Conformal Field Theory

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Chapter 1

Conformal Symmetry

1.1 Introduction

Conformal symmetry is defined as an angle-preserving symmetry in the complex plane. However, this definition is not wrong but isn't very useful either. It requires us to define what constitutes an angle to fully comprehend it. There has been discussion in the literature where physicists have argued that such a definition only makes sense when we are only concerned with spatial directions. In special relativity and beyond, one must incorporate time as well, and the notion of an angle with time is not a very well-defined object. Therefore, an alternative that also holds for angle-preserving transformations is given by:

$$x \to \lambda x$$

Note that, this is **not** a general coordinate transformation which relabels the coordinates but it is actually changing the underlying geometry. Under this transformation, we observe:

$$\begin{aligned}\cos\theta &= \frac{\vec{x}\cdot\vec{y}}{|x||y|} \\ &= \frac{g_{\mu\nu}x^\mu y^\nu}{\sqrt{g_{\mu\nu}x^\mu x^\nu}\sqrt{g_{\mu\nu}y^\mu y^\nu}}\end{aligned}$$

under conformal transformation $g_{\mu\nu} \to \Omega(x) g_{\mu\nu}$

$$\begin{aligned}&= \frac{\Omega(x)g_{\mu\nu}x^\mu y^\nu}{\sqrt{\Omega(x)g_{\mu\nu}x^\mu x^\nu}\sqrt{\Omega(x)g_{\mu\nu}y^\mu y^\nu}} \\ &= \cos\theta\end{aligned}$$

Therefore, conformal transformation could be defined as one which scales the metric and as such preserves the angles.

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \qquad \text{(with } \Omega(x) > 0\text{)}$$

The approach to studying a Conformal Field Theory (CFT) is very different from that of Quantum Field Theory (QFT). In QFT, we typically write the Lagrangian first and then study the equations of motion. However, this is rarely the case in CFT. Instead, we focus little on writing the Lagrangian but rather utilize the conformal symmetry directly to guess the functional form of the correlation function. This approach is known as the conformal bootstrap. We will clarify the difference between conformal transformation, weyl scaling and diffeomorphism now:

General coordinate invariance (diffeomorphism)

Classical field theories may have a variety of symmetries. One symmetry that we will assume them to have is general coordinate invariance. Using the action principle this can be used to show that the energy momentum tensor is conserved. In general, this tensor is defined in terms of the variation of the action S under changes of the space-time metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}\,.$$

Then the definition of the energy momentum tensor is

$$\delta S = \frac{1}{2}\int d^d x \sqrt{g}\, T^{\mu\nu}\delta g_{\mu\nu}\,.$$

If the theory is invariant under general coordinate transformations one can show that

$$(T^{\mu\nu})_{;\nu} = 0\,.$$

Here (as usual in general relativity) "$;\nu$" denotes a covariant derivative. In flat coordinates the condition reads $\partial_\nu T^{\mu\nu} = 0$.

Weyl invariance

We are not interested in general coordinate invariance, but in a different symmetry which can also be formulated in terms of the metric and the energy momentum tensor. This symmetry is called *Weyl invariance*. The transformation we consider is

$$g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x)\,,$$

or in infinitesimal form

$$g_{\mu\nu} \to g_{\mu\nu}(x) + \omega(x) g_{\mu\nu}(x)\,.$$

The condition for invariance of an action under such a symmetry can also be phrased in terms of the energy momentum tensor. Substituting $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}(x)$ into the previous expression we find

$$\delta S = \frac{1}{2}\int d^d x \sqrt{g}\, T^\mu_{\ \mu}\omega(x)\,.$$

Since this must be true for arbitrary functions ω we conclude that the condition for Weyl invariance is

$$T^\mu_{\ \mu} = 0\,.$$

Conformal invariance

A *conformal transformation* can now be defined as a coordinate transformation which acts on the metric as a Weyl transformation. Consider a general coordinate transformation $x \to x'$, such that $x^\mu = f^\mu(x')$. This has the following effect on the metric

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial f^\rho}{\partial x'^\mu}\frac{\partial f^\sigma}{\partial x'^\nu} g_{\rho\sigma}(f(x'))\,.$$

We are going to require that the left hand side is proportional to $g_{\mu\nu}$. Rotations and translations do not change the metric at all, and hence preserve all inner products $v \cdot w \equiv v^\mu g_{\mu\nu} w^\nu$. They are thus part of the group of conformal transformations. A coordinate transformation satisfying the equation above preserves all angles, $\frac{v\cdot w}{\sqrt{v^2 w^2}}$ (hence the name 'conformal'). Later in this chapter we will determine all such transformations.

If a field theory has a conserved, traceless energy momentum tensor, it is invariant both under general coordinate transformations and Weyl transformations. Suppose the action has the form

$$S = \int d^d x \mathcal{L}(\partial_x, g_{\mu\nu}(x), \phi(x))\,.$$

Here ϕ denotes generically any field that might appear, except for the metric which we have indicated separately since it plays a special rôle. We have also explicitly indicated space-time derivatives. General coordinate invariance implies that

$$S = S' \equiv \int d^d x' \mathcal{L}(\partial_{x'}, g'_{\mu\nu}(x'), \phi'(x'))\,.$$

Here $g'_{\mu\nu}$ is as defined above, and the transformations of a field ϕ depends on its spin. If it is a tensor of rank n one has

$$\phi'_{\mu_1,\ldots,\mu_n}(x') = \left|\frac{\partial f}{\partial x'}\right|^{\frac{\Delta}{d}} \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}} \phi_{\nu_1,\ldots,\nu_n}(f(x'))\,. \tag{1.1}$$

In particular, for a scalar function $\phi(x)$ we find $\phi'(x') = \phi(f(x'))$ and for the derivative of a scalar function we get

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial x'^\mu}\phi'(x') = \frac{\partial}{\partial x'^\mu}\phi(f(x')) = \frac{\partial f^\nu}{\partial x'^\mu}\frac{\partial}{\partial f^\nu}\phi(f(x'))\,,$$

i.e. it transforms like a vector (note, however, that nth order ordinary derivatives do not transform like a tensor of rank n; this is only true if one uses covariant derivatives). If the coordinate transformation $x \to x'$ is of the same type as above, we can **use Weyl invariance of the action to change the metric back into its original form**. Then we have

$$S = S'' \equiv \int d^d x' \mathcal{L}(\partial_{x'}, g_{\mu\nu}(f(x')), \phi'(x')) = \int d^d x' \mathcal{L}(\partial_{x'}, g_{\mu\nu}(x), \phi'(x'))\,.$$

This is the conformal symmetry of the action. Note that the background metric[1] now remains unchanged if we start with a flat space metric $g_{\mu\nu} = \eta_{\mu\nu}$. This means that we can define the conformal transformation for theories in flat space that are not coupled to gravity. We may then forget about general coordinate invariance and start with an action in which no "dynamical" metric appears.

The statement of conformal invariance is then that the action of such a theory is unchanged if we integrate the same Lagrangian (or other physical scalar) expressed in terms of the new fields $\phi'(x')$ over the new coordinates x'. The restriction to flat space is not really a restriction if we are in two dimensions. Then a general metric is given by three functions, $g_{11}(x)$, $g_{22}(x)$ and $g_{12}(x) = g_{21}(x)$. A general coordinate transformation allows us to change this using two functions, $f^1(x)$ and $f^2(x)$, and we can – generically – use this freedom to set $g_{12}(x) = 0$ and $g_{11}(x) = \pm g_{22}(x)$ (depending on the signature of the metric), so that the metric has the form $g(x)\eta_{\mu\nu}$. This is called *conformal gauge*. Then, using a Weyl transformation, we can remove the function $g(x)$ and bring the metric to the form $\eta_{\mu\nu}$. In more than two dimensions we do not have enough freedom to do this, and then the assumption made here is really a restriction to non-gravitational theories in flat space.

On a given two-dimensional manifold the conformal gauge choice can be made locally, but usually not globally. This means that we will be able to use conformal field theory in some coordinate patch, but that additional data may be needed to describe the theory globally.

Fields that transform like Eqn. (1.1) above under conformal transformations are called *conformal fields*, or also *primary fields*.

1.2 Infinitesimal Conformal Transformation

The fundamental essence of conformal transformations resides in their infinitesimal form, which serves as a crucial tool for investigating how fields transform under these symmetries. It plays a pivotal role in defining the generator of the conformal group and, subsequently, constraining the set of possible correlators that are compatible with conformal symmetry. Any infinitesimal transformation can be expressed as:

$$x'^{\mu} = x^{\mu} + \underset{\uparrow\,\text{infinitesimal}}{\epsilon}{}^{\mu}(x)$$

and subsequently,

$$x^{\mu} = x'^{\mu} - \epsilon^{\mu}(x)$$

therefore, the metric transforms like:

$$\begin{aligned}
g'_{\mu\nu} &= \underbrace{\frac{\partial x^{\alpha}}{\partial x'^{\mu}}}_{\delta^{\alpha}_{\mu} - \partial_{\mu}\epsilon^{\alpha}(x)} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \\
&= \left[\delta^{\alpha}_{\mu} - \frac{\partial \epsilon^{\alpha}(x)}{\partial x'^{\mu}}\right]\left[\delta^{\beta}_{\nu} - \frac{\partial \epsilon^{\beta}(x)}{\partial x'^{\nu}}\right] g_{\alpha\beta} \\
&= \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} g_{\alpha\beta} - \delta^{\beta}_{\nu}\partial_{\mu}\epsilon^{\alpha}(x) g_{\alpha\beta} - \delta^{\alpha}_{\mu}\partial_{\nu}\epsilon^{\beta}(x) g_{\alpha\beta} + \mathcal{O}(\epsilon^2) \\
\Omega(x) g_{\mu\nu} &= g_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu}
\end{aligned}$$

In the third step, we used chain rule on $\epsilon^{\alpha}(x)$ and ignored $\mathcal{O}((\partial\epsilon)^2)$ terms. From the last line, it is reasonable to expect that:

$$\begin{aligned}
g_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} &= [1 + f(x)] g_{\mu\nu} \\
\partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) &= f(x) g_{\mu\nu} \qquad (1.2)
\end{aligned}$$

Contracting Indicies

$$\partial^{\mu}\epsilon_{\mu}(x) + \partial^{\mu}\epsilon_{\mu}(x) = f(x)\delta^{\mu}_{\mu}$$

$$2(\partial\cdot\epsilon) = \underset{\uparrow\,\text{dimension of spacetime}}{d}\; f(x)$$

[1] when we do perturbation theory we define the physics on perturbed manifold as physics of tensor fields living on background manifold. In such case the conformal transformation is done on background manifold and the way perturbation changes is dictated again by killing equation. So we just do the coordinate transformation of full metric by weyl scaling is done to bring the background metric back to original form.

$$f(x) = \frac{2}{d}\frac{\partial \epsilon_\mu(x)}{\partial x_\mu}$$

Substituing back in (1.2)

$$\boxed{\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \frac{2}{d}(\partial \cdot \epsilon)\ g_{\mu\nu}} \tag{1.3}$$

Now, we operate by ∂^ν

$$\begin{aligned}
\frac{\partial}{\partial x'_\nu}\left[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)\right] &= \frac{\partial}{\partial x'_\nu}\left(\frac{2}{d}\partial \cdot \epsilon(x)\ \overset{\text{assuming flat metric}}{g_{\mu\nu}}\right) \\
\partial_\mu \underbrace{\partial^\nu \epsilon_\nu}_{\partial\cdot\epsilon} + \underbrace{\partial^\nu \partial_\nu}_{\Box} \epsilon_\mu &= \frac{2}{d} g_{\mu\nu}\partial^\nu \partial \cdot \epsilon \\
\partial_\mu(\partial \cdot \epsilon) + \Box\epsilon &= \frac{2}{d}\partial_\mu(\partial \cdot \epsilon)
\end{aligned}$$

Operating by ∂_ν

$$\partial_\nu[\partial_\mu(\partial \cdot \epsilon) + \Box\epsilon] = \partial_\nu\left[\frac{2}{d}\partial_\mu(\partial \cdot \epsilon)\right]$$

$$\left(1 - \frac{2}{d}\right)\partial_\nu \partial_\mu(\partial \cdot \epsilon) + \Box(\partial_\nu \epsilon_\mu) = 0 \tag{1.4}$$

under relabeling $\mu \leftrightarrow \nu$

$$\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box(\partial_\mu \epsilon_\nu) = 0 \tag{1.5}$$

adding (1.4) and (1.5)

$$\begin{aligned}
2\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box\underbrace{[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)]}_{\frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu}} &= 0 \\
\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \frac{1}{d}\Box(\partial \cdot \epsilon)g_{\mu\nu} &= 0 \\
\left[g_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu\right](\partial \cdot \epsilon) &= 0
\end{aligned} \tag{1.6}$$

Contracting the indicies

$$\begin{aligned}
[d\Box + (d-2)\Box](\partial \cdot \epsilon) &= 0 \\
2(d-1)\Box(\partial \cdot \epsilon) &= 0
\end{aligned}$$

hence,

$$\boxed{(d-1)\Box(\partial \cdot \epsilon) = 0} \tag{1.7}$$

if $d = 1 \implies$ any $\epsilon^\mu(x)$ satisfies (1.7), therefore, is conformal transformation. It is interesting to note that any 1D QFT is conformal field theory, but for our purpose it's not very useful. We will be concerned with $d \neq 1$ for the rest of this notes unless stated otherwise. Consider the action of ∂_α on (1.3) and then cyclic relabeling of incidies as $\alpha \to \mu \to \nu$

$$\partial_\alpha[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)] = \frac{2}{d}\partial_\alpha g_{\mu\nu}(\partial \cdot \epsilon) \tag{1.8}$$

$$\partial_\mu[\partial_\nu \epsilon_\alpha(x) + \partial_\alpha \epsilon_\nu(x)] = \frac{2}{d}\partial_\mu g_{\nu\alpha}(\partial \cdot \epsilon) \tag{1.9}$$

$$\partial_\nu[\partial_\alpha \epsilon_\mu(x) + \partial_\mu \epsilon_\alpha(x)] = \frac{2}{d}\partial_\nu g_{\alpha\mu}(\partial \cdot \epsilon) \tag{1.10}$$

Adding the first two equation and subtracting from the last, we get [(1.8) + (1.9) − (1.10)]:

$$\cancel{2}\partial_\alpha \partial_\mu \epsilon_\nu = \frac{\cancel{2}}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial \cdot \epsilon)$$

$$\partial_\alpha \partial_\mu \epsilon_\nu = \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial \cdot \epsilon) \tag{1.11}$$

Referring to eqn 1.7.11 of "Ideas and Methods of Supersymmetry and Supergravity" by Sergio M. Kuzenko, we find that the 3rd order derivation of $\epsilon^\mu(x)$ vanishes. Therefore, the most general conformal transformation is of the type:

$$x'^\mu = x^\mu + \underbrace{a^\mu + b^\mu{}_\nu x^\nu + c^\mu{}_{\nu\alpha} x^\nu x^\alpha}_{\epsilon^\mu}$$

Where, $a^\mu, b^\mu{}_\nu$ and $c^\mu{}_{\nu\alpha}$ are parameters relevant to their transformation. The goal here is simple:

- First find the relevant transformations
- Then based on the transformation rule, find the generators.

For $\epsilon^\mu = a^\mu$:

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu \\ &= x^\mu + \delta^\mu_\nu a^\nu \\ &= x^\mu + (\partial_\nu x^\mu) a^\nu \\ &= [1 + i a^\nu(-i\partial_\nu)]x^\mu \end{aligned}$$

Thus, the generator of translation is $P_\mu - i\partial_\mu$[2]. For $\epsilon^\mu = b^\mu{}_\alpha x^\alpha$, we refer to (1.3)

$$\begin{aligned} \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) &= \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu} \\ \partial_\mu(b_{\nu\alpha}x^\alpha) + \partial_\nu(b_{\mu\alpha}x^\alpha) &= \frac{2}{d}(\partial^\mu b_{\mu\alpha}x^\alpha) g_{\mu\nu} \\ b_{\nu\alpha}\delta^\alpha_\mu + b_{\mu\alpha}\delta^\alpha_\nu &= \frac{2}{d}(b_{\mu\alpha}g^{\alpha\mu}) g_{\mu\nu} \\ b_{\nu\mu} + b_{\mu\nu} &= \frac{2}{d} b^\alpha{}_\alpha g_{\mu\nu} \\ \frac{b_{\nu\mu} + b_{\mu\nu}}{2} &= \frac{1}{d} b^\alpha{}_\alpha g_{\mu\nu} \end{aligned}$$

now,

$$\begin{aligned} b_{\mu\nu} &= \frac{b_{\mu\nu} - b_{\nu\mu}}{2} + \frac{b_{\mu\nu} + b_{\nu\mu}}{2} \\ &= M_{\mu\nu} + \lambda g_{\mu\nu} \end{aligned}$$

If $b_{\mu\nu} = \lambda g_{\mu\nu} (M_{\mu\nu} = 0)$

$$\begin{aligned} x'^\mu &= x_\mu + b^\mu{}_\nu x^\nu \\ &= x^\mu + \lambda g^{\mu\alpha} \underbrace{g_{\alpha\nu} x^\nu}_{x_\alpha} \\ &= x^\mu + \lambda x^\mu \\ &= x^\mu + \lambda x^\nu \delta^\mu_\nu \\ &= x^\mu + \lambda x^\nu(\partial_\nu x^\mu) \\ &= x^\mu + i\lambda x^\nu(-i\partial_\nu x^\mu) \\ &= (1 + i\lambda(-ix^\nu\partial_\nu))x^\mu \end{aligned}$$

Thus, the generator of dilatation is $D = -ix^\mu \partial_\mu$. For $b_{\mu\nu} = M_{\mu\nu}(\lambda = 0)$.

$$\begin{aligned} x'^\mu &= x^\mu + M^\mu{}_\nu x^\nu \\ &= x^\mu + M^\alpha{}_\nu \delta^\mu_\alpha x^\nu \\ &= x^\mu + M^\alpha{}_\nu(\partial_\alpha x^\mu) x^\nu \\ &= x^\mu + M_{\alpha\nu}(\partial^\alpha x^\mu) x^\nu \\ &= x^\mu + \frac{M_{\alpha\nu} - M_{\nu\alpha}}{2}(\partial^\alpha x^\mu) x^\nu \end{aligned}$$

[2] if we use $[1 - a^\nu(\partial_\nu)]x^\mu$ as the definition, then $P_\mu = i\partial_\mu$ would be the generator

$$
\begin{aligned}
&= x^\mu + \frac{1}{2}M_{\alpha\nu}(\partial^\alpha x^\mu)x^\nu - \overset{\text{relabeling } \nu\leftrightarrow\alpha}{\boxed{\frac{1}{2}M_{\nu\alpha}(\partial^\alpha x^\mu)x^\nu}} \\
&= x^\mu + \frac{1}{2}M_{\alpha\nu}(\partial^\alpha x^\mu)x^\nu - \frac{1}{2}M_{\alpha\nu}(\partial^\nu x^\mu)x^\alpha \\
&= x^\mu + \frac{1}{2}M_{\alpha\nu}(x^\nu\partial^\alpha - x^\alpha\partial^\nu)x^\mu \\
&= x^\mu + \frac{i}{2}M_{\alpha\nu}\left\{-i(x^\nu\partial^\alpha - x^\alpha\partial^\nu)\right\}x^\mu \\
&= x^\mu + \frac{i}{2}M_{\alpha\nu}\{\underbrace{i(x^\alpha\partial^\nu - x^\nu\partial^\alpha)}_{L^{\alpha\nu}}\}x^\mu
\end{aligned}
$$

Thus, the generator of rotation is $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$. Now, the last part $\epsilon^\mu = c^\mu{}_{\nu\alpha}x^\nu x^\alpha = c^\mu{}_{\alpha\nu}x^\nu x^\alpha$, we refer to (1.11):

$$
\begin{aligned}
\partial_\alpha\partial_\mu\epsilon_\nu &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon) \\
\partial_\alpha\partial_\mu(c_{\nu\sigma\beta}x^\sigma x^\beta) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu]\partial^\mu(c_{\mu\sigma\beta}x^\sigma x^\beta) \\
c_{\nu\sigma\beta}\partial_\alpha(\delta^\sigma_\mu x^\beta + x^\sigma\delta^\beta_\mu) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu]c_{\mu\sigma\beta}(g^{\sigma\mu}x^\beta + x^\sigma g^{\beta\mu}) \\
c_{\nu\sigma\beta}(\delta^\sigma_\mu\delta^\beta_\alpha + \delta^\sigma_\alpha\delta^\beta_\mu) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](c^\sigma{}_{\sigma\beta}x^\beta + c^\beta{}_{\sigma\beta}x^\sigma) \\
2c_{\nu\mu\alpha} &= \frac{2}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu]c^\sigma{}_{\sigma\beta}x^\beta \\
c_{\nu\mu\alpha} &= \underbrace{\frac{1}{d}c^\sigma{}_{\sigma\beta}}_{b_\beta}[g_{\mu\nu}\delta^\beta_\alpha + g_{\nu\alpha}\delta^\beta_\mu - g_{\alpha\mu}\delta^\beta_\nu] \\
&= g_{\nu\mu}b_\alpha + g_{\nu\alpha}b_\mu - g_{\mu\alpha}b_\nu
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\epsilon_\mu &= c_{\mu\alpha\beta}x^{\alpha\beta} \\
&= (g_{\mu\alpha}b_\beta + g_{\mu\beta}b_\alpha - g_{\alpha\beta}b_\mu)x^\alpha x^\beta \\
&= x_\mu(b\cdot x) + x_\mu(b\cdot x) - b_\mu(x\cdot x) \\
&= 2x_\mu(b\cdot x) - x^2 b_\mu
\end{aligned}
$$

Hence, the Special Conformal Transformation looks like:

$$
\begin{aligned}
x'^\mu &= x^\mu + 2x^\mu(b\cdot x) - x^2 b^\mu \\
&= x^\mu + 2(b\cdot x)x^\nu\delta^\mu_\nu - x^2 b^\nu\delta^\mu_\nu \\
&= x^\mu + 2(b\cdot x)x^\nu\partial_\nu x^\mu - x^2 b^\nu\partial_\nu x^\mu \\
&= [1 + 2(b\cdot x)x^\nu\partial_\nu - x^2 b^\nu\partial_\nu]x^\mu \\
&= [1 + \{2b^\alpha x_\alpha x^\nu\partial_\nu - x^2 b^\alpha\partial_\alpha\}]x^\mu \\
&= [1 + ib^\alpha\{\underbrace{-i(2x_\alpha x^\nu\partial_\nu - x^2\partial_\alpha)}_{K_\alpha}\}]x^\mu
\end{aligned}
$$

Hence, the generator for Special Conformal Transformations (SCT) takes the form $K^\mu = -i(2x^\mu x\cdot\partial - x^2\partial^\mu)$. We will now list all the **infinitesimal** transformations and their generators we found in this section.

1. Translation

$$
x'^\mu = x^\mu + a^\mu \qquad\qquad P_\mu = -i\partial_\mu \tag{1.12}
$$

2. Rotation

$$
x'^\mu = x^\mu + M^\mu{}_\nu x^\nu \qquad\qquad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{1.13}
$$

3. Dilatation

$$
x'^\mu = (1+\lambda)x^\mu \qquad\qquad D = -ix^\mu\partial_\mu \tag{1.14}
$$

4. Special Conformal Transformation

$$x'^{\mu} = x^{\mu} + 2x^{\mu}(b\cdot x) - x^2 b^{\mu} \qquad\qquad K^{\mu} = -i(2x^{\mu} x\cdot\partial - x^2\partial^{\mu}) \tag{1.15}$$

In the above listed transfomations, the parameters $a^{\mu}, M^{\mu}{}_{\nu}, \lambda$ and b^{μ} are all infinitesimal.

1.3 Finite Conformal Transformation

In the previous section, we considered the infinitesimal conformal transformation, however in this section we will consider the finite conformal transformation.

1. Translation

$$x'^{\mu} = x^{\mu} + \underbrace{a^{\mu}}_{\text{finite vector}} = e^{ia^{\nu}P_{\nu}} x^{\mu}$$

2. Dilatation

$$x'^{\mu} = \left(1 + \frac{\lambda}{N}\right) x^{\mu}$$

In order to achieve the finite dilatation, we use the infinitesimal transformation recursively by dividing the finite λ into infinitely many λ/N pieces and then transforming

$$\begin{aligned} x'^{\mu} &= \left(1+\frac{\lambda''}{N}\right)\underbrace{\left(1+\frac{\lambda'}{N}\right)\left(1+\frac{\lambda}{N}\right)x^{\mu}}_{x''^{\mu}} \\ &= \lim_{N\to\infty}\left(1+\frac{\lambda}{N}\right)^{N} x^{\mu} \\ &= e^{\lambda}x^{\mu} = e^{i\lambda D}x^{\mu} \end{aligned}$$

3. Rotation

$$\begin{aligned} x'^{\mu} &= \left[1 + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right]^{\mu}{}_{\nu} x^{\nu} \\ &= \left[e^{\frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}}\right]^{\mu}{}_{\nu} x^{\nu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \end{aligned}$$

4. The special conformal transformation

$$x'^{\mu} = x^{\mu} + 2x^{\mu}(b\cdot x) - x^2 b^{\mu}$$

let $b^{\mu} = \underbrace{t}_{\text{infinitesimal parameter, i.e. } t \text{ is small.}} e^{\mu}$

$$x'^{\mu}(t) \equiv x^{\mu}(t) = x^{\mu} + 2t(e\cdot x)x^{\mu} - x^2 t e^{\mu}$$

To find the finite form of the transformation we have to recursively apply the above equation multiple times (Lie Algebra sence). The usual way is to integrate the infinitesimal form. The other way, and since we know that the transformations satisfy the conformal Killing equation, is to find the integral curve of the corresponding conformal Killing vector field as they are equivalent (Differential Geometry sense). Consider the $t-$derivative of the above[3].

$$\frac{\mathrm{d}x^{\mu}(t)}{\mathrm{d}t} = 2(e\cdot x)x^{\mu} - x^2 e^{\mu} \tag{1.16}$$

defining $y^{\mu}(t) = \frac{x^{\mu}(t)}{x^2(t)}$

$$\dot{y}^{\mu}(t) \overset{\text{quotient rule}}{=} \frac{x^2\dot{x}^{\mu} - 2(\dot{x}\cdot x)x^{\mu}}{(x^2)^2}$$

[3]when we consider the differential equation, we are no longer thinking of it as transformation but rather flow along a trajectory parameterized by t. This part was taken from pg 16 of “Four point function in momentum spaces and topological terms in gravity”

$$
\begin{aligned}
&= \frac{x^2[2(e\cdot x)x^\mu - x^2e^\mu] - 2[2(e\cdot x)x^\nu - x^2e^\nu]x_\nu x^\mu}{x^4} \\
&= \frac{x^2[2(e\cdot x)x^\mu - x^2e^\mu] - 2[2(e\cdot x)x^2 - x^2(e\cdot x)]x^\mu}{x^4} \\
&= \frac{x^2[\cancel{2(e\cdot x)x^\mu} - x^2e^\mu] - \cancel{2(e\cdot x)x^2x^\mu}}{x^4} \\
\dot{y}^\mu(t) &= -e^\mu
\end{aligned}
$$

Solving the above differential equation

$$
\begin{aligned}
y^\mu(t) &= y^\mu(0) - te^\mu \\
\frac{x^\mu(t)}{x^2(t)} &= \frac{x^\mu(0)}{x^2(0)} - te^\mu
\end{aligned}
$$

going back to the old notation $x'^\mu \equiv x^\mu(t)$

$$
\begin{aligned}
\frac{x'^\mu}{x'^2} &= \frac{x^\mu}{x^2} - te^\mu \\
&= \frac{x^\mu}{x^2} - b^\mu
\end{aligned}
\tag{1.17}
$$

Squaring both sides

$$
\begin{aligned}
\left(\frac{x'^\mu}{x'^2}\right)^2 \equiv \frac{x'^\mu}{x'^2}\frac{x'_\mu}{x'^2} &= \left(\frac{x^\mu}{x^2} - b^\mu\right)^2 \\
\frac{x'^2}{x'^4} &= \left(\frac{x^\mu}{x^2}\right)^2 + b^2 - \frac{2(x\cdot b)}{x^2} \\
\frac{1}{x'^2} &= \frac{1 + b^2x^2 - 2(x\cdot b)}{x^2} \\
x'^2 &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned}
\tag{1.18}
$$

referring to (1.17)

$$
x'^\mu = x'^2\left[\frac{x^\mu}{x^2} - b^\mu\right]
$$

and substituting (1.18)

$$
\begin{aligned}
x'^\mu &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}\left[\frac{x^\mu}{x^2} - b^\mu\right] \\
&= \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned}
$$

Above procedure also suggests that, finite SCT could be described as a sequence of inversion→translation→inversion. Where inversion is defined as:

$$
I(x^\mu) = \frac{x^\mu}{x^2}
$$

First we note that the inversion is a global conformal transformation and since it is undefined at origin, it does not have an infinitesimal part i.e. we can not expect inversion to be obtained by exponentiating an element from the conformal Lie algebra. It is not the connected element of conformal group and in embedding space formalism, inversion is related to parity. Another interesting point to note is that there is no parameter associated with the transformation here such as λ for dilatation or b^μ for SCT. Lastly, it is also closely related to the stereographic projection. To show this let us study the stereographic projection of sphere onto a plane. Consider $x \in \mathbb{R}^n$, and define stereographic projection from the **north pole** of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as:

$$
X^i = \frac{2x^i}{1+|x|^2}, \quad X^{n+1} = \frac{|x|^2 - 1}{1+|x|^2}
$$

where x^i are coordinates on the projected plane and X^i are the coordiantes on the sphere in embedding space. Now project this point on the sphere back to $\mathbb{R}^n$ via stereographic projection from the **south pole**:

$$x'^i = \frac{X^i}{1 + X^{n+1}}$$

Substituting:

$$x'^i = \frac{\frac{2x^i}{1+|x|^2}}{1 + \frac{|x|^2-1}{1+|x|^2}} = \frac{2x^i}{(1+|x|^2) + (|x|^2 - 1)} = \frac{2x^i}{2|x|^2} = \frac{x^i}{|x|^2}$$

Hence, the composition gives:

$$x^i \mapsto \frac{x^i}{|x|^2}$$

which is the **inversion** in the unit sphere. Even though this inversion does not have a killing vector associated with it, but it is reasonable to look for the killing vector associated with stereographic projection. In general, we note that these two transformation would have the following form:

$$x'^\mu = \Omega(x) x^\mu$$

If $\partial_\mu \partial_\nu (\frac{1}{\Omega}) \propto g_{\mu\nu}$. The killing vector associated with it will have the form:

$$K^A{}_\mu = \frac{1}{\Omega^2} \frac{\partial x'^A}{\partial x^\mu}$$

Coming back to special conformal transformation which was the topic at hand, we now look at how they scale the metric tensor.

$$\frac{\partial x'^\mu}{\partial x^\nu} = \left\{ \frac{\delta^\mu_\nu - 2b^\mu x_\nu}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\nu + 2b^2 x_\nu)}{\Lambda^2} \right\}$$

$$\begin{aligned}
g_{\alpha\beta}(x) = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g'_{\mu\nu}(x') \bigg|_{x'=x'(x)} &= \left\{ \frac{\delta^\mu_\alpha - 2b^\mu x_\alpha}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^2} \right\} \\
&\times \left\{ \frac{\delta^\nu_\beta - 2b^\nu x_\beta}{\Lambda} - \frac{(x^\nu - b^\nu x^2)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^2} \right\} g'_{\mu\nu}(x') \\
&= \left\{ \frac{\delta^\mu_\alpha - 2b^\mu x_\alpha}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\alpha + 2b^2 x'_\alpha)}{\Lambda^2} \right\} \\
&\times \left\{ \frac{g'_{\mu\beta} - 2b_\mu x_\beta}{\Lambda} - \frac{(x_\mu - b_\mu x^2)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^2} \right\} \\
&= \frac{g'_{\alpha\beta} - 2b_\beta x_\alpha - 2b_\alpha x_\beta + 4b^2 x_\alpha x_\beta}{\Lambda^2} - \frac{(x_\beta - b_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&+ \frac{(2(b \cdot x) x_\beta - 2b^2 x_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&- \frac{(x_\alpha - 2b_\alpha x^2)(-2b_\alpha + 2b^2 x_\beta)}{\Lambda^3} \\
&+ \frac{(2(b \cdot x) x_\alpha - 2b^2 x^2 x_\alpha)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^3} \\
&+ \frac{(x^\mu - b^\mu x^2)(x_\mu - b_\mu x^2)(-2b_\alpha + 2b^2 x_\alpha)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^4} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{(-2b_\beta x_\alpha - 2_\alpha x_\beta + 4b^2 x_\alpha x_\beta)(1 - 2b \cdot x + b^2 x^2)}{\Lambda^3} \\
&+ \frac{1}{\Lambda^3} \left\{ 2b_\alpha x_\mu - 2b_\alpha b_\beta x^2 - 2b^2 x_\alpha x_\beta + 2b^2 x^2 x_\alpha b_\beta \right. \\
&\quad \left. -4(b \cdot x) b_\alpha x_\beta + 4b^2 x^2 b_\alpha x_\beta + 4b^2 (b \cdot x) x_\alpha x_\beta - 4b^\mu x^2 x_\alpha x_\beta \right\} + (\alpha \leftrightarrow \beta) \\
&+ \frac{\left\{ x^2 - 2(b \cdot x) x^2 + b^2 x^4 \right\} \left\{ 4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta \right\}}{\Lambda^4} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3} \\
&\times \left(-\cancel{2b_\beta x_\alpha} - \cancel{2b_\alpha x_\beta} + \cancel{4b^2 x_\alpha x_\beta} + \cancel{4(b \cdot x) b_\alpha x_\beta} - \cancel{8b^2 (b \cdot x) x_\alpha x_\beta} \right.
\end{aligned}$$

$$
\begin{aligned}
&-\cancel{2b^2x^2b_\beta x_\alpha} - \cancel{2b^2x^2b_\alpha x_\beta} + 4b^\mu x^2 x_\alpha x_\beta + \cancel{2b_\alpha x_\beta} + \cancel{2b_\beta x_\alpha} \\
&-4b_\alpha b_\beta x^2 - \cancel{4b^2x_\alpha x_\beta} + \cancel{2b^2x^2x_\alpha b_\beta} + \cancel{2b^2x^2x_\beta x_\alpha} - \cancel{4bxb_\alpha x_\beta} \\
&-\cancel{4bxb_\beta x_\alpha} + 4b^2x^2b_\alpha x_\beta + 4b^2x^2b_\beta x_\alpha + \cancel{8b^2b^2x_\alpha x_\beta} - 8b^4x^2x_\alpha x_\beta\big) \\
&+ x^2\frac{\Lambda}{\Lambda^4}\left\{4b_\alpha b_\beta - 4b^2b_\beta x_\alpha - 4b^2b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3}\left\{-4b_\alpha b_\beta x^2 + 4b^2x^2b_\alpha x_\beta + 4b^2x^2b_\beta x_\alpha - 4b^4x^2x_\alpha x_\beta\right\} \\
&+ x^2\frac{1}{\Lambda^3}\left\{4b_\alpha b_\beta - 4b^2b_\beta x_\alpha - 4b^2b_\alpha x_\beta + 4b^4x_\alpha x_\beta\right\}
\end{aligned}
$$

$$
g'_{\alpha\beta}(x') = \Lambda^2 g_{\alpha\beta}(x)
$$

Jacobian of the Transformation

The following part is taken from “Conformal Field Theory Primer in $D \geq 3$” by Andrew Evans, pg 36:

$$
\begin{aligned}
\text{Translation:} \quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1 \\
\text{Rotation:} \quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1 \\
\text{Dilataion:} \quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \lambda^{-d} \\
\text{Inversion:} \quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \left(\frac{1}{\tilde{x}^2}\right)^d
\end{aligned}
$$

Since the rest are easier to show, we will only focus on showing the last part:

$$
\begin{aligned}
\frac{\partial x^\mu}{\partial \tilde{x}^\nu} &= \frac{1}{\tilde{x}^2}\left[\delta^\mu_\nu - 2\frac{\tilde{x}^\mu\tilde{x}_\nu}{\tilde{x}^2}\right] \\
\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) &= \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}}\frac{\partial x^{\mu_2}}{\partial \tilde{x}^{\nu}_2}\cdots\frac{\partial x^{\mu_d}}{\partial \tilde{x}^{\nu}_d} \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\left[\delta^{\mu_1}_{\nu_1} - 2\frac{\tilde{x}^{\mu_1}\tilde{x}_{\nu_1}}{\tilde{x}^2}\right]\left[\delta^{\mu_2}_{\nu_2} - 2\frac{\tilde{x}^{\mu_2}\tilde{x}_{\nu_2}}{\tilde{x}^2}\right]\cdots\left[\delta^{\mu_d}_{\nu_d} - 2\frac{\tilde{x}^{\mu_d}\tilde{x}_{\nu_d}}{\tilde{x}^2}\right] \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{i=1}^{d}\delta^{\mu_i}_{\nu_i} - 2\sum_{j=1}^{d}\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\tilde{x}^{\mu_j}\tilde{x}_{\nu_j}}{\tilde{x}^2}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} + 0
\end{aligned}
$$

Now we use the identity

$$
\epsilon_{\color{red}{i_1\ldots i_k}\, i_{k+1}\ldots i_n}\epsilon^{\color{red}{i_1\ldots i_k}\, j_{k+1}\ldots j_n} = \delta^{i_1\ldots i_k\, j_{k+1}\ldots j_n}_{i_1\ldots i_k\, i_{k+1}\ldots i_n} = k!\,\delta^{j_{k+1}\ldots j_n}_{i_{k+1}\ldots i_n}
$$

which in our case becomes

$$
\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} = (d-1)!\delta^{\mu_j}_{\nu_j}
$$

Hence

$$
\begin{aligned}
\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) &= \left(\frac{1}{\tilde{x}^2}\right)^d\left(\frac{d! - 2\sum_{j=1}^{d}(d-1)!}{d!}\right) \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d\left(1 - \frac{2d(d-1)!}{d!}\right) = -\left(\frac{1}{\tilde{x}^2}\right)^d
\end{aligned}
$$

How distances transform

Under translation

$$
x'^\mu = x^\mu + a^\mu
$$

So,

$$
x'^\mu_a - x'^\mu_b = x^\mu_a + a^\mu - x^\mu_b - a^\mu
$$

$$= x_a^\mu - x_b^\mu$$

Thus, the distances are invariant under translation:

$$|x'_a - x'_b| = |x_a^\mu - x_b^\mu|$$

Under dilatation

$$x'^\mu = (1+\lambda)x^\mu$$

So,

$$\begin{aligned} x_a'^\mu - x_b'^\mu &= (1+\lambda)x_a^\mu - (1+\lambda)x_b^\mu \\ &= (1+\lambda)(x_a^\mu - x_b^\mu) \end{aligned}$$

We find that the distances between two point scales under dilatation, therefore the natural quantity which is invariant under both translation and dilatation is

$$\begin{aligned} \frac{\left|x_a'^\mu - x_b'^\mu\right|}{\left|x_c'^\mu - x_d'^\mu\right|} &= \frac{\cancel{1+\lambda}}{\cancel{1+\lambda}} \frac{|x_a^\mu - x_b^\mu|}{|x_c^\mu - x_d^\mu|} \\ &= \frac{|x_a^\mu - x_b^\mu|}{|x_c^\mu - x_d^\mu|} \end{aligned}$$

Under special conformal transformation

$$\begin{aligned} x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2x^2} \\ &= \frac{x^\mu - b^\mu x^2}{\Lambda^2(x)} \end{aligned}$$

So,

$$\begin{aligned} x_a'^\mu &= \frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} \\ x_b'^\mu &= \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)} \end{aligned}$$

and,

$$x_a'^\mu - x_b'^\mu = \frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}$$

squaring both sides

$$\begin{aligned} (x_a'^\mu - x_b'^\mu)^2 &= \left(\frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}\right)^2 \\ &= \frac{x_a^2 + b^2(x_a^2)^2 - 2x_a^2(x_a\cdot b)}{\Lambda^4(x_a)} + \frac{x_b^2 + b^2(x_b^2)^2 - 2x_b^2(x_b\cdot b)}{\Lambda^4(x_b)} \\ &\quad - \frac{2}{\Lambda^2(x_a)\Lambda^2(x_b)}\left[x_a\cdot x_b - x_b^2(x_a\cdot b) - x_a^2(b\cdot x_b) + b^2x_a^2x_b^2\right] \\ &= x_a^2\left[\frac{1-2(x_a\cdot b)}{\Lambda^4(x_a)} + \frac{\overbrace{2(b\cdot x_b) - b^2x_b^2}^{1-\Lambda^2(x_b)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1-2(x_b\cdot b)}{\Lambda^4(x_b)} + \frac{\overbrace{2(b\cdot x_a) - b^2x_a^2}^{1-\Lambda^2(x_a)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] \\ &\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\ &= x_a^2\left[\frac{1-2(x_a\cdot b) - \Lambda^2(x_a)}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1-2(x_b\cdot b) - \Lambda^2(x_b)}{\Lambda^4(x_b)}\right. \\ &\quad \left. + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \end{aligned}$$

$$
\begin{aligned}
&= x_a^2\left[\frac{-b^2x_a^2}{\Lambda^4(x_a)}+\frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right]+x_b^2\left[\frac{-b^2x_b^2}{\Lambda^4(x_b)}+\frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right]\\
&\quad+\frac{b^2(x_a^2)^2}{\Lambda^4(x_a)}+\frac{b^2(x_b^2)^2}{\Lambda^4(x_b)}-\frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)}\\
&= \frac{(x_a-x_b)^2}{\Lambda^2(x_a)\Lambda^2(x_b)}
\end{aligned}
$$

Thus, we find that the ratio of distances are not invariant under SCT.

$$
\frac{\left|x_a'^{\mu}-x_b'^{\mu}\right|}{|x_a-x_b|}=\frac{1}{\Lambda(x_a)\Lambda(x_b)}
$$

where $\Lambda(x_a)=\sqrt{1-2(x_a\cdot b)+b^2x_a^2}$ We can however, construct another quantity which is invariant under SCT.

$$
\begin{aligned}
\frac{|x_a'-x_b'|}{|x_b'-x_d'|}\frac{|x_d'-x_c'|}{|x_c'-x_a'|}&=\frac{\frac{|x_a-x_b|}{\Lambda(x_a)\Lambda(x_b)}}{\frac{|x_b-x_d|}{\Lambda(x_b)\Lambda(x_d)}}\frac{\frac{|x_d-x_c|}{\Lambda(x_d)\Lambda(x_c)}}{\frac{|x_c-x_a|}{\Lambda(x_c)\Lambda(x_a)}}\\
&=\frac{|x_a-x_b|}{|x_b-x_d|}\frac{|x_d-x_c|}{|x_c-x_a|}
\end{aligned}
$$

Such expressions are called, anharmonic ratios or cross-ratios.

1.4 Lie Algebra of Generators

$$
\begin{aligned}
[P_\mu,P_\nu]&=[-i\partial_\mu,-i\partial_\nu]\\
&=-[\partial_\mu,\partial_\nu]=0
\end{aligned}
$$

Some useful identities

$$
\begin{aligned}
[x_\alpha,\partial_\beta]f&=x_\alpha\partial_\beta f-\underbrace{\partial_\beta(x_\alpha f)}_{(\partial_\beta x_\alpha)f+x_\alpha\partial_\beta f}\\
&=x_\alpha\partial_\beta f-x_\alpha\partial_\beta f-(\partial_\beta x_\alpha)f\\
&=-(\partial_\beta x_\alpha)f
\end{aligned}
$$

$$
\begin{aligned}
[x_\alpha,\partial_\beta]&=-\partial_\beta x_\alpha=-g_{\beta\alpha}\partial^\mu x_\alpha\\
&=g_{\beta\alpha}
\end{aligned}
\tag{1.19}
$$

next is,

$$
\begin{aligned}
[x^2,\partial_\beta]&=[x^\alpha x_\alpha,\partial_\beta]\\
&=x^\alpha[x_\alpha,\partial_\beta]+[x^\alpha,\partial_\beta]x_\alpha\\
&=-x^\alpha g_{\beta\alpha}-\delta^\alpha_\beta x_\alpha\\
&=-x_\beta-x_\beta\\
&=-2x_\beta
\end{aligned}
\tag{1.20}
$$

and the last one is,

$$
\begin{aligned}
[x_\mu x^\nu,\partial_\beta]&=x_\mu[x^\nu,\partial_\beta]+[x_\mu,\partial_\beta]x^\nu\\
&=-x_\nu\delta^\nu_\beta-x^\nu g_{\beta\alpha}
\end{aligned}
\tag{1.21}
$$

We will now consider, the lie algebra of different operators one by one.

$$
\begin{aligned}
[P_\mu,D]&=[-i\partial_\mu,-ix^\alpha\partial_\alpha]\\
&=-[\partial_\mu,x^\alpha\partial_\alpha]\\
&=-x^\alpha[\partial_\mu,\partial_\alpha]-[\partial_\mu,x^\alpha]\partial_\alpha\\
&=-\delta^\alpha_\mu\partial_\alpha=-\partial_\mu=-i(-i\partial_\mu)
\end{aligned}
$$

$$= -iP_\mu$$

$$\begin{aligned}
[P_\mu, L_{\alpha\beta}] &= [-i\partial_\mu, -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\
&= -[\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha] \\
&= -[\partial_\mu, x_\alpha]\partial_\beta + [\partial_\mu, x_\beta]\partial_\alpha \\
&= g_{\alpha\mu}\partial_\beta - g_{\beta_\mu}\partial_\alpha \\
&= i(g_{\alpha\mu}P_\beta - g_{\beta_\mu}P_\alpha)
\end{aligned}$$

$$\begin{aligned}
[P_\mu, K_\nu] &= [-i\partial_\mu, -i(2x_\mu x^\alpha\partial_\alpha - x^2\partial_\nu)] \\
&= -[\partial_\mu, 2x_\nu x^\alpha\partial_\alpha - x^2\partial_\nu] \\
&= -2x_\nu x^\alpha[\partial_\mu, \partial_\alpha] - 2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + x^2[\partial_\mu, \partial_\nu] + [\partial_\mu, x^2]\partial_\nu \\
&= -2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + [\partial_\mu, x^2]\partial_\nu \\
&= -2(g_{\mu\nu}x^\alpha + \delta^\alpha_\mu x_\nu)\partial_\alpha + 2x_\mu\partial_\nu \\
&= -2g_{\mu\nu}x^\alpha\partial_\alpha - 2(x_\nu\partial_\mu - x_\mu\partial_\nu) \\
&= -2ig_{\mu\nu}D - 2iL_{\mu\nu} \\
&= -2i(g_{\mu\nu}D - L_{\mu\nu})
\end{aligned}$$

$$\begin{aligned}
[D, K_\mu] &= -[x^\alpha\partial_\alpha, 2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu] \\
&= -2[x^\alpha\partial_\alpha, x_\mu x^\beta\partial_\beta] + [x^\alpha\partial_\alpha, x^2\partial_\mu] \\
&= -2\left\{x^\alpha[\partial_\alpha, x_\mu x^\beta]\partial_\beta + x_\mu x^\beta[x^\alpha, \partial_\beta]\partial_\alpha\right\} \\
&\qquad + x^\alpha[\partial_\alpha, x^2]\partial_\mu + x^2[x^\alpha, \partial_\mu]\partial_\alpha \\
&= -2\left\{x^\alpha\left(g_{\alpha\mu}x^\beta + \delta^\beta_\alpha x_\mu\right)\partial_\beta + x_\mu x^\beta(-\delta^\alpha_\beta)\partial_\alpha\right\} \\
&\qquad + 2x^2\partial_\mu - \cancel{x^\alpha x^2\partial_\alpha\partial_\mu} + \cancel{x^2x^\alpha\partial_\alpha\partial_\mu} - x^2\partial_\mu \\
&= -\cancel{2x_\mu x^\beta\partial_\beta} - 2x^\beta x_\mu\partial_\beta + \cancel{2x_\mu x^\beta\partial_\beta} + x^2\partial_\mu \\
&= -(2x^\beta x_\mu\partial_\beta - x^2\partial_\mu) \\
&= -iK_\mu
\end{aligned}$$

$$\begin{aligned}
[K_\mu, L_{\alpha\beta}] &= [-i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu), i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\
&= [2x_\mu x^\nu\partial_\nu - x^2\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha] \\
&= 2[x_\mu x^\nu\partial_\nu, x_\alpha\partial_\beta] - [x^2\partial_\mu, x_\alpha\partial_\beta] + \underbrace{2[x_\mu x^\nu\partial_\nu, x_\beta\partial_\alpha] - [x^2\partial_\mu, x_\beta\partial_\alpha]}_{\alpha\leftrightarrow\beta} \\
&= 2\left\{x_\mu x^\nu[\partial_\nu, x_\alpha]\partial_\beta + x_\alpha[x_\mu x^\nu, \partial_\beta]\partial_\nu\right\} - x^2[\partial_\mu, x_\alpha]\partial_\beta - x_\alpha[x^2, \partial_\beta]\partial_\mu - (\alpha \leftrightarrow \beta) \\
&= \cancel{2x_\mu x^\nu(g_{\nu\alpha})\partial_\beta} - 2x_\alpha\left(g_{\mu\beta}x^\nu + \cancel{\delta^\nu_\beta x_\mu}\right)\partial_\nu - x^2g_{\mu\alpha}\partial_\beta + 2x_\alpha x_\beta\partial_\mu - (\alpha \leftrightarrow \beta) \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2g_{\mu\alpha}\partial_\beta + \cancel{2x_\alpha x_\beta\partial_\mu} + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2g_{\mu\beta}\partial_\alpha - \cancel{2x_\beta x_\alpha\partial_\mu} \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2g_{\mu\alpha}\partial_\beta + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2g_{\mu\beta}\partial_\alpha \\
&= -g_{\mu\beta}(2x_\alpha x^\nu\partial_\nu - x^2\partial_\alpha) + g_{\mu\alpha}(2x_\beta x^\nu\partial_\nu - x^2\partial_\beta) \\
&= ig_{\mu\alpha}K_\beta - ig_{\mu\beta}K_\alpha = i(g_{\mu\alpha}K_\beta - g_{\mu\beta}K_\alpha)
\end{aligned}$$

$$\begin{aligned}
[K_\mu, K_\nu] &= -[2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu, 2x_\nu x^\beta\partial_\beta - x^2\partial_\nu] \\
&= -4[x_\mu x^\alpha\partial_\alpha, x_\nu x^\beta\partial_\beta] + 2[x_\mu x^\alpha\partial_\alpha, x^2\partial_\nu] + 2[x^2\partial_\mu, x_\nu x^\beta\partial_\beta] - [x^2\partial_\mu, x^2\partial_\nu] \\
&= -4x_\nu x^\beta\left[x_\mu x^\alpha, \partial_\beta\right]\partial_\alpha - 4x_\mu x^\alpha\left[\partial_\alpha, x_\nu x^\beta\right]\partial_\beta + 2x_\mu x^\alpha\left[\partial_\alpha, x^2\right]\partial_\nu + 2x^2\left[x_\mu x^\alpha, \partial_\nu\right]\partial_\alpha \\
&\qquad + 2x^2\left[\partial_\mu, x_\nu x^\beta\right]\partial_\beta + 2x_\nu x^\beta\left[x^2, \partial_\beta\right]\partial_\mu - x^2[\partial_\mu, x^2]\partial_\nu - x^2[x^2, \partial_\nu]\partial_\mu \\
&= \cancel{4x_\nu x^\beta(g_{\mu\beta}x^\alpha + \delta^\alpha_\beta x_\mu)\partial_\alpha} - \cancel{4x_\mu x^\alpha(g_{\alpha\nu}x^\beta + \delta^\beta_\alpha x_\nu)\partial_\beta} + 4x_\mu x^2\partial_\nu - 2x^2(\cancel{g_{\mu\nu}x^\alpha} + \delta^\alpha_\nu x_\mu)\partial_\alpha \\
&\qquad + 2x^2(\cancel{g_{\mu\nu}x^\beta} + \delta^\beta_\mu x_\nu)\partial_\beta - 4x_\nu x^2\partial_\mu - 2x^2x_\mu\partial_\nu + 2x^2x_\nu\partial_\mu \\
&= \cancel{4x_\mu x^2\partial_\nu - 2x_\mu x^2\partial_\nu} + \cancel{2x_\nu x^2\partial_\mu - 4x_\nu x^2\partial_\mu} - \cancel{2x^2x_\mu\partial_\nu} + \cancel{2x^2x_\nu\partial_\mu} \\
&= 0
\end{aligned}$$

Next, we will see that Conformal Algebra in d dimensions is isomorphic to the Lie algebra of the Lorentz group in $d+2$ dimensions, any conformal covariant correlator in d dimensions should be obtainable from Lorentz covariant expressions in $d+2$ dimensions via some kind of dimensional reduction procedure. This is essentially the idea behind **Embedding Formalism**. We define the following set of new operators:

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} \\ J_{0,\mu} &= \frac{1}{2}\left(P_\mu + K_\mu\right) \\ J_{-1,\mu} &= \frac{1}{2}\left(P_\mu - K_\mu\right) \\ J_{-1,0} &= D \end{aligned}$$

with the property that

$$J_{ab} = -J_{ba}$$

where

$$a, b \in \{-1, 0, 1, \cdots, \overset{\text{d is dimension of spacetime}}{d}\}$$

These new generators, obey $SO(d+1,1)$ lie algebra:

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{bc}J_{ad}\right) \tag{1.22}$$

In this section, we will explicitly assume the form of flat metric as being euclidean, and given as:

$$g_{\mu\nu} = \eta_{\mu\nu} = (\underbrace{1, 1, \cdots, 1}_{d})$$

Our metric in (1.22) would be given as:

$$\eta_{ab} = (\underset{\eta_{-1-1}=-1}{-1}, \underset{\eta_{00}=1}{1}, \underbrace{1, \cdots, 1}_{\mu,\nu}) \tag{1.23}$$

If our original metric was Minkowski, we would have had:

$$\eta_{ab} = (-1, 1, \underbrace{-1, \cdots, 1}_{d})$$

We will now check, if (1.22) holds true:

$$\begin{aligned} [J_{\mu\nu}, J_{0,\alpha}] &= \left[L_{\mu\nu}, \frac{1}{2}(P_\alpha + K_\alpha)\right] \\ &= \frac{1}{2}[L_{\mu\nu}, P_\alpha] + \frac{1}{2}[L_{\mu\nu}, K_\alpha] \\ &= -\frac{1}{2}[P_\alpha, L_{\mu\nu}] - \frac{1}{2}[K_\alpha, L_{\mu\nu}] \\ &= -\frac{1}{2}\left(\eta_{\alpha\mu}P_\mu - \eta_{\alpha\nu}P_\mu\right) - \frac{1}{2}(\eta_{\alpha\mu}K_\nu - \eta_{\alpha\nu}K_\nu) \\ &= -\eta_{\alpha\mu}\left[\frac{1}{2}(P_\nu + K_\nu)\right] + \eta_{\alpha,\nu}\left[\frac{1}{2}(P_\mu + K_\mu)\right] \\ &= -i\eta_{\alpha\mu}J_{0,\nu} + i\eta_{\alpha\nu}J_{0,\mu} \end{aligned}$$

$$\begin{aligned} [J_{0,\mu}, J_{-1,0}] &= \left[\frac{1}{2}(P_\mu + K_\mu), D\right] \\ &= \frac{1}{2}[P_\mu, D] + \frac{1}{2}[K_\mu, D] \\ &= -\frac{1}{2}iP_\mu - \frac{1}{2}(-iK_\mu) = \frac{-i}{2}(P_\mu - K_\mu) = -iJ_{-1,\mu} \end{aligned}$$

If we assume that the metric in (1.22) is indeed given by (1.23). Then, the algebra (1.22) holds true. This shows the isomorphism between the conformal group in $d-$dimensions and the group $SO(d+1,1)$ with $1/2(d+1)(d+2)$ parameters.

Conformal Generators on the Field

Finite form of conformal transformation ($x' = \Lambda x$)[4]

$$
\begin{aligned}
\Phi'_a(x') &= U(\Lambda)\Phi_a(x)U^{-1}(\Lambda) \\
\Phi'_a(\Lambda x) &= \sum_b \pi_{ab}(\Lambda)\Phi_b(x) \\
&= \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x') \\
&= \pi_{ab}(e^{i\omega_g c_g})\Phi_b(e^{-i\omega_g c_g}x')
\end{aligned} \tag{1.24}
$$

We have dropped the $\sum$ sign and summation over repeated indices are implied. Infinitesimal form of (1.24):

$$
\begin{aligned}
\Phi'_a(x') &= (1 - i\omega_g \overset{\text{generator only acting on field}}{T_g})_{ab}\Phi_b(\Lambda^{-1}x') \\
&= (1 - i\omega_g T_g)_{ab}\, \underbrace{\Phi_b[(1 - i\omega_g \overset{\text{generator which only acts on } x'^{\mu}}{c_g})x'^{\mu}]}_{\Phi_b(x')+\{(1-i\omega_g c_g)x'^{\mu}-x'^{\mu}\}\partial_\mu\Phi_b(x')} \\
&= (1 - i\omega_g T_g)_{ab}[\Phi_b(x') - i\omega_g c_g x'^{\mu}\partial_\mu\Phi_b(x')] \\
\Phi'(x') &= \Phi(x') - i\omega_g\left[T_g + \underset{\text{accounts for the change in argument of field}}{\boxed{c_g x'^{\mu}\frac{\partial}{\partial x'^{\mu}}}}\right]\Phi(x') + \mathcal{O}(\omega_g^2)
\end{aligned}
$$

However, we will not use this approach but rather we will consider the transformations at origin and then translate it to every other point. This approach is based on studying the stabilizer subgroup of the Conformal Symmetry.[5] So, if we study the same at origin:

$$
\begin{aligned}
\Phi'(0) &= \Phi(x') - i\omega_g\left[T_g + c_g x'^{\mu}\frac{\partial}{\partial x'^{\mu}}\right]\Phi(x')\bigg|_{x'=0} \\
&= \Phi(0) - i\omega_g T_g \Phi(0)
\end{aligned}
$$

using translation operator

$$
\begin{aligned}
e^{ix^{\lambda}P_{\lambda}}\Phi'(0)e^{-ix^{\alpha}P_{\alpha}} &= e^{ix^{\lambda}P_{\lambda}}\Phi(0)e^{-ix^{\alpha}P_{\alpha}} - e^{ix^{\lambda}P_{\lambda}}i\omega_g T_g\Phi(0)e^{-ix^{\alpha}P_{\alpha}} \\
\Phi'(x) &= \Phi(x) - e^{ix^{\lambda}P_{\lambda}}i\omega_g T_g e^{-ix^{\sigma}P_{\sigma}}e^{ix^{\beta}P_{\beta}}\Phi(0)e^{-ix^{\alpha}P_{\alpha}} \\
&= \Phi(x) - i\omega_g \underbrace{e^{ix^{\lambda}P_{\lambda}}T_g e^{-ix^{\sigma}P_{\sigma}}}_{\text{we will find these ``translated operators'' later}}\Phi(x)
\end{aligned}
$$

For translation

$$
\begin{aligned}
\Phi'(x + a) &= e^{ia^{\lambda}P_{\lambda}}\Phi(x)e^{-ia^{\alpha}P_{\alpha}} \\
&= e^{ia^{\lambda}[P_{\lambda},\ ]}\Phi(x)
\end{aligned}
$$

using (1.26)

$$
= e^{a\cdot\partial}\Phi(x)
$$

For rotation, at $x'^{\mu} = 0. \implies x^{\mu} = 0$

$$
\Phi'_a(0) = \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}0) = \pi_{ab}(\Lambda)\Phi_b(0)
$$

Now, assuming the generator of rotation $T_g = L_{\mu\nu}$ acts like[6]

$$
L_{\mu\nu}\Phi_a(0) = S_{\mu\nu}\Phi_a(0) \tag{1.25}
$$

[4] tobias osborne's lecture notes pg 18

[5] pg 7 of "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications"

[6] pg 10, paragraph 2 of "Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions"

at origin. At any other point, it will behave as:

$$\begin{aligned} L_{\mu\nu}\Phi_a(x) &= e^{ix^\beta P_\beta} L_{\mu\nu}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\ &= \underbrace{e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma}}_{?} \underbrace{e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}}_{\Phi_a(x)} \end{aligned}$$

by taking the derivative of second term, we obtain the following commutator

$$\begin{aligned} \Phi_a(x) &= e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\ \partial_\mu\Phi_a(x) &= (\partial_\mu e^{ix^\lambda P_\lambda})\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)(\partial_\mu e^{-ix^\alpha P_\alpha}) \\ &= iP^\mu e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}(-iP^\mu) \\ &= iP^\mu\Phi_a(x) - i\Phi_a(x)P^\mu \\ &= i[P^\mu, \Phi_a(x)] \end{aligned} \tag{1.26}$$

We will now derive the form of $e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma}$[7]:

$$\begin{aligned} e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma} &= L_{\mu\nu} + [L_{\mu\nu}, -ix^\alpha P_\alpha] + \frac{1}{2!}[[L_{\mu\nu}, -ix^\alpha P_\alpha], -ix^\alpha P_\alpha] + \dots \\ &= L_{\mu\nu} + ix^\alpha \underbrace{[P_\alpha, L_{\mu\nu}]}_{i(g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu)} + \dots \\ &= L_{\mu\nu} + i^2x^\alpha(g_{\alpha\mu}P_\nu - g_{\alpha\nu}uP_\mu) \\ &= L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \\ &= L_{\mu\nu} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{we found in section 1.2}} \end{aligned}$$

we know, at $x' = 0$ we have $L_{\mu\nu} = S_{\mu\nu}$, so for the sake of consistency we get

$$e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma} = \underbrace{S_{\mu\nu}}_{\text{Spin Operator}\uparrow} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\uparrow\text{transforms the argument of field}}$$

The exponential map of above can be found in any textbook on QFT which describes rotation or Lorentz transformation.[8] If we ignore $S_{\mu\nu}$, then we can see how the last part acts on field:

$$\begin{aligned} x'^\mu &= \left(\delta^\mu_\nu + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right)x^\mu \\ &= x^\mu + \frac{i}{2}\omega_{\alpha\beta}i\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right)x^\nu \\ \Phi'(x) &= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\Phi(x) \\ &= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\Phi(x) \\ &= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha g^{\beta\sigma}\partial_\sigma - x^\beta g^{\alpha\sigma}\partial_\sigma)\Phi(x) \\ &= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)x^\sigma\partial_\sigma\Phi(x) \\ &= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}x\cdot\partial\Phi(x) \\ &\approx \Phi\left(x^\mu - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}x^\nu\right) \\ \Phi'(x') &= \Phi(x) \end{aligned}$$

For dilatation, at $x'^\mu = 0$, $x'^\mu = (1+\lambda)x^\mu = 0 \implies x^\mu = 0$. We have $\omega_g = \lambda$ and $T_g = D$:

$$D\Phi_a(0) = \tilde{\Delta}\Phi_a(0) \tag{1.27}$$

corresponding commutator (by operating it on eigenstate of dilatation)

$$D\left|\Delta\right\rangle = [D, \Phi_\Delta(0)]\left|0\right\rangle + \Phi_\Delta(0)D\left|0\right\rangle$$

---

[7] using BCH lemma $e^A B e^{-A} = e^{[A,\ ]}B$

[8] check eqn 1.141 and 1.150 of "QFT in curved spacetime" by Leonard Parker

$$\begin{aligned}
\tilde{\Delta}\,|\Delta\rangle &= [D, \Phi_\Delta(0)]\,|0\rangle + 0 \\
\tilde{\Delta}\Phi_\Delta(0)\,|0\rangle &= [D, \Phi_\Delta(0)]\,|0\rangle
\end{aligned}$$

Applying the same procedure, we consider:

$$\begin{aligned}
e^{ix^\beta P_\beta} D e^{-ix^\sigma P_\sigma} &= D + [D, -ix^\beta P_\beta] + \frac{1}{2!}[[D, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= D - ix^\alpha(iP_\alpha) \\
&= D + x^\alpha P_\alpha \\
&= D - ix^\alpha \partial_\alpha
\end{aligned} \tag{1.28}$$

for the sake of consistency at $x' = 0$

$$= \tilde{\Delta} - ix^\alpha \partial_\alpha$$

Now, we consider

$$D\Phi_a(x) = (\tilde{\Delta} - ix^\alpha \partial_\alpha)\Phi_a(x)$$

redefining $\tilde{\Delta} \equiv -i\Delta$, we get

$$D\Phi_a(x) = -i(\Delta + x^\alpha \partial_\alpha)\Phi_a(x)$$

Similarly,[9]

$$\begin{aligned}
[D, \Phi_a(x)] &= De^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma} D \\
&= e^{ix^\lambda P_\lambda}\underbrace{e^{ix^\alpha P_\alpha} D e^{-ix^\beta P_\beta}}_{=D + x^\alpha P_\alpha}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)\underbrace{e^{-ix^\sigma P_\sigma} D e^{-ix^\alpha P_\alpha}}_{=D + x^\alpha P_\alpha} e^{-ix^\lambda P_\lambda} \\
&= e^{ix^\beta P_\beta}[D + x^\alpha P_\alpha, \Phi_a(0)]e^{-ix^\sigma P_\sigma} \\
&= e^{ix^\beta P_\beta}\underbrace{[D, \Phi_a(0)]}_{\tilde{\Delta}\Phi_a(0)}e^{-ix^\sigma P_\sigma} + e^{ix^\beta P_\beta}\underbrace{[x^\alpha P_\alpha, \Phi_a(0)]}_{=x^\alpha[P_\alpha, \Phi_a(0)]}e^{-ix^\sigma P_\sigma} \\
&= \tilde{\Delta}\Phi_a(x) - ix \cdot \partial\Phi_a(x) \\
&= -i(\Delta + x \cdot \partial)\Phi_a(x)
\end{aligned}$$

Finite Dilatation[10], we consider

$$x' = e^\lambda x = e^{i\lambda D}x = \left(1 + i\frac{\lambda}{N}\overset{Dx^\mu = -ix\cdot\partial x^\mu}{D}\right)\dots\left(1 + i\frac{\lambda}{N}D\right)x$$

then at origin, the field transforms (active transformation) as:

$$\begin{aligned}
\Phi'_a(0) &= \left(1 + i\frac{\lambda}{N}D\right)\dots\left(1 + i\frac{\lambda}{N}D\right)\Phi_a(0) \\
&= \left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) && (\text{using } D\Phi(0) = \tilde{\Delta}\Phi) \\
&= e^{i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{-\lambda\Delta_a}\Phi_a(0)
\end{aligned}$$

In passive transformation

$$\begin{aligned}
\Phi'_a(0) &= \left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) \\
&= e^{-i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{\lambda\Delta}\Phi_a(0)
\end{aligned}$$

For arbitrary point (ignoring the change in argument of field and thus generator c_g):

$$\Phi'_a(x') = \pi_{ab}(e^{i\lambda D})\Phi_b(x)$$

[9] from pg 31 of 2309.10107, and x is not an operator here but a number

[10] look up *Lectures Notes For An Introduction to Conformal Field Theory A Course Given By Dr. Tobias Osborne*, pg 19

$$= [e^{i\lambda\tilde{\Delta}}]_{ab}\Phi_b(x)$$
$$\Phi'_a(e^\lambda x) = [e^{-\lambda\Delta}]_{ab}\Phi_b(x) = e^{-\lambda\Delta}\Phi_a(x)$$

For SCT, $x'^\mu = 0 \implies x^\mu = 0$. Hence, we will consider the same equations, but in this context:

$$K_\mu\Phi_a(0) = \kappa_\mu\Phi_a(0)$$

Again, applying the same procedure,

$$\begin{aligned}
e^{ix^\beta P_\beta}K_\mu e^{-ix^\sigma P_\sigma} &= K_\mu + [K_\mu, -ix^\beta P_\beta] + \frac{1}{2!}[[K_\mu, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu - ix^\beta[K_\mu, P_\beta] + \frac{1}{2}[-ix^\beta[K_\mu, P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu + 2x^\beta(g_{\mu\beta}D - L_{\mu\beta}) + \frac{1}{2}[2x^\beta(g_{\mu\beta}D - L_{\mu\beta}), -ix^\alpha P_\alpha] \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} - ix_\mu x^\alpha[D, P_\alpha] + ix^\beta x^\alpha \underbrace{[L_{\mu\beta}, P_\alpha]}_{-i(g_{\alpha\mu}P_\beta - g_{\alpha\beta}P_\mu)} \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + x_\mu x^\alpha P_\alpha + x_\mu x^\beta P_\beta - x_\alpha x^\alpha P_\mu \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + 2x_\mu x^\alpha P_\alpha - x_\alpha x^\alpha P_\mu
\end{aligned}$$

From the generator of dilatation and SCT, we have[11]

$$[D, K_\mu] = -iK_\mu \qquad \text{at } x'^\mu = 0 \implies \qquad [\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu$$

and

$$[D, L_{\mu\nu}] = 0 \qquad \text{at } x'^\mu = 0 \implies \qquad [\tilde{\Delta}, S_{\mu\nu}] = 0$$

For primary fields:

$$K_\mu\Phi_a(0) = 0$$

Since, for primary field $\tilde{\Delta}$ commutes with all other operators which belong to the stability subgroup. By Schur's lemma $\tilde{\Delta} \propto I$, where I is an identity operator. The SCT and momentum generator acts as ladder operator for Dilatation.

$$[D, [P_\mu, \Phi(0)]] = [P_\mu, [D, \Phi(0)]] + [[D, P_\mu], \Phi(0)] = -i(\Delta + 1)[P_\mu, \Phi(0)]$$
$$[D, [K_\mu, \Phi(0)]] = [K_\mu, [D, \Phi(0)]] + [[D, K_\mu], \Phi(0)] = -i(\Delta - 1)[K_\mu, \Phi(0)]$$

Finite Conformal Transformation of Fields

We begin by noting that *translation* and *rotation* do not introduce any new thing that we hadn't encountered in QFT, it is only the dilatation which does. Upon exponentiating the infinitesimal dilatation:

$$\begin{aligned}
\Phi'(x') &= e^{-i\omega_g\left[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}\right]}\Phi(x') \\
&= e^{-i\omega_g T_g}e^{-i\omega_g c_g x'\cdot\frac{\partial}{\partial x'}}\Phi(x') \\
&= e^{-i\omega_g T_g}\Phi(e^{-i\omega_g c_g}x')
\end{aligned}$$

This section is taken from "advanced mathematical methods - conformal field theory" by David Duffins.[12] (verify the statement, footnote can be verified from weinbegs' QFT pg 191)

$$\begin{aligned}
\Phi'_a(x') = U(\Lambda)\Phi_a(x')U^{-1}(\Lambda) &= e^{-i\omega_g T_g}\Phi_a e^{i\omega_g T_g} \\
&= e^{-i\omega_g [T_g,\ ]}\Phi_a(x')
\end{aligned}$$

For translation

$$\Phi(x) = e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} = e^{x\partial}\Phi(0)$$

Or,

$$\Phi'(x') = \Phi(x)$$

[11] same notes, look at eqn 65 to 70 (pg 18-19), all these commutators are for T_g

[12] Active coordinate transformation is given as: $\Phi(x') = U(\Lambda)\Phi(x)U^{-1}(\Lambda)$ whereas passive transformation is given as $\Phi(x') = U^{-1}(\Lambda)\Phi(x)U(\Lambda)$

$$
\begin{aligned}
&= \Phi(x' - a) \\
&= e^{-a\frac{\partial}{\partial x'}}\Phi(x') \\
&= e^{-iaP}\Phi(x')
\end{aligned}
$$

For dilatation ($x' = e^{\lambda}x$)

$$
\begin{aligned}
\Phi'_a(x') &= e^{-i\lambda D}\Phi_a(x') \\
&= e^{-\lambda(\Delta + x'\cdot\partial)}\Phi_a(x') \\
&= e^{-\lambda\Delta}\underbrace{e^{-\lambda x\cdot\partial}\Phi_a(x')}_{\Phi_a[e^{-\lambda}x']} \\
&= e^{-\lambda\Delta}\Phi_a(x)
\end{aligned}
$$

The last part could be understood as:

$$
\begin{aligned}
\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right] &= e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a(x) \\
\Phi_a\left[\underbrace{\left(1-\frac{\lambda}{N}\right)\dots\left(1-\frac{\lambda}{N}\right)}_{N\text{ terms}}x\right] &= e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a\left[\underbrace{\left(1-\frac{\lambda}{N}\right)\dots\left(1-\frac{\lambda}{N}\right)}_{N-1\text{ terms}}x\right] \\
\Phi_a(e^{-\lambda}x) &= e^{-\lambda x\cdot\partial}\Phi_a(x)
\end{aligned}
$$

or, alternatively

$$
\begin{aligned}
e^{-\lambda x\cdot\partial}\Phi_a(x) &= \left(1-\frac{\lambda}{N}x\cdot\partial\right)^N\Phi_a(x) \\
&= \left(1-\frac{\lambda}{N}x\cdot\partial\right)\dots\underbrace{\left(1-\frac{\lambda}{N}x\cdot\partial\right)\Phi_a(x)}_{\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right]} \\
&= \Phi_a\left[\left(1-\frac{\lambda}{N}\right)^N x\right] \\
&= \Phi_a(e^{-\lambda}x)
\end{aligned}
$$

Chapter 2

Embedding coordinates for Euclidean Space

Consider the embedding space coordinates

$$X^{-1}, X^0, \underbrace{X^1, X^2, \ldots, X^d}_{X^\mu}$$

we introduce the following null coordinates, $X^M = (X^+, X^-, X^\mu)$, where[1]

$$\left.\begin{aligned} X^+ &= X^{-1} + X^0 \\ X^- &= X^{-1} - X^0 \end{aligned}\right\} X^{-1} = \frac{X^+ + X^-}{2}; \quad X^0 = \frac{X^+ - X^-}{2}$$

with the mostly plus metric in $\mathbb{R}^{d+1,1}$, reads as

$$ds^2 = -(dX^{-1})^2 + (dX^0)^2 + \sum_{\mu=1}^{d}(dX^\mu)^2 = \sum_{\mu=1}^{d}(dX^\mu)^2 - dX^+dX^-$$

with the metric given as:

$$\eta_{MN} = \begin{pmatrix} 0 & -1/2 & 0 & 0 & \cdots \\ -1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 0 & 1 & \\ & & & & \ddots \end{pmatrix}$$

We can easily show that the generators in $d+2$ dimensional space reduces to d dimensional conformal generators in Euclidean space. We go from (X_{-1}, X_0, X_μ) coordinates to (ρ, η, x_μ) by following coordinate transformation.

$$\begin{aligned} X_{-1} &= \frac{\rho(1 - \eta^2 + \vec{x}^2)}{2} \\ X_0 &= \frac{\rho(1 + \eta^2 - \vec{x}^2)}{2} \\ X_\mu &= \rho x_\mu \end{aligned}$$

We can note that $X'^A = \lambda X^A$ corresponds to the same transformation as given above with $\rho' \equiv \lambda\rho$. We will see that it has a profound implication. For now, let us invert the above mentioned transformation as following:

$$\begin{aligned} \rho &= X_{-1} + X_0 \\ \eta &= \frac{\sqrt{\eta_{MN}X^M X^N}}{X_{-1} + X_0} \leftarrow \rho\eta = \sqrt{\eta_{MN}X^M X^N} \end{aligned}$$

[1] the index with lowest numeric value has the same sign in both $X^\pm$. If we had considered, X^{d+1} rather than X^{-1} then the definition would have been something like

$$X^\pm = X^0 \pm X^{d+1}$$

$$x_\mu = \frac{X_\mu}{X_{-1}+X_0}$$

We can write the change of basis as:

$$\begin{aligned}
\frac{\partial}{\partial X_{-1}} &= \frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\
\frac{\partial}{\partial X_0} &= \frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\
\frac{\partial}{\partial X_\mu} &= \frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}
\end{aligned}$$

Then, the generators for Lorentz transformation in $d+2$ dimensional space transforms as:

$$\begin{aligned}
P_\mu = J_{-1,\mu} + J_{0,\mu} &= (X_{-1}+X_0)\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} - \frac{\partial}{\partial X_{-1}}\right) \\
&= \rho\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - \rho x_\mu\left(\frac{1}{\rho\eta}\frac{\partial}{\partial \eta}\right) \\
&= \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x^\mu}
\end{aligned}$$

$$\begin{aligned}
K_\mu = J_{0,\mu} - J_{-1,\mu} &= (X_0 - X_{-1})\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_{-1}}\right) \\
&= \rho(\eta^2-\vec{x}^2)\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - 2\rho x_\mu\left[\frac{\partial}{\partial \rho} - \left(\frac{\eta^2+\vec{x}^2}{2\rho\eta}\right) - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&= \frac{\eta^2-\vec{x}^2}{\eta}x_\mu\frac{\partial}{\partial \eta} + (\eta^2-\vec{x}^2)\frac{\partial}{\partial x_\mu} - 2\rho x_\mu\frac{\partial}{\partial \rho} + x_\mu\frac{\eta^2+\vec{x}^2}{\eta}\frac{\partial}{\partial \eta} + 2x_\mu(x\cdot\partial) \\
&= 2x_\mu(x\cdot\partial) - \vec{x}^2\partial_\mu + \eta^2\partial_\mu - 2\rho x_\mu\frac{\partial}{\partial \rho} + 2x_\mu\eta\frac{\partial}{\partial \eta}
\end{aligned}$$

$$\begin{aligned}
D = J_{-10} = X_{-1}\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X^{-1}} &= X_{-1}\frac{\partial}{\partial X_0} + X_0\frac{\partial}{\partial X_{-1}} \\
&= \frac{\rho(1-\eta^2+\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&\quad + \frac{\rho(1+\eta^2-\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&= \rho\frac{\partial}{\partial \rho} - x^\mu\frac{\partial}{\partial x^\mu} - \eta\frac{\partial}{\partial \eta}
\end{aligned}$$

For $\eta = 0$, $\rho =$ constant

$$\begin{aligned}
P_\mu &= \partial_\mu \\
K_\mu &= 2x_\mu(x\cdot\partial) - x^\nu x_\nu\partial_\mu \\
D &= -x^\mu\partial_\mu
\end{aligned}$$

Note that conformal algebra is satisfied by both $\pm P_\mu$ and $\pm K_\mu$. The null cone corresponds to $\eta = 0$ but no condition imposed on ρ. Thus, there's a gauge redundancy: different values of ρ acting as scale factor for the coordinates correspond to the same physical point.

Next important to understand now is how the tensor fields transform under conformal transformation. We can use the embedding space to deduce their transformation law which is often more illuminating than the algebra gymnastics.

2.1 Tensor field under conformal transformation

We begin with the embedding space formalism to derive how tensors transform under conformal transformations. In this approach, physical spacetime coordinates x^μ are understood as projections from a higher-dimensional embedding space with coordinates $X^A \in \mathbb{R}^{d,2}$, where the conformal group $SO(d,2)$ acts linearly. Tensors in physical space are then obtained by pulling back embedding space tensors via this projection. Specifically, a tensor in physical space is related to its embedding space counterpart as follows:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \frac{\partial x^{\mu_1}}{\partial X^{A_1}}\cdots\frac{\partial x^{\mu_n}}{\partial X^{A_n}}\frac{\partial X^{B_1}}{\partial x^{\nu_1}}\cdots\frac{\partial X^{B_m}}{\partial x^{\nu_m}}\,T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)$$

The map from embedding to physical space is given by

$$x^\mu = \frac{X^\mu}{X^+} \equiv \frac{X^\mu}{X^0 + X^{-1}}$$

but due to the projective nature of this construction—i.e., physical points correspond to rays in the embedding space—we are free to rescale $X \sim \lambda X$, which is a manifestation of dilatation symmetry. Under this rescaling, the tensor should satisfy

$$T(\lambda X) = \lambda^{-\Delta} T(X)$$

which defines its conformal weight Δ. Using this, the projected tensor can be written as

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \left(\frac{1}{X^+}\right)^{\Delta}\frac{\partial x^{\mu_1}}{\partial X^{A_1}}\cdots\frac{\partial x^{\mu_n}}{\partial X^{A_n}}\frac{\partial X^{B_1}}{\partial x^{\nu_1}}\cdots\frac{\partial X^{B_m}}{\partial x^{\nu_m}}\,T^{A_1\cdots A_n}_{B_1\cdots B_m}\left(\frac{X}{X^+}\right)$$

This includes both the Jacobian factors from the change of variables and the prefactor from conformal weight.

If we choose the embedding slice $X^+ = 1$, then the projection simplifies significantly. In this gauge, we define frame fields (also known as projectors or vielbeins) by

$$e^\mu_A = \frac{\partial x^\mu}{\partial X^A}$$

and the projected tensor becomes

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = e^{\mu_1}_{A_1}\cdots e^{\mu_n}_{A_n} e^{B_1}_{\nu_1}\cdots e^{B_m}_{\nu_m} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)$$

Using the explicit form of the projection $x^\mu = X^\mu/X^+$, we compute

$$\frac{\partial x^\mu}{\partial X^A} = \frac{\delta^\mu_A(X^0 + X^{-1}) - X^\mu(\delta^0_A + \delta^{-1}_A)}{(X^0 + X^{-1})^2}$$

and setting $X^+ = 1$, we get

$$e^\mu_A = \delta^\mu_A, \qquad e^0_A = e^{-1}_A = -x^\mu$$

However, if we don't choose the slice $X^+ = 1$, then the projected tensor carries an additional scaling dependence:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \left(\frac{1}{X^+}\right)^{\Delta+n-m} \underbrace{e^{\mu_1}_{A_1}\cdots e^{\mu_n}_{A_n} e^{B_1}_{\nu_1}\cdots e^{B_m}_{\nu_m} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)}_{\text{depends only on physical point}}$$

This shows that tensors projected from different embedding space sections—i.e., different choices of X^+—differ by a power of X^+. So if two representations x and $\tilde{x}$ correspond to the same physical point but lie on different sections (i.e., with different values of X^+), then the corresponding tensors are related as

$$T(\tilde{x}) = \left(\frac{X^+}{\tilde{X}^+}\right)^{\Delta+n-m} T(x)$$

Next, consider how the tensor transforms under a conformal change of coordinates $x \mapsto x'$. This corresponds to a Lorentz transformation $X \mapsto X' = \Lambda X$ in embedding space. The tensor at x' is then:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x') = \frac{\partial x'^{\mu_1}}{\partial X'^{A_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial X'^{A_n}}\frac{\partial X'^{B_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial X'^{B_m}}{\partial x'^{\nu_m}} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X')$$

Now applying the chain rule, we insert identities:

$$\frac{\partial x'^\mu}{\partial X'} = \frac{\partial x'^\mu}{\partial x^\alpha}\cdot\frac{\partial x^\alpha}{\partial X'}, \qquad \frac{\partial X'}{\partial x'} = \frac{\partial X'}{\partial x^\beta}\cdot\frac{\partial x^\beta}{\partial x'}$$

This yields:

$$T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_m}(x') = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right)\left[\frac{\partial x^{\alpha_1}}{\partial X^{A_1}}\cdots\frac{\partial x^{\alpha_n}}{\partial X^{A_n}}\frac{\partial X^{B_1}}{\partial x^{\beta_1}}\cdots\frac{\partial X^{B_m}}{\partial x^{\beta_m}}\right]\underbrace{T^{A_1A_2\cdots A_n}_{B_1B_2\cdots B_m}(X)}_{\text{evaluated in different section}}$$

$$
\begin{aligned}
&= \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(\tilde{x}) \\
&= \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right)\left(\frac{X^+}{\tilde{X}^+}\right)^{\Delta+n-m} T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)
\end{aligned}
$$

The factor $(X^+/\tilde{X}^+)^{\Delta+n-m}$ arises because x and x' are physically the same point, but obtained by projecting from different embedding sections.

To express this ratio in terms of the coordinate Jacobian $|\partial x'/\partial x|$, note that from the projection $x^\mu = X^\mu/X^+$, one finds

$$
\left|\frac{\partial x'}{\partial x}\right| = \left|\frac{\partial X}{\partial x} \cdot \frac{\partial x'}{\partial X}\right| = \left|\frac{X^+}{\tilde{X}^+}\right|^d
$$

and therefore,

$$
\left(\frac{X^+}{\tilde{X}^+}\right)^{\Delta+n-m} = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta+n-m}{d}}
$$

Finally, combining everything, the full transformation law for the projected tensor under a conformal coordinate transformation is:

$$
T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x') = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta+n-m}{d}} \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)
$$

This expression makes it manifest that the projected tensor transforms as a tensor under general coordinate transformations, but with an additional *conformal weight* $\Delta + n - m$ that reflects both the homogeneity of the embedding space tensor and the number of upper and lower indices involved in the projection.

2.2 Finding Correlators from Embedding Space

Needless to say, it is significantly easier to construct Lorentz covariant expressions than conformally covariant ones. Therefore, the natural question arises: once we have constructed Lorentz covariant expressions in $d+2$ dimensions, **how do we descend to d dimensions without breaking covariance?**

Since we have already fixed $\eta = 0$ in our derivation of the conformal generators, we now focus on the structure preserved by Lorentz transformations: the null light cone $X^2 = 0$ in embedding space. This cone, defined in $\mathbb{R}^{d+1,1}$ as the space of null rays through the origin, is given by:

$$
\begin{aligned}
X^2 &= -(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 \\
&= -X^+X^- + \sum_{\mu=1}^{d}(X^\mu)^2 = 0
\end{aligned}
$$

Although correlators are initially constructed as Lorentz-invariant functions over the full $d+2$-dimensional ambient space, we now restrict them to the null cone. This constraint effectively reduces the support of such correlators to a $d+1$-dimensional submanifold, since one of the coordinate dependencies—say, X^-—can be eliminated using the condition $X^2 = 0$ (in our case it leads to $\eta = 0$).

Next, we reinterpret embedding space as a fiber bundle over the physical d-dimensional spacetime (where the CFT is defined). Each fiber consists of null lines in the $(d+2)$-dimensional space, and each point in the base space corresponds to an equivalence class of null vectors $X^A \sim \lambda X^A$, for any non-zero λ. This reflects the earlier observation regarding the arbitrariness of ρ: all such rescalings represent the same physical point in d dimensions.

This identification has an important consequence: As mentioned earlier, this introduces gauge redundancy in our description. To eliminate another coordinate, say $X^+ = \rho$, we fix the gauge by selecting a section of the bundle with specific choice of the slice on the embedding space, typically the *Euclidean section*, defined by:

$$
X^+ = \rho = f(X^\mu) \equiv f(x^\mu)
$$

Although Lorentz transformations may take a null vector outside this section (as they mix time-like and space-like directions), theu can always be brought back by utilising the scaling equivalence $X^A \sim \lambda X^A$. This choice anchors us to physical d-dimensional spacetime, completing the descent from $d+2$ dimensions while maintaining conformal covariance inherited from Lorentz invariance in the higher-dimensional space. With these prescriptions in place we can now identify X^μ with the Euclidean space coordinates x^μ by stripping ρ dependence.

$$
X^\mu \equiv x^\mu
$$

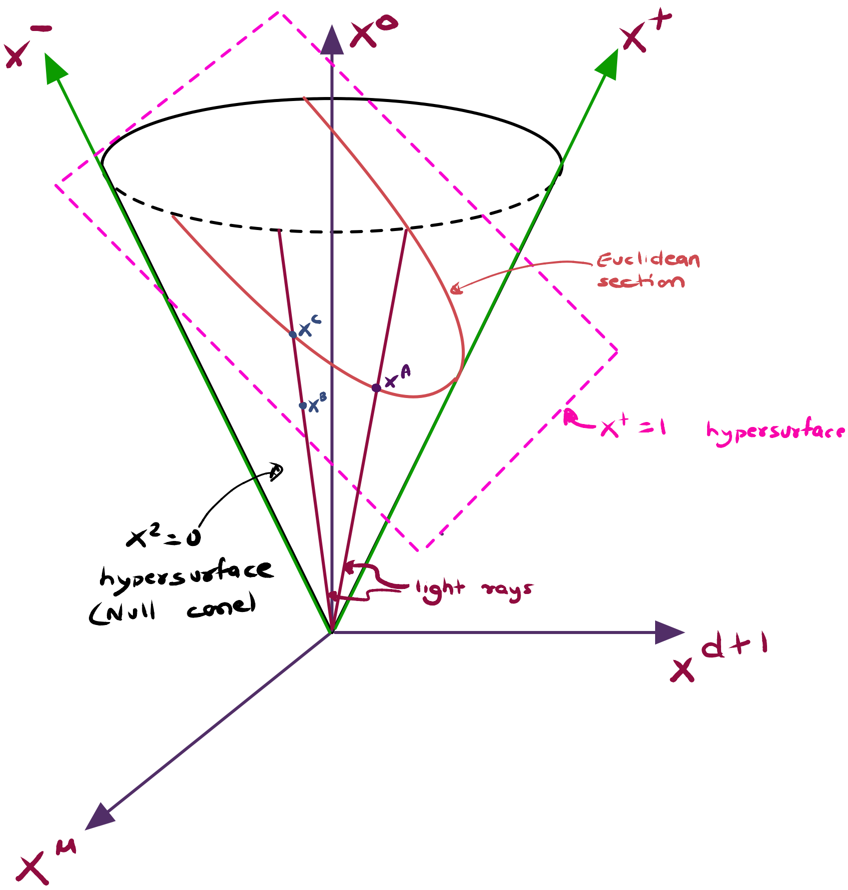


Figure 2.1: The hypersurface perpendicular to X^+ axis cutting at $X^+ = 1$ is shown as a plane and the null hypersurface is shown as the cone. The intersection of these two hypersurfaces describes the Euclidean Section. Dilatations are rotation in the X^0X^{d+1} plane and SCT or momentum generators are rotations in $X^\mu X^{d+1}$ with X^0X^{d+1} plane.

This leads to definition of X^- based on null condition as:

$$X^- = \frac{\sum_{\mu=1}^{d}(X^\mu)^2}{X^+} = \frac{X^\mu X_\mu}{X^+} = \frac{x^2}{f(X^\mu)}$$

or, equivalently[2]

$$\rho(-\eta^2 + \vec{x}^2) = \frac{\rho^2\vec{x}^2}{\rho} \implies \eta = 0$$

The spacetime interval on this section is given as:

$$ds^2 = dx^2 - dX^+dX^-\Big|_{X^+=f(X^i),X^-=\frac{x^2}{X^+}}$$

This section satisfies two of the following conditions:

- section intersects each of the light rays at some point
- maps each point in d dimensional Euclidean space to a point on the null cone in Embedding space.

We have shown how to get generators of conformal transformation from Lorentz generators by embedding a null cone in ambient space. Let us now analyze how Lorentz Transformation acts on a generic section on that null cone. The Lorentz transformation acting as rotation on the point X^A in the null-cone will move it to another point on the null cone outside the the section $X^B = \Lambda^B{}_A X^A$. However, suppose via some conformal

[2] in our notation $x^\mu = \rho\vec{x}^\mu$

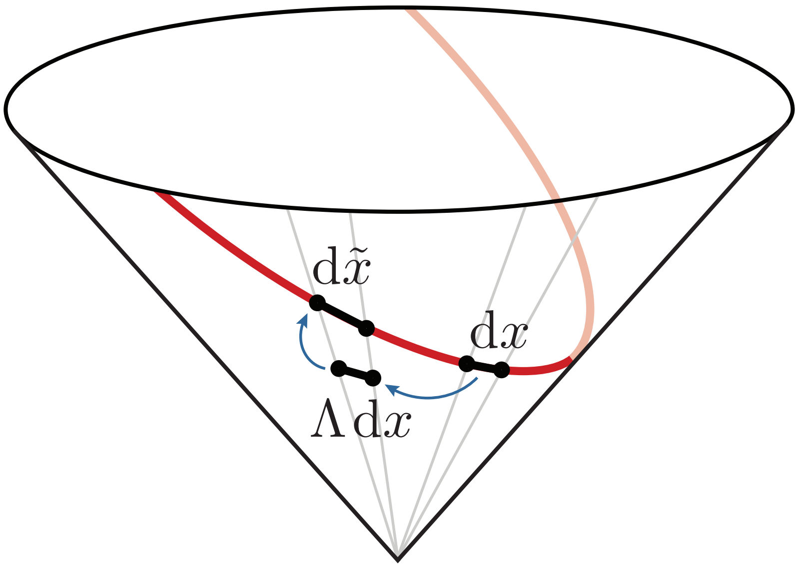


Figure 2.2: Upon Lorentz transformation, the points get mapped to different section however by utilizing the dilatation, we bring it back inside the original Euclidean section.

transformation (dilatation) in d dimensional Euclidean Space, we can move X^B to X^C back into the section. In all these steps, the only thing that changes the metric is the rescaling to get back into the section.

$$\begin{aligned}
ds_B^2 &= dX^M dX_M \\
&= d(\lambda(X)X^M)d(\lambda(X)X_M) \\
&= [\lambda dX^M + X^M(\boldsymbol{\nabla}\lambda\cdot dX)][\lambda dX_M + X_M(\boldsymbol{\nabla}\lambda\cdot dX)] \\
&= \lambda^2 dX^M dX_M + 2\lambda\underbrace{dX^M X_M}_{=0}(\boldsymbol{\nabla}\lambda\cdot dX) + \underbrace{X^M X_M}_{=0}(\boldsymbol{\nabla}\lambda\cdot dX)^2 \\
&= \lambda^2 dX^M dX_M = \lambda^2 ds_C^2
\end{aligned}$$

where we used, $X^2 = 0$ and $X^\mu dX_\mu = 0$ for restricting it to null cone. Assuming the three conditions we used for simplification applies, the Lorentz Transformation in $d+2$-dimensional spacetime is equivalent to conformal transformation in $d-$dimensional spacetime iff metric in $d-$dimensional space is Euclidean thus, dX_+ in ds^2 has to vanish. It gives us the condition for defining the Euclidean section as $X^+ = \rho = constant$ and thus, for the sake of simplificity, we take it as 1. Thus, we have two conditions which we can use to eliminate two extra degree of freedom.

In the embedding space formalism, choosing an Euclidean section corresponds to picking a specific way to embed the $d-$dimensional space in the $(d+2)-$dimensional space. We define the following map between d dimensional Euclidean Space with conformal symmetry to null cone in $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$

$$(X^+, X^-, X^\mu) \equiv \left(1, x^2, x^\mu\right)$$

Here, we note that choosing a constant value for X^+ would give us a section on the cone on which the induced metric is Euclidean.

2.3 Tensors in Embedding Space

In this section, we will only concern ourselves with traceless and symmetric fields in $\mathbb{R}^d$ and leave the anti-symmetric tensors for future. Consider a symmetric and traceless tensor[3] $O_{M_1\ldots M_S}$ defined on the cone $X^2 = 0$ in $\mathbb{R}^{d+1,1}$. Under the rescaling $X \to \lambda X$, the tensor transforms as

$$O_{M_1\ldots M_S}(\lambda X) = \lambda^{-\Delta} O_{M_1\ldots M_S}(X)$$

[3]symmetric tensors with spin s under $SO(d)$ form irreducible representations that correspond to integer spin particles (bosons). Anti-symmetric tensor fields have interpretation like they correspond to bivector of spinors etc.

i.e. it is a homogeneous function of degree Δ. We expect $O_{M_1 \ldots M_S}$ to get mapped to traceless and symmetric primary field in $\mathbb{R}^d$. Since, each index go from 0 to $d+1$, in $\mathbb{R}^{d+1,1}$ we find that, for $d+2-$dimensional fields other than scalar have 2 more degree of freedom per index than $d-$dimensional fields. In order to remove the extra degree of freedom, we consider the transversality condition.

$$X^{M_1} O_{M_1 \ldots M_S} = 0$$

We define the physical field to be:

$$\phi_{\mu\nu\lambda\ldots}(x) = \frac{\partial X^{M_1}}{\partial x^\mu} \frac{\partial X^{M_2}}{\partial x^\nu} \frac{\partial X^{M_3}}{\partial x^\lambda} \ldots O_{M_1 \ldots M_S}(X) \Bigg|_{X=X(x)}$$

Note that, this definition implies a redundancy. Indeed, anything proportional to X^M gives zero since

$$X^2 = 0 \implies X_M \frac{\partial X^M}{\partial x^\mu} = 0$$

Therefore, $O_{M_1 \ldots M_S}(X) \to O_{M_1 \ldots M_S}(X) + X_{M_1} F_{M_2 \ldots M_S}(X)$ gets mapped to the same physical field. This $SO(d+1,1)$ tensor is sometimes referred to as pure gauge in the literature. It is this gauge redundancy that reduces another degree of freedom per index by making it unphysical.

2.4 Examples: Two point and Three point correlator

In the last section we showed how the embedding space formalism put in place could be used to deduce the conformally invariant correlator. In this section we will utilize the formalism and explicitly construct two point and three point function using the formalism developed thus far. From 1.3, we know that the ratios are only invariant under dilatation and translation. Therefore, we seek to construct an invariant out of these ratios and metric tensor which is also invariant under SCT and the exchange of indices $\mu \leftrightarrow \nu$. First we will derive the form of scalar one point correlator. A scalar primary is denoted by $\hat{\mathcal{O}}_\Delta$ (notice the absence of the Lorentz index, which indicates that it is a scalar operator). We want to enforce the invariance of correlator under conformal transformation. For a one-point function, this reduces to

$$\langle \hat{\mathcal{O}}_\Delta(x'^\mu) \rangle = \langle \hat{\tilde{\mathcal{O}}}_\Delta(x'^\mu) \rangle \tag{2.1}$$

This condition must be enforced for all four conformal transformations. We will begin by enforcing translation.

Translation: $\tilde{x}^\mu = x^\mu + a^\mu$

It was given previously that the Jacobian for a translation is one. Therefore, our operator simply does not change under this translation

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \hat{\mathcal{O}}_\Delta(x^\mu)$$

enforcing this in (2.1), we are left with

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu) \rangle$$

where the operators are now the same on both sides of the equation. Notice that a correlation function is just a function. Therefore, this is equivalent to saying

$$f(\tilde{x}^\mu) = f(x^\mu)$$

Since this must be true for every possible translation, this tells us that the function has the same output regardless of what the input is, which means the function must just be some constant. Therefore, by enforcing translation we can conclude that

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu) \rangle = \text{constant} = C$$

We are not yet done. We need to make sure that all four transformations leave the one-point function invariant. Let's see what we can learn when we enforce dilatation.

Dilatation: $\tilde{x}^\mu = \lambda x^\mu$

Applying the Jacobian for dilitation, we see our Primary Scalar Operator transforms as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right|^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \lambda^{-\Delta} \hat{\mathcal{O}}_\Delta(x^\mu)$$

We want to enforce this in (2.1) and use our results from enforcing translation invariance. This gives us

$$\begin{aligned}\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu)\rangle &= \langle \hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu)\rangle \\ &= \langle \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta(x^\mu)\rangle \\ &= \lambda^{-\Delta}\langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle\end{aligned}$$

We found previously, by enforcing translation, that

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu)\rangle = C$$

which means

$$C = \lambda^{-\Delta}C$$

This equation must be true for arbitrary scale factor λ. Therefore, unless $\Delta = 0$, we can conclude that $C = 0$.

For unitary CFTs, the only $\Delta = 0$ operator is the identity operator. So, with the exception of the identity, all one-point functions must vanish!

$$\boxed{\langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle = 0 \text{ for } \Delta \neq 0} \tag{2.2}$$

We said that we must impose all four conformal transformations, but the others are trivially satisfied at this point. So we are done with one-point correlators! Again, we'd like to highlight the fact that this is the result for ALL CFTs. You don't need to know anything else about the system, only that it has conformal symmetry.

2.4.1 Two-point Scalar Primary

For two-point functions, we need to enforce

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle \tag{2.3}$$

Again, this must be done for all four conformal transformations. As with the one-point function, we will begin by enforcing translation.

Translation:

First, notice that $\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle$ is an object that takes two positions as inputs and gives back a number, so we can just write this as a function of $\tilde{x}_1^\mu$ and $\tilde{x}_2^\mu$

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = f(\tilde{x}_1^\mu, \tilde{x}_2^\mu)$$

We found previously that under translations, scalar primary operators transform as

$$\hat{\tilde{\mathcal{O}}}(\tilde{x}^\mu) = \hat{\mathcal{O}}(x^\mu)$$

Putting this result in the right side of equation (2.3), we find that

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle$$

Notice that the only differences between the left and right side of this equation are the inputs. The function on each side is the same

$$f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) = f(x_1^\mu, x_2^\mu)$$

Under translation, we have $\tilde{x}^\mu = x^\mu + a^\mu \to x^\mu = \tilde{x}^\mu - a^\mu$. If we put this into the previous equation it becomes

$$f(x_1^\mu - a^\mu, x_2^\mu - a^\mu) = f(x_1^\mu, x_2^\mu)\ ,\ \forall a^\mu$$

This must be true regardless of the value of a^μ which means that a^μ must somehow cancel out. This is only satisfied if it is a function of $x_1^\mu - x_2^\mu$ so we have

$$f(x_1^\mu, x_2^\mu) = f(x_1^\mu - x_2^\mu)$$

That is, our function cannot be any function of the two positions. Rather, it can only depend on the displacement between the two positions. Therefore,

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(x_1^\mu - x_2^\mu)$$

Let's now see what we can learn by enforcing rotation.

Rotation $\tilde{x}^\mu = \Lambda^\mu_\nu x^\mu$

The Jacobian for rotation is the same as for translation, 1. Therefore, scalar primary operators transform the same under rotation as they do under translation

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle$$

Re-expressing this in a more familiar form, as functions, we have

$$f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) = f(x_1^\mu, x_2^\mu)$$

Next, we impose what we found by imposing translational invariance

$$f(\tilde{x}_1^\mu - \tilde{x}_2^\mu) = f(x_1^\mu - x_2^\mu)$$

Expressing the transformed coordinates in terms of our original coordinate system, we find

$$f(\Lambda^\mu_\nu(x_1^\nu - x_2^\nu)) = f(x_1^\mu - x_2^\mu)$$

This tells us that applying a rotation has no effect on the output. This means that the function must depend only on the magnitude of the separation $|x_1^\mu - x_2^\mu|$ (Recall, rotating a vector changes it, but rotating a scalar does nothing). So, from applying translational and rotational invariance, we can conclude that

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(|x_1^\mu - x_2^\mu|)$$

We will now continue by enforcing invariance under dilatation.

Dilatation

Recall, under dilatation, scalar primary operators transform as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta(x^\mu)$$

Substituting this into our two-point function condition, eqn. (2.3), we have

$$\begin{aligned}
\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle &= \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle \\
&= \langle \lambda^{-\Delta_1}\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\lambda^{-\Delta_2}\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle \\
&= \lambda^{-\Delta_1}\lambda^{-\Delta_2}\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle \\
&= \lambda^{-(\Delta_1+\Delta_2)}f(|x_1^\mu - x_2^\mu|)
\end{aligned}$$

where we are able to pull the λ's out of the correlator because they are just scalars. Notice also that we used what we already learned from translational and rotational invariance. This tells us that

$$f(|\tilde{x}_1^\mu - \tilde{x}_2^\mu|) = \lambda^{-(\Delta_1+\Delta_2)}f(|x_1^\mu - x_2^\mu|)$$

We can apply the transformation to the coordinates on the left-hand-side, which gives

$$f(\lambda|x_1^\mu - x_2^\mu|) = \lambda^{-(\Delta_1+\Delta_2)}f(|x_1^\mu - x_2^\mu|)$$

What does this mean? We can consider expanding our function in a power series

$$f(|x_1^\mu - x_2^\mu|) = \sum_n c_n|x_1^\mu - x_2^\mu|^n$$

Substituting this in above gives

$$\sum_n c_n\lambda^n|x_1^\mu - x_2^\mu|^n = \lambda^{-(\Delta_1+\Delta_2)}\sum_n c_n|x_1^\mu - x_2^\mu|^n$$

This is only satisfied for all λ, if all $n = 0$ except $n = -(\Delta_1 + \Delta_2)$. Therefore, after enforcing translation, rotation, and dilatation symmetry we have

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = C|x_1^\mu - x_2^\mu|^{-(\Delta_1+\Delta_2)} \tag{2.4}$$

where C is some undetermined constant.

Special Conformal Transformation

Enforcing special conformal symmetry directly is a very messy business. Luckily for us, as discussed previously, a special conformal transformation is equivalent to performing an inversion, followed by a translation, followed by another inversion. Since we have already enforced translational invariance, this means it is sufficient to enforce inversion invariance, which is much easier. Recall, an inversion is given by

$$x^\mu = \frac{\tilde{x}^\mu}{\tilde{x}^2}$$

As with the other transformations, we need the Jacobian for inversion in order to see how the operators will transform. This is given by

$$\left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \frac{1}{\tilde{x}^{2D}}$$

Therefore, under inversion, scalar primary operators transform as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left(\frac{1}{\tilde{x}^{2D}}\right)^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \frac{1}{(\tilde{x}^2)^\Delta}\hat{\mathcal{O}}_\Delta(x^\mu)$$

As usual, we will now go put this into equation (2.3) to enforce the symmetry

$$\begin{aligned}\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle &= \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle \\ &= \left\langle \frac{1}{(\tilde{x}_1^2)^{\Delta_1}}\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\frac{1}{(\tilde{x}_2^2)^{\Delta_2}}\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\right\rangle \\ &= \frac{1}{(\tilde{x}_1^2)^{\Delta_1}}\frac{1}{(\tilde{x}_2^2)^{\Delta_2}}\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle\end{aligned}$$

Now, we can use our result from enforcing dilitation to replace $\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle$ on the left and $\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle$ on the right of this equation to get

$$\frac{C}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \frac{1}{(\tilde{x}_1^2)^{\Delta_1}}\frac{1}{(\tilde{x}_2^2)^{\Delta_2}}\frac{C}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}} \tag{2.5}$$

With a bit of algebra, this is equivalent to

$$\frac{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \frac{1}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}} \tag{2.6}$$

In order to put this in a more friendly form, we will use the following identity for inversions. Note: verifying this relationship requires substituting in the inversion transformation and some algebra. The reader is highly encouraged to check it.

$$\frac{\tilde{x}_1^2\tilde{x}_2^2}{(\tilde{x}_1^\mu - \tilde{x}_2^\mu)^2} = \frac{1}{(x_1^\mu - x_2^\mu)^2} \tag{2.7}$$

Using this identity in eqn. (2.6), we find

$$\frac{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \left[\frac{\tilde{x}_1^2\tilde{x}_2^2}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^2}\right]^{\frac{\Delta_1+\Delta_2}{2}}$$

This is only satisfied if

$$\Delta_1 = \Delta_2$$

Therefore, we find that the two-point function vanishes, unless the dimensions of the two operators are the same. In summary, the two-point function for scalar primaries in ANY CFT is given by

$$\boxed{\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = \frac{C\delta_{\Delta_1\Delta_2}}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}}} \tag{2.8}$$

Note that it is standard convention to choose to normalize your operators so that $C = 1$, so you will often see this without the C constant included. We leave it here for complete generality.

2.4.2 Three-point Scalar Primary

For the three-point function, we need to enforce

$$\langle \hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = \langle \hat{\tilde{\mathcal{O}}}_1(x_1^\mu)\hat{\tilde{\mathcal{O}}}_2(x_2^\mu)\hat{\tilde{\mathcal{O}}}_3(x_3^\mu)\rangle \tag{2.9}$$

Enforcing the symmetries for the three-point function follows in a very similar way to the two-point function, so we will not include as much detail. The reader is encouraged to work through any excluded details on their own.

Poincaré

For translations and rotations, the same line of argumentation that was used for two-point functions can be applied. However, instead of two points at our disposal, we have three. Therefore, our function can be a function of the magnitude of the separations between any parings of three points.

$$\langle \hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|)$$

where $|x_{12}^\mu| = |x_1^\mu - x_2^\mu|$, $|x_{23}^\mu| = |x_2^\mu - x_3^\mu|$, and $|x_{31}^\mu| = |x_3^\mu - x_1^\mu|$.

Dilatation

Enforcing dilitation invariance with our three-point function, we have

$$\begin{aligned}
\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(\tilde{x}_3^\mu)\rangle &= \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_3}(\tilde{x}_3^\mu)\rangle \\
&= \langle \lambda^{-\Delta_1}\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\lambda^{-\Delta_2}\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\lambda^{-\Delta_3}\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu)\rangle \\
&= \lambda^{-\Delta_1}\lambda^{-\Delta_2}\lambda^{-\Delta_3}\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu)\rangle
\end{aligned}$$

Using our results from enforcing Poincaré invariance, this becomes

$$f(|\tilde{x}_{12}^\mu|, |\tilde{x}_{23}^\mu|, |\tilde{x}_{31}^\mu|) = \lambda^{-(\Delta_1+\Delta_2+\Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) \tag{2.10}$$

Subsituting in the dilitation transformation on the LHS, this is

$$f(\lambda|x_{12}^\mu|, \lambda|x_{23}^\mu|, \lambda|x_{31}^\mu|) = \lambda^{-(\Delta_1+\Delta_2+\Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) \tag{2.11}$$

As with the two-point function, we can expand our function in a power series.

$$f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) = \sum_{nmp} c_{nmp}|x_{12}^\mu|^n |x_{23}^\mu|^m |x_{31}^\mu|^p \tag{2.12}$$

Substituting this in, we find that all terms must vanish, unless

$$n + m + p = -(\Delta_1 + \Delta_2 + \Delta_3)$$

Therefore, dilitation and Poincaré invariance tell us

$$\langle \hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = \sum_{nmp=-(\Delta_1+\Delta_2+\Delta_3)} c_{nmp}|x_{12}^\mu|_n |x_{23}^\mu|^m |x_{31}^\mu|^p \tag{2.13}$$

Special Conformal Transformation

Again, to find the effect of imposing special conformal symmetry, we need only to impose inversion symmetry, which is much easier. Although easier, the algebra is still quite nasty and will not be shown here. Ultimately, inversion (therefore special conformal) invariance leads to the additional constraint that all terms vanish, unless

$$\begin{aligned}
n &= \Delta_1 + \Delta_2 - \Delta_3 \\
m &= \Delta_1 + \Delta_3 - \Delta_2 \\
p &= \Delta_2 + \Delta_3 - \Delta_1
\end{aligned} \tag{2.14}$$

Therefore, after enforcing all of the conformal symmetries on the 3-point function of scalar primaries, we find

$$\boxed{\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu) = \frac{C_{123}}{|x_{12}^\mu|^n |x_{23}^\mu|^m |x_{31}^\mu|^p}} \tag{2.15}$$

where n, m, and p are given by (2.14). We find that, as was the case with the two-point scalar primaries, the spatial dependence of 3-point scalar primaries are completely determined. We are left only with a set of constants C_{123}. It turns out that this set of constants is vitally important to defining any particular conformal field theory and they tell you how much your given operators interact. This set of constants goes by various names including the *3-point coefficients*, the *OPE coefficients*, and the *structure constants*.

2.4.3 Going beyond scalars

Moving on, we next consider the two point corrector of vector field. The ansatz for such a correlator is:[4]

$$\langle J_\mu(x_1)J_\nu(x_2)\rangle = C\underbrace{\frac{1}{|x_1-x_2|^{2\Delta}}}_{\text{same as scalar case}}\left[g_{\mu\nu}+\delta\frac{(x_1-x_2)_\mu(x_1-x_2)_\nu}{(x_1-x_2)^2}\right]$$

The correlation function is invariant under translation, therefore we will consider following redefinition:

$$\begin{aligned}\langle J_\mu(x_1)J_\nu(x_2)\rangle &= \langle J_\mu(x_1-x_2)J_\nu(0)\rangle\\ &= \langle J_\mu(x_{12})J_\nu(0)\rangle = \langle J_\mu(x)J_\nu(0)\rangle\end{aligned}$$

Since SCT is just inversion→translation→inversion, we can this property to our advantage. As the correlation function is already invariant under translations, it suffices to verify its invariance under inversions. If this property holds, then by extension, the correlation function will also be invariant under SCT. The inversion transformation is given as[5]:

$$x'_\mu = \frac{x_\mu}{x^2} \qquad\qquad |x'|^2 = \frac{1}{|x|^2}$$

and

$$\frac{\partial x'_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\frac{x_\nu}{x^2} = \frac{1}{x^2}\left[g_{\mu\nu}-2\frac{x_\mu x_\nu}{x^2}\right] = x'^2\underbrace{\left[g_{\mu\nu}-2\frac{x'_\mu x'_\nu}{x'^2}\right]}_{I_{\mu\nu}}$$

The vector field would transform as

$$\langle J_\mu(x'_1)J_\nu(x'_2)\rangle = \underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{\Delta/d}\left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{\Delta/d}}_{\substack{\text{this was used to derive}\\ \text{the correlation function for scalar case}}}\overbrace{\frac{\partial x'_\alpha}{\partial x^\mu}\frac{\partial x'_\beta}{\partial x^\nu}}^{\text{without conformal factor}}\left\langle J^\alpha(x_1)J^\beta(x_2)\right\rangle$$

we see that

$$\underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{\Delta/d}_{x=x_1}}_{|x'_1|^{-2\Delta}}\left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{\Delta/d}_{x=x_2}\frac{1}{|x_{12}|^{2\Delta}} = |x'_1|^{-2\Delta}|x'_2|^{-2\Delta}\frac{1}{|x_{12}|^{2\Delta}} = \frac{1}{|x'_{12}|^{2\Delta}}$$

where we used

$$\begin{aligned}|x'_{12}|^2 &= \left(\frac{x_1^\mu}{x_1^2}-\frac{x_2^\mu}{x_2^2}\right)^2\\ &= \frac{|x_1|^2}{x_1^4}+\frac{|x_2|^2}{x_2^4}-2\frac{x_1^\mu}{x_1^2}\frac{x_{2\mu}}{x_2^2}\\ &= \frac{1}{x_1^2}-2\frac{x_1}{x_1^2}\cdot\frac{x_2}{x_2^2}+\frac{1}{x_2^2}\\ &= \frac{(x_1-x_2)^2}{x_1^2x_2^2} = \frac{|x_{12}|^2}{|x_1|^2|x_2|^2} = \frac{|x_{12}|^2}{|x'_1|^{-2}|x'_2|^{-2}}\end{aligned}$$

[4] pg 24 of "CFT with boundary and defects" by Herzog

[5] pg 17-18 of "Quantum Gravity and Cosmology based on Conformal Field Theory" and section 4.5 of "A conformal field theory primer in $D \geq 3$" by Andrew Evans

Then, we only have to ensure that $g^{\mu\nu} + \delta \frac{x'^{\mu}_{12} x'^{\nu}_{12}}{x'^2_{12}}$ is invariant under inversion.[6]

$$
\begin{aligned}
g^{\mu\nu} + \delta \frac{(x'_{12})^{\mu}(x'_{12})^{\nu}}{(x'_{12})^2} &= \left(\delta^{\mu}_{\alpha} - 2\frac{x^{\mu}_1 x_{1\alpha}}{x_1^2}\right)\left(\delta^{\nu}_{\beta} - 2\frac{x^{\nu}_2 x_{2\beta}}{x_2^2}\right)\left[g^{\alpha\beta} + \delta \frac{x^{\alpha}_{12} x^{\beta}_{12}}{x^2_{12}}\right]\Bigg|_{x^{\mu} = \frac{x'^{\mu}}{|x'|^2}} \\
&= \left(\delta^{\mu}_{\alpha} - 2\frac{x'^{\mu}_1 x'_{1\alpha}}{|x_1|'^2}\right)\left(\delta^{\nu}_{\beta} - 2\frac{x'^{\nu}_2 x'_{2\beta}}{|x'_2|^2}\right)\left[g^{\alpha\beta} + \delta \frac{(x'_1|x'_2|^2 - x'_2|x'_1|^2)^{\alpha}(x'_1|x'_2|^2 - x'_2|x'_1|^2)^{\beta}}{x'^2_{12}|x'_1|^2|x'_2|^2}\right] \\
&= \Bigg[g^{\mu\beta} + \delta \frac{(x'_1|x'_2|^2 - x'_2|x'_1|^2)^{\mu}(x'_1|x'_2|^2 - x'_2|x'_1|^2)^{\beta}}{x'^2_{12}|x'_1|^2|x'_2|^2} - 2\frac{x'^{\mu}_1 x'^{\beta}_1}{|x_1|'^2} \\
&\quad - 2\delta \frac{x'^{\mu}_1(\cancel{|x'_1|^2}|x'_2|^2 - x'_1\cdot x'_2\cancel{|x'_1|^2})(x'_1|x'_2|^2 - x'_2|x'_1|^2)^{\beta}}{x'^2_{12}|x'_1|^2\cancel{|x'_1|^2}|x'_2|^2}\Bigg]\left(\delta^{\nu}_{\beta} - 2\frac{x'^{\nu}_2 x'_{2\beta}}{|x'_2|^2}\right) \\
&= \left[g^{\mu\beta} - \delta \frac{(x'_1|x'_2|^2 + x'_2|x'_1|^2 - 2x'_1(x'_1\cdot x'_2))^{\mu}(x'_1|x'_2|^2 - x'_2|x'_1|^2)^{\beta}}{x'^2_{12}|x'_1|^2|x'_2|^2} - 2\frac{x'^{\mu}_1 x'^{\beta}_1}{|x_1|'^2}\right]\left(\delta^{\nu}_{\beta} - 2\frac{x'^{\nu}_2 x'_{2\beta}}{|x'_2|^2}\right) \\
&= \left[g^{\mu\nu} - \delta \frac{(x'_1|x'_2|^2 + x'_2|x'_1|^2 - 2x'_1(x'_1\cdot x'_2))^{\mu}(x'_1|x'_2|^2 - x'_2|x'_1|^2)^{\nu}}{x'^2_{12}|x'_1|^2|x'_2|^2} - 2\frac{x'^{\mu}_1 x'^{\nu}_1}{|x_1|'^2}\right] \\
&\quad - 2\frac{x'^{\nu}_2 x'^{\mu}_2}{|x'_2|^2} + 2\delta \frac{(x'_1|x'_2|^2 + x'_2|x'_1|^2 - 2x'_1(x'_1\cdot x'_2))^{\mu}(x'_2\cdot x'_1\cancel{|x'_2|^2} - \cancel{|x'_2|^2}|x'_1|^2)x'^{\nu}_2}{x'^2_{12}|x'_1|^2|x'_2|^2\cancel{|x'_2|^2}} + 4\frac{x'^{\mu}_1 x'^{\nu}_2(x'_1\cdot x'_2)}{|x_1|'^2} \\
&= g^{\mu\nu} - 2\frac{x'^{\mu}_1 x'^{\nu}_1}{|x_1|'^2} - 2\frac{x'^{\nu}_2 x'^{\mu}_2}{|x'_2|^2} + 4\frac{x'^{\mu}_1 x'^{\nu}_2(x'_1\cdot x'_2)}{|x_1|'^2} \\
&\quad - \delta \frac{\{x'_1|x'_2|^2 + x'_2|x'_1|^2 - 2x'_1(x'_1\cdot x'_2)\}^{\mu}\{x'_1|x'_2|^2 + x'_2|x'_1|^2 - 2(x'_1\cdot x'_2)x'_2\}^{\nu}}{x'^2_{12}|x'_1|^2|x'_2|^2} \\
&= g^{\mu\nu} - 2\frac{x'^{\mu}_1 x'^{\nu}_1}{|x_1|'^2} - 2\frac{x'^{\nu}_2 x'^{\mu}_2}{|x'_2|^2} + 4\frac{x'^{\mu}_1 x'^{\nu}_2(x'_1\cdot x'_2)}{|x_1|'^2} \\
&\quad - \delta \frac{\{x'_2|x'_1|^2 + |x'_1 - x'_2|^2 x'_1 - |x'_1|^2 x'_1\}^{\mu}\{x'_1|x'_2|^2 + |x'_1 - x'_2|^2 x'_2 - |x'_2|^2 x'_2\}^{\nu}}{x'^2_{12}|x'_1|^2|x'_2|^2} \\
&= g^{\mu\nu} - 2\frac{x'^{\mu}_1 x'^{\nu}_1}{|x_1|'^2} - 2\frac{x'^{\nu}_2 x'^{\mu}_2}{|x'_2|^2} + 4\frac{x'^{\mu}_1 x'^{\nu}_2(x'_1\cdot x'_2)}{|x_1|'^2} \\
&\quad + \delta\frac{x'^{\mu}_{12} x'^{\nu}_{12}}{x'^2_{12}} - \delta\frac{x'^{\mu}_1}{|x'_1|^2}x'^{\nu}_{12} + \delta\frac{x'^{\nu}_2}{|x'_2|^2}x'^{\mu}_{12} - \delta|x'_{12}|^2\frac{x'^{\mu}_1 x'^{\nu}_2}{|x'_1|^2|x'_2|^2} \\
&= g^{\mu\nu} - (\delta+2)\frac{x'^{\mu}_1 x'^{\nu}_1}{|x_1|'^2} - (\delta+2)\frac{x'^{\nu}_2 x'^{\mu}_2}{|x'_2|^2} + 2(\delta+2)\frac{x'^{\mu}_1 x'^{\nu}_2(x'_1\cdot x'_2)}{|x_1|'^2} + \cancel{\delta\frac{x'^{\mu}_1 x'^{\nu}_2}{|x'_1|^2}} + \cancel{\delta\frac{x'^{\mu}_1 x'^{\nu}_2}{|x'_2|^2}} \\
&\quad - \cancel{\delta|x'_1|^2\frac{x'^{\mu}_1 x'^{\nu}_2}{|x'_1|^2|x'_2|^2}} - \cancel{\delta|x'_2|^2\frac{x'^{\mu}_1 x'^{\nu}_2}{|x'_1|^2|x'_2|^2}} + \delta\frac{x'^{\mu}_{12} x'^{\nu}_{12}}{x'^2_{12}}
\end{aligned}
$$

which implies $\delta = -2$. Hence, the two point function is given as

$$
\langle J_{\mu}(x) J_{\nu}(0)\rangle = \frac{C}{|x|^{2\Delta}}\left[g_{\mu\nu} - 2\frac{x_{\mu}x_{\nu}}{x^2}\right]
$$

The embedding space formalism gives the same answer[7]: Considering a tensor field of $SO(d+1,1)$ denoted as $O_{A_1\dots A_n}(X)$, with the properties

[6] the conformal factor is there following eqn 55 of TASI Lectures on the Conformal Bootstrap. The tensor operator under inversion transforms as mentioned in eqn 3.18 of Conformal Field Theory with Boundaries and Defects or eqn 1.55 and 1.60 of EPFL Lectures on Conformal Field Theory in D¿= 3 Dimensions

$$
O'^{\mu}(x') = \left|\frac{\partial x^{\alpha}}{\partial x'^{\beta}}\right|^{\frac{\Delta+1}{d}} \frac{\partial x'^{\mu}}{\partial x^{\nu}} O^{\nu}(x') = \left|\frac{\partial x^{\alpha}}{\partial x'^{\beta}}\right|^{\Delta/d} I^{\mu}_{\nu}(x') O^{\nu}(x')
$$

$$
O'_{\mu}(x') = \left|\frac{\partial x^{\alpha}}{\partial x'^{\beta}}\right|^{\frac{\Delta-1}{d}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} O_{\nu}(x')
$$

[7] section 2.4 of “Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions” or section 5.2.2 of “Conformal Field Theory” by Loriano Bonora

- defined on the null-cone $X^2 = 0$,
- traceless and symmetric,
- homogeneous of degree $-\Delta$ in X, i.e., $O_{A_1 \ldots A_n}(\lambda X) = \lambda^{-\Delta} O_{A_1 \ldots A_n}(X)$,
- transverse $X^{A_i} O_{A_1 \ldots A_n}(X) = 0$, with $i = 1, \ldots, n$

It is clear that those are conditions rendering $O_{A_1 \ldots A_n}(X)$ manifestly invariant under $SO(d+1,1)$. In order to find the corresponding tensor in $\mathbb{R}^d$, one has to restrict $O_{A_1 \ldots . A_n}(X)$ to the Poincaré section and project the indices as

$$\langle O^\mu(x_1) O^\nu(x_2) \rangle = \frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_1^B}{\partial x_2^\nu} \langle O_A(X_1) O_B(X_2) \rangle$$

For example, the most general form of the two-point function of two operators with spin-1 and dimension Δ can be derived as:[8]

$$\left\langle O^A(X_1) O^B(X_2) \right\rangle = \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} \left[\eta^{AB} + \alpha \frac{X_2^A X_1^B}{X_1 \cdot X_2} + \beta \frac{X_1^A X_2^B}{X_1 \cdot X_2} \right]$$

We will drop the last term as it projects to zero anyways.

$$\left\langle O^A(X_1) O^B(X_2) \right\rangle = \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} \left[\eta^{AB} + \alpha \frac{X_2^A X_1^B}{X_1 \cdot X_2} \right]$$

According to the transverse condition

$$X_{A1} \left\langle O^A(X_1) O^B(X_2) \right\rangle = \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} [X_1^B + \alpha X_1^B] = 0 \implies \alpha = -1$$

we now use the projection to find the correlation function in $\mathbb{R}^d$:

$$\begin{aligned}
\langle O_\mu(x_1) O_\nu(x_2) \rangle &= \frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \langle O_A(X_1) O_B(X_2) \rangle \\
&= \frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} \left[\eta_{AB} - \frac{X_{A2} X_{B1}}{X_1 \cdot X_2} \right] \\
&= \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} \left[\frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \eta_{AB} - \frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \frac{X_{A2} X_{B1}}{X_1 \cdot X_2} \right] \\
&= \frac{C_{12}}{(x_1 - x_2)^{2\Delta}} \left[g_{\mu\nu} - 2 \frac{(x_1 - x_2)_\mu (x_1 - x_2)_\nu}{(x_1 - x_2)^2} \right]
\end{aligned}$$

where we used $X^A = (X^a, X^+, X^-) = (x^a, 1, x^2)$, $X_B = \left(x_a, -\frac{1}{2} x^2, -\frac{1}{2} \right)$ and $\eta_{ab} = I_{d \times d}$ with $\eta_{+-} = \eta_{-+} = -1/2$

$$\frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \eta_{AB} = g_{\mu\nu}$$

$$\begin{aligned}
\frac{\partial X_1^A}{\partial x_1^\mu} X_{A2} &= \eta_{ab} \frac{\partial x_1^a}{\partial x_1^\mu} x_2^b - \frac{1}{2} \cancelto{0}{\frac{\partial 1}{\partial x_1^\mu}} x_2^2 - \frac{1}{2} \frac{\partial x_1^2}{\partial x_1^\mu} 1 \\
&= \eta_{ab} \delta_\mu^a x_2^b - x_{1\mu} = (x_2 - x_1)_\mu = -(x_1 - x_2)_\mu
\end{aligned}$$

$$\frac{\partial X_2^B}{\partial x_2^\nu} X_{B1} = \eta_{ab} \frac{\partial x_2^a}{\partial x_2^\mu} x_1^b - \frac{1}{2} \cancelto{0}{\frac{\partial 1}{\partial x_2^\mu}} x_1^2 - \frac{1}{2} \frac{\partial x_2^2}{\partial x_2^\mu} 1 = (x_1 - x_2)_\nu$$

$$(X_1 - X_2)^A (X_1 - X_2)_A = (x_1 - x_2)^a (x_1 - x_2)_a - \frac{1}{2}(1-1)(x_1^2 - x_2^2) - (x_1^2 - x_2^2)(\tfrac{1}{2} - \tfrac{1}{2})$$

$$\implies X_1 \cdot X_2 = -\frac{1}{2}(x_1 - x_2)^2$$

[8] here the terms in bracket is chosen such that they are invariant under the replacement $x \to \lambda x$. We are not using the transformation law for any of them. Under which, even the metric will change to $\eta_{AB} \to \lambda^{-2} \eta_{AB}$.

Next, we boostrap three point correlator. [9] On the null cone we will have

$$\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle = \frac{W_M}{(-2X_1\cdot X_2)^{\alpha_{123}}(-2X_1\cdot X_3)^{\alpha_{132}}(-2X_2\cdot X_3)^{\alpha_{231}}}$$

where the powers α_{ijk} of the scalar factor are determined by the dilatation as in case of scalar operators and the tensor structure W_M equals to

$$W_M = \frac{(-2X_2\cdot X_3)X_{1M} - (-2X_1\cdot X_3)X_{2M} - (-2X_1\cdot X_2)X_{3M}}{(-2X_1\cdot X_2)^{\frac{1}{2}}(-2X_1\cdot X_3)^{\frac{1}{2}}(-2X_2\cdot X_3)^{\frac{1}{2}}}.$$

Let us comment a few things on the tensor structure. The relative sign is, as before, fixed by transversality.

$$\begin{aligned}(X_1)^M W_M &= 0\\ (X_2)^M W_M &= 0\\ (X_3)^M W_M &= 0\end{aligned}$$

We drop the term proportional to X_{3M}, since would project to zero anyway. The scaling behavior of correlation function under dilatation is completely determined in the scalar part so the tensor structure have scaling 0 in all variables ($X\to\lambda X \implies W_\mu \to \lambda^0 W_\mu$). Finally, it is immediate to check that the tensor structure is transverse, i.e. $(X_3)_M W_M = 0$. Projecting to physical space as:

$$\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle = \frac{\partial X_3^M}{\partial x_3^\mu}\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle$$

we find, as explicitly computed before,

$$\begin{aligned}\frac{\partial X_3^M}{\partial x_3^\mu}X_{iM} &= (x_i - x_3)_\mu, \quad i=1,2\\ -2X_i\cdot X_j &= (x_i-x_j)^2, \quad i=1,2,3\ (i<j),\end{aligned}$$

so that we end up with the tensor structure

$$W_\mu = \frac{|x_2-x_3|^2(x_1-x_3)_\mu - |x_1-x_3|^2(x_2-x_3)_\mu}{|x_1-x_2||x_1-x_3||x_2-x_3|} = \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}$$

Therefore, the three-point function of two scalars and one spin 1 operators in physical space is given by

$$\begin{aligned}\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle &= \frac{\frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}\\ &= \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2+1}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1+1}}\end{aligned}$$

The three-point function of higher-spin operators $J_{\mu_1\ldots\mu_\ell}$ is constructed from the above, analogously as what we did for the two-point functions, since it turns out that W_μ is the only indexed object for three points that is conformal invariant.

2.5 Fermions in Embedding Space

Following is taken from section 3.2 of Lectures on Conformal Field Theories by Hugh Osborn. To discuss spinor fields in the embedding formalism requires extending the usual $d-$dimensional gamma matrices to $d+2$ dimensions. For $d=2n$, we define[10]

$$\begin{aligned}a_0^\pm &= \frac{1}{2}(\pm\gamma^0+\gamma^1)\\ a_1^\pm &= \frac{1}{2}(\gamma^2 \pm i\gamma^3)\\ a_2^\pm &= \frac{1}{2}(\gamma^4 \pm i\gamma^5)\end{aligned}$$

[9] pg 30 of Masters Thesis on "Spinning Correlators at Finite Temperature" of Oscar Arandes Tejerina

[10] we have abused notation for the sake of avoiding cluttering of indices and $\pm$

$$\vdots$$

$$a^{\pm}_{\frac{d-2}{2}} = \frac{1}{2}(\gamma^{d-2} \pm i\gamma^{d-1})$$

where gamma matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

One can show that:

$$\begin{aligned} \{a_i^-, a_j^-\} &= \{a_i^+, a_j^+\} = 0 \\ \{a_i^-, a_j^+\} &= \delta_{ij} \qquad i, j = 0, 1, 2 \ldots {}^{d-2}/_2. \end{aligned} \tag{2.16}$$

In the literature ${}^{d-2}/_2$ is defined as another variable labeled by k, but for the sake for clarity we will keep it explicit. This is the algebra of raising and lowering operators for ${}^{d}/_2$ independent two-level systems. We ask how many basis vectors are there (including lowest weight state) which could be formed by operating ${}^{d}/_2$ raising a_i^+ on lowest weight state:[11]

$$\sum_{r=0}^{d/2} {}^{d/2}C_r = 2^{d/2}$$

It implies that in $d-$dimensions, we have $2^{d/2} \times 2^{d/2}$ dimensional matrix representation for $\gamma-$matrices. We will use the highest weight representation to determine $a_j^{\pm}$ and then use them to construct γ_μ. From (2.16), we quickly observe that

$$(a_i^-)^2 = 0 = (a_i^+)^2$$

It implies that we can only act a_i or $a_i^\dagger$ once on a state, the second time it acts the state is annihilation. We will build off our intuition from harmonic oscillator (fermionic) and assume that there is a lowest weight state $|\xi\rangle$ such that

$$a_i^- |\xi\rangle = 0 \quad \text{for all } i$$

Similarly, acting on it once by each $a_i^\dagger$ for all i, we can construct states in the representation. The states can be labeled $s = (s_0, s_1, ..., s_{\frac{d-2}{2}})$, where each of the $s_a = \pm\frac{1}{2}$:

$$|\xi^{(s)}\rangle = (a^+_{\frac{d-2}{2}})^{s_{\frac{d-2}{2}} + \frac{1}{2}} \ldots (a_0^+)^{s_0 + \frac{1}{2}} |\xi\rangle \tag{2.17}$$

The lowest weight state $|\xi\rangle$ corresponds to all $s_a = -\frac{1}{2}$. Taking the $|\xi^s\rangle$ as a basis, we derive the matrix elements of γ_μ from the definitions and the anti-commutation relation. Starting with $d = 2$, we have a single two-level system:

$$|\xi^{(\frac{1}{2})}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\xi^{(-\frac{1}{2})}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we can construct the raising and lowering operator connecting these two matrices as:

$$a_0^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a_0^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we find:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $d = 4$, we have 2 independent fermionic oscillator:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we construct, the following a_i^+ and a_i^- operators for $i = 0, 1$.

$$a_0^+ = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_0^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

[11] we would like to remind ourselves that number of linearly independent basis is defined as the dimension of space.

and

$$a_1^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_1^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

From (2.17) we see that[12]

$$a_0^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \qquad a_1^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \qquad a_0^+ a_1^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, we conclude that the gamma matrices are gives as:

$$\gamma^0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

The above choice of gamma matrices satisfy the clifford algebra, however the chosen basis is not familiar from QFT textbooks. Given a representation γ^μ in d dimensions, we can construct a representation Γ^μ in $d+2$ dimensions using the prescription,

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\gamma^\mu \otimes \sigma^3, \quad \mu = 0, \dots, d-3,$$

$$\Gamma^{d-2} = \mathbb{I} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^1, \quad \Gamma^{d-1} = \mathbb{I} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^2$$

where the σ^i obey

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

The 2×2 matrices that we add act on the index $s_{d-2/2}$, which newly appears in going from $d = 2k$ to $2k+2$ dimensions. In odd dimensions the first $d-1$ gamma matrices can be constructed as above, and $\Gamma_d = \pm \Gamma_1 \Gamma_2 \dots \Gamma_{d-1}$ completes the gamma matrix algebra. There are two independent representations of the gamma matrix algebra in odd dimensions, differing in the sign of Γ_d. These representations are exchanged by parity, and both representations appear in a parity-conserving theory.

We now move onto calculating the correlation function involving spinors in embedding space formalism. To define spinor fields on null cone in embedding space requires that the number of component in $\mathbb{R}^d$ is half the number of components in $\mathbb{R}^{d+1,1}$.

$$\psi(x) \to \Psi(X), \qquad \bar{\psi}(x) \to \bar{\Psi}(X)$$

which satisfies the following homogeneity condition:

$$\Psi(\lambda X) = \lambda^{-\Delta + \frac{1}{2}} \Psi(X), \qquad \bar{\Psi}(\lambda X) = \lambda^{-\Delta + \frac{1}{2}} \bar{\Psi}(X)$$

The degrees of freedom of $\Psi; \bar{\Psi}$ are reduced to those for $\psi; \bar{\psi}$ by imposing the transversality condition like before:

$$\bar{\Gamma}_A X^A \Psi(X) = 0 \qquad \bar{\Psi}(X) \Gamma_A X^A = 0$$

It introduces the gauge invariance and thus the degrees of freedom are now halved by imposing the equivalence relations

$$\Psi'(X) \sim \Psi' + \bar{\Gamma}_A X^A \zeta(X) \qquad \bar{\Psi}'(X) \sim \bar{\Psi}' + \bar{\zeta}(X) \Gamma_A X^A \tag{2.18}$$

for arbitrary spinor $\zeta(X); \bar{\zeta}(X)$ of appropriate homogeneity. From standard QFT, we are familiar that

$$V_A = \bar{\Psi} \Gamma_A \Psi'$$

[12] $a_0^\pm$ acts like raising and lowering operator in the same oscillator while $a_1^\pm$ changes the oscillator.

transforms like a vector and we also have

$$\Psi(X) = \Gamma_B X^B \Psi'(X),$$
$$\bar{\Psi}(X) = \bar{\Psi}'(X)\bar{\Gamma}_B X^B.$$

Now compute the contraction:

$$V_A = \bar{\Psi}(X)\Gamma_A \Psi'(X) = \left(\bar{\Psi}'(X)\bar{\Gamma}_B \overset{\text{B-th component of coordinate (number)}}{X^B}\right)\Gamma_A \Psi'(X)$$
$$= \bar{\Psi}'(X)\bar{\Gamma}_B X^B \Gamma_A \Psi'(X)$$

Use the Clifford algebra identity:

$$\bar{\Gamma}_A \Gamma_B = -\bar{\Gamma}_B \Gamma_A + 2\eta_{AB}$$
$$\Gamma_A \bar{\Gamma}_B = -\Gamma_B \bar{\Gamma}_A + 2\eta_{AB},$$

from (2.18), and above we rewrite:

$$V_A = \bar{\Psi}'\bar{\Gamma}_B X^B \Gamma_A \Psi' = -\bar{\Psi}'\bar{\Gamma}_A \Gamma_B X^B \Psi' + 2X_A \bar{\Psi}'\Psi'$$

The second term, $2X_A\bar{\Psi}'\Psi'$, is proportional to X_A and is hence pure gauge under the equivalence relation:

$$V_A \sim V_A + X_A f(X)$$

so it can be discarded in physical quantities. Therefore, we obtain:

$$\bar{\Psi}\Gamma_A \Psi' + X_A f(X) \sim -\bar{\Psi}'\bar{\Gamma}_A \Gamma_B X^B \Psi' + 2X_A \bar{\Psi}'\Psi'$$
$$\bar{\Psi}(X)\Gamma_A \Psi'(X) \sim -\bar{\Psi}'(X)\bar{\Gamma}_A \Psi(X)$$

Which also satisfies $X^A V_A = 0$ due to transversality

$$\bar{\Psi}\Gamma_A \Psi' \sim -\bar{\Psi}'\bar{\Gamma}_A \Psi$$

whereas,

$$\bar{\Psi}\Psi$$

transforms like a scalar. However, the above is only under (2.18) in odd dimensions. So it does not correspond to a scalar on the projective null cone in even dimensions.

Chapter 3

de Sitter

3.1 Introduction

de Sitter spaces are the simplest solution of Einstein Field Equation with non zero cosmological constant. Similar to the case where $\Lambda = 0$ corresponds to a unique constant curvature ($\mathcal{R} = 0$) spacetime, de Sitter spacetime corresponds to unique positive constant curvature geometry corresponding to $\Lambda > 0$. This de Sitter geometry also has its origin in embedding space and we will use this embedding to express the resulting induced metric in various coordinate systems. In this way we will, in particular, also recover the metrics encountered in the cosmological contexts. This will be a good exercise to show that the solution of the Friedmann equations in the cosmological constant dominated phase is unique (for a given cosmological constant Λ) and uniquely given by the maximally symmetric dS space. de Sitter space is embedded in $\mathbb{R}^{4,1}$ as:[1]

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = l^2$$

It describes a time-like hyperboloid with topology $\mathbb{R} \times S^3$, the S^3 arising from the slicing at fixed X^0,

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \underbrace{l^2 + (X^0)^2}_{\text{constant}} > 0 \tag{3.1}$$

The spacetime interval in $\mathbb{R}^{4,1}$ is given as:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2$$

We can clearly see that the isometry group for $\mathbb{R}^{(4,1)}$ is $SO(4,1)$. Thus, the induced metric on the de Sitter space will be invariant under the same. However the way these isometries of $\mathbb{R}^{4,1}$ acted on embedding space will be different from how they will act on de Sitter space $\mathbb{M}^{3,1}$. The list of isometries for de Sitter are given as:

- 3 Rotation + 3 Translation
$$x_i \to a_i + R_{ij}x_j$$
- 1 Dilatation
$$x_\mu \to \lambda x_\mu$$
- 3 Special Conformal Transformation ($\eta \to 0$ or $b^\mu = (0, \vec{b})$)
$$x_i \to \frac{x_i - b_i(-\eta^2 + \vec{x}^2)}{1 - 2\vec{b}\cdot\vec{x} + b^2(-\eta^2 + \vec{x}^2)}$$

We will derive them in next section. We can notice that de Sitter in 4D has a total of $5(5-1)/2 = 10$ isometries. The conformal symmetry of d-dimensional Euclidean space is closely related to the isometry groups of $(d+1)$-dimensional de Sitter space and $(d+2)$-dimensional Minkowski space.

Coordinatizing de Sitter

There are many ways to introduce coordinate system on de Sitter space. We will start by setting a cartesian like coordinate system in ambient space and then introduce $d-2$ dimensional hypersurfaces which will slice the de Sitter space into space+time. The family of curve orthogonal to this hypersurface will become our timelike

[1] we define it like that because it helps us realize the de Sitter isometries in linear fashion.

direction on de Sitter and points on this $d-2$ dimensional hypersurface will be spacelike. This will lead to natural introduction of inducing embedding space coordinate to label the points on de Sitter hypersurface. Let us first define n^a as normal vectors on $3-$sphere (S^3). We can parameterize (3.1) as following

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = l^2 + (X^0)^2$$
$$l^2 \underbrace{\sum_{a,b=1}^{4} \delta_{ab} n^a n^b}_{1} \cosh\left(\frac{\tau}{l}\right) = l^2 + l^2 \sinh\left(\frac{\tau}{l}\right)$$

i.e. we can introduce following global coordinates on de Sitter space as[2]:

$$\left.\begin{aligned} X^0 &= l^2 \sinh\left(\tfrac{\tau}{l}\right) \\ X^a &= l^2 n^a \cosh\left(\tfrac{\tau}{l}\right); \qquad a = 1,2,3,4 \end{aligned}\right\} \text{closed slicing}$$

we also define following notation:

$$\sum_{a,b=1}^{d-1} \delta_{ab} dn^a dn^b = d\Omega^2_{d-1}$$

For $\mathbb{S}^3$, the normal vector n^a looks like

$$\begin{aligned} n^1 &= \cos\chi \\ n^2 &= \sin\chi \cos\theta \\ n^3 &= \sin\chi \sin\theta \cos\phi \\ n^4 &= \sin\chi \sin\theta \sin\phi \end{aligned}$$

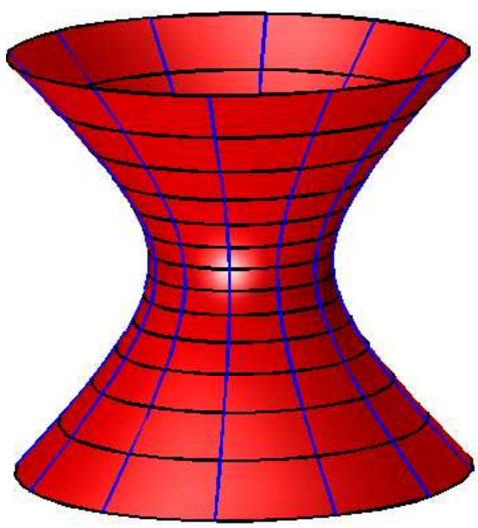
where

$$\chi \in [0,\pi], \quad \theta \in [0,\pi], \quad \phi \in [0,2\pi]$$

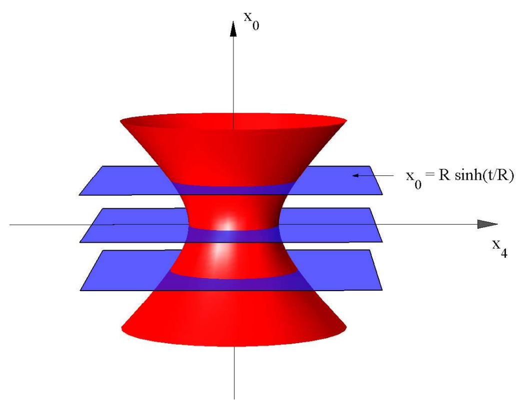
We quickly note that the above definition of coordinates gives rise to following induced metric on the hyperboloid.

$$ds^2 = -d\tau^2 + l^2 \cosh^2\left(\frac{\tau}{l}\right) d\Omega_3^2$$

where $d\Omega_3^2$ is metric on S^3. Constant time slices are then compact. For $\tau > 0$, this is the typical picture of a closed Universe whose size is expanding exponentially as time evolves forward. The minimal size of the sphere is at $\tau = 0$, where the radius of the sphere is one (in units of the dS radius). In these coordinates dS_4 looks like a $3-$sphere which starts out infinitely large at $\tau = -\infty$, then shrinks to a minimal finite size at $\tau = 0$, then grows again to infinite size as $\tau = \infty$. Note that this metric depends explicitly on the global time; dS does not have a **global** timelike Killing vector.



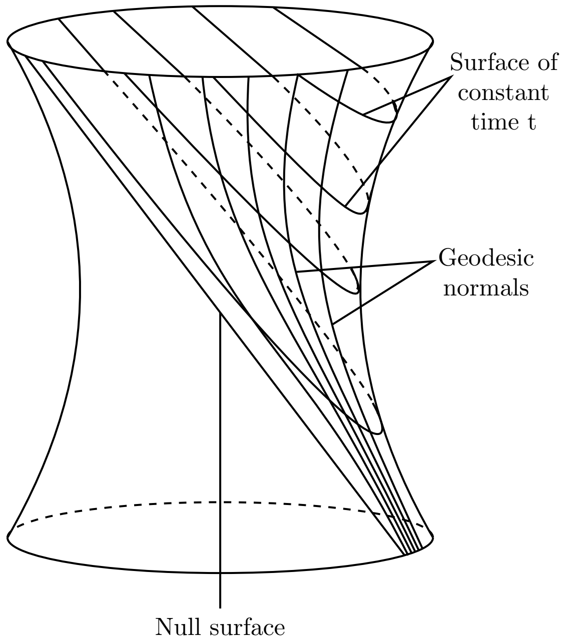
(a)



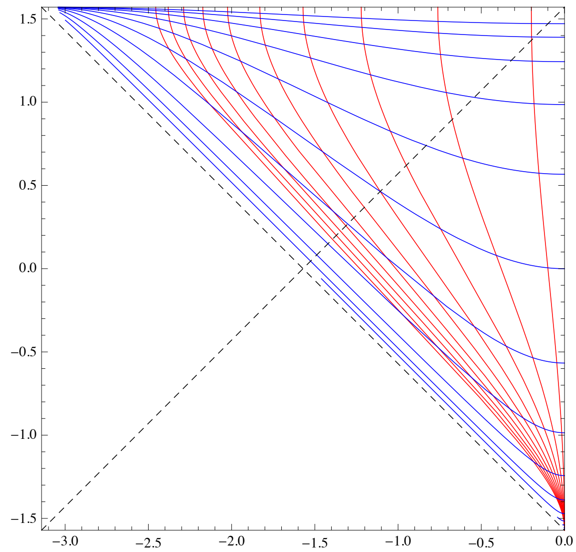
(b)

[2] chapter 4 of "Jerry B. Griffiths, Jirí Podolský - Exact Space-Times in Einstein's General Relativity"

3.1.1 Flat coordinates



(c)



(d)

Figure 3.1: (a) de Sitter geodesic. (b) Flat slicing of de Sitter, drawn on the Penrose diagram. Blue curves are constant-t flat slices, and red curves are the surfaces of constant-r. Intersections of the blue curves with the dashed line are the cross-sections of the cosmological horizon.

These are the coordinates $t, \mathbf{x}$, defined by

$$\begin{aligned} X^0 &= l \sinh\left(\frac{t}{l}\right) + \frac{1}{2l} x^2 e^{\frac{t}{l}}, \\ X^1 &= l \cosh\left(\frac{t}{l}\right) - \frac{1}{2l} x^2 e^{\frac{t}{l}}, \\ X^i &= x^i e^{\frac{t}{l}}, \quad i = 2, 3, 4. \end{aligned}$$

where $x^2 = \delta_{ab} x^a x^b$. These coordinates do not cover the full de Sitter space, but only the patch

$$X^0 + X^1 = l e^{t/l} > 0.$$

observable by observer at south pole ($\theta = \pi$) in penrose diagram. What it implies is that the space is coordinatized by a collection of plane in the ambient space with slope -1 and X^d intercept given by $He^{t/H} > 0$. The flat slicing covers only half the Penrose diagram, so this metric by itself is past-geodesically incomplete. Timelike worldlines, unless they are specially chosen to sit at the South pole, will exit the flat slicing in the past, in finite affine time. In these coordinates, the metric reads

$$ds^2 = -dt^2 + e^{2t/l} \sum_{i=1}^{3} dx_i^2.$$

This metric can be regarded as a FRW cosmology with an exponential function $a(t)$ by replacing $l = \frac{1}{H}$:

$$a(t) = e^{Ht}. \tag{3.2}$$

If we draw a diagonal plane through the embedding diagram, this is the 'upper triangle.' Similar coordinates can be chosen to cover just the 'lower triangle.'

3.1.2 Conformally flat coordinates

There is another coordinate which is conformal to flat Cartesian coordinates and covers more parts of de Sitter space than flat coordinates. Using familiar cartesian-like coordinates (η, x, y, z), the de Sitter hyperboloid is covered by:

$$\begin{aligned} x^0 &= \frac{l^2 + s}{2\eta} \\ x^1 &= \frac{l^2 - s}{2\eta} \\ x^2 &= l\frac{x}{\eta} \\ x^3 &= l\frac{y}{\eta} \\ x^4 &= l\frac{z}{\eta} \end{aligned}$$

where $s = -\eta^2 + x^2 + y^2 + z^2$ with $\eta, x, y, z \in (-\infty, \infty)$. Note that these coordinates still do not cover the full de Sitter space but only the patch

$$x^0 + x^1 \neq 0$$

In these coordinates, the de Sitter metric is

$$ds^2 = \frac{l^2}{\eta^2}(-d\eta^2 + d\vec{x}^2)$$

where η is usually referred to as conformal time. This is usually the preferred frame for the computation of cosmological correlators. In these coordinates the time η is not a Killing vector, and the only manifest symmetries are translations and rotations of the x^i coordinate.

3.1.3 Static coordinates

A very important aspect of dS space is that no single observer has access to the full spacetime. This is clear by just looking at the Penrose diagram. An important set of coordinates are those that describe the region accessible to a single observer. This is the intersection between the region of space that can affect the observer and the region that can be affected by them. In terms of embedding coordiantes, they are given by:

$$\begin{aligned} X^0 &= l\sqrt{1 - \frac{r^2}{l^2}}\sinh\left(\frac{t}{l}\right), \\ X^1 &= l\sqrt{1 - \frac{r^2}{l^2}}\cosh\left(\frac{t}{l}\right), \\ X^a &= rn^a, \quad a = 2,3,4 \end{aligned}$$

where

$$0 \leq r < l$$

They only cover the region

$$x^1 > 0, \quad r^2 < l^2.$$

These are accelerating observers in the ambient space:

$$-(X^0)^2 + (X^1)^2 + (X^a)^2 = (l^2 - \cancel{r^2}) + \cancel{r^2} = \frac{1}{\alpha^2}$$

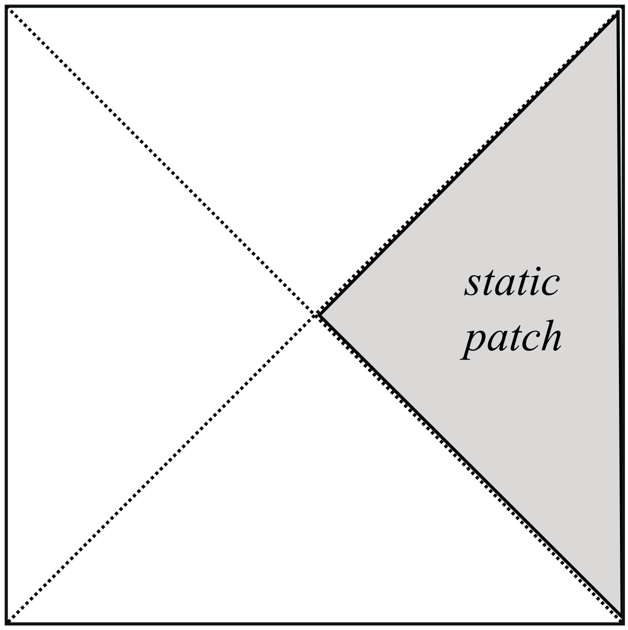


Figure 3.2: Static patch, on the Penrose diagram. This is the causal patch of an observer sitting at the north pole, i.e. $\theta = 0$ in global coordinates, i.e. $r = 0$ in static coordinates. The right edge of the diagram is $r_{\text{static}} = H$; the bifurcate Killing horizon is $r_{\text{static}} =$. The other three patches can also be covered by (independent) static coordinate systems, much like the four regions of the Penrose diagram for Schwarzschild black holes

In these coordinates, the metric reads

$$ds^2 = \left(1 - \frac{r^2}{l^2}\right) dt^2 - \frac{dr^2}{1 - \frac{r^2}{l^2}} - r^2 d\Omega_3^2.$$

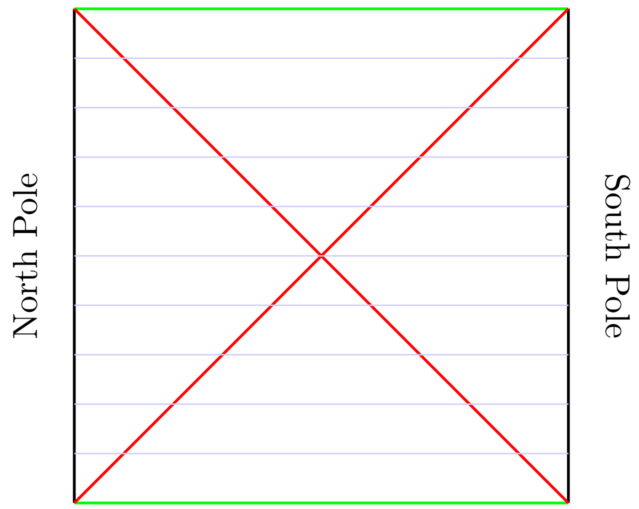
Notice the presence of an explicit horizon at $r = l$, which makes manifest the presence of event horizons for observers in de Sitter space.

3.1.4 Conformal Structure

The boundary of de Sitter space-time, given by $d\Omega = 0$, is located at $\eta = 0$ and $\eta = \pi$ which corresponds to past and future infinity. In contrast to Minkowski space, the infinities $\mathcal{I}^-$ and $\mathcal{I}^+$ now have a spacelike character. To look at the causal structure of dS space, it is convenient to define a conformal time T such that $\cos T = {}^1/_{\cosh(\frac{\tau}{H})}$. From here, it follows that $-\pi/2 \leq T \leq \pi/2$ and writing the metric on 3-sphere S^3 as $d\theta + \sin^2(\theta) d\Omega_2^2$ we see that the de Sitter is conformal to:

$$ds^2 = \frac{1}{\cos(T)}(-dT^2 + d\theta^2 + \cos^2(\theta) d\Omega_2^2)$$

where, θ spans from 0 to π. Suppressing the transverse 2-sphere and adding the points with $T = \pm\pi/2$ (future and past infinity), we end up with the simple Penrose diagram of de Sitter space in (T, θ) plane. Each horizontal line in the diagram (in blue) corresponds to a 3$-$sphere, whose radius is given by ${}^1/_{\cosh(T)}$.



Each point in each line corresponds to a 2−sphere with the exception of both vertical edges corresponding to $\theta = 0, \pi$ are not spheres but single points. We usually call those points the North and South pole of the sphere, and we like to think about inertial observers sitting at those points.

3.2 Killing Vectors

de Sitter space is maximally symmetric space with 10 killing vector associated with translation (3), rotation (3), dilatation (1) and SCT (3). There are two ways to find the killing vector

- Either find the coordianate transformation such that metric is independent of one or more coordinates.
- Solve the killing equation in some coordinate system

Since finding all the relevant coordiante transformation is very much dependent on luck, we choose to solve the killing equation in conformal coordinate. The metric for de Sitter space in conformal coordinates is given as:

$$ds^2 = \frac{1}{(H\eta)^2}(-d\eta^2 + d\vec{x}^2) = \frac{1}{(H\eta)^2}\eta_{\mu\nu}dx^\mu dx^\nu$$

The killing equation would be given as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

It is easier to first focus on the case where $\Gamma^\lambda_{\mu\nu} = 0$, so will we start with that.

3.2.1 Translation

The first thing to note is that metric is already independent of x^i, therefore it has 3 killing vector associated with translation. We can show that they are $\xi^\mu = \delta^\mu_i$, and

$$\xi_\mu = g_{\mu\nu}\xi^\nu = g_{\mu\nu}\delta^\nu_i = g_{\mu i}$$

From killing equation:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = \xi_{\mu,\nu} + \xi_{\nu,\mu} - 2\Gamma^\rho_{\mu\nu}\xi_\rho$$

Substituting $\xi_\rho = g_{\rho i}$, we get:

$$g_{\mu i,\nu} + g_{\nu i,\mu} - g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})g_{\rho i}$$

Now, using $g_{\rho i}g^{\rho\sigma} = \delta^\sigma_i$, the Killing condition simplifies to:

$$g_{\mu i,\nu} + g_{\nu i,\mu} - (g_{i\mu,\nu} + g_{i\nu,\mu} - g_{\mu\nu,i}) = 0$$

Hence,

$$\xi^x = (0,1,0,0) \qquad \xi^y = (0,0,1,0) \qquad \xi^z = (0,0,0,1)$$

or more compactly

$$\boxed{\xi = \xi^i \partial_i = \partial_i}$$

To generate translation from them, we just exponentiate the generators:

$$e^{a\partial_x} x = x + a\partial_x x + 0 = x + a$$

3.2.2 Rotation

Next, we solve for $\mu \neq \nu$ case where $\Gamma^\lambda_{\mu\nu} = 0$:

$$\begin{aligned}
\partial_x \xi_y + \partial_y \xi_x &= 0 \\
\partial_x \xi_z + \partial_z \xi_x &= 0 \\
\partial_y \xi_z + \partial_z \xi_y &= 0
\end{aligned}$$

Since $\partial_i g^{\mu\nu} = 0$, we can rewrite them as

$$\begin{aligned}
\partial_x \xi^y + \partial_y \xi^x &= 0 \\
\partial_x \xi^z + \partial_z \xi^x &= 0 \\
\partial_y \xi^z + \partial_z \xi^y &= 0
\end{aligned}$$

These are same as killing vector for flat space rotation.

$$\xi = \epsilon^{(i)jk} x_j \partial_k$$

we can just exponentiate them and find rotation. We will do it for $i = 3$ which describes the rotation about z-axis

$$\xi = (0, -y, x, 0) \equiv -y\partial_x + x\partial_y$$

The relevant transformation is:

$$\begin{aligned}
e^{\theta\xi} x &= \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n \left(-y\partial_x + x\partial_y\right)^n x \\
&= x + \theta\left(-y\partial_x + x\partial_y\right) x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 x + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 x \ldots \\
&= x - \theta y - \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right) y - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right) x \ldots \\
&= \left(1 - \frac{\theta^2}{2!} + \ldots\right) x - \left(\theta - \frac{\theta^3}{3!} + \ldots\right) y \\
&= \cos(\theta) x - \sin(\theta) y
\end{aligned}$$

Similarly,

$$\begin{aligned}
e^{\theta\xi} y &= \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n \left(-y\partial_x + x\partial_y\right)^n y \\
&= y + \theta\left(-y\partial_x + x\partial_y\right) y + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 y + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 y \ldots \\
&= y + \theta x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right) x - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right) y \ldots \\
&= \left(1 - \frac{\theta^2}{2!} + \ldots\right) y + \left(\theta - \frac{\theta^3}{3!} + \ldots\right) x \\
&= \cos(\theta) y + \sin(\theta) x
\end{aligned}$$

Represented in matrix form:

$$\begin{bmatrix} \eta' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta \\ x \\ y \\ z \end{bmatrix}$$

3.2.3 Dilatation

Now to proceed further we need to write down the non vanishing connection terms:

$$\Gamma^x_{\eta x} = -\frac{1}{\eta} \qquad \Gamma^y_{\eta y} = -\frac{1}{\eta}$$
$$\Gamma^z_{\eta z} = -\frac{1}{\eta} \qquad \Gamma^\eta_{xx} = -\frac{1}{\eta}$$
$$\Gamma^\eta_{yy} = -\frac{1}{\eta} \qquad \Gamma^\eta_{zz} = -\frac{1}{\eta}$$
$$\Gamma^\eta_{\eta\eta} = -\frac{1}{\eta}$$

We can write the killing equation as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$
$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\Gamma^\lambda_{\mu\nu}\xi_\lambda = 0$$

for $\mu = \nu$, we have following four expressions:

$$\partial_x \xi_x = -\frac{1}{\eta}\xi_\eta \qquad \partial_y \xi_y = -\frac{1}{\eta}\xi_\eta$$
$$\partial_z \xi_z = -\frac{1}{\eta}\xi_\eta \qquad \partial_\eta \xi_\eta = -\frac{1}{\eta}\xi_\eta$$

We can solve the last equation as:

$$\frac{d\xi_\eta}{\xi_\eta} = -\frac{d\eta}{\eta} \quad \Rightarrow \quad \ln \xi_\eta = -\ln \eta + \ln f(x,y,z)$$

which leads to

$$\boxed{\xi_\eta(\eta, x, y, z) = \frac{f(x,y,z)}{\eta}}$$

Now we determine $f(x,y,z)$:

$$\xi_\eta = \frac{f(x,y,z)}{\eta} \Rightarrow \partial_\eta \xi_\eta = -\frac{1}{\eta}\xi_\eta = -\frac{f(x,y,z)}{\eta^2}$$

So,

$$\partial_x \xi_x = -\frac{f(x,y,z)}{\eta^2}; \qquad \partial_y \xi_y = -\frac{f(x,y,z)}{\eta^2}; \qquad \partial_z \xi_z = -\frac{f(x,y,z)}{\eta^2}$$

For now let us focus on the specific case where,

$$f(x,y,z) = \text{const} = A \Rightarrow \xi_\eta = \frac{A}{\eta}$$

We will determine the specific form of $f(x,y,z)$ later. Then:

$$\partial_x \xi_x = -\frac{A}{\eta^2} \Rightarrow \xi_x = -\frac{A}{\eta^2}x + \phi_1(y,z,\eta)$$

$$\partial_y \xi_y = -\frac{A}{\eta^2} \Rightarrow \xi_y = -\frac{A}{\eta^2}y + \phi_2(x,z,\eta)$$

$$\partial_z \xi_z = -\frac{A}{\eta^2} \Rightarrow \xi_z = -\frac{A}{\eta^2}z + \phi_3(x,y,\eta)$$

For now lets set $\phi_i = 0$ which describes rotation and fix coefficient A such that in the component form:

$$\boxed{\xi^\mu = g^{\mu\nu}\xi_\nu = (\eta, x, y, z)}$$

is the killing vector associated with dilatation. With the basis, it looks like:

$$D = \xi^\mu \partial_\mu = x^\mu \partial_\mu$$

Exponentiating them

$$
\begin{aligned}
e^{\lambda D} x = e^{\lambda x\cdot\partial_x} x &= \lim_{N\to\infty}\left(1+\frac{\lambda}{N}x\cdot\partial\right)^N x \\
&= \lim_{N\to\infty}\underbrace{\left(1+\frac{\lambda}{N}x\cdot\partial\right)\ldots\left(1+\frac{\lambda}{N}x\cdot\partial\right)}_{N\text{ terms}} x \\
&= \lim_{N\to\infty}\underbrace{\left(1+\frac{\lambda}{N}x\cdot\partial\right)\ldots\left(1+\frac{\lambda}{N}x\cdot\partial\right)}_{N-1\text{ terms}}\left(x+\frac{\lambda}{N}x\right) \\
&= \lim_{N\to\infty}\left(1+\frac{\lambda}{N}\right)\underbrace{\left(1+\frac{\lambda}{N}x\cdot\partial\right)\ldots\left(1+\frac{\lambda}{N}x\cdot\partial\right)}_{N-1\text{ terms}} x \\
&= e^{\lambda} x
\end{aligned}
$$

or, more simply we can utilize $x\cdot\partial_x x = x$ and then

$$
\begin{aligned}
e^{\lambda D} x &= \sum_{ni=0}^{\infty}\frac{(\lambda x\cdot\partial_x)^n}{n!}x \\
&= 1+\frac{\lambda}{1!}x\cdot\partial_x x+\frac{\lambda^2}{2!}(x\cdot\partial_x)(x\cdot\partial_x x)+\ldots \\
&= \left(1+\lambda+\frac{\lambda^2}{2!}+\ldots\right)x = e^{\lambda}x
\end{aligned}
$$

3.3 Spatial special conformal transformation

The remaining killing equation which lead to SCT are as follows:

$$\partial_i\xi_\eta+\partial_\eta\xi_i+\frac{2}{\eta}\xi_i=0$$

Going back we found the solution to

$$\partial_\eta\,\xi_\eta \;=\; -\,\frac{1}{\eta}\,\xi_\eta,$$

as

$$\xi_\eta(\eta,x,y,z) \;=\; \frac{f(x,y,z)}{\eta} \quad \text{for some function } f(x,y,z).$$

This time we won't assume it to just be a constant. Therefore the killing equation to be solved becomes:

$$\frac{1}{\eta}\,\partial_i f(x,y,z) \;+\; \partial_\eta\,\xi_i \;+\; \frac{2}{\eta}\,\xi_i=0,$$

which is a first-order linear ordinary differential equation in η for each spatial component ξ_i. The homogeneous part $\partial_\eta\xi_i+(2/\eta)\xi_i=0$ integrates immediately to $\xi_i^{(\mathrm{hom})}(\eta)=C_i(x,y,z)/\eta^2$. A particular solution of the inhomogeneous equation can be found by considering power series in η, defined as $\xi_i=A(x,y,z)\eta^\alpha$

$$
\begin{aligned}
A\alpha\eta^{\alpha-1}+2A\eta^{\alpha-1} &= -\eta^{-1}\partial_i f(x,y,z) \\
(\alpha+2)A\eta^{\alpha-1} &= -\eta^{-1}\partial_i f(x,y,z) \implies \alpha=0, A=-\frac{1}{2}\partial_i f(x,y,z)
\end{aligned}
$$

Therefore, $\xi_i^{(\mathrm{part})}=-\frac{1}{2}\,\partial_i f(x,y,z)$. Hence the full solution is

$$
\begin{aligned}
\xi_\eta &= \frac{f(x,y,z)}{\eta} \\
\xi_i(\eta,x,y,z) &= \frac{C_i(x,y,z)}{\eta^2} \;-\; \frac{1}{2}\,\partial_i f(x,y,z)
\end{aligned}
$$

Meanwhile, from the $i\neq j$ Killing equations:

$$\partial_i\xi_j+\partial_j\xi_i=\frac{1}{\eta^2}(\partial_i C_j+\partial_j C_i)-\partial_i\partial_j f=0$$

we infer $\partial_i\partial_j f = 0$ and $\partial_i C_j + \partial_j C_i = 0$. From $\partial_i\partial_j f = 0$ one concludes f is at most linear in (x, y, z), so

$$f(x,y,z) = a_x\,x + a_y\,y + a_z\,z + A$$

with constant coefficients a_i, A. There are further constraints on C_i as:

$$\begin{aligned}\partial_i\xi_i &= \frac{1}{\eta^2}\partial_i C_i(x,y,z) - \frac{1}{2}\partial_i\partial_i f(x,y,z) = -\frac{f(x,y,z)}{\eta^2}\\ \partial_i C_i(x,y,z) &= -f(x,y,z)\end{aligned}$$

Since we have already studied dilatation, where we studied the effect of $A \neq 0$, therefore let us set A to zero,

$$C_i = -\int f(x,y,z)dx_i = -(a\cdot x)x_i + \frac{1}{2}a_i(x^i)^2 + g(x_j \forall j \neq i)$$

Then

$$\partial_i C_j + \partial_j C_i = 0 \implies C_i = -\left(\sum_{j=1}^{3} a_j x^j\right)x_i + \frac{1}{2}a_i\left(\sum_{j=1}^{3} x^j x^j\right)$$

Let choose $(a_x, a_y, a_z) = -\frac{2}{H^2}(b_x, b_y, b_z)$ so ξ_i appears without fraction. Then

$$\xi_\eta(\eta,x,y,z) = -\frac{2b_i x^i}{H^2\eta} \qquad \xi_i(\eta,x,y,z) \begin{aligned}[t] &= \frac{2(b\cdot x)x_i - b_i\vec{x}^2}{H^2\eta^2} - \frac{1}{2H^2}\partial_i\left(-2b_j x^j\right)\\ &= \frac{2(b\cdot x)x_i - b_i\vec{x}^2}{H^2\eta^2} + \frac{b_i}{H^2}\end{aligned}$$

Raising indices gives[3]

$$\xi^\eta = 2\,\eta\left(b_i x^i\right), \qquad \xi^i = (\eta^2 - \vec{x}^2)b^i + 2(b\cdot x)x^i$$

Since $x^\mu = (\eta, x^i)$ and $b^\mu = (0, b^i)$, this coincides exactly with

$$\xi^\mu = 2(b\cdot x)x^\mu - b^\mu\left(-\eta^2 + \vec{x}^2\right), \quad b^0 = 0$$

These are the generators of the isometry of de Sitter space not the conformal transformations of de Sitter space. We should be careful in regard to not interpret these killing vectors as conformal killing vectors. Since b^μ is spacelike with $b^0 = 0$ it does not generate SCT for $\mathbb{R}^4$ but for $\mathbb{R}^3$ on $\eta = 0$ hypersurface. We can see that these generators of de Sitter isometry has similar form as the expressions in section 1.12

- Translation
$$P_i = \partial_i$$
- Rotation
$$L_{ij} = x_i\partial_j - x_j\partial_i$$
- Dilatation
$$D = x^\mu\partial_\mu$$
- SCT
$$K^i = [2(b^j x_j)x^i - b^i(\eta_{kl}x^k x^l)]\partial_i$$

3.4 Ambient space

de Sitter space is constant curvature space which makes it possible to study them by embedding them in higher dimensional flat spacetime. The special relativity of higher dimension will apppear as general relativity on d e sitter space. The generators of $D+1$ dimensional lorentz transformations in such embedding space are given as:

$$J_{MN} = X_M\partial_N - X_N\partial_M$$

[3] indices of b_i and x_i are raised and lowered by $\eta_{\mu\nu}$

where $M, N = 0, 1, 2, \ldots D$. There are several ways to coordinatize the embedded de Sitter space, for this section we choose to work in flat slicing. The coordinate transformation between the two given as:

$$\left.\begin{aligned} X^0 &= \tfrac{\rho}{2(-\eta)}(1-s) \\ X^D &= \tfrac{\rho}{2(-\eta)}(1+s) \end{aligned}\right\} \implies \rho = -\eta(X^D + X^0)$$

$$X^i = \frac{\rho}{-\eta}x^i \implies x^i = \frac{-\eta}{\rho}X^i = \frac{X^i}{X^0 + X^D}$$

where

$$s = \eta^2 - \delta_{ij}x^i x^j$$

and $i = 1, 2, \ldots D-1$. We can note from below that $\rho =$constant hypersurface are hyperbolic in nature.

$$g_{AB}X^A X^B = -(X^0)^2 + (X^D)^2 + \delta_{ij}X^i X^j = \frac{\rho^2 s^2}{\eta^2} + \frac{\rho^2}{\eta^2}\delta_{ij}x^i x^j = \rho^2$$

Inverting above relation,

$$\begin{aligned} \rho &= \sqrt{g_{AB}X^A X^B} \implies \frac{\partial\rho}{\partial X^0} = -\frac{X^0}{\rho} = \frac{s-1}{2(-\eta)} \\ \eta &= -\frac{\rho}{X^0 + X^D} \implies \frac{\partial\eta}{\partial X^0} = \frac{X_0 X_D + X_D^2 + X^i X_i}{(X^D + X^0)^2\sqrt{g_{AB}X^A X^B}} = \frac{\frac{1+s}{2} + \delta_{ij}x^i x^j}{\rho} \\ x^i &= \frac{X^i}{X^0 + X^D} \end{aligned}$$

we also have

$$\begin{aligned} \frac{\partial}{\partial X^0} &= \frac{1}{2(-\eta)}(-1 + \eta^2 - \delta_{ij}x^i x^j)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(1 + \eta^2 + \delta_{ij}x^i x^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i \\ \frac{\partial}{\partial X^D} &= \frac{1}{2(-\eta)}(1 + \eta^2 - \delta_{ij}x^i x^j)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(-1 + \eta^2 + \delta_{ij}x^i x^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i \\ \frac{\partial}{\partial X^i} &= \frac{x_i}{-\eta}\frac{\partial}{\partial\rho} - \frac{x_i}{\rho}\frac{\partial}{\partial\eta} - \frac{\eta}{\rho}\partial_i \end{aligned}$$

In this coordinate

$$ds^2 = -(dX^0)^2 + (dX^D)^2 + \delta_{ij}dX^i dX^j = d\rho^2 + \rho^2\frac{(-d\eta^2 + \delta_{ij}dx^i dx^j)}{\eta^2}$$

Then

$$\begin{aligned} P_i = J_{Di} - J_{0i} &= X_D\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^D} - X_0\frac{\partial}{\partial X^i} + X_i\frac{\partial}{\partial X^0} \\ &= (X_D - X_0)\frac{\partial}{\partial X^i} - X_i\left(\frac{\partial}{\partial X^D} - \frac{\partial}{\partial X^0}\right) \\ &= (X^D + X^0)\frac{\partial}{\partial X^i} - X^i\left(\frac{\partial}{\partial X^D} - \frac{\partial}{\partial X^0}\right) \\ &= \frac{\rho}{-\eta}\left(\frac{x^i}{-\eta}\frac{\partial}{\partial\rho} - \frac{x^i}{\rho}\frac{\partial}{\partial\eta} - \frac{\eta}{\rho}\partial_i\right) - \frac{\rho}{-\eta}x^i\left(\frac{1}{-\eta}\frac{\partial}{\partial\rho} - \frac{1}{\rho}\frac{\partial}{\partial\eta}\right) \\ &= \partial_i \end{aligned}$$

since $i = 1, 2, \ldots D-1$, there are $D-1$ momenta and SCT generator.

$$\begin{aligned} K_i = J_{Di} + J_{0i} &= X_D\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^D} + X_0\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^0} \\ &= (X_D + X_0)\frac{\partial}{\partial X^i} - X_i\left(\frac{\partial}{\partial X^D} + \frac{\partial}{\partial X^0}\right) \\ &= (X^D - X^0)\frac{\partial}{\partial X^i} - X^i\left(\frac{\partial}{\partial X^D} + \frac{\partial}{\partial X^0}\right) \\ &= \frac{\rho s}{-\eta}\left(\cancel{\frac{x^i}{-\eta}\frac{\partial}{\partial\rho}} - \frac{x^i}{\rho}\frac{\partial}{\partial\eta} - \frac{\eta}{\rho}\partial_i\right) - \frac{\rho}{-\eta}x^i\left(\cancel{\frac{s}{-\eta}\frac{\partial}{\partial\rho}} + \frac{\eta^2 + \delta_{ij}x^i x^j}{\rho}\frac{\partial}{\partial\eta} + \frac{2\eta}{\rho}x^j\partial_j\right) \end{aligned}$$

$$
\begin{aligned}
&= \cancel{\frac{x^i s}{\eta}} \frac{\partial}{\partial \eta} + s\partial_i + \frac{2\eta^2 - \cancel{s}}{\eta} x_i \frac{\partial}{\partial \eta} + 2x_i(x^j\partial_j) \\
&= (\eta^2 - \delta_{jk}x^j x^k)\partial_i + 2x_i \eta \partial \eta + 2x_i(x^j\partial_j) \\
&= 2x_i(x\cdot\partial) - (-\eta^2 + \vec{x}^2)\partial_i
\end{aligned}
$$

$$
\begin{aligned}
D = J_{D0} &= X_D \frac{\partial}{\partial X^0} - X_0 \frac{\partial}{\partial X^D} = X_D \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^D} \\
&= \frac{\rho}{2(-\eta)}(1+s)\left[\frac{1}{2(-\eta)}(-1+s)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(1+\eta^2+\delta_{ij}x^i x^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i\right] \\
&\quad + \frac{\rho}{2(-\eta)}(1-s)\left[\frac{1}{2(-\eta)}(1+s)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(-1+\eta^2+\delta_{ij}x^i x^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i\right] \\
&= -\eta\frac{\partial}{\partial\eta} - x^i\partial_i
\end{aligned}
$$

These generators do not have any dependence on de Sitter radius l. Therefore, we could work in sterographic projection of de sitter coordinates onto minkowski space (sterographic projection as conformal transformation). We begin by recalling that de Sitter space can be realized as a hyperboloid embedded in $d+1$ dimensions:

$$-X_0^2 + X_1^2 + \cdots + X_{d-1}^2 + X_d^2 = l^2.$$

The normal to this surface could be given as:

$$n_A = \frac{\partial}{\partial x^A}\left(X^B X_B - l^2\right) = 2X_A$$

Therefore, any tensor field K which is tangent to this hypersurface satisfies the following transversality condition.

$$X^A K_A = 0$$

We define the streographic projection by the following coordinate transformation:

$$
\begin{aligned}
X^\mu &\equiv r^\mu = \Omega(x)x^\mu \\
X^4 &\equiv r^4 = l\Omega(x)\left(1 - \frac{x^2}{4l^2}\right) \\
\Omega(x) &= \frac{1}{1+\frac{x^2}{4l^2}}
\end{aligned}
$$

Here, r^a is the cartesian coordinate on de Sitter space and x^a is the coordinate on projected minkowski space. We can note that $r_A r^A = l^2$ is imposed on de Sitter coordinates but no such condition is imposed on $x^2 = \eta_{\mu\nu}x^\mu x^\nu$.Then we define the following object:[4]

$$
\begin{aligned}
K^\mu_A &= \frac{1}{\Omega(x)^2}\frac{\partial r_A}{\partial x_\mu} = \frac{\delta^\mu_A}{\Omega} - r_A\frac{\partial}{\partial x_\mu}\left(\frac{1}{\Omega(x)}\right) \\
&= \left(1 + \frac{x^2}{4l^2}\right)^2 \frac{\partial r_A}{\partial x_\mu}
\end{aligned}
$$

and we can derive their explicit form by using:

$$\frac{\partial r_\nu}{\partial x_\mu} = \Omega(x)\delta^\mu_\nu - x_\nu\Omega(x)^2\frac{x^\mu}{2l^2} \qquad\qquad \frac{\partial r_5}{\partial x^\mu} = -\Omega(x)^2\frac{x_\mu}{l}$$

which satisfies the transversality condition and as such can be used as projection tensor. However, we will derive their explicit form in somewhat lengthy but illuminating way which will reveal their true nature. These K^μ_A satisfy the conformal killing equation

$$\frac{\partial}{\partial x^\mu}K^A{}_\nu + \frac{\partial}{\partial x^\nu}K^A{}_\mu = \delta^A_\mu\partial_\nu\left(\frac{1}{\Omega}\right) - \delta^A_\nu\partial_\mu\left(\frac{1}{\Omega}\right) - x^A\partial_\mu\partial_\nu\left(\frac{1}{\Omega}\right)$$

[4]this insight was provided in A unified construction of Skyrme-type non-linear sigma models via the higher dimensional Landau models

$$+\delta^A_\nu \partial_\mu \left(\frac{1}{\Omega}\right) - \delta^A_\mu \partial_\nu \left(\frac{1}{\Omega}\right) - x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right)$$
$$= -2x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) = 2f(x) g_{\mu\nu}$$

and

$$\begin{aligned} f(x) g_{\mu\nu} &= -x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) \\ f(x) g^{\mu\nu} g_{\mu\nu} &= -x^A \partial_\mu \partial^\mu \left(\frac{1}{\Omega}\right) = \partial_\mu K^{A\mu} \\ f(x) &= \frac{\partial_\mu K^{A\mu}}{d} \end{aligned}$$

So, ${K^A}_\mu$ could be interpreted as killing vector as long as $\partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) \propto g_{\mu\nu}$. There is additional property that ${K^A}_\mu$ satisfies, namely transversality condition:[5]

$$r^A {K_A}^\mu \Big|_{r_A r^A = l^2} = 0 \implies x^\nu {K_\nu}^\mu + l\left(1 - \frac{x^2}{4l^2}\right) {K_4}^\mu = 0$$

Since we can write the general form of conformal killing vectors as following,

$$K^\mu_a = t^\mu_a + \epsilon_a x^\mu + \omega^{\mu\nu}_a x_\nu + \lambda^\mu_a x^2 - 2\lambda^\sigma_a x_\sigma x^\mu$$

inserting above in transverality condition leads to:

$$x^\nu (t^\mu_\nu + \epsilon_\nu x^\mu + \omega^{\mu\sigma}_\nu x_\sigma + \lambda^\mu_\nu x^2 - 2\lambda^\sigma_\nu x_\sigma x^\mu) + l\left(1 - \frac{x^\nu x_\nu}{4l^2}\right)(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) = 0$$

Or,

$$\begin{aligned} x^\nu t^\mu_\nu + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + \lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu) + l(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) \\ - \frac{x^\nu x_\nu}{4l}(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) = 0 \end{aligned}$$

It can be conveniently written as:

$$\begin{aligned} l t^\mu_4 + (x^\nu t^\mu_\nu + l\epsilon_4 x^\mu + l\omega^{\mu\nu}_4 x_\nu) + \left[-\frac{x^2}{4l} t^\mu_4 + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + l\lambda^\mu_4 x^2 - 2l\lambda^\sigma_4 x_\sigma x^\mu\right] \\ + \left[\lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu - \frac{1}{4l}\epsilon_4 x^2 x^\mu - \frac{1}{4l}\omega^{\mu\nu}_4 x^2 x_\nu\right] + \left[-\frac{x^4}{4l}\lambda^\mu_4 + \frac{2x^2 x_\sigma x^\mu}{4l}\lambda^\sigma_4\right] = 0 \end{aligned}$$

Let us set coefficient of x^ν to zero, order by order.

$$\begin{aligned} t^\mu_4 &= 0 \\ x^\nu t^\mu_\nu + l\epsilon_4 x^\mu + l\omega^{\mu\nu}_4 x_\nu &= 0 \\ -\frac{x^2}{4l} t^\mu_4 + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + l\lambda^\mu_4 x^2 - 2l\lambda^\sigma_4 x_\sigma x^\mu &= 0 \\ \lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu - \frac{1}{4l}\epsilon_4 x^2 x^\mu - \frac{1}{4l}\omega^{\mu\nu}_4 x^2 x_\nu &= 0 \\ -\frac{x^4}{4l}\lambda^\mu_4 + \frac{2x^2 x_\sigma x^\mu}{4l}\lambda^\sigma_4 &= 0 \end{aligned}$$

contracting the last four equations with x_μ:

$$t^\mu_\nu x_\mu x^\nu + l\epsilon_4 x^2 = 0$$

[5] following the procedure outlined in section 2 of Gauge Theories on de Sitter space and Killing Vectors

$$\begin{aligned}
\epsilon_\nu x^\nu x^2 + l\lambda_4^\mu x_\mu x^2 - 2l\lambda_4^\sigma x_\sigma x^2 &= 0 \\
\lambda_\nu^\mu x_\mu x^\nu x^2 - 2\lambda_\nu^\sigma x^\nu x_\sigma x^2 - \frac{1}{4l}\epsilon_4 x^4 &= 0 \\
x^4 \lambda_4^\mu &= 0
\end{aligned}$$

Hence,

$$\begin{aligned}
t_\nu^\mu x_\mu x^\nu + l\epsilon_4 x^2 &= 0 \\
\epsilon_\nu x^\nu x^2 &= 0 \\
-\lambda_\nu^\sigma x^\nu x_\sigma x^2 - \frac{1}{4l}\epsilon_4 x^4 &= 0
\end{aligned}$$

Since t generates translation, we have $t_\nu^\mu = \delta_\nu^\mu$:

$$\begin{aligned}
\epsilon_4 &= -\frac{1}{l} \\
\lambda_{\mu\nu} &= \frac{1}{4l^2} g_{\mu\nu}
\end{aligned}$$

Substituting them in Killing vector, we get:

$$\begin{aligned}
K_\nu^\mu &= \left(1 + \frac{x^2}{4l^2}\right)\delta_\nu^\mu - \frac{x_\nu x^\mu}{2l^2} \\
&= \delta_\nu^\mu + \frac{1}{4l^2}(x^2\delta_\nu^\mu - 2x_\nu x^\mu) \\
K_4^\mu &= -\frac{x^\mu}{l}
\end{aligned}$$

These satisfy few properties:

$$\begin{aligned}
K_A^\mu K^{A\nu} &= \left(1 + \frac{x^2}{4l^2}\right)^2 g^{\mu\nu} = \frac{1}{\Omega(x)^2} g^{\mu\nu} \\
K_B^\mu K_{C\mu} &= \left(1 + \frac{x^2}{4l^2}\right)^2 (\delta_{BC} - r_B r_C)
\end{aligned}$$

verify above part, there could be missing scaling on the last $r_B r_C$ term. It could be used to make projection operator like[6]

$$\begin{aligned}
A_\mu &= \frac{\partial r^B}{\partial x^\mu}\hat{A}_B = \Omega(x)^2 K^{B\mu}\hat{A}_B \\
K_C^\mu A_\mu &= \Omega(x)^2 K^{B\mu} K_C^\mu \hat{A}_B = (\delta_C^B - r^B r_C)\hat{A}_B \\
\hat{A}_B &= K_B^\mu A_\mu \qquad \text{(where we used } r^B\hat{A}_B = 0\text{)}
\end{aligned}$$

We can define the derivative on de Sitter space as:

$$\begin{aligned}
\hat{\partial}_A &= \frac{\partial}{\partial r^A} - \frac{r_A r_B}{l^2}\frac{\partial}{\partial r_B} \implies r^A\hat{\partial}_A = 0 \\
&\equiv K_A^\mu \frac{\partial}{\partial x^\mu}
\end{aligned}$$

using $r_A r^A = l^2$ and $\eta_{AB} = (-,+,+\dots)$

$$\begin{aligned}
J_{AB} &= r_A\frac{\partial}{\partial r^B} - r_B\frac{\partial}{\partial r^A} \\
&= r_A\left(\hat{\partial}_B + \frac{r_B r_C}{l^2}\frac{\partial}{\partial r_C}\right) - r_B\left(\hat{\partial}_A + \frac{r_A r_C}{l^2}\frac{\partial}{\partial r_C}\right) \\
&= (r_A\hat{\partial}_B - r_B\hat{\partial}_A) \\
&= (r_A K_B^\mu - r_B K_A^\mu)\partial_\mu
\end{aligned}$$

[6] hat tensors are on de Sitter and without hat ones are on Minkowski

in a more compact form:

$$r^A J_{AB} = l^2 K_B^\mu \partial_\mu$$

Then, for $A, B \neq 4$

$$\begin{aligned} L_{ab} &= \Omega(x) \left[x_a \left(1 + \frac{x^2}{4l^2} \right) \delta_b^\mu - \frac{x_a x_b x^\mu}{4l^2} - x_b \left(1 + \frac{x^2}{4l^2} \right) \delta_a^\mu + \frac{x_a x_b x^\mu}{4l^2} \right] \partial_\mu \\ &= (x_a \delta_b^\mu - x_b \delta_a^\mu) \partial_\mu \end{aligned}$$

These are spatial rotation getting mapped to spatial rotation in Minkowski space. Boosts in the ambient space could be given as

$$K_i = J_{0i}$$

Now we calculate the de Sitter Momenta

$$\begin{aligned} lP_\nu = J_{4\nu} &= \Omega(x) \left[l \left(1 - \frac{x^2}{4l^2} \right) \left\{ \left(1 + \frac{x^2}{4l^2} \right) \delta_\nu^\mu - \frac{x_\nu x^\mu}{2l^2} \right\} + \frac{x_\nu x^\mu}{l} \right] \partial_\mu \\ &= \Omega(x) \left[l \left(1 - \frac{x^2}{4l^2} \right) \left(1 + \frac{x^2}{4l^2} \right) \delta_\nu^\mu - \frac{x_\nu x^\mu}{2l} + \frac{x^2}{4l^2} \frac{x_\nu x^\mu}{2l} + \frac{x_\nu x^\mu}{l} \right] \partial_\mu \\ &= \Omega(x) \left[l \left(1 - \frac{x^2}{4l^2} \right) \left(1 + \frac{x^2}{4l^2} \right) \delta_\nu^\mu + \frac{x^2}{4l^2} \frac{x_\nu x^\mu}{2l} + \frac{x_\nu x^\mu}{2l} \right] \partial_\mu \\ &= \left[l \left(1 - \frac{x^2}{4l^2} \right) \delta_\nu^\mu + \frac{x_\nu x^\mu}{2l} \right] \partial_\mu \\ &= l \partial_\nu + \frac{1}{4l} \left[-x^2 \delta_\nu^\mu + 2 x_\nu x^\mu \right] \partial_\mu \end{aligned}$$

The zeroth component of this momenta (which used to be dilatation) becomes our new Hamiltonian:

$$H = \frac{J_{40}}{l}$$

Since under Wigner Innou contraction the rotations about these planes will appear as translation and they are generated by Momentum operator. We will often refer to these as de Sitter momenta and in the limit $l \to \infty$, they begin to commute and we recover Lorentz algebra.

$$\begin{aligned} [J_{\mu\nu}, J_{\lambda\rho}] &= \eta_{\mu\lambda} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\lambda} + \eta_{\nu\rho} J_{\mu\lambda} - \eta_{\nu\lambda} J_{\mu\rho} \\ [J_{\mu\nu}, P_\lambda] &= \frac{1}{l} [J_{\mu\nu}, J_{4\lambda}] = \eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu \\ [P_\mu, P_\nu] &= \frac{1}{c^2 l^2} [J_{4\mu}, J_{4\nu}] = \frac{1}{c^2 l^2} J_{\mu\nu} \end{aligned}$$

The scalar curvature of de Sitter space is given by:

$$R = \frac{d(d-1)}{l^2} = \frac{2d}{d-2} \Lambda$$

3.5 Wigner Innou Contraction

We will first use the stereographic projection to see how the Newton Hook limit could be achieved.

$$\begin{aligned} P_\mu &= \frac{J_{4\mu}}{l} \\ L_{ab} &= J_{ab} = x_a \partial_b - x_b \partial_a \\ -K_a &= L_{a0} = \frac{J_{a0}}{c} \end{aligned}$$

where $a = 1, 2, 3$ and $\mu = 0, 1, 2, 3$ with mostly plus metric signature:

$$[J_{AB}, J_{CD}] = \eta_{AC} J_{BD} - \eta_{AD} J_{BC} + \eta_{BD} J_{AC} - \eta_{BC} J_{AD}$$

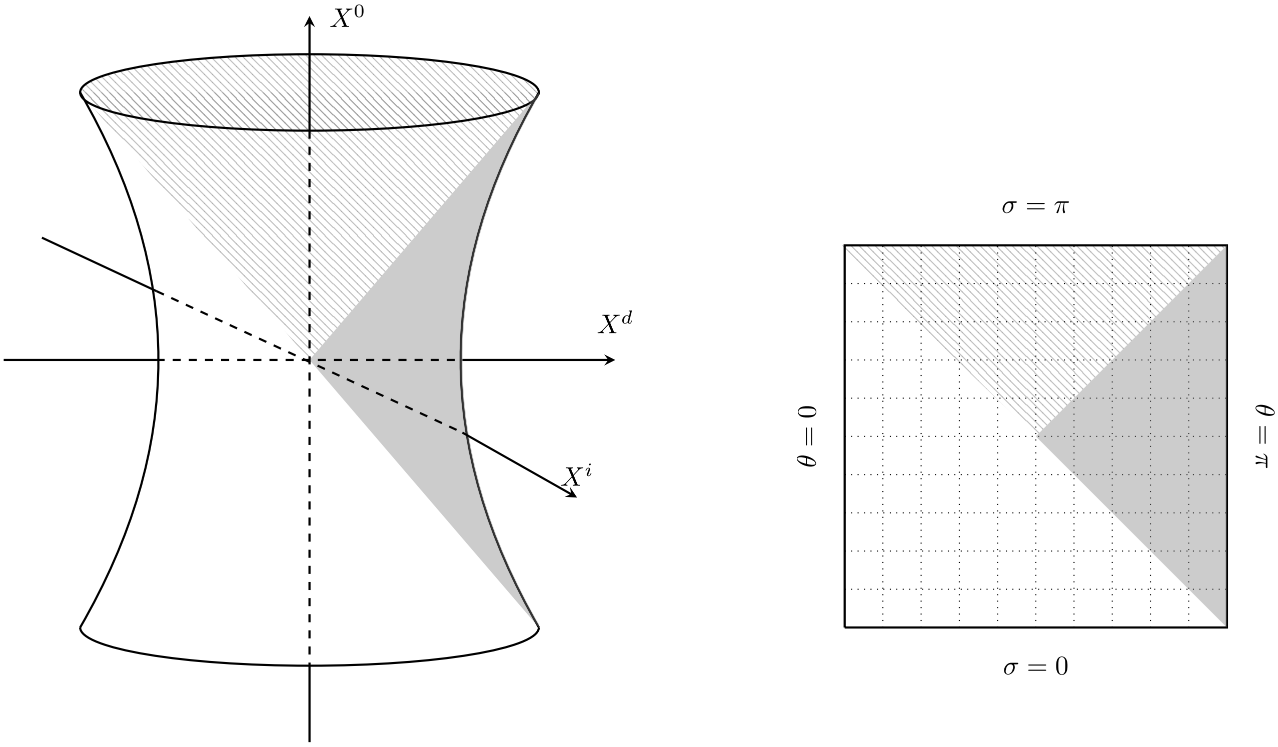


Figure 3.3: To take the $l \to \infty$ limit, imagine placing a plane at $x^d = l$. The plane will be coordinatized by x^μ and the rotation in x^0x^d plane as well as x^ix^d plane will appear as translation. That's why we split the generators of the ambient space into 6 lorentz generator + 4 translation. Taking the limit basically implies that we are confining ourselves to origin of the plane near the $x^d = l$ plane.

which leads to

$$
\begin{aligned}
[J_{\mu\nu}, J_{\lambda\rho}] &= \eta_{\mu\lambda}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\lambda} + \eta_{\nu\rho}J_{\mu\lambda} - \eta_{\nu\lambda}J_{\mu\rho} \\
[J_{\mu\nu}, P_\lambda] &= \frac{1}{l}[J_{\mu\nu}, J_{4\lambda}] = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu \\
[P_\mu, P_\nu] &= \frac{1}{c^2l^2}[J_{4\mu}, J_{4\nu}] = \frac{1}{c^2l^2}J_{\mu\nu}
\end{aligned}
$$

Then, the algebra becomes:[7]

$$
\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{ad}L_{be} + \delta_{be}L_{ad} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0} \\
[L_{0b}, L_{0e}] &= \frac{1}{c^2}[J_{0b}, J_{0e}] = \eta_{00}\frac{1}{c^2}J_{be}, \\
[L_{ab}, P_d] &= \frac{1}{l}[J_{ab}, J_{4d}] = \delta_{bd}\frac{J_{a4}}{l} - \delta_{ad}\frac{J_{b4}}{l} = \delta_{ad}P_b - \delta_{bd}P_a \\
[L_{a0}, P_b] &= \frac{1}{cl}[J_{a0}, J_{4b}] = -\frac{1}{cl}\delta_{ab}J_{04} = \frac{1}{c}\delta_{ab}P_0, \\
[L_{a0}, P_0] &= \frac{1}{cl}[J_{a0}, J_{40}] = \eta_{00}\frac{1}{cl}J_{a4} = \frac{1}{c}P_a, \\
[L_{ab}, P_0] &= \frac{1}{l}[J_{ab}, J_{40}] = 0, \\
[P_a, P_b] &= \frac{1}{l^2}[J_{4a}, J_{4b}] = \frac{\eta_{44}}{l^2}J_{ab} = \frac{1}{l^2}L_{ab}, \\
[P_a, P_0] &= \frac{1}{l^2}[J_{4a}, J_{40}] = \frac{\eta_{44}}{l^2}L_{a0} = -\frac{c}{l^2}K_a \\
[P_0, P_0] &= \frac{1}{l^2}[J_{40}, J_{40}] = 0.
\end{aligned}
$$

[7]This form of the analysis was suggested in section 2.4.2 of de Sitter Relativity: Foundation and some physical implications

We find that unless we scale P_0 by c. The resulting algebra will be ill defined as $[L_{a0}, P_0] \to 0$ and it will not match with Galilean group where $[L_{a0}, P_0] = P_a$.

$$\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{be} L_{ad} - \delta_{ad} L_{be} - \delta_{bd} L_{ae} - \delta_{ae} L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad} L_{b0} - \delta_{bd} L_{a0} \\
[L_{0b}, L_{0e}] &= -\frac{1}{c^2} L_{be}, \\
[L_{ab}, P_d] &= \delta_{ad} P_b - \delta_{bd} P_a \\
[L_{a0}, P_b] &= \frac{1}{c^2} \delta_{ab} P_0, \\
[L_{a0}, P_0] &= P_a, \\
[L_{ab}, P_0] &= 0, \\
[P_a, P_b] &= \frac{1}{\tau^2 c^2} L_{ab}, \\
[P_a, P_0] &= \frac{1}{\tau^2} L_{a0}, \\
[P_0, P_0] &= 0.
\end{aligned}$$

where $\tau = \frac{l}{c}$ is kept constant during the process of taking the limit $l \to \infty$ and $c \to \infty$:

$$\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{be} L_{ad} - \delta_{ad} L_{be} - \delta_{bd} L_{ae} - \delta_{ae} L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad} L_{b0} - \delta_{bd} L_{a0} \\
[L_{0b}, L_{0e}] &= 0 \\
[L_{ab}, P_d] &= \delta_{ad} P_b - \delta_{bd} P_a \\
[L_{a0}, P_b] &= 0, \\
[L_{a0}, P_0] &= P_a, \\
[L_{ab}, P_0] &= 0, \\
[P_a, P_b] &= 0 \\
[P_a, P_0] &= -\frac{1}{\tau^2} K_a, \\
[P_0, P_0] &= 0.
\end{aligned}$$

which is same as

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, P_j] = \epsilon_{ijk} P_k, && [J_i, K_j] = \epsilon_{ijk} K_k, \\
&[P_i, P_j] = 0, && [K_i, K_j] = 0, && [K_i, P_j] = 0, \\
&[H, J_i] = 0, && [H, P_i] = \frac{c^2 \Lambda}{3} K_i, && [H, K_i] = P_i,
\end{aligned}$$

for

$$J_i = \frac{1}{2} \epsilon_{ijk} J_{jk} \qquad\qquad P_0 = H$$

Alternatively, we can derive it by considering the lie algebra of the killing vector fields in the ambient space in the following form:

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, P_j] = \epsilon_{ijk} P_k, && [J_i, K_j] = \epsilon_{ijk} K_k, \\
&[P_i, P_j] = \epsilon_{ijk} J_k, && [K_i, K_j] = -\epsilon_{ijk} J_k, && [K_i, P_j] = -\delta_{ij} H, \\
&[H, J_i] = 0, && [H, P_i] = K_i, && [H, K_i] = P_i,
\end{aligned}$$

Under the Wigner-Innou contraction to be considered here, we only have to examine Lie brackets which doesn't involve J_i. Therefore, we only have to worry about the following 5 commutators:

$$\begin{aligned}
&[P_i, P_j] = \epsilon_{ijk} J_k, && [K_i, K_j] = -\epsilon_{ijk} J_k, && [K_i, P_j] = -\delta_{ij} H, \\
&[H, P_i] = K_i, && [H, K_i] = P_i,
\end{aligned}$$

We will consider the following rescaling as suggested in section 2 of covariant formulation of newton-hooke particle and its canonical analysis[8]. The ambient space boost is scaled down by c for taking non relativistic limit, H is scaled up by c for consistency and P_i is scaled down by l for small Λ limit:

$$\tilde{P}_i = \frac{P_i}{cl} = \frac{J_{4i}}{cl} \qquad \tilde{P}_0 \equiv \tilde{H} = \frac{H}{l} = \frac{J_{40}}{l} \qquad \tilde{K}_i = \frac{K_i}{c} = \frac{J_{0i}}{c}$$

Note that Hamiltonian is nothing but the Dilatation operator.

