \frac{d(d+2)}{k^4} \delta^d(k^\mu) k_\alpha k_\beta - \frac{d}{k^2} \delta^d(k^\mu) g_{\alpha\beta}
\end{aligned}
$$

We will now discuss the same for SCT. We perform the same substitution

$$
\begin{aligned}
-iK &= -2x_\mu \Delta - 2x_\mu \underbrace{\vec{x} \cdot \partial_{\vec{x}}}_{-i\frac{\partial}{\partial k^\alpha}(-ik^\alpha)} + |\vec{x}|^2 \partial_\mu \\
&= 2i\Delta \frac{\partial}{\partial k^\mu} - 2i \frac{\partial^2}{\partial k^\mu \partial k^\alpha} k^\alpha + i \frac{\partial^2}{\partial k^\alpha \partial k_\alpha} k_\mu \\
&= 2i\Delta \frac{\partial}{\partial k^\mu} \underbrace{- 2i \frac{\partial}{\partial k^\mu} \left(d + k^\alpha \frac{\partial}{\partial k^\alpha} \right)}_{2i\left(d\frac{\partial}{\partial k^\mu} + \frac{\partial}{\partial k^\mu} + k^\alpha \frac{\partial^2}{\partial k^\mu \partial k^\alpha}\right)} + \underbrace{i \frac{\partial}{\partial k^\alpha} \left(\delta^\alpha_\mu + k_\mu \frac{\partial}{\partial k_\alpha} \right)}_{2i\frac{\partial}{\partial k^\mu} + ik_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha}} \\
&= 2i\Delta \frac{\partial}{\partial k^\mu} - 2id \frac{\partial}{\partial k^\mu} - 2ik^\alpha \frac{\partial^2}{\partial k^\mu \partial k^\alpha} + ik_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha} \\
&= i \left[2(\Delta - d) \frac{\partial}{\partial k^\mu} - 2k^\alpha \frac{\partial^2}{\partial k^\mu \partial k^\alpha} + k_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha} \right]
\end{aligned}
$$

Then,

$$
\begin{aligned}
K^\mu \delta^d(\textstyle\sum_{i=1}^n p_i^k) &= \sum_{j=1}^n \left[k_j^\mu \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\alpha}} - 2k_j^\alpha \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\mu}} + 2(\Delta_j - d) \frac{\partial}{\partial k_{j\mu}} \right] \delta^d\Big(\overbrace{\textstyle\sum_{i=1}^n k_i^\mu}^{P^\mu = \sum_{i=1}^n k_i^\mu} \Big) \\
&= \sum_{j=1}^n \left[k_j^\mu \frac{\partial^2}{\partial P^\alpha \partial P_\alpha} - 2k_j^\alpha \frac{\partial^2}{\partial P^\alpha \partial P_\mu} + 2(\Delta_j - d) \frac{\partial}{\partial P_\mu} \right] \delta^d(P) \\
&= \left[\left(\sum_{j=1}^n k_j^\mu \right) \frac{\partial^2}{\partial P^\alpha \partial P_\alpha} - 2 \left(\sum_{j=1}^n k_j^\alpha \right) \frac{\partial^2}{\partial P^\alpha \partial P_\mu} + 2 \sum_{j=1}^n (\Delta_j - d) \frac{\partial}{\partial P_\mu} \right] \delta^d(P) \\
&= \left[P^\mu \frac{\partial^2}{\partial P^\alpha \partial P_\alpha} - 2P^\alpha \frac{\partial^2}{\partial P^\alpha \partial P_\mu} + 2(\Delta - nd) \frac{\partial}{\partial P_\mu} \right] \delta^d(P) \\
&= \frac{2d}{P^2} \delta^d(P) P^\mu - 2\frac{d^2 + d}{P^2} \delta^d(P) P^\mu + \frac{2d(\Delta - nd)}{P^2} \delta^d(P) P^\mu \\
&= 2d \left[nd - d - \Delta \right] \frac{\delta^d(P) P^\mu}{P^2} \\
&= -2[(n-1)d - \Delta] \frac{\partial \delta^d(P)}{\partial P^\mu}
\end{aligned}
$$

In the fourth line, we used $\sum_{j=1}^n \Delta_j = \Delta$. Now, when K operates on the correlation function, it produces three kinds of terms,

- All operators in K acting purely on $\langle O_1(p_1) \ldots O_n(p_n) \rangle$
- All operators in K acting purely on $\delta^d(\sum_{i=1}^n p_i)$
- Operators acting on both $\langle O_1(p_1) \ldots O_n(p_n) \rangle$ and $\delta^d(\sum_{i=1}^n p_i)$

We will consider the action of

$$
k_j^\mu \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\alpha}} - 2k_j^\alpha \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\mu}}
$$

then, they will operate like:

$$2\frac{\partial \delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\alpha}\underbrace{\left[k_j^\mu \frac{\partial}{\partial k_{j\alpha}} - k_j^\alpha \frac{\partial}{\partial k_{j\mu}}\right]}_{iL_{\mu\alpha}}\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

$$-2\frac{\partial \delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\mu} k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

First part vanishes due to rotational invariance. Therefore the extra terms would be:

$$\delta'_{\text{terms}} = -2\frac{\partial \delta^d(P)}{\partial P}\left[(n-1)d - \Delta + \sum_{j=1}^n k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\right]\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

The above vanishes from (4.3). Therefore the SCT ward identity is given as:

$$K_\mu\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle = -\left[2(\Delta-d)\frac{\partial}{\partial k^\mu} - 2k^\alpha\frac{\partial^2}{\partial k^\mu \partial k^\alpha} + k_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha}\right]\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle \quad (4.4)$$