Cosmology from Scratch

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February 6, 2026

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Chapter 1

FRW Cosmology

1.1 Einstein's Field Equation

The energy-momentum tensor for a perfect fluid can be represented as a covariant tensor as

$$T^{\mu\nu} = (P + \rho)u^{\mu}u^{\nu} - g^{\mu\nu}P$$

In a co-moving reference frame, u^{μ}, the four-velocity is $u^{\mu} = (1, 0, 0, 0)$ and

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \tag{1.1}$$

where ρ and P represent the energy density and pressure respectively.

1.1.1 RW Metric and Friedmann Solution

The Robertson-Walker (RW) metric for an isotropic and homogeneous spacetime can be represented in spherical coordinate (r, θ, ϕ) as

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

where $a(t)$ is the scale factor and k is the spatial curvature parameter of space

- For a non-static universe, the physical distance $d(t)$ between two point at time t can be written as

  $$d(t) = d(t_i)\frac{a(t_i)}{a(t)}$$

  where t_i is some earlier time. The covariant metric tensor of the above RW metric is

  $$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(1 - kr^2)^{-1} & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2\sin^2\theta \end{bmatrix} \tag{1.2}$$

- **Friedmann Equations**

  Using the metric tensor (1.2) and the energy-momentum tensor (1.1) in the Einstein's field equation, gives the following set of equations :

  $$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{1.3}$$

  $$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = -8\pi GP \tag{1.4}$$

  above expressions are known as 1^{st} and 2^{nd} Friedmann equation respectively

- **Energy-Momentum Conservation Equation**

  In an adiabatic system, $dQ = 0$. Now $E = \rho V$ and $V \sim a^3$. From the 1^{st} law of thermodynamics,

$$\begin{aligned} dQ = 0 &\Rightarrow dE + PdV = 0 \\ &\Rightarrow d(\rho a^3) + Pd(a^3) = 0 \\ &\Rightarrow 3a^2\dot{a}\rho + a^3\dot{\rho} + 3Pa^2\dot{a} = 0 \\ &\Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho P) = 0 \end{aligned} \tag{1.5}$$

  equation (1.3), equation (1.4) and equation (1.5) are related by Bianchi identity. Generally we take equation (1.3) and equation (1.5) as fundamental equations.

- **Derivation of equation (1.4) using equation(1.3) and equation(1.5)**

  Differentiating equation (1.3) with respect to time gives

$$\frac{2\dot{a}\ddot{a}}{a^2} - 2\frac{\dot{a}^3}{a^3} - 2\frac{k\dot{a}}{a^3} = \frac{8\pi G}{3}\dot{\rho} \tag{1.6}$$

  Replacing $\dot{\rho}$ by $-3\frac{\dot{a}}{a}(\rho + P)$ in (7) results in

$$\frac{\ddot{a}}{a} - \left[\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] = -4\pi G(\rho + P) \tag{1.7}$$

  Using equation (1.3) in equation (1.7) gives

$$\frac{\ddot{a}}{a} = -4\pi G(\rho + P) = -\frac{1}{2}\left(\frac{8\pi G}{3}\rho\right) - 4\pi Gp = -\frac{4\pi G}{3}(\rho + 3P) \tag{1.8}$$

  Again using equation (1.3) in equation (1.8) results in

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = -8\pi GP$$

  which is nothing but the 2^nd Friedmann equation. From equation (1.8), we can see that if $P < -\frac{1}{3}\rho$, one gets an accelerated universe.

1.1.2 Non-Static Models of the Universe

In this section we will consider the time evolution of $a(t)$ for a matter dominated universe with spatial curvature parameter $k = -1, 0, 1$. In a matter dominated universe, the energy density is dominated by that of non-relativistic particles and pressure $P = 0$. Friedmann equations for a matter dominated universe take the form

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\rho \tag{1.9}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = 0 \tag{1.10}$$

Substituting equation (1.9) in equation (1.10) gives

$$2\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\rho \tag{1.11}$$

We can write equation (1.9) as

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

or, equivalently

$$\frac{k}{a^2} = \frac{8\pi G}{3}\left[\rho - \frac{3H^2}{8\pi G}\right] = \frac{8\pi G}{3}[\rho - \rho_c]$$

where ρ_c, the critical density, defined as

$$\rho_c = \frac{3H^2}{8\pi G} \tag{1.12}$$

is the energy density for a flat universe. For an isolated system of homogeneous non-relativistic particles, ρ, the energy density is given by $\rho = mn$,where m is the mass of the particles and n is the number density of the particles. Since the volume varies as a^2, we observe that n varies as a^{-3} which implies $\rho(t) \sim a^{-3}(t)$. Another useful quantity, the deceleration parameter is defined as

$$q(t) = -\frac{\ddot{a}a(t)}{\dot{a}^2(t)} \tag{1.13}$$

Using equation (1.11) and equation (1.13), the decelerating parameter $q(t)$ can be written as

$$q(t) = -\frac{\ddot{a}a(t)}{\frac{\dot{a}^2}{a^2(t)}a^2(t)} = \frac{1}{H^2(t)}\frac{4\pi G\rho_0 a_0^3(t)}{3a^3(t)} = \frac{8\pi G\rho_0 a_0^3}{3H^2(t)a^3(t)} \tag{1.14}$$

For the present time, we have

$$q_0 = \frac{8\pi G\rho_0 a_0^3}{3H^2(t)a^3(t)} = \frac{\rho_0}{2\rho_c}$$

- **Case 1- Closed Universe**
  For a closed universe, we have $k = 1, \rho > \rho_c, q > \frac{1}{2}$. For $k = 1$ equation (1.3) gives

$$\begin{aligned}\frac{\dot{a}^2 + 1}{a^2} &= \frac{8\pi G\rho_0 a_0^3}{3a^3}\\ \dot{a}^2 + 1 &= \frac{8\pi G\rho_0 a_0^3}{3a}\\ \dot{a}^2 + 1 &= \frac{B}{a}\end{aligned} \tag{1.15}$$

  where we define

$$B = \frac{8\pi G\rho_0 a_0^3}{3}$$

  Now from equation (1.15),

$$\begin{aligned}\dot{a}^2 + 1 = \frac{B}{a} &\Rightarrow \frac{da}{dt} = \sqrt{\frac{B-a}{a}}\\ &\Rightarrow \int_0^t dt = \int_0^a \sqrt{\frac{a}{B-a}}da\end{aligned} \tag{1.16}$$

  Using an angular parameter η, we write

$$\begin{aligned}a = B\sin^2\frac{\eta}{2} &= \frac{B}{2}(1-\cos\eta)\\ \Rightarrow da &= B\sin\frac{\eta}{2}\cos\frac{\eta}{2}d\eta\end{aligned}$$

  equation (1.16) on simplification gives

$$a = \frac{B}{2}(1-\cos\eta)$$

  and

$$t = \frac{B}{2}(\eta - \sin\eta)$$

- **Case 2- Flat Universe**
  For a flat universe we have $k = 0,\ \rho = \rho_c,\ q = \frac{1}{2}$. Now from equation (1.3), with $k = 0$, we have

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0 a_0^3}{3a^3} \tag{1.17}$$

  From equation (1.12), we have

$$H^2 = \frac{8\pi G\rho_0}{3}$$

  So equation (1.17) becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0 a_0^3}{3a^3} = \frac{H^2 a_0^3}{a^3}$$

or

$$a\dot{a}^2 = H^2 a_0^3 \Rightarrow \sqrt{a}\dot{a} = (H^2 a_0^3)^{\frac{1}{2}} \tag{1.18}$$

Integrating equation (1.18) we get

$$a(t) = \left[\frac{3}{2}(H^2 a_0^3)^{\frac{1}{2}}\right]^{\frac{3}{2}} t^{\frac{3}{2}}$$

This gives

$$t = \frac{2}{4}(H^2 a_0^3)^{\frac{1}{2}} a^{\frac{3}{2}}(t)$$

So we have

$$t_0 = \frac{2}{3H_0}$$

For a universe dominated by relativistic particles, i.e., a radiation dominated universe, $a(t) \sim t^{\frac{1}{2}}$ and $t_0 = \frac{1}{2H_0}$

- **Case 3- Open Universe**
  For a open universe, we have $k = -1,\ \rho < \rho_c,\ q < \frac{1}{2}$. For $k = -1$ equation (1.3) gives

$$\dot{a}^2 - 1 = \frac{8\pi G \rho_0 a_0^3}{3a} = \frac{B}{a}$$
$$\Rightarrow \frac{da}{dt} = \sqrt{\frac{B+a}{a}}$$
$$\Rightarrow \int_0^t dt \sqrt{\frac{a}{B+a}} da \tag{1.19}$$

Introducing an angular parameter η , we write

$$a = B \sinh^2 \frac{\eta}{2} = \frac{B}{2}(1 - \cosh \eta)$$
$$\Rightarrow da = B \sinh \frac{\eta}{2} \cosh \frac{\eta}{2} d\eta$$

equation(1.19) on simplification gives

$$a = \frac{B}{2}(\cosh \eta - 1)$$

and

$$t = \frac{B}{2}(\sinh \eta - \eta)$$

1.2 Inflation

The theory of inflation tries to provide a solution to the above problems. It postulates the existence of a scalar field ϕ, the **inflaton** , with which was associated a potential $V(\phi)$. The energy density associated with ϕ, represented as ρ was the dominant form of energy density at some early time. Since the evolution of the universe at any time is determined by the dominant energy density, $\rho(\phi)$ determined the evolution of the universe as long as it remained dominant. We can write the energy density of ϕ as the contribution of both the kinetic energy density ρ_K and potential energy density ρ_P , i.e.,

$$\rho(\phi) = \overbrace{\rho_K}^{\text{Kinetic energy density}} + \underbrace{\rho_P}_{\text{Potential energy density}}$$

Initially, for some reason, the scalar field was not at the minimum of its potential. It was displaced. Then, because its potential was flat, it rolled slowly to the minimum of its potential, the state of its lowest energy

During the slow roll period, we can neglect the contribution of ρ_K and write

$$\rho(\phi) \approx \rho_P$$

Inflation says, for a short period of time, when ϕ was slowly rolling down, $\rho(\phi)$ was nearly constant. Now from the 1^{st} Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \overset{k=0}{\frac{8\pi G}{3}} \rho_\phi$$

Taking the RHS as a constant quantity, the above equation can be integrated to give

$$a(t) = a(t_i)e^{A(t-t_i)}$$

where t_i is the initial time of inflation, and $A^2 = \frac{8\pi G}{3}\rho_\phi$. This short period of exponential increase in the value of scale factor $R(t)$, is known as the period of **inflation**.

Condition for Inflation : This part is taken from "Physics of Inflation" by Daniel Baumann. The period of inflation is marked by the condition that

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0$$

However, there is another alternative condition which means exactly the same thing. The upside being, it give us much needed insights into the problem that inflation can solve. One such definition is made via *shrinking hubble sphere*.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a(t)H}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\dot{a}(t)^{-1} = -\frac{1}{[\dot{a}(t)]^2}\ddot{a}(t)$$

Since, $\ddot{R} > 0$ for inflation and $R^2 > 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(RH)^{-1} < 0$$

but

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a(t)H}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{a(t)^{-1}}{H} = \frac{-\frac{\dot{a}}{a^2}H - a^{-1}\dot{H}}{H^2}$$

$$-\frac{1}{a^2}\frac{\dot{a}}{H} - \frac{1}{a}\frac{\dot{H}}{H^2} < 0$$

$$-\frac{\dot{a}}{a}\frac{1}{H} - \frac{\dot{H}}{H^2} < 0$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2} < 1$$

The shrinking hubble sphere corresponds to the condition that during inflation, the Hubble scale as measured with respect to the expansion (so-called comoving coordinates) is shrinking because the expansion proceeds at a greater rate than the proper Hubble distance.

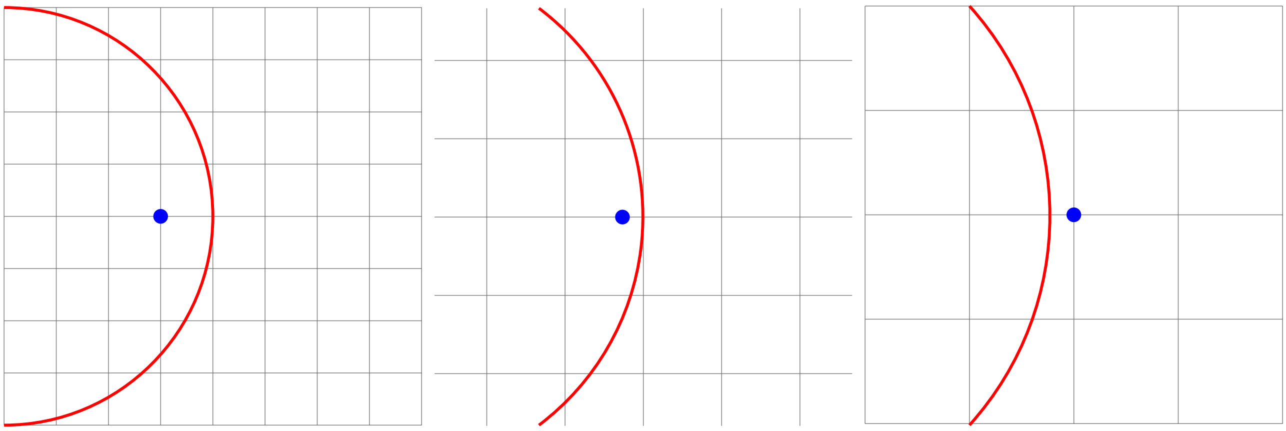


Figure 1.1: The red circle marks the comoving horizon, which grows more slowly than the background (grid) during inflation. An object at rest with respect to the background (comoving) eventually overtakes the horizon.

Using (1.28)[1]

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{1}{H}\frac{\mathrm{d}\ln H}{\mathrm{d}t} = -\frac{\mathrm{d}\ln H}{\mathrm{d}\ln a} < 1$$

The exact $\epsilon = 0$ case corresponds to having a **de Sitter background**, which is a vacuum solution of Einstein Field equation with positive cosmological constant. The other parameter corresponding to the inflation is:

$$\eta \equiv \frac{1}{H}\frac{\mathrm{d}\ln\epsilon}{\mathrm{d}t} = \frac{\mathrm{d}\ln\epsilon}{\mathrm{d}\ln a} = \frac{\dot{\epsilon}}{H\epsilon}$$

For $|\eta| < 1$, the fractional change of ϵ per Hubble time ($t = 1/H$) is small and inflation persists.

[1] also seen from equation 1.47, 1.49 and 1.50 of "Field Theory in Cosmology" by Enrico Pajer.

1.2.1 Causal Contact and the Horizon Problem

In the early universe, during the period of cosmic inflation, the concept of causal contact and the behavior of the comoving horizon are crucial to understanding the dynamics of the universe. The Horizon Problem, which is usually defined as the question of how the observable temperature inhomogeneties in CMB is of the order,

$$\frac{\delta T}{T} \sim 10^{-5}$$

requires the early universe to be in thermal equilibrium and thus demands causal contact.

Solution to Horizon Problem

Imagine at every point in space there was a light bulb, and they all illuminated only once. Initially, many of these light bulbs were outside our light cone because space was expanding. This expansion made it difficult for the light from these bulbs to reach us. However, as the eventually, light bulbs that were once outside our light cone began to come within our observable universe. When their light finally reaches us, it will appear as if these distant light bulbs at the edge of our vision have just illuminated.

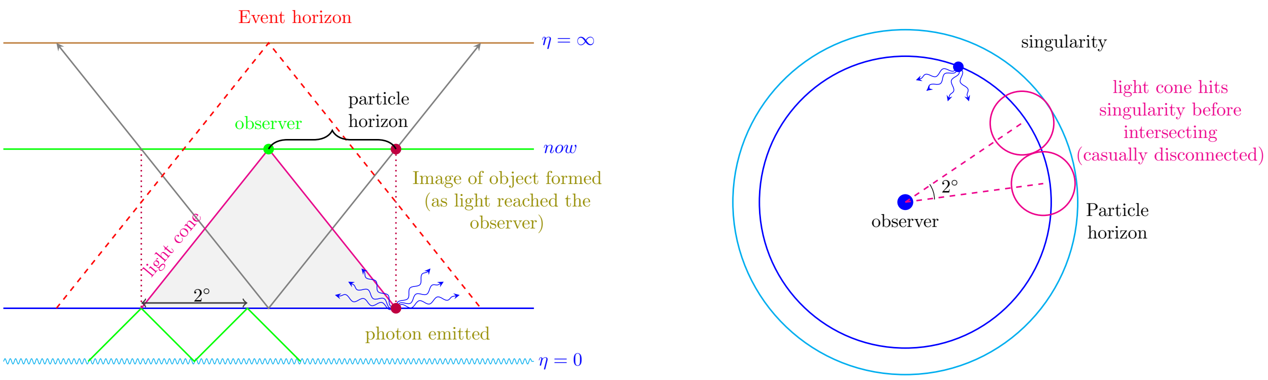


Figure 1.2: Spacetime diagram illustrating the concept of event horizon and particle horizon. Here $\eta = 0$ represents the Big Bang singularity without inflation. The event horizon is the light cone at future infinity, while the particle horizon is the maximum distance we can observe, marking the boundary of the observable universe. Points in the sky with an angular separation of 2° or more are causally disconnected.

Effectively, the farther we look, the older we look into the past. At the current time, the farthest object we see is the oldest object that just came inside our light cone. Thus, we define the largest comoving distance from which an observer at time t (now) is able to receive signals traveling at the speed of light as follows:

$$\begin{aligned} \chi_p = \eta - \eta_i &= \int_{t_i}^{t} \frac{dt}{a(t)} \\ &= \int_{a_i}^{a} \frac{da}{a\dot{a}} \\ &= \int_{\ln a_i}^{\ln a} \frac{1}{aH} d\ln a \end{aligned}$$

It defines the observable universe and has often been referred to as Particle Horizon or Comoving Horizon or Cosmological Horizon. Similar to comoving horizon, one can also define Hubble Horizon as

$$\begin{aligned} v &= xH \\ c &= R_H H \\ R_H &= \frac{c}{H} \end{aligned}$$

This is physical hubble radius, which can be expressed as:

$$R_H = a(t) R_H^{\text{comoving}} \implies R_H^{\text{comoving}} = \frac{c}{a(t)H}$$

For a universe, dominated by fluid:

$$\frac{1}{aH} = \frac{1}{H_0} a^{\frac{1}{2}(1+3w)}$$

Using the above, we find

$$\chi_p(a) = \frac{2H_0^{-1}}{(1+3w)}\left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)}\right] \equiv \eta - \eta_i.$$

The fact that the comoving horizon receives its largest contribution from $\eta \to \infty$ can be made manifest by defining

$$\eta_i \equiv \frac{2H_0^{-1}}{(1+3w)} a_i^{\frac{1}{2}(1+3w)} \xrightarrow{a_i \to 0, w > -\frac{1}{3}} 0.$$

as the time of Big Bang. The comoving horizon is finite,

$$\chi_p(t) = \eta - \eta_i = \frac{2H_0^{-1}}{(1+3w)} a(t)^{\frac{1}{2}(1+3w)} = \frac{2}{(1+3w)}(aH)^{-1}.$$

Thus, there is a particle horizon[2]. If we now, assume that the Strong Energy Conditions ($w > -1/3$) are violated during inflation. Then, the conformal time corresponding to big bang i.e. $a \to 0$ corresponds to:

$$\eta_i \to -\infty$$

and the comoving horizon distance diverges[3]. This implies that there was much more conformal time between the singularity and decoupling than we had thought"! In inflationary cosmology, $\eta = 0$ isn't the initial singularity, but instead becomes only a transition point between inflation and the standard Big Bang evolution. There is time both before and after $\eta = 0$.

1.2.2 Dynamics and Relevant Parameters

As has been first proposed by Guth and Linde, The theory of inflation solves the **Horizon**, **Flatness** and **Monopole** problem associated with the FRW cosmology. Additionally provides the seed for the **Large Scale Structure (LSS)** formation in the universe.

Inflation postulates the existence of a scalar field ϕ, the inflaton. Associated with inflaton was the potential $V(\phi)$. The essential condition for the inflation to occur is the **"Slow Roll"** . To extract some observable/detectable evidence, one needs a particular form of inflaton. Unfortunately, inflation says nothing about the form of $V(\phi)$. The potential $V(\phi)$ can be any suitable potential. This is the reason today there exist hundreds of models for inflation, each with a different form of $V(\phi)$. The good thing is, they have different predictions which, when analyzed with the experimental/observational data, helps us the discard/accept any model.

As we have learned, the essential condition for the inflation to occur is the slow roll of $V(\phi)$. Now onwards We will denote $V(\phi)$ as V . Mathematically, the **S**low **R**oll **A**pproximations (**SRA**) are

1. V was nearly constant
2. $\dot{V}$ and $\ddot{V}$ can be taken to be 0 or can be safely neglected in comparison to V
3. We can take gradient of the potential $V' = \frac{\partial V}{\partial \phi}$

We have the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{1.20}$$

The energy-momentum tensor from Noether's theorem can be written as

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu}\mathcal{L}$$

where the Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

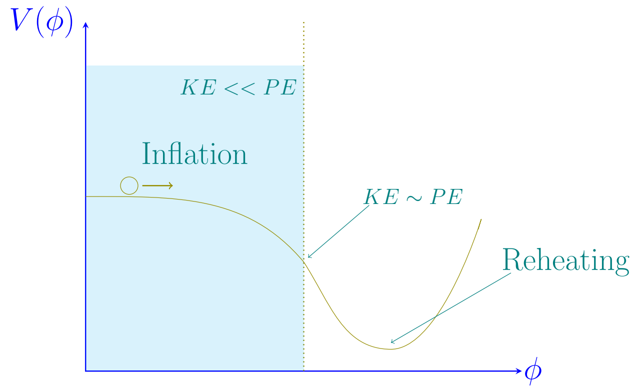


Figure 1.3: Inflation is kickstarted once inflaton begins to slowlly roll towards the minima.

For a perfect fluid

$$T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$$

[2]Weinberg's Cosmology pg 98
[3]Weinberg's Cosmology pg 99

This gives

$$\rho = T^{00} = \overbrace{\frac{1}{2}\dot{\phi}^2}^{\text{Kinetic energy density(to be neglected)}} + V(\phi) \tag{1.21}$$

and

$$p = \frac{T^{11} + T^{22} + T^{33}}{3} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.22}$$

From (1.8), we have

$$\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) \\
&= -\frac{4\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{3}{2}\dot{\phi}^2 - 3V(\phi)\right) \\
&= -\frac{8\pi G}{3}\left[\dot{\phi}^2 - V(\phi)\right] > 0
\end{aligned}$$

From above we can easily see that the inflation ends when $\dot{\phi} \approx V(\phi)$. Invoking the **SRA**, we can easily verify $\rho = -\rho$. Using equation(1.21) and equation (1.22) in equation (1.20), we get

$$\begin{aligned}
{}^{4}\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) + 3H\left(\frac{1}{2}\dot{\phi}^2 + \cancel{V(\phi)} + \frac{1}{2}\dot{\phi}^2 - \cancel{V(\phi)}\right) &= 0 \\
\ddot{\phi}\dot{\phi} + (\partial_\phi V)\dot{\phi} + 3H\dot{\phi}^2 &= 0 \\
\ddot{\phi} + 3H\dot{\phi} &= -\partial_\phi V
\end{aligned} \tag{1.23}$$

Invoking the **SRA** we can write, equation (1.23) as

$$3H\dot{\phi} = -\partial_\phi V \tag{1.24}$$

Slow Roll Parameters

The slow roll parameters $\epsilon(\phi)$ and $\eta(\phi)$ are defined as

$$\epsilon(\phi) = \frac{M_{\rm pl}^2}{16\pi}\left(\frac{\partial_\phi V}{V}\right)^2 \tag{1.25}$$

$$\eta(\phi) = \frac{M_{\rm pl}^2}{8\pi}\left(\frac{\partial_\phi^2 V}{V}\right) \tag{1.26}$$

where $M_{\rm pl}$ is the plank mass, also written in terms of reduced plank mass (m_p) as $m_{\rm p} = \frac{M_{\rm pl}}{\sqrt{8\pi}}$. These are not exactly the same as previous slow roll parameter but are related via[5]:

$$\begin{aligned}
\epsilon &\approx \epsilon(\phi) \\
\eta &\approx \eta(\phi) - \epsilon(\phi)
\end{aligned}$$

Inflation will occur as long as ϵ and η are < 1. Now, Friedmann equation can be written as

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right]$$

Using **SRA** we write,

$$H^2 = \frac{8\pi G}{3}V(\phi) \tag{1.27}$$

Slow roll parameter and Inflation

In this section we will derive the relation between the slow roll parameter and inflation. The Hubble parameter H is defined as

$$H = \frac{\dot{a}}{a} = \frac{\mathrm{d}\ln a}{\mathrm{d}t} \tag{1.28}$$

[5] eqn 29 and 30 of Slow-roll inflation and the Hamilton-Jacobi Formalism

This gives,

$$\dot{H} = \frac{\ddot{a}}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 \Rightarrow \dot{H} + H^2 = \frac{\ddot{a}}{a^2}$$

Since, $\ddot{R}$ & $R > 0$, we get $\dot{H} + H^2 > 0$, i.e., and $-\frac{\dot{H}}{H^2} < 1$. Substituting G by $\frac{1}{M_{\rm pl}^2}$ in equation(1.27) we get

$$H^2 = \frac{8\pi V(\phi)}{3M_{\rm pl}^2} \tag{1.29}$$

On differentiation (1.29) gives,

$$\begin{aligned} 2H\dot{H} &= \frac{8\pi}{3M_{\rm pl}^2}\frac{d}{dt}V \\ &= \frac{8\pi}{3M_{\rm pl}^2}\frac{dV}{d\phi}\frac{d\phi}{dt} \\ &= \frac{8\pi\dot{\phi}V'}{3M_{\rm pl}^2} \end{aligned}$$

defining $\frac{\partial V}{\partial \phi} = V'$. So we have

$$\dot{H} = \frac{8\pi\dot{\phi}V'}{6HM_{\rm pl}^2} \tag{1.30}$$

We have seen that, for inflation to occur we need

$$-\frac{\dot{H}}{H^2} < 1$$

Using equation (1.29) and equation (1.30) we get

$$-\frac{8\pi\dot{\phi}V'}{6HM_{\rm pl}^2 H^2} < 1$$

$$\Rightarrow -\frac{\dot{\phi}V'}{2HV} < 1$$

Also, substituting equation (1.24), we get

$$-\frac{\dot{H}}{H^2} = -\frac{\dot{\phi}V'}{2HV} = \frac{V'^2}{6H^2V} < 1$$

Using equation (1.29) in the above expression gives,

$$\frac{V'^2}{6V\left(\frac{8\pi}{3M_{\rm pl}^2}\right)V} < 1$$

$$\Rightarrow \frac{M_{\rm pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 < 1$$

which is essentially the 1^{st} slow roll parameter. Hence, we see that inflation will take place until ϵ reaches 1.

Ultra slow roll

The other slow roll parameter

$$\epsilon_2 = \frac{\dot{\epsilon}}{\epsilon H} = -6 - \frac{2V'}{H\dot{\phi}} + 2\epsilon = \frac{-6H\dot{\phi} - 2(-\ddot{\phi} - 3H\ddot{\phi})}{H\dot{\phi}} + 2\epsilon = \frac{2\ddot{\phi}}{H\dot{\phi}} + 2\epsilon$$

using we used (1.23). When $\partial_\phi V \approx 0$, we have

$$\epsilon_2 = -6 + 2\epsilon$$

and since $\epsilon \ll 1$, we have $|\epsilon_2| \approx 6$. Thus, if during inflation, the potential be comes suddenly very flat, $|\epsilon_2|$, which is initially small grows to larger than unity, SR is applicable no-more and a period of USR begins. Refer this paper for more.

Number of e-fold increase

During inflation the scale factor of the universe changes as

$$a(t) = a(t_0)e^{Ht}$$

where Ht is the number of e-fold increase in the scale factor which tells us that $a(t)$ evolved by the factor of e^{Ht} from its initial value. Therefore, the no. of e-fold increase is defined as:[6]

$$N = \int_{a(t_i)}^{a(t_f)} d\ln a = H\int_{t_i}^{t_f} dt = \ln\left(\frac{a(t_f)}{a(t_i)}\right) \implies a(t_f) = e^N a(t_i)$$

Now, dividing equation (1.24) by equation (1.29) gives

$$\frac{3\dot{\phi}}{H} = -\frac{3M_{\rm pl}^2}{8\pi}\left(\frac{V'}{V}\right) \Rightarrow \frac{\dot{\phi}}{H} = -\frac{M_{\rm pl}^2}{8\pi}\left(\frac{V'}{V}\right) \tag{1.31}$$

So we have

$$N = \int_{t_i}^{t_f} Hdt = \int_{t_i}^{t_f} H\frac{dt}{d\phi}d\phi = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}}d\phi \tag{1.32}$$

from equation(1.31) and equation(1.32),

$$N = \int_{\phi_i}^{\phi_f} -\frac{8\pi}{M_{\rm pl}^2}\left(\frac{V}{V'}\right)d\phi = \int_{\phi_f}^{\phi_i} \frac{8\pi}{M_{\rm pl}^2}\left(\frac{V}{V'}\right)d\phi$$

gives the number of e-fold increase in the size of the universe during the period of inflation in terms of the potential and its derivative.

1.2.3 How much Inflation is enough?

To solve the horizon problem, the largest scales observed today $\lambda(t_0) \approx {}^1/_{H_0}$ should be within the horizon at the beginning of inflation, i.e

$$\frac{1}{a_0H_0} < \frac{1}{a_iH_i}$$

which can be rewritten as

$$\frac{1}{H_0}\frac{a_f}{a_0}\frac{a_i}{a_f} < \frac{1}{H_i}$$

where ${}^{a_i}/_{a_f} = e^{-N}$. Since the temperature of photons drops with $T \sim {}^1/_a$, we can express ${}^{a_f}/_{a_0} = {}^{T_0}/_{T_f}$ through the CMB temperature today T_0 and the temperature after reheating T_f . Solving for N, we find

$$N > \ln({}^{T_0}/_{H_0}) + \ln({}^{H_i}/_{T_f}) \approx 67 + \ln({}^{H_i}/_{T_f})$$

The limits on N are always dependent on the energy scale of inflation, one typically assumes $H_i \approx 10^{15}$GeV. The reheating temperature depends typically on the decay rate of the inflaton, but is of similar order $T_f \sim 10^8 - 10^{12}$GeV, so the second term would be of order $\mathcal{O}(10)$.

1.3 Hamilton Jacobi Equation

Consider a system with configuration variable q and Hamiltonian $H(q,p,t)$. The Hamilton-Jacobi equation seeks a *principal function* $S(q,P,t)$ satisfying:

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{1.33}$$

The momenta are given by:

$$p = \frac{\partial S}{\partial q} \tag{1.34}$$

For time-independent Hamiltonians ($\partial H/\partial t = 0$), we can separate time:

$$S(q,P,t) = W(q,P) - Et \tag{1.35}$$

where E is constant energy, and W satisfies:

$$H\left(q, \frac{\partial W}{\partial q}\right) = E \tag{1.36}$$

[6]For any exponentially growing quantity $A(t)$, the natural quantity to consider for the growth is e-folding. It is defined as

$$A(t_f) = e^N A(t_i).$$

1 e-fold is the measure of time taken by the exponentially growing quantity to increase by a factor of e.

1.4 Cosmological Action for Scalar Field

Consider a homogeneous scalar field $\phi(t)$ in a flat FRW universe with metric:

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\vec{x}^2 \tag{1.37}$$

The total action in this case can be given as follows:

$$S = \frac{1}{16\pi G_N}\int \sqrt{-g}\, R\, d^4x - \int \sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right] d^4x \tag{1.38}$$

Under the parametrization of FRW metric, one can write:

$$S = \int dt\left[-\frac{3}{8\pi G_N}\frac{a\dot{a}^2}{N} + \frac{a^3\dot{\phi}^2}{2N} - Na^3V(\phi)\right] \tag{1.39}$$

Choosing cosmic time gauge $N = 1$:

$$S = \int dt\left[-\frac{3}{8\pi G_N}a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 - a^3V(\phi)\right] \tag{1.40}$$

The conjugate momenta of the system with canonical variable $q^1 = a(t)$ and $q^2 = \phi(t,x)$ can be deduced as:

$$\pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3}{4\pi G_N}a\dot{a} \tag{1.41}$$

$$\pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = a^3\dot{\phi} \tag{1.42}$$

This allows us to write the Hamiltonian:

$$H_{\text{total}} = \pi_a\dot{a} + \pi_\phi\dot{\phi} - L = -\frac{2\pi G_N}{3}\frac{\pi_a^2}{a} + \frac{\pi_\phi^2}{2a^3} + a^3V(\phi) \tag{1.43}$$

In GR, this is a constrained system. The Hamiltonian constraint (from variation with respect to N) gives:

$$\mathcal{H} \equiv -\frac{2\pi G_N}{3}\frac{\pi_a^2}{a} + \frac{\pi_\phi^2}{2a^3} + a^3V(\phi) = 0 \tag{1.44}$$

This is equivalent to the Friedmann equation. We seek a principal function $S(a,\phi)$ such that:

$$\pi_a = \frac{\partial S}{\partial a}, \quad \pi_\phi = \frac{\partial S}{\partial \phi} \tag{1.45}$$

The Hamilton-Jacobi equation is $\mathcal{H}(a,\phi,\partial S/\partial a,\partial S/\partial\phi) = 0$:

$$-\frac{2\pi G_N}{3a}\left(\frac{\partial S}{\partial a}\right)^2 + \frac{1}{2a^3}\left(\frac{\partial S}{\partial \phi}\right)^2 + a^3V(\phi) = 0 \tag{1.46}$$

$$-\frac{2\pi G_N}{3}a^2\left(\frac{\partial S}{\partial a}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial \phi}\right)^2 + a^6V(\phi) = 0 \tag{1.47}$$

Since the last term contains the product of a and ϕ. We can use the following as separation ansatz:

$$S(a,\phi) = -\frac{1}{4\pi G_N}a^3H(\phi) \tag{1.48}$$

where $H(\phi)$ is the Hubble parameter as a function of ϕ. This gives us

$$\frac{\partial S}{\partial a} = -\frac{3}{4\pi G_N}a^2H(\phi) \tag{1.49}$$

$$\frac{\partial S}{\partial \phi} = -\frac{1}{4\pi G_N}a^3H_{,\phi}(\phi) \tag{1.50}$$

Putting this back in (1.47), the first and second term can be simplified to:

$$-\frac{2\pi G_N}{3}a^2\left(-\frac{3}{4\pi G_N}a^2H\right)^2 = -\frac{2\pi G_N}{3}a^2\cdot\frac{9}{16\pi^2G_N^2}a^4H^2 = -\frac{3}{8\pi G_N}a^6H^2 \tag{1.51}$$

$$\frac{1}{2}\left(-\frac{1}{4\pi G_N}a^3 H_{,\phi}\right)^2 = \frac{1}{2}\cdot\frac{1}{16\pi^2 G_N^2}a^6(H_{,\phi})^2 = \frac{1}{32\pi^2 G_N^2}a^6(H_{,\phi})^2 \tag{1.52}$$

and hence the equation (1.47) becomes:

$$a^6\left[-\frac{3}{8\pi G_N}H^2 + \frac{1}{32\pi^2 G_N^2}(H_{,\phi})^2 + V\right] = 0 \tag{1.53}$$

Which could be rewritten as:

$$-12\pi G_N H^2 + (H_{,\phi})^2 + 32\pi^2 G_N^2 V = 0 \tag{1.54}$$

or

$$\boxed{(H_{,\phi})^2 - 12\pi G_N H^2 = -32\pi^2 G_N^2 V(\phi)} \tag{1.55}$$

From $\pi_\phi = \partial S/\partial\phi$ and the ansatz (1.48):

$$\pi_\phi = -\frac{1}{4\pi G_N}a^3 H_{,\phi} \tag{1.56}$$

But from the definition: $\pi_\phi = a^3\dot\phi$ (with $N = 1$), so:

$$a^3\dot\phi = -\frac{1}{4\pi G_N}a^3 H_{,\phi} \implies \boxed{\dot\phi = -\frac{1}{4\pi G_N}H_{,\phi}} \tag{1.57}$$

From (1.57), we have:

$$\dot\phi^2 = \frac{1}{16\pi^2 G_N^2}(H_{,\phi})^2 \tag{1.58}$$

The Friedmann equation $H^2 = \frac{8\pi G_N}{3}\left(\frac{1}{2}\dot\phi^2 + V\right)$ becomes:

$$H^2 = \frac{8\pi G_N}{3}\left[\frac{1}{2}\cdot\frac{1}{16\pi^2 G_N^2}(H_{,\phi})^2 + V\right] \tag{1.59}$$

$$3H^2 = 8\pi G_N\left[\frac{1}{32\pi^2 G_N^2}(H_{,\phi})^2 + V\right] \tag{1.60}$$

$$(H_{,\phi})^2 - 12\pi G_N H^2 = -32\pi^2 G_N^2 V \tag{1.61}$$

which is exactly (1.55). The second Friedmann equation $\dot H = -4\pi G_N\dot\phi^2$ follows from differentiating $H(\phi(t))$:

$$\dot H = H_{,\phi}\dot\phi = H_{,\phi}\left(-\frac{1}{4\pi G_N}H_{,\phi}\right) = -\frac{1}{4\pi G_N}(H_{,\phi})^2 = -4\pi G_N\dot\phi^2 \tag{1.62}$$

using $\dot\phi^2 = \frac{1}{16\pi^2 G_N^2}(H_{,\phi})^2$. Here $S(a,\phi) = -\frac{1}{4\pi G_N}a^3 H(\phi)$ is the principal Generating function. S has no explicit time dependence because the system is autonomous (Hamiltonian constraint $\mathcal{H} = 0$ replaces $H + \partial S/\partial t = 0$). In the HJ formalism, slow-roll parameters become geometric:

$$\epsilon_H \equiv -\frac{\dot H}{H^2} = \frac{1}{16\pi G_N}\left(\frac{H_{,\phi}}{H}\right)^2 \tag{1.63}$$

$$\eta_H \equiv -\frac{\ddot\phi}{H\dot\phi} = \frac{1}{8\pi G_N}\frac{H_{,\phi\phi}}{H} \tag{1.64}$$

Slow-roll inflation occurs when $\epsilon_H \ll 1$ and $|\eta_H| \ll 1$.

1.5 Background evolution

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right). \tag{1.65}$$

This together with Klein–Gordon equation coming from $\nabla_\mu T^{\mu\nu} = 0$ forms a closed dynamical system for $\phi(t)$. Let us introduce a mass scale m and define the dimensionless time variable

$$z \equiv mt, \tag{1.66}$$

with $\phi(t) \equiv \phi(z)$. The derivatives in these coordinates can be given as:

$$\dot{\phi} = m\phi'(z), \qquad \ddot{\phi} = m^2\phi''(z), \tag{1.67}$$

where primes denote derivatives with respect to z. The Friedmann equation becomes

$$H = \frac{m}{\sqrt{3}M_{\rm pl}}\sqrt{\frac{1}{2}\phi'^2 + \frac{V(\phi)}{m^2}}. \tag{1.68}$$

Substituting into the Klein–Gordon equation gives the dimensionless equation of motion,

$$\boxed{\phi'' + \sqrt{3}\sqrt{\frac{1}{2}\phi'^2 + \frac{V(\phi)}{m^2}}\,\phi' + \frac{1}{m^2}V'(\phi) = 0} \tag{1.69}$$

This equation governs the background evolution for an arbitrary potential $V(\phi)$. The first Hubble slow-roll parameter is

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2/2}{\dot{\phi}^2/2 + V(\phi)}. \tag{1.70}$$

In dimensionless variables,

$$\epsilon(z) = \frac{\phi'^2/2}{\frac{1}{2}\phi'^2 + \frac{V(\phi)}{m^2}}. \tag{1.71}$$

Inflation ends dynamically when $\epsilon = 1$. This can be implemented in mathematica as following:

```
ClearAll["Global`*"]
V0 = 1;

m = 1;
V[\[Phi]_] := V0*\[Phi]^2/2;
DV[\[Phi]_] := D[V[\[Phi]], \[Phi]];

eom = \[Phi]''[z] +
Sqrt[(3 (\[Phi]'[z])^2)/2 + 3 V[\[Phi][z]]/m^2]*\[Phi]'[z] +
DV[\[Phi][z]]/m^2 == 0;

zi = 0.0001;
\[Phi]i = 1;

ics = {\[Phi][zi] == \[Phi]i, \[Phi]'[zi] == -(Sqrt[V[\[Phi]i]]/m)};

sol = NDSolve[{eom, ics}, \[Phi], {z, zi, 100}, MaxSteps -> Infinity];
Plot[Evaluate[\[Phi][z] /. sol], {z, zi, 100}, Frame -> True,
FrameLabel -> {"mt", "\[Phi]"}, PlotRange -> All,
ScalingFunctions -> {"Log"}]
```

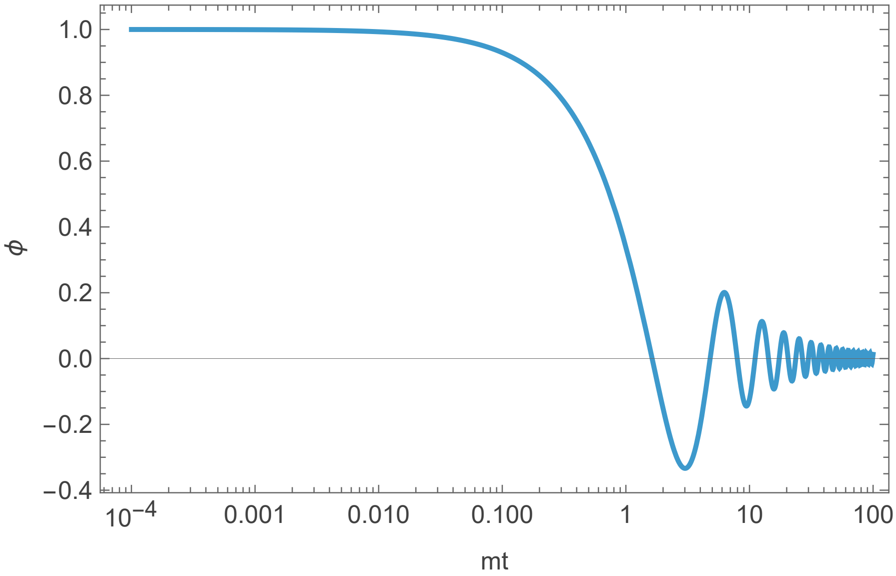


Figure 1.4: Evolution of the inflaton and scale factor during chaotic inflation for the $V(\phi) = \frac{\phi^2}{2}$ model. The scalar field is initially in a slow-roll regime and the expansion of the Universe is accelerated. At the end of this regime, it starts oscillating at the minimum of its potential and it is equivalent to a pressureless fluid

1.6 de Sitter spacetime

de Sitter spaces are the simplest solution of Einstein Field Equation with non zero cosmological constant. Similar to the case where $\Lambda = 0$ corresponds to a unique constant curvature ($\mathcal{R} = 0$) spacetime, de Sitter spacetime corresponds to unique positive constant curvature geometry corresponding to $\Lambda > 0$. This de Sitter geometry also has its origin in embedding space and we will use this embedding to express the resulting induced metric in various coordinate systems. In this way we will, in particular, also recover the metrics encountered in the cosmological contexts. This will be a good exercise to show that the solution of the Friedmann equations in the cosmological constant dominated phase is unique (for a given cosmological constant Λ) and uniquely given by the maximally symmetric dS space. de Sitter space is embedded in $\mathbb{R}^{4,1}$ as:[7]

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = H^2$$

It describes a time-like hyperboloid with topology $\mathbb{R} \times S^3$, the S^3 arising from the slicing at fixed x^0,

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \underbrace{H^2 + (x^0)^2}_{\text{constant}} > 0 \tag{1.72}$$

The spacetime interval in $\mathbb{R}^{4,1}$ is given as:

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$$

We can clearly see that the isometry group for $\mathbb{R}^{(4,1)}$ is $SO(4,1)$. Thus, the induced metric on the de Sitter space will be invariant under the same. However the way these isometries acted on embedding space will be different from how they will act on de Sitter space. The list of isometries for de Sitter are given as:

- 3 Rotation + 3 Translation

$$x_i \to a_i + R_{ij}x_j$$

- 1 Dilatation

$$x_\mu \to \lambda x_\mu$$

- 3 Special Conformal Transformation ($\eta \to 0$ or $b^\mu = (0, \vec{b})$)

$$x_i \to \frac{x_i - b_i(-\eta^2 + \vec{x}^2)}{1 - 2\vec{b}\cdot\vec{x} + b^2(-\eta^2 + \vec{x}^2)}$$

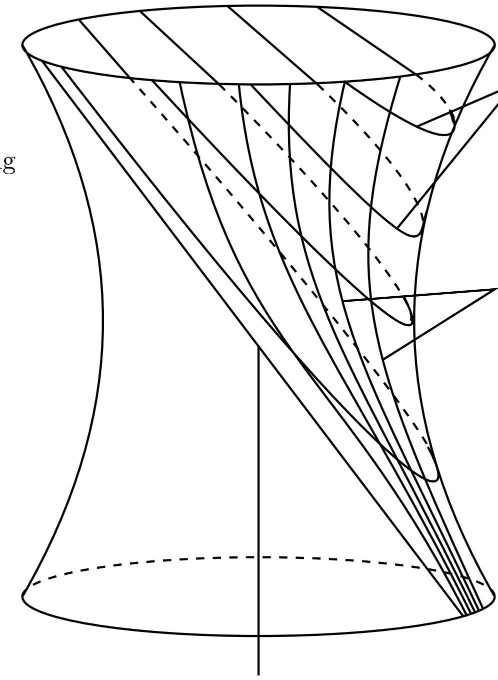
[7] we define it like that because it helps us realize the de Sitter isometries in linear fashion.

We conclude that de Sitter in 4D has $5(5-1)/2 = 10$ isometries. The conformal symmetry of d-dimensional Euclidean space is closely related to the isometry groups of $(d+1)$-dimensional de Sitter space and $(d+2)$-dimensional Minkowski space. We define n^a as normal vectors on 3−sphere (S^3). We can parameterize (1.72) as following

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = H^2 + (x^0)^2$$
$$H^2 \underbrace{\sum_{a,b=1}^{4} \delta_{ab} n^a n^b}_{1} \cosh\Big(\frac{\tau}{H}\Big) = H^2 + H^2 \sinh\Big(\frac{\tau}{H}\Big)$$

i.e. we can introduce following global coordinates on de Sitter space as[8]:

$$\left.\begin{aligned} x^0 &= H^2 \sinh\big(\tfrac{\tau}{H}\big) \\ x^a &= H^2 n^a \cosh\big(\tfrac{\tau}{H}\big); \qquad a = 1,2,3,4 \end{aligned}\right\} \text{closed slicing}$$



we also define following notation:

$$\sum_{a,b=1}^{d-1} \delta_{ab} dn^a dn^b = d\Omega_{d-1}^2$$

We quickly note that the above definition of coordinates gives rise to following induced metric on the hyperboloid.

$$ds^2 = -d\tau^2 + H^2 \cosh^2\Big(\frac{\tau}{H}\Big) d\Omega_3^2$$

where $d\Omega_3^2$ is metric on S^3. Constant time slices are then compact. For $\tau > 0$, this is the typical picture of a closed Universe whose size is expanding exponentially as time evolves forward. The minimal size of the sphere is at $\tau = 0$, where the radius of the sphere is one (in units of the dS radius). In these coordinates dS_4 looks like a 3−sphere which starts out infinitely large at $\tau = -\infty$, then shrinks to a minimal finite size at $\tau = 0$, then grows again to infinite size as $\tau = \infty$.

[8] chapter 4 of "Jerry B. Griffiths, Jirí Podolský - Exact Space-Times in Einstein's General Relativity"

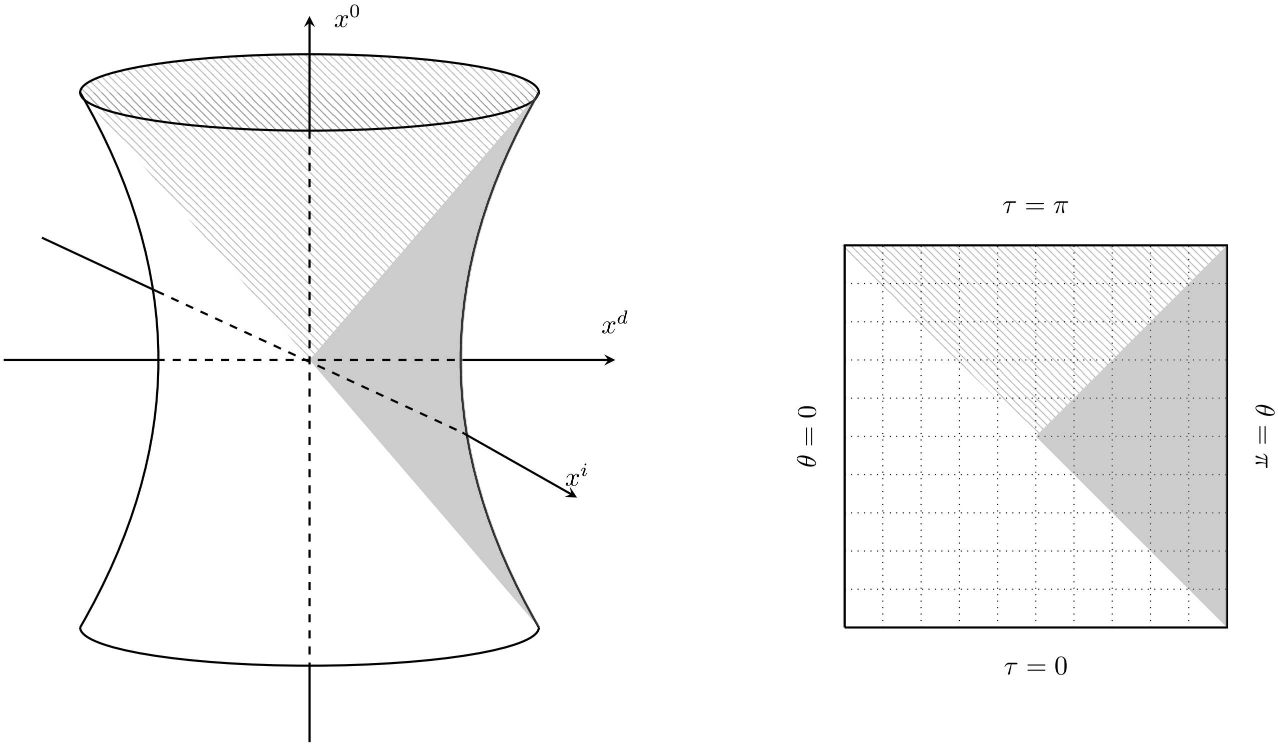


Figure 1.5: de Sitter space can be visualized as hyperbolic hypersurface embedded in higher dimensional Minkowski space. The static patch can be thought of as region of hyperbola contained within the lightcone

Note that this metric depends explicitly on the global time; dS does not have a global timelike Killing vector.

1.6.1 Flat coordinates

These are the coordinates $t, \mathbf{x}$, defined by

$$
\begin{aligned}
x^0 &= H \sinh\left(\frac{t}{H}\right) + \frac{1}{2H} x^2 e^{\frac{t}{H}}, \\
x^1 &= H \cosh\left(\frac{t}{H}\right) - \frac{1}{2H} x^2 e^{\frac{t}{H}}, \\
x^i &= x^i e^{\frac{t}{H}}, \quad i = 2, 3, 4.
\end{aligned}
$$

where $x^2 = \delta_{ab} x^a x^b$. These coordinates do not cover the full de Sitter space, but only the patch

$$x^0 + x^1 = He^{t/H} > 0.$$

observable by observer at south pole ($\theta = \pi$) in penrose diagram.

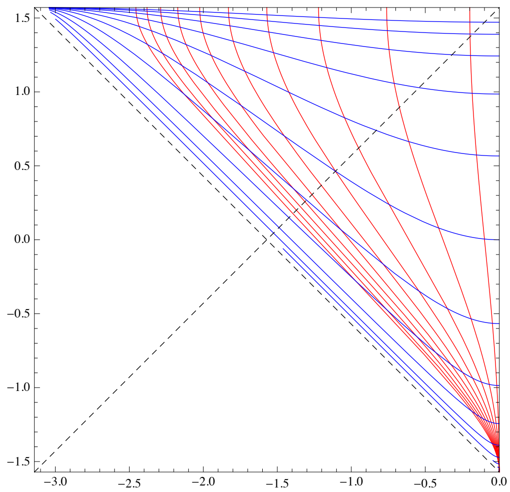


Figure 1.6: Flat slicing of de Sitter, drawn on the Penrose diagram. Blue curves are constant$-t$ flat slices, and red curves are the surfaces of constant$-r$. Intersections of the blue curves with the dashed line are the cross-sections of the cosmological horizon

The flat slicing covers only half the Penrose diagram, so this metric by itself is past-geodesically incomplete. Timelike worldlines, unless they are specially chosen to sit at the South pole, will exit the flat slicing in the past, in finite affine time. In these coordinates, the metric reads

$$ds^2 = -dt^2 + e^{2t/H} \sum_{i=1}^{3} dx_i^2.$$

This metric can be regarded as a FRW cosmology with an exponential function $a(t)$ by replacing $H \to \frac{1}{H}$:

$$a(t) = e^{Ht}. \tag{1.73}$$

If we draw a diagonal plane through the embedding diagram, this is the ‘upper triangle.’ Similar coordinates can be chosen to cover just the ‘lower triangle.’

1.6.2 Conformally flat coordinates

There is another coordinate which is conformal to flat Cartesian coordinates and covers more parts of de Sitter space than flat coordinates. Using familiar cartesian-like coordinates (η, x, y, z), the de Sitter hyperboloid is covered by:

$$\begin{aligned} x^0 &= \frac{H^2+s}{2\eta} \\ x^1 &= \frac{H^2-s}{2\eta} \\ x^2 &= H\frac{x}{\eta} \\ x^3 &= H\frac{y}{\eta} \\ x^4 &= H\frac{z}{\eta} \end{aligned}$$

where $s = -\eta^2 + x^2 + y^2 + z^2$ with $\eta, x, y, z \in (-\infty, \infty)$. Note that these coordinates still do not cover the full de Sitter space but only the patch

$$x^0 + x^1 \neq 0$$

In these coordinates, the de Sitter metric is

$$ds^2 = \frac{H^2}{\eta^2}(-d\eta^2 + d\vec{x}^2)$$

where η is usually referred to as conformal time. This is usually the preferred frame for the computation of cosmological correlators. In these coordinates the time η is not a Killing vector, and the only manifest symmetries are translations and rotations of the x^i coordinate.

1.6.3 Static coordinates

A very important aspect of dS space is that no single observer has access to the full spacetime. This is clear by just looking at the Penrose diagram. An important set of coordinates are those that describe the region accessible to a single observer. This is the intersection between the region of space that can affect the observer and the region that can be affected by them. In terms of embedding coordiantes, they are given by:

$$\begin{aligned} x^0 &= H\sqrt{1-\frac{r^2}{H^2}}\sinh\left(\frac{t}{H}\right), \\ x^1 &= H\sqrt{1-\frac{r^2}{H^2}}\cosh\left(\frac{t}{H}\right), \\ x^a &= rn^a, \quad a=2,3,4 \end{aligned}$$

where

$$0 \le r < H.$$

They only cover the region

$$x^1 > 0, \quad r^2 < H^2.$$

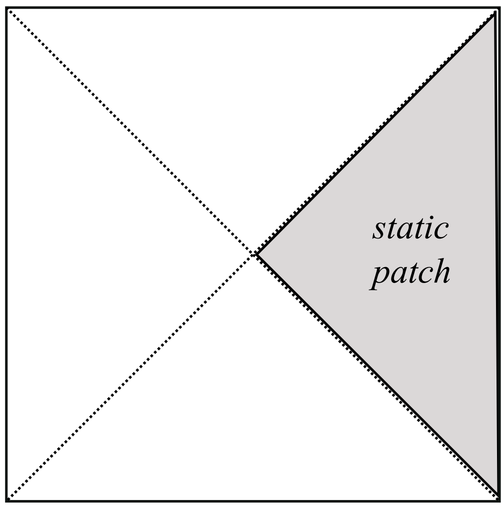


Figure 1.7: Static patch, on the Penrose diagram. This is the causal patch of an observer sitting at the north pole, i.e. $\theta = 0$ in global coordinates, i.e. $r = 0$ in static coordinates. The right edge of the diagram is $r_{\text{static}} = H$; the bifurcate Killing horizon is $r_{\text{static}} =$. The other three patches can also be covered by (independent) static coordinate systems, much like the four regions of the Penrose diagram for Schwarzschild black holes

In these coordinates, the metric reads

$$ds^2 = \left(1-\frac{r^2}{H^2}\right)dt^2 - \frac{dr^2}{1-\frac{r^2}{H^2}} - r^2 d\Omega_3^2.$$

Notice the presence of an explicit horizon at $r = H$, which makes manifest the presence of event horizons for observers in de Sitter space.

1.6.4 Conformal Structure

The boundary of de Sitter space-time, given by $d\Omega = 0$, is located at $\eta = 0$ and $\eta = \pi$ which corresponds to past and future infinity. In contrast to Minkowski space, the infinities $\mathcal{I}^-$ and $\mathcal{I}^+$ now have a spacelike character.

1.7 Penrose Diagram

Penrose diagram graphically represents the causal structure of 2-dimensional spacetime (and of any spacetime whose main features can be reduced to 2 dimensions). The reduction would be done by supressing certain coordinates, therefore the component of metric tensor along those axes won't be of concern. The idea of a

Penrose diagram is this. First, we use a coordinate transformation on the spacetime $(M;g)$ to bring "infinity" to a finite coordinate distance, so that we can draw the entire spacetime on a sheet of paper. The metric will typically diverge as we approach the "points at infinity", i.e. the edges of the finite diagram. To remedy this, we perform a conformal transformation on g to obtain a new metric $\tilde{g}$ that is regular on the edges.

Note that the curvature tensors are in general not preserved under conformal transformation e.g. $R^{\mu}_{\nu\rho\sigma} \neq \tilde{R}^{\mu}_{\nu\rho\sigma}, R \neq \tilde{R}$. It provides a good visual picture of what is happening but should not be considered as depiction of physical processes. In what follows next, we will first transform to null coordinates then perform certain transformation to bring infinity to finite coordinate distance and then the last transformation will deform it to look more like flat metric (only the part of metric whose coordinates we'll use in sketching conformal diagram).

The penrose time and space coordinates are defined via conformal transformation of kruskal coordinates. The definition of conformal metric is given as:

$$\tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}$$

The role of $\Omega(x)^2$ in general is to takeout any divergent part that's encoded in $g_{\mu\nu}$ so that we can include the points where $g_{\mu\nu}$ was divergent. As $\Omega(x)^2 > 0$, it leaves the spacetime interval invariant thus preserving the causal structure.

1.7.1 Minkowski Spacetime

The flat spacetime metric is given as:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega \qquad (d\Omega = d\theta^2 + \sin^2\theta d\phi^2)$$

switching to null coordinates defined by ($u = t - r$ and $v = t + r$) where u is outgoing null coordinate and v is ingoing null coordinate, we have:

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\Omega \\ &= -\left[d\left(\frac{u+v}{2}\right)\right]^2 + \left[d\left(\frac{u-v}{2}\right)\right]^2 + \left(\frac{u-v}{2}\right)^2 d\Omega \\ &= -\left[\left(\frac{du+dv}{2}\right)\right]^2 + \left[\left(\frac{du-dv}{2}\right)\right]^2 + \left(\frac{u-v}{2}\right)^2 d\Omega \\ &= -dudv + \left(\frac{u-v}{2}\right)^2 d\Omega \end{aligned}$$

now, we will do a transformation to bring the points from infinity to finite distance

$$u = \tan U \qquad\qquad v = \tan V$$

Here I would like to remind ourselves that, u =constant and v =constant describes the null geodesic in flat spacetime. Since the new transformed coordinates solely depend upon individual coordinate, U =constant and V =constant will describe the null geodesic here as well.

$$\begin{aligned} ds^2 &= -dudv + \left(\frac{u-v}{2}\right)^2 d\Omega \\ &= -\sec^2 U \sec^2 V dUdV + \frac{\sin(U-V)}{4\cos^2 U \cos^2 V} d\Omega \\ &= \frac{1}{4\cos^2 U \cos^2 V}\left[-4dUdV + \sin(U-V)d\Omega\right] \end{aligned}$$

Now, we will do the conformal scaling and get rid of the $1/4\cos^2 U \cos^2 V$. Since the null trajectories aren't at $45°$, we propose to do another coordinate transformation given via:

$$\begin{aligned} T &= V + U \\ R &= V - U \end{aligned}$$

The final metric would look like:

$$ds^2 = -dT^2 + dR^2 + \sin^2 R d\Omega$$

now, before we go ahead we have to check the limit of T and R.

$$-\frac{\pi}{2} < U, V < +\frac{\pi}{2}$$

So the extremities would be,

$$-\pi < T < \pi \qquad\qquad 0 < R < \pi$$

This implies that all the events in the minkowski spacetime are being labelled by (T, R, θ, ϕ). The past null infinity would be given via:

$$\mathcal{I}^- = T - R = 2U\Big|_{u=-\infty \implies t=-\infty} = -\pi$$

and future null infinity is:

$$\mathcal{I}^- = T + R = 2V\Big|_{v=\infty \implies t=\infty} = \pi$$

Light rays would be described by

$$T \pm R = constant$$

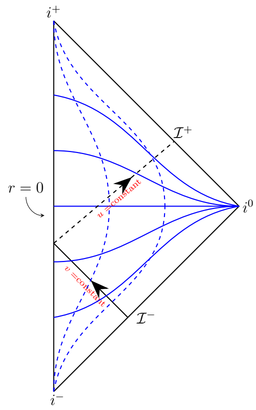


Figure 1.8: Penrose diagram for minkowski spacetime

1.7.2 Schwartzschild Spacetime

Much like previous case, we will begin from null coordinate i.e. Eddington Finkelstein coordinate. The transformation here will describe the schwartzschild patch in penrose diagram but in order to extend the spacetime to explore other regions we will have to do the same transformation in Kruskal Coordinates. The metric (in tortoise coordinates) would look like:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right)(dt^2 - dr^{\star 2}) + r^2 d\Omega \\ &= -\left(1 - \frac{2GM}{r}\right) du dv + r^2 d\Omega \end{aligned}$$

where we used $u = t - r^\star$ and $v = t + r^\star$[9] with

$$r^\star = r + 2GM \ln\Big(\frac{r}{2GM} - 1\Big)$$

Performing the transformation

$$u = \tan U \qquad\qquad v = \tan V$$

we get:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right) du dv + r^2 d\Omega \\ &= -\frac{1}{\cos^2 U \cos^2 V}\left(1 - \frac{2GM}{r}\right) dU dV + r^2 d\Omega \\ &= \frac{1}{4\cos^2 U \cos^2 V}\left(1 - \frac{2GM}{r}\right)[-4dUdV + 4r^2 \cos^2 U \cos^2 V \left(1 - \frac{2GM}{r}\right)^{-1} d\Omega] \end{aligned}$$

like earlier case,

$$-\frac{\pi}{2} < U, V < +\frac{\pi}{2}$$

We can now do the following transformation ($T = U + V$ and $R = V - U$) after removing the $1/4\cos^2 U \cos^2 V$ from the metric.

$$\begin{aligned} ds^2 &= -4dUdV + 4r^2 \cos^2 U \cos^2 V \left(1 - \frac{2GM}{r}\right)^{-1} d\Omega \\ &= -dT^2 + dR^2 + 4r^2 \cos^2\left(\frac{T-R}{2}\right)\cos^2\left(\frac{T+R}{2}\right)\left(1 - \frac{2GM}{r}\right)^{-1} d\Omega \end{aligned}$$

[9]Here there's a problem, the coordinate singularity at $r = 2GM$ is now shifted to $u = \infty$ or $v = -\infty$, note that $\lim_{x\to 0} \ln(x) = -\infty$.

$$= -dT^2 + dR^2 + r^2 \left[\cos(T) + \cos(R)\right]^2 \left(1 - \frac{2GM}{r}\right)^{-1} d\Omega$$

Here, the limits on coordinate T and R describing the extremes are as:

$$-\pi < T < \pi \qquad\qquad 0 < R < \pi$$

The surface describing the event horizon is given via:

$$\mathcal{H}^+ = U\Big|_{r=2GM} = \frac{\pi}{2} \implies \frac{T-R}{2} = \frac{\pi}{2}$$
$$\mathcal{H}^- = V\Big|_{r=2GM} = -\frac{\pi}{2} \implies \frac{T+R}{2} = -\frac{\pi}{2}$$

The past and future null infinity are described via:

$$\mathcal{I}^- = U\Big|_{u=-\infty,\text{v finite}} = -\frac{\pi}{2} \implies \frac{T-R}{2} = -\frac{\pi}{2}$$
$$\mathcal{I}^+ = V\Big|_{v=\infty,\text{u finite}} = \frac{\pi}{2} \implies \frac{T+R}{2} = \frac{\pi}{2}$$

The specific definition of past and future null infinity is based on the choice of ingoing ($v = constant$) and outgoing rays ($u = constant$). Since the boundary of this region will be described by $\frac{T+R}{2} = \pm\frac{\pi}{2}$ and $\frac{T-R}{2} = \pm\frac{\pi}{2}$, the overall shape would be rhombus which doesn't include the singularity or the event horizon. These boundaries would certain region from the triangular zone.

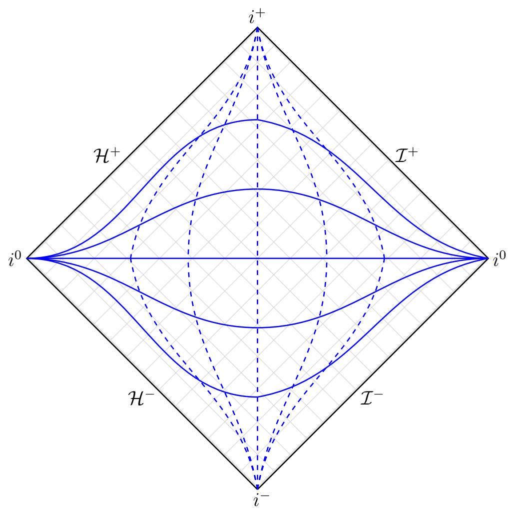


Figure 1.9: Penrose diagram corresponding to schwartzschild patch which covers the region outside the event horizon

Since, this only encompasses the part of spacetime outside the event horizon we need to perform the conformal transformation in coordinate system where event horizon is already at finite distance. One such coordinate is Kruskal Szekeres coordinate. We perform the transformation given via:

$$r^\star = \frac{v-u}{2} = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$
$$t = \frac{u+v}{2}$$

$$U = \mp e^{-u/4GM} = \mp e^{r^\star - t/4GM} = \mp \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r-t/4GM} \tag{1.74}$$

$$V = \pm e^{v/4GM} = \pm e^{r^\star + t/4GM} = \pm \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r+t/4GM} \tag{1.75}$$

These transformation gives us:

$$dU = \pm \frac{1}{4GM} e^{-u/4GM} du$$

$$\begin{aligned} dV &= \pm\frac{1}{4GM}e^{v/4GM}dv \\ dUdV &= \frac{1}{16G^2M^2}e^{v-u/4GM}dudv \\ 16G^2M^2 &= e^{r^\star/2GM}dudv \end{aligned}$$

The metric in these coordinates become:

$$\begin{aligned} ds^2 &= -\left(1-\frac{2GM}{r}\right)dudv + r^2d\Omega \\ &= -\frac{2GM}{r}\left(\frac{r}{2GM}-1\right)dudv + r^2d\Omega \\ &= -\frac{2GM}{r}e^{\ln(r/2GM-1)}dudv + r^2d\Omega \\ &= -\frac{2GM}{r}e^{r^\star - r/2GM}dudv + r^2d\Omega \\ &= -\frac{32G^3M^3}{r}e^{-r/2GM}dUdV + r(U,V)^2d\Omega \end{aligned} \tag{1.76}$$

We now do the following transformation: $U = \tan\tilde{U}$ and $V = \tan\tilde{V}$

$$\begin{aligned} ds^2 &= -\frac{32G^3M^3}{r}e^{-r/2GM}dUdV + r(U,V)^2d\Omega \\ &= -\frac{1}{4\cos^2\tilde{U}\cos^2\tilde{V}}4\frac{32G^3M^3}{r}e^{-r/2M}d\tilde{U}d\tilde{V} + r(\tilde{U},\tilde{V})^2d\Omega \\ &= \frac{32G^3M^3}{r}e^{-r/2GM}(-dT^2+dR^2) + r^2\left[\cos(T)+\cos(R)\right]^2d\Omega \\ &= -dT^2+dR^2+\frac{r}{32G^3M^3}e^{r/2GM}r^2\left[\cos(T)+\cos(R)\right]^2d\Omega \end{aligned}$$

Where we did the transformation $T = \tilde{U}+\tilde{V}$ and $R = \tilde{V}-\tilde{U}$ and then got rid of the $1/4\cos^2\tilde{U}\cos^2\tilde{V}$ factor in the denominator. The curvature singularity in the Kruskal coordinate lies at:

$$\begin{aligned} UV &= \mp e^{\frac{r^\star}{2M}} \\ &= \mp e^{r/2GM}\left|\frac{r}{2Gm}-1\right| = e^{r/2GM}\left(1-\frac{r}{2GM}\right) \\ UV &= 1 \qquad (\text{as } r\to 0^+) \end{aligned}$$

thus,

$$\begin{aligned} UV = 1 \implies &\tan\tilde{U}\tan\tilde{V} = 1 \\ &\sin\tilde{U}\sin\tilde{V} = \cos\tilde{U}\cos\tilde{V} = \cos\left(\tilde{U}+\tilde{V}\right) = 0 \\ &\tilde{U}+\tilde{V} = T = \pm\frac{\pi}{2} \end{aligned}$$

The light rays are described by

$$T \pm R = constant$$

The limits on $\tilde{U}$ and $\tilde{V}$ are given as:

$$-\frac{\pi}{2} < \tilde{U},\tilde{V} < \frac{\pi}{2}$$

The boundaries of this conformal diagram are given via

$$\begin{aligned} \tilde{U} &= \frac{T-R}{2} = \pm\frac{\pi}{2} \\ T-R &= \pm\pi \end{aligned}$$

and,

$$\begin{aligned} \tilde{V} &= \frac{T+R}{2} = \pm\frac{\pi}{2} \\ T+R &= \pm\pi \end{aligned}$$

The location of horizon is given via:

$$\mathcal{H}^+ = U\Big|_{r=2GM} = 0 \implies \tilde{U} = 0$$
$$\tilde{U} = \frac{T-R}{2} \implies T-R=0$$

then

$$\mathcal{H}^- = V\Big|_{r=2GM} = 0 \implies \tilde{V} = 0$$
$$\tilde{V} = \frac{T+R}{2} \implies T+R=0$$

The future and past infinity are at:

$$\mathcal{I}^-{}_1 = U\Big|_{u=-\infty} = -\infty \implies \tilde{U} = -\frac{\pi}{2}$$
$$\tilde{U} = \frac{T-R}{2} \implies T-R=-\pi$$

then

$$\mathcal{I}^+{}_1 = V\Big|_{v=\infty} = \infty \implies \tilde{V} = \frac{\pi}{2}$$
$$\tilde{V} = \frac{T+R}{2} \implies T+R=\pi$$

There is actually another future and past infinity which exploit the lower limit of $\tilde{U}$ and $\tilde{V}$ but in order to explore that we have to consider spacelike geodesic. We use the other sign convention for u and v in the definition (1.74), then:

$$\mathcal{I}^+{}_2 = \tilde{U}\Big|_{U=\infty} = \frac{\pi}{2}$$
$$\tilde{U} = \frac{T-R}{2} \implies T-R=\pi$$

then

$$\mathcal{I}^-{}_2 = \tilde{V}\Big|_{V=-\infty} = -\frac{\pi}{2}$$
$$\tilde{V} = \frac{T+R}{2} \implies T+R=-\pi$$

Taking all these into consideration, the final penrose diagram looks like this:

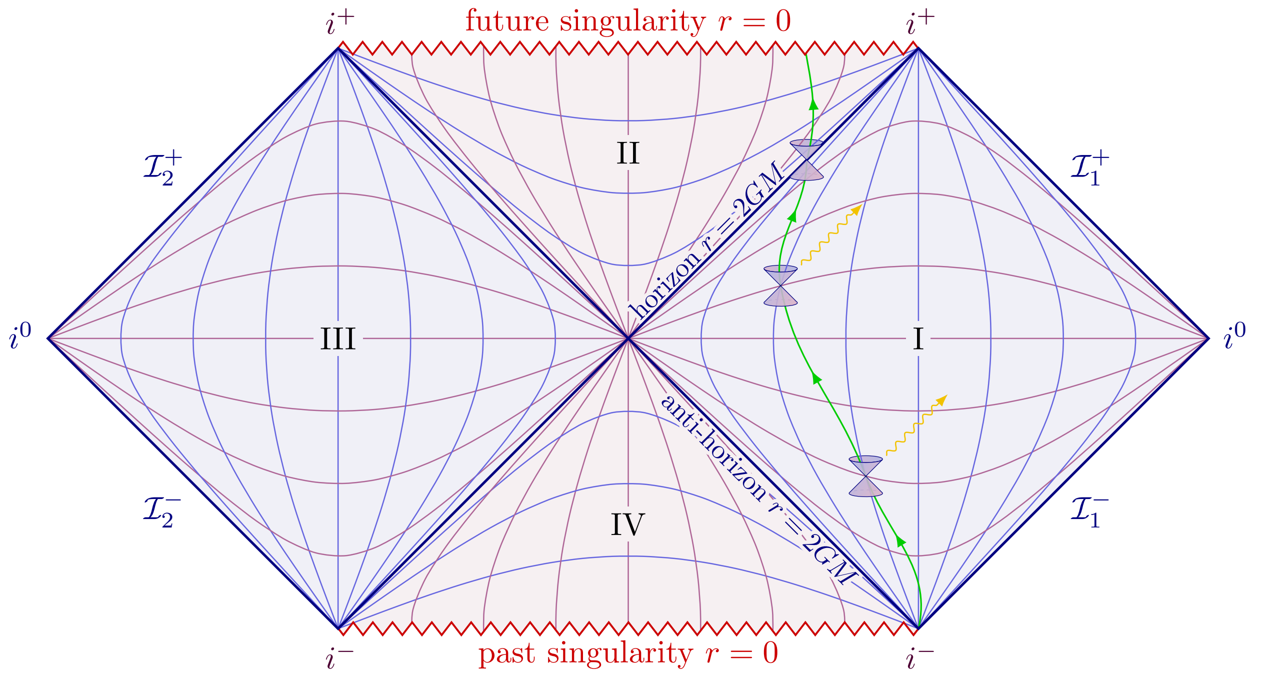


Figure 1.10: Penrose diagram of maximally extended schwarzschild spacetime

1.7.3 Reissner Nordström Solution

We will see that unlike the Schwarzchild metric, which leads to a finite Penrose diagram, the Reissner-Nordstrom metric leads to a tower of asymptotically flat universes, each connected via wormholes, black holes and white holes. The metric tensor for Reissner Nordström solution in eddington finkelstein coordinate is given as:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega \\ &= -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\Omega \\ &= -\frac{\Delta}{r^2}(dt^2 - dr^{\star 2}) + r^2 d\Omega && \text{(defining } {}^{r^2}\!/\!{}_{\Delta} dr^2 = {}^{\Delta}\!/\!{}_{r^2} dr^{\star 2}) \\ &= -\frac{\Delta}{r^2} du dv + r^2 d\Omega \end{aligned}$$

where $\Delta = r^2 - 2Mr + Q^2$. The Reissner Nordstrom blackhole has two horizon given via:

$$\begin{aligned} \Delta &= 0 \\ &= r^2 - 2GMr + Q^2 \\ &= (r - r_+)(r - r_-) \end{aligned}$$

There are three regions:

$$\begin{aligned} \Delta &> 0 && \text{(for } r > r_+ \text{ or } r < r_-) \\ \Delta &< 0 && \text{(for } r_- < r < r_+) \end{aligned}$$

The tortoise coordinate can be related to original r coordinate via:

$$\begin{aligned} dr^\star &= \frac{r^2}{\Delta} dr \\ &= \frac{r^2}{(r - r_+)(r - r_-)} dr \\ r^\star &= r + \frac{1}{2\kappa_+} \ln|r - r_+| + \frac{1}{2\kappa_-} \ln|r - r_-| + C \end{aligned}$$

where,

$$\kappa_+ = \frac{r_+ - r_-}{2r_+^2} \qquad \kappa_- = -\frac{r_+ - r_-}{2r_-^2}$$

Here we note that

$$r^\star\Big|_{r=0} \neq 0$$

as we had in Schwarzschild case, therefore we set the boundary condition so that it is. The new tortoise coordinate which satisfies this condition looks like:

$$r^\star = r + \frac{1}{2\kappa_+}\ln\left|\frac{r}{r_+} - 1\right| + \frac{1}{2\kappa_-}\ln\left|\frac{r}{r_-} - 1\right|$$

From the integral, we see that it has bifurcation points at $r = r_+$ and $r = r_-$. The tortoise coordinate $r^\star$ is undefined at, $r = r_+$ and $r = r_-$, therefore we have to consider three different regions and then patch them together. The modulus in $r^\star$ will open with different sign based on the region we are in. Assuming we label roots of Δ via the condition $r_+ > r_-$, we see that

$$k_+ > 0 \qquad\qquad k_- < 0$$

The transformation to null coordinates are as:

$$v = t + r^\star \qquad\qquad u = t - r^\star$$

thus, we see:

$$r^\star(r_\pm) = -\infty$$

The above defined null coordinates are valid only in the region where $\Delta > 0$. So, we don't expect the conformal scaling to map across the event horizons. Therefore to bring the horizon back at finite distance we have to perform another transformation. The metric in Kruskal-like coordinate given by

$$\begin{aligned}
U = \mp e^{-\kappa_\pm u} &\implies dU = \pm\kappa_\pm e^{-\kappa_\pm u} du\\
V = \pm e^{\kappa_\pm v} &\implies dV = \pm\kappa_\pm e^{\kappa_\pm v} dv\\
dUdV = \kappa_\pm^2 e^{\kappa_\pm(v-u)} &dudv = \kappa_\pm^2 e^{2\kappa_\pm r^\star} dudv
\end{aligned}$$

where $dUdV$ simplifies to:

$$\begin{aligned}
dUdV &= \kappa_\pm^2 e^{2\kappa_\pm r^\star} dudv\\
&= \kappa_\pm^2 e^{2\kappa_\pm\left(r + \ln\left|\frac{r}{r_+}-1\right|^{1/2\kappa_+} + \ln\left|\frac{r}{r_-}-1\right|^{1/2\kappa_-}\right)} dudv\\
&= \kappa_\pm^2 e^{2\kappa_\pm r + \ln\left|\frac{r}{r_+}-1\right|^{\kappa_\pm/\kappa_+} + \ln\left|\frac{r}{r_-}-1\right|^{\kappa_\pm/\kappa_-}} dudv\\
&= \kappa_\pm^2 e^{2\kappa_\pm r}\left|\frac{r}{r_+} - 1\right|^{\kappa_\pm/\kappa_+}\left|\frac{r}{r_-} - 1\right|^{-\kappa_\pm/|\kappa_-|} dudv
\end{aligned}$$

The final metric for κ_+ would look like:

$$\begin{aligned}
ds^2 &= -\frac{(r-r_+)(r-r_-)}{r^2} dudv + r^2 d\Omega\\
&= -\frac{r_+ r_- e^{-2\kappa_+ r}}{\kappa_+^2 r^2}\left|\frac{r - r_-}{r_-}\right|^{1+\kappa_+/|\kappa_-|} dUdV + r^2 d\Omega
\end{aligned}$$

This looks somewhat like the case for schwarzschild metric, thus we will now consider following transformation:

$$U = \tan\tilde{U} \qquad\qquad V = \tan\tilde{V}$$

Thus,

$$\begin{aligned}
ds^2 &= -\frac{r_+ r_- e^{-2\kappa_+ r}}{\kappa_+^2 r^2}\left|\frac{r - r_-}{r_-}\right|^{1+\kappa_+/|\kappa_-|} dUdV + r^2 d\Omega\\
&= -\frac{1}{\cos^2\tilde{U}\cos^2\tilde{V}}\frac{r_+ r_- e^{-2\kappa_+ r}}{\kappa_+^2 r^2}\left|\frac{r - r_-}{r_-}\right|^{1+\kappa_+/|\kappa_-|} d\tilde{U}d\tilde{V} + r^2 d\Omega\\
&= -dT^2 + dR^2 + \cos^2\tilde{U}\cos^2\tilde{V} r_+ r_- e^{2\kappa_+ r}\kappa_+^2 r^2\left|\frac{r - r_-}{r_-}\right|^{-\kappa_+/|\kappa_-|-1} r^2 d\Omega
\end{aligned}$$

In the last step we have considered $T = \tilde{U} + \tilde{V}$ and $R = \tilde{V} - \tilde{U}$. This can now be used to explore the conformal structure of the Reissner Nordstrom solution. We start with:

$$\begin{aligned}
U &= \mp e^{-\kappa_{\pm} u} = \mp e^{-\kappa_{\pm}(t-r^{\star})} = \mp e^{-\kappa_{\pm}(t-r)} \left| \frac{r}{r_+} - 1 \right|^{\kappa_{\pm}/2\kappa_+} \left| \frac{r}{r_-} - 1 \right|^{-\kappa_{\pm}/2|\kappa_-|} \\
V &= \pm e^{\kappa_{\pm} v} = \pm e^{\kappa_{\pm}(t+r^{\star})} = \pm e^{\kappa_{\pm}(t+r)} \left| \frac{r}{r_+} - 1 \right|^{\kappa_{\pm}/2\kappa_+} \left| \frac{r}{r_-} - 1 \right|^{-\kappa_{\pm}/2|\kappa_-|} \\
UV &= -e^{\kappa_{\pm}(v-u)} = -e^{2\kappa_{\pm} r} \left| \frac{r}{r_+} - 1 \right|^{\kappa_{\pm}/\kappa_+} \left| \frac{r}{r_-} - 1 \right|^{-\kappa_{\pm}/|\kappa_-|}
\end{aligned}$$

The outer horizon lies at $r = r_+$,

$$\begin{aligned}
U\Big|_{r=r_+, t=\infty} &= 0 \implies \tilde{U} = 0 \\
&= \frac{T-R}{2} = 0
\end{aligned}$$

similarly

$$\begin{aligned}
V\Big|_{r=r_+, t=-\infty} &= 0 \implies \tilde{V} = 0 \\
&= \frac{T+R}{2} = 0
\end{aligned}$$

The inner horizon $r = r_-$ in region $r < r_-$, lies at (no conditions are imposed over t):

$$\begin{aligned}
U\Big|_{r=r_-} &= \pm\infty \implies \tilde{U} = \pm\frac{\pi}{2} \\
&= \frac{T-R}{2} = \pm\frac{\pi}{2}
\end{aligned}$$

similarly

$$\begin{aligned}
V\Big|_{r=r_-} &= \mp\infty \implies \tilde{V} = \mp\frac{\pi}{2} \\
&= \frac{T+R}{2} = \mp\frac{\pi}{2}
\end{aligned}$$

We observe that once the observer crosses $r = r_+$, he or she is in the range $r_- < r < r_+$. Therefore, the null coordinates could have been defined in that region as:

$$u = t + r^{\star} \qquad\qquad v = -t + r^{\star}$$

Since in the region $r_- < r < r_+$ we have $\Delta < 0$ so,

$$\begin{aligned}
ds^2 &= -\frac{\Delta}{r^2}(dt^2 - dr^{\star 2}) + r^2 d\Omega \\
&= \frac{|\Delta|}{r^2}(dt^2 - dr^{\star 2}) + r^2 d\Omega \\
&= -\frac{|\Delta|}{r^2} du dv + r^2 d\Omega
\end{aligned}$$

The above null coordinate preserves the metric signature. Thus we proceed like the earlier case only to replace the final results with new definition of u and v.

$$\begin{aligned}
U &= \mp e^{-\kappa_{\pm}(t+r^{\star})} = \mp e^{-\kappa_{\pm}(t+r)} (r - r_+)^{-\kappa_{\pm}/2\kappa_+} (r - r_-)^{-\kappa_{\pm}/2\kappa_-} \\
V &= \pm e^{\kappa_{\pm}(-t+r^{\star})} = \pm e^{\kappa_{\pm}(-t+r)} (r - r_+)^{\kappa_{\pm}/2\kappa_+} (r - r_-)^{\kappa_{\pm}/2\kappa_-}
\end{aligned}$$

The inner horizon is at (no conditions are imposed over t):

$$U\Big|_{r=r_-} = \pm\infty \implies \tilde{U} = \pm\frac{\pi}{2}$$

$$
\begin{aligned}
&= \frac{T-R}{2} = \pm\frac{\pi}{2} \\
V\Big|_{r=r_-} &= \mp\infty \implies \tilde{V} = \mp\frac{\pi}{2} \\
&= \frac{T+R}{2} = \mp\frac{\pi}{2}
\end{aligned}
$$

The outer horizon ($r = r_+$) in this region would be defined via,

$$
\begin{aligned}
U\Big|_{r=r_+,t=-\infty} &= 0 \implies \tilde{U} = 0 \\
&= \frac{T-R}{2} = 0
\end{aligned}
$$

similarly

$$
\begin{aligned}
V\Big|_{r=r_+,t=\infty} &= 0 \implies \tilde{V} = 0 \\
&= \frac{T+R}{2} = 0
\end{aligned}
$$

The region $r_- < r < r_+$ will be glued with $r < r_-$ and $r > r_+$. The future and past infinity are at:

$$
\begin{aligned}
\mathcal{I}^-{}_1 = U\Big|_{r=\infty,t=-\infty \implies u=-\infty} &= -\infty \implies \tilde{U} = -\frac{\pi}{2} \\
\tilde{U} &= \frac{T-R}{2} \implies T - R = -\pi
\end{aligned}
$$

then

$$
\begin{aligned}
\mathcal{I}^+{}_1 = V\Big|_{r=t=\infty \implies v=\infty} &= \infty \implies \tilde{V} = \frac{\pi}{2} \\
\tilde{V} &= \frac{T+R}{2} \implies T + R = \pi
\end{aligned}
$$

the remaining two null infinity horizon are defined as

$$
\begin{aligned}
\mathcal{I}^+{}_2 = \tilde{U}\Big|_{U=\infty} &= \frac{\pi}{2} \\
\tilde{U} = \frac{T-R}{2} &\implies T - R = \pi
\end{aligned}
$$

then

$$
\begin{aligned}
\mathcal{I}^-{}_2 = \tilde{V}\Big|_{V=-\infty} &= -\frac{\pi}{2} \\
\tilde{V} = \frac{T+R}{2} &\implies T + R = -\pi
\end{aligned}
$$

The singularity lies at:

$$
\begin{aligned}
UV\Big|_{r=0} &= -e^{\kappa_\pm(v-u)} = -e^{2\kappa_\pm r}\left|\frac{r}{r_+} - 1\right|^{\kappa_\pm/\kappa_+}\left|\frac{r}{r_-} - 1\right|^{-\kappa_\pm/|\kappa_-|}\Bigg|_{r=0} \\
&= -1 \\
\implies \tan\tilde{U}\tan\tilde{V} &= -1 \\
\sin\tilde{U}\sin\tilde{V} &= -\cos\tilde{U}\cos\tilde{V} \\
\cos\left(\tilde{U} - \tilde{V}\right) &= 0 \\
\tilde{U} - \tilde{V} = R &= \pm\frac{\pi}{2}
\end{aligned}
$$

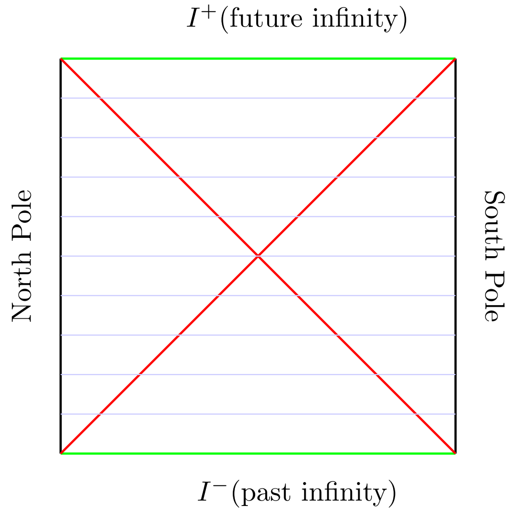
The case we just studied is for $r_+ \neq r_-$, i.e. $G^2M^2 > Q^2$.

1.7.4 de Sitter

To look at the causal structure of dS space, it is convenient to define a conformal time T such that $\cos T = {}^1/_{\cosh(\frac{\tau}{H})}$. From here, it follows that ${}^{-\pi}/_2 \leq T \leq {}^{\pi}/_2$ and writing the metric on 3-sphere S^3 as $d\theta + \sin^2(\theta)d\Omega_2^2$ we see that the de Sitter is conformal to:

$$ds^2 = \frac{1}{\cos(T)}(-dT^2 + d\theta^2 + \cos^2(\theta)d\Omega_2^2)$$

where, θ spans from 0 to π. Suppressing the transverse 2-sphere and adding the points with $T = \pm{}^{\pi}/_2$ (future and past infinity), we end up with the simple Penrose diagram of de Sitter space in (T, θ) plane. Each horizontal line in the diagram (in blue) corresponds to a 3−sphere, whose radius is given by ${}^1/_{\cosh(T)}$.



Each point in each line corresponds to a 2−sphere with the exception of both vertical edges corresponding to $\theta = 0, \pi$ are not spheres but single points. We usually call those points the North and South pole of the sphere, and we like to think about inertial observers sitting at those points.

Chapter 2

Quantum Field Theory in Flat spacetime

2.1 Scalar Field Theory

The propagator of the theory depends on the spin of the particles and remaining form of feynman diagram depends on the interaction terms. For example in ϕ^3 theory, there are three leg at each vertex whereas in ϕ^4 theory there are four leg at each vertex. Now at the vertex level, it's just classical physics, the loop correction and vertex corrections arise from quantum mechanical effects. The self energy diagrams are quantum corrections in propagators. Now, in the feynman diagrams the order of term being interpreted as diagram is represented by number of vertices. For $0th$ order there are no, vertex, for the first order there is one vertex with three legs (for ϕ^3 theory) for second order diagram, we have two vertex and joining them together leads to loop diagram. For third order, we have three vertex

2.2 Contractions

When we are evaluating S matrix elements, it is important to keep in mind the time ordering operator. Wick Contraction gives us a systematic way to consider them into our calculation. Each contraction leads to creation of propagators and the uncontracted terms act on vacuum to create particles which appear as external leg in the feynman diagram and the contractions appear as propagator or internal legs in the feynman diagram. Every possible contraction will lead to a feynman diagram and thus the need for evaluating Symmetry factor arises.

$$S = 1 + i\mathcal{T}$$

and

$$i\mathcal{M} = \langle f|i\mathcal{T}|i\rangle$$

2.3 ϕ^3 Theory

The interaction Lagrangian of this theory is given as

$$\mathcal{L}_{\text{interaction}} = \frac{g}{3!}\phi^3$$

and each of the vertex, would then look like:

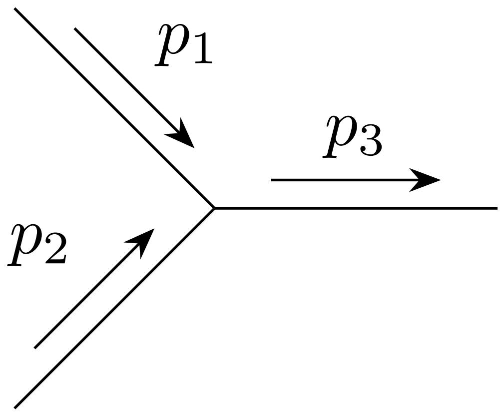
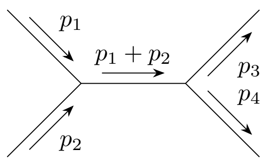


Figure 2.1: The following diagram will be combined in various ways to give us different physical scenarios.

with different momenta, based on the frame of reference chosen.

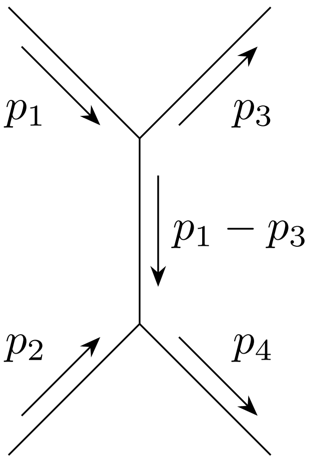
2.3.1 $2 \to 2$ Scattering Channels

In ϕ^3 theory, there are three diagrams describing scattering:
The first one is:



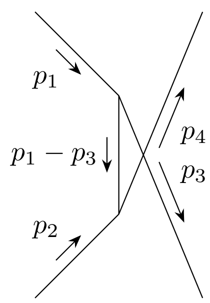
$$= (ig)^2 \frac{i}{(p_1+p_2)^2 - m^2 + i\epsilon}$$

Second one is:



$$= (ig)^2 \frac{i}{(p_1-p_3)^2 - m^2 + i\epsilon}$$

and the last one is:



$$= (ig)^2 \frac{i}{(p_1-p_4)^2 - m^2 + i\epsilon}$$

The scattering cross section is given as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\mathrm{CM}^2}} |\mathcal{M}|^2$$

$$= \frac{g^2}{64\pi^2 E_{\text{CM}^2}} \left[\frac{1}{(p_1 - p_2)^2 - m^2} + \frac{1}{(p_1 - p_3)^2 - m^2} + \frac{1}{(p_1 - p_4)^2 - m^2} \right]^2$$

2.3.2 Self Energy correction in s channel

The transfer matrix elements can be evaluated as:

$$i\mathcal{M}^0 = (ig)^2 \frac{i}{p^\mu p_\mu - m^2}$$
$$i\mathcal{M}^1 = (ig)^2 \frac{i}{p^\mu p_\mu - m^2} i\mathcal{M}_{\text{loop}} \frac{i}{p^\mu p_\mu - m^2}$$
$$i\mathcal{M}^2 = (ig)^2 \frac{i}{p^\mu p_\mu - m^2} i\mathcal{M}_{\text{loop}} \frac{i}{p^\mu p_\mu - m^2} i\mathcal{M}_{\text{loop}} \frac{i}{p^\mu p_\mu - m^2}$$

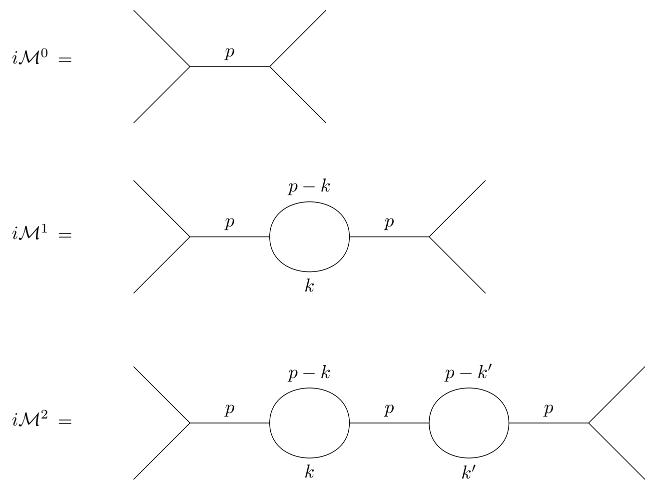


Figure 2.2: The above matrix elements represent these feynman diagrams

The loop diagram with the symmetry factor of 1/2 can be evaluated as:

$$i\mathcal{M}_{\text{loop}} = \frac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2 - m^2 + i\epsilon} \frac{i}{(k)^2 - m^2 + i\epsilon}$$

Through feynman reparametrization trick

$$i\mathcal{M}_{\text{loop}} = \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{[\{k - p(1-x)\}^2 + p^2 x(1-x) - m^2 + i\epsilon]^2}$$

Doing a translation in momentum space which is equivalent to boost and the Jacobian corresponding to boost is 1. So the integration measure will be unaffected. Under the transformation, $k^\mu \to k^\mu + p^\mu(1-x)$

$$i\mathcal{M}_{\text{loop}} = \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{[k^2 - \{m^2 - p^2 x(1-x)\} + i\epsilon]^2} \tag{2.1}$$

Let $\Delta = m^2 - p^2 x(1-x)$ and use Pauli Viller Regulator, which we will use to evaluate the integral in wick rotated spherical polar coordinate. The idea behind Pauli Viller Regularisation is that,

$$\int_0^\infty dk_E \frac{k^n}{k^{n+1}} \to \infty \qquad\qquad \int_0^\infty dk \frac{k^n}{k^{n+2}} \to \text{finite}$$

Thus, we will consider loop level contribution from pauli viller ghost term:

$$\int_0^\infty \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^2} \to \int_0^\infty \frac{d^4k}{(2\pi)^4} \left[\frac{1}{(k^2 - \Delta + i\epsilon)^2} - \frac{1}{(k^2 - \Lambda^2 + i\epsilon)^2} \right]$$

$$= i \int_0^\infty \frac{d^4k_E}{(2\pi)^4} \left[\frac{1}{(-k_E^2 - \Delta)^2} - \frac{1}{(-k_E^2 - \Lambda^2)^2} \right]$$

$$
\begin{aligned}
&= -i \int_0^\infty \frac{d^4k_E}{(2\pi)^4} \frac{(\Delta - \Lambda^2)(2k_E^2 + \Delta + \Lambda^2)}{(k_E^2 + \Delta)^2 (k_E^2 + \Lambda^2)^2} \\
&= -i \int \frac{d\Omega}{(2\pi)^4} \int_0^\infty d|k_E||k_E|^3 \frac{(\Delta - \Lambda^2)(2k_E^2 + \Delta + \Lambda^2)}{(k_E^2 + \Delta)^2 (k_E^2 + \Lambda^2)^2} \\
&= -i \int \frac{d\Omega}{(2\pi)^4} \frac{\ln(\Delta/\Lambda^2)}{2} \\
&= -i \frac{2\pi^2}{(2\pi)^4} \frac{\ln(\Delta/\Lambda^2)}{2} \\
&= -\frac{i}{16\pi^2} \ln\left(\frac{\Delta}{\Lambda^2}\right)
\end{aligned}
$$

Note that the ln has branch point at $\Delta = 0$, so discontinuity may arise because of it which will give off non-zero imaginary part, anyways we will ignore such consideration for now and focus on the x integral:

$$
i\mathcal{M}_{\text{loop}} = -\frac{g^2}{2} \int_0^1 dx \frac{i}{16\pi^2} \ln\left(\frac{\Delta}{\Lambda^2}\right) \tag{2.2}
$$

for $m = 0$

$$
i\mathcal{M}_{\text{loop}} = -i \frac{g^2}{32\pi^2} \left[-2 + \ln\left(\frac{-p^2}{\Lambda^2}\right) \right]
$$

for $m \neq 0$

$$
i\mathcal{M}_{\text{loop}} = -i \frac{g^2}{32\pi^2} \left[-2 + 2\sqrt{4m^2 - p^2} \frac{\tan^{-1} \frac{p}{\sqrt{4m^2 - p^2}}}{p} + \ln\left(\frac{m^2}{\Lambda^2}\right) \right]
$$

Here, note that p is the momentum of virtual photon and isn't on shell and hence for the log to be well defined, has to be spacelike. In the massless limit, the regularised propagator is given as:

$$
\begin{aligned}
\mathcal{M} &= -\frac{g^2}{p^2} + \frac{g^2}{p^2} \left[\frac{1}{32\pi^2} \frac{g^2}{p^2} \ln\left(\frac{-p^2}{\Lambda^2}\right) \right] + \dots \\
&= -\frac{g^2}{p^2} \left[1 - \frac{1}{32\pi^2} \frac{g^2}{p^2} \ln\left(\frac{-p^2}{\Lambda^2}\right) + \dots \right] \\
&= -\frac{\tilde{g}^2}{p^2}
\end{aligned}
$$

Here we demand that the result resemble that of tree level amplitude and hence the coupling is renormalized. For sixth order term, we had two loop but the delta function integrals would have separated both of them and we would have ended with

$$
\frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \frac{i}{(p-k)^2} \frac{1}{k^2} \frac{1}{p^2} \frac{i}{(p-k')^2} \frac{1}{k'^2} = i\mathcal{M}(k) i\mathcal{M}(k')
$$

2.4 ϕ^4 Theory

The Lagrangian for the interacting scalar field is given by:

$$
\mathcal{L} = \frac{\hbar^2}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m^2c^2}{2} \phi^2 + \frac{g}{4!} \phi^4
$$

The equation of motion then becomes:

$$
(\hbar^2 \partial^\mu \partial_\mu + m^2c^2)\phi = \frac{g}{3!} \phi^3
$$

We solve the above by considering power series expansion in coupling[1]:

$$
\phi = \sum \phi_n g^n
$$

[1] This part is taken from "Solving Classical Field Equations" by Robert C. Helling or section 6.2 of No Nonsense Quantum Field Theory by Jakob Schwichtenberg

Then, we have

$$(\hbar^2\partial^\mu\partial_\mu + m^2c^2)\sum \phi_n g^n = \frac{g}{3!}\phi^3$$
$$(\hbar^2\partial^\mu\partial_\mu + m^2c^2)\sum \phi_n g^n = \frac{g}{6}\sum \phi_k g^k \sum \phi_l g^l \sum \phi_m g^m$$
$$\sum(\hbar^2\partial^\mu\partial_\mu + m^2c^2)\phi_n g^n = \frac{1}{6}\sum_{n=k+l+m+1}(\phi_k\phi_l\phi_m)g^n$$

For $n = 0$:

$$(\hbar^2\partial^\mu\partial_\mu + m^2c^2)\phi_0 = 0$$
$$\phi_0(x) = \int \frac{d^4k}{(2\pi\hbar)^4}\tilde{\phi_0}(k)\delta\left(k^2 - \frac{m^2c^2}{\hbar^2}\right)e^{ikx}$$

For $n = 1$:

$$(\hbar^2\partial^\mu\partial_\mu + m^2c^2)\phi_1 = \frac{1}{6}\phi_0^3$$
$$\phi_1(x) = \frac{1}{6}\int d^4y G(x-y)\phi_0^3(y)$$

but

$$G(x) = \int \frac{d^4k}{(2\pi\hbar)^4}\frac{e^{ikx}}{-\hbar^2k^2 + m^2c^2}$$

Thus, we have:

$$\begin{aligned}
\phi_1(x) &= \frac{1}{6}\int d^4y \frac{d^4k}{(2\pi\hbar)^4}\frac{e^{ik(x-y)}}{-\hbar^2k^2+m^2c^2}\prod_{i=1}^{3}\int \frac{d^4k_i}{(2\pi\hbar)^4}\tilde{\phi_0}(k_i)\delta\left(k_i^2 - \frac{m^2c^2}{\hbar^2}\right)e^{ik_iy} \\
&= \frac{1}{6}\int \frac{d^4k}{(2\pi\hbar)^4}\frac{e^{ikx}}{-\hbar^2k^2+m^2c^2}\int d^4y e^{i(-k+k_1+k_2+k_3)y}\prod_{i=1}^{3}\frac{d^4k_i}{(2\pi\hbar)^4}\tilde{\phi_0}(k_i)\delta\left(k_i^2 - \frac{m^2c^2}{\hbar^2}\right) \\
&= \prod_{i=1}^{3}\frac{d^4k_i}{(2\pi\hbar)^4}\tilde{\phi_0}(k_i)\delta\left(k_i^2 - \frac{m^2c^2}{\hbar^2}\right)\times\frac{1}{6}\int d^4k\frac{e^{ikx}}{-\hbar^2k^2+m^2c^2}\delta(-k+k_1+k_2+k_3) \\
&= \frac{1}{6}\prod_{i=1}^{3}\frac{d^4k_i}{(2\pi\hbar)^4}\tilde{\phi_0}(k_i)\delta\left(k_i^2 - \frac{m^2c^2}{\hbar^2}\right)\times\frac{e^{i(k_1+k_2+k_3)x}}{-\hbar^2(k_1+k_2+k_3)^2+m^2c^2} \\
&= \frac{\frac{1}{6}\prod_{i=1}^{3}\frac{d^4k_i}{(2\pi\hbar)^4}\tilde{\phi_0}(k_i)\delta\left(k_i^2 - \frac{m^2c^2}{\hbar^2}\right)e^{ik_ix}}{-\hbar^2(k_1+k_2+k_3)^2+m^2c^2} \\
&= \frac{1}{-\hbar^2(k_1+k_2+k_3)^2+m^2c^2}\times\frac{1}{6}\prod_{i=1}^{3}\frac{d^3k_i}{(2\pi\hbar)^3\sqrt{2\hbar\omega_{k_i}}}(a_{k_i}e^{-ik_ix} + a_{k_i}^\dagger e^{ik_ix})
\end{aligned}$$

This term is pictorially represented in fig 2.3 and describes the classical interacting ϕ^4 theory (time ordering and canonical quantization condition is missing). In this theory, each vertex has 4 leg coming out of it and each one of them are identical.

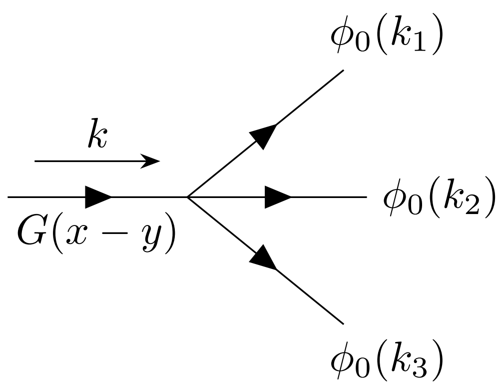


Figure 2.3: The general schematic of each vertex in ϕ^4 theory.

The only purpose of showing the above calculation was to entail that a formalism of feynman diagram does exist in classical field theory. Now without developing the same for quantum field theory, we wish to calculate the following diagram and corresponding self energy corrections to the internal propagator. The only difference here is that in QFT we focus on solving SDE perturbatively rather than the EOM for quantum fields.

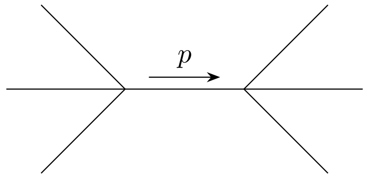
$i\mathcal{M}^0 =$ 

Figure 2.4

The self energy correction to the internal lines is given as:

$$
\begin{aligned}
i\mathcal{M}^0 &= (ig)^2 \frac{i}{p^\mu p_\mu - m^2} \\
i\mathcal{M}^1 &= (ig)^2 \frac{i}{p^\mu p_\mu - m^2} i\mathcal{M}_{\text{loop}} \frac{i}{p^\mu p_\mu - m^2}
\end{aligned}
$$

2.4.1 Pauli Viller in ϕ^4

The symmetry factor of this diagram is 1/2 and the integral over propagator momentum is divergent. Thus we do this trick and take the limit $\Lambda \to \infty$

$$
i\mathcal{M}_{\text{loop}} \quad = \quad \frac{ig}{2}\int_0^\infty \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon} \quad \to \quad -\frac{g}{2}\int_0^\infty \frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2 - m^2 + i\epsilon} - \sum_{j=1,2}\frac{a_j}{k^2 - \Lambda_j^2 + i\epsilon}\right]
$$

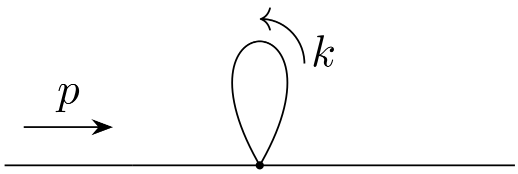


Figure 2.5: One loop feynman diagram in ϕ^4 theory

Notice here that the ghost particle has same momentum. Now, we need to simplify the integrand.

$$
\begin{aligned}
&\frac{1}{k^2 - m^2 + i\epsilon} - \frac{a_1}{k^2 - \Lambda_1^2 + i\epsilon} - \frac{a_2}{k^2 - \Lambda_2^2 + i\epsilon} \\
&\qquad = \frac{(1 + a_1 + a_2)k^4 - [a_1(m^2 + \Lambda_2^2) + a_2(m^2 + \Lambda_1^2) + \Lambda_1^2 + \Lambda_2^2]k^2 + m^2(a_2\Lambda_1^2 + a_1\Lambda_2^2) + \Lambda_1^2\Lambda_2^2}{(k^2 - m^2 + i\epsilon)(k^2 - \Lambda_1^2 + i\epsilon)(k^2 - \Lambda_2^2 + i\epsilon)}
\end{aligned}
$$

Now, for the sake of convergence we don't want k term in the numerator thus we set:

$$
\begin{aligned}
1 + a_1 + a_2 &= 0 \\
a_1(m^2 + \Lambda_2^2) + a_2(m^2 + \Lambda_1^2) + \Lambda_1^2 + \Lambda_2^2 &= 0
\end{aligned}
$$

and solving them we obtain:

$$
\begin{aligned}
a_1 &= \frac{m^2 - \Lambda_2^2}{\Lambda_1^2 - \Lambda_2^2} \\
a_2 &= -\frac{m^2 - \Lambda_1^2}{\Lambda_1^2 - \Lambda_2^2}
\end{aligned}
$$

using these, we get:

$$
m^2(a_2\Lambda_1^2 + a_1\Lambda_2^2) + \Lambda_1^2\Lambda_2^2 = (\Lambda_1^2 - m^2)(\Lambda_2^2 - m^2)
$$

Thus

$$\frac{1}{k^2-m^2+i\epsilon}-\sum_{j=1,2}\frac{a_j}{k^2-\Lambda_j^2+i\epsilon}=\frac{(\Lambda_1^2-m^2)(\Lambda_2^2-m^2)}{(k^2-m^2+i\epsilon)(k^2-\Lambda_1^2+i\epsilon)(k^2-\Lambda_2^2+i\epsilon)}$$

After we set $\Lambda_1=\Lambda_2=\Lambda$, we perform the wick rotation. Upon integration, we find :

$$\begin{aligned}-\frac{g}{2}\int_0^\infty\frac{d^4k}{(2\pi)^4}\frac{(\Lambda^2-m^2)(\Lambda^2-m^2)}{(k^2-m^2+i\epsilon)(k^2-\Lambda^2+i\epsilon)(k^2-\Lambda^2+i\epsilon)}&=\frac{ig}{16\pi^2}\frac{\left[(m^2-\Lambda^2)+m^2\ln\left(\frac{\Lambda^2}{m^2}\right)\right]}{2}\\&\simeq-\frac{ig}{32\pi^2}\left[\Lambda^2-m^2\ln\left(\frac{\Lambda^2}{m^2}\right)\right]\end{aligned}$$

The following integral can be reabsorbed into the definition of mass renormalization in the on-shell scheme. The reason why this case is so special that the whole contribution is getting cancelled by mass counter term. It is so because the result is independent of 'p' momenta unlike ϕ^3 case, refer (2.2).

2.4.2 Four Point function in ϕ^4

Four Point functions have 4 external legs, they would describe the ϕ^4 scattering between two particles. The diagrams in s, t and u channels are given as:

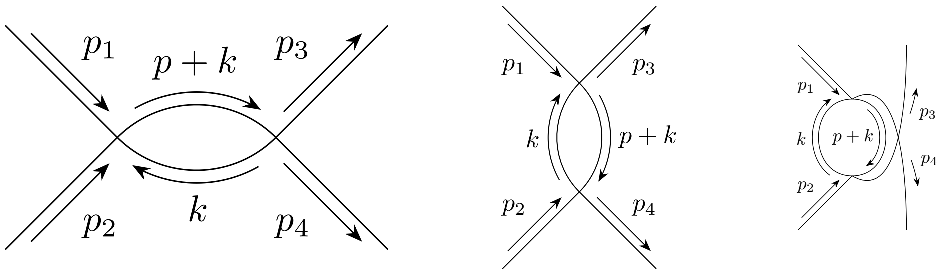
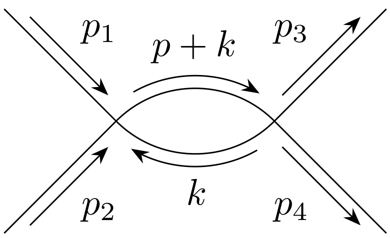


Figure 2.6: These are the s channel, t channel and u channel for ϕ^4 theory.

We will first calculate the s channel contribution.



$$=\frac{1}{2}(ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p+k)^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}$$

This integral looks similar to ϕ^3 case with the distinction that $p=p_1+p_2$. However, unlike earlier we would use hard cutoff to show how the result would still be independent once renormalization is done.

$$\begin{aligned}i\mathcal{M}_s&=\frac{1}{2}(ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p+k)^2-m^2+i\epsilon}\frac{i}{(k)^2-m^2+i\epsilon}\\&=\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\int_0^1dx\frac{1}{[k^2-\{m^2-p^2x(1-x)\}+i\epsilon]^2}\\&=\frac{ig^2}{2}\int_0^\Lambda\frac{d\Omega}{(2\pi)^4}dk_E\frac{k_E^3}{[-k_E^2-\Delta]^2}\\&=\frac{ig^2}{32\pi^2}\int_0^1dx\left[\ln\left(1+\frac{\Lambda^2}{m^2-p^2x(1-x)}\right)+\frac{m^2-x(1-x)p^2}{m^2-x(1-x)p^2+\Lambda^2}-1\right]\end{aligned}$$

In the limit $\Lambda\to\infty$, second term would not contribute. Thus, we would drop it here. To proceed further, we need to introduce counter terms to absorb the cutoff parameter so that the final result is independent of choice of regularization. Using on shell renormalization, we have:

$$i\mathcal{M}^R=ig_R+i\mathcal{M}_s+i\mathcal{M}_t+i\mathcal{M}_u+i\delta_g$$

From the renormalization condition, it is seen that

$$\delta_g = -\mathcal{M}\Big|_{p^2=4m^2} - 2\mathcal{M}\Big|_{p^2=0}$$

The above condition comes from the statement that $\mathcal{M}(p_0^2) = g_R$ then

$$ig_R = ig_R + i\mathcal{M}_s\Big|_{p^2=4m^2} + i\mathcal{M}_t\Big|_{p^2=0} + i\mathcal{M}_u\Big|_{p^2=0} + i\delta_g$$

Since, we are focusing mainly on $s-$channel, we will ignore the contribution from t and u channels. Thus, we see that

$$\delta_g = -\mathcal{M}\Big|_{p^2=4m^2}$$

So, the final result if we only consider first order and second order corrections (from s-channel only) would be:

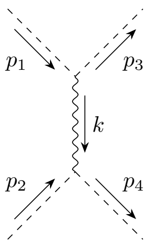
$$\mathcal{M}(p^2) = g_R + \frac{g_R^2}{32\pi^2}\ln\left(\frac{p^2}{p_0^2}\right)$$

2.5 Scalar QED

The interaction lagrangian is given as

$$\mathcal{L}_{\text{interaction}} = -ieA_\mu(\phi^\star\partial_\mu\phi - \phi\partial_\mu\phi^\star) + e^2A_\mu^2|\phi|^2$$

Here we will first focus on the scattering events being described by the first term in the interaction lagrangian. Note that, **this will not have $s-$channel contribution** as that would describe electron positron annihilation. Thus, we will only evaluate $t-$channel and $u-$channel scattering amplitude and associated cross-section. First we evaluate the $t-$channel:



$$= (-ie)^2(p_1^\mu + p_3^\mu)\frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{k^2}(p_2^\nu + p_4^\nu)$$

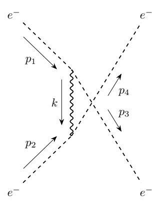
From the delta function at each vertex, we see that $k_\mu = p_1^\mu - p_3^\mu = p_2^\mu - p_4^\mu$. We see that

$$\begin{aligned}
k_\mu(p_1^\mu + p_3^\mu) &= (p_1^\mu - p_3^\mu)(p_1^\mu + p_3^\mu) \\
&= p_1^2 + p_3^2 \\
&= m^2 - m^2 \\
&= 0
\end{aligned}$$

So, the final result with this simplification becomes:

$$i\mathcal{M}_t = ie^2(p_1^\mu + p_3^\mu)\frac{g_{\mu\nu}}{k^2}(p_2^\nu + p_4^\nu)$$

For $u-$channel:



$$= (-ie)^2(p_1^\mu + p_4^\mu)\frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{k^2}(p_2^\nu + p_3^\nu)$$

Thus, similarly the final expression would look like:

$$i\mathcal{M}_u = ie^2(p_1^\mu + p_4^\mu)\frac{g_{\mu\nu}}{k^2}(p_2^\nu + p_3^\nu)$$

2.6 QED: Lagrangian

The Lagrangian for QED is given as:

$$\mathcal{L} = \bar{\psi}(\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

A natural question that arises, is where are the interaction term and why don't we have higher order terms in the lagrangian such as

$$(\bar{\psi}\psi)^2$$

or

$$(F_{\mu\nu}F^{\mu\nu})^2$$

The reason behind is that once we do that, the EOM for free theory (in the absense of interaction terms which will come from the covariant derivative) will become different[2]. We could have added higher order terms into the Lagrangian in the absence of interaction, but that would not describe the free theory[3]: free field Lagrangian mean linear equations of motion. Anyways, the interaction term in the Lagrangian is:

$$\mathcal{L}_{\text{interaction}} = e\bar{\psi}\not{A}\psi$$

which dictates that the spinor polarization $\bar{u}$ will always be on the left of γ^μ in all Feynman diagrams.

2.7 QED: fermion propagator

We will first start with the Schwinger Dyson Equation for fermion and find the green's function for that and interpret it as the fermion propagator. Alternatively, it could be calculated using cluster decomposition theorem by decomposing the fields in terms of creation and annihilation operators which obey anti-commutation relationship. These two different approaches lead to the same final result. First we will discuss how the time ordering of fermions are defined and how to take derivative of the time ordered product.

$$\begin{aligned}
\mathcal{T}[\psi(x)\bar{\psi}(y)] &= \theta(x_0 - y_0)\psi(x)\bar{\psi}(y) - \theta(y_0 - x_0)\bar{\psi}(y)\psi(x) \\
\partial_{x_0}\mathcal{T}[\psi(x)\bar{\psi}(y)] &= \theta(x_0 - y_0)\partial_{x_0}\psi(x)\bar{\psi}(y) - \theta(y_0 - x_0)\bar{\psi}(y)\partial_{x_0}\psi(x) \\
&\quad + \partial_{x_0}\theta(x_0 - y_0)\psi(x)\bar{\psi}(y) - \partial_{x_0}\theta(y_0 - x_0)\bar{\psi}(y)\psi(x) \\
&= \mathcal{T}[\partial_{x_0}\psi(x)\bar{\psi}(y)] + \delta(x_0 - y_0)\psi(x)\bar{\psi}(y) + \delta(y_0 - x_0)\bar{\psi}(y)\psi(x)
\end{aligned}$$

Now, we will derive the EOM (Schwinger Dyson Equation) for fermionic correlation function:

$$\begin{aligned}
i\not{\partial}\langle\Omega|\,\mathcal{T}[\psi(x)\bar{\psi}(y)]\,|\Omega\rangle &= \langle\Omega|\,\mathcal{T}[\cancelto{m\psi(x)}{i\not{\partial}\psi(x)}\bar{\psi}(y)]\,|\Omega\rangle \\
&\quad + i\gamma^0\delta(x^0 - y^0)(\langle\Omega|\,\psi(x)\bar{\psi}(y)\,|\Omega\rangle + \langle\Omega|\,\bar{\psi}(y)\psi(x)\,|\Omega\rangle)
\end{aligned}$$

Since we used free particle EOM and

$$\not{\partial}_{\mu\alpha}\theta(x^0 - y^0) = \gamma^\beta_{\mu\alpha}\partial_\beta\theta(x^0 - y^0) = \gamma^0_{\mu\alpha}\delta(x^0 - y^0)$$

the interacting vacuum has to be replaced by free vacuum. For the sake of simplification, we now switch to index notation:

$$\begin{aligned}
(i\not{\partial} - m)_{\mu\alpha}\langle 0|\,\mathcal{T}[\psi(x)\bar{\psi}(y)]\,|0\rangle_{\alpha\nu} &= i\gamma^0_{\mu\alpha}\delta(x^0 - y^0)(\langle\Omega|\,\psi_\alpha(x)\bar{\psi}_\nu(y)\,|\Omega\rangle \\
&\qquad + \langle\Omega|\,\bar{\psi}_\nu(y)\psi(x)_\alpha\,|\Omega\rangle) \\
&= i\gamma^0_{\mu\alpha}\delta(x^0 - y^0)(\langle\Omega|\,\psi_\alpha(x)\psi^\dagger_\beta(y)\,|\Omega\rangle\,\gamma^0_{\beta\nu} \\
&\qquad + \langle\Omega|\,\psi^\dagger_\beta(y)\psi(x)_\alpha\,|\Omega\rangle\,\gamma^0_{\beta\nu}) \\
&= i\gamma^0\,\langle 0|\,\{\psi(x), \psi^\dagger(y)\}\,|0\rangle\,\gamma^0 \\
(i\not{\partial} - m)\,\langle 0|\,\mathcal{T}[\psi(x)\bar{\psi}(y)]\,|0\rangle &= i\delta^4(x - y)
\end{aligned}$$

In the first line we used dirac equation for simplification and in last step we have used $\{\psi(x), \psi^\dagger(y)\} = \delta^4(x-y)$. By following redefinition, we can reinterpret the SDE as green's equation for classical propagator

$$iD(x, y) = \langle 0|\,\mathcal{T}\psi(x)\bar{\psi}(y)]\,|0\rangle$$

[2] check section 8.7.2 of A Modern Introduction to Classical Electrodynamics by Michele Maggiore for electromagnetic lagrangian and the effect of metric signature

[3] pg 556 of QFT by Sidney Coleman.

$$(i\partial\!\!\!/ - m)D(x,y) = \delta^4(x-y)$$

Consider the same for scalar field

$$\begin{aligned}
(\partial^\mu\partial_\mu + m^2)\langle 0|\mathcal{T}[\phi(x)\phi(y)]|0\rangle &= -i\delta^4(x-y)\\
(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\langle 0|\mathcal{T}[\phi(x)\phi(y)]|0\rangle &= -i\delta^4(x-y)\\
-\left[-\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu - m^2\right]\langle 0|\mathcal{T}[\phi(x)\phi(y)]|0\rangle &= -i\delta^4(x-y)\\
\left[-\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu - m^2\right]\langle 0|\mathcal{T}[\phi(x)\phi(y)]|0\rangle &= i\delta^4(x-y)\\
\left[(i\partial\!\!\!/)^2 - m^2\right]\langle 0|\mathcal{T}[\phi(x)\phi(y)]|0\rangle &= i\delta^4(x-y)\\
-i(i\partial\!\!\!/ - m)(i\partial\!\!\!/ + m)\langle 0|\mathcal{T}[\phi(x)\phi(y)]|0\rangle &= \delta^4(x-y)\\
(i\partial\!\!\!/ - m)D(x,y) &= \delta^4(x-y)
\end{aligned}$$

Thus, we see:

$$D(x,y) = -i(i\partial\!\!\!/ + m)\langle\Omega|\mathcal{T}[\phi(x)\phi(y)]|\Omega\rangle$$

Since, we will mostly be working in momentum space, we can express the fermion propagator in momentum space as:

$$\begin{aligned}
D(x,y) &= -i(i\partial\!\!\!/ + m)\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k e^{-ik_\mu(x^\mu - y^\mu)}\frac{i}{k^2 - m^2 + i\epsilon}\\
&= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k e^{-ik(x-y)}\frac{(k\!\!\!/ + m)}{k^2 - m^2 + i\epsilon}\\
\langle\Omega|\psi(x)\bar\psi(y)|\Omega\rangle &= iD(x,y)\\
&= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k e^{-ik(x-y)}\frac{i(k\!\!\!/ + m)}{k^2 - m^2 + i\epsilon}
\end{aligned}$$

If we did not wish to use the propagator of scalar field, we could have alternatively used the following procedure:

$$(i\partial\!\!\!/ - m)\langle 0|\mathcal{T}[\psi(x)\bar\psi(y)]|0\rangle = i\delta(x-y)$$

Performing the Fourier transform

$$\begin{aligned}
\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k e^{-ik(x-y)}(k\!\!\!/ - m)D(\tilde p) &= \frac{i}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k e^{-ik(x-y)}\\
\int_{-\infty}^{\infty} d^4k e^{-ik(x-y)}(k\!\!\!/ - m)D(\tilde p) &= i\int_{-\infty}^{\infty} d^4k e^{-ik(x-y)}
\end{aligned}$$

thus we get

$$\begin{aligned}
\langle 0|\mathcal{T}[\psi(x)\bar\psi(y)]|0\rangle &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k e^{-ik(x-y)}D(\tilde p)\\
&= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k e^{-ik(x-y)}\frac{i}{k\!\!\!/ - m}\\
&= \lim_{\epsilon\to 0}\frac{i(k\!\!\!/ - m)}{k^2 - m^2 + i\epsilon}
\end{aligned}$$

2.7.1 Wick Rotated Euclidean Space

In wick rotated euclidean space, the canonical quantization is given via:

$$[\psi(x), \psi^\dagger(y)] = -i\delta(x-y)$$

Using this, we find

$$\begin{aligned}
i\partial\!\!\!/\langle\Omega|\mathcal{T}[\psi(x)\bar\psi(y)]|\Omega\rangle &= \langle\Omega|\mathcal{T}[\cancelto{m\psi(x)}{i\partial\!\!\!/\psi(x)}\bar\psi(y)]|\Omega\rangle\\
&\quad + i\gamma^0\delta(x^0 - y^0)(\langle\Omega|\psi(x)\bar\psi(y)|\Omega\rangle + \langle\Omega|\bar\psi(y)\psi(x)|\Omega\rangle)\\
&= i\langle 0|\{\psi(x), \psi^\dagger(y)\}|0\rangle
\end{aligned}$$

$$(i\partial\!\!\!/ - m) \langle 0| \mathcal{T}[\psi(x)\bar{\psi}(y)] |0\rangle = \delta^4(x-y)$$

Now, we solve it to find the wick rotated propagator:

$$D(x,y)_E = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d^4k e^{-ik(x-y)} \frac{(k\!\!\!/ + m)}{k^2 - m^2 + i\epsilon}$$

2.7.2 Propagator from Quantization of Free Field

This part of the derivation is something which can be found in many textbook and is standard derivation which derives the propagator using free vacuum. We start with the cluster decomposition theorem and express the quantum field in terms of creation and annihilation operator where the constraints imposed over the fields get passed down to these operators. We don't have to know their exact functional form to extract what we wish to know about the dynamics. Thus, we leave these operators in the symbolic form without worrying about their representation.

$$\psi(x) = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \left\{ a_s(k)e^{-ikx}u_s(k) + a_s^\dagger(k)e^{ikx}v_s(k) \right\}$$

Consider the two point correlation function

$$\langle 0| \mathcal{T}\{\psi(x), \bar{\psi}(y)\} |0\rangle = \langle 0| \psi(x)\bar{\psi}(y) |0\rangle \theta(x_0 - y_0) - \langle 0| \bar{\psi}(y)\psi(x) |0\rangle \theta(y_0 - x_0)$$

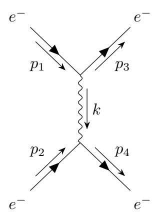
2.8 Classification of Dirac Fermions

In this section, we discuss the concept somewhat similar to that of Casimir operator. The casimir operators are used to classify the states as they commute with all the other generator of the algebra. Similarly, when we look at gamma matrices, there is a special matrix which commutes with all other matrices. So, the eigenstate of this matrix can be used to label states and thus, also be used to construct a projection operator which splits the Dirac fermion into smaller irreducible representation.

$$P = \frac{1+\gamma_5}{2}$$

2.9 QED: Möller Scattering

This is simply $e^-e^- \to e^-e^-$ scattering in QED. The diagram looks similar to scalar QED but the integrals are slight different due to the fact that interaction Lagrangian is different.



$$= (-ie)^2 \bar{u}_3\gamma^\mu u_1 \frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{k^2} \bar{u}_4\gamma^\nu u_2$$

with the choice of notation $u_3 = u(p_3)$. The scattering amplitude can be simplified further before integration. We will treat k^μ as a number in the following simplification.

$$\begin{aligned}
\bar{u}_3\gamma^\mu u_1 k_\mu &= \bar{u}_3\gamma^\mu u_1(p_1^\mu - p_3^\mu) \\
&= \bar{u}_3\gamma^\mu p_1^\mu u_1 - \bar{u}_3\gamma^\mu p_3^\mu u_1 \\
&= \bar{u}_3 m_e u_1 - \bar{u}_3 m_e u_1 \\
&= 0 = \partial^\mu J_\mu
\end{aligned}$$

Thus, the final expression for scattering amplitude is:

$$i\mathcal{M}_t = ie^2\bar{u}_3\gamma^\mu u_1 \frac{g_{\mu\nu}}{k^2} \bar{u}_4\gamma^\nu u_2$$

Here, we will average out the spin for practical experimental purpose to take care of our ignorance regarding the electron spin orientation for each particle. First we will average it out and then in the later part we will use the above expression to derive coulomb's law.

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}_t|^2 = \frac{e^4}{4k^4}\sum_{\text{spin}} [(\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2)][(\bar{u}_3\gamma^\nu u_1)(\bar{u}_4\gamma_\nu u_2)]^\star$$

The above expression can be simplified using $(a \cdot B \cdot c)^\star = (a \cdot B \cdot c)^{\dagger}$[4]:

$$
\begin{aligned}
\sum_s \sum_{s'} (\bar{u}_3^s \gamma^\mu u_1^{s'})(\bar{u}_3^s \gamma^\nu u_1^{s'})^\star &= \sum_s \sum_{s'} (\bar{u}_3^s \gamma^\mu u_1^{s'})(u_1^{s'} \gamma^\nu \bar{u}_3^s) \\
&= \sum_s \bar{u}_3^s \gamma^\mu \left(\sum_{s'} u_1^{s'} u_1^{s'} \right) \gamma^\mu \bar{u}_3^s \\
&= \sum_s \bar{u}_3^s \gamma^\mu (\not{p}_1 + m) \gamma^\nu \bar{u}_3^s \\
&= \Big[\gamma^\mu (\not{p}_1 + m) \gamma^\nu \Big]_{\alpha\beta} \left\{ \sum_s \bar{u}_3^s \bar{u}_3^s \right\}_{\beta\alpha} \\
&= \mathrm{Tr}\Big[\gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_3 + m) \Big]
\end{aligned}
$$

Thus, we get

$$
\frac{1}{4} \sum_{\text{spin}} |\mathcal{M}_t|^2 = \frac{e^4}{4k^4} \mathrm{Tr}\Big[(\not{p}_1 + m) \gamma^\mu (\not{p}_3 + m) \gamma^\nu \Big] \mathrm{Tr}\Big[(\not{p}_2 + m) \gamma_\nu (\not{p}_4 + m) \gamma_\mu \Big]
$$

The above traces would be evaluated using the following properties:

$$
\begin{aligned}
\mathrm{Tr}[A + B] &= \mathrm{Tr}[A] + \mathrm{Tr}[B] \\
\mathrm{Tr}[\alpha A] &= \alpha \, \mathrm{Tr}[A] \\
\mathrm{Tr}[ABC...K] &= \mathrm{Tr}[KABC...]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathrm{Tr}\Big[\gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_3 + m) \Big] &= \mathrm{Tr}\Big[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^\mu \not{p}_1 \gamma^\nu m + \gamma^\mu m \gamma^\nu \not{p}_3 + \gamma^\mu m \gamma^\nu m \Big] \\
&= \mathrm{Tr}\Big[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 \Big] + \cancelto{0}{m \, \mathrm{Tr}\Big[\gamma^\mu \not{p}_1 \gamma^\nu \Big]} + \cancelto{0}{m \, \mathrm{Tr}\Big[\gamma^\mu \gamma^\nu \not{p}_3 \Big]} + m^2 \, \mathrm{Tr}[\gamma^\mu \gamma^\nu] \\
&= (p_1)_\alpha (p_3)_\beta \, \mathrm{Tr}\big[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \big] + m^2 \, \mathrm{Tr}[\gamma^\mu \gamma^\nu] \\
&= 4 (p_1)_\alpha (p_3)_\beta [g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\alpha\nu}] + 4m^2 g^{\mu\nu} \\
&= 4[p_1^\mu p_3^\nu - g^{\mu\nu} p_1 \cdot p_3 + p_1^\nu p_3^\mu + m^2 g^{\mu\nu}]
\end{aligned}
$$

Similarly, we get

$$
\mathrm{Tr}\Big[\gamma_\nu (\not{p}_2 + m) \gamma_\mu (\not{p}_4 + m) \Big] = 4[p_{2\nu} p_{4\mu} - g_{\nu\mu} p_2 \cdot p_4 + p_{2\mu} p_{4\nu} + m^2 g_{\nu\mu}]
$$

The final expression will have all the indices contracted over, therefore the final expression would be expressed in terms of norms and inner product between momenta.

$$
\begin{aligned}
&[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + g^{\mu\nu}(m^2 - p_1 \cdot p_3)] \times [p_{2\nu} p_{4\mu} + p_{2\mu} p_{4\nu} + g_{\nu\mu}(m^2 - p_2 \cdot p_4)] \\
&\quad = [2(p_1 \cdot p_4)(p_2 \cdot p_3) + 2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2 p_1 \cdot p_3 (m^2 - p_2 \cdot p_4) \\
&\qquad + 2 p_2 \cdot p_4 (m^2 - p_1 \cdot p_3) + 4(m^2 - p_1 \cdot p_3)(m^2 - p_2 \cdot p_4)] \\
&\quad = 2[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4) + p_1 \cdot p_3 (m^2 - p_2 \cdot p_4) \\
&\qquad + p_2 \cdot p_4 (m^2 - p_1 \cdot p_3) + 2(m^2 - p_1 \cdot p_3)(m^2 - p_2 \cdot p_4)] \\
&\quad = 2[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4) + p_1 \cdot p_3 (m^2 - p_2 \cdot p_4) \\
&\qquad + p_2 \cdot p_4 (m^2 - p_1 \cdot p_3) + 2(m^4 - m^2 p_1 \cdot p_3 - m^2 p_2 \cdot p_4 + p_1 \cdot p_3 p_2 \cdot p_4)] \\
&\quad = 2[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4) - m^2 p_1 \cdot p_3 - m^2 p_2 \cdot p_4 + 2m^4]
\end{aligned}
$$

We now conveniently introduce Mandelstam variable to simply the resulting expression.

$$
\begin{aligned}
s = (p_1 + p_2)^2 &\implies s = p_1^2 + p_2^2 + 2 p_1 \cdot p_2 \\
t = (p_1 - p_3)^2 &\implies t = p_1^2 + p_3^2 - 2 p_1 \cdot p_3
\end{aligned}
$$

[4] since the complex conjugate is same as the hermitian conjugate as the quantity inside bracket is 1×1 matrix. This is known as Casimir trick

$$u = (p_1 - p_4)^2 \implies u = p_1^2 + p_4^2 - 2p_1 \cdot p_4$$

using the on shell condition, we simplify the product of traces using:

$$\begin{aligned}
p_3 \cdot p_4 &= p_1 \cdot p_2 = \frac{s}{2} - m^2 \\
p_2 \cdot p_4 &= p_1 \cdot p_3 = m^2 - \frac{t}{2} \\
p_2 \cdot p_3 &= p_1 \cdot p_4 = m^2 - \frac{u}{2}
\end{aligned}$$

Therefore, finally

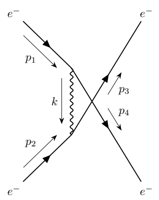
$$\mathcal{M} = \frac{8e^4}{t^2}[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4) - m^2 p_1 \cdot p_3 - m^2 p_2 \cdot p_4 + 2m^4]$$

upon substitution we find

$$\begin{aligned}
\mathcal{M} &= \frac{8e^4}{t^2}\left[\left(m^2 - \frac{u}{2}^2\right) + \left(\frac{s}{2} - m^2\right)^2 - 2m^2\left(m^2 - \frac{t}{2}\right) + 2m^4\right] \\
&= \frac{2e^4}{t^2}[s^2 + u^2 - 4m^2(s - t + u) + 8m^4] \\
&= \frac{2e^4}{t^2}[s^2 + u^2 - 8m^2(s + u) + 24m^2]
\end{aligned}$$

2.9.1 $u-$channel

First thing to note here is that there would be relative sign between t and u channel diagrams. So, since we took $t-$channel contribution to be positive, we will take $u-$channel contribution to be negative.



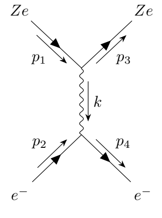
$$= -(-ie)^2 \bar{u}_3 \gamma^\mu u_2 \frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{k^2} \bar{u}_4 \gamma^\nu u_1$$

Similar to $t-$channel, the contribution from $u-$channel diagram would be given as:

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}_u|^2 &= \frac{e^4}{4k^4}\sum_{\text{spin}}[(\bar{u}_3\gamma^\mu u_2)(\bar{u}_4\gamma_\mu u_1)][(\bar{u}_3\gamma^\nu u_2)(\bar{u}_4\gamma_\nu u_1)]^\star \\
&= \frac{e^4}{4k^4}\sum_{\text{spin}}[(\bar{u}_3\gamma^\mu u_2)(\bar{u}_4\gamma_\mu u_1)][(\bar{u}_1\gamma_\nu u_4)(\bar{u}_2\gamma^\nu u_3)] \\
&= \frac{e^4}{4k^4}\,\mathrm{Tr}\left[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\right]\mathrm{Tr}\left[(\not{p}_2 + m)\gamma_\nu(\not{p}_3 + m)\gamma_\mu\right] \\
&= \frac{4e^4}{k^4}[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) - m^2 p_2 \cdot p_3 - m^2 p_1 \cdot p_4 + 2m^4] \\
&= \frac{8e^4}{u^2}[s^2 + t^2 - 8m^2(s + t) + 24m^4]
\end{aligned}$$

2.10 Rutherford Scattering

This would be the perfect example for us to deduce the potential. The non-relativistic version of this process had been well studied in quantum mechanics and since the outgoing particles are not identical. At second order there is only one diagram $t-$channel.



$$= (iZe)\bar{u}_3\gamma^\mu u_1 \frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{k^2}(-ie)\bar{u}_4\gamma^\nu u_2$$

This ends up the same final expression:

$$
\begin{aligned}
i\mathcal{M}_t &= i\frac{Ze^2}{k^2}(\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2) \\
&= i\frac{Ze^2}{k^2}(\bar{u}_3\gamma^0 u_1)(\bar{u}_4\gamma_0 u_2) + \mathcal{O}(p) \\
&\approx i\frac{Ze^2}{k^2}(u_3^\dagger\gamma_0\gamma_0 u_1)(u_4^\dagger\gamma_0\gamma_0 u_1) \\
&= i\frac{Ze^2}{k^2}(2E_p)(2E_e) \approx i\frac{Ze^2}{k^2}(2m_N)(2m_e)
\end{aligned}
$$

Now from born approximation, we have

$$
\begin{aligned}
\frac{\partial\sigma}{\partial\Omega} &= \frac{m_{\text{reduced}}^2}{4\pi^2}\tilde{V}^2(k) = \frac{1}{64\pi^2 E_{COM}}|\mathcal{M}|^2 \\
\tilde{V}(k) &= \frac{\mathcal{M}}{4m_{\text{reduced}}E_{COM}}
\end{aligned}
$$

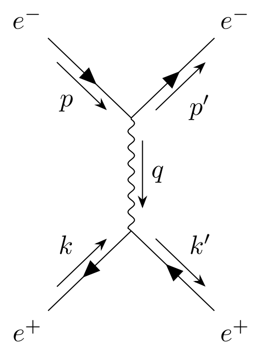
Since, $m_{\text{reduced}} = m_e$, $E_{COM} = m_N$, thus

$$
\begin{aligned}
\tilde{V}(k) &= \frac{\mathcal{M}}{4m_e m_N} \\
&= \frac{Ze^2}{k^2}
\end{aligned}
$$

which is the coulomb potential in momentum space.

2.11 QED: Bhabha Scattering

Bhabha Scattering describes $e^-e^+ \to e^+e^-$, so it has both $s-$channel and $t-$channel (we only need t and $u-$channel, when outgoing particles are identical). In the non-relativistic limit $s \approx (2m)^2$ and will drop out. Therefore in the non relativistic limit, only $t-$channel will contribute and therefore we will begin our calculation from there. Here, we call (p,k) the incoming momenta and (p',k') the outgoing momenta. The $t-$channel diagram looks like:



$$
= \bar{u}(p')(-ie\gamma^\mu)u(p)\frac{-i}{q^2}\left[g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]\bar{v}(k)\,(ie\gamma^\nu)\,v(k')
$$

where (p,p') are incoming and outgoing electron,respectively; (k,k') are the incoming and outgoing proton (or any other fermion of charge Q, except the electron which would also require the u channel), respectively; $q = p - p' = k' - k$ is the t-channel 4-momentum; ξ allows a choice of gauge.

The gauge term does not immediately drop in this case. We have:

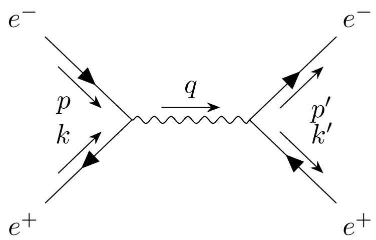
$$
\begin{aligned}
&\bar{u}(p')(-ie\gamma^\mu)u(p)\frac{-i}{q^2}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right)\bar{v}(k)\,(ie\gamma^\nu)\,v(k') \\
&= \{\dots\} - \frac{i}{t^2}(1-\xi)e^2\bar{u}(p')(i\not{q})u(p)\bar{v}(k)(i\not{q})v(k') \\
&= \{\dots\} - \frac{i}{t^2}(1-\xi)e^2\bar{u}(p')(i\not{p} - i\not{p}')u(p)\bar{v}(k)(i\not{k}' - i\not{k})v(k') \\
&= \{\dots\} - \frac{i}{t^2}(1-\xi)e^2\bar{u}(p')2mu(p)\bar{v}(k)2mv(k') \\
&= \{\dots\} - \frac{4im^2e^2}{t^2}(1-\xi)e^2\bar{u}(p')u(p)\bar{v}(k)v(k')
\end{aligned}
$$

ng $\gamma^\mu q_\mu = \not{q}$

where we use the Dirac and conjugate Dirac equations in the middle step. The matrix element becomes

$$i\mathcal{M}^t = -\frac{ie^2}{t} g_{\mu\nu} \bar{u}(p')\gamma^\alpha u(p)\bar{u}(k')\gamma^\beta u(k) - \frac{4im^2e^2}{t^2}(1-\xi)e^2\bar{u}(p')u(p)\bar{\nu}(k)\nu(k')$$

where we set $q^2 = t$. We should get the same result in any gauge, so we could just set $\xi = 1$, but in any case the term is of order m^2 so we drop it. The other $s-$channel contribution is written as:



$$= \bar{\nu}(k)(-ie\gamma^\alpha)u(p)\frac{-i}{q^2}\left[g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]\bar{u}(p')\left(-ie\gamma^\beta\right)\nu(k')$$

the gauge terms vanish immediately. We replace $q^2 = s$, leaving

$$i\mathcal{M}^s = \frac{ie^2}{s} g_{\mu\nu}\bar{v}(k)\gamma^\mu u(p)\bar{u}(p')\gamma^\nu v(k')$$

The resulting matrix element is

$$\begin{aligned} i\mathcal{M} &= i\mathcal{M}^s + i\mathcal{M}^t \\ &= \frac{ie^2}{s} g_{\mu\nu}\bar{\nu}(k)\gamma^\mu u(p)\bar{u}(p')\gamma^\nu v(k') - \frac{ie^2}{t} g_{\mu\nu}\bar{u}(p')\gamma^\mu u(p)\bar{\nu}(k)\gamma^\nu v(k') \end{aligned}$$

Squaring the matrix element leads to four terms,

$$|i\mathcal{M}|^2 = |i\mathcal{M}^s|^2 + \mathcal{M}^s\mathcal{M}^{t*} + \mathcal{M}^t\mathcal{M}^{s*} + |i\mathcal{M}^t|^2$$

We work them out one at a time.

2.11.1 Matrix squared: s channel

We already have $|i\mathcal{M}^s|^2$ from the muon case. Averaging over initial spins and summing over final spins, we found

$$\begin{aligned} \frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}^s|^2 &= \frac{8e^4}{s^2}\Big(\left((p\cdot p')(k\cdot k') - (k'\cdot p')(k\cdot p) + (p'\cdot k)(p\cdot k') - m^2(k\cdot p)\right) \\ &\qquad + (k\cdot p)(k'\cdot p') + m^2(k'\cdot p') + 2m(k\cdot p) + 2m\Big) \\ &= \frac{8e^4}{s^2}\left((p\cdot p')(k\cdot k') + (p'\cdot k)(p\cdot k') + m^2(k'\cdot p') + m^2(k\cdot p) + 2m^4\right) \end{aligned}$$

where some simplification occurs because of the equal masses.

2.11.2 Matrix squared: t channel

For the final term, we have

$$\begin{aligned} \frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}^t|^2 &= \frac{1}{4}\sum_{\text{all spins}} \left(\frac{ie^2}{t} g_{\mu\nu}\bar{u}(p')\gamma^\mu u(p)\bar{\nu}(k)\gamma^\nu v(k')\right)\left(-\frac{ie^2}{t} g_{\rho\sigma}\bar{\nu}(k')\gamma^\rho \nu(k)\bar{u}(p)\gamma^\sigma u(p')\right) \\ &= \sum_{\text{all spins}} \frac{e^4}{4t^2} g_{\mu\nu}g_{\rho\sigma}\bar{u}(p')\gamma^\mu u(p)\bar{\nu}(k)\gamma^\nu v(k')\bar{\nu}(k')\gamma^\rho \nu(k)\bar{u}(p)\gamma^\sigma u(p') \\ &= \sum_{\text{all spins}} \frac{e^4}{4t^2} g_{\mu\nu}g_{\rho\sigma}\bar{u}_a(p')\gamma^u_{ab}u_b(p)\bar{\nu}_c(k)\gamma^\nu_{cd}\nu_d(k')\bar{\nu}_e(k')\gamma^\rho_{ef}\nu_f(k)\bar{u}_g(p)\gamma^\rho_{gh}u_h(p') \\ &= (-1)^{14}\sum_{\text{all spins}} \frac{e^4}{4t^2} g_{\mu\nu}g_{\rho\sigma}\gamma^u_{ab}u_b(p)\bar{u}_g(p)\gamma^\rho_{gh}u_h(p')\bar{u}_a(p')\gamma^\nu_{cd}\nu_d(k')\bar{\nu}_e(k')\gamma^\rho_{ef}\nu_f(k)\bar{\nu}_c(k) \\ &= \frac{e^4}{4t^2} g_{\mu\nu}g_{\rho\sigma}\gamma^u_{ab}(\not{p}+m)_{bg}\gamma^\sigma_{gh}(\not{p}'+m)_{ha}\gamma^\nu_{cd}(\not{k}'-m)_{de}\gamma^\rho_{ef}(\not{k}-m)_{fc} \end{aligned}$$

$$= \frac{e^4}{4t^2} g_{\mu\nu} g_{\rho\sigma} \operatorname{tr}\left(\gamma^\mu(\not{p}+m)\gamma^\sigma(\not{p}'+m)\right) \operatorname{tr}\left(\gamma^\nu(\not{k}'-m)\gamma^\rho(\not{k}-m)\right)$$

where the factor $(-1)^{14}$ keeps track of the number of times we commute the fermion fields. The traces give

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\mu(\not{p}+m)\gamma^\sigma(\not{p}'+m)\right) &= \operatorname{tr}\left(\gamma^\mu\not{p}\gamma^\sigma\not{p}' + m\gamma^\mu\gamma^\sigma\not{p}' + m\gamma^\mu\not{p}\gamma^\sigma + m^2\gamma^\mu\gamma^\sigma\right) \\
&= \operatorname{tr}\left(\gamma^\mu\not{p}\gamma^\sigma\not{p}'\right) + m^2 \operatorname{tr}\left(\gamma^\mu\gamma^\sigma\right) \\
&= p_\lambda p'_\tau \operatorname{tr}\left(\gamma\mu\gamma^\lambda\gamma^\sigma\gamma^\tau\right) + 4m^2 g^{\mu\sigma} \\
&= 4p_\lambda p'_\lambda \left(g^{\mu\lambda}g^{\sigma\tau} - g^{\mu\sigma}g^{\lambda\tau} + g^{\mu\tau}g^{\lambda\sigma}\right) + 4m^2 g^{\mu\sigma} \\
&= 4\left(p^\mu p'^\sigma - (p\cdot p')g^{\mu\sigma} + p'^\mu p\sigma + m^2 g^{\mu\sigma}\right) \\
\operatorname{tr}\left(\gamma^\nu(\not{k}'-m)\gamma^\rho(\not{k}-m)\right) &= 4\left(k'^\nu k^\rho - (k\cdot k')g^{\nu\rho} + k^\nu k'\rho + m^2 g^{\nu\rho}\right)
\end{aligned}$$

Combining these results,

$$\begin{aligned}
\frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}^t|^2 &= \frac{e^4}{4t^2} g_{\mu\nu} g_{\rho\sigma} \left(p^\mu p'^\sigma - (p\cdot p')g^{\mu\sigma} + p'^\mu p\sigma + m^2 g^{\mu\sigma}\right)\left(k'^\nu k^\rho - (k\cdot k')g^{\nu\rho} + k^\nu k'\rho + m^2 g^{\nu\rho}\right) \\
&= \frac{e^4}{4t^2} p_\nu p'_\rho \left(k'^\nu k^\rho - (k\cdot k')g^{\nu\rho} + k^\nu k'\rho + m^2 g^{\nu\rho}\right) \\
&\quad - \frac{e^4}{4t^2}(p\cdot p')\left(k'^\nu k^\rho - (k\cdot k')g^{\nu\rho} + k^\nu k'\rho + m^2 g^{\nu\rho}\right) \\
&\quad + \frac{e^4}{4t^2} p'_\nu p_\rho \left(k'^\nu k^\rho - (k\cdot k')g^{\nu\rho} + k^\nu k'\rho + m^2 g^{\nu\rho}\right) \\
&\quad + \frac{4m^2e^4}{t^2} g_{\nu\rho} \left(k'^\nu k^\rho - (k\cdot k')g^{\nu\rho} + k^\nu k'\rho + m^2 g^{\nu\rho}\right)
\end{aligned}$$

Performing the remaining contractions,

$$\begin{aligned}
\frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}^t|^2 &= \frac{e^4}{4t^2}\left[(p\cdot k')(p'\cdot k) - (k\cdot k')(p\cdot p') + (p\cdot k)(p'\cdot k') + m^2(p\cdot p')\right] \\
&\quad + \frac{e^4}{4t^2}(p\cdot p')\left[(k\cdot k') - 4(k\cdot k') + (k\cdot k') + 4m^2\right] \\
&\quad + \frac{e^4}{4t^2}\left[(p'\cdot k')(p\cdot k) - (k\cdot k')(p\cdot p') + (p'\cdot k)(p\cdot k') + m^2(p\cdot p')\right] \\
&\quad + \frac{4m^2e^4}{t^2}\left[(k\cdot k') - 4(k\cdot k') + (k\cdot k') + 4m^2\right] \\
&= \frac{8e^4}{t^2}[(p\cdot k')(p'\cdot k) - (k\cdot k')(p\cdot p') + (p\cdot k)(p'\cdot k') + m^2(p\cdot p') + (m^2 - (p\cdot p')) \\
&\qquad\qquad (-(k\cdot k') + 2m^2)] \\
&= \frac{8e^4}{t^2}\left[(p\cdot k')(p'\cdot k) + (p\cdot k)(p'\cdot k') - m^2(p\cdot p') - m^2(k\cdot k') + 2m^4\right]
\end{aligned}$$

2.11.3 Matrix squared:first cross term

Now consider the first of the cross terms. We need to make sure we've assigned the same momenta to the four particles.But for the s channel we've set p'as the initial positron momentum instead of the final.

$$\begin{aligned}
\frac{1}{4}\sum_{\text{all spins}} \mathcal{M}^s\mathcal{M}^{t*} &= \frac{1}{4}\sum_{\text{all spins}} \left(\frac{ie^2}{s} g_{\alpha\beta}\bar{\nu}(k)\gamma^\alpha u(p)\bar{u}(p')\gamma^\beta v(k')\right)\left(-\frac{ie^2}{t} g_{\mu\nu}\bar{\nu}(k')\gamma^\nu \nu(k)\bar{u}(p)\gamma^\mu u(p')\right) \\
&= \frac{e^4}{4st} g_{\alpha\beta} g_{\mu\nu} \sum_{\text{all spins}} \bar{\nu}_a(k)\gamma^\alpha_{ab} u_b(p)\bar{u}_c(p')\gamma^\beta_{cd}\nu_d(k')\bar{\nu}_e(k')\gamma^\nu_{ef}\nu_f(k)\bar{u}_g(p)\gamma^\mu_{gh}u_h(p') \\
&= (-1)^{15}\frac{e^4}{4st} g_{\alpha\beta} g_{\mu\nu} \sum_{\text{all spins}} \gamma^\alpha_{ab} u_b(p)\bar{u}_g(p)\gamma^\beta_{cd}\nu_e(k')\bar{\nu}_e(k')\gamma^\nu_{ef}\nu_f(k)\bar{\nu}_a(k)\gamma^u_{gh}u_h(p')\bar{\nu}_c(p') \\
&= -\frac{e^4}{4st} g_{\alpha\beta} g_{\mu\nu} \not{k}_{fa}\gamma^\alpha_{ab}\not{p}_{bg}\gamma^\mu_{gh}\not{p}'_{hc}\gamma^\beta_{cd}\not{k}'_{de}\gamma^\nu_{ef} \\
&= -\frac{e^4}{4st} g_{\alpha\beta} g_{\mu\nu} \operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{k}'\gamma^\nu\right)
\end{aligned}$$

where we have dropped all mass terms. Notice that the fermion exchanges have introduced an overall sign.

Now evaluate the trace. First, we eliminate the free γ-matrices, using $\operatorname{tr}\left(g_{\alpha\beta}\gamma^\alpha\gamma^\beta\right) = \frac{1}{2}\operatorname{tr}\left(g_{\alpha\beta}\left\{\gamma^\alpha,\gamma^\beta\right\}\right) = 4$. To do this, we must bring the two relevant gamma matrices next to each other. Also, notice that

$$\begin{aligned}
\gamma^\alpha \not{p} &= p_\beta\gamma^\alpha\gamma^\beta\left(2p^\alpha - \not{p}\gamma^\alpha\right)\\
&= p_\beta\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right)\\
&= 2p^\alpha - \not{p}\gamma^\alpha
\end{aligned}$$

and for any of p, k, p', k'

$$\begin{aligned}
\not{p}\not{p} &= p_\alpha p_\beta\gamma^\alpha\gamma^\beta\\
&= \frac{1}{2}p_\alpha p_\beta\left\{\gamma^\alpha,\gamma^\beta\right\}\\
&= \frac{1}{2}p_\alpha p_\beta 2g^{\alpha\beta}\\
&= p^2\\
&= 0
\end{aligned}$$

The trace is

$$\frac{1}{4}\sum_{\text{all spins}}\mathcal{M}^s\mathcal{M}^{t*} = -\frac{e^4}{4st}g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{k}'\gamma^\nu\right)$$

Using conservation of momentum, we replace $k' = p + k - p'$,

$$\begin{aligned}
\frac{1}{4}\sum_{\text{all spins}}\mathcal{M}^s\mathcal{M}^{t*} &= -\frac{e^4}{4st}g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\left(\not{p}+\not{k}-\not{p}'\right)\gamma^\nu\right)\\
&= -\frac{e^4}{4st}(T_p + T_k - T_{p'})
\end{aligned}$$

where

$$\begin{aligned}
T_p &= g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{p}\gamma^\nu\right)\\
T_k &= g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{k}\gamma^\nu\right)\\
T_{p'} &= g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{p}'\gamma^\nu\right)\\
&= g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{p}'\gamma^\nu\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\right)
\end{aligned}$$

where we used the cyclic property on $T_{p'}$ to check that it is related to T_k by the cyclic substitution, $k \to p' \to p \to k$, and renaming the dummy Lorentz indices. We therefore only need to compute the first two.
For T_p

$$\begin{aligned}
T_p &= g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{p}\gamma^\nu\right)\\
&= g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\left(2p^\mu - \gamma^\mu\not{p}\right)\not{p}'\left(2p^\beta - \not{p}\gamma^\beta\right)\gamma^\nu\right)\\
&= 4tr(\not{k}\not{p}\not{p}'\not{p}) - 2g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}'\not{p}\gamma^\beta\not{p}\right)\\
&\quad - 2g_{\mu\nu}\operatorname{tr}\left(\not{k}\not{p}\gamma^\mu\not{p}\not{p}'\gamma^\nu\right) + g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\gamma^\mu\not{p}\not{p}'\not{p}\gamma^\beta\gamma^\nu\right)\\
&= 4\operatorname{tr}\left(\not{k}\not{p}(2(p\cdot p') - \not{p}\not{p}')\right) - 2g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\not{p}'\left(2p^\beta - \gamma^\beta\not{p}\right)\not{p}\right)\\
&\quad - 2g_{\mu\nu}\operatorname{tr}\left(\not{k}\not{p}(2p^\mu - \not{p}\gamma^\mu)\not{p}'\gamma^\nu\right) + g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\gamma^\mu(2(p\cdot p') - \not{p}\not{p}')\not{p}\gamma^\beta\gamma^\nu\right)
\end{aligned}$$

Simplifying, and using $p^2 = m^2 = 0$,

$$\begin{aligned}
T_p &= 8(p\cdot p')tr(\not{k}\not{p}) - 4\operatorname{tr}\left(\not{k}p^2\not{p}'\right) - 4\operatorname{tr}\left(\not{k}\not{p}\not{p}'\not{p}\right) + 2g_{\mu\nu}\left(\not{k}\gamma^\alpha\not{p}'\gamma^\beta p^2\right)\\
&\qquad - 4\operatorname{tr}\left(\not{k}\not{p}\not{p}'\not{p}\right) + 2g_{\mu\nu}\operatorname{tr}\left(\not{k}p^2\gamma^\mu\not{p}'\gamma^\nu\right) + g_{\alpha\beta}g_{\mu\nu}2(p\cdot p')\operatorname{tr}\left(\not{k}\gamma^\alpha\gamma^\mu\not{p}\gamma^\beta\gamma^\nu\right)\\
&\qquad - g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\gamma^\mu\not{p}'\not{p}\not{p}\gamma^\beta\gamma^\nu\right)\\
&= 32(p\cdot p')(k\cdot p) - 4\operatorname{tr}\left(\not{k}\not{p}\not{p}'\not{p}\right) - 4\operatorname{tr}\left(\not{k}\not{p}\not{p}'\not{p}\right) + 2(p\cdot p')g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\gamma^\mu\not{p}\gamma^\beta\gamma^\nu\right)\\
&= 32(p\cdot p')(k\cdot p) - 8\operatorname{tr}\left(\not{k}(2(p\cdot p') - \not{p}'\not{p})\not{p}\right) + 2(p\cdot p')g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\gamma^\mu\not{p}\gamma^\beta\gamma^\nu\right)
\end{aligned}$$

look at

$$g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left(\not{k}\gamma^\alpha\gamma^\mu\not{p}\gamma^\beta\gamma^\nu\right) = g_{\alpha\beta}g_{\mu\nu}\operatorname{tr}\left((2k^\alpha - \gamma^\alpha\not{k})\gamma^\mu\not{p}\left(2g^{\beta\nu} - \gamma^\nu\gamma^\beta\right)\right)$$

$$
\begin{aligned}
&= 4\,\mathrm{tr}\big(\not{k}\not{p}\big) - 2g_{\mu\nu}\,\mathrm{tr}\left(\gamma^\mu \not{p}\gamma^\nu \not{k}\right) - 2g_{\alpha\mu}\,\mathrm{tr}\left(\gamma^\alpha \not{k}\gamma^\mu \not{p}\right) \\
&\quad + 4g_{\mu\nu}\,\mathrm{tr}\left(\not{k}\gamma^\mu \not{p}\gamma^\nu\right) \\
&= 4\,\mathrm{tr}\big(\not{k}\not{p}\big) - (2+2-4)g_{\mu\nu}\,\mathrm{tr}\left(\gamma^\mu \not{p}\gamma^\nu \not{k}\right) \\
&= 16(k\cdot p)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
T_p &= 32(p\cdot p')(k\cdot p) - 16(p\cdot p')tr(\not{k}\not{p}) + 8tr(\not{k}\not{p}'\not{p}\not{p}) + 32(p\cdot p')(k\cdot p) \\
&= 32(p\cdot p')(k\cdot p) - 64(p\cdot p')(p\cdot k) + 32(p\cdot p')(k\cdot p) \\
&= 0
\end{aligned}
$$

Now compute T_k, which is easier because the repeated ks are closer:

$$
\begin{aligned}
T_k &= g_{\alpha\beta}g_{\mu\nu}\,\mathrm{tr}\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \not{k}\gamma^\nu\right) \\
&= g_{\alpha\beta}g_{\mu\nu}\,\mathrm{tr}\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta (2k^\nu - \gamma^\nu \not{k})\right) \\
&= 2g_{\alpha\beta}\,\mathrm{tr}\left(\not{k}\gamma^\alpha \not{p}\not{k}\not{p}'\gamma^\beta\right) - g_{\alpha\beta}g_{\mu\nu}\,\mathrm{tr}\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \gamma^\nu \not{k}\right) \\
&= 2g_{\alpha\beta}\,\mathrm{tr}\left((2k^\alpha - \gamma^\alpha \not{k})\not{p}\not{k}\not{p}'\gamma^\beta\right) \\
&= 4\,\mathrm{tr}\big(\not{p}\not{k}\not{p}'\not{k}\big) - 2g_{\alpha\beta}\,\mathrm{tr}\left(\gamma^\alpha \not{k}\not{p}\not{k}\not{p}'\gamma^\beta\right) \\
&= 4\,\mathrm{tr}\big(\not{p}\not{k}\not{p}'\not{k}\big) - 8\,\mathrm{tr}\big(\not{k}\not{p}\not{k}\not{p}'\big) \\
&= -4\,\mathrm{tr}\left(\not{p}\not{k}(2(p'\cdot k) - \not{k}\not{p}')\right) \\
&= -8(p'\cdot k)tr(\not{p}\not{k}) + 4tr(\not{p}\not{k}\not{k}\not{p}') \\
&= -32(p'\cdot k)(p\cdot k)
\end{aligned}
$$

and cycling $k \to p' \to p \to k$ we have

$$
T_{p'} = -32(p\cdot p')(k\cdot p')
$$

The full form of this cross term contribution to the squared matrix element it therefore,

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{all spins}} \mathcal{M}^s\mathcal{M}^{t*} &= -\frac{e^4}{4st}(T_p + T_k - T_{p'}) \\
&= -\frac{e^4}{4st}\left(0 - 32(p'\cdot k)(p\cdot k) + 32(p\cdot p')(k\cdot p')\right) \\
&= -\frac{8e^4}{st}\left((p\cdot p')(k\cdot p') - (p'\cdot k)(p\cdot k)\right)
\end{aligned}
$$

2.11.4 Matrix squared: second cross term

The second cross term, averaged and summed over spins is

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{all spins}} \mathcal{M}^t\mathcal{M}^{s*} &= \frac{1}{4}\sum_{\text{all spins}} \left(\frac{ie^2}{t}g_{\mu\nu}\bar{u}(p')\gamma^\mu u(p)\bar{\nu}(k)\gamma^\nu v(k')\right)\left(-\frac{ie^2}{s}g_{\mu\nu}\bar{\nu}(k')\gamma^\beta u(p')\bar{u}(p)\gamma^\alpha \nu(k)\right) \\
&= \frac{e^4}{4st}g_{\alpha\beta}g_{\mu\nu}\sum_{\text{all spin}} \bar{u}_a(p')\gamma^\nu_{ab}u_b(p)\bar{\nu}_c(k)\gamma^\nu_{cd}\nu_d(k')\bar{\nu}_e(k')\gamma^\beta_{ef}u_f(p')\bar{u}_g(p)\gamma^\alpha_{gh}\nu_h(k) \\
&= (-1)^{15}\frac{e^4}{4st}g_{\alpha\beta}g_{\mu\nu}\sum_{\text{all spins}} \gamma^\mu_{ab}u_b(p)\bar{u}_g(p)\gamma^\nu_{cd}\nu_d(k')\bar{\nu}_e(k')\gamma^\beta_{ef}u_f(p')\bar{u}_g(p)\gamma^\alpha_{gh}\nu_h(k) \\
&= -\frac{e^4}{4st}g_{\alpha\beta}g_{\mu\nu}\gamma^\mu_{ab}u_b(p)\bar{u}_g(p)\gamma^\alpha_{gh}\nu_h(k)\bar{\nu}_c(k)\gamma^\nu_{cd}\nu_d(k')\bar{\nu}_e(k')\gamma^\beta_{ef}u_f(p')\bar{u}_a(p') \\
&= -\frac{e^4}{4st}g_{\alpha\beta}g_{\mu\nu}\,\mathrm{tr}\left(\gamma^\mu \not{p}\gamma^\alpha \not{k}\gamma^\nu \not{k}'\gamma^\beta \not{p}'\right)
\end{aligned}
$$

We need

$$
\mathrm{tr}\left(\gamma^\mu \not{p}\gamma^\alpha \not{k}\gamma^\nu \not{k}'\gamma^\beta \not{p}'\right) = \mathrm{tr}\left(\not{p}\gamma^\alpha \not{k}\gamma^\nu \not{k}'\gamma^\beta \not{p}'\gamma^\mu\right)
$$

Compare this to the first cross term, where we calculated

$$
\frac{e^4}{4st}g_{\alpha\beta}g_{\mu\nu}\,\mathrm{tr}\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \not{k}'\gamma^\nu\right) = \frac{8e^4}{st}\left((p\cdot p')(k\cdot p') - (p'\cdot k)(p\cdot k)\right)
$$

These differ only by the exchange $p \leftrightarrow k, p' \leftrightarrow k'$. Therefore,

$$\frac{1}{4}\sum_{all\ spins}\mathcal{M}^t\mathcal{M}^{s*} = -\frac{8e^4}{st}\left((k\cdot k')(p\cdot k') - (k'\cdot p)(k\cdot p)\right)$$

2.11.5 Final matrix squared

The final matrix squared, averaged/summed over spins, is

$$\begin{aligned}
|i\mathcal{M}|^2 &= |i\mathcal{M}^s|^2 + \mathcal{M}^s\mathcal{M}^{t*} + \mathcal{M}^t\mathcal{M}^{s*} + |i\mathcal{M}^t|^2 \\
&= \frac{8e^4}{s^2}\left((p\cdot p')\,(k\cdot k') + (p'\cdot k)\,(p\cdot k') + m^2\,(k'\cdot p') + m^2\,(k\cdot p) + 2m^4\right) \\
&- \frac{8e^4}{st}\left((p\cdot p')(k\cdot p') - (p'\cdot k)(p\cdot k)\right) \\
&- \frac{8e^4}{st}\left((k\cdot k')(p\cdot k') - (k'\cdot p)(k\cdot p)\right) \\
&+ \frac{8e^4}{t^2}\left((p\cdot k')(p'\cdot k) + (p\cdot k)(p'\cdot k') - m^2(p\cdot p') - m^2(k\cdot k') + 2m^4\right)
\end{aligned}$$

Dropping the mass terms and combining,

$$\begin{aligned}
|i\mathcal{M}|^2 = &\frac{8e^4}{s^2}\left((p\cdot p')\,(k\cdot k') + (p'\cdot k)\,(p\cdot k')\right) + \frac{8e^4}{t^2}\left((p\cdot k')(p'\cdot k) + (p\cdot k)(p'\cdot k')\right) \\
&- \frac{8e^4}{st}\left((p\cdot p')(k\cdot p') - (p'\cdot k)(p\cdot k) + (k\cdot k')(p\cdot k') - (k'\cdot p)(k\cdot p)\right)
\end{aligned}$$

Something is wrong with the cross term

$$\begin{aligned}
s &= (k+p)^2 = (k'+p')^2 \approx 2k\cdot p \approx 2k'\cdot p' \\
t &= (k-k')^2 = (p-p')^2 \approx -2k\cdot k' \approx -2p\cdot p' \\
u &= (k-p')^2 = (p-k')^2 \approx -2k\cdot p' \approx -2k'\cdot p
\end{aligned}$$

where the approximations are for the high-energy (relativistic) limit.

2.11.6 Relativistic kinematics

Consider the collision of an electron on a positron in the CM frame. Then the momenta are

$$\begin{aligned}
p &= (E, \mathbf{p}) \\
k &= (E, -\mathbf{p}) \\
p' &= (E, \mathbf{p}') \\
k' &= (E, -\mathbf{p}')
\end{aligned}$$

Then, dropping masses, and setting $\mathbf{p}^2 = E^2 - m^2 = E^2$, all quantities may be expressed in terms of E and θ:

$$\begin{aligned}
p\cdot k &= E^2 + \mathbf{p}^2 \\
&= 2E^2 - m^2 \\
&= 2E^2 \\
p'\cdot k &= E^2 + \mathbf{p}\cdot\mathbf{p}' \\
&= E^2 + \mathbf{p}^2\cos\theta \\
&= E^2(1+\cos\theta) \\
k'\cdot k &= E^2 - \mathbf{p}\cdot\mathbf{p}' \\
&= E^2(1-\cos\theta) \\
p\cdot p' &= E^2(1-\cos\theta) \\
p\cdot k' &= E^2(1+\cos\theta) \\
p'\cdot k' &= E^2 + \mathbf{p}'\cdot\mathbf{p}' \\
&= 2E^2
\end{aligned}$$

$$\begin{aligned}
t &= (p-p')^2 \\
&= 2m^2 - 2p\cdot p' \\
&= -2E^2(1-\cos\theta) \\
s &= (p+k)^2 \\
&= 4E^2
\end{aligned}$$

Substituting into the squared matrix element,

$$\begin{aligned}
|i\mathcal{M}|^2 &= \frac{8e^4}{s^2}\left((p\cdot p')\,(k\cdot k') + (p'\cdot k)\,(p\cdot k')\right) + \frac{8e^4}{t^2}\left((p\cdot k')(p'\cdot k) + (p\cdot k)(p'\cdot k')\right) \\
&- \frac{8e^4}{st}\left((p\cdot p')(k\cdot p') - (p'\cdot k)(p\cdot k)\right) - \frac{8e^4}{st}\left((k\cdot k')(p\cdot k') - (k'\cdot p)(k\cdot p)\right) \\
&= \frac{8e^4}{4E^2 4E^2}\left(E^2(1-\cos\theta)E^2(1-\cos\theta) + E^2(1+\cos\theta)E^2(1+\cos\theta)\right) \\
&+ \frac{8e^4}{2E^2(1-\cos\theta)2E^2(1-\cos\theta)}\left(E^2(1+\cos\theta)E^2(1+\cos\theta) + 2E^2 2E^2\right) \\
&- \frac{8e^4}{4E^2(-2E^2(1-\cos\theta))}\left(E^2(1-\cos\theta)E^2(1+\cos\theta) - E^2(1+\cos\theta)2E^2\right) \\
&- \frac{8e^4}{4E^2(-2E^2(1-\cos\theta))}\left(E^2(1-\cos\theta)E^2(1+\cos\theta) - E^2(1+\cos\theta)2E^2\right)
\end{aligned}$$

Then the differential cross section is

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}|\mathcal{M}_{ji}|^2 \\
&= \frac{e^4}{64\pi^2 4E^2}\left[(1+\cos^2\theta) + \frac{2\left\{4 + (1+\cos\theta)^2\right\}}{(1-\cos\theta)^2}\right] \\
&\qquad\qquad + \frac{e^4}{64\pi^2 4E^2}\left[\frac{1}{(1-\cos\theta)}\left\{2(1-\cos^2\theta) - 4(1+\cos\theta)\right\}\right] \\
&= \frac{e^4}{64\pi^2 4E^2}\left[(1+\cos^2\theta) + \frac{2\left\{4+(1+\cos\theta)^2\right\}}{(1-\cos\theta)^2} + \frac{2(1-\cos^2\theta)}{1-\cos\theta} - \frac{4(1+\cos\theta)}{1-\cos\theta}\right] \\
&= \frac{e^4}{64\pi^2 4E^2}\left[\frac{1}{2}(1+\cos^2\theta) + \frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} + 2\cos^2\frac{\theta}{2} - \frac{2\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right] \\
&= \frac{e^4}{64\pi^2 4E^2}\left[\frac{1}{2}(1+\cos^2\theta) + \frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} - \frac{2\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right]
\end{aligned}$$

This is the ultra-relativistic limit of the Bhabha cross section (1936).

2.11.7 Traces of gamma matrices

Now compute the traces using the fundamental relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, and the cyclic property of the trace, $\mathrm{tr}(A...BC) = \mathrm{tr}(CA...B)$. First, we can show that the trace of the product of any odd number of γ-matrices vanishes by using $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma^\alpha\} = 0$,

$$\begin{aligned}
\mathrm{tr}\left(\underbrace{\gamma^\alpha ... \gamma^\beta}_{2n+1}\right) &= \mathrm{tr}\left(1\gamma^\alpha...\gamma^\beta\right) \\
&= \mathrm{tr}\left(\gamma_5\gamma_5\gamma^\alpha...\gamma^\beta\right) \\
&= -\,\mathrm{tr}\left(\gamma_5\gamma^\alpha\gamma_5...\gamma^\beta\right) \\
&= (-1)^{2n+1}\,\mathrm{tr}\left(\gamma_5\gamma^\alpha...\gamma^\beta\gamma_5\right) \\
&= (-1)^{2n+1}\,\mathrm{tr}\left(\gamma_5\gamma_5\gamma^\alpha...\gamma^\beta\right) \\
&= -\,\mathrm{tr}\left(\gamma^\alpha...\gamma^\beta\right) \\
&= 0
\end{aligned}$$

For even products, we will need traces of products of 2,4,6 and 8 gamma matrices.

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\right) &= \operatorname{tr}\left(-\gamma^\beta\gamma^\alpha + 2g^{\alpha\beta}I\right)\\
&= -\operatorname{tr}\left(\gamma^\beta\gamma^\alpha\right) + 2g^{\alpha\beta}\operatorname{tr}(I)\\
&= -\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\right) + 8g^{\alpha\beta}\\
\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\right) &= 4g^{\alpha\beta}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= \operatorname{tr}\left(\left(-\gamma^\beta\gamma^\alpha + 2g^{\alpha\beta}1\right)\gamma^\mu\gamma^\nu\right)\\
&= -\operatorname{tr}\left(\gamma^\beta\gamma^\alpha\gamma^\mu\gamma^\nu\right) + 2g^{\alpha\beta}\operatorname{tr}(\gamma^\mu\gamma^\nu)\\
&= -\operatorname{tr}\left(\gamma^\beta\left(-\gamma^\mu\gamma^\alpha + 2g^{\mu\alpha}\right)\gamma^\nu\right) + 2g^{\alpha\beta}\operatorname{tr}(\gamma^\mu\gamma^\nu)\\
&= \operatorname{tr}\left(\gamma^\beta\gamma^\mu\gamma^\alpha\gamma^\nu\right) - 2g^{\mu\alpha}\operatorname{tr}(\gamma\beta\gamma\nu) + 2g^{\alpha\beta}\operatorname{tr}(\gamma^\mu\gamma^\nu)\\
&= \operatorname{tr}\left(\gamma^\beta\gamma^\mu(-\gamma^\nu\gamma^\alpha) + 2g^{\nu\alpha}\right) - 2g^{\mu\alpha}\operatorname{tr}(\gamma\beta\gamma\nu) + 2g^{\alpha\beta}\operatorname{tr}(\gamma^\mu\gamma^\nu)\\
&= -\operatorname{tr}\left(\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha\right) + 2g^{\nu\alpha}\operatorname{tr}\left(\gamma^\beta\gamma^\mu\right) - 2g^{\mu\alpha}tr(\gamma\beta\gamma\nu) + 2g^{\alpha\beta}\operatorname{tr}(\gamma^\mu\gamma^\nu)\\
2\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= 2g^{\nu\alpha}\operatorname{tr}\left(\gamma^\beta\gamma^\mu\right) - 2g^{\mu\alpha}tr(\gamma\beta\gamma\nu) + 2g^{\alpha\beta}\operatorname{tr}(\gamma^\mu\gamma^\nu)\\
\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= 4g^{\nu\alpha}g^{\beta\mu} - 4g^{\mu\alpha}g^{\beta\nu} + 4g^{\alpha\beta}g^{\mu\nu}
\end{aligned}$$

For six, we use the simple pattern to more quickly find

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= \operatorname{tr}\left(\left(-\gamma^\beta\gamma^\alpha + 2g^{\alpha\beta}I\right)\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)\\
&= \operatorname{tr}\left(-\gamma^\beta\left(2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha\right)\gamma^\nu\gamma^\rho\gamma^\sigma + 2g^{\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)\\
&= \operatorname{tr}\Big(-\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha + 2g^{\alpha\sigma}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho - 2g^{\alpha\rho}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma + 2g^{\alpha\nu}\gamma^\beta\gamma^\mu\gamma^\rho\gamma^\sigma\\
&\qquad -2g^{\alpha\mu}\gamma^\beta\gamma^\nu\gamma^\rho\gamma^\sigma + 2g^{\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\Big)
\end{aligned}$$

ed commutation relation of α with ν, then with ρ and finally with σ

and from here we can use the result for the trace of four,

$$\begin{aligned}
2\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 2\operatorname{tr}\left(g^{\alpha\sigma}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho - g^{\alpha\rho}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma + g^{\alpha\nu}\gamma^\beta\gamma^\mu\gamma^\rho\gamma^\sigma - g^{\alpha\mu}\gamma^\beta\gamma^\nu\gamma^\rho\gamma^\sigma\right.\\
&\qquad \left. -g^{\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)\\
\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4g^{\alpha\sigma}\left(g^{\beta\mu}g^{\nu\rho} - g^{\beta\nu}g^{\mu\rho} + g^{\rho\beta}g^{\mu\nu}\right) - 4g^{\alpha\rho}\left(g^{\beta\mu}g^{\nu\sigma} - g^{\beta\nu}g^{\mu\sigma} + g^{\sigma\beta}g^{\mu\nu}\right)\\
&+ 4g^{\alpha\nu}\left(g^{\beta\mu}g^{\rho\sigma} - g^{\beta\rho}g^{\mu\sigma} + g^{\beta\sigma}g^{\mu\rho}\right) - 4g^{\alpha\mu}\left(g^{\beta\nu}g^{\rho\sigma} - g^{\beta\rho}g^{\nu\sigma} + g^{\beta\sigma}g^{\mu\rho}\right)\\
&+ 4g^{\alpha\beta}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)
\end{aligned}$$

or, perhaps more mnemonically,

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4g^{\alpha\beta}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) - 4g^{\alpha\mu}\left(g^{\beta\nu}g^{\rho\sigma} - g^{\beta\rho}g^{\nu\sigma} + g^{\beta\sigma}g^{\mu\rho}\right)\\
&+ 4g^{\alpha\nu}\left(g^{\beta\mu}g^{\rho\sigma} - g^{\beta\rho}g^{\mu\sigma} + g^{\beta\sigma}g^{\mu\rho}\right) - 4g^{\alpha\rho}\left(g^{\beta\mu}g^{\nu\sigma} - g^{\beta\nu}g^{\mu\sigma} + g^{\sigma\beta}g^{\mu\nu}\right)\\
&+ 4g^{\alpha\sigma}\left(g^{\beta\mu}g^{\nu\rho} - g^{\beta\nu}g^{\mu\rho} + g^{\rho\beta}g^{\mu\nu}\right)
\end{aligned}$$

Chapter 3

Statistical Mechanics in the Flat Spacetime

We will model the expanding universe problem as adiabatically expanding system. The choice of metric signature here would be $(+---)$. The quantity of interest will be given as:

$$\begin{aligned}
n &= J^{\mu}u_{\mu} \\
&= u_{\mu}\frac{g}{(2\pi\hbar)^3}\int_0^{\infty} f(\vec{p})\frac{d^3p}{p_0}p^{\mu} \\
&= \frac{g}{(2\pi\hbar)^3}\int f(\vec{p})d^3p \qquad \text{(using } E = p^{\mu}u_{\mu}\text{)}
\end{aligned}$$

Refer problem 5.34 of Lightman-Problem book in relativity or Chapter 7 of intro to relativistic stat mech by remi hakim. Ignoring interaction between the particles

$$\begin{aligned}
\rho &= \frac{g}{(2\pi\hbar)^3}\int E(\vec{p})f(\vec{p})d^3p \\
P &= \frac{g}{(2\pi\hbar)^3}\int \frac{|\vec{p}|^2c^2}{3E}f(\vec{p})d^3p
\end{aligned}$$

where g is the internal degree of freedom such as the number of spin states, color states etc, n is the number density of that particle species, ρ is the energy density and P is the pressure. All these quantities will depend upon the choice of distribution function $f(P)$ which we will assume to be fermi dirac distribution. Here we'd like to first notice that the functional form of fermi dirac distribution is same in all ensemble. What changes based on ensemble is functional form of other parameter such as Energy, Pressure etc. Therefore, we will first defined what we mean by pressure and then derive the fermi dirac distribution in all three ensemble.

Consider a particle bouncing back and forth between two walls parallel to $yz-$plane separated by length L_x. The momentum of the particle in the normalized direction of the walls is p_x, so a momentum change of $2p_x$ occurs on every collision. The velocity of the particle in the direction is $v_x = c^2p_x/E$ because

$$p_x = \gamma m v_x, \quad E = \gamma m c^2.$$

Then the time between collisions is $2L_x/v_x$. The average force[1] on the wall is

$$F_{\text{av}} = \frac{\int F_x dt}{\int dt} = \frac{\Delta p_x}{\Delta t} = \frac{2p_x}{2L_xE/p_xc^2} = \frac{1}{L_x}\frac{c^2p_x^2}{E}.$$

Dividing both sides by the area of the walls, L_yL_z, we have the pressure

$$P = \frac{1}{V}\frac{c^2p_x^2}{E} = \frac{1}{V}\frac{c^2p^2}{3E}.$$

Here, we use the relation which holds for average: $\langle p^2\rangle = \langle p_x^2\rangle + \langle p_y^2\rangle + \langle p_z^2\rangle = 3\langle p_x^2\rangle$. Finally, we sum over all the particles by integrating over the phase space density, giving

$$P_{\text{tot}} = \int d\mathbf{p}\,\frac{f(\mathbf{p})}{(2\pi\hbar)^3}\,\frac{c^2p_x^2}{E} = \int d\mathbf{p}\,\frac{f(\mathbf{p})}{(2\pi\hbar)^3}\,\frac{c^2p^2}{3E}.$$

[1]These are not average four-force

Since the integration sums over all the particles, we can use the relation which only holds for average. The approach we used here is that, we first observed the dependence of Pressure on momentum and energy, we then decided to evaluate the average of p^2/E, rather than using the Thermodynamic connection through partition function. In the derivation above, we have ignored the expansion of the universe.

The covarient form of the above quantities in general relativity can be found in chapter 4 of "Introduction to Relativistic Statistical Mechanics" by Remi Hakim

$$T^{\mu\nu} = \int_{p^2=m^2} \sqrt{|g|}\frac{p^\mu p^\nu}{p_0} d^3p f(p)$$

Sometimes it is also written as[2]

$$T^{\mu\nu} \int_{p^2=m^2} \frac{p^\mu p^\nu}{\sqrt{|g|}p^0} dp_1 dp_2 dp_3 f(p)$$

where $\sqrt{|g|}$ is the absolute value of determinant of metric tensor $g_{\mu\nu}$. Difference between the two integration measure is in index placement.

3.1 Ensembles

Now we derive the fermi dirac distribution in all three ensemble even though we will approximate the Early Universe via Grand Canonical Ensemble in locally flat spacetime. It is useful to see that the distribution is same in all three ensemble so that we could appreciate how using different ensemble wouldn't have made much difference. Following part are taken from wikipedia:

3.1.1 Grand Canonical Ensemble

In this ensemble, the system is able to exchange energy and exchange particles with a reservoir (temperature T and chemical potential μ fixed by the reservoir).

Due to the non-interacting quality, each available single-particle level (with energy level ϵ) forms a separate thermodynamic system in contact with the reservoir. In other words, each single-particle level is a separate, tiny grand canonical ensemble. By the Pauli exclusion principle, there are only two possible microstates for the single-particle level: no particle (energy $E = 0$), or one particle (energy $E = \epsilon$). The resulting partition function for that single-particle level therefore has just two terms:

$$\begin{aligned}\mathcal{Z} &= \exp\big(0(\mu-\epsilon)/k_\mathrm{B}T\big) + \exp\big(1(\mu-\epsilon)/k_\mathrm{B}T\big)\\ &= 1 + \exp\big((\mu-\epsilon)/k_\mathrm{B}T\big),\end{aligned}$$

and the average particle number for that single-particle level substate is given by

$$\begin{aligned}\langle N\rangle &= k_\mathrm{B}T\frac{1}{\mathcal{Z}}\left(\frac{\partial\mathcal{Z}}{\partial\mu}\right)_{V,T}\\ &= \frac{1}{\exp\big((\epsilon-\mu)/k_\mathrm{B}T\big)+1}.\end{aligned}$$

This result applies for each single-particle level, and thus gives the Fermi–Dirac distribution for the entire state of the system. The variance in particle number (due to thermal fluctuations) may also be derived (the particle number has a simple Bernoulli distribution):

$$\big\langle(\Delta N)^2\big\rangle = k_\mathrm{B}T\left(\frac{d\langle N\rangle}{d\mu}\right)_{V,T} \tag{3.1}$$

$$= \langle N\rangle\,(1-\langle N\rangle). \tag{3.2}$$

This quantity is important in transport phenomena such as the Mott relations for electrical conductivity and thermoelectric coefficient for an electron gas, where the ability of an energy level to contribute to transport phenomena is proportional to $\big\langle(\Delta N)^2\big\rangle$.

[2] exercise 15, pg 56 of Modern Cosmology by Scott Dodelson

3.1.2 Canonical ensemble

It is also possible to derive Fermi–Dirac statistics in the canonical ensemble. Consider a many-particle system composed of N identical fermions that have negligible mutual interaction and are in thermal equilibrium. Since there is negligible interaction between the fermions, the energy E_R of a state R of the many-particle system can be expressed as a sum of single-particle energies,

$$E_R = \sum_r n_r \varepsilon_r$$

where n_r is called the occupancy number and is the number of particles in the single-particle state r with energy ε_r. The summation is over all possible single-particle states r.

The probability that the many-particle system is in the state R, is given by the normalized canonical distribution,

$$P_R = \frac{e^{-\beta E_R}}{\sum_{R'} e^{-\beta E_{R'}}}$$

where $\beta = 1/k_\mathrm{B}T$ in $e^{-\beta E_R}$ is called the Boltzmann factor, and the summation is over all possible states R' of the many-particle system. The average value for an occupancy number n_i is

$$\langle n_i \rangle = \sum_R n_i P_R$$

Note that the state R of the many-particle system can be specified by the particle occupancy of the single-particle states, i.e. by specifying n_1, $n_2, \ldots$, so that

$$P_R = P_{n_1,n_2,\ldots} = \frac{e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \cdots)}}{\displaystyle\sum_{n_1{}',n_2{}',\ldots} e^{-\beta(n_1{}'\varepsilon_1 + n_2{}'\varepsilon_2 + \cdots)}} \tag{3.3}$$

and the equation for $\langle n_i \rangle$ becomes

$$\begin{aligned}\langle n_i \rangle &= \sum_{n_1,n_2,\ldots} n_i\ P_{n_1,n_2,\ldots} \\ &= \frac{\displaystyle\sum_{n_1,n_2,\ldots} n_i\ e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \cdots + n_i\varepsilon_i + \cdots)}}{\displaystyle\sum_{n_1,n_2,\ldots} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \cdots + n_i\varepsilon_i + \cdots)}}\end{aligned}$$

where the summation is over all combinations of values of $n_1, n_2, \ldots$ which obey the Pauli exclusion principle, and $n_r = 0$ or 1 for each r. Furthermore, each combination of values of $n_1, n_2, \ldots$ satisfies the constraint that the total number of particles is N,

$$\sum_r n_r = N.$$

Rearranging the summations,

$$\langle n_i \rangle = \frac{\displaystyle\sum_{n_i=0}^{1} n_i\ e^{-\beta(n_i\varepsilon_i)} \qquad \sum_{n_1,n_2,\ldots} {}^{(i)} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \cdots)}}{\displaystyle\sum_{n_i=0}^{1} e^{-\beta(n_i\varepsilon_i)} \qquad \sum_{n_1,n_2,\ldots} {}^{(i)} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \cdots)}}$$

where the ${}^{(i)}$ on the summation sign indicates that the sum is not over n_i and is subject to the constraint that the total number of particles associated with the summation is $N_i = N - n_i$. Note that $\sum^{(i)}$ still depends on n_i through the N_i constraint, since in one case $n_i = 0$ and $\sum^{(i)}$ is evaluated with $N_i = N$, while in the other case $n_i = 1$ and $\sum^{(i)}$ is evaluated with $N_i = N - 1$. To simplify the notation and to clearly indicate that $\sum^{(i)}$ still depends on n_i through $N - n_i$, define

$$Z_i(N - n_i) \equiv \sum_{n_1,n_2,\ldots} {}^{(i)} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \cdots)}. \tag{3.4}$$

So that the previous expression for $\bar{n}_i$ can be rewritten and evaluated in terms of the Z_i,

$$\bar{n}_i = \frac{\sum_{n_i=0}^{1} n_i\ e^{-\beta(n_i\varepsilon_i)}\ Z_i(N - n_i)}{\sum_{n_i=0}^{1} e^{-\beta(n_i\varepsilon_i)}\ Z_i(N - n_i)}$$

$$= \frac{0 + e^{-\beta\varepsilon_i} Z_i(N-1)}{Z_i(N) + e^{-\beta\varepsilon_i} Z_i(N-1)}$$
$$= \frac{1}{[Z_i(N)/Z_i(N-1)]e^{\beta\varepsilon_i} + 1}.$$

The following approximation will be used to find an expression to substitute for $Z_i(N)/Z_i(N-1)$:

$$\ln Z_i(N-1) \simeq \ln Z_i(N) - \frac{\partial \ln Z_i(N)}{\partial N}$$
$$= \ln Z_i(N) - \alpha_i$$

where $\alpha_i \equiv \frac{\partial \ln Z_i(N)}{\partial N}$. If the number of particles N is large enough so that the change in the chemical potential μ is very small when a particle is added to the system, then $\alpha_i \simeq -\mu/k_\mathrm{B}T$. Taking the base e antilog of both sides, substituting for α_i, and rearranging,

$$Z_i(N)/Z_i(N-1) = e^{-\mu/k_B T}$$

Substituting the above into the equation for $\bar{n}_i$, and using a previous definition of β to substitute $1/k_B T$ for β, results in the Fermi–Dirac distribution.

$$\langle n_i \rangle = \frac{1}{e^{(\varepsilon_i - \mu)/k_B T} + 1}$$

Like the Maxwell–Boltzmann distribution and the Bose–Einstein distribution the Fermi–Dirac distribution can also be derived by the Darwin–Fowler method of mean values. **Unlike** Grand Canonical Ensemble, the chemical potential in canonical ensemble is not an independent property. It becomes a function of temperature. Let me explain

Let us say we keep μ fixed and change the temperature. The mean number of particles in the system changes when temperature changes. To restore the value of $\langle N \rangle$ to the fixed value N, we change the chemical potential. This implies μ is not any more an independent property; it is a function of T. I must say there is nothing unphysical about this strategy. We are studying a physical system enclosed by a non-permeable wall - a wall that does not permit particle exchange. The chemical potential μ, is a well defined property of the system.It is just that μ is not any more under our control. The system automatically selects the value of μ depending on the temperature.

3.1.3 Microcanonical ensemble

A result can be achieved by directly analyzing the multiplicities of the system and using Lagrange multipliers

Suppose we have a number of energy levels, labeled by index i, each level having energy ε_i and containing a total of n_i particles. Suppose each level contains g_i distinct sublevels, all of which have the same energy, and which are distinguishable. For example, two particles may have different momenta (i.e., their momenta may be along different directions), in which case they are distinguishable from each other, yet they can still have the same energy. The value of g_i associated with level i is called the "degeneracy" of that energy level. The Pauli exclusion principle states that only one fermion can occupy any such sublevel.

The number of ways of distributing n_i indistinguishable particles among the g_i sublevels of an energy level, with a maximum of one particle per sublevel, is given by the binomial coefficient, using its combinatorial interpretation:

$$w(n_i, g_i) = \frac{g_i!}{n_i!(g_i - n_i)!}.$$

For example, distributing two particles in three sublevels will give population numbers of 110, 101, or 011 for a total of three ways which equals 3!/(2!1!).

The number of ways that a set of occupation numbers n_i can be realized is the product of the ways that each individual energy level can be populated:

$$W = \prod_i w(n_i, g_i) = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}.$$

Following the same procedure used in deriving the Maxwell–Boltzmann statistics, we wish to find the set of n_i for which W is maximized, subject to the constraint that there be a fixed number of particles, and a fixed energy. We constrain our solution using Lagrange multipliers forming the function:

$$f(n_i) = \ln(W) + \alpha\left(N - \sum n_i\right) + \beta\left(E - \sum n_i \varepsilon_i\right).$$

Using Stirling's approximation for the factorials, taking the derivative with respect to n_i, setting the result to zero, and solving for n_i yields the Fermi–Dirac population numbers:

$$n_i = \frac{g_i}{e^{\alpha+\beta\varepsilon_i}+1}.$$

By a process similar to that outlined in the Maxwell–Boltzmann statistics article, it can be shown thermodynamically that $\beta = \frac{1}{k_\mathrm{B}T}$ and $\alpha = -\frac{\mu}{k_\mathrm{B}T}$, so that finally, the probability that a state will be occupied is:

$$\langle n_i\rangle = \frac{n_i}{g_i} = \frac{1}{e^{(\varepsilon_i-\mu)/k_\mathrm{B}T}+1}.$$

The chemical potential has different meanings in different ensembles. For the microcanonical ensemble, μ/T is a Lagrange multiplier (related to the constraint of having a fixed number of particles N) and represents the change of entropy S when a particle is added (at constant energy E and volume V). It is given by

$$\frac{\mu}{T} = -\left(\frac{\partial S}{\partial N}\right)_{E,V}$$

where T is temperature. In the canonical ensemble μ represents a change of free energy F when a particle is added (at constant T and V), and is given by

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$

Note that in the canonical and microcanonical ensembles N is usually an integer, so the derivatives are differences and are exactly valid in the thermodynamic limit.

Now, instead of doing the same for Bose Einstein statistics, we will assume the same to be true. Therefore the Bose-Einstein distribution is given as:

$$\langle n_i\rangle = \frac{1}{e^{(\varepsilon_i-\mu)/k_\mathrm{B}T}-1}.$$

Chapter 4

Thermal History of the Early Universe

In the beginning, all forces were unified, resulting in effectively one force. As gravity split off during the Planck era, there were two forces to consider: gravity and the GUT (Grand Unified Theory) force. Since gravity is sensitive to everything, the slow roll of inflaton or symmetry breaking of the GUT triggered inflation.

By the end of inflation or shortly thereafter, the strong force had broken away from the GUT. At this moment, the standard big bang cosmology begins, with three fundamental forces: gravity, the strong force, and the electroweak force.

The early universe can be characterized by different eras based on physical processes and dominant forms of energy or matter. These classifications can be done in two main ways: chronologically, based on significant events, and based on the dominant energy or matter content. Here is a breakdown of both methods:

Chronological Classification

1. **Plank Era**: The earliest phase of the universe, where quantum gravitational effects are significant, and our current understanding of physics is insufficient to describe it accurately.

2. **Grand Unification**: The era when the strong, weak, and electromagnetic forces are believed to have been unified into a single force i.e. Strong Interaction, Weak Interaction and Electromagnetic interaction behaved the same. Later on Strong force splits off and inflation kickstarts[1] which leads to particle creation via pair production setting the primordial matter-antimatter asymmetry as the initial condition.

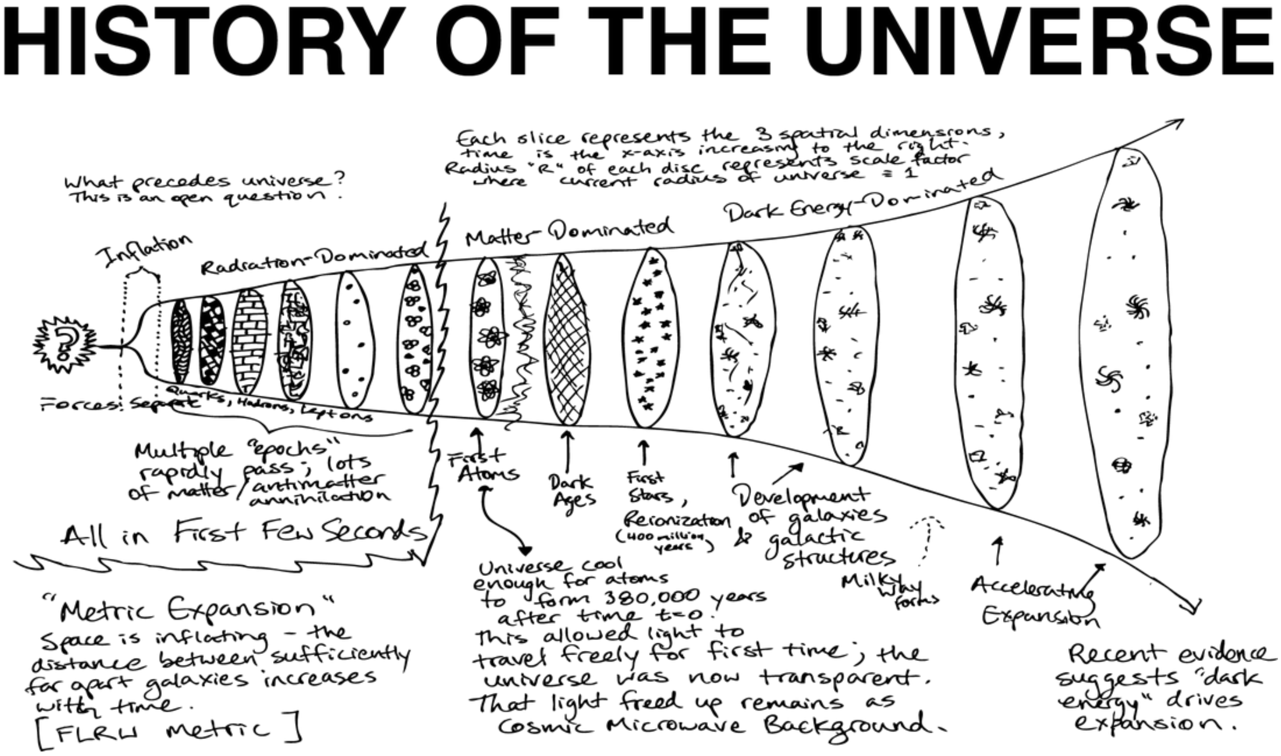
3. **Inflation**: A period of extremely rapid exponential expansion of the universe driven by a high-energy vacuum state which made the universe colder (but still incredibly hot), with a rapid expansion (first 10^{-12} seconds). Inflation was triggered by GUT symmetry breaking and at the end of inflation, strong force became distinct from electroweak force.

4. **Reheating**: Made it hotter again, but at a much lower temperature than the early universe (first 20 minutes) because the large potential energy of the inflaton field decays into particles and fills the Universe with Standard Model particles, including electromagnetic radiation, starting the radiation dominated phase of the Universe. This process established thermal equilibrium and hence allows us to talk about temperature. In the ideal, instaneous reheating case, the temperature of the universe was about 10^{15}GeV. Hence the electroweak symmetry breaking hasn't taken place yet.

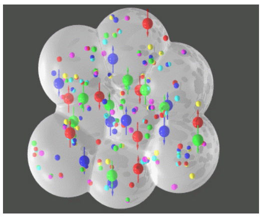
5. **Early Thermalization**: Some time after inflation, the created particles went through thermalization, where mutual interactions lead to thermal equilibrium. The electromagnetic and weak interaction have not yet separated, and the gauge bosons and fermions have not yet gained mass through the Higgs mechanism.

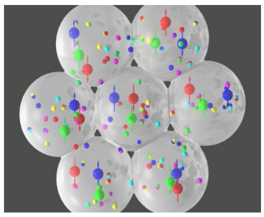
6. **Baryogensis**: Since, QFT assumes that particle and anti-particles are always created in equal amounts which is in contradiction with the cosmological observation. Thus, the baryogenesis tries to propose a process through which baryon asymmetry (an excess of matter over antimatter) is generated without assuming a primordial matter-antimatter asymmetry as an initial condition.

[1] Alan Guth believed that the symmetry breaking transition when the strong force separated from the electroweak force. If this is a first order transition then it is possible to get supercooling so the temperature of the universe fell to below where the symmetry breaking would normally occur. When this happens it's easy to show (see Guth's book) that the supercooled state forms a false vacuum that behaves like a negative pressure and causes exponential expansion



7. **QCD Phase Transition**: The quark epoch begins with the strong force being "frozen out" or separating off from the electroweak force. The strong force is responsible for quarks binding together into nucleons. When the strong force became frozen out, the matter transformed from a quark-gluon plasma (left diagram) into hadronic matter (right diagram), made of individual nucleons, each separately made up of quarks and gluons.





8. **Electroweak Era** or **Spontaneous symmetry breaking**: Strong interaction and weak interaction started behaving differently. At 100 GeV particles receive their masses through the Higgs mechanism. Above we have seen how this leads to a drastic change in the strength of the weak interaction. The spontaneous symmetry breaking introduced the mass scale and thus relativistic and non-relativistic particles entered the picture. Massless particles are always relativistic but massive are not.

9. **Decoupling**: As the universe expands the temperature of the thermal bath begins to drop and as the temperature of the thermal bath goes below the mass scale of the paricle, the particle species decouple from the thermal bath.

   - Since dark matter is very weakly interacting with ordinary matter it was the first to decouple from the thermal bath
   - Then neutrinos decoupled from the thermal bath since they only interact via weak interaction.
   - As the neutrino decouples, the temperature of thermal bath drops below the electron mass and electron-positron annihilation is trigged. Which transfers their energy to photons leading to higher photon temperature compared to neutrino temperature.

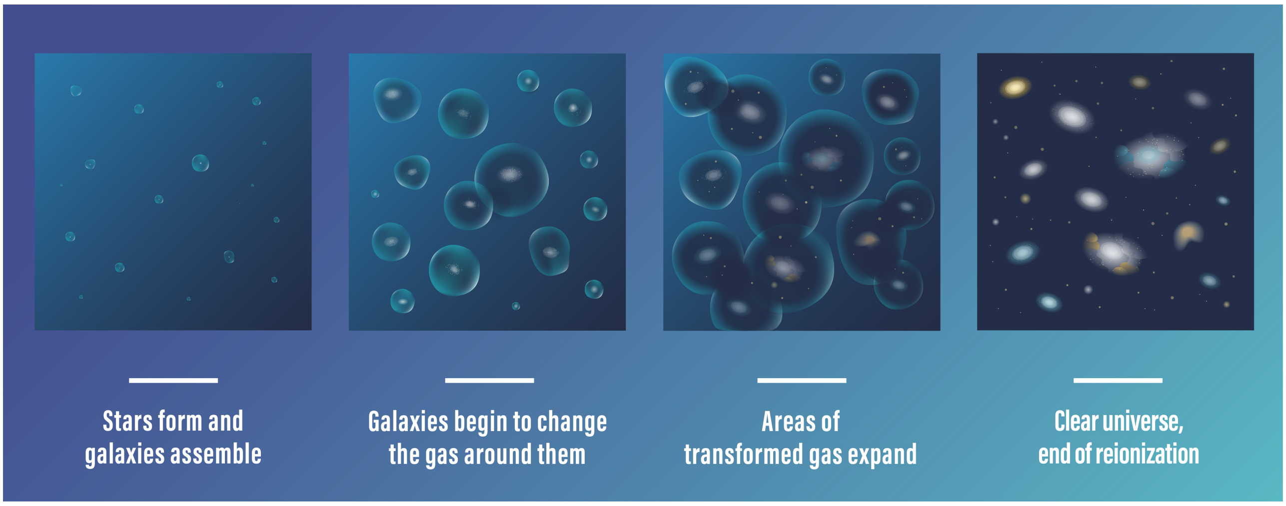
10. **Big Bang Nucleosynthesis**: Formation of light element nucleus, such as hydrogen and helium (first 20 minutes).

11. **Recombination** (misnomer): Formation of neutral atoms from ions (around 370,000 years).

12. **Photon Decoupling** or **Surface of last scattering**: Before recombination the strongest coupling between the photons and the rest of the plasma was through Thomson scattering. The sharp drop in the free electron density after recombination means that this process becomes inefficient and the photons decouple from the thermal bath. They have since streamed freely through the universe and are today observed as the cosmic microwave background (CMB).

13. **Dark Ages**: The universe was mostly plasma (ionized state), with no stars or galaxies (around 370,000 years to 1 billion years).

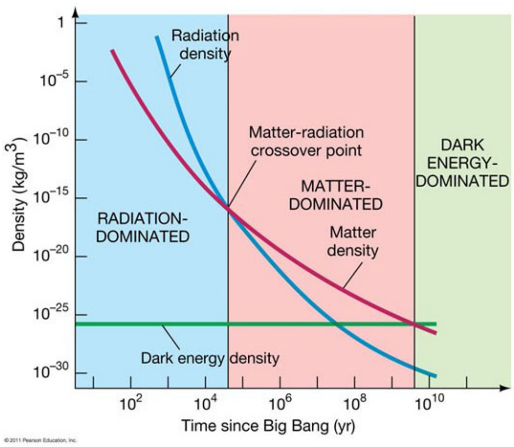
14. **Reionization**: The first star formed, whose ionizing radiation illuminated the universe (exact timeline being researched).



15. **Galaxy, Solar System Formation**: The formation of galaxies, stars, and our Solar System (around 1 billion years to present).

Classification Based on Dominant Energy or Matter:

This is the classification, which we will use to specify the state of the universe where we will be doing computations.



1. **Inflation**: Dominated by the inflaton field and associated vacuum energy, leading to rapid expansion i.e. $\ddot{a} > 0$.

2. **Reheating**: End of inflation, and beginning of Radiation dominated era where spacetime transition from exponentially expanding state to a slower, power-law expansion typical of a radiation-dominated universe.

3. **Radiation Dominated**: The universe's energy density is dominated by radiation (photons and neutrinos). This era includes the epochs of reheating, baryogenesis, and nucleosynthesis.

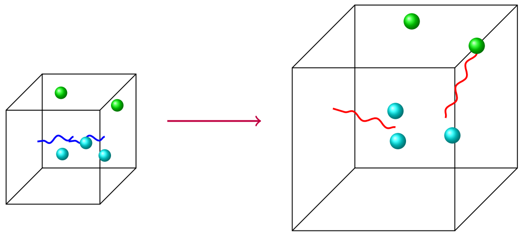


Figure 4.1: The idea here is that as the universe expanded, the density of matter decreased, since the matter was dispersed over a wider and wider region. The radiation energy density from light waves decreased as well, but it was also redshifted, so the radiation density decreased faster than the matter density. The matter-radiation crossover point on the timeline defines the point in time where matter began to dominate the universe.

4. **Matter Dominated**: The energy density of the universe becomes dominated by non-relativistic matter[2], primarily dark matter and ordinary matter (baryons). This era includes the formation of galaxies and large-scale structures.
5. **Dark Energy Dominated**: Dominated by dark energy, the expansion of the universe accelerates i.e. $\ddot{a} > 0$

We begin our study in Radiation Dominated Era, soon after reheating. The key concept in understanding the thermal history of the early universe is the interaction rate. First we brush up the definition of cross-section:

$$\begin{aligned}\sigma &= \frac{\text{number of particles scattered}}{\text{time} \times \text{number density in beam} \times \text{velocity of beam}}\\ &= \frac{1}{t}\frac{1}{n \times v_{\text{rel}}} N_{\text{scattered}}\end{aligned}$$

If we assume that there was only one indicent particle, then $P = \frac{N_{\text{scattered}}}{N_{\text{incident}}} = N_{\text{scattered}}$. Thus,

$$d\sigma = \frac{1}{t}\frac{1}{n \times v_{\text{rel}}} dP \implies dP = t(n \times v_{\text{rel}}) d\sigma$$

We define the interaction rate as:

$$\begin{aligned}\Gamma &= \frac{\mathrm{d}P}{\mathrm{d}t}\\ &= n\sigma v_{_{\text{rel}}}\end{aligned}$$

where, $v_{_{\text{rel}}}$ is the relative velocity between target and projectile. Now, we see that the typical timescale of interaction is given as

$$t_c \equiv \frac{1}{\Gamma} << t_H \equiv \frac{1}{H} \tag{coupled}$$

i.e. we are assuming that the time required for particles to interact is much smaller than the time universe takes to expand taking the particles away from each other. Particles are taking less time to interact with each other than the time universe requires to expand. Local thermal equilibrium is then reached before the effect of the expansion becomes relevant. As the universe cools, the rate of interactions may decrease faster than the expansion rate.

$$t_c \equiv \frac{1}{\Gamma} >> t_H \equiv \frac{1}{H} \tag{decoupled}$$

At $t_c \sim t_H$, the particles decouple[3] (falls out of equilibrium) from the primordial plasma/thermal bath by breaking the chemical equilibrium,

$$A + B \leftrightarrow C + D \xrightarrow{to} A + B \rightarrow C + D$$

In the equilibrium condition, once the temperature falls below the mass of the particle (non-relativistic limit) due to expansion of the universe, then we observe the exponential decay in number density which is interpreted

[2] If Higgs Mechanism hadn't taken place, both radiation and matter would have been composed of massless particles and both would have experienced redshift.

[3] just like coupling meant interaction and the coupling constant would decide the interaction strength, once there are no more interaction of that kind taking place, we call it decoupling

as particle-anti particle annihilation taking place as there isn't enough energy for particle-anti particle creation . At some point as the universe expands and cools, the particle and anti-particle become so dilute that they cannot find each other to annihilate. This is called "freeze-out". Different particle species may have different interaction rates and so may decouple at different times. A key note regarding decoupling and freeze-out is that, decoupling is an event when $t_c \sim t_H$ for each particle species respectively and freeze-out is a process that begins after decoupling. In the non-relativistic equilibrium thermodynamics, the number density sees an exponential decay however, once the decoupling takes place the equilibrium thermodynamics no longer applies. Then, based on Boltzmann Transport equation, in the limit $\Gamma_i << H$ for each particle species, we see that the number density eventually becomes constant. This is referred as freezeout.

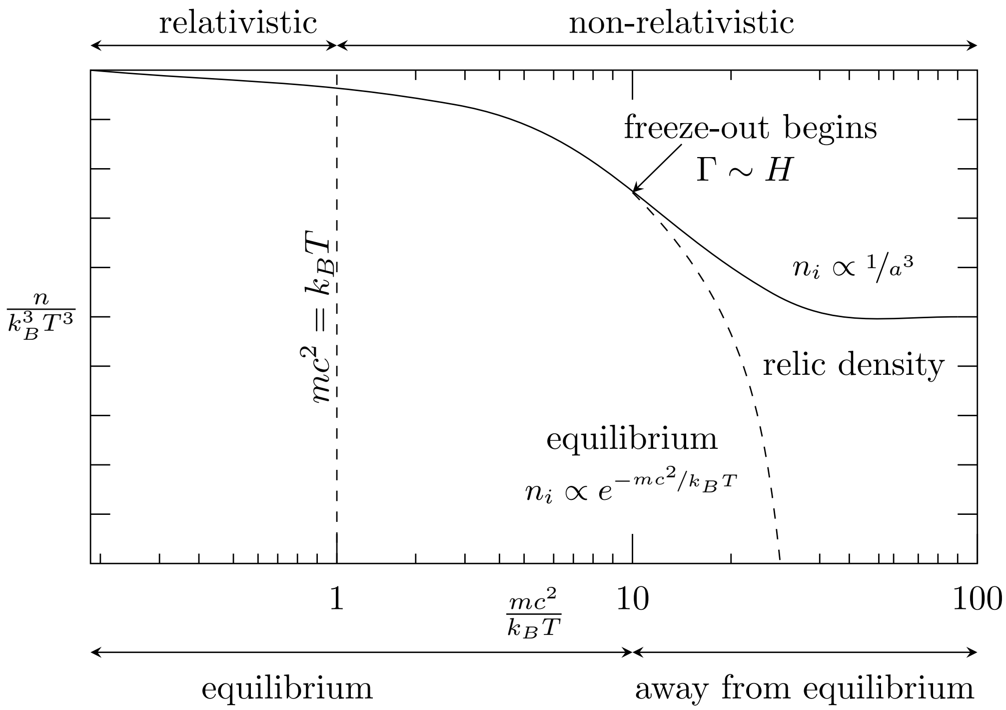


Figure 4.2: A schematic illustration of particle freeze-out. At high temperatures, $k_B T >> mc^2$, the particle abundance tracks its equilibrium value. At low temperatures, $k_B T << mc^2$, the particles freeze out and maintain a density that is much larger than the Boltzmann-suppressed equilibrium abundance.

We can numerically calculate the same using mathematica.[4] First we define the variables [5]

```
yeq0[z_] := 0.145*(g0/gs0)*z^(3/2)*E^(-z)
lam0 = (1.32)*(Sqrt[gs0])*m0*2.44*(10^18)*\[Sigma]0;
lam1 = (1.32)*(Sqrt[gs0])*m0*2.44*(10^18)*\[Sigma]1;
lam2 = (1.32)*(Sqrt[gs0])*m0*2.44*(10^18)*\[Sigma]2;
gs0 = 100;
g0 = 2;
c = 0.1;
c1 = 0.5;
c2 = 0.05;
m0 = 1000; \[Sigma]0 = c^2/m0^2;
\[Sigma]1 = c1^2/m0^2;
\[Sigma]2 = c2^2/m0^2;
```

Then, we solve the Boltzmann Transport Equation (from equation 3.34 of that paper)

```
sol1211 = NDSolve[{Y'[z] + lam0/z^2 (Y[z]^2 - yeq0[z]^2) == 0,
Y[0.01] == yeq0[0.01]}, {Y}, {z, 0.01, 5 10^2},
WorkingPrecision -> 20, AccuracyGoal -> 4, PrecisionGoal -> 1,
MaxStepSize -> 0.01, MaxSteps -> 10^6(*,MaxStepSize->0.0001,
MaxSteps->Infinity*)(*,AccuracyGoal->10,PrecisionGoal->10,
```

[4] the calculations are taken from "Defense Against the Dark Arts Notes on dark matter and particle physics" by Flip Tanedo
[5] yeq0 is defined in equation 3.38 and lam0 is defined in equation 3.36 of that paper

```
MaxStepSize->0.1,MaxSteps->10^5,WorkingPrecision->20*)]
sol1212 =
NDSolve[{Y'[z] + lam1/z^2 (Y[z]^2 - yeq0[z]^2) == 0,
Y[0.01] == yeq0[0.01]}, {Y}, {z, 0.01, 5 10^2},
WorkingPrecision -> 20, AccuracyGoal -> 4, PrecisionGoal -> 1,
MaxStepSize -> 0.01, MaxSteps -> 10^6(*,MaxStepSize->0.0001,
MaxSteps->Infinity*)(*,AccuracyGoal->10,PrecisionGoal->10,
MaxStepSize->0.1,MaxSteps->10^5,WorkingPrecision->20*)]
sol1213 =
NDSolve[{Y'[z] + lam2/z^2 (Y[z]^2 - yeq0[z]^2) == 0,
Y[0.01] == yeq0[0.01]}, {Y}, {z, 0.01, 5 10^2},
WorkingPrecision -> 20, AccuracyGoal -> 4, PrecisionGoal -> 1,
MaxStepSize -> 0.01, MaxSteps -> 10^6(*,MaxStepSize->0.0001,
MaxSteps->Infinity*)(*,AccuracyGoal->10,PrecisionGoal->10,
MaxStepSize->0.1,MaxSteps->10^5,WorkingPrecision->20*)]
```

After that, the last step is to plot.

```
ptl = Placed[
LineLegend[{Directive[Darker[Green, 0.15], Thickness[0.005]],
Directive[Darker[Blue, 0.15], Thickness[0.005]],
Directive[Darker[Magenta, 0.15],
Thickness[0.005]]}, {"\[Lambda] = 0.05", "\[Lambda] = 0.1",
"\[Lambda] = 0.5"},
LegendFunction -> (Framed[#, RoundingRadius -> 6,
FrameStyle -> Black] &), LegendLayout -> "Column",
LegendLabel -> Placed[Text[Style["", Directive[Black]]], Above],
LabelStyle ->
Directive[Black, FontFamily -> "Arial", Bold, FontSize -> 10],
LegendMarkerSize -> 20], {0.9, 0.8}];
LogLogPlot[{Y[z] /. sol1213, Y[z] /. sol1211, Y[z] /. sol1212,
yeq0[z]}, {z, 0.01, 500}, Frame -> True,
PlotStyle -> {{Green, Thick}, {Blue, Thick}, {Magenta,
Thick}, {Black, Dashed}}, AspectRatio -> 0.6, ImageSize -> Large,
Frame -> True, FrameStyle -> Directive[Black, 20],
FrameLabel -> {Style["z", Black, FontFamily -> "Arial",
FontSize -> 17, Bold],
Style["Y", Black, FontFamily -> "Arial", FontSize -> 17, Bold]},
PlotRange -> {{0.01, 5 10^2}, {10^-20, 10^-2}},
PlotLegends -> (# & /@ {ptl})]
```

The output becomes:

4.1 Equilibrium Thermodynamics

4.1.1 Using Fermi-Dirac or Bose-Einstein statistic

Energy Density

We will now, explicitly find the energy density, pressure and number density in the early universe.

$$
\begin{aligned}
\rho^{\text{eq}} &= \sum_{\text{all species}} \frac{g_i}{(2\pi\hbar)^3} \int_0^\infty \frac{\sqrt{p^2c^2+m^2c^4}}{\exp\left(\sqrt{p^2c^2+m^2c^4}-\mu/k_BT\right) \pm 1} 4\pi p^2 dp \\
&= \sum_{\text{boson}} \frac{g_i}{2\pi^2\hbar^3} \int_0^\infty dp \frac{p^2\sqrt{p^2c^2+m^2c^4}}{\exp\left(\sqrt{p^2c^2+m^2c^4}-\mu/k_BT\right) - 1} \\
&\qquad + \sum_{\text{fermion}} \frac{g_i}{2\pi^2\hbar^3} \int_0^\infty dp \frac{p^2\sqrt{p^2c^2+m^2c^4}}{\exp\left(\sqrt{p^2c^2+m^2c^4}-\mu/k_BT\right) + 1}
\end{aligned}
$$

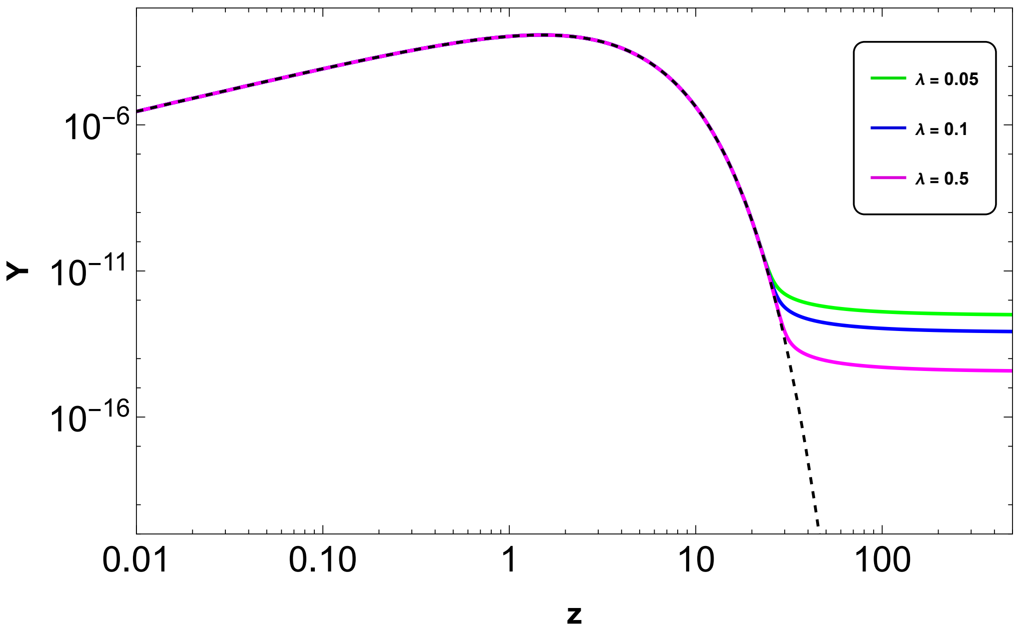


Figure 4.3: Plot of Yield as a function of redshift describing sensitivity of relic density over parameters of the theory.

defining $x \equiv {}^{mc^2}/_{k_BT}$ and $\xi \equiv {}^{pc}/_{k_BT}$, we get

$$= \sum_{\text{boson}} \frac{g_i}{2\pi^2\hbar^3} \int_0^\infty \frac{k_BT}{c} d\xi \frac{\frac{k_B^2T^2}{c^2}\xi^2\sqrt{k_B^2T^2\xi^2 + k_B^2T^2x^2}}{\exp\left(\sqrt{\xi^2+x^2} - {}^{\mu}/_{k_BT}\right) - 1} + \sum_{\text{fermion}} \frac{g_i}{2\pi^2\hbar^3} \int_0^\infty \frac{k_BT}{c} d\xi \frac{\frac{k_B^2T^2}{c^2}\xi^2\sqrt{k_B^2T^2\xi^2 + k_B^2T^2}}{\exp\left(\sqrt{\xi^2+x^2} - {}^{\mu}/_{k_BT}\right) + 1}$$

$$= \frac{k_B^4T^4}{2\pi^2\hbar^3c^3}\left[\sum_{\text{boson}} g_i \int_0^\infty d\xi \frac{\xi^2\sqrt{\xi^2+x^2}}{\exp\left(\sqrt{\xi^2+x^2} - {}^{\mu}/_{k_BT}\right) - 1} + \sum_{\text{fermion}} g_i \int_0^\infty d\xi \frac{\xi^2\sqrt{\xi^2+x^2}}{\exp\left(\sqrt{\xi^2+x^2} - {}^{\mu}/_{k_BT}\right) + 1}\right]$$

Ultra-Relativistic Limit In the early universe, $\mu \approx 0$. However, if we assume that $x \to 0$, then the integrals can be evaluated exactly and this condition is referred to as ultra-relativistic limit.

$$\begin{aligned}
\rho^{\text{eq}} &= \frac{k_B^4T^4}{2\pi^2\hbar^3c^3}\left[\sum_{\text{boson}} g_i \int_0^\infty d\xi \frac{\xi^3}{e^\xi - 1} + \sum_{\text{fermion}} g_i \int_0^\infty d\xi \frac{\xi^3}{e^\xi + 1}\right] \\
&= \frac{k_B^4T^4}{2\pi^2\hbar^3c^3}\left[\sum_{\text{boson}} g_i \int_0^\infty d\xi \frac{\xi^3}{e^\xi - 1} + \sum_{\text{fermion}} g_i \int_0^\infty d\xi \frac{\xi^3}{e^\xi - 1} - \sum_{\text{fermion}} g_i \int_0^\infty d\xi \frac{2\xi^3}{e^{2\xi} - 1}\right] \\
&= \frac{k_B^4T^4}{2\pi^2\hbar^3c^3}\left\{\sum_{\text{boson}} g_i\zeta(3+1)\Gamma(3+1) + \sum_{\text{fermion}} g_i\zeta(3+1)\Gamma(3+1) - \frac{g_i}{8}\zeta(3+1)\Gamma(3+1)\right\} \\
&= \frac{k_B^4T^4}{2\pi^2\hbar^3c^3}\left\{\sum_{\text{boson}} g_i \frac{\pi^4}{15} + \sum_{\text{fermion}} g_i \left(1 - \frac{1}{8}\right)\frac{\pi^4}{15}\right\} \\
&= \frac{k_B^4T^4}{2\pi^2\hbar^3c^3}\frac{\pi^4}{15}\underbrace{\left\{\sum_{\text{boson}} g_i + \frac{7}{8}\sum_{\text{fermion}} g_i\right\}}_{g^\star} \\
&= \frac{\pi^2k_B^4T^4}{30\hbar^3c^3}g^\star
\end{aligned}$$

where $g^\star$ is referred as effective degree of freedom. Note that the above derivation only holds if $k_B T_i >> m_i c^2$ for the species of particle under consideration therefore only relativistic bosons and fermions contribute to it. As the universe was cooling down due to adiabatic expansion. Temperature of universe would eventually fall below the energy scale associated with the mass of particle, in that condition the relativistic particle approximation would start getting violated. Thus those non-relativistic species of particle would stop contributing to the effective degree of freedom.

Non-Relativistic Limit Here, we have to note that, when we convert the integral from momentum space to integral over E via $E^2 = (mc^2)^2 + (Pc)^2$, the lower limit can't be set to $E = 0$ since, the smallest value of E is mc^2. Here we assume $\xi << x$, so $\xi/x << 1$.

$$\begin{aligned}
\rho^{\text{eq}} &= \sum_{\text{all species}} g_i \frac{k_B^4 T^4}{2\pi^2\hbar^3 c^3} \int_0^\infty d\xi \frac{\xi^2\sqrt{\xi^2+x^2}}{e^{\sqrt{\xi^2+x^2}} \pm 1} \\
&= \sum_{\text{all species}} g_i \frac{k_B^4 T^4}{2\pi^2\hbar^3 c^3} \int_0^\infty d\xi \frac{\xi^2 x\sqrt{1+\xi^2/x^2}}{e^{x(1+\xi^2/x^2)^{1/2}}} \\
&= \sum_{\text{all species}} g_i \frac{k_B^4 T^4}{2\pi^2\hbar^3 c^3} \left[\int_0^\infty d\xi \xi^2 x e^{-x-\xi^2/2x} + \int_0^\infty d\xi \frac{\xi^4}{2x} e^{-x-\xi^2/2x}\right] \\
&\approx \sum_{\text{all species}} g_i \frac{k_B^4 T^4}{2\pi^2\hbar^3 c^3} x e^{-x} \int_0^\infty d\xi \xi^2 e^{-\xi^2/2x} \\
&= \sum_{\text{all species}} g_i mc^2 \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3} e^{-mc^2/k_B T} \int_0^\infty d\xi \xi^2 e^{-\xi^2/2x} \qquad (\text{using } x = mc^2/k_B T) \\
&= \sum_{\text{all species}} mc^2 g_i \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-mc^2/k_B T}
\end{aligned}$$

The Number Density

The number density, we will define using $n_+ - n_-$, but first do the calculation without assuming the particle and anti-particle nature:

$$\begin{aligned}
n_i^{\text{eq}} &= \frac{g_i}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{\exp\left(\sqrt{p^2c^2+m^2c^4} - \mu/k_B T\right) \pm 1} 4\pi p^2 dp \\
&= \frac{g_i}{2\pi^2\hbar^3} \int_0^\infty dp \frac{p^2}{\exp\left(\sqrt{p^2c^2+m^2c^4} - \mu/k_B T\right) \pm 1}
\end{aligned}$$

defining $x \equiv mc^2/k_B T$ and $\xi \equiv pc/k_B T$, we get

$$\begin{aligned}
&= \frac{g_i}{2\pi^2\hbar^3} \int_0^\infty \frac{k_B T}{c} d\xi \frac{\frac{k_B^2 T^2}{c^2}\xi^2}{\exp\left(\sqrt{\xi^2+x^2} - \mu/k_B T\right) \pm 1} \\
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3} \int_0^\infty d\xi \frac{\xi^2}{\exp\left(\sqrt{\xi^2+x^2} - \mu/k_B T\right) \pm 1}
\end{aligned}$$

Ultra-relativistic Limit In the early universe, $\mu \approx 0$. At temperatures above about 10^{16} K, all particles of the Standard Model are highly relativistic, so that their masses can be neglected. Therefore we assume that $x \to 0$, the integrals can then be evaluated exactly and this condition is referred to as ultra-relativistic limit.

$$\begin{aligned}
n^{\text{eq}} &= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3} \int_0^\infty d\xi \frac{\xi^2}{\exp\left(\sqrt{\xi^2+x^2} - \mu/k_B T\right) \pm 1} \\
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3} \int_0^\infty d\xi \frac{\xi^2}{e^{\xi - \mu/k_B T} \pm 1}
\end{aligned}$$

for $\mu/k_B T << 1$

$$= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3} \begin{cases} \frac{3}{2}\zeta(3) & \text{for fermions} \\ \\ 2\zeta(3) & \text{for bosons} \end{cases}$$

we will now, evaluate the net number density:

$$n_+^{\rm eq} - n_-^{\rm eq} = g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\int_0^\infty d\xi \frac{\xi^2}{e^{\xi-\mu/k_BT}\pm 1} - \int_0^\infty d\xi \frac{\xi^2}{e^{\xi+\mu/k_BT}\pm 1}\right]$$

doing shift of variable $y = \xi - \mu/k_BT$ in first integral and $y = \xi + \mu/k_BT$ in the second integral, we have

$$\begin{aligned}
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\int_{-\mu/k_BT}^\infty dy \frac{(y+\mu/k_BT)^2}{e^y \pm 1} - \int_{\mu/k_BT}^\infty dy \frac{(y-\mu/k_BT)^2}{e^y \pm 1}\right] \\
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\int_{-\mu/k_BT}^\infty dy \frac{(y+\mu/k_BT)^2}{e^y \pm 1} - \int_{\mu/k_BT}^\infty dy \frac{(y-\mu/k_BT)^2}{e^y \pm 1}\right] \\
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\int_{-\mu/k_BT}^0 dy \frac{(y+\mu/k_BT)^2}{e^y \pm 1} + \int_0^\infty dy \frac{(y+\mu/k_BT)^2}{e^y \pm 1}\right. \\
&\qquad \left. - \left\{\int_0^\infty dy \frac{(y-\mu/k_BT)^2}{e^y \pm 1} - \int_0^{\mu/k_BT} dy \frac{(y-\mu/k_BT)^2}{e^y \pm 1}\right\}\right] \\
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\int_{-\mu/k_BT}^0 dy \frac{(y+\mu/k_BT)^2}{e^y \pm 1} + \int_0^{\mu/k_BT} dy \frac{(y-\mu/k_BT)^2}{e^y \pm 1}\right. \\
&\qquad \left. + \int_0^\infty dy \frac{(y+\mu/k_BT)^2 - (y-\mu/k_BT)^2}{e^y \pm 1}\right]
\end{aligned}$$

redefining $y \to -y$ in the first integral and using $a^2 - b^2 = (a+b)(a-b)$ in the last integral

$$\begin{aligned}
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\int_0^{\mu/k_BT} dy \frac{(-y+\mu/k_BT)^2}{e^{-y} \pm 1} + \int_0^{\mu/k_BT} dy \frac{(y-\mu/k_BT)^2}{e^y \pm 1}\right. \\
&\qquad \left. + \int_0^\infty dy \frac{(\mu/k_BT + \mu/k_BT)(y+y)}{e^y \pm 1}\right] \\
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\int_0^{\mu/k_BT} dy (y-\mu/k_BT)^2 + 4\frac{\mu}{k_BT}\int_0^\infty dy \frac{y}{e^y \pm 1}\right] \\
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\left.\frac{(y-\mu/k_BT)^3}{3}\right|_{y=0}^{y=\mu/k_BT} + 4\frac{\mu}{k_BT}\int_0^\infty dy \frac{y}{e^y \pm 1}\right] \\
&= g_i \frac{k_B^3 T^3}{2\pi^2\hbar^3 c^3}\left[\frac{\mu^3}{3k_B^3T^3} + \mathcal{I}\right]
\end{aligned}$$

We will now, explictely evaluate the last integral for fermion and boson. We use the following identity:

$$\begin{aligned}
\frac{1}{e^y+1} &= \frac{1}{e^y-1} + \text{something} \\
\text{something} &= \frac{1}{e^y+1} - \frac{1}{e^y-1} \\
&= \frac{e^y - 1 - e^y - 1}{(e^y+1)(e^y-1)} \\
&= -\frac{2}{e^{2y}-1}
\end{aligned}$$

hence,

$$\frac{1}{e^y+1} = \frac{1}{e^y-1} - \frac{2}{e^{2y}-1}$$

therefore,

$$\begin{aligned}
4\frac{\mu}{k_BT}\int_0^\infty dy \frac{y}{e^y+1} &= 4\frac{\mu}{k_BT}\left\{\int_0^\infty dy \frac{y}{e^y-1} - \int_0^\infty dy \frac{2y}{e^{2y}-1}\right\} \\
&= 4\frac{\mu}{k_BT}\left\{\Gamma(2)\zeta(2) - \frac{1}{2}\Gamma(2)\zeta(2)\right\} \\
&= 2\frac{\mu}{k_BT}\Gamma(2)\zeta(2) = \frac{\pi^2\mu}{3k_BT}
\end{aligned}$$

and for bosons,

$$4\frac{\mu}{k_BT}\int_0^\infty dy\frac{y}{e^y-1}=4\frac{\mu}{k_BT}\Gamma(2)\zeta(2)=\frac{2\pi^2\mu}{3k_BT}$$

Since, the matter surrounding us are mostly fermions, we get the net fermion density as

$$n^{\text{eq}}_{\text{fermion}}=g_i\frac{k_B^3T^3}{6\pi^2\hbar^3c^3}\left[\frac{\mu^3}{k_B^3T^3}+\frac{\pi^2\mu}{k_BT}\right]$$

and

$$n^{\text{eq}}_{\text{boson}}=g_i\frac{k_B^3T^3}{6\pi^2\hbar^3c^3}\left[\frac{\mu^3}{k_B^3T^3}+\frac{2\pi^2\mu}{k_BT}\right]$$

Non-Relativistic Limit Here we assume $\xi << x$, so ${}^{\xi}\!/\!{}_{x} << 1$.

$$\begin{aligned}
n^{\text{eq}} &= g_i\frac{k_B^3T^3}{2\pi^2\hbar^3c^3}\int_0^\infty d\xi\frac{\xi^2}{e^{\sqrt{\xi^2+x^2}+\mu/k_BT}\pm 1}\\
&= g_i\frac{k_B^3T^3}{2\pi^2\hbar^3c^3}\int_0^\infty d\xi\frac{\xi^2}{e^{x(1+\xi^2/x^2)^{1/2}+\mu/k_BT}}\\
&= g_i\frac{k_B^3T^3}{2\pi^2\hbar^3c^3}\int_0^\infty d\xi\xi^2e^{-x-\xi^2/2x+\mu/k_BT}\\
&= g_i\frac{k_B^3T^3}{2\pi^2\hbar^3c^3}e^{-x+\mu/k_BT}\int_0^\infty d\xi\xi^2e^{-\xi^2/2x}\\
&= g_i\frac{k_B^3T^3}{2\pi^2\hbar^3c^3}\sqrt{\frac{\pi}{2}}x^{3/2}e^{-x+\mu/k_BT} && \text{(where } x = mc^2/k_BT\text{)}\\
&= g_i\frac{k_B^3T^3}{2\pi^2\hbar^3c^3}\sqrt{\frac{\pi}{2}}\left(\frac{mc^2}{k_BT}\right)^{3/2}e^{-mc^2/k_BT+\mu/k_BT}\\
&= g_i\frac{1}{\hbar^3}\left(\frac{mk_BT}{2\pi}\right)^{3/2}e^{-mc^2-\mu/k_BT}
\end{aligned}$$

Thus we arrive at, with $\mu=0$

$$\boxed{n^{\text{eq}}=g_i\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}e^{-mc^2/k_BT}}$$

Pressure

The quantity of interest is pressure:

$$\begin{aligned}
P^{\text{eq}} &= \sum_{\text{all species}}\frac{g_i}{(2\pi\hbar)^3}\int\frac{|\vec{p}|^2c^2}{3E}f(\vec{p})d^3p\\
&= \sum_{\text{all species}}\frac{g_i}{2\pi^2\hbar^3}\int_0^\infty\frac{p^2c^2}{3\sqrt{p^2c^2+m^2c^4}}\frac{1}{e^{\sqrt{p^2c^2+m^2c^4}-\mu/k_BT}\pm 1}p^2dp
\end{aligned}$$

defining $x\equiv mc^2/k_BT$ and $\xi = pc/k_BT$, we get

$$\begin{aligned}
&= \sum_{\text{all species}}\frac{g_i}{2\pi^2\hbar^3}\int_0^\infty\frac{k_B^2T^2\xi^2}{3k_BT\sqrt{\xi^2+x^2}}\frac{1}{\exp\left(\sqrt{\xi^2+x^2}-\mu/k_BT\right)\pm 1}\left(\frac{k_BT}{c}\xi\right)^2\frac{k_BT}{c}d\xi\\
&= \sum_{\text{all species}}\frac{g_i}{2\pi^2\hbar^3}\int_0^\infty\frac{k_BT\xi^2}{3\sqrt{\xi^2+x^2}}\frac{1}{\exp\left(\sqrt{\xi^2+x^2}-\mu/k_BT\right)\pm 1}\left(\frac{k_BT}{c}\xi\right)^2\frac{k_BT}{c}d\xi\\
&= \frac{1}{3}\sum_{\text{all species}}g_i\frac{k_B^4T^4}{2\pi^2\hbar^3c^3}\int_0^\infty\frac{\xi^4}{\sqrt{\xi^2+x^2}}\frac{1}{\exp\left(\sqrt{\xi^2+x^2}-\mu/k_BT\right)\pm 1}d\xi
\end{aligned}$$

Ultra-relativistic limit In the ultra-relativistic limit $x \to 0$ and $\mu \approx 0$, we observe that

$$\begin{aligned}
P^{\text{eq}} &= \frac{1}{3} \sum_{\text{all species}} g_i \frac{k_B^4 T^4}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\xi^4}{\sqrt{\xi^2}} \frac{1}{\exp\left(\sqrt{\xi^2} - {}^{\mu}/_{k_B T}\right) \pm 1} d\xi \\
&= \frac{1}{3} \underbrace{\frac{k_B^4 T^4}{2\pi^2 \hbar^3 c^3} \left[\sum_{\text{boson}} g_i \int_0^\infty d\xi \frac{\xi^3}{e^{\xi - {}^{\mu}/_{k_B T}} - 1} + \sum_{\text{fermion}} g_i \int_0^\infty d\xi \frac{\xi^3}{e^{\xi - {}^{\mu}/_{k_B T}} + 1} \right]}_{\rho^{\text{eq}}} \\
&= \frac{1}{3} \rho^{\text{eq}}
\end{aligned}$$

Non-relativistic Limit In the limit $\xi << x \implies {}^{\xi}/_{x} << 1$

$$\begin{aligned}
P^{\text{eq}} &= \frac{1}{3} \sum_{\text{all species}} g_i \frac{k_B^4 T^4}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\xi^4}{\sqrt{\xi^2 + x^2}} \frac{1}{\exp\left(\sqrt{\xi^2 + x^2} - {}^{\mu}/_{k_B T}\right) \pm 1} d\xi \\
&= \frac{1}{3} \sum_{\text{all species}} g_i \frac{k_B^4 T^4}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\xi^4}{x\sqrt{1 + \frac{\xi^2}{x^2}}} \frac{1}{\exp\left(x\sqrt{1 + \frac{\xi^2}{x^2}} - {}^{\mu}/_{k_B T}\right) \pm 1} d\xi \\
&\approx \frac{1}{3} \sum_{\text{all species}} g_i \frac{k_B^4 T^4}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\xi^4}{x + \frac{\xi^2}{2x}} \frac{1}{\exp\left(x + \frac{\xi^2}{2x} - {}^{\mu}/_{k_B T}\right) \pm 1} d\xi \\
&= \frac{k_B T}{3} \left[\sum_{\text{all species}} g_i \frac{k_B^3 T^3}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\xi^4}{x + \frac{\xi^2}{2x}} \frac{1}{\exp\left(x + \frac{\xi^2}{2x} - {}^{\mu}/_{k_B T}\right) \pm 1} d\xi \right] \\
&\approx \frac{k_B T}{3} \left[\sum_{\text{all species}} g_i \frac{k_B^3 T^3}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\xi^4}{x} e^{-x - {}^{\xi^2}/_{2x} + {}^{\mu}/_{k_B T}} d\xi \right] \\
&= \frac{k_B T}{3} \left[\sum_{\text{all species}} g_i \frac{k_B^3 T^3}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\xi^4}{x} e^{-x - {}^{\xi^2}/_{2x} + {}^{\mu}/_{k_B T}} d\xi \right] \\
&= \frac{k_B T}{3} \left[\sum_{\text{all species}} g_i \frac{k_B^3 T^3}{2\pi^2 \hbar^3 c^3} \frac{1}{x} \times 3\sqrt{\frac{\pi}{2}} x^{5/2} e^{-x - {}^{\mu}/_{k_B T}} \right] \\
&= k_B T \underbrace{\left[\sum_{\text{all species}} g_i \left(\frac{m k_B T}{2\pi \hbar^2} \right)^{3/2} e^{-mc^2 - {}^{\mu}/_{k_B T}} \right]}_{n^{\text{eq}}} \\
&= n^{\text{eq}} k_B T
\end{aligned}$$

4.1.2 Using Maxwell-Boltzmann distribution

Now that we have seen how keeping the explicit $\hbar$ and c, makes the calculation look somewhat clustered, we will switch to natural units for this part. Following the previous section

Equilibrium Number density

The equilibrium number density is given by,

$$n^{\text{eq}} = \frac{g_i}{(2\pi\hbar)^3} \int_0^\infty d^3 p e^{-\frac{\sqrt{p^2 c^2 + m^2 c^4}}{k_B T}} = \frac{g}{(2\pi)^3} \int d^3 p \, e^{-\frac{\sqrt{p^2 + m^2}}{T}},$$

where $p = |\mathbf{p}|$ and m is the mass of the particle. Since the integrand depends only on the magnitude of the momentum, it is convenient to switch to spherical coordinates in momentum space. In three dimensions, the volume element in spherical coordinates is

$$d^3 p = p^2 \, dp \, d\Omega = 4\pi p^2 \, dp$$

Thus, the integral reduces to

$$n^{\text{eq}} = \frac{g}{(2\pi)^3} 4\pi \int_0^\infty p^2 e^{-\frac{\sqrt{p^2 + m^2}}{T}} \, dp.$$

To proceed, we use the substitution

$$p = m \sinh t, \quad \text{so that} \quad dp = m \cosh t \, dt.$$

With this substitution, the energy becomes

$$\sqrt{p^2 + m^2} = \sqrt{m^2 \sinh^2 t + m^2} = m \cosh t,$$

and the factor $p^2 dp$ transforms as

$$p^2 dp = (m \sinh t)^2 \cdot (m \cosh t \, dt) = m^3 \sinh^2 t \cosh t \, dt.$$

Substituting these expressions into the integral, we obtain

$$n^{\mathrm{eq}} = \frac{g}{2\pi^2} m^3 \int_0^\infty \sinh^2 t \cosh t \, e^{-\frac{m \cosh t}{T}} \, dt.$$

Using the identity $\sinh^2 t = \cosh^2 t - 1$, we have

$$\sinh^2 t \cosh t = (\cosh^2 t - 1) \cosh t = \cosh^3 t - \cosh t.$$

Hence, the integral becomes

$$n^{\mathrm{eq}} = \frac{g}{2\pi^2} m^3 \int_0^\infty (\cosh^3 t - \cosh t) \, e^{-\frac{m \cosh t}{T}} \, dt.$$

We can relate powers of $\cosh t$ to $\cosh(3t)$ using the identity

$$\cosh 3t = 4 \cosh^3 t - 3 \cosh t \quad \Rightarrow \quad \cosh^3 t - \cosh t = \frac{1}{4}(\cosh 3t - \cosh t).$$

Thus, the integral can be rewritten as

$$I = \frac{g}{2\pi^2} m^3 \int_0^\infty \frac{1}{4}(\cosh 3t - \cosh t) \, e^{-\frac{m \cosh t}{T}} \, dt = I = \frac{g}{8\pi^2} m^3 \int_0^\infty (\cosh 3t - \cosh t) \, e^{-\frac{m \cosh t}{T}} \, dt.$$

Next, we recall the integral representation of the modified Bessel function of the second kind:

$$K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh(\nu t) \, dt, \quad \mathrm{Re}(z) > 0.$$

Comparing with our integral, we identify

$$\int_0^\infty \cosh 3t \, e^{-m \cosh t / T} \, dt = K_3 \left(\frac{m}{T} \right), \quad \int_0^\infty \cosh t \, e^{-m \cosh t / T} \, dt = K_1 \left(\frac{m}{T} \right).$$

Hence, we can write

$$I = \frac{g}{8\pi^2} m^3 \left[K_3 \left(\frac{m}{T} \right) - K_1 \left(\frac{m}{T} \right) \right].$$

Finally, using the recurrence relation for modified Bessel functions

$$K_{\nu+1}(z) - K_{\nu-1}(z) = \frac{2\nu}{z} K_\nu(z),$$

with $\nu = 2$, we get

$$K_3(z) - K_1(z) = \frac{4}{z} K_2(z).$$

Therefore, the final result of the integral is

$$n^{\mathrm{eq}} = \frac{g}{8\pi^2} m^3 \cdot \frac{4T}{m} K_2 \left(\frac{m}{T} \right) = \frac{g}{(2\pi)^3} 4\pi m^2 T \, K_2 \left(\frac{m}{T} \right)$$

Equilibrium Energy Density

We consider a particle species of mass m and internal degrees of freedom g, in thermal equilibrium at temperature T, obeying Maxwell–Boltzmann statistics. The equilibrium energy density is defined as

$$\rho_{\text{eq}}(T) = g \int \frac{d^3p}{(2\pi)^3}\, E\, e^{-E/T}, \qquad E = \sqrt{p^2 + m^2}. \tag{4.1}$$

Using spherical symmetry in momentum space,

$$d^3p = 4\pi p^2 dp, \tag{4.2}$$

we obtain

$$\rho_{\text{eq}} = \frac{g}{2\pi^2} \int_0^\infty dp\, p^2 \sqrt{p^2 + m^2}\, e^{-\sqrt{p^2+m^2}/T}. \tag{4.3}$$

Introduce the change of variables

$$p = m \sinh u, \qquad E = \sqrt{p^2 + m^2} = m \cosh u. \tag{4.4}$$

Then

$$dp = m \cosh u\, du, \tag{4.5}$$

and

$$p^2 E\, dp = m^4 \sinh^2 u \cosh^2 u\, du. \tag{4.6}$$

Substituting into (4.3),

$$\rho_{\text{eq}} = \frac{gm^4}{2\pi^2} \int_0^\infty du\ \sinh^2 u \cosh^2 u\, e^{-m\cosh u/T}. \tag{4.7}$$

Using the identity

$$\sinh^2 u = \cosh^2 u - 1, \tag{4.8}$$

we find

$$\sinh^2 u \cosh^2 u = \cosh^4 u - \cosh^2 u. \tag{4.9}$$

Thus,

$$\rho_{\text{eq}} = \frac{gm^4}{2\pi^2} \int_0^\infty du\, \big(\cosh^4 u - \cosh^2 u\big) e^{-m\cosh u/T}. \tag{4.10}$$

We use the identities

$$\cosh^2 u = \frac{1}{2}\big(\cosh 2u + 1\big), \tag{4.11}$$

$$\cosh^4 u = \frac{1}{8}\big(\cosh 4u + 4\cosh 2u + 3\big). \tag{4.12}$$

Subtracting,

$$\cosh^4 u - \cosh^2 u = \frac{1}{8}\cosh 4u + \frac{1}{4}\cosh 2u - \frac{1}{8}. \tag{4.13}$$

Substituting into (4.10),

$$\rho_{\text{eq}} = \frac{gm^4}{16\pi^2} \int_0^\infty du\, \Big[\cosh 4u + 2\cosh 2u - 1\Big] e^{-m\cosh u/T}. \tag{4.14}$$

The modified Bessel function of the second kind is defined by

$$K_\nu(z) = \int_0^\infty du\, e^{-z\cosh u} \cosh(\nu u). \tag{4.15}$$

Using this definition, equation (4.14) becomes

$$\rho_{\text{eq}} = \frac{gm^4}{16\pi^2} \left[K_4(x) + 2K_2(x) - K_0(x)\right], \qquad x \equiv \frac{m}{T}. \tag{4.16}$$

The modified Bessel functions satisfy the recursion relation

$$K_{\nu+1}(x) + K_{\nu-1}(x) = \frac{2\nu}{x} K_\nu(x). \tag{4.17}$$

Applying this relation:

$$K_4 + K_2 = \frac{6}{x} K_3, \tag{4.18}$$

$$K_2 + K_0 = \frac{2}{x} K_1. \tag{4.19}$$

Subtracting the second from the first,

$$K_4 - K_0 = \frac{6}{x} K_3 - \frac{2}{x} K_1. \tag{4.20}$$

Using the identity

$$K_3 - K_1 = \frac{4}{x} K_2, \tag{4.21}$$

we obtain

$$K_4 + 2K_2 - K_0 = \frac{4}{x} K_1 + \frac{12}{x^2} K_2. \tag{4.22}$$

Substituting into (4.16),

$$\rho_{\text{eq}} = \frac{gm^4}{16\pi^2} \left(\frac{4}{x} K_1 + \frac{12}{x^2} K_2 \right). \tag{4.23}$$

Restoring $x = m/T$,

$$\boxed{\rho_{\text{eq}}(T) = \frac{gm^2T^2}{2\pi^2} \left[mK_1\left(\frac{m}{T}\right) + 3TK_2\left(\frac{m}{T}\right) \right].} \tag{4.24}$$

4.2 Decay Rate and Relative Velocity

This derivation follows *Landau & Lifshitz, The Classical Theory of Fields*, Section 12, where we obtain a Lorentz-invariant expression for the relative velocity between two particles in special relativity. Let the four-momenta of the two particles be

$$p_1^\mu = (E_1, \mathbf{p}_1) = (\gamma_1 m_1, \gamma_1 m_1 \mathbf{v}_1), \qquad p_2^\mu = (E_2, \mathbf{p}_2) = (\gamma_2 m_2, \gamma_2 m_2 \mathbf{v}_2), \tag{4.25}$$

where natural units $c = 1$ are used and $\gamma_i = (1 - v_i^2)^{-1/2}$.

To determine the relative velocity between the two particles, we consider the scalar product of their four-momenta in the rest frame of the second particle. In this frame, the momentum of particle 2 is

$$p_2^\mu = (m_2, \mathbf{0}),$$

while particle 1 moves with velocity $\mathbf{v}_{\text{rel}}$ and Lorentz factor $\gamma_{\text{rel}} = (1 - v_{\text{rel}}^2)^{-1/2}$. Thus,

$$p_1^\mu = (\gamma_{\text{rel}} m_1, \gamma_{\text{rel}} m_1 \mathbf{v}_{\text{rel}}).$$

The Lorentz scalar product between the two momenta is

$$p_1 \cdot p_2 = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 = \gamma_1 \gamma_2 m_1 m_2 (1 - \mathbf{v}_1 \cdot \mathbf{v}_2) = E_1 E_2 (1 - \mathbf{v}_1 \cdot \mathbf{v}_2) \tag{4.26}$$

where we substituted the definitions $E_i = \gamma_i m_i$ and $\mathbf{p}_i = \gamma_i m_i \mathbf{v}_i$. Upon squaring, we have

$$(p_1 \cdot p_2)^2 = (m_1 m_2 \gamma_1 \gamma_2)^2 (1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2. \tag{4.27}$$

Expanding the square of the scalar product, we find

$$(p_1 \cdot p_2)^2 = (m_1 m_2 \gamma_1 \gamma_2)^2 (1 - 2\mathbf{v}_1 \cdot \mathbf{v}_2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2). \tag{4.28}$$

Recall the vector identity

$$(\mathbf{a} \times \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2,$$

We may write

$$(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 = v_1^2 v_2^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2.$$

Substituting this, we obtain

$$(p_1 \cdot p_2)^2 = (m_1 m_2 \gamma_1 \gamma_2)^2 \left[1 - 2\mathbf{v}_1 \cdot \mathbf{v}_2 + v_1^2 v_2^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2 \right]. \tag{4.29}$$

We also have

$$2\mathbf{v}_1 \cdot \mathbf{v}_2 = v_1^2 + v_2^2 - (\mathbf{v}_1 - \mathbf{v}_2)^2$$

Substituting this form back, we have

$$\begin{aligned}(p_1 \cdot p_2)^2 &= (m_1 m_2 \gamma_1 \gamma_2)^2 \left[(\mathbf{v}_1 - \mathbf{v}_2)^2 + 1 - v_1^2 - v_2^2 + v_1^2 v_2^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2\right] \\ &= (m_1 m_2 \gamma_1 \gamma_2)^2 \left[(\mathbf{v}_1 - \mathbf{v}_2)^2 + (1 - v_1^2)(1 - v_2^2) - (\mathbf{v}_1 \times \mathbf{v}_2)^2\right]\end{aligned}$$

Using $1 - v_i^2 = 1/\gamma_i^2$, this becomes

$$(p_1 \cdot p_2)^2 = (m_1 m_2 \gamma_1 \gamma_2)^2 \left[(\mathbf{v}_1 - \mathbf{v}_2)^2 + \frac{1}{\gamma_1^2 \gamma_2^2} - (\mathbf{v}_1 \times \mathbf{v}_2)^2\right]. \tag{4.30}$$

Subtracting $m_1^2 m_2^2$ from both sides gives

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = (m_1 m_2 \gamma_1 \gamma_2)^2 \left[(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2\right]. \tag{4.31}$$

Diving by (4.27), we get our expression for the magnitude of relative velocity v_{rel}:

$$v_{\text{rel}} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1 \cdot p_2} = \frac{\sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}}{1 - \mathbf{v}_1 \cdot \mathbf{v}_2}. \tag{4.32}$$

This is the general Lorentz-invariant expression for the relative velocity between two particles moving with arbitrary velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. We can express the above using the mandelstam variable

$$p_1 \cdot p_2 = \frac{s - m_1^2 - m_2^2}{2}$$

Then,

$$\begin{aligned} v_{\text{rel}} &= \frac{\sqrt{\left(\frac{s - m_1^2 - m_2^2}{2}\right)^2 - m_1^2 m_2^2}}{\frac{s - (m_1^2 + m_2^2)}{2}} = \frac{\sqrt{(s - m_1^2 - m_2^2)^2 - 4 m_1^2 m_2^2}}{[s - (m_1^2 + m_2^2)]} \\ &= \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{[s - (m_1^2 + m_2^2)]} \end{aligned} \tag{4.33}$$

For colinear motion, $\mathbf{v}_1 \parallel \mathbf{v}_2$, the cross product vanishes and the expression reduces to the familiar relativistic law of velocity addition,

$$v_{\text{rel}} = \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{1 - \mathbf{v}_1 \mathbf{v}_2} = \frac{E_1 E_2}{p_1 \cdot p_2} |\mathbf{v}_1 - \mathbf{v}_2| \tag{4.34}$$

4.2.1 Decay Rate

The thermal average of σv_{rel} following Mirco Cannoni is

$$\langle \sigma |\mathbf{v}_1 - \mathbf{v}_2| \rangle = \frac{1}{(4\pi)^2 T^2 m_1^2 m_2^2 K_2(m_1/T) K_2(m_2/T)} \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 \frac{(p_1 \cdot p_2)}{E_1 E_2} v_{\text{rel}} \, \sigma(s) \, e^{-(E_1 + E_2)/T}. \tag{4.35}$$

This is a complicated integral which can be simplified by transforming it into a single integral over the invariant mass squared $s = (p_1 + p_2)^2$. All calculations are performed in the rest frame of the heat bath, $u^\mu = (1, \mathbf{0})$. The integration measure could be simplified as:

$$d^3\mathbf{p}_i = p_i^2 dp_i \, d\Omega_i, \qquad E_i = \sqrt{p_i^2 + m_i^2}, \qquad i = 1, 2.$$

Let us perform the following transformation $\theta_1 \to \theta_1$ and $\theta_2 \to (\theta_2 - \theta_1) \equiv \theta$ which is equivalent to choosing $\mathbf{p}_1$ along the z-axis and allowing θ be the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. Then $d\Omega_1 = 4\pi$ and $d\Omega_2 = 2\pi \, d(\cos\theta)$, giving

$$d^3\mathbf{p}_1 \, d^3\mathbf{p}_2 = 8\pi^2 \, |\mathbf{p_1}|^2 |\mathbf{p_2}|^2 dp_1 dp_2 \, d(\cos\theta).$$

Also, using $d\mathbf{p}_i = (E_i/\mathbf{p}_i) \, dE_i$, we have

$$\frac{d^3\mathbf{p}_1}{E_1} \frac{d^3\mathbf{p}_2}{E_2} = 8\pi^2 \, |\mathbf{p_1}||\mathbf{p_2}| dE_1 \, dE_2 \, d(\cos\theta).$$

We recall

$$p_{1\mu} p^{2\mu} = E_1 E_2 - |\mathbf{p_1}||\mathbf{p_2}| \cos\theta, \qquad s = m_1^2 + m_2^2 + 2(E_1 E_2 - |\mathbf{p_1}||\mathbf{p_2}| \cos\theta).$$

Hence,

$$p_{1\mu}p^{2\mu} = \frac{s - (m_1^2 + m_2^2)}{2}.$$

Let's perform the change of variable as suggested by P. Gondolo and G. Gelmini, Nucl. Phys. B 360

$$Y = E_1 + E_2, \qquad Z = E_1 - E_2, \qquad s = (p_1 + p_2)^2.$$

The associated Jacobian for the transformation can be found as:

$$\frac{\partial(Y, Z, s)}{\partial(E_1, E_2, \cos\theta)} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ \dfrac{\partial s}{\partial E_1} & \dfrac{\partial s}{\partial E_2} & \dfrac{\partial s}{\partial \cos\theta} \end{pmatrix}.$$

The remaining derivatives can be computed from $s = m_1^2 + m_2^2 + 2(E_1E_2 - p_1p_2\cos\theta)$:

$$\begin{aligned}
\frac{\partial s}{\partial E_1} &= 2\left(E_2 - \cos\theta\,\frac{E_1|\mathbf{p_2}|}{|\mathbf{p_1}|}\right) \\
\frac{\partial s}{\partial E_2} &= 2\left(E_1 - \cos\theta\,\frac{E_2|\mathbf{p_1}|}{|\mathbf{p_2}|}\right), \\
\frac{\partial s}{\partial \cos\theta} &= 2|\mathbf{p_1}||\mathbf{p_2}|
\end{aligned}$$

Now the determinant:

$$\det\frac{\partial(Y, Z, s)}{\partial(E_1, E_2, \cos\theta)} = -4|\mathbf{p_1}||\mathbf{p_2}|$$

Hence,

$$dE_1\,dE_2\,d(\cos\theta) = \frac{1}{4|\mathbf{p_1}||\mathbf{p_2}|}\,dY\,dZ\,ds$$

Plugging this into the momentum-space measure:

$$\frac{d^3\mathbf{p}_1}{E_1}\frac{d^3\mathbf{p}_2}{E_2} = 8\pi^2|\mathbf{p_1}||\mathbf{p_2}|dE_1\,dE_2\,d(\cos\theta) = 2\pi^2\,dY\,dZ\,ds.$$

Inserting the above results into (4.35), we obtain

$$\langle\sigma|\mathbf{v}_1 - \mathbf{v}_2|\rangle = \frac{\pi^2}{(4\pi)^2T^2\,m_1^2m_2^2K_2(m_1/T)K_2(m_2/T)}\int dY\,dZ\,ds\,\sigma(s)\,[s - (m_1^2 + m_2^2)]v_{\rm rel}\,e^{-Y/T}. \tag{4.36}$$

The variables Y and Z are restricted by kinematics: for fixed s, not every pair (Y, Z) is possible. The restriction arises from the condition $|\cos\theta| \le 1$. From $s = m_1^2 + m_2^2 + 2(E_1E_2 - p_1p_2\cos\theta)$ we isolate $\cos\theta$:

$$\cos\theta = \frac{2E_1E_2 + m_1^2 + m_2^2 - s}{2p_1p_2}.$$

Substitute $E_1 = (Y + Z)/2$, $E_2 = (Y - Z)/2$, and $p_i = \sqrt{E_i^2 - m_i^2}$. Then

$$\cos\theta = \frac{Y^2 - Z^2 + 2(m_1^2 + m_2^2) - 2s}{4\sqrt{\left(\frac{Y+Z}{2}\right)^2 - m_1^2}\sqrt{\left(\frac{Y-Z}{2}\right)^2 - m_2^2}}.$$

The condition $|\cos\theta| \le 1$ determines the allowed range of Z for each (Y, s). Squaring and solving for Z gives the two roots $Z_\pm$ satisfying

$$Z_\pm = \frac{Y(m_1^2 - m_2^2) \pm 2\sqrt{s}\,p'(s)\,\sqrt{Y^2 - s}}{s},$$

where

$$p'(s) = \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{2\sqrt{s}} \tag{4.37}$$

Hence the integration interval in Z is $Z \in [Z_-, Z_+]$, and its length is

$$Z_+ - Z_- = \frac{4p'(s)}{\sqrt{s}}\sqrt{Y^2 - s}.$$

We note that the integrand in (4.36) is independent of Z, so the Z-integration is simply:

$$\int_{Z_-}^{Z_+} dZ\, e^{-Y/T} = e^{-Y/T}\,(Z_+ - Z_-) = e^{-Y/T}\,\frac{4p'(s)}{\sqrt{s}}\sqrt{Y^2-s}.$$

Substitute this back into (4.36):

$$\langle\sigma v\rangle = \frac{1}{4T\,m_1^2 m_2^2 K_2(m_1/T)K_2(m_2/T)}\int_{(m_1+m_2)^2}^{\infty} ds\,\sigma(s)\,[s-(m_1^2+m_2^2)]v_{\rm rel}\,\frac{p'(s)}{\sqrt{s}}\int_{\sqrt{s}}^{\infty} dY\,e^{-Y/T}\sqrt{Y^2-s}.$$

Let $y = Y/\sqrt{s}$, so $dY = \sqrt{s}\,dy$ and $Y^2 - s = s(y^2-1)$. Then

$$\int_{\sqrt{s}}^{\infty} dY\,e^{-Y/T}\sqrt{Y^2-s} = s\int_1^{\infty} dy\,e^{-(\sqrt{s}/T)y}\sqrt{y^2-1}.$$

The standard integral representation of the modified Bessel function K_1 is

$$K_1(a) = a\int_1^{\infty} dy\,e^{-ay}\sqrt{y^2-1}, \qquad \mathrm{Re}\,a>0.$$

Thus

$$\int_{\sqrt{s}}^{\infty} dY\,e^{-Y/T}\sqrt{Y^2-s} = T\sqrt{s}\,K_1(\sqrt{s}/T)$$

Using above with (4.33) and (4.37), we find

$$\boxed{\langle\sigma|\mathbf{v}_1-\mathbf{v}_2|\rangle = \frac{1}{8T\,m_1^2 m_2^2 K_2(m_1/T)K_2(m_2/T)}\int_{(m_1+m_2)^2}^{\infty} ds\,\frac{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}{\sqrt{s}}\sigma(s)\,K_1(\sqrt{s}/T).}$$

4.3 Boltzmann Transport Equation

Boltzmann Equation, in theory, can be used to derive the distribution function and its time evolution. However, the problem is the need for boundary condition. Which means, we need to know the distribution at some time t. But, in the framework where the system begins to move away from the equilibrium, we already know the distribution at some time t from the equilibrium statistical mechanics. Thus, it can be used to inquire more about the system.

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{m}\cdot\nabla_{\vec{q}}f + \vec{F}\cdot\nabla_{\vec{p}}f = \left(\frac{\partial f}{\partial t}\right)_{\text{collision}}$$

Example

Let us consider Maxwell-Boltzmann distribution[6]:

$$f(\vec{p},\vec{q}) = Ae^{-\beta\left(\frac{|\vec{p}|^2}{2m}+U(\vec{q})\right)}, \text{ where } \beta = \frac{1}{k_BT}.$$

Since this distribution describes equilibrium distribution the collision term would vanish i.e., $\partial_t f = 0$, whereas the gradients are:

$$\nabla_{\vec{q}}f = -\beta f\nabla_{\vec{q}}U = \beta\vec{F}f, \qquad\qquad \nabla_{\vec{p}}f = -\beta\frac{\vec{p}}{m}f,$$

that is the sum of the two gradient term in the Boltzmann Transport Equation is zero. The density distribution under this distribution is not uniform, unless the potential is a constant, in which the distribution becomes Maxwell distribution

$$f(\vec{p}) = Ae^{-\beta\frac{|\vec{p}|^2}{2m}},$$

that gives a non-zero gradient $\nabla_{\vec{p}}f = -\beta\frac{\vec{p}}{m}f$, but still satisfies the Boltzmann equation, since now $\vec{F} = -\nabla_{\vec{q}}U = 0$.

[6]It can be derived for dilute gases using Boltzmann Transport Equation, see chapter 4 of Kerson Huang's Statistical Mechanics

4.3.1 Derivation

The derivation that we will follow is from Kerson Huang's Statistical Mechanics as well as Donald McQuarrie's Stat Mech. We assume binary collision and ignore the impact of any external force on the collision. We will model the interaction taking place between the particles as sort of collision to get a better visualization into the problem.

In the absense of collision, all the molecule that start out at (r, v, t) will end up at $(r + vt, v + \delta v, t + dt)$. However in the presence of collision, some molecule may leave the stream. Let the number of $j-$molecule lost due to collision be represented by $\Gamma^-_{ji} dr dv_j dt$. Similary let the number of $j-$molecule that join the stream due to collision with $i-$molecule be represented by $\Gamma^+_{ji} dr dv_j dt$. Where i and j labels the species of molecule.

$$\left(\frac{\partial f_j}{\partial t}\right)_{\text{collision}} = \sum_i \Gamma^+_{ji} - \Gamma^-_{ji}$$

Consider a $12 \to 1'2'$ scattering process where we consider the '$2'$ as target and '$1'$ as projectile. The number of collision taking place in the time interval is given as

$$\text{Number of collision} = \underbrace{dN_{12}}_{\text{initial number of colliding particles}} \; \underbrace{dP_{12\to 1'2'}}_{\text{rate of collision}} \; \delta t$$

Where dN_{12} is the initial number of colliding pair $(\vec{p}_1, \vec{p}_2)$. We introduce the two-particle correlation function F which encodes their initial correlated distribution information by

$$dN_{12} = F(\vec{r}, \vec{p}_1, \vec{p}_2, t) d^3r d^3p_1 d^3p_2.$$

This can be rewritten in terms of cross-section σ, since a collision is just transition from the initial state to a set of final states via interaction. If we assume binary collision. Then,

$$\frac{\text{number of collision}}{\text{volume of phase space}} = \frac{\text{net change in the number of particle}}{\text{volume of phase space}} = (\Gamma^+ - \Gamma^-)\delta t$$

$$\begin{aligned}
\Gamma^+ \cancel{d^3 r d^3 p_1 \delta t} &= dN_{12} dP_{12\to 1'2'} \delta t \\
&= F(\vec{r}, \vec{p}_1, \vec{p}_2, t) \cancel{d^3 r d^3 p_1} d^3p_2 dP_{12\to 1'2'} \cancel{\delta t} \\
\Gamma^+ &= d^3p_2 dP_{12\to 1'2'} F(\vec{r}, \vec{p}_1, \vec{p}_2, t)
\end{aligned}$$

Thus, we have

$$\left(\frac{\partial f_1}{\partial t}\right)_{\text{collision}} = d^3p_2 \; \overbrace{|\vec{v}_1 - \vec{v}_2| d\sigma}^{\text{from transition rate for 1 particle}} \; (F_{1'2'} - F_{12})$$

Now, we will make a crucial assumption, also known as molecular chaos assumption or stosszahlansatz. This assumption is generally not true and implies that position and velocities are uncorrelation. It is the precisely why we apply the Boltzmann Equation for dilute gases.

$$\frac{\mathrm{d}f_1}{\mathrm{d}t} = \frac{\partial f_1}{\partial t} + \frac{\vec{p}_1}{m} \cdot \nabla_{\vec{q}} f_1 + \vec{F} \cdot \nabla_{\vec{p_1}} f_1 = d^3p_2 |\vec{v}_1 - \vec{v}_2| \; \overset{\text{will be integrated over all final state momenta}}{d\sigma} \; (f_{1'} f_{2'} - f_1 f_2) \tag{4.38}$$

The above equation holds true in flat spacetime, however in curved spacetime the LHS of Boltzmann Transport Equation needs modification as follows:

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}\lambda} &= \frac{\partial f}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}\lambda} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} + \frac{\partial f}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}\lambda} + \frac{\partial f}{\partial p^i}\frac{\mathrm{d}p^i}{\mathrm{d}\lambda} \\
&= p^0 \frac{\partial f}{\partial t} + p^i \frac{\partial f}{\partial x^i} + \frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda}\frac{\partial f}{\partial p^\mu} \\
&= p^\mu \frac{\partial f}{\partial x^\mu} + \frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda}\frac{\partial f}{\partial p^\mu} \\
&= p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^\mu} = \mathcal{L}[f]
\end{aligned}$$

We have substituted ${dp^\mu}/{d\lambda}$ from the geodesic equation. Now, we will simplify the above in case of FLRW geometry.

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = p^\mu \frac{\partial f}{\partial x^\mu} + \frac{p^\mu}{p^\mu}\frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda}\frac{\partial f}{\partial p^\mu}$$

$$
\begin{aligned}
&= p^\mu \frac{\partial f}{\partial x^\mu} + \frac{\mathrm{d}\ln p^\mu}{\mathrm{d}\lambda}\frac{\partial f}{\partial \ln p^\mu} \\
&= p^\mu \frac{\partial f}{\partial x^\mu} - \frac{\mathrm{d}\ln a}{\mathrm{d}\lambda}\frac{\partial f}{\partial \ln p^\mu} \qquad \text{(used } p^\mu a = \text{constant)} \\
\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}f}{\mathrm{d}t} &= p^\mu \frac{\partial f}{\partial x^\mu} - \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{\partial f}{\partial \ln p^\mu}
\end{aligned}
$$

assuming $f \equiv f(t,p)$

$$
\begin{aligned}
p^0\frac{\mathrm{d}f}{\mathrm{d}t} &= p^0\frac{\partial f}{\partial t} - p^0\frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{\partial f}{\partial \ln p^\mu} = C[f] \\
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{C[f]}{p^0}
\end{aligned}
$$

We can express the Boltzmann Equation simply as:[7]

$$
\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \mathcal{L}[f] = C[f]
$$

Now, we will integrate f with respect to d^3p

$$
\begin{aligned}
g\int \frac{\mathrm{d}f}{\mathrm{d}t}\frac{d^3p}{(2\pi\hbar)^3} &= g\int \left[\frac{\partial f}{\partial t} - \frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{\partial f}{\partial \ln p^\mu}\right]\frac{d^3p}{(2\pi\hbar)^3} \\
&= \frac{\partial}{\partial t}\frac{g}{(2\pi\hbar)^3}\int f(t,p)d^3p - \frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{g}{(2\pi\hbar)^3}\int \frac{\partial f}{\partial \ln p^\mu}d^3p \\
&= \frac{\mathrm{d}n}{\mathrm{d}t} - \frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{g}{(2\pi\hbar)^3}\left\{ f(t,p)\times 4\pi p^3\Big|_{p=0}^{p=\infty} - \int 3f(t,p)d^3p\right\} \\
&= \frac{\mathrm{d}n}{\mathrm{d}t} + \frac{\mathrm{d}\ln a}{\mathrm{d}t}\times 3n = \frac{1}{a^3}\frac{\mathrm{d}na^3}{\mathrm{d}t}
\end{aligned} \tag{4.39}
$$

where we evaluated the second integral as follows:

$$
\begin{aligned}
\int \frac{\partial f}{\partial \ln p^\mu}d^3p &= \int \frac{\partial f}{\partial \ln p^\mu}4\pi p^2\frac{p}{p}dp \\
&= \int_0^\infty \frac{\partial f}{\partial \ln p^\mu}d\ln p \times 4\pi p^3 - \int_0^\infty f d(4\pi p^3) \\
&= f4\pi p^3\Big|_{p=0}^{p=\infty} - 3\int_0^\infty f4\pi p^2 dp
\end{aligned}
$$

4.3.2 Away from Equilibrium

We first note that, the number density given as

$$
n = \frac{g}{(2\pi\hbar)^3}\int d^3p f(\vec{p}) \tag{4.40}
$$

can be used in Boltzmann Transport Equation to express the evolution of number density:

$$
\begin{aligned}
\frac{g}{(2\pi\hbar)^3}\int d^3p\frac{\mathrm{d}f(\vec{p})}{\mathrm{d}t} &= \overbrace{\frac{g}{(2\pi\hbar)^3}\int C[f]\frac{d^3p}{E}}^{\text{Lorentz Invariant}} = \tilde{C}[f] \\
\frac{1}{a^3}\frac{\mathrm{d}na^3}{\mathrm{d}t} &= \prod_{i=1}^{4}\int \frac{g_i d^3p_i}{(2\pi\hbar)^3 2E_i}\left|\tilde{\mathcal{M}}_{12\to 34}\right|^2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \\
&\quad\times\frac{1}{S}\left[f_3f_4(1\pm f_1)(1\pm f_2) - f_1f_2(1\pm f_3)(1\pm f_4)\right]
\end{aligned}
$$

where $\tilde{C}$ is the new collision term which is directly defined in the second line. g is the internal degree of freedom and S is the appropriate symmetry factors for identical particles in the initial or final states, taking into account that there can be multiple particles of the same species in the final state. δ corresponds to the delta function,

[7] better derivation for collision term can be found in "Boltzmann equation and its cosmological applications" by Seishi Enomoto

and $\left|\tilde{\mathcal{M}}_{12\to 34}\right|$ is the amplitude of specific process, averaged over initial and final spins as well as internal degree of freedom.

$$\left|\tilde{\mathcal{M}}_{12\to 34}\right| = \frac{1}{\prod_i g_i} \sum_{\text{spins } i} |\mathcal{M}_i|$$

The plus and minus sign in $1 \pm f_i$ corresponds to the particle species being bosons and fermions respectively. We now assume, the particles are approximately obeying Maxwell Boltzmann distribution for further simplification.

$$\frac{1}{e^{E-\mu/k_BT} \pm 1} \simeq e^{-E+\mu/k_BT}$$

Under this assumption, the Bose enhancement and Pauli blocking term $1 \pm f_i$ can be dropped upto first order in perturbation theory.

$$\begin{aligned} f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) &\approx f_3 f_4 - f_1 f_2 \\ &= e^{-E_1+E_2/k_BT} \left\{ e^{\mu_3+\mu_4/k_BT} - e^{\mu_1+\mu_2/k_BT} \right\} \end{aligned}$$

where we used $E_1 + E_2 = E_3 + E_4$ for simplification. The above approximation, also leads to:

$$\left.\begin{aligned} n_i &= g_i e^{\mu_i/k_BT} \int \frac{d^3p}{(2\pi\hbar)^3} e^{-E_i/k_BT} \\ n_i^{(0)} &= g_i \int \frac{d^3p}{(2\pi\hbar)^3} e^{-E_i/k_BT} \end{aligned}\right\} \frac{n_i}{n_i^{(0)}} = e^{\mu_i/k_BT} \tag{4.41}$$

Let us define thermally averaged cross section as[8]

$$\begin{aligned} \langle \sigma v \rangle &\equiv \frac{1}{n_1 n_2} \prod_{i=1}^{4} \int \frac{d^3 p_i}{(2\pi\hbar)^3 2E_i} \overset{\text{distribution function}}{e^{-E_1+\mu_1-E_2+\mu_2/k_BT}} \left|\tilde{\mathcal{M}}\right|^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \\ &= \frac{1}{n_1^{(0)} n_2^{(0)}} \prod_{i=1}^{4} \int \frac{d^3 p_i}{(2\pi\hbar)^3 2E_i} e^{-(E_1+E_2)/k_BT} \left|\tilde{\mathcal{M}}\right|^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \end{aligned}$$

Then, finally:

$$\frac{1}{a^3} \frac{\mathrm{d}n_1 a^3}{\mathrm{d}t} = \langle \sigma v \rangle\, n_1^{(0)} n_2^{(0)} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}$$

$$\frac{1}{a^3} \frac{\mathrm{d}n_1 a^3}{\mathrm{d}\ln a} \frac{\mathrm{d}\ln a}{\mathrm{d}t} = - \overset{\text{thermally averaged cross section}}{\langle \sigma v \rangle}\, n_1 n_2 \left\{ 1 - \frac{n_3 n_4}{n_1 n_2} \left(\frac{n_1^{(0)} n_2^{(0)}}{n_3^{(0)} n_4^{(0)}} \right) \right\}$$

$$\frac{1}{n_1 a^3} \frac{\mathrm{d}n_1 a^3}{\mathrm{d}\ln a} \frac{\dot{a}}{a} = -n_2 \langle \sigma v \rangle \left\{ 1 - \frac{n_3 n_4}{n_1 n_2} \left(\frac{n_1^{(0)} n_2^{(0)}}{n_3^{(0)} n_4^{(0)}} \right) \right\}$$

The Boltzmann Equation can be rewritten in more suggestive form:

$$\frac{1}{n_1 a^3} \frac{\mathrm{d}n_1 a^3}{\mathrm{d}x} = -\frac{\Gamma_1}{H} \left\{ 1 - \frac{n_3 n_4}{n_1 n_2} \left(\frac{n_1 n_2}{n_3 n_4} \right)_{eq} \right\}$$

where $\Gamma_1 \equiv n_2 \langle \sigma v \rangle$ is the interaction rate and H is the expansion rate. and the second factor just tells us how close we are to equilibrium. Lets try to understand the solution of this general equation a bit better which will reveal a few concepts like decouping and freeze-out that will be very useful later on. If $\Gamma_1 >> H$ then the prefactor on the right hand side is huge and the system is efficiently driven towards chemical equlibrium. This is not hard to see from the form of the equation and is also easy to understand physically: if the interaction rate is large relative to the expansion rate of the Universe then we have many interactions in the time it takes the Universe to double in size and we are easily able to maintain equilibrium $n_i \sim n_i^{\text{eq}}$ even though the number density of each species gets smaller and smaller. However this cannot last forever. Remember that the equilibrium distribution for massive partices $n_i \propto e^{-mc^2/k_BT}$ gets exponensialy suppressed when the temperature goes down (so if equilibrium had lasted we would have no more massive particles left in the Universe today). Thus the equilibrium we have must eventually fail. Once Γ_1 drops below H then the interactions are no longer

[8] The thermal averaging will come from the integral over p_1 and p_2 whereas p_3 and p_4 will be stored inside cross-section.

efficient enough to maintain equilibrium and we say that it decouples. After it decouples the interaction is no longer relevant, so knowing when ($\Gamma_1 \sim H$) this happens already gives us important information. If a particle species has no more efficient interactions that ties it to the rest of the primordial plasma then it can no longer stay in equilibrium with it and will evolve on its own like all the other stuff wasn't there (and if anything happens in the plasma that changes, say, its temperature then that change will not be reflected in stuff that has decoupled).

As we reach $\Gamma_1 << H$ the right hand side is practically zero and in this regime the equation reduces to $d(n_1 a^3)dx \approx 0 \implies n_1 \propto 1/a^3$ which is just normal volume dilution in an expanding Universe (i.e. the number of particles in a co-moving volume stays constant, its the physical volume that gets bigger $\propto a^3$) which is what we would expect if there was no interactions at all. When this happens for a massive particle species we say that it **freezes out**. The evolution is exponential decay of $n_1 a^3$ until decoupling, $\Gamma_1 \sim H$, and then it flattens out us and leaves us with a freeze-out abundance $n_1 a^3 \approx$constant. What this freeze-out abundance really is typucally requires numerical solutions of the Boltzmann equation (though we shall see an example of dark matter production where we can do some analytical estimates). We will encounter freeze-out in two main situations 1) when we look at recombination we see that not all of the free electrons and protons will end up in atoms 2) freeze-out is one of the most common production mechanisms for producing dark matter (for example WIMP's).

4.3.3 Saha approximation

The discussion above shows how we can qualitatively understand the solution to the Boltzmann equation, but we can also get a bit more quantitative understanding by extracting a very useful equation. Recall that if the interactions are efficient then we will be close to equilibrium and $\left[1 - \frac{n_3 n_4}{n_1 n_2}\left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}\right] \approx 0$. This gives us to the Saha approximation

$$\frac{n_3 n_4}{n_1 n_2} \approx \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}$$

and the right hand side we know how to compute in terms of the temperature and the particle masses. This equation tells us that each of the n_i's may change, but this particular combination must stay roughly constant. One of the most important interaction we will encounter is $e^- + p^+ \leftrightarrow H + \gamma$ and we will use this equation to tell us how the number density of free electrons will evolve in time, i.e. *how fast the Universe goes from a plasma to a transparent Universe with atoms.* The Saha approximation is great in that it's simple, doesn't require detailed knowledge of the interactions (QFT calculations) and allows us to get a better understanding for how the number density evolves in the presence of interactions. However it can only do so much, it will eventually break down and when the Saha approximation is no longer valid then we have no other option than solving the full Boltzmann equation above to be able to accurately pin down the evolution.

4.3.4 Dark Matter

Another derivation for boltzmann equation is given in lecture notes 2, of David Morrisey. Consider a process $\chi\bar{\chi} \to f\bar{f}$ with χ and $\bar{\chi}$ being the same in the non-relativistic limit. It implies that, for χ chemical potential should vanish i.e. $\mu = 0$.

$$\begin{aligned}
\tilde{C} &= \prod_{i=1}^{4} \int \frac{g_i d^3 p_i}{(2\pi\hbar)^3 2E_i} \left|\tilde{\mathcal{M}}_{12\to 34}\right|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \left(\frac{1}{2} \times 2\right) (f_3 f_{-} f_1 f_2) \\
&= \prod_{i=1}^{2} \int \frac{g_i d^3 p_i}{(2\pi\hbar)^3 2\cancel{E_i}} \sigma v \cancel{2E_1 2E_2} \times (f_3 f_4 - f_1 f_2) \\
&= \prod_{i=1}^{2} \int \frac{g_i d^3 p_i}{(2\pi\hbar)^3} (f_3 f_4 - f_1 f_2) \times \sigma v
\end{aligned}$$

now, let us consider $f_3 f_4 = f_f f_{\bar{f}} = (f_\chi f_{\bar{\chi}})_{eq}$ and $f_1 f_2 = f_\chi f_{\bar{\chi}}$ and thermal averaging of cross-section.

$$\begin{aligned}
\frac{1}{a^3}\frac{\mathrm{d}n_\chi a^3}{\mathrm{d}t} &= \prod_{i=1}^{2} \int \frac{g_i d^3 p_i}{(2\pi\hbar)^3} (f_f f_{\bar{f}} - f_\chi f_{\bar{\chi}}) \times \sigma v \\
&= \prod_{i=1}^{2} \int \frac{g_i d^3 p_i}{(2\pi\hbar)^3} (f_\chi f_{\bar{\chi}})_{eq} \times \sigma v - \prod_{i=1}^{2} \int \frac{g_i d^3 p_i}{(2\pi\hbar)^3} f_\chi f_{\bar{\chi}} \times \sigma v
\end{aligned}$$

$$\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3Hn_\chi = \langle \sigma v \rangle \left(n^2_{\chi\ \text{equilibrium}} - n^2_\chi\right)$$

where we have assumed that the $(f_f)_{eq}$ and $(f_{\bar{f}})_{eq}$ are the distribution function of χ and $\bar{\chi}$, if they were in thermal equilibrium. In the last step we used the definition of thermally averaged cross section from where the n_χ terms came into the expression.

4.4 Optical Depth and Last Scattering

Light that travels through a medium can be absorbed by the medium. If we have a source emiting an intensity I_0 then an observer a distance r away from the source will observe an intensity

$$I(r) = I_0 e^{-\tau(r)}.$$

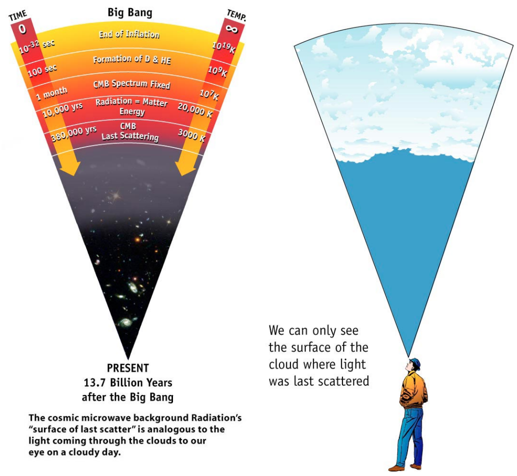


Figure 4.4: Last Scattering surface (credit: NASA/WMAP science team)

Chapter 5

Quantum Field Theory in curved spacetime

5.1 Diffeomorphism Invariance and Vacuum

The following discussion is adapted from Section 4.5 of *Advances in Algebraic Quantum Field Theory* and related literature.

In quantum field theory on Minkowski spacetime, Poincaré covariance is invaluable in specifying the vacuum state. It allows for the characterization of physical states based on a vacuum defined as a translationally invariant state annihilated by all Poincaré Group generators. However, in curved spacetimes, global Poincaré invariance is unavailable, and other criteria must be devised.[1] In flat spacetime, the vacuum is unique; in curved spacetime, this definition is not valid globally, necessitating a new definition.

We adopt an operational or "For-All-Practical-Purposes" (FAPP) perspective, focusing on local measurements by specific observers.

5.1.1 The FAPP Approach

We rely on three core principles to define the vacuum in this context:

1. The Energy (Hamiltonian) operator is the generator of time translations.
2. In quantum field theory, the **vacuum** state is the state of lowest energy; i.e., it is the lowest eigenvalue eigenstate of the Hamiltonian.[2] While typically unique in free field theories on flat space, in curved spacetime with time-dependent fields, a state minimizing energy at t_0 is not guaranteed to satisfy the minimization condition at a later time.
3. Proper time is an observer-dependent, **local** concept, applicable only within a neighborhood of that observer.

Critiques often suggest that definition (1) requires a spacetime with global time-translation symmetry. However, in the neighborhood of any observer, we can define an *effective* Hamiltonian:

$$H = \int_R d^3x\, T^{00} \tag{5.1}$$

where T^{ab} is the energy-momentum tensor and R is a spatial neighborhood of the observer. The coordinate system is chosen such that the "time" index (0) aligns with the observer's proper time. This operator H acts as the generator of time translations within the neighborhood R. Consequently, any state $|0\rangle$ that minimizes the expectation value of H qualifies as a vacuum state **locally, for that observer**.

> **Warning:** The integration region R must be of finite spatial extent (macroscopic). If R is microscopic, the energy density T^{00} is not bounded below due to quantum inequalities (see Epstein-Glaser renormalization). We assume here that spacetime curvature and observer acceleration are mild enough to allow R to be sufficiently large.

This local energy operator illustrates the concept of a **splittable symmetry**—a symmetry applied locally to a subsystem. This is the relevant form of time translation when considering localized observers.

[1] See p. 141 of "Advances in Algebraic Quantum Field Theory".
[2] Refer to Section 6.3.2 of "Introduction to Quantum Effects in Gravity" by Sergei Winitzki for a detailed discussion.

Application to Inertial Observers in Flat Spacetime

In flat spacetime, thanks to global time-translation symmetry, the vacuum state can be defined globally. Thanks to Poincaré symmetry, the vacuum state defined by global energy minimization is identical for all inertial observers, even though their energy operators differ. Therefore, all inertial observers in flat spacetime agree on the definition of the vacuum and, consequently, the definition of "particle."

Accelerating Observer in Flat Spacetime: The Unruh Effect

For a uniformly accelerating observer in flat spacetime, we use the **boost generator**, denoted by K, as the operator that generates translations of the observer's proper time.

While K is globally defined, its role here is local: it generates time translations in the neighborhood of the accelerating observer. We can define a set of states that minimize the expectation value of a local version of K. These states appear as the vacuum **locally to that observer**. However, none of these states coincide with the Minkowski vacuum. This discrepancy explains why an accelerating observer detects particles (thermal radiation) in the Minkowski vacuum, and conversely, why inertial observers perceive particles in the accelerating observer's vacuum. This is a direct application of the principles 1, 2, and 3.

The Hawking Effect

This framework extends naturally to the Hawking effect. The vacuum defined by the generator of time translations at asymptotic infinity differs from the local vacuum defined by the generator of proper time for an observer hovering near the horizon. This mismatch leads to particle production.

The Ambiguity of Instantaneous Particles

A critical limitation of the particle concept in curved spacetime is the ambiguity of an "instantaneous" particle number. As pointed out by Fulling in section **Particle Observables at finite times** of *Aspects of Quantum Field Theory in Curved Spacetime* and his 1979 work.

5.1.2 Quantization

The primary objective of this section is to solve the Klein-Gordon (K.G.) equation and subsequently impose the commutator constraints. We ensure that these constraints are transferred directly to the creation and annihilation operators. This discussion is based on Chapter 1 of "Lectures on Quantum Gravity" by Donald Marolf and "QFT in Curved Spaces" by Marcos Mariño.

Assuming the K.G. equation possesses two solutions, f and g, the canonical commutation relation is defined via the inner product:

$$[a(f), a^\dagger(g)] = \langle f, g\rangle$$

where[3] the scalar product is given by:

$$\langle f, g\rangle = i\int d^3x\sqrt{|h|}\{f^\star(\partial_t g) - (\partial_t f^\star)g\} = \langle g, f\rangle^\star = -\langle g^\star, f^\star\rangle \tag{5.2}$$

The final equality in the equation above can be explicitly verified as follows:

$$\langle g^\star, f^\star\rangle = i\int d^3x\sqrt{|h|}\{g(\partial_t f^\star) - (\partial_t g)f^\star\} = -\langle f, g\rangle$$

We begin by defining the field expansion:

$$\phi(t, x) = \sum_{\boldsymbol{k}}(a_{\boldsymbol{k}}u_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}u^\star_{\boldsymbol{k}})$$

Using the orthogonality of the modes, we isolate the annihilation operator:

$$a_{\boldsymbol{k}} = \langle u_{\boldsymbol{k}}, \phi(t, x)\rangle$$

[3]Refer to pg. 44 of Birrell and Davies, which notes that the value of $\langle f, g\rangle$ is independent of spacetime foliation, or pg. 103 of *Advanced GR and QFT in Curved Spacetime* by Stefano Liberati.

5.1.3 Bogoliubov Transformation

In classical mechanics, canonical transformations are defined as coordinate changes that preserve the structure of Poisson Brackets, i.e.:

$$\{P, Q\}_{\text{PB}} = \{P', Q'\}_{\text{PB}}$$

An analogous concept exists in quantum field theory for creation and annihilation operators. We start by imposing the condition that the commutation relations must remain invariant under the transformation:

$$[a, a^\dagger] = [b, b^\dagger]$$

Consider the general linear transformation:

$$\begin{aligned} b &= c_1 a + c_2 a^\dagger \\ b^\dagger &= d_1 a + d_2 a^\dagger \end{aligned}$$

Since $b^\dagger$ is the Hermitian conjugate of b, we determine the relationship between coefficients:

$$\begin{aligned} b^\dagger &= (c_1 a + c_2 a^\dagger)^\dagger \\ &= c_1^\star a^\dagger + c_2^\star a \\ \implies d_1 &= c_2^\star \\ d_2 &= c_1^\star \end{aligned}$$

Now, we compute the commutator for the new operators:

$$\begin{aligned} [b, b^\dagger]_\pm &= [c_1 a + c_2 a^\dagger, c_1^\star a^\dagger + c_2^\star a]_\pm \\ &= c_1^\star [c_1 a + c_2 a^\dagger, a^\dagger]_\pm + c_2^\star [c_1 a + c_2 a^\dagger, a]_\pm \\ &= c_1^2 [a, a^\dagger]_\pm + c_1^\star c_2 \cancelto{0}{[a^\dagger, a^\dagger]} + c_2^\star c_1 \cancelto{0}{[a, a]} + c_2^2 [a^\dagger, a]_\pm \\ &= (c_1^2 \mp c_2^2)[a, a^\dagger]_\pm \end{aligned}$$

where $\pm$ refers to fermionic (anticommutator) or bosonic (commutator) operators respectively. We will apply this concept to the study of Unruh radiation. The procedure involves three steps:

1. Solve the K.G. equation in Minkowski coordinates.
2. Solve the K.G. equation in Rindler coordinates.
3. Find the transformation of creation and annihilation operators between the two coordinate systems which preserves the commutation relationship.

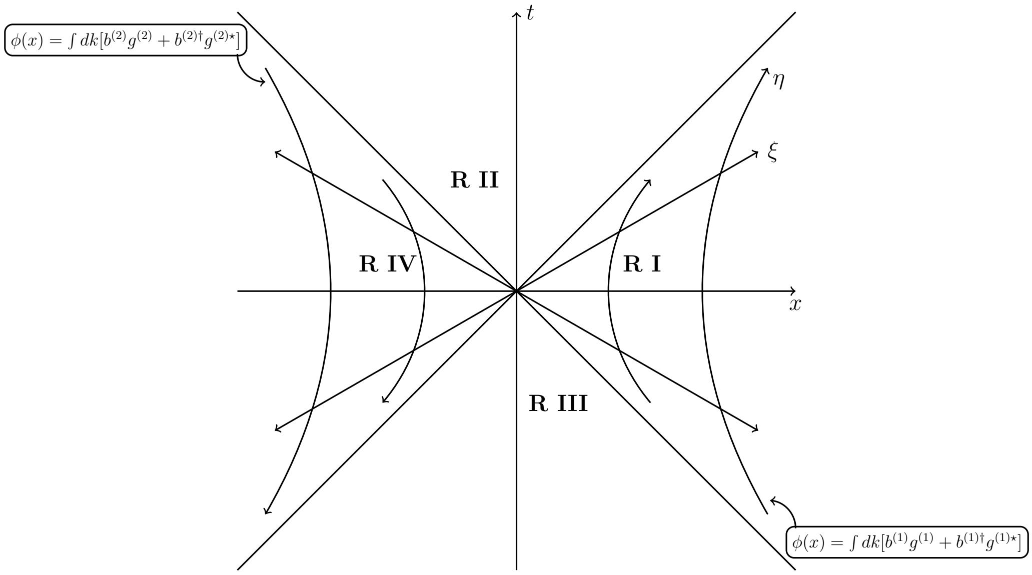


Figure 5.1: Rindler Wedge

A crucial task is to determine how the operators transform given the transformation of the field modes. **How can we find the corresponding transformation between the creation and annihilation operators** in the two coordinate systems? We start with the field expansions in the two bases:

$$\phi(x) = \sum_i [\hat{a}_i f_i + \hat{a}_i^\dagger f_i^\star] \qquad\qquad \phi(x) = \sum_i [\hat{b}_i g_i + \hat{b}_i^\dagger g_i^\star]$$

Let us consider the following linear transformation between the mode functions:

$$f_i = \sum_j \alpha_{ij} g_j + \beta_{ij} g_j^\star$$

Substituting this relation into the first field expansion, we derive:

$$\begin{aligned}
\phi(x) &= \sum_i [\hat{a}_i f_i + \hat{a}_i^\dagger f_i^\star] \\
&= \sum_i \left[\hat{a}_i \sum_j (\alpha_{ij} g_j + \beta_{ij} g_j^\star) + \hat{a}_i^\dagger \sum_j (\alpha_{ij}^\star g_j^\star + \beta_{ij}^\star g_j) \right] \\
&= \sum_i \left[\sum_j (\alpha_{ij} \hat{a}_i + \beta_{ij}^\star \hat{a}_i^\dagger) g_j + \sum_j (\beta_{ij} \hat{a}_i + \alpha_{ij}^\star \hat{a}_i^\dagger) g_j^\star \right] \\
&= \sum_j \Bigg(\underbrace{\sum_i (\alpha_{ij} \hat{a}_i + \beta_{ij}^\star \hat{a}_i^\dagger)}_{\hat{b}_j} g_j + \underbrace{\sum_i (\beta_{ij} \hat{a}_i + \alpha_{ij}^\star \hat{a}_i^\dagger)}_{\hat{b}_j^\dagger} g_j^\star \Bigg) \\
&= \sum_j [\hat{b}_j g_j + \hat{b}_j^\dagger g_j^\star]
\end{aligned}$$

If we express this in matrix form, the mode transformation is:

$$\begin{bmatrix} f_i \\ f_i^\star \end{bmatrix} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} \\ \beta_{ij}^\star & \alpha_{ij}^\star \end{bmatrix} \begin{bmatrix} g_j \\ g_j^\star \end{bmatrix}$$

And the corresponding operator transformation is:

$$\begin{bmatrix} \hat{b}_i \\ \hat{b}_i^\dagger \end{bmatrix} = \begin{bmatrix} \alpha_{ji} & \beta_{ji}^\star \\ \beta_{ji} & \alpha_{ji}^\star \end{bmatrix} \begin{bmatrix} \hat{a}_j \\ \hat{a}_j^\dagger \end{bmatrix}$$

Similarly, we consider the inverse transformation for the modes:

$$g_i = \sum_j \lambda_{ij} f_j + \mu_{ij} f_j^\star$$

The coefficients can be extracted by utilizing the normalization condition:

$$\begin{aligned}
\langle f_i, f_j \rangle &= \delta_{ij} \\
\langle f_i^\star, f_j^\star \rangle &= -\langle f_i, f_j \rangle^\star = -\delta_{ij}
\end{aligned}$$

The last condition is a direct consequence of the definition of the Klein-Gordon inner product. We compute:

$$\begin{aligned}
\langle g_i, f_j \rangle &= \left\langle \sum_k \lambda_{ik} f_k + \mu_{ik} f_k^\star, f_j \right\rangle \\
&= \sum_k \lambda_{ik} \langle f_k, f_j \rangle + \mu_{ik} \cancelto{0}{\langle f_k^\star, f_j \rangle} \\
&= \sum_k \lambda_{ik} \delta_{kj} = \lambda_{ij}
\end{aligned}$$

and

$$\langle g_i, f_j^\star \rangle = \left\langle \sum_k \lambda_{ik} f_k + \mu_{ik} f_k^\star, f_j^\star \right\rangle$$

$$
\begin{aligned}
&= \sum_k \lambda_{ik} \langle f_k, f_j^\star \rangle + \mu_{ik} \langle f_k^\star, f_j^\star \rangle \\
&= \sum_k -\mu_{ik}\delta_{kj} = -\mu_{ij}
\end{aligned}
$$

Utilizing the skew-symmetry property:

$$\langle a, b\rangle = \langle b, a\rangle^\star = -\langle a^\star, b^\star\rangle$$

we identify the coefficients:

$$
\begin{aligned}
\langle g_i, f_j\rangle &= \langle f_j, g_i\rangle^\star = \alpha_{ji}^\star \\
\langle g_i, f_j^\star\rangle &= \langle f_j^\star, g_i\rangle^\star = -\underbrace{\langle f_j, g_i^\star\rangle}_{-\beta_{ji}} \\
&= \beta_{ji}
\end{aligned}
$$

Hence, the inverse mode transformation is:

$$g_i = \sum_j (\alpha_{ji}^\star f_j - \beta_{ji} f_j^\star)$$

Finally, the inverse transformation for the operators can be derived by substituting back into the field expansion:

$$
\begin{aligned}
\phi(x) &= \sum_i [\hat{b}_i g_i + \hat{b}_i^\dagger g_i^\star] \\
&= \sum_i \left[\hat{b}_i \sum_j (\alpha_{ji}^\star f_j - \beta_{ji} f_j^\star) + \hat{b}_i^\dagger \sum_j (\alpha_{ji} f_j^\star - \beta_{ji}^\star f_j) \right] \\
&= \sum_i \left[\sum_j (\alpha_{ji}^\star \hat{b}_i - \beta_{ji}^\star \hat{b}_i^\dagger) f_j + \sum_j (\alpha_{ji} \hat{b}_i^\dagger - \beta_{ij} \hat{b}_i) f_j^\star \right] \\
&= \sum_j \Bigg(\underbrace{\sum_i (\alpha_{ji}^\star \hat{b}_i - \beta_{ji}^\star \hat{b}_i^\dagger)}_{\hat{a}_j} f_j + \underbrace{\sum_i (\alpha_{ji} \hat{b}_i^\dagger - \beta_{ij} \hat{b}_i)}_{\hat{a}_j^\dagger} f_j^\star \Bigg) \\
&= \sum_j [\hat{a}_j f_j + \hat{a}_j^\dagger f_j^\star]
\end{aligned}
$$

Expressed in matrix form:

$$
\begin{bmatrix} g_i \\ g_i^\star \end{bmatrix} = \begin{bmatrix} \alpha_{ji}^\star & -\beta_{ji} \\ -\beta_{ji}^\star & \alpha_{ji} \end{bmatrix} \begin{bmatrix} f_j \\ f_j^\star \end{bmatrix}
$$
$$
\begin{bmatrix} \hat{a}_i \\ \hat{a}_i^\dagger \end{bmatrix} = \begin{bmatrix} \alpha_{ij}^\star & -\beta_{ij}^\star \\ -\beta_{ij} & \alpha_{ij} \end{bmatrix} \begin{bmatrix} \hat{b}_j \\ \hat{b}_j^\dagger \end{bmatrix}
$$

The above transformation corresponds to bosons, since:

$$\alpha^2 - \beta^2 = 1$$

In the case of fermions, we would have:

$$\alpha^2 + \beta^2 = 1$$

though the inner product would be defined in an equivalent manner.

Ambiguity of Particle Content

Suppose the system is in the vacuum state $|0_a\rangle$ defined by the a-operators (where $\hat{a}_i |0_a\rangle = 0$). The "b-observer" (e.g., an accelerating observer) measures the number of particles using the operator $\hat{N}_j^{(b)} = \hat{b}_j^\dagger \hat{b}_j$. The expectation value of this number operator in the a-vacuum is:

$$\langle 0_a | \hat{N}_j^{(b)} |0_a\rangle = \langle 0_a | \hat{b}_j^\dagger \hat{b}_j |0_a\rangle$$

$$= \langle 0_a | \sum_{k,l} (\beta_{jk}\hat{a}_k + \alpha^\star_{jk}\hat{a}^\dagger_k)(\alpha_{jl}\hat{a}_l + \beta^\star_{jl}\hat{a}^\dagger_l) |0_a\rangle$$
$$= \sum_k |\beta_{jk}|^2$$

If $\beta_{jk} \neq 0$, the a-vacuum contains a non-zero density of b-particles. This result demonstrates that the particle concept is observer-dependent.

5.1.4 Rindler Spacetime

The accelerated observer is described by the trajectory:

$$ct(\tau) = \frac{c}{\alpha}\sinh(\alpha\tau) \qquad\qquad x(\tau) = \frac{c}{\alpha}\cosh(\alpha\tau)$$

and, we observe that

$$x^2 - c^2t^2 = \frac{c^2}{\alpha^2}$$

where α is the norm of 4-acceleration.

$$\alpha^\mu\alpha_\mu =$$

We can define the new hyperbolic coordinate, more acquainted to study accelerated motion.

$$ct = \frac{ce^{a\xi/c}}{a}\sinh(a\eta) \qquad\qquad x = \frac{ce^{a\xi/c}}{a}\cosh(a\eta) \tag{5.3}$$

Let us for a minute look back at the coordinate transformation. Clearly this covers only the region $x > 0$ and as

$$x - t = \frac{c}{a}e^{a\xi/c}[\cosh(a\eta) - \sinh(a\eta)] = \frac{c}{a}e^{a\xi/c}e^{a\eta} > 0$$

we have that our coordinate transformation **only covers the region $\mathbf{x} > |\mathbf{t}|$ and not the entire space-time**. In order to have a coordinate system over the whole of space-time we also need to cover the three other regions. This is R I in Fig 5.2, **R II and R III are space-like regions, so we are not to worried about them**. In fact they can be reached by analytical continuation if we would so desire. For R IV we can simply flip the signs and we then have

$$ct = -\frac{ce^{a\xi/c}}{a}\sinh(a\eta) \qquad x = -\frac{ce^{a\xi/c}}{a}\cosh(a\eta) \qquad \text{for } x < |t| \tag{5.4}$$

Strictly speaking we are abusing notation as in both R I and R IV the coordinates go from $-\infty$ to $+\infty$. However we will solve this by carefully identifying which region we are working in. The only point we have not covered then is the origin, but that is a point of zero measure and should not bother us. We will come back to this issue of analytic continuation in section 5.1.6 when we study bogoliubov transformation. Let us move past this ambiguity for the moment. In the new $(c\eta, \xi)$ coordinates, metric would look like:

$$\begin{aligned}
ds^2 &= c^2dt^2 - dx^2 \\
&= \left(d\left[\frac{ce^{a\xi/c}}{a}\sinh(a\eta)\right]\right)^2 - \left(d\left[\frac{ce^{a\xi/c}}{a}\cosh(a\eta)\right]\right)^2 \\
&= \left[\frac{\cancel{c}e^{a\xi/c}}{\cancel{a}}\frac{\cancel{a}}{\cancel{c}}\sinh(a\eta)d\xi + \frac{ce^{a\xi/c}}{\cancel{a}}\cancel{a}\cosh(a\eta)d\eta\right]^2 \\
&\qquad - \left[\frac{\cancel{c}e^{a\xi/c}}{\cancel{a}}\frac{\cancel{a}}{\cancel{c}}\cosh(a\eta)d\xi + \frac{ce^{a\xi/c}}{\cancel{a}}\cancel{a}\sinh(a\eta)d\eta\right]^2 \\
&= e^{2a\xi/c}[c^2d\eta^2 - d\xi^2]
\end{aligned}$$

[4] The proper acceleration in this coordinate is given by

$$x^2 - c^2t^2 = \frac{c^2}{\alpha^2}$$

[4] under the definition $d\bar{\xi} = \left(1 + \frac{a}{c}\bar{\xi}\right)d\xi \implies \xi = \frac{c}{a}\ln(1 + \frac{a}{c}\bar{\xi}) \implies e^{a\bar{\xi}/c} = 1 + \frac{a}{c}\xi$, we get:

$$ds^2 = \left(1 + \frac{a}{c}\bar{\xi}\right)^2 c^2d\eta^2 - d\bar{\xi}^2$$

$$\frac{c^2 e^{2a\xi/c}}{a^2}\left[\cosh^2(a\eta) - \sinh^2(a\eta)\right] = \frac{c^2}{\alpha^2}$$
$$\implies \alpha = a e^{-a\xi/c}$$

We can easily spot that the metric tensor is independent of $c\eta$ and thus $\partial_{c\eta} \equiv (1, 0)$ is the killing vector in this coordinate. The norm of this killing vector field is given by

$$\begin{aligned} \boldsymbol{g}(\partial_\eta, \partial_\eta) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{bmatrix} g^{\eta\eta} & g^{\eta\xi} \\ g^{\eta\xi} & g^{\xi\xi} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= g^{\eta\eta} \\ &= e^{2a\xi/c} > 0 \end{aligned} \tag{5.5}$$

This is timelike killing vector. Switching to natural units with $c = 1$. The Klein Gordon Equation of massless scalar field for the Rindler observer is:

$$\begin{aligned} \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi) &= 0 \\ e^{-2a\xi}\partial_\mu(e^{2a\xi}g^{\mu\nu}\partial_\nu)\phi &= 0 \\ e^{-2a\xi}\partial_\eta(\underbrace{e^{2a\xi}g^{\eta\eta}}_{1}\partial_\eta)\phi + e^{-2a\xi}\partial_\xi(\underbrace{e^{2a\xi}g^{\xi\xi}}_{-1}\partial_\xi)\phi &= 0 \\ e^{-2a\xi}(\partial_\eta^2 - \partial_\xi^2)\phi &= 0 \end{aligned}$$

These are differential equation with constant coefficient, thus the general solution is linear combination of plane wave solutions.

This minus sign can be absorbed if we use $(-+++)$ metric signature

$$\phi = \sum A e^{-\,ik_\mu x^\mu} + B e^{ik_\mu x^\mu} = \sum A e^{-ik_\eta \eta + ik_\xi \xi} + B e^{ik_\eta \eta - ik_\xi \xi}$$

with $k^\mu = (k_\eta, k_\xi)$. The solution to Klein Gordon Equation in minkowski coordinate is given via:

$$\phi = \sum\ A\ e^{-ik_t t + ik_x x} +\ B\ e^{ik_t t - ik_x x}$$

will be interpreted as annihilation operator

will be interpreted as creation operator

5.1.5 Killing Vector Field

In general relativity, killing vector fields are defined as the generator of isometry in the same sense, as we define momentum as generator of translation in Quantum Field Theory. The group theoretic structure of these killing vector fields emerges from their lie algebra. I'd like to remind us that from weinberg's QFT volume 1, we learnt that the eigenvalue of the generator of symmetry is the conserved quantity. Therefore one can expect a version of the same to be true even here. We will use the killing vector field to define the basis, but first we need to revise certain things.

A general coordinate transformation can be expressed as[5]:

$$\begin{aligned} y^\alpha &= e^{\epsilon \mathcal{L}_\xi} x^\alpha = e^{\epsilon\xi} x^\alpha \\ &\approx x^\alpha - \epsilon\xi^\alpha \end{aligned} \qquad (\xi = \xi^\mu \partial_\mu)$$

In the exact above sense, we interpret the killing vector ξ as the generator of isometry i.e. they preserve the norm of 4-vectors. Killing vectors are quite useful in the General Relativity as they lead to certain conserved quantity along the geodesic as follows:

$$\begin{aligned} \frac{\mathrm{d}K_\mu p^\mu}{\mathrm{d}\lambda} &= p^\nu \nabla_\nu (K_\mu p^\mu) \\ &= p^\nu(\nabla_\nu K_\mu)p^\mu + p^\nu K_\mu(\nabla_\nu p^\mu) \\ &= \frac{1}{2}p^\nu p^\mu(\nabla_\nu K_\mu + \nabla_\mu K_\nu) + K_\mu(p^\nu \nabla_\nu p^\mu) \\ &= 0 \end{aligned}$$

[5] M. Fecko's "Differential Geometry and Lie groups for Physicists" defines the coordinate x^α as "scalar-functions". In ch. 4 he defines the exponential map of Lie derivative as pullback. We borrowed the same idea here

The first term vanishes due to Killing equation and the second term due to Geodesic Equation. Another insight can be borrowed from studying the exponential map of killing vector fields in Minkowski spacetime. We begin by first finding out the killing vectors:

$$\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = 0$$

Note that $\Gamma^i_{jk} = 0$ in Minkowski coordinates. For $\mu = \nu = t$

$$\nabla_t \chi_t + \nabla_t \chi_t = 0 \implies \chi_t = f(x)$$

for $\mu = \nu = x$

$$\nabla_x \chi_x + \nabla_x \chi_x = 0 \implies \chi_x = g(t)$$

for $\mu = t$ and $\nu = x$

$$\begin{aligned} \nabla_t \chi_x + \nabla_x \chi_t &= 0 \\ \frac{1}{c}\frac{\partial g(t)}{\partial t} + \frac{\partial f(x)}{\partial x} &= 0 \end{aligned}$$

This implies that the solution are

$$f(x) = -x \qquad\qquad g(t) = ct$$

Therefore, in component form:

$$\chi^\mu = (x, ct, 0, 0) \qquad\qquad \chi_\mu = (-x, ct, 0, 0)$$

or, with $c = 1$

$$\chi = x\partial_t + t\partial_x \tag{5.6}$$

We will use this representation to derive the Lorentz Transformation. Consider the following:

$$\begin{aligned} e^{\beta\chi} x &= \sum_{n=0}^{\infty} \frac{1}{n!}\beta^n \left(x\partial_t + t\partial_x\right)^n x \\ &= x + \beta\left(x\partial_t + t\partial_x\right) x + \frac{\beta^2}{2!}\left(x\partial_t + t\partial_x\right)^2 x + \frac{\beta^3}{3!}\left(x\partial_t + t\partial_x\right)^3 x \dots \\ &= x + \beta t + \frac{\beta^2}{2!}\left(x\partial_t + t\partial_x\right) t + \frac{\beta^3}{3!}\left(x\partial_t + t\partial_x\right) x \dots \\ &= \left(1 + \frac{\beta^2}{2!} + \dots\right) x + \left(\beta + \frac{\beta^3}{3!} + \dots\right) t \\ &= \cosh(\beta) x + \sinh(\beta) t \end{aligned}$$

Similarly

$$\begin{aligned} e^{\beta\chi} t &= \sum_{n=0}^{\infty} \frac{1}{n!}\beta^n \left(x\partial_t + t\partial_x\right)^n t \\ &= t + \beta\left(x\partial_t + t\partial_x\right) t + \frac{\beta^2}{2!}\left(x\partial_t + t\partial_x\right)^2 t + \frac{\beta^3}{3!}\left(x\partial_t + t\partial_x\right)^3 t \dots \\ &= t + \beta x + \frac{\beta^2}{2!}\left(x\partial_t + t\partial_x\right) x + \frac{\beta^3}{3!}\left(x\partial_t + t\partial_x\right) t \dots \\ &= \left(1 + \frac{\beta^2}{2!} + \dots\right) t + \left(\beta + \frac{\beta^3}{3!} + \dots\right) x \\ &= \cosh(\beta) t + \sinh(\beta) x \end{aligned}$$

and

$$\begin{aligned} e^{\beta\chi} y &= y \\ e^{\beta\chi} z &= z \end{aligned}$$

Represented in matrix form:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cosh(\beta) & \sinh(\beta) & 0 & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

The above transformation describes boost. Since, it is used to jump between the frame of references. The conserved quantity associated with this is:

$$\chi_\mu p^\mu = -xE + p_x t$$

is due to $\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = 0$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\chi_\mu p^\mu &= -E\frac{\mathrm{d}x}{\mathrm{d}\tau} + p_x\frac{\mathrm{d}t}{\mathrm{d}\tau} - x\cancelto{0}{\frac{\mathrm{d}E}{\mathrm{d}\tau}} + t\cancelto{0}{\frac{\mathrm{d}p_x}{\mathrm{d}\tau}} \\ &= -\gamma E v + \gamma^2 m v \\ &= -\gamma^2 m v + \gamma^2 m v && (\text{using } E = \gamma m) \\ &= 0 \end{aligned}$$

We have another killing vector field which describes the rotation about z-axis

$$\xi = (0, -y, x, 0) \equiv -y\partial_x + x\partial_y$$

The relevant transformation is:

$$\begin{aligned} e^{\theta\xi}x &= \sum_{n=0}^{\infty} \frac{1}{n!}\theta^n \left(-y\partial_x + x\partial_y\right)^n x \\ &= x + \theta\left(-y\partial_x + x\partial_y\right)x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 x + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 x \ldots \\ &= x - \theta y - \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)y - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)x \ldots \\ &= \left(1 - \frac{\theta^2}{2!} + \ldots\right)x - \left(\theta - \frac{\theta^3}{3!} + \ldots\right)y \\ &= \cos(\theta)x - \sin(\theta)y \end{aligned}$$

Similarly,

$$\begin{aligned} e^{\theta\xi}y &= \sum_{n=0}^{\infty} \frac{1}{n!}\theta^n \left(-y\partial_x + x\partial_y\right)^n y \\ &= y + \theta\left(-y\partial_x + x\partial_y\right)y + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 y + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 y \ldots \\ &= y + \theta x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)x - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)y \ldots \\ &= \left(1 - \frac{\theta^2}{2!} + \ldots\right)y + \left(\theta - \frac{\theta^3}{3!} + \ldots\right)x \\ &= \cos(\theta)y + \sin(\theta)x \end{aligned}$$

Represented in matrix form:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

The conserved quantity associated with this killing vector is:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_\mu p^\mu = 0 \implies \xi_\mu p^\mu = -yp_x + xp_y = J_z$$

Since, the metric in Minkowski spacetime is independent of all coordinates, there are 4 Killing vectors along each of those axis[6].[7][1cm]

$$
\begin{aligned}
K_t &= \partial_t \equiv (1,0,0,0) & K_t p^t &= E \\
K_x &= \partial_x \equiv (0,1,0,0) & K_x p^x &= p_x \\
K_y &= \partial_y \equiv (0,0,1,0) & K_y p^y &= p_y \\
K_z &= \partial_z \equiv (0,0,0,1) & K_z p^z &= p_z
\end{aligned}
$$

The transformation associated with them are:

$$
\begin{aligned}
e^{t' K_t} x(t) &= e^{t'\partial_t} x(t) \\
&= \sum_{n=0}^{\infty} \frac{(t')^n}{n!} \frac{\partial^n x(t)}{\partial t^n} = x(t+t')
\end{aligned}
$$

which is the taylor series expansion of position of particle. The general form of Killing vector field in Minkowski spacetime is given as

$$\xi_\alpha^{(A)} = c_\alpha^{(A)} + \epsilon_{\alpha\beta}^{(A)} x^\beta$$

where $x^\mu = \left(x^0, x^1, x^2, x^3\right)$ and the killing vectors associated with translation are:

$$
\begin{aligned}
c_\alpha^{(1)} &= (1,0,0,0) & \epsilon_{\alpha\beta}^{(1)} &= 0 \\
c_\alpha^{(2)} &= (0,1,0,0) & \epsilon_{\alpha\beta}^{(2)} &= 0 \\
c_\alpha^{(3)} &= (0,0,1,0) & \epsilon_{\alpha\beta}^{(3)} &= 0 \\
c_\alpha^{(4)} &= (0,0,0,1) & \epsilon_{\alpha\beta}^{(4)} &= 0
\end{aligned}
$$

These above generators lead to conserved quantities such as Energy and Momentum. The Killing vectors associated with boost and rotation are:

$$
c_\alpha^{(5)} = 0 \qquad \epsilon_{\alpha\beta}^{(5)} = \begin{array}{c} \begin{array}{cccc} x^0 & x^1 & x^2 & x^3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(5)} x^\beta = \left(x^1, -x^0, 0, 0\right)
$$

$$
c_\alpha^{(6)} = 0 \qquad \epsilon_{\alpha\beta}^{(6)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(6)} x^\beta = \left(x^2, -x^1, 0, 0\right)
$$

$$
c_\alpha^{(7)} = 0 \qquad \epsilon_{\alpha\beta}^{(7)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(7)} x^\beta = \left(x^3, -x^0, 0, 0\right)
$$

$$
c_\alpha^{(8)} = 0 \qquad \epsilon_{\alpha\beta}^{(8)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(8)} x^\beta = \left(0, x^2, -x^1, 0\right)
$$

$$
c_\alpha^{(9)} = 0 \qquad \epsilon_{\alpha\beta}^{(9)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(9)} x^\beta = \left(0, x^3, -x^1, 0\right)
$$

$$
c_\alpha^{(10)} = 0 \qquad \epsilon_{\alpha\beta}^{(10)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(10)} x^\beta = \left(0, 0, x^3, -x^2\right)
$$

The above matrices are the matrix representation of killing vectors which are the generators of Poincaré Group.

---

[6] This is the exact reason why killing vector fields are useful. They don't just provide us the constant of motion but also indicate that there exists a special coordinate system in which the metric is independent of those coordinates

[7] we are defining

$$p^\mu = (E, p_x, p_y, p_z)$$

5.1.6 Back to Wave Equation

Reverting back to the KG equation in Rindler coordinate:

$$\begin{aligned}\partial_\eta &= \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x \\ &= e^{a\xi}\left(\cosh(a\eta)\partial_t + \sinh(a\eta)\partial_x\right) \\ &= a(x\partial_t + t\partial_x)\end{aligned}$$

We quickly observe that this is the generator of boost along x-axis. Referring to (5.5) and (5.6), we find that it is timelike killing vector field. Reminding ourselves that in Quantum Field Theory, all these generators are actually operators and thus, it makes sense to consider their eigenfunctions as the preferred basis[8] for calculations and interpretation of solutions. We consider the eigenvalue equation for plane waves in Minkowski spacetime.

$$\begin{aligned}\partial_t A e^{-ik_t t + ik_x x} &= -ik_t A e^{-ik_t t + ik_x x} \\ \partial_t B e^{ik_t t - ik_x x} &= ik_t B e^{ik_t t - ik_x x}\end{aligned}$$

In the unitary representation, the above equation looks like:

$$\begin{aligned}i\partial_t A e^{-ik_t t + ik_x x} &= k_t A e^{ik_t t - ik_x x} \\ i\partial_t B e^{ik_t t - ik_x x} &= -k_t B e^{-ik_t t + ik_x x}\end{aligned}$$

hence, $Be^{ik_t t - ik_x x}$ are the negative energy solution and $Ae^{-ik_t t + ik_x x}$ are positive energy solution. Therefore, it is reasonable to expect that A will be interpreted as 'annihilation' operator which destroys particle and B will be referred as 'creation' operator[9]. For Rindler observer[10]:

$$\begin{aligned}\partial_\eta A e^{-ik_\eta \eta + ik_\xi \xi} &= -ik_\eta A e^{-ik_\eta \eta + ik_\xi \xi} \\ \partial_\eta B e^{ik_\eta \eta - ik_\xi \xi} &= ik_\eta B e^{ik_\eta \eta - ik_\xi \xi}\end{aligned}$$

Thus, based on the argument of identifying the coefficient of positive eigenvalue plane wave solution as the annihilation operator. We interpret A as the annihilation operator and B as the creation operator. Hence,

$$\phi_{\text{minkowski}} = \int_{-\infty}^{\infty} \frac{dk_x}{\sqrt{2\pi(2k_t)}}[a_{k_x} e^{-ik_t t + ik_x x} + a^\dagger_{k_x} e^{ik_t t - ik_x x}]$$

and

$$\phi_{\text{rindler}} = \int_{-\infty}^{\infty} \frac{dk_\xi}{\sqrt{2\pi(2k_\eta)}}[b_{k_\xi} e^{-ik_\eta \eta + ik_\xi \xi} + b^\dagger_{k_\xi} e^{ik_\eta \eta - ik_\xi \xi}]$$

We will follow the method described in "Particle Creation in Vacuum states" for finding the relationship between the two creation and annihilation operator in different basis. The key point is the following: since the wave equation is conformally invariant, and Rindler is conformal to Minkowski, we can write the mode expansion in terms of plane waves just as we do in the flat space case. The additional simplification afforded by the use of lightcone coordinates is that the contribution from positive-frequency modes in this expansion can be directly matched to the corresponding positive-frequency contribution in flat space (similarly for the negative frequencies). The lack of mode mixing in these coordinates greatly shortens the computation of the Bogolyubov coefficients, because — in addition to avoiding the case-by case treatment — it enables us to isolate the desired modes via a simple Fourier transform, rather than needing to perform the full Klein-Gordon inner product. We begin by first splitting the field in positive mode and negative mode.

$$\begin{aligned}\phi_{\text{rindler}} = \int_{0}^{\infty} &\frac{dk_\xi}{\sqrt{2\pi(2k_\eta)}}[b_{k_\xi} e^{-ik_\eta \eta + ik_\xi \xi} + b^\dagger_{k_\xi} e^{ik_\eta \eta - ik_\xi \xi}] \\ &+ \int_{-\infty}^{0} \frac{dk_\xi}{\sqrt{2\pi(2|k_\eta|)}}[b_{k_\xi} e^{ik_\eta \eta + ik_\xi \xi} + b^\dagger_{k_\xi} e^{-ik_\eta \eta - ik_\xi \xi}] \qquad (5.7)\end{aligned}$$

From the wave equation

$$({}^{1}\!/{}_{c^2}\partial_\eta^2 - \partial_\xi^2)e^{-ik_\eta \eta + ik_\xi \xi} = 0$$

[8]pg 14, QFT in curved space by Marcos Marino says "One possible, "natural" choice of modes occurs when the spacetime has a Killing vector. Then,one can choose the modes in such a way that they are eigenfunctions of the Killing vector. This is indeed what one does in Minkowski space, where ∂_t is Killing."

[9]QFT in Curved space by Marcos Marino, pg 10

[10]we should remind ourselves that here we are using the eigenbasis of generator of boost for mode expansion

$$(-k_\eta^2/c^2 + k_\xi^2)e^{-ik_\eta\eta + ik_\xi\xi} = 0 \implies k_\eta = |k_\xi c|$$

Defining $k \equiv |k_\xi|$ and using $u_\pm = \eta \pm \xi$ with $c = 1$, to switch to null/lightcone coordinates:

$$\phi_{\text{rindler}} = \int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_-} + b_k^\dagger e^{iku_-}] + \int_0^\infty \frac{dk}{\sqrt{2\pi(2|k|)}}[b_{-k} e^{-iku_+} + b_{-k}^\dagger e^{iku_+}] \tag{5.8}$$

doing the same $x_\pm = ct \pm x$ with $\omega = |k_x|$ in Minkowski spacetime.[11]

$$\begin{aligned}\phi_{\text{minkowski}} &= \int_{-\infty}^\infty \frac{dk_x}{\sqrt{2\pi(2k_t)}}[a_{k_x} e^{-ik_t t + ik_x x} + a_{k_x}^\dagger e^{ik_t t - ik_x x}] \\ &= \int_{-\infty}^\infty \frac{dk_x}{\sqrt{2\pi(2|k_x|)}}[a_{k_x} e^{-i|k_x c|t + ik_x x} + a_{k_x}^\dagger e^{i|k_x c|t - ik_x x}] \qquad (\text{using } k_t = |k_x c|)\end{aligned}$$

using $\omega = |k_x|$

$$= \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i\omega x_-} + a_\omega^\dagger e^{i\omega x_-}] + \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2|\omega|)}}[a_{-\omega} e^{-i\omega x_+} + a_{-\omega}^\dagger e^{i\omega x_+}]$$

Now comes the chief advantage of this approach mentioned above: since the notion of positive/negative momenta is preserved under the conformal transformation from Minkowski to Rindler space, we can directly identify

$$\int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_-} + b_k^\dagger e^{iku_-}] = \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i\omega x_-} + a_\omega^\dagger e^{i\omega x_-}]$$

$$\int_0^\infty \frac{dk}{\sqrt{2\pi(2|k|)}}[b_{-k} e^{-iku_+} + b_{-k}^\dagger e^{iku_+}] = \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2|\omega|)}}[a_{-\omega} e^{-i\omega x_+} + a_{-\omega}^\dagger e^{i\omega x_+}]$$

Peforming the fourier transform: [12]

$$\begin{aligned}&\int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}} e^{i\omega' x_-} \int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_-} + b_k^\dagger e^{iku_-}] \\ &\qquad = \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}} e^{i\omega' x_-} \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i\omega x_-} + a_\omega^\dagger e^{i\omega x_-}] \\ &\qquad = \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i(\omega-\omega')x_-} + a_\omega^\dagger e^{i(\omega+\omega')x_-}] \\ &\qquad = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}}[a_\omega \delta(\omega - \omega') + a_\omega^\dagger \delta(\omega + \omega')]\end{aligned}$$

Since the above mode corresponds to positive frequencies i.e. $\omega' > 0$:

$$\begin{aligned}\frac{a_\omega}{\sqrt{2\omega}} &= \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_- + i\omega' x_-} + b_k^\dagger e^{iku_- + i\omega' x_-}] \\ a_\omega &= \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\omega}{k}}[b_k e^{-iku_- + i\omega' x_-} + b_k^\dagger e^{iku_- + i\omega' x_-}]\end{aligned}$$

Since the above mode corresponds to positive frequencies i.e. $\omega' > 0$:

$$\frac{a_\omega}{\sqrt{2\omega}} = \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_- + i\omega' x_-} + b_k^\dagger e^{iku_- + i\omega' x_-}]$$

[11] with signature $(+---)$

$$\begin{aligned}e^{-ik_\mu x^\mu} &= e^{-i\frac{k_t}{c}ct + ik_x x} \\ &= e^{-ik_t t + ik_x x}\end{aligned}$$

[12] using

$$\delta(x) = \int_{-\infty}^\infty \frac{dk}{\sqrt{2\pi}} e^{-ikx} \frac{1}{\sqrt{2\pi}}$$

$$a_\omega = \int_{-\infty}^{\infty} \frac{dx_-}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\omega}{k}} [b_k e^{-iku_- + i\omega' x_-} + b_k^\dagger e^{iku_- + i\omega' x_-}]$$

we can express the above entirely in terms of minkowski lightcone coordinate as:

$$\begin{aligned} ds^2_{\text{minkowski}} &= ds^2_{\text{rindler}} \\ dt^2 - dx^2 &= e^{2a\xi}(d\eta^2 - d\xi^2) \\ dx_- dx_+ &= e^{-a(u_- - u_+)} du_- du_+ \end{aligned} \tag{5.9}$$

From, above we see that (since there's no mixing of coordinates)

$$\begin{aligned} x_- &= \frac{e^{-au_-}}{-a} \\ x_+ &= \frac{e^{au_+}}{a} \end{aligned}$$

or, using (5.3)

$$\begin{aligned} x_\pm &= t \pm x \\ &= \frac{e^{a\xi}}{a}[\sinh(a\eta) \pm \cosh(a\eta)] \\ &= \frac{e^{a\xi}}{a}\left[\frac{e^{a\eta} - e^{-a\eta}}{2} \pm \frac{e^{a\eta} + e^{-a\eta}}{2}\right] \\ &= \frac{e^{a\xi}}{a}\left[\frac{e^{a\eta} \pm e^{a\eta}}{2} - \frac{e^{-a\eta} \mp e^{-a\eta}}{2}\right] \\ &= \pm\frac{e^{a(\xi\pm\eta)}}{a} = \pm\frac{e^{\pm au_\pm}}{a} \end{aligned} \tag{5.10}$$

Thus,

$$a_\omega = \int_{-\infty}^{\infty} \frac{dx_-}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\omega}{k}} \left[b_k e^{-iku_- + i\omega' \frac{e^{-au_-}}{a}} + b_k^\dagger e^{iku_- + i\omega' \frac{e^{-au_-}}{a}}\right]$$

Similarly, the inverse transformation is given as:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} e^{ik'u_-} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi(2\omega)}} [a_\omega e^{-i\omega x_-} + a_\omega^\dagger e^{i\omega x_-}] \\ = \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} e^{ik'u_-} \int_0^{\infty} \frac{dk}{\sqrt{2\pi(2k)}} [b_k e^{-iku_-} + b_k^\dagger e^{iku_-}] \\ = \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2\pi(2k)}} [b_k e^{-i(k-k')u_-} + b_k^\dagger e^{i(k+k')u_-}] \\ = \int_0^{\infty} \frac{dk}{\sqrt{2k}} [b_k \delta(k-k') + b_k^\dagger \delta(k+k')] \\ = \frac{b_{k'}}{\sqrt{2k'}} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{b_k}{\sqrt{2k}} &= \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi(2\omega)}} [a_\omega e^{-i\omega x_- + iku_-} + a_\omega^\dagger e^{i\omega x_- + iku_-}] \\ b_k &= \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{k}{\omega}} \left[a_\omega e^{i\omega \frac{e^{-au_-}}{a} + iku_-} + a^\dagger e^{-i\omega \frac{e^{-au_-}}{a} + iku_-}\right] \\ &= \int_0^{\infty} d\omega \left[a_\omega \sqrt{\frac{k}{\omega}} F(\omega, k) + a_\omega^\dagger \sqrt{\frac{k^\star}{\omega^\star}} F(-\omega, k)\right] \end{aligned} \tag{5.11}$$

using above we can also get:

$$(b_k)^\dagger = b_k^\dagger = \int_0^{\infty} d\omega \left[a_\omega^\dagger \sqrt{\frac{k^\star}{\omega^\star}} F(-\omega, -k) + a_\omega \sqrt{\frac{k}{\omega}} F(\omega, -k)\right] \tag{5.12}$$

Similarly, performing the Fourier transformation on negative modes, we get:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} e^{ik'u_+} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi(2\omega)}} [a_{-\omega} e^{-i\omega x_+} + a^\dagger_{-\omega} e^{i\omega x_+}] \\
&= \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} e^{ik'u_+} \int_0^{\infty} \frac{dk}{\sqrt{2\pi(2|k|)}} [b_{-k} e^{-iku_+} + b^\dagger_{-k} e^{iku_+}] \\
&= \int_0^{\infty} \frac{dk}{\sqrt{2|k|}} [b_{-k}\delta(k-k') + b^\dagger_{-k}\delta(k+k')]
\end{aligned}$$

Since, here $k' < 0$, we get:

$$\begin{aligned}
\frac{b^\dagger_{-k}}{\sqrt{2|k|}} &= \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi(2|\omega|)}} [a_{-\omega} e^{-i\omega x_+ + iku_+} + a^\dagger_{-\omega} e^{i\omega x_+ + iku_+}] \\
&= \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi(2|\omega|)}} \left[a_{-\omega} e^{-i\omega \frac{e^{au_+}}{a} + iku_+} + a^\dagger_{-\omega} e^{i\omega \frac{e^{au_+}}{a} + iku_+}\right] \\
b^\dagger_{-k} &= \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{k}{\omega}} \left[a_{-\omega} e^{-i\omega \frac{e^{au_+}}{a} + iku_+} + a^\dagger_{-\omega} e^{i\omega \frac{e^{au_+}}{a} + iku_+}\right]
\end{aligned} \tag{5.13}$$

and thus:

$$b_{-k} = \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{k}{\omega}} \left[a^\dagger_{-\omega} e^{i\omega \frac{e^{au_+}}{a} - iku_+} + a_{-\omega} e^{-i\omega \frac{e^{au_+}}{a} - iku_+}\right]$$

From (5.12), we get

$$b^\dagger_{-k} = \int_0^{\infty} d\omega \left[a^\dagger_\omega \sqrt{\frac{k^\star}{\omega^\star}} F(-\omega, k) + a_\omega \sqrt{\frac{k}{\omega}} F(\omega, k)\right] \tag{5.14}$$

We observe that (5.13) and (5.14) are same with the identification $a^\dagger_{-\omega} = a_\omega$.

Analytic Continuation

There is another way to derive the Bogoliubov transformation, originally due to Unruh by first writing the field in complete basis set in the entire Minkowski spacetime. We will find a set of modes that share the same vacuum state as the Minkowski modes, albeit with different description of the excited modes. But these new modes will have a simpler overlap with the Rindler modes and hence the Bogoliubov parameters will be easier to calculate. The way to do this is to analytically continue the Rindler modes over all of space-time and express this extension in terms of the original Rindler modes. Here, we need to be careful, because in Region I, we have

$$\partial_\eta = a(x\partial_t + t\partial_x)$$

whereas in Region IV, we have

$$\partial_\eta = -a(x\partial_t + t\partial_x)$$

Therefore,

$$\phi(x) = \begin{cases} \int \frac{dk_\xi}{\sqrt{2\pi(2k_\eta)}} [b^{(1)} \underbrace{e^{-ik_\eta \eta + ik_\xi \xi}}_{\sqrt{4\pi k_\eta} g^{(1)}} + b^{(1)\dagger} \underbrace{e^{+ik_\eta \eta - ik_\xi \xi}}_{\sqrt{4\pi k_\eta} g^{(1)\star}}] & \text{In region } I \\ \\ \int \frac{dk_\xi}{\sqrt{2\pi(2k_\eta)}} [b^{(2)} \underbrace{e^{ik_\eta \eta + ik_\xi \xi}}_{\sqrt{4\pi k_\eta} g^{(2)}} + b^{(2)\dagger} \underbrace{e^{-ik_\eta \eta - ik_\xi \xi}}_{\sqrt{4\pi k_\eta} g^{(2)\star}}] & \text{In region } IV \end{cases} \tag{5.15}$$

As we will see, this approach is similar to the one described in section 5.1.3. We start by writing the solution in Minkowski spacetime

$$\phi(x)_{\text{minkowski}} = \int dk [a_k f + a_k^\dagger f^\star]$$

In Rindler spacetime, the solution in the combined region I and region IV is written as:

$$\phi(x)_{\text{rindler}} = \int dk [\underbrace{b^{(1)} g^{(1)} + b^{(1)\dagger} g^{(1)\star}}_{\text{defined in region 1}} + \underbrace{b^{(2)} g^{(2)} + b^{(2)\dagger} g^{(2)\star}}_{\text{defined in region 4}}]$$

In region I:

$$a(x-t) = e^{a\xi}(\cosh a\eta - \sinh a\eta)$$

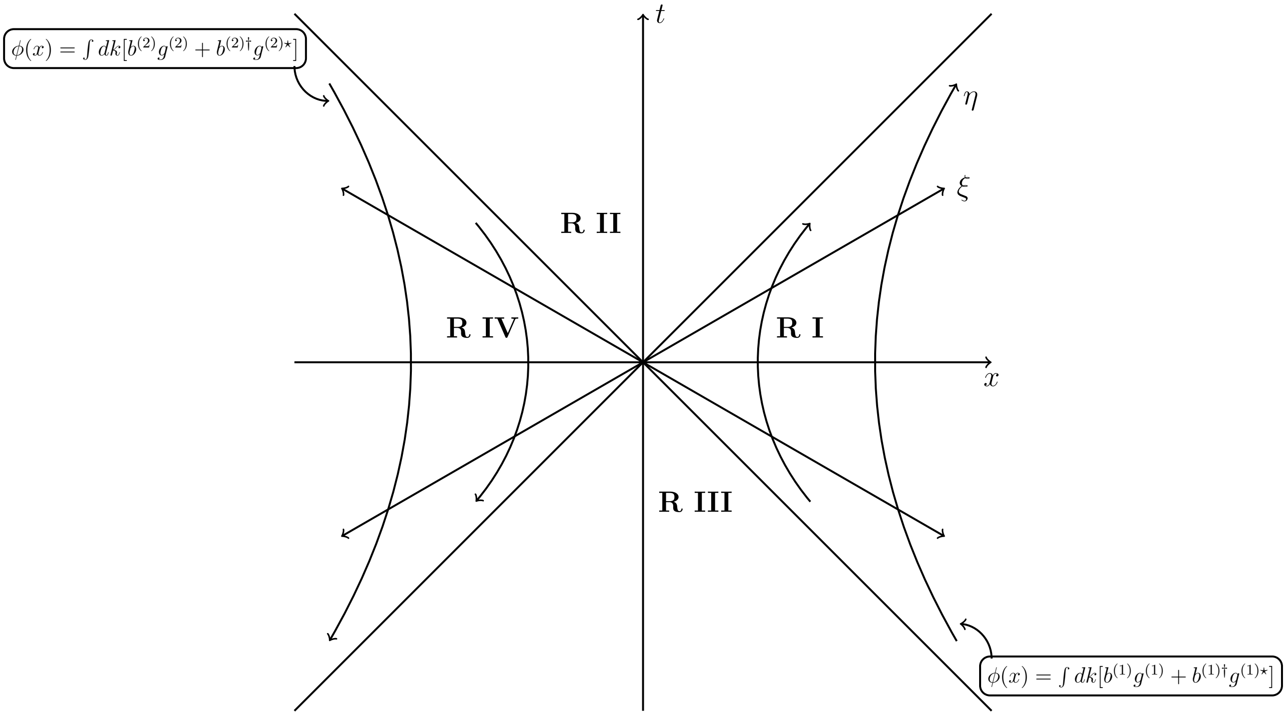


Figure 5.2: We can write the field in the region of overlap ($x > |t|$) of Minkowski space and Rindler space using complete basis set as $\phi(x) = \int dk[b^{(1)}g^{(1)} + b^{(1)\dagger}g^{(1)\star} + b^{(2)}g^{(2)} + b^{(2)\dagger}g^{(2)\star}]$.

$$
\begin{aligned}
&= e^{a\xi}\left(\frac{e^{a\eta}+e^{-a\eta}}{2} - \frac{e^{a\eta}-e^{-a\eta}}{2}\right) \\
&= e^{-a(\eta-\xi)} \\
a(x+t) &= e^{a\xi}(\cosh a\eta + \sinh a\eta) \\
&= e^{a\xi}\left(\frac{e^{a\eta}+e^{-a\eta}}{2} + \frac{e^{a\eta}-e^{-a\eta}}{2}\right) \\
&= e^{a(\eta+\xi)}
\end{aligned}
$$

In region IV, using (5.4):

$$
\begin{aligned}
a(-x+t) &= -e^{a\xi}(-\cosh a\eta + \sinh a\eta) \\
&= e^{a\xi}\left(\frac{e^{a\eta}+e^{-a\eta}}{2} - \frac{e^{a\eta}-e^{-a\eta}}{2}\right) \\
&= e^{-a(\eta-\xi)} \\
a(-x-t) &= -e^{a\xi}(-\cosh a\eta - \sinh a\eta) \\
&= e^{a\xi}\left(\frac{e^{a\eta}+e^{-a\eta}}{2} + \frac{e^{a\eta}-e^{-a\eta}}{2}\right) \\
&= e^{a(\eta+\xi)}
\end{aligned}
$$

A quick observation tells us that region I and region IV are related via a transformation of the form $\eta \to \eta \pm i\pi$. Since, the solution with $k_\eta > 0$ is only convergent in the region where $\mathrm{Im}\{\eta - \xi\} < 0$, we have to take the branch cut in the upper half plane. Therefore, the function $g^{(1)}_{k_\xi}$ is analytic only in the lower half plane which implies the transformation to be considered has to be $\eta \to \eta - i\pi$. Alternatively, If we now assume $k_\eta > 0$ and so $k_\eta = k_\xi c \equiv k$ then we can write for $g^{(1)}_{k_\xi}$ in region I:

$$
\begin{aligned}
\sqrt{4\pi k_\eta}\, g^{(1)}_{k_\xi} &= e^{-ik_\eta\eta + ik_\xi\xi} = e^{-ik_\xi(\eta-\xi)} = \left[e^{-a(\eta-\xi)}\right]^{ik/a} = [a(x-t)]^{ik/a} \\
&= a^{ik/a}(x-t)^{ik/a}
\end{aligned}
\tag{5.16}
$$

In region IV

$$
\sqrt{4\pi k_\eta}\, g^{(2)\star}_{-k_\xi} = e^{-ik_\eta\eta + ik_\xi\xi} = e^{-ik_\xi(\eta-\xi)} = \left[e^{-a(\eta-\xi)}\right]^{ik/a} = [a(-x+t)]^{ik/a}
$$

$$= [ae^{\ln(-x+t)}]^{ik/a}$$

Since the above expression involves $z^{ik/n}$ and the exponent can be non-integer. It is a multivalued function and therefore we have to choose the branch-cut before we proceed.

$$\sqrt{4\pi k_\eta}g^{(2)}_{-k_\xi} = [ae^{\lim_{\epsilon\to 0}\ln\{x-(t-i\epsilon)\}-i\pi}]^{ik/a} = [ae^{\ln|x-t|-i\pi}]^{ik/a} = a^{ik/a}(x-t)^{ik/a}e^{\frac{\pi k}{a}}$$

In the above, we made the choice of branch cut in the upper half of the complex plane therefore, we had to consider $t \to \lim_{\epsilon\to 0}(t - i\epsilon)$ i.e. approach the real t axis from below for the sake of convergence. The choice of

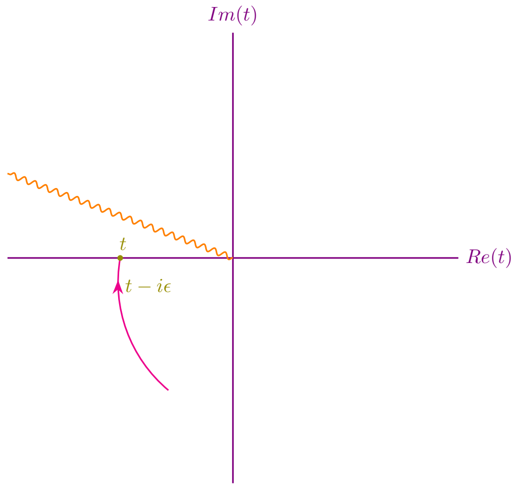


Figure 5.3: Analytic continuation is to be thought off as performing certain transformation which either extends the region of convergence or changes it. We then patch all the different analytically continued solutions and form an extended atlas. Here, we choose the branch cut to lie in the upper half of complex t plane, therefore we approach the real t axis by taking the limit in the region of analyticity $\lim_{\epsilon\to 0}(t - i\epsilon)$.

branch cut controls weather we get $e^{\frac{\pi k}{a}}$ or $e^{-\frac{\pi k}{a}}$ in the above equation. Since we considered the branch cut in the upper half of the complex plane, we are effectively doing the analytic continuation in the lower half. The linear combination:

$$\sqrt{4\pi k_\eta}[g^{(1)}_{k_\xi} + e^{-\frac{\pi k}{a}}g^{(2)\star}_{-k_\xi}] = 2a^{ik/a}(x-t)^{ik/a}$$

is similar to (5.16) and is to be interpreted as the analytic continuation of $g^{(1)}_{k_\xi}$. We can consider a normalized version with $k_\eta = k_\xi \equiv k$:

$$\begin{aligned} h^{(1)}_k &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi k}{a}\right)}}\left[e^{\frac{\pi k}{2a}}g^{(1)}_k + e^{\frac{\pi k}{a}}e^{-\frac{\pi k}{2a}}g^{(1)}_k\right] \\ &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi k}{a}\right)}}\left[e^{\frac{\pi k}{2a}}g^{(1)}_k + e^{-\frac{\pi k}{2a}}g^{(2)\star}_{-k}\right] \end{aligned}$$

Coming back to Minkowski coordinate

$$\phi_{\text{minkowski}} = \int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i\omega(t-x)} + a^\dagger_\omega e^{i\omega(t-x)}]$$

We observe that $e^{-i\omega(t-x)}$ solution with $\omega > 0$ is only convergent in the region where $\mathrm{Im}\{t-x\} < 0$. Since it is convergent in this region, we need to ensure that it is also analytic here, which means we put the branch cut in the upper half complex plane.[13]

Same procedure can also be applied to find the conjugate mode. In region I for $k_\eta > 0$:

$$\sqrt{4\pi k_\eta}g^{(1)\star}_{-k_\xi} = e^{ik_\eta\eta + ik_\xi\xi} = e^{ik_\xi(\eta+\xi)} = \left[e^{a(\eta+\xi)}\right]^{ik/a} = [a(x+t)]^{ik/a}$$

[13] $\ln z = \ln re^{i\theta} = \ln r + i\theta$ has a branch cut; as $z(r;\theta+2\pi) = z(r;\theta)$ but $\ln z(r;\theta+2\pi) = \ln z(r;\theta) + 2\pi$, here periodicity of z and $\ln z$ aren't same. It is traditional to put this branch cut on the real axis, $\theta = 0$, but we can of course put it anywhere starting at the origin. The important point being that if we rotate θ by 2π we cross the branch cut.

$$= a^{ik/a}(x+t)^{ik/a} \tag{5.17}$$

In region IV

$$\begin{aligned}\sqrt{4\pi k_\eta} g^{(2)}_{k_\xi} &= e^{ik_\xi \eta + ik_\xi \xi} = e^{ik_\eta(\eta+\xi)} = \left[e^{a(\eta+\xi)}\right]^{ik/a} = [a(-x-t)]^{ik/a} \\ &= [ae^{\lim\limits_{\epsilon\to 0}\ln\{-x-(t-i\epsilon)\}}]^{ik/a} = [ae^{\ln|x+t|+i\pi}]^{ik/a} = a^{ik/a}e^{-\pi k/a}(x+t)^{ik/a}\end{aligned}$$

Thus,

$$\begin{aligned}h^{(2)}_k &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi k}{a}\right)}}\left[e^{\frac{\pi k}{2a}}g^{(2)}_k + e^{\frac{\pi k}{a}}e^{-\frac{\pi k}{2a}}g^{(2)}_k\right] \\ &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi k}{a}\right)}}\left[e^{\frac{\pi k}{2a}}g^{(2)}_k + e^{-\frac{\pi k}{2a}}g^{(1)\star}_{-k}\right]\end{aligned}$$

One could wonder, what about region II and region III? Actually, region II and region III are spacelike, so we aren't worried about them. Also, we expect the fields to commute in that region for the sake of causality.

5.1.7 Relativity of Vacuum

In this section, we will evaluate the number of particles between Minkowski observer and Rindler observer. We start by noting that $b^{(1)}_k |0\rangle_M \neq 0$ since, $|0\rangle_M$ is the minkowski vacuum. The way to think about it is this, $b^{(1)}_k$ is the ladder operator corresponding to generator of boost, whereas $a^{(1)}_k$ is the ladder operator corresponding to generator of time translation. Therefore, the shift in eigenvectors that ladder operators are supposed to do is only for their respective eigenstates. We use the notation $_M\langle 0|\ldots|0\rangle_M \equiv \langle\ldots\rangle_M$, where M denotes minkowski. [14][1cm]

$$\begin{aligned}\langle N_k\rangle &= \left\langle b^\dagger_k b_k\right\rangle_M \\ &= \int d\omega' d\omega \quad {}_M\langle 0|\left[a^\dagger_{\omega'}\sqrt{\frac{k^\star}{\omega^{\star\prime}}}F(-\omega',-k) + a_{\omega'}\sqrt{\frac{k}{\omega'}}F(\omega',-k)\right] \\ &\qquad\qquad \times\left[a_\omega\sqrt{\frac{k}{\omega}}F(\omega,k) + a^\dagger_\omega\sqrt{\frac{k^\star}{\omega^\star}}F(-\omega,k)\right]|0\rangle_M\end{aligned}$$

Only non zero contribution comes from:

$$\begin{aligned}&= \int d\omega' d\omega\sqrt{\frac{k}{\omega'}}\sqrt{\frac{k^\star}{\omega^\star}}F(\omega',-k)F(-\omega,k)\underbrace{\left\langle a_{\omega'}a^\dagger_\omega\right\rangle}_{\delta(\omega'-\omega)} \\ &= \int d\omega\left|\frac{k}{\omega}\right||F(-\omega,k)|^2\end{aligned}$$

Now, we focus on integrating $F(-\omega,k)$:

$$\begin{aligned}F(\omega,k) &= \int_{-\infty}^{\infty}\frac{du_-}{2\pi}\exp\left(i\omega\frac{e^{-au_-}}{a} + iku_-\right) \\ &= \int_{-\infty}^{\infty}\frac{du_-}{2\pi}\exp\left(i\omega\frac{e^{-au_-}}{a}\right)e^{iku_-}\end{aligned}$$

making the substitution $x = e^{-au_-}$, we get

$$-ae^{-au_-}du_- \to dx$$
$$\int_{-\infty}^{\infty} \to \int_{\infty}^{0}$$

[14] much like we use $[a, a^\dagger] = 1$

$$\langle 0|[a,a^\dagger]|0\rangle = \langle 0|0\rangle$$
$$\langle 0|aa^\dagger|0\rangle = 1$$

$$\exp\left(i\omega\frac{e^{-au_-}}{a}\right) \to e^{i\omega x/a}$$
$$e^{iku_-} \to [e^{-au_-}]^{\frac{-ik}{a}}$$

Therefore

$$\begin{aligned} F(\omega,k) &= \frac{1}{2\pi}\int_0^\infty \frac{dx}{ax} e^{i\omega x/a} x^{-ik/a} \\ &= \int_0^\infty \frac{dx}{2\pi a} x^{-ik/a-1} e^{i\omega x/a} \end{aligned}$$

expressing in more familiar form:

$$F(\omega,k) = \int_0^\infty \frac{dx}{2\pi a} x^{s-1} e^{bx}, \qquad s = -\frac{ik}{a}, \qquad b = -\frac{i\omega}{a}.$$

We can utilize the identity

$$\int_0^\infty dx x^{s-1} e^{-bx} = e^{-s\ln(b)}\Gamma(s) \qquad \text{(with Re}\{b\} > 0 \text{ and Re}\{s\} \in (0,1))$$

Since, our s and b are purely imaginary, we will use shift them along real axis ($\epsilon > 0$) and then take the limit $\epsilon \to 0$.

$$\begin{aligned} F(\omega,k) &= \lim_{\epsilon\to 0}\int_0^\infty \frac{dx}{2\pi a} x^{(s+\epsilon)-1} e^{(b+\epsilon)x} \\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi a} e^{-(s+\epsilon)\ln(b+\epsilon)}\Gamma(s+\epsilon) \end{aligned}$$

Since, logarithm is multivalued function in complex plane. We consider the branch cut along $-ve$ x-axis:

$$\begin{aligned} \lim_{\epsilon\to 0}\ln(b+\epsilon) &= \lim_{\epsilon\to 0}\ln\left|-i\frac{\omega}{a}+\epsilon\right| + \lim_{\epsilon\to 0} i\arg\left(-i\frac{\omega}{a}+\epsilon\right) \\ &= \ln\left|\frac{\omega}{a}\right| + \lim_{\epsilon\to 0} i\tan^{-1}\left(\frac{-\omega/a}{\epsilon}\right) \\ &= \ln\left|\frac{\omega}{a}\right| - i\frac{\pi}{2}\underbrace{\text{sgn}}_{\text{signum function}}\left(\frac{\omega}{a}\right) \end{aligned}$$

Therefore, we get:

$$\begin{aligned} F(\omega,k) &= \frac{1}{2\pi a} e^{-s\lim_{\epsilon\to 0}\ln(b+\epsilon)}\Gamma(s) \\ &= \frac{1}{2\pi a}\exp\left[i\frac{k}{a}\left\{\ln\left|\frac{\omega}{a}\right| - i\frac{\pi}{2}\text{sgn}\left(\frac{\omega}{a}\right)\right\}\right]\Gamma\left(-\frac{i\omega}{a}\right) \\ &= \frac{1}{2\pi a}\exp\left[i\frac{k}{a}\ln\left|\frac{\omega}{a}\right| + \frac{\pi k}{2a}\text{sgn}\left(\frac{\omega}{a}\right)\right]\Gamma\left(-\frac{i\omega}{a}\right) \end{aligned}$$

Since, we had already defined $a > 0$, assuming $\omega > 0$:

$$\begin{aligned} F(\omega,k) &= \frac{1}{2\pi a}\exp\left[i\frac{k}{a}\ln\left|\frac{\omega}{a}\right| + \frac{\pi k}{2a}\right]\Gamma\left(-\frac{i\omega}{a}\right) \\ &= \frac{1}{2\pi a}\exp\left[i\frac{k}{a}\ln\left|\frac{\omega}{a}\right| + \frac{\pi k}{a} - \frac{\pi k}{2a}\right]\Gamma\left(-\frac{i\omega}{a}\right) \end{aligned}$$

since,

$$F(-\omega,k) = \frac{1}{2\pi a}\exp\left[i\frac{k}{a}\ln\left|\frac{\omega}{a}\right| - \frac{\pi k}{2a}\right]\Gamma\left(-\frac{i\omega}{a}\right)$$
$$F(\omega,k) = F(-\omega,k)e^{\pi k/a} \tag{5.18}$$

Here we have two choice, either evaluate the integral explicitly or use an alternative trick by utilizing (5.11):[15]

$$[b_k, b_{k'}^\dagger] = \int_0^\infty d\omega\int_0^\infty d\omega'\sqrt{\frac{kk'^\star}{\omega\omega'^\star}}\left\{F(\omega,k)F(-\omega',-k')\right.$$

[15] The canonical quantization condition over creation and annihilation operators depends on the convention used for Fourier transform.

$$-F(-\omega,k)F(\omega',-k')\}\underbrace{[a_\omega,a^\dagger_{\omega'}]}_{\delta(\omega-\omega')}$$

$$=\delta(k-k')$$

using (5.18) and setting $k=k'$

$$\delta(0)=\int_0^\infty d\omega\left|\frac{k}{\omega}\right|\left[e^{2\pi k/a}|F(-\omega,k)|^2-|F(-\omega,k)|^2\right]$$

$$\int_0^\infty d\omega\left|\frac{k}{\omega}\right||F(-\omega,k)|^2=\frac{\delta(0)}{e^{2\pi k/a}-1}$$

Thus,

$$\langle N_k\rangle=\frac{\delta(0)}{e^{2\pi k/a}-1}$$

We can absorb the delta function by using hard cutoff as:

$$\begin{aligned}\delta(k-k')&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx e^{i(k-k')x}\\&=\frac{1}{\sqrt{2\pi}}\lim_{L\to\infty}\int_{-L}^{L}dx e^{i(k-k')x}\frac{1}{\sqrt{2\pi}}\end{aligned}$$

at $k=k'$

$$\delta(0)=\frac{V}{2\pi}$$

and then express the particle number in terms of number density.

$$\langle n_k\rangle=\frac{2\pi}{e^{2\pi k/a}-1}=\frac{2\pi}{e^{E/T}-1}$$

where $E=\sqrt{k^2+m^2}|_{m=0}$ and we identify $T={}^{a}\!/\!{}_{2\pi}$. Alternatively, we could also derive the same calculation using analytic continuation. Let us expand the field in terms the analytically continued basis set:

$$\phi(x)=\int dk[c_k^{(1)}h_k^{(1)}+c_k^{(1)\dagger}h_k^{(1)\star}+c_k^{(2)}h_k^{(2)}+c_k^{(2)\dagger}h_k^{(2)\star}]$$

The same was originally expressed in terms of rindler coordinates as:

$$\phi(x)=\int dk[b_k^{(1)}g_k^{(1)}+b_k^{(1)\dagger}g_k^{(1)\star}+b_k^{(2)}g_k^{(2)}+b_k^{(2)\dagger}b_k^{(2)\star}]$$

Therefore, we consider the bogoliubov transformation which connects the $b_k^{(1)}$ and $b_k^{(2)}$ with $c_k^{(1)}$ and $c_k^{(2)}$. Since the transformation matrix considered here is real and symmetric, we can rewrite the relationship between the creation and annihilation as if:

$$\begin{bmatrix}g_k^{(1)}\\g_{-k}^{(2)\star}\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}h_k^{(1)}\\h_{-k}^{(2)\star}\end{bmatrix}$$

then,

$$\begin{bmatrix}b_k^{(1)}\\b_{-k}^{(2)\dagger}\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}^{-1}\begin{bmatrix}c_k^{(1)}\\c_{-k}^{(2)\dagger}\end{bmatrix}$$

Earlier we derived:

$$\begin{bmatrix}h_k^{(1)}\\h_{-k}^{(2)\star}\end{bmatrix}=\underbrace{\frac{1}{\sqrt{2\sinh({}^{\pi k}\!/\!{}_{a})}}\begin{bmatrix}e^{\pi k/2a}&e^{-\pi k/2a}\\e^{-\pi k/2a}&e^{\pi k/2a}\end{bmatrix}}_{\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}^{-1}}\begin{bmatrix}g_k^{(1)}\\g_{-k}^{(2)\star}\end{bmatrix}$$

Therefore, we get:

$$\begin{bmatrix} b_k^{(1)} \\ b_{-k}^{(2)\dagger} \end{bmatrix} = \frac{1}{\sqrt{2\sinh(\pi k/a)}} \begin{bmatrix} e^{\pi k/2a} & e^{-\pi k/2a} \\ e^{-\pi k/2a} & e^{\pi k/2a} \end{bmatrix} \begin{bmatrix} c_k^{(1)} \\ c_{-k}^{(2)\dagger} \end{bmatrix}$$

Now, we can evaluate the number density of $b_k^{(1)}$ modes noting that $b_k^{(1)} \left|0\right\rangle_R = 0$ where $\left|0\right\rangle_R$ is the rindler vacuum. But $b_k^{(1)} \left|0\right\rangle_M \neq 0$ unless $\left|0\right\rangle_M$ and $\left|0\right\rangle_R$ coincide. We note that $c_k^{(1)} \left|0\right\rangle_M = 0$, then:

$$\begin{aligned}
\left\langle N_k^{(1)} \right\rangle &= {}_M\left\langle 0\right| b_k^{(1)\dagger} b_k^{(1)} \left|0\right\rangle_M \\
&= \frac{1}{2\sinh\left(\frac{\pi k}{a}\right)} {}_M\left\langle 0\right| (e^{\pi k/2a} c_k^{(1)\dagger} + e^{-\pi k/2a} c_{-k}^{(2)})(e^{\pi k/2a} c_k^{(1)} + e^{-\pi k/2a} c_{-k}^{(2)\dagger}) \left|0\right\rangle_M \\
&= \frac{1}{2\sinh\left(\frac{\pi k}{a}\right)} {}_M\left\langle 0\right| e^{-\pi k/2a} c_{-k}^{(2)\dagger} e^{-\pi k/2a} c_{-k}^{(2)} \left|0\right\rangle_M \\
&= \frac{e^{-\pi k/a}}{2\sinh(\pi k/a)} {}_M\left\langle 0\right| \ c_{-k}^{(2)\dagger} c_{-k}^{(2)} \ \left|0\right\rangle_M
\end{aligned}$$

$c_{-k}^{(2)} c_{-k}^{(2)\dagger} + [c_{-k}^{(2)}, c_{-k}^{(2)\dagger}]$

$$\begin{aligned}
&= \frac{e^{-\pi k/a}}{2\sinh(\pi k/a)} {}_M\left\langle 0\right| \delta(0) \left|0\right\rangle_M \\
&= \frac{e^{-\pi k/a}}{e^{\pi k/a} - e^{-\pi k/a}} \delta(0) \\
&= \frac{\delta(0)}{e^{2\pi k/a} - 1}
\end{aligned}$$

where we have used $[c_k^{(1)}, c_k^{(2)}] = 0$, because it can be shown that $\left\langle h_k^{(1)}, h_k^{(2)} \right\rangle = 0$. The above result is to be thought of as, average value of occupation number of rindler modes $b_k^{(1)}$ in minkowski vacuum. [16]

5.1.8 Particle creation in Bernard and Duncan Model

This part is taken from Birrel and Davis, section 3.4. In the previous section, we saw that for an accelerated observer, the vacuum is filled with particles. According to the equivalence principle, the physics for an accelerated observer and an observer in curved spacetime are equivalent. Therefore, it is natural to investigate the same phenomenon in curved spacetime. For simplicity, we will consider a time-dependent metric that is Minkowskian at $t = \pm\infty$ and is varying in between. As we will show, a vacuum state in $t = -\infty$ will be changed into a non-vacuum state for $t = +\infty$. The metric tensor in FLRW spacetime is given as:

$$ds^2 = dt^2 - a^2(t) dx^2,$$

where the function $a(t)$ determines the relative expansion of the universe, and is called the scale factor. We introduce a new time parameter η, called conformal time, via $d\eta = dt/a(t)$. Using the new parameter, the metric becomes

Time dependence of metric is equivalent to conformal transformation.

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2) = \ C(\eta) \ (d\eta^2 - dx^2),$$

where $C(\eta) = a^2(\eta)$ is defined as the conformal scale factor. We will in this section choose the form of the conformal scale factor to be

$$C(\eta) = A + B\tanh(\rho\eta),$$

where A, B and ρ are constants. A conformal scale factor is constant in the limit $\eta \to \pm\infty$ since

$$\lim_{\eta\to\pm\infty} C(\eta) = A \pm B.$$

[16] In the language of "Introduction to quantum effects in gravity" by Sergei Winitzki, pg 72– For instance, the Minkowski vacuum defined above is a state without minkowski particles but with rindler particle density N_k in each mode ϕ_k

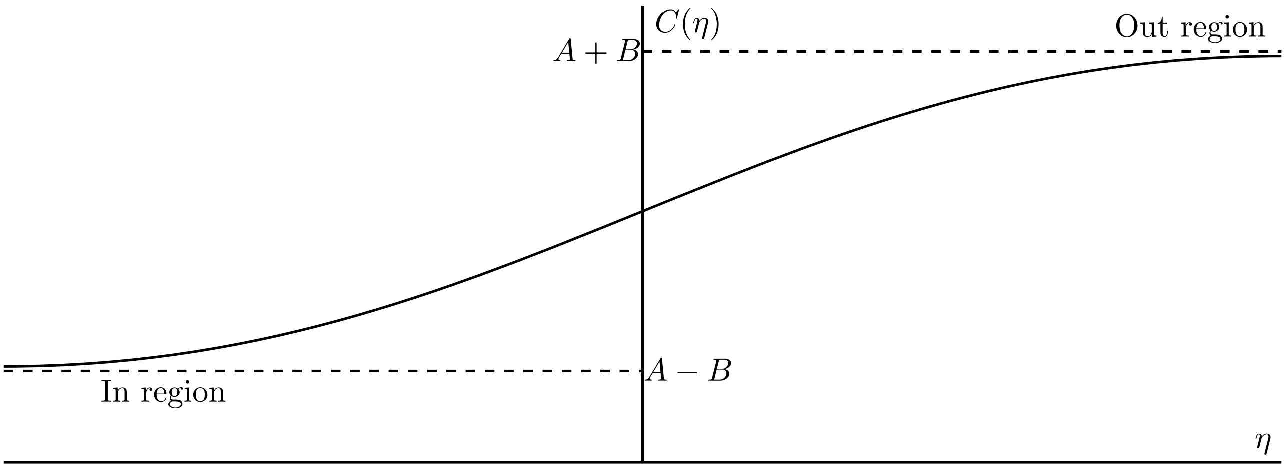


Figure 5.4: A qualitative picture of the evolution of the scale factor in conformal time emerges, depicting a scenario where both the in-region and out-region are flat, but feature **distinct** metric tensors. For particle creation to take place, the vacuum has to transform rather than stay invariant under time translation of metric.

The asymptotic behaviour of the conformal scale factor is also noticeable in a plot, shown in figure 5.4. The static universe corresponding to $\eta \to -\infty$ is called the in-region, while the static universe corresponding to $\eta \to \infty$ is called the out-region. Both the in-region and out-region resemble Minkowski spacetime however the definition of vacuum in both of them will be different.

Introducing a scalar field

The metric described thus far in this section is now applied to a scalar field.[17]

$$(\Box + m^2)\phi(x,\eta) = 0 \tag{5.19}$$

Note that we replaced the standard time t with conformal time η. We introduce a complete set of orthonormal mode solutions $u_k(x,\eta)$ of above that obey the properties

$$\begin{aligned}(u_k, u_l) &= \delta_{kl},\\ (u_k^*, u_l^*) &= -\delta_{kl},\\ (u_k, u_l^*) &= 0.\end{aligned}$$

The scalar field $\phi(x,\eta)$ can be expanded by a linear combination of the orthonormal mode solutions as

$$\phi(x,\eta) = \sum_k \left[a_k u_k(x,\eta) + a_k^\dagger u_k^*(x,\eta)\right],$$

where a_k and $a_k^\dagger$ are annihilation and creation operators, respectfully.

The conformal scale factor $C(\eta)$ is not a function of the spatial coordinate x, and thus spatial translation invariance is still a symmetry in this spacetime. This means that we can separate the space and time variables in the scalar mode functions u_k as

$$u_k(x,\eta) = \frac{1}{\sqrt{2\pi}} e^{ikx} \chi_k(\eta) \tag{5.20}$$

Substituting the mode functions (5.20) into the scalar field equation (5.19) gives

$$\frac{d^2}{d\eta^2}\chi_k(\eta) + \left[k^2 + C(\eta)m^2\right]\chi_k(\eta) = 0,$$

which is an ordinary differential equation for $\chi_k(\eta)$. The differential equation can be solved in terms of hypergeometric functions, which are functions that can be defined in the form as a hypergeometric series. The mode function in the in-region ($\eta \to -\infty$), taken directly from Birrel and Davis, are

$$\lim_{\eta\to-\infty} u_k^{\text{in}}(x,\eta) = \frac{1}{\sqrt{4\pi\omega_{\text{in}}}} e^{ikx - i\omega_{\text{in}}\eta}, \quad \omega_{\text{in}} = [k^2 + m^2(A-B)]^{1/2}\text{[18]}$$

[17] at $\eta \to -\infty$ the EOM takes the form on Klein Gordon Equation in minkowski spacetime, refer (5.26).

[18] the mode function evolves from ω_{in} to ω_{out}

Similarly, the mode function for the out-region are

$$\lim_{\eta\to\infty} u_k^{\text{out}}(x,\eta) = \frac{1}{\sqrt{4\pi\omega_{\text{out}}}} e^{ikx - i\omega_{\text{out}}\eta}, \quad \omega_{\text{out}} = [k^2 + m^2(A+B)]^{1/2}$$

The two mode solutions in the different static regions are clearly different for **massive** particles.[19] However, we can be express $u_k^{\text{in}}(x,\eta)$ as a linear combination of the real and imaginary part of $u_k^{\text{out}}(x,\eta)$:[20]

$$u_k^{\text{in}}(x,\eta) = \alpha_k u_k^{\text{out}}(x,\eta) + \beta_k u_k^{\text{out}*}(x,\eta).$$

To find the Bogoliubov coefficients α_k and β_k, we use the full solution:

$$^{21}u_k^{\text{in}}(x,\eta) = \frac{1}{\sqrt{4\pi\omega_{\text{in}}}} e^{i\{kx - \omega_+\eta - \frac{\omega_-}{\rho}\ln[2\cosh(\rho\eta)]\}}$$
$$ {}_2F_1\left(1 + \frac{i\omega_-}{\rho}, \frac{i\omega_-}{\rho}, 1 - \frac{i\omega_{\text{in}}}{\rho}; \frac{1}{2}[1+\tanh(\rho\eta)]\right)$$

$$^{22}u_k^{\text{out}}(x,\eta) = \frac{1}{\sqrt{4\pi\omega_{\text{out}}}} e^{i\{kx - \omega_+\eta - \frac{\omega_-}{\rho}\ln[2\cosh(\rho\eta)]\}}$$
$$ {}_2F_1\left(1 + \frac{i\omega_-}{\rho}, \frac{i\omega_-}{\rho}, 1 + \frac{i\omega_{\text{out}}}{\rho}; \frac{1}{2}[1-\tanh(\rho\eta)]\right)$$

where

$$\omega_\pm = \frac{1}{2}(\omega_{\text{out}} \pm \omega_{\text{in}})$$

We explicitly make use of the following properties of hypergeometric functions[23].

$$\begin{aligned}
{}_2F_1(a,b,c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1(a,b,1+a+b-c;1-z) \\
&\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ {}_2F_1(c-a,c-b,1+c-a-b;1-z) \\
{}_2F_1(a,b,c;z) &= (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b,c;z)
\end{aligned}$$

we get:

$$\begin{aligned}
&{}_2F_1\left(1 + \frac{i\omega_-}{\rho}, \frac{i\omega_-}{\rho}, 1 - \frac{i\omega_{\text{in}}}{\rho}; \frac{1}{2}[1+\tanh(\rho\eta)]\right) \\
&= \frac{\Gamma\left(1 - \frac{i\omega_{\text{in}}}{\rho}\right)\Gamma\left(i\frac{\omega_{\text{in}}+2\omega_-}{\rho}\right)}{\Gamma\left(-i\frac{\omega_{\text{in}}+\omega_-}{\rho}\right)\Gamma\left(1 - i\frac{\omega_{\text{in}}+\omega_-}{\rho}\right)}\ {}_2F_1\left(1 + \frac{i\omega_-}{\rho}, \frac{i\omega_-}{\rho}, 1 + i\frac{2\omega_- + \omega_{\text{in}}}{\rho}; \frac{1}{2}[1-\tanh(\rho\eta)]\right) \\
&\quad + \underbrace{\left(\frac{1}{2}[1-\tanh(\rho\eta)]\right)^{-i\frac{\omega_{\text{in}}+2\omega_-}{\rho}} \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(i\omega_{\text{out}}/\rho)}{\Gamma(i\omega_-/\rho)\Gamma(1 + i\omega_-/\rho)} \times {}_2F_1\left(-i\frac{\omega_{\text{in}}+\omega_-}{\rho}, 1 - i\frac{\omega_{\text{in}}+\omega_-}{\rho}, 1 - i\frac{\omega_{\text{in}}+2\omega_-}{\rho}; \frac{1}{2}[1-\tanh(\rho\eta)]\right)}_{\left(\frac{1}{2}[1+\tanh(\rho\eta)]\right)^{i\frac{\omega_{\text{in}}}{\rho}}\ {}_2F_1\left(1 - i\frac{\omega_-}{\rho}, -i\frac{\omega_-}{\rho}, 1 - i\frac{\omega_{\text{in}}+2\omega_-}{\rho}; \frac{1}{2}[1-\tanh(\rho\eta)]\right)} \\
&= \frac{\Gamma\left(1 - \frac{i\omega_{\text{in}}}{\rho}\right)\Gamma\left(i\frac{\omega_{\text{out}}}{\rho}\right)}{\Gamma\left(-i\frac{\omega_+}{\rho}\right)\Gamma\left(1 - i\frac{\omega_+}{\rho}\right)}\ {}_2F_1\left(1 + \frac{i\omega_-}{\rho}, \frac{i\omega_-}{\rho}, 1 + i\frac{2\omega_- + \omega_{\text{in}}}{\rho}; \frac{1}{2}[1-\tanh(\rho\eta)]\right) \\
&\quad + \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(i\omega_{\text{out}}/\rho)}{\Gamma(i\omega_-/\rho)\Gamma(1 + i\omega_-/\rho)}\ {}_2F_1\left(1 - \frac{i\omega_-}{\rho}, -\frac{i\omega_-}{\rho}, 1 - i\frac{2\omega_- + \omega_{\text{in}}}{\rho}; \frac{1}{2}[1-\tanh(\rho\eta)]\right) \underbrace{e^{2i\omega_+\eta}(2\cosh(\rho\eta))^{2i\omega_-/\rho}}_{e^{2i[\omega_+\eta + (\omega_-/\rho)\ln(2\cosh\rho\eta)]}}
\end{aligned}$$

[19] For massless particles, the in-mode and out-mode are identical, which means that there is no mixing and therefore no particle creation at late time. One might initially expect that due to the conformal invariance of the massless scalar fields, the state before and after scattering would respect this symmetry and remain unchanged, thus precluding any particle creation from occurring in free theory. However, contrary to this expectation in interacting theory, it has been argued that conformal-symmetry breaking can still lead to cosmological particle creation in $\lambda\phi^4$ theory.. This happens because of RG flow.

[20] this feels much like we are imposing continuity condition at some time η.

[23] using $\omega_{\text{out}} = 2\omega_\pm \mp \omega_{\text{in}}$, and $\omega_{\text{in}} + \omega_- = \omega_{\text{in}} + \frac{1}{2}(\omega_{\text{out}} - \omega_{\text{in}}) = \frac{2\omega_{\text{in}} + \omega_{\text{out}} - \omega_{\text{in}}}{2} = \frac{\omega_{\text{out}} + \omega_{\text{in}}}{2} = \omega_+$

where we used, $z = 1/2(1 + \tanh(\rho\eta))$

$$\begin{aligned}(1-z)^{-i\omega_{\text{out}}/\rho} z^{i\omega_{\text{in}}/\rho} &= (1-z)^{-i(\omega_+ + \omega_-)/\rho} z^{i(\omega_+ - \omega_-)/\rho} \\ &= \left(\frac{1-z}{z}\right)^{-i\omega_+/\rho} [z(1-z)]^{-i\omega_-/\rho} \\ &= (e^{-2\rho\eta})^{-i\omega_+/\rho} \left(\frac{1}{4\cosh^2(\rho\eta)}\right)^{-i\omega_-/\rho} \\ &= e^{2i\omega_+\eta}(2\cosh(\rho\eta))^{2i\omega_-/\rho}\end{aligned}$$

Thus we find

$$\alpha_k = \left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{1/2} \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(-i\omega_{\text{out}}/\rho)}{\Gamma(-i\omega_+/\rho)\Gamma(1 - i\omega_+/\rho)}, \tag{5.21}$$

and

$$\beta_k = \left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{1/2} \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(i\omega_{\text{out}}/\rho)}{\Gamma(i\omega_-/\rho)\Gamma(1 + i\omega_-/\rho)} e^{2ikx} \tag{5.22}$$

Using Eqs. (5.21) and (5.22) and Euler's reflection formula for Gamma function, we find:

$$|\alpha_k|^2 = \frac{\sinh^2(\pi\omega_{\text{out}}/\rho)}{\sinh(\pi\omega_{\text{in}}/\rho)\sinh(\pi\omega_{\text{out}}/\rho)}$$

and

$$|\beta_k|^2 = \frac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_{\text{in}}/\rho)\sinh(\pi\omega_{\text{out}}/\rho)}.$$

we can check that it satisfies the normalization condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

The mean paricle density in the out mode would be given as:

$${}_{\text{in}}\langle 0| n_k |0\rangle_{\text{in}} = |\beta_k|^2$$

This is to be interpreted as following: the $|0\rangle_{\text{in}}$ are not populated with in modes but only out-modes. As such, we need detector for out-mode to see particles in the $|0\rangle_{\text{in}}$ vacuum.

5.1.9 Hawking Radiation

This part is taken from chapter 9 of "Introduction to Quantum Effects in gravity" by Sergei Winitzki. We start from Eddington Finkelstein coordinate with a slight difference in how we define the tortoise coordinate.

$$r^\star = r + 2GM\ln\left(\frac{r}{2GM} - 1\right) - 2GM$$

The metric with this redefinition, $u = t - r^\star$ and $v = t + r^\star$, takes the form of:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right) dudv + r^2 d\Omega \\ &= -\frac{2GM}{r}\left(\frac{r}{2GM} - 1\right) dudv + r^2 d\Omega \\ &= -\frac{2GM}{r} e^{\ln(r/2GM - 1)} dudv + r^2 d\Omega \\ &= -\frac{2GMe^{1-r/2GM}}{r} e^{v-u/4GM} dudv + r^2 d\Omega \end{aligned} \tag{5.23}$$

If we define[24]:

$$dU = e^{-u/4GM} du \qquad\qquad dV = e^{v/4GM} dv$$

we get:

$$ds^2 = -\frac{2GM}{r} e^{1-r/2GM} dUdV + r^2 d\Omega^2 \tag{5.24}$$

[24] earlier we had defined $dU = \frac{-1}{4GM} e^{-u/4GM} du$ and $dV = \frac{1}{4GM} e^{v/4GM} dv$

In the near horizon limit:

$$ds^2 = dUdV$$

so the Kruskal vacuum is the appropriate one for an observer sitting next to the black hole horizon. On the other hand, since Kruskal–Szekeres coordinates cover the whole of spacetime, they correspond to the Minkowski vacuum that we studied in the quantization of a scalar field in Rindler space. In the asymptotic limit:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega$$

Therefore, we assume the existence of two observer, one locally inertial observer and the other static or accelerated observer near the horizon. From (5.23) and (5.24), we have:

$$-\frac{2GM}{r} e^{1-r/2GM} e^{v-u/4GM} dudv + r^2 d\Omega = -\frac{2GM}{r} e^{1-r/2GM} dUdV + r^2 d\Omega^2$$

Near the horizon,

$$e^{v-u/4GM} dudv + r^2 d\Omega = dUdV + r^2 d\Omega^2$$

The above expression is similar to (5.9), therefore, the rest of the steps are same as in Unruh radiation with the replacement

$$a = \frac{1}{4GM} \qquad u_+ = v \qquad u_- = u$$
$$x_+ = V \qquad x_- = U$$

in equation (5.10). Therefore, the temperature of the black hole is then, given as:

$$T = \frac{1}{8\pi GM}$$

5.2 QFT in de Sitter Space

In this section, we will expand our understanding by exploring the effects of gravity on particle production in de Sitter space. In contrast to the previous section where we studied field theory near a black hole and in a conformally flat FLRW universe, we will now examine the impact of the gravitational field on particle production in de Sitter space.

As we discussed earlier, in the context of a massless, conformally coupled scalar in a conformally flat spacetime, there was no particle creation. However, this section will reveal that even conformally coupled scalars can experience particle production in de Sitter space due to the presence of the de Sitter horizon. To begin our investigation, we will first consider massless scalar fields.

5.2.1 Massless Scalar field in deSitter space

The metric tensor during inflation is approximately that of de Sitter space ($a(t) \approx e^{Ht}$) in flat slicing is given as:[25]

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 d\vec{x}^2 \\ &= a(t)^2(-d\eta^2 + d\vec{x}^2) \\ &= \underbrace{e^{2Ht}}_{1/\eta^2 H^2}(-d\eta^2 + d\vec{x}^2) \\ &= \frac{1}{H^2\eta^2}(-d\eta^2 + d\vec{x}^2) \end{aligned}$$

[25] In Bernad and Duncan Model, $C(\eta)$ was given as

$$A + B\tanh(\rho\eta),$$

however, here $C(\eta) = -1/H\eta$ with Ricci Scalar given as:

$$R = 12H^2$$

where we defined conformal time as $d\eta = {}^{dt}/_{a(t)} = -d({}^{e^{-Ht}}/_H)$. The Lagrangian for classical scalar field is given as:

$$L = \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - V(\phi)\right] \tag{5.25}$$

The equation of motion becomes:

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\left[\frac{\partial L}{\partial(\partial_\mu\phi)}\right] - \frac{\partial L}{\partial\phi} &= 0\\ \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + m^2\sqrt{-g}\phi + \sqrt{-g}V'(\phi) &= 0\\ \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + m^2\phi + V'(\phi) &= 0\\ \Box\phi + m^2\phi + V'(\phi) &= 0\end{aligned}$$

for massless and free theory, we have $m = 0$ and $V(\phi) = 0$:

$$\begin{aligned}\Box\phi = \partial_\mu\partial^\mu\phi &= 0\\ \underbrace{\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t}}_{\frac{1}{e^{3Ht}}\partial_t(e^{3Ht}\dot{\phi})} - \frac{1}{a^2}\nabla^2\phi &= 0\\ \frac{1}{a}\frac{\partial}{\partial\eta}\left(\frac{1}{a}\frac{\partial}{\partial\eta}\right)\phi + 3H\frac{1}{a}\frac{\partial}{\partial\eta}\phi - \frac{1}{a^2}\nabla^2\phi &= 0\\ \frac{1}{a}\left(\frac{\partial}{\partial\eta}\frac{1}{a}\right)\frac{\partial\phi}{\partial\eta} + \frac{1}{a^2}\frac{\partial^2\phi}{\partial\eta^2} + \frac{3H}{a}\frac{\partial}{\partial\eta}\phi - \frac{1}{a^2}\nabla^2\phi &= 0\end{aligned}$$

using $a = -{}^1/_{\eta H}$

$$\begin{aligned}-\frac{H}{a}\frac{\partial\phi}{\partial\eta} + \frac{1}{a^2}\frac{\partial^2\phi}{\partial\eta^2} + \frac{3H}{a}\frac{\partial\phi}{\partial\eta} - \frac{1}{a^2}\nabla^2\phi &= 0\\ \frac{1}{a^2}\frac{\partial^2\phi}{\partial\eta^2} + \frac{2H}{a}\frac{\partial\phi}{\partial\eta} - \frac{1}{a^2}\nabla^2\phi &= 0\end{aligned}$$

Thus,

$$\boxed{H^2\eta^2\frac{\partial^2\phi}{\partial\eta^2} - 2H^2\eta\frac{\partial\phi}{\partial\eta} - H^2\eta^2\nabla^2\phi = 0}$$

In momentum space: [26]

$$\ddot{\phi}_{\vec{k}} - \frac{2}{\eta}\dot{\phi}_{\vec{k}} + k^2\phi_{\vec{k}} = 0$$

where $\dot{}$ means derivative with respect to η. It has the solution of the form: [27]

$$\phi_k = c_1\underbrace{\frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta}}_{f_k} + c_2\underbrace{\frac{H}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta}}_{\bar{f}_k}$$

with the normalization of mode function, chosen so that

$$\begin{aligned}\bar{f}_k(\partial_t f_k) - (\partial_t\bar{f}_k)f_k &= -H\eta[\bar{f}_k(\partial_\eta f_k) - (\partial_\eta\bar{f}_k)f_k]\\ &= -iH^3\eta^3\end{aligned}$$

[26] we have dropped the $e^{i\vec{k}\cdot\vec{x}}$ factors

[27] for continuous case:

$$\phi(x,\eta) = \int dk[c_k f_k e^{i\vec{k}\cdot\vec{x}} + c.c]$$

for descrete case:

$$\phi(x,\eta) = \sum_k c_k \underbrace{f_k e^{i\vec{k}\cdot\vec{x}}}_{f_k(x)}$$

from (5.2), we get (with the $x-$dependence of mode function coming from $e^{i\vec{k}\cdot\vec{x}}$):

$$\langle f_k, f_{k'}\rangle = i\int \frac{d^3x}{H^3\eta^3} \times -iH^3\eta^3 e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3\delta(\vec{k}-\vec{k}')$$

Before we discuss the alternative derivation of the mode function, I'd like to discuss the time evolution of superhorizon modes characterized by $k << aH$:

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} - \frac{1}{a^2}\nabla^2\phi = 0$$

in momentum space

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} + \frac{1}{a^2}k^2\phi = 0$$

In the limit ${}^k/_a << H \implies k\eta << 1 \implies k\eta \to 0$, the mode function becomes constant. However, from EOM we get:

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} = 0 \implies \phi = A\frac{e^{-Ht}}{H} + B$$

We see that modes outside the comoving horizon suffer exponential decay and eventually as $\eta \to 0$ it become constant. The second way to derive the mode function is via redefinition $\phi_{\vec{k}} = -H\eta u_{\vec{k}}$[28]. Then, we get:

$$\begin{aligned}
\ddot{\phi}_{\vec{k}} - \frac{2}{\eta}\dot{\phi}_{\vec{k}} + k^2\phi_{\vec{k}} &= 0 \\
-H\frac{\partial}{\partial\eta}\left(u_{\vec{k}} + \eta\dot{u}_{\vec{k}}\right) + \frac{2H}{\eta}\left(u_{\vec{k}} + \eta\dot{u}_{\vec{k}}\right) - k^2H\eta u_{\vec{k}} &= 0 \\
\ddot{u}_{\vec{k}} + \left(k^2 - \frac{2}{\eta^2}\right)u_{\vec{k}} &= 0
\end{aligned}$$

which is much easier to solve. It has the solution given as:

$$u_{\vec{k}} = \frac{1}{\sqrt{2k}}\left(1 \mp \frac{i}{k\eta}\right)e^{\pm ik\eta}$$

In the early time limit $\eta \to -\infty$, which essentially turns the de Sitter Klein Gordon Equation into Minkowski one. We have

$$\begin{aligned}
\ddot{u}_{\vec{k}} + \left(k^2 - \cancelto{0}{\frac{2}{\eta^2}}\right)u_{\vec{k}} &= 0 \\
\ddot{u}_{\vec{k}} + k^2u_{\vec{k}} &= 0
\end{aligned} \tag{5.26}$$

which has the solution

$$u_{\vec{k}} = \frac{1}{\sqrt{2k}}e^{\pm ik\eta}$$

where we have chosen the overall normalization, so that the solutions have Wronskian $\pm i$. So that the mode functions at infinite past are orthonormal based on (5.2) i.e. $\langle u_{\vec{k}}, u_{\vec{k}'}\rangle = (2\pi)^3\delta(\vec{k}-\vec{k}')$. At late time ($\eta \to 0$), we can ignore terms independent of η:

$$\begin{aligned}
\ddot{u}_{\vec{k}} + \left(k^2 - \frac{2}{\eta^2}\right)u_{\vec{k}} &= 0 \\
\ddot{u}_{\vec{k}} - \frac{2}{\eta^2}u_{\vec{k}} &= 0
\end{aligned}$$

Hence, the solution would be:

$$u_{\vec{k}}(x,\eta) = u_{\vec{k}}(x)\eta^{-1} + \overline{u}_{\vec{k}}(x)\eta^2$$

Therefore, we can express the field at late time as:

$$\phi_{\vec{k}} \approx \sum_{\Delta=0,3} O_\Delta(x)\eta^\Delta \tag{5.27}$$

where

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}$$

for general mass.

[28]because we don't know $\dot{\phi}$ at infinite past, so we'd like to get rid of it before we can take the limit

Particle Production

This part is taken from "Field Theory in Cosmology" by Enrico Pajer, section 2.3. In the infinite past we defined the mode function as positive energy solution[29] (minkowski mode), we can express the field in de Sitter space at some finite η as:

$$\phi_k = b_k u_k + b_k^\dagger u_k^\star$$

with $\{b_k, b_k^\dagger\}$, a new set of creation and annihilation operators which defines the new vacuum as $b_k |0\rangle = 0$. The bunch davis mode function is given as:

$$u_k = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}$$

and

$$\frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} = \left[1 - \frac{i}{k\eta} - \frac{1}{2}\left(\frac{1}{k\eta}\right)^2\right]\frac{1}{\sqrt{2k}}e^{-ik\eta} + \frac{1}{2(k\eta)^2}e^{-2ik\eta}\frac{1}{\sqrt{2k}}e^{ik\eta}$$

with

$$\left.\begin{aligned}\alpha &= 1 - \tfrac{i}{k\eta} - \tfrac{1}{2(k\eta)^2}\\ \beta &= \tfrac{1}{2(k\eta)^2}e^{-2ik\eta}\end{aligned}\right\}|\alpha|^2 - |\beta|^2 = 1$$

Assuming, at early time, the total number $n_{\text{minkowski}}$ is

$$n_{\text{minkowski}} = 0$$

However, at later time, the number density, $n_{_{BD}}$, of massless particles in de Sitter space is given as:

$$n_{_{BD}} = |\beta|^2 = \frac{1}{4k^4\eta^4} \neq 0$$

The probability of particle production is given as[30]

$$P = 1 - \left|\frac{\beta}{\alpha}\right|^2$$

5.3 Massive particles in de Sitter space

The action for a massive scalar field in an FRW background is

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}m^2\chi^2\right]\\ &= \frac{1}{2}\int d\eta d^3x\left[a^2(\eta)\left((\chi')^2 - (\nabla\chi)^2\right) - m^2a^4(\eta)\chi^2\right].\end{aligned}$$

Introducing the canonically-normalized field $u = a\chi$, and substituting $a(\eta) = -(H\eta)^{-1}$, the action becomes

$$\begin{aligned} S &= \frac{1}{2}\int d\eta d^3x\left[(u')^2 - (\nabla u)^2 - \left(m^2a^2 - \frac{a''}{a}\right)u^2\right]\\ &= \frac{1}{2}\int d\eta d^3x\left[(u')^2 - (\nabla u)^2 - \left(\frac{m^2/H^2 - 2}{\eta^2}\right)u^2\right].\end{aligned}$$

The equation of motion (in Fourier space) then is

$$u_k'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2}\right)u_k = 0.$$

Defining $x \equiv -k\eta$, this becomes

$$x^2\frac{d^2u_k}{dx^2} + \left(x^2 - \nu^2 + \frac{1}{4}\right)u_k = 0, \quad \text{where } \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}.$$

[29] described as adiabatic vacuum at zeroth order in A note on inflation and transplanckian physics pg 6

[30] pg 59 of "Quantum Field Theory in deSitter spacetime" by S. Sunil Kumar.

Writing $u_k(\eta) \equiv \sqrt{x}\, g(x)$, this takes the form of a Bessel equation

$$x^2 \frac{d^2 g}{dx^2} + x\frac{dg}{dx} + (x^2 - \nu^2) g(x) = 0,$$

which has the following solution in terms of Hankel functions:

$$g(x) = c_1 H_\nu^{(1)}(x) + c_2 H_\nu^{(2)}(x).$$

Using

$$H_\nu^{(1)}(x) \xrightarrow{x\to\infty} \sqrt{\frac{2}{\pi x}}\, e^{i(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)},$$
$$H_\nu^{(2)}(x) \xrightarrow{x\to\infty} \sqrt{\frac{2}{\pi x}}\, e^{-i(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)},$$

the early-time limit of the canonically-normalized field becomes

$$\lim_{-k\eta\to\infty} u_k(\eta) = \sqrt{\frac{2}{\pi}} \left(c_1 e^{-\frac{i}{4}\pi(1+2\nu)} e^{-ik\eta} + c_2 e^{\frac{i}{4}\pi(1+2\nu)} e^{ik\eta} \right).$$

This matches the Bunch-Davies initial condition,

$$\lim_{k\eta\to-\infty} u_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta},$$

if $c_2 = 0$ and $c_1 = e^{i\frac{1}{4}\pi(1+2\nu)}\sqrt{\pi/(4k)}$, which fixes the solution to be

$$u_k(\eta) = e^{\frac{i}{4}\pi(1+2\nu)} \sqrt{\frac{\pi}{4}} \sqrt{-\eta} H_\nu^{(1)}(-k\eta).$$

The de Sitter mode function for the massive field, $f_k = u_k/a$, then is

$$f_k = H\frac{\sqrt{\pi}}{2} e^{\frac{i}{4}\pi(1+2\nu)} (-\eta)^{3/2} H_\nu^{(1)}(-k\eta) \qquad \left(\text{where } \nu = \sqrt{\tfrac{9}{4} - \tfrac{m^2}{H^2}}\right)$$

At late time:

$$\begin{aligned} f_k &= -H\frac{\sqrt{\pi}}{2} e^{\frac{i}{4}\pi(1+2\nu)} (-\eta)^{3/2} \frac{i}{\pi} \left[\Gamma(\nu) \left(\frac{-k\eta}{2}\right)^{-\nu} + e^{-i\pi\nu}\Gamma(-\nu)\left(\frac{-k\eta}{2}\right)^{\nu} \right] \\ &= -iH\sqrt{\frac{2}{\pi k^3}} e^{\frac{i\pi}{4}} \left[e^{i\frac{\pi\nu}{2}} \Gamma(\nu) \left(\frac{-k\eta}{2}\right)^{\frac{3}{2}-\nu} + e^{-i\frac{\pi\nu}{2}} \Gamma(-\nu) \left(\frac{-k\eta}{2}\right)^{\frac{3}{2}+\nu} \right] \end{aligned}$$

For very massive fields $m >> H$, we have $\nu \approx {}^{im}/_{H}$

$$f_k = -iH\sqrt{\frac{2}{\pi k^3}} e^{\frac{i\pi}{4}} \left[e^{-\pi m/2H} \Gamma\left(\frac{im}{H}\right) \left(\frac{-k\eta}{2}\right)^{\frac{3}{2} - \frac{im}{H}} + c.c \right]$$

In coordinate time:

$$\begin{aligned} f_k &= iH\sqrt{\frac{2}{\pi k^3}} e^{\frac{i\pi}{4}} \left[e^{-\pi m/2H} \Gamma\left(\frac{im}{H}\right) \left(\frac{ke^{-Ht}}{2H}\right)^{\frac{3}{2} - \frac{im}{H}} + e^{\pi m/2H} \Gamma\left(-\frac{im}{H}\right) \left(\frac{ke^{-Ht}}{2H}\right)^{\frac{3}{2} + \frac{im}{H}} \right] \\ &= iH\sqrt{\frac{2}{\pi k^3}} e^{-\frac{i\pi}{4}} \left[e^{-\pi m/2H} \Gamma\left(\frac{im}{H}\right) \left(\frac{k}{2H}\right)^{\frac{3}{2} - \frac{im}{H}} e^{-3Ht/2 + imt} + e^{\pi m/2H} \Gamma\left(-\frac{im}{H}\right) \left(\frac{k}{2H}\right)^{\frac{3}{2} + \frac{im}{H}} e^{-3Ht/2 - imt} \right] \end{aligned}$$

Then, the coefficient of e^{iEt} is recognized as β_k:

$$\begin{aligned} |\beta|^2 &= H^2 \frac{2}{\pi k^3} \left(\frac{k}{2H}\right)^3 |\Gamma(i\mu)|^2 e^{-\pi\mu} \\ &= \frac{1}{2\pi}\frac{1}{H}\frac{\pi}{(m/H)\sinh(\pi m/H)} e^{-\pi m/H} \approx e^{-2\pi\mu} \\ &= e^{-2\pi m/H} \end{aligned}$$

Look up "Thermodynamics of de Sitter" to learn how temperature is defined. We first consider the Euclideanization and then make the space periodic. The periodicity defines the temperature.

5.4 Horizon exit and horizon entry

Taken from "Coherent Phase Argument for Inflation - Scott Dodelson"

We typically model inflation using de Sitter space and as such, the modes which go outside the comoving horizon should experience, exponential dampening($m \neq 0$) or become constant ($m = 0$)

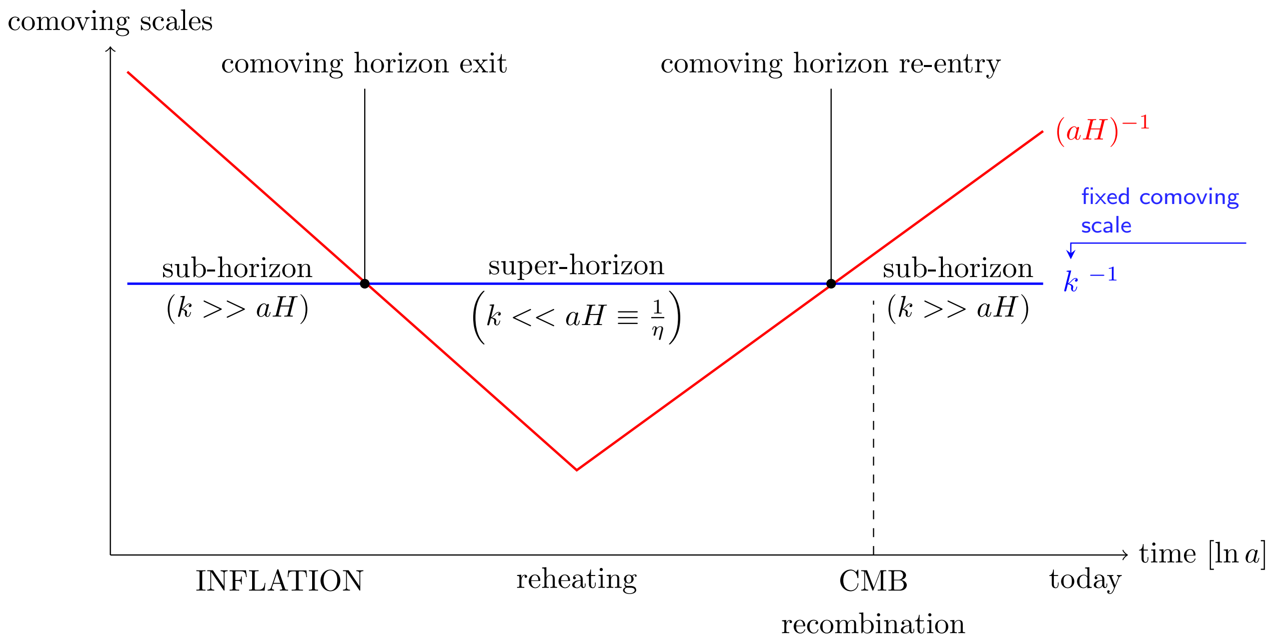


Figure 5.5: Scales (here density fluctuation) of cosmological interest were larger than the Hubble radius until $a \sim 10^{-5}$ (where today is at $a(t_0) = 1$). However, before inflation, all scales of interest were smaller than the Hubble radius. Similarly, at very late times, the scales of cosmological interest are back within the Hubble radius.

The way to interpret this is following:

Before inflation ends, though, all modes leave the horizon, that is, their wavelengths get stretched so much that no causal physics can alter them. Once this happens, their amplitudes remain constant. They stay constant up until the time much later on (for modes of interest this might typically happen when the universe is 100,000 years old) when the modes re-enter the horizon, at which time causal physics can once again affect their amplitudes.

The next question, one asks is How do these perturbations evolve once they re-enter the horizon? A very simplified toy model to look at this is following:

$$\ddot{\delta} - c_s^2 \nabla^2 \delta = F$$

where c_s is the sound speed and F is the gravitational source. The perturbations obey the wave equation as one expects physically: a region which is very overdense is driven by gravity to become more overdense, but driven toward the average density by pressure.

At this point, you might come to the conclusion that the spectrum of anisotropies in the radiation today will exhibit a series of peaks and troughs just as a guitar string produces a series of higher harmonics. In fact, the spectrum of the CMB looks remarkably like that of a guitar string. However, underlying the similarity is a pair of differences which are essential to the argument that inflation is the origin of the perturbations. A guitar string produces a set of harmonics because it is tied down at its ends. So there are only a discrete set of frequencies at which it can oscillate. There is no such restriction for perturbations in the early universe, so why do we see anisotropies at certain frequencies but not at others?

5.5 Schwinger effect in de Sitter space

Taken from Schwinger effect in de Sitter space by Markus B. Fr¨ob, Jaume Garriga, Sugumi Kanno, Misao Sasaki, Jiro Soda, Takahiro Tanaka, Alexander Vilenkin. In flat spacetime, the pair production by the Electric field is described by

$$\Gamma = \frac{q^2 EB}{8\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \coth\left(\frac{B}{E} n\pi\right) e^{-nm^2\pi/|q|E}$$

In the limit $|q| \to 0, E \to 0$ with $B = 0$, we have

$$\Gamma \approx \frac{q^2 E^2}{8\pi^3} e^{-m^2\pi/|q|E}$$

The main result of this section is the amplitude for pair production which goes like $e^{-\pi\mu}$ with $\mu = m^2/|q|E$.

5.6 Statistics of fluctuations

This part has been taken from 'Lectures on cosmological correlators' by Daniel Baumann. Inflation predicts gaussianity, which stems from the fact that we study the quantum fluctuations in the Bunch-Davies vacuum (corresponding to the ground state of the harmonic oscillator). To show this, we consider the Schrödinger picture and ask: *What is the probability distribution corresponding to each perturbation mode?* To answer this question, we need to solve the Schrödinger equation. We begin by considering the eigenstate of the quantum field operator. $\hat{\Phi}$:

$$\hat{\Phi}(t, x) |\phi\rangle = \phi(t, x) |\phi\rangle \tag{5.28}$$

where $\phi(t, x)$ is the spatial profile of the field at a given time t. For a particle in potential V, the Hamiltonian is given as:

$$H = \frac{p^2}{2m} + V(x)$$

We start by beginning with a wave function, which is defined as

$$\psi(x) \equiv \langle x|\psi\rangle \qquad \text{with} \qquad \hat{x}|x\rangle = x|x\rangle$$

and, we have (setting $\hbar = 1$)

$$[\hat{x}, \hat{p}] = i$$

The schrodinger equation (with flat metric) becomes:

$$i\frac{\partial}{\partial t}\psi = -\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$$

Similarly, in quantum field theory, the Hamiltonian for a scalar field is given as:[31]

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}\omega^2\phi^2$$

where, we have

$$\psi[\phi(t, x)] \equiv \langle\phi|\psi\rangle \qquad \text{with} \qquad \hat{\Phi}(t, x)|\phi\rangle = \phi(t, x)|\phi\rangle$$

and,

$$[\hat{\Phi}(x), \hat{\Pi}(y)] = i\delta(x - y)$$

The schrodinger equation then becomes:[32]

$$i\frac{\partial}{\partial\eta}\psi[\phi(t, x)] = \left(-\frac{1}{2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2}\omega^2\phi\right)\psi[\phi(t, x)]$$

This looks like the Schroedinger equation for harmonic oscillator (with $m = 1$) and thus we know that the wavefunction for ground state is gaussian i.e.

$$\langle\phi|0\rangle = Ce^{-\frac{\omega}{2}\phi(t,x)^2} e^{-i \underset{\uparrow \text{ground state energy}}{E_0} \eta}$$

As long as the primordial density fluctuation depends linearly on the above quantum field, it should obey a Gaussian distribution. There are other ways to detect whether the distribution is Gaussian or not. By measuring odd-order moments of the distribution function. The bispectrum is the first non-zero moment for a non-Gaussian distribution, in the sense that if the bispectrum is non-zero, the distribution is non-Gaussian. Note that the converse is not necessarily true. Thus, if we measure a non-zero bispectrum, we can be sure that the distribution is non-Gaussian and there are some **interaction** or **particle production** taking place. Measurements of higher-order correlations (or non-Gaussianity) are the analog of measuring collisions in particle physics. The study of non-Gaussianity therefore also goes by the name of "cosmological collider physics". We

[31] section 14.7.1 of Gravitation by Padmanabhan
[32] using section 10.1 of QFT of point particles and strings by Hatfield

can see the same from path integral approach as well. The wavefunction at a given time (which we can set to $t_* = 0$ without loss of generality) is formally computed by the following path integral

$$\Psi[\phi] = \int \mathcal{D}\Phi \, e^{iS[\Phi]} \overset{\text{using perturbation theory at tree level}}{\approx} e^{iS[\Phi_{\rm cl}]},$$

with

$$\Phi(0) = \phi \qquad \Phi(-\infty^+) = 0$$

Consider a scalar field, whose action is

$$S[\Phi] = \int dt \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2 \Phi^2 \right),$$

where Φ is the deviation from equilibrium and ω is the constant frequency of the oscillator. On-shell, the action can equivalently be written as a pure boundary term

$$S[\Phi_{\rm cl}] = \int_{t_i}^{t^*} dt \left[\frac{1}{2}\partial_t(\dot{\Phi}_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl}) \right] = \frac{1}{2}\dot{\Phi}_{\rm cl}\Phi_{\rm cl}\Big|_{t=t^*},$$

where we have integrated by parts and used the fact that the classical solution satisfies the equation of motion $\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl} = 0$. To determine the classical solution $\Phi_{\rm cl}(t)$ we have to specify two boundary conditions: We require: 1) the late-time value of the oscillator position is $\Phi_{\rm cl}(t_* = 0) = \phi$, and 2) the early-time limit is the minimum-energy solution $\Phi_{\rm cl}(t) \sim e^{i\omega t}$. The unique solution satisfying these boundary conditions is $\Phi_{\rm cl}(t) = \phi e^{i\omega t}$. Substituting this into above, we get

$$S = \frac{i\omega}{2}\phi^2 \quad \Longrightarrow \quad \exp(iS) = \exp\left(-\frac{\omega}{2}\phi^2\right) \quad \Longrightarrow \quad |\Psi[\phi]|^2 = e^{-\omega\phi^2}.$$

We see the familiar fact that the ground state wavefunction of the simple harmonic oscillator is a Gaussian. The width of this Gaussian determines the size of the zero-point fluctuations of the oscillator:

$$\langle \phi^2 \rangle = \frac{1}{2\omega}.$$