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, \tilde{P}_j] = \epsilon_{ijk} \tilde{P}_k, && [J_i, \tilde{K}_j] = \epsilon_{ijk} \tilde{K}_k,\\
&[\tilde{P}_i, \tilde{P}_j] = \frac{1}{c^2 l^2} \epsilon_{ijk} J_k && [\tilde{K}_i, \tilde{K}_j] = -\frac{1}{c^2} \epsilon_{ijk} J_k && [\tilde{K}_i, \tilde{P}_j] = -\frac{1}{c^2} \delta_{ij} \tilde{H}\\
&[\tilde{H}, J_i] = 0, && [\tilde{H}, \tilde{P}_i] = \frac{1}{l^2} K_i, && [\tilde{H}, \tilde{K}_i] = \tilde{P}_i,
\end{aligned}$$

Here the parameter l is defined as $l = \frac{c^2 \Lambda}{3}$. In the limit $c \to \infty$ and $\Lambda \to 0$ while keeping $c^2\Lambda$.

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k, && [J_i, P_j] = \epsilon_{ijk} P_k, && [J_i, K_j] = \epsilon_{ijk} K_k,\\
&[P_i, P_j] = 0, && [K_i, K_j] = 0, && [K_i, P_j] = 0,\\
&[H, J_i] = 0, && [H, P_i] = \frac{c^2 \Lambda}{3} K_i, && [H, K_i] = P_i,
\end{aligned}$$

It is often a good idea to consider the central extension of Newton Hooke' group by redefining $\tilde{H} = \frac{H}{l} - mc^2$ The only thing it changes is the

$$[\tilde{K}_i, \tilde{P}_j] = \lim_{c \to \infty} \frac{1}{c^2 l} [K_i, P_j] = -\lim_{c \to \infty} \frac{1}{c^2} \delta_{ij} \frac{H}{l} = -\lim_{c \to \infty} \left(m + \frac{\tilde{H}}{c^2} \right) \delta_{ij} = -m \delta_{ij}$$

where m appearing as central charge is the mass of the particle. One more way to find the Wigner Innou Contraction is by studying how the generators change under scaling. At $X_D = l$, we have

$$\begin{aligned}
J_{D\mu} &= \lim_{l \to \infty} \frac{1}{l} \left[X_D \frac{\partial}{\partial X^\mu} - X_\mu \frac{\partial}{\partial X^D} \right] = \lim_{l \to \infty} \frac{1}{l} \left[l \frac{\partial}{\partial X^\mu} - X_\mu \times 0 \right]\\
&= \frac{\partial}{\partial X^\mu}
\end{aligned}$$

and using (1.19)

$$\begin{aligned}
[P_\mu, J_{\alpha\beta}] &= [\partial_\mu, (x_\alpha \partial_\beta - x_\beta \partial_\alpha)]\\
&= [\partial_\mu, x_\alpha] \partial_\beta - [\partial_\mu, x_\beta] \partial_\alpha\\
&= -g_{\alpha\mu} \partial_\beta + g_{\beta_\mu} \partial_\alpha\\
&= g_{\beta_\mu} P_\alpha - g_{\alpha\mu} P_\beta
\end{aligned}$$

The Boost

$$\begin{aligned}
K_i &= \lim_{c \to \infty} \frac{1}{c} J_{0i}\\
&= \lim_{c \to \infty} \frac{1}{c} \left(X_0 \frac{\partial}{\partial X^i} - X^i \frac{\partial}{\partial X^0} \right)\\
&= \eta \partial_i
\end{aligned}$$

This is the form of representation that could be found by taking $l \to \infty$ and $X_\mu = 0, X_D = l$.

3.6 Ward Identity

Representation of the generator associated with central extension of Newton-Hooke algebra could be given as:[9]

$$J_i = \epsilon_{ijk} x_j \partial_k$$

[8]Boosts are scaled down by c for Galilean contraction.

[9]borrowed from equation 30 of Covariant Formulation of the Newton-Hooke Particle and its Canonical Analysis

$$\begin{aligned}
H &= \partial_\eta \\
P_i &= \left(\frac{m}{l}\right) x_i \sin\left(\frac{\eta}{l}\right) - \cos\left(\frac{\eta}{l}\right) \partial_i \\
K_i &= -m \left[x_i \cos\left(\frac{\eta}{l}\right) + \left(\frac{l}{m}\right) \sin\left(\frac{\eta}{l}\right) \partial_i \right]
\end{aligned}$$

In the limit $\eta \to 0$:

$$\begin{aligned}
H &= 0 \\
P_i &= -\partial_i \\
K_i &= -m x_i \\
J_i &= \epsilon_{ijk} x^k \partial_j = x^k \partial_j - x^j \partial_k
\end{aligned}$$

The relevant ward identities could be given as:

$$\begin{aligned}
\sum_{a=1}^{n} k_a^i \left\langle O(\vec{k}_1) \dots O(\vec{k}_n) \right\rangle &= 0 \\
\sum_{a=1}^{n} m_a \frac{\partial}{\partial k_a^i} \left\langle O(\vec{k}_1) \dots O(\vec{k}_n) \right\rangle &= 0 \\
\sum_{a=1}^{n} \left[k_a^j \frac{\partial}{\partial k_a^i} - k_a^j \frac{\partial}{\partial k_a^i} \right] \left\langle O(\vec{k}_1) \dots O(\vec{k}_n) \right\rangle &= 0
\end{aligned}$$

The momentum ward identity is trivial, as it enforces the delta function conservation i.e.

$$\left(\sum_{a=1}^{n} \vec{k}_a^i \right) \left\langle O(\vec{k}_1) \dots O(\vec{k}_n) \right\rangle = 0 \implies \left\langle O(\vec{k}_1) \dots O(\vec{k}_n) \right\rangle = (2\pi)^3 \delta(\textstyle\sum_{a=1}^{n} k_a^i) \mathcal{A}(\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n)$$

The last one enforces rotational invariance, implying $\mathcal{A}(\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n)$ depends only on the scalar product of $k_i \cdot k_j$ where i and j can be same.

3.7 Conformal Boundary

The Wigner Innou Contraction studied in preceding section was basically flattening out the de Sitter geometry to make it a cylinder of infinite radius. The conformal boundary already sits at $X^d = \infty$. So rather than taking the $\eta \to \infty$ at the end, we will now consider studying the conformal field theory on this boundary and try to see how one can find where this point lies in the stereographic projection. The goal is simple, find where the conformal boundary in projected minkowski plane lies, take the non relativistic limit. First, we ask **how the structure of conformal boundary depends on the de Sitter radius l.** Then we will take non relativistic limit.

Chapter 4

Ward Identities in de Sitter space

The infinitesimal conformal transformation which respects the de Sitter isometry will look like

$$\begin{aligned} \text{dilation:} &\quad \eta \to \eta(1+\lambda), & \mathbf{x} &\to \mathbf{x}(1+\lambda), \\ \text{SCT:} &\quad \eta \to \eta(1-2\mathbf{b}\cdot\mathbf{x}), & \mathbf{x} &\to \mathbf{x} - 2(\mathbf{b}\cdot\mathbf{x})\mathbf{x} + (\mathbf{x}^2-\eta^2)\mathbf{b}, \end{aligned}$$

We consider the following quantity:

$$\begin{aligned} e^{i\lambda D}\langle\ldots\rangle &= \langle\ldots\rangle \implies D\langle\ldots\rangle = 0 \\ e^{i\vec{b}\cdot\vec{K}}\langle\ldots\rangle &= \langle\ldots\rangle \implies \vec{b}\cdot\vec{K}\langle\ldots\rangle \equiv \underbrace{\mathbf{b}\cdot\mathbf{K}}_{\text{function, not an operator}}\langle\ldots\rangle = 0 \end{aligned}$$

Since, at late time ($\eta \to 0$) we will be decomposing our fields like:[1]

$$\Phi = \sum_{\{\Delta\}} \eta^\Delta O_\Delta(\vec{x})$$

where Δ is the scaling dimension of O_Δ. We deduce how the generators act on the boundary operator O_Δ by using the fact that scaling dimension of Φ is zero in de sitter space. Under $x \to x' \equiv \lambda x$, $\partial'_\mu = \lambda^{-1}\partial_\mu$ and $\phi'(x') = \lambda^{-\Delta}\phi(x)$:[2]

$$\begin{aligned} S &= \int d^4x' \frac{(\partial'_0\phi')^2 - (\nabla'\phi')^2}{(H\eta')^2} \\ &= \lambda^4 \int d^4x \frac{\lambda^{-2(1+\Delta)}[(\partial_0\phi)^2 - (\nabla\phi)^2]}{\lambda^2(H\eta)^2} \implies \Delta = 0 \end{aligned}$$

Let us now come back to dilatation and study how it acts on O:

$$\begin{aligned} D\Phi &= -x\cdot\partial\Phi \\ &= -x\cdot\partial\sum_{\{\Delta\}}\eta^\Delta O_\Delta \\ &= \sum_{\{\Delta\}}(-\eta\partial_\eta - \vec{x}\partial_{\vec{x}})\eta^\Delta O_\Delta \\ &= \sum_{\{\Delta\}}\eta^\Delta\underbrace{(-\Delta - \vec{x}\partial_{\vec{x}})}_{D}O_\Delta \\ &= D\sum_{\{\Delta\}}\eta^\Delta O_\Delta \\ b\cdot K\Phi &= b^\mu[-2x_\mu\eta\partial_\eta - 2x_\mu\vec{x}\cdot\partial_{\vec{x}} - (\eta^2-|\vec{x}|^2)\partial_\mu]\Phi \\ &= [-2(\vec{b}\cdot\vec{x})\eta\partial_\eta - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} - (\eta^2-|\vec{x}|^2)\vec{b}\cdot\partial_{\vec{x}}]\Phi \\ &= \sum_{\{\Delta\}}\underbrace{[-2(\vec{b}\cdot\vec{x})\Delta - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} - (\eta^2-|\vec{x}|^2)\vec{b}\cdot\partial_{\vec{x}}]}_{b\cdot K}\eta^\Delta O_\Delta \end{aligned}$$

[1] $\eta = \lim_{t\to\infty} -\frac{e^{-Ht}}{H} = 0$

[2] note that we used the weyl scaling to remove the λ^2 factor from metric under change of coordinate. So this basically amounts to substituting the coordinate transformation in the metric.

in the limit $\eta \to 0$

$$= [-2(\vec{b}\cdot\vec{x})\Delta - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} + |\vec{x}|^2\vec{b}\cdot\partial_{\vec{x}}]\Phi$$

In momentum space, the above operators take the following form:

$$\begin{aligned} D &: (3-\eta\partial_\eta) + k^i\partial_{k_i}, \\ \mathbf{b}\cdot\mathbf{K} &: (3-\eta\partial_\eta)\, 2b^i\partial_{k_i} - \mathbf{b}\cdot\mathbf{k}\partial_{k_i}\partial_{k_i} + 2k^i\partial_{k_i}b^j\partial_{k_j}. \end{aligned}$$

or, simply

$$\begin{aligned} D &: (3-\Delta) + k^i\partial_{k_i}, \\ \mathbf{b}\cdot\mathbf{K} &: (3-\Delta)\, 2b^i\partial_{k_i} - \mathbf{b}\cdot\mathbf{k}\partial_{k_i}\partial_{k_i} + 2k^i\partial_{k_i}b^j\partial_{k_j}. \end{aligned}$$

There are two ways to derive this expression, we will discuss both of them. The first is based on using Fourier Transform:

$$f(\vec{x}) = \int d^3x e^{i\vec{x}\cdot\vec{k}} f(\vec{k})$$

In our case

$$\begin{aligned} D\underbrace{\int d^3k e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k})}_{O_\Delta(\vec{x})} &= -\left(\Delta + x^j\frac{\partial}{\partial x^j}\right)\int d^3k e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k}) \\ &= \int d^3k\left[-\Delta - x^j\frac{\partial}{\partial x^j}\right]e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k}) \\ &= \int d^3k[-\Delta e^{i\vec{x}\cdot\vec{k}} - x^j e^{i\vec{x}\cdot\vec{k}}(ik_j)]O_\Delta(\vec{k}) \end{aligned}$$

we have to get rid of x^j, so we consider:

$$\begin{aligned} &= \int d^3k\left[-\Delta e^{i\vec{x}\cdot\vec{k}} - \left(-i\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right)ik_j\right]O_\Delta(\vec{k}) \\ &= \int d^3k\left[-\Delta e^{i\vec{x}\cdot\vec{k}} - \left(\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right)k_j\right]O_\Delta(\vec{k}) \end{aligned}$$

integrating by parts the second term

$$\begin{aligned} &= \int d^3k e^{i\vec{x}\cdot\vec{k}}\left[-\Delta + \left(\frac{\partial}{\partial k_j}\right)k_j\right]O_\Delta(\vec{p}) \\ &= \int d^3k e^{i\vec{x}\cdot\vec{k}}\left(-\Delta + \cancelto{3}{\frac{\partial k_j}{\partial k_j}} + k_j\frac{\partial}{\partial k_j}\right)O_\Delta(k)). \end{aligned}$$

Thus the action of the dilatation generator in momentum space is

$$DO_\Delta(\vec{k}) = \left(3 - \Delta + k^j\frac{\partial}{\partial k^j}\right)O_\Delta(\vec{k})$$

The second way is to replace the following in the expression in coordinate space (it can be seen operating the corresponding derivative operator on $e^{ix^\mu k_\mu}$ and the corresponding integration by parts).

$$\begin{aligned} x_\mu &\to -i\frac{\partial}{\partial k^\mu} \\ \frac{\partial}{\partial x_\mu} &\to -ik^\mu \end{aligned}$$

Substituting in

$$DO_\Delta(\vec{x}) = -\left(\Delta + x^j\frac{\partial}{\partial x^j}\right)O_\Delta(\vec{x})$$

we get

$$DO_\Delta(\vec{k}) = -\left[\Delta - i\frac{\partial}{\partial k^j}\left(-ik^j\right)\right]O_\Delta(\vec{k}) = -\left(\Delta - 3 - k^j\frac{\partial}{\partial k^j}\right)O_\Delta(\vec{k})$$

4.1 Conformal Ward Identity in momentum space

The following is taken from "Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex". We start by assuming that the generator of conformal transformation annihilates the correlation function.

$$\begin{aligned} D\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle &= 0 \\ e^{ix_n\cdot P}D\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle e^{-ix_n\cdot P} &= 0 \\ \underbrace{e^{ix_n\cdot P}De^{-ix_n\cdot P}}_{D+\vec{x}_n\cdot\partial_{\vec{x}_n}}\underbrace{e^{ix_n\cdot P}\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle e^{-ix_n\cdot P}}_{\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\rangle} &= 0 \end{aligned}$$

now, in Fourier space:

$$\begin{aligned} \langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\rangle &= \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n-1}ik_j\cdot(x_j-x_n)+0} \\ &\quad\times\delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\ &= \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n-1}ik_j\cdot x_j-ix_n\cdot(\sum_{j=1}^{n-1}k_j)} \\ &\quad\times\delta^d\left(k_n+\sum_j^{n-1}k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\ &\equiv \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n}ik_j\cdot x_j} \\ &\quad\times\delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \end{aligned}$$

From first and third equality, we observe that the replacement we need to make are:

$$\begin{aligned} (x_j-x_n)_\mu &\to -i\frac{\partial}{\partial k_j^\mu} \\ \frac{\partial}{\partial x_\mu} &\to -ik^\mu \end{aligned}$$

Thus,

$$\begin{aligned} \underbrace{(D+\vec{x}_n\cdot\partial_{\vec{x}_n})}_{-(\sum_{j=1}^n\Delta_j+x_j\cdot\partial_{x_j})+x_n\partial_{x_n}}\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\rangle &= 0 \\ -\left[\sum_j^n\Delta_j+\sum_j^{n-1}(x_j-x_n)\cdot\partial_{x_j}\right]\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\rangle &= 0 \\ \underbrace{-\left[\sum_j^n\Delta_j+\sum_j^{n-1}\left(-i\frac{\partial}{\partial k_j^\mu}\right)\cdot(-ik_j^\mu)\right]}_{-[\Delta-(n-1)d-\sum_{j=1}^{n-1}k_j\cdot\partial_{k_j}]}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle\delta^d\left(\sum_j^n k_j\right) &= 0 \end{aligned}$$

The alternate way to do the same is by explicitly doing it. We then will have to use

$$\int dx f(x)\partial_x\delta(x-a) = -\int dx\partial_x f(x)\delta(x-a) = -\partial_x f(a)$$

Consider the following integral:

$$\begin{aligned} I_{\alpha\beta} &= \int d^dk\left[\frac{\partial}{\partial k^\alpha}\delta^d(k^\mu)\right]k_\beta \\ &= -\int d^dk\delta^d(k^\mu)\underbrace{\frac{\partial k_\beta}{\partial k^\alpha}}_{g_{\alpha\beta}} = -g_{\alpha\beta} \end{aligned} \tag{4.1}$$

However, we also know that:

$$\begin{aligned}
\int d^dk\delta^d(k^\mu) &= 1 \\
\int d^dk\delta^d(k^\mu)\frac{k^2}{k^2} &= 1 \\
\int d^dk\delta^d(k^\mu)\frac{g^{\alpha\beta}k_\alpha k_\beta}{k^2} &= 1 \\
\int d^dk\frac{\delta^d(k^\mu)}{k^2}k_\alpha k_\beta &= \frac{1}{d}g_{\alpha\beta}
\end{aligned} \tag{4.2}$$

from (4.1) and (4.2), we get

$$\frac{\partial}{\partial k^\alpha}\delta^d(k^\mu) = -\frac{d}{k^2}k_\alpha\delta^d(k^\mu)$$

using the above, we derive:

$$k^\alpha\frac{\partial}{\partial k^\alpha}\delta^3(\vec{k}) = -\frac{3}{k^2}k^\alpha k_\alpha\delta^3(\vec{k}) = -3\delta^3(\vec{k})$$

Then, we have:

$$\begin{aligned}
&-\sum_{j=1}^{n}\left(\Delta_j + x_j^\alpha\frac{\partial}{\partial x_j^\alpha}\right)\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle \\
&\quad= -\sum_{j=1}^{3}\left(\Delta_j + x_j^\alpha\frac{\partial}{\partial x_j^\alpha}\right)\int\prod_{l=1}^{n}d^dk_l\delta^d(\textstyle\sum_{i=1}^n\vec{k}_i)e^{ik_lx_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\
&\quad= -\int\prod_{l=1}^{n}d^dk_l\delta^d(\textstyle\sum_{i=1}^n\vec{k}_i)e^{ik_lx_l}\left(\displaystyle\sum_{j=1}^{n}\Delta_j - \underbrace{\sum_{j=1}^{n}d}_{nd} - \sum_{j=1}^{n}k_j^\alpha\frac{\partial}{\partial k_j^\alpha}\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle + \delta'_{\text{term}}
\end{aligned}$$

where

$$\delta'_{\text{term}} = \sum_{j=1}^{n}\int\prod_{l=1}^{n}d^dk_l\left[k_j^\alpha\frac{\partial}{\partial k_j^\alpha}\delta^d(\textstyle\sum_{i=1}^n\vec{k}_i)\right]e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle$$

defining $P = \sum_{i=1}^n\vec{k}_i$

$$\begin{aligned}
&= \sum_{j=1}^{n}\int\prod_{l=1}^{n}d^dk_l\left[k_j^\alpha\frac{\partial}{\partial P^\alpha}\delta^d(P)\right]e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\
&= \int\prod_{l=1}^{n}d^dk_l\left[P^\alpha\frac{\partial}{\partial P^\alpha}\delta^d(P)\right]e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\
&= -d\int\prod_{l=1}^{n}d^dk_l\delta^d(\textstyle\sum_{i=1}^n\vec{k}_i)e^{ik_lx_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle
\end{aligned}$$

Thus, finally

$$\begin{aligned}
-\sum_{j=1}^{n}\left(x_j^\alpha\frac{\partial}{\partial x_j^\alpha} + \Delta_j\right)\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle &= -\int\prod_{l=1}^{n}d^dk_l\delta^d(\textstyle\sum_{i=1}^n k_i)e^{ik_lx_l} \\
&\left(\sum_{j=1}^{n}\Delta_j - (n-1)d - \sum_{j=1}^{n}k_j^\alpha\frac{\partial}{\partial k_j^\alpha}\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle
\end{aligned} \tag{4.3}$$

Before we proceed, there is another identity we'd like to derive which would be very helpful. [3]

$$
\begin{aligned}
\partial_\alpha\partial_\beta\delta^d(k^\mu) &= \partial_\alpha\left[-\frac{d}{k^2}\delta^d(k^\mu)k_\beta\right] \\
&= -d\left(\frac{\partial}{\partial k^\alpha}k^{-2}\right)\delta^d(k^\mu)k_\beta - \frac{d}{k^2}[\partial_\alpha\delta^d(k)]k_\beta - \frac{d}{k^2}\delta^d(k^\mu)\partial_\alpha k_\beta \\
&= \frac{2d}{k^3}\frac{k_\alpha}{k}\delta^d(k^\mu)k_\beta + \frac{d^2}{k^4}\delta^d(k^\mu)k_\alpha k_\beta - \frac{d}{k^2}\delta^d(k^\mu)g_{\alpha\beta} \\
&= \frac{d(d+2)}{k^4}\delta^d(k^\mu)k_\alpha k_\beta - \frac{d}{k^2}\delta^d(k^\mu)g_{\alpha\beta}
\end{aligned}
$$

We will now discuss the same for SCT. We perform the same substitution

$$
\begin{aligned}
-iK &= -2x_\mu\Delta - 2x_\mu\underbrace{\vec{x}\cdot\partial_{\vec{x}}}_{-i\frac{\partial}{\partial k^\alpha}(-ik^\alpha)} + |\vec{x}|^2\partial_\mu \\
&= 2i\Delta\frac{\partial}{\partial k^\mu} - 2i\frac{\partial^2}{\partial k^\mu\partial k^\alpha}k^\alpha + i\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}k_\mu \\
&= 2i\Delta\frac{\partial}{\partial k^\mu} - \underbrace{2i\frac{\partial}{\partial k^\mu}\left(d+k^\alpha\frac{\partial}{\partial k^\alpha}\right)}_{2i\left(d\frac{\partial}{\partial k^\mu}+\frac{\partial}{\partial k^\mu}+k^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha}\right)} + \underbrace{i\frac{\partial}{\partial k^\alpha}\left(\delta^\alpha_\mu + k_\mu\frac{\partial}{\partial k_\alpha}\right)}_{2i\frac{\partial}{\partial k^\mu}+ik_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}} \\
&= 2i\Delta\frac{\partial}{\partial k^\mu} - 2id\frac{\partial}{\partial k^\mu} - 2ik^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha} + ik_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha} \\
&= i\left[2(\Delta-d)\frac{\partial}{\partial k^\mu} - 2k^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha} + k_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}\right]
\end{aligned}
$$

Then,

$$
\begin{aligned}
K^\mu\delta^d(\textstyle\sum_{i=1}^n p_i^k) &= \sum_{j=1}^n\left[k_j^\mu\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\alpha}} - 2k_j^\alpha\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\mu}} + 2(\Delta_j-d)\frac{\partial}{\partial k_{j\mu}}\right]\delta^d(\underbrace{\textstyle\sum_{i=1}^n k_i^\mu}_{P^\mu=\sum_{i=1}^n k_i^\mu}) \\
&= \sum_{j=1}^n\left[k_j^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha} - 2k_j^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu} + 2(\Delta_j-d)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \left[\left(\sum_{j=1}^n k_j^\mu\right)\frac{\partial^2}{\partial P^\alpha\partial P_\alpha} - 2\left(\sum_{j=1}^n k_j^\alpha\right)\frac{\partial^2}{\partial P^\alpha\partial P_\mu} + 2\sum_{j=1}^n(\Delta_j-d)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \left[P^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha} - 2P^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu} + 2(\Delta-nd)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \frac{2d}{P^2}\delta^d(P)P^\mu - 2\frac{d^2+d}{P^2}\delta^d(P)P^\mu + \frac{2d(\Delta-nd)}{P^2}\delta^d(P)P^\mu \\
&= 2d\left[nd-d-\Delta\right]\frac{\delta^d(P)P^\mu}{P^2} \\
&= -2[(n-1)d-\Delta]\frac{\partial\delta^d(P)}{\partial P^\mu}
\end{aligned}
$$

In the fourth line, we used $\sum_{j=1}^n\Delta_j = \Delta$. Now, when K operates on the correlation function, it produces three kinds of terms,

- All operators in K acting purely on $\langle O_1(p_1)\ldots O_n(p_n)\rangle$
- All operators in K acting purely on $\delta^d(\sum_{i=1}^n p_i)$

[3] In the third line, we have used

$$
\begin{aligned}
\frac{\partial k}{\partial k^\alpha} &= \frac{\partial\sqrt{k^\alpha k_\alpha}}{\partial k^\alpha} \\
&= \frac{1}{2\sqrt{k^\alpha k_\alpha}}\left[\frac{\partial k^\alpha}{\partial k^\alpha}k_\alpha + k_\alpha\frac{\partial k^\alpha}{\partial k^\alpha}\right] = \frac{k^\alpha}{k}
\end{aligned}
$$

- Operators acting on both $\langle O_1(p_1)\dots O_n(p_n)\rangle$ and $\delta^d(\sum_{i=1}^n p_i)$

We will consider the action of

$$k_j^\mu \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\alpha}} - 2k_j^\alpha \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\mu}}$$

then, they will operate like:

$$2\frac{\partial \delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\alpha} \underbrace{\left[k_j^\mu \frac{\partial}{\partial k_{j\alpha}} - k_j^\alpha \frac{\partial}{\partial k_{j\mu}}\right]}_{iL_{\mu\alpha}} \Big\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\Big\rangle - 2\frac{\partial \delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\mu} k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \Big\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\Big\rangle$$

First part vanishes due to rotational invariance. Therefore the extra terms would be:

$$\delta'_{\text{terms}} = -2\frac{\partial \delta^d(P)}{\partial P}\left[(n-1)d - \Delta + \sum_{j=1}^n k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\right]\Big\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\Big\rangle$$

The above vanishes from (4.3). Therefore the SCT ward identity is given as:

$$K_\mu \langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle = -\left[2(\Delta - d)\frac{\partial}{\partial k^\mu} - 2k^\alpha \frac{\partial^2}{\partial k^\mu \partial k^\alpha} + k_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha}\right]\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle \tag{4.4}$$