Conformal Field Theory

Chandra Prakash
email : chandra.pp@alumni.iitg.ac.in

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Chapter 1

Conformal Symmetry

1.1 Introduction

Conformal symmetry is, in essence, the symmetry of **shapes without scales**. In Euclidean geometry, this is often described as "angle-preserving" symmetry. While this definition is correct in a purely spatial setting, it becomes less illuminating once we step into relativistic physics, where time is on equal footing with space. The notion of an "angle" between two events separated in time is not geometrically well-defined in the same way, so we need a more general formulation.

A more robust way to think about conformal transformations is that they are transformations which preserve the metric **up to a local rescaling**. That is, they preserve the light-cone structure of spacetime, and hence the causal relations, but may change distances by position-dependent factors:

$$g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x), \quad \Omega(x) > 0.$$

Here, $\Omega(x)$ is the local scaling factor, and $\Omega(x) = 1$ corresponds to an isometry. This definition works in both Euclidean and Lorentzian settings, and it makes clear why conformal transformations generalize ordinary Poincaré symmetries: they allow stretching of spacetime while preserving angles and null directions.

In flat space, an important special case is the global scaling transformation:

$$x^\mu \;\longrightarrow\; \lambda\, x^\mu,$$

which is not a mere relabeling of coordinates (i.e., not just a diffeomorphism), but a genuine change in the geometry. Under such a transformation, the ratios of lengths along a direction are preserved, and therefore angles are unchanged. This property underlies the term "conformal."

The study of **Conformal Field Theory (CFT)** leverages this symmetry in a very different way from conventional Quantum Field Theory (QFT). In QFT, one typically begins with a Lagrangian and derives correlation functions from the equations of motion and perturbation theory. In CFT, by contrast, the symmetry itself is so constraining that it often determines the form of correlation functions without reference to a specific Lagrangian. This leads naturally to the **conformal bootstrap** program, in which the consistency conditions of conformal symmetry, unitarity, and the operator product expansion are used to solve the theory.

Before we can use these powerful tools, it is important to distinguish conformal transformations from two related but conceptually different ideas: **Weyl rescalings** and **diffeomorphisms**. We will examine these one by one in the next section.

General coordinate invariance (diffeomorphism)

Classical field theories can possess a variety of symmetries. One symmetry we will assume here is **general coordinate invariance**. Using the action principle, this symmetry can be used to show that the energy–momentum tensor is conserved.

In general, the energy–momentum tensor is defined through the variation of the action S under changes in the space–time metric:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}\,.$$

By definition,

$$\delta S = \frac{1}{2} \int d^d x\, \sqrt{g}\, T^{\mu\nu}\, \delta g_{\mu\nu}\,.$$

If the theory is invariant under general coordinate transformations, one can show that

$$(T^{\mu\nu})_{;\nu} = 0\,,$$

where, as usual in general relativity, "$;\nu$" denotes the covariant derivative. In flat coordinates, this condition reduces to

$$\partial_\nu T^{\mu\nu} = 0\,.$$

Weyl invariance

In addition to general coordinate invariance, many field theories possess another powerful symmetry: **Weyl invariance**. While diffeomorphism invariance constrains how the metric responds to arbitrary coordinate changes, Weyl invariance instead concerns how the theory behaves under local rescalings of the metric. Under a Weyl transformation, the metric changes as

$$g_{\mu\nu}(x) \to \Omega(x)\, g_{\mu\nu}(x)\,,$$

or, in infinitesimal form,

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \omega(x)\, g_{\mu\nu}(x)\,.$$

The condition for the action to remain invariant under such a transformation can be expressed in terms of the energy–momentum tensor. Substituting $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}(x)$ into the earlier definition, we find

$$\delta S = \frac{1}{2}\int d^d x\, \sqrt{g}\, T^\mu_{\ \mu}\, \omega(x)\,.$$

Since this must hold for arbitrary functions $\omega(x)$, we conclude that the condition for Weyl invariance is

$$T^\mu_{\ \mu} = 0\,.$$

Thus, just as diffeomorphism invariance implies the covariant conservation of the energy–momentum tensor, Weyl invariance implies that the energy-momentum tensor must be traceless.

Conformal invariance

1.1.1 Conformal Transformations

A *conformal transformation* can be defined as a coordinate transformation that acts on the metric as a Weyl transformation. Consider a general coordinate transformation

$$x \to\ x', \quad x^\mu = f^\mu(x')\,.$$

The metric then transforms as

$$g_{\mu\nu}(x) \to\ g'_{\mu\nu}(x') = \frac{\partial f^\rho}{\partial x'^\mu}\frac{\partial f^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}\big(f(x')\big)\,.$$

We now require that the transformed metric be proportional to the original one. Rotations and translations clearly satisfy this condition: they leave the metric unchanged and hence preserve all inner products

$$v\cdot w \ \equiv\ v^\mu g_{\mu\nu} w^\nu\,.$$

They are therefore part of the conformal group. More generally, any coordinate transformation satisfying the above proportionality preserves all angles,

$$\frac{v\cdot w}{\sqrt{v^2\, w^2}}\,,$$

which is the origin of the term "conformal." Later in this chapter we will determine all such transformations explicitly.

If a field theory has a conserved and traceless energy–momentum tensor, it is invariant under both general coordinate transformations and Weyl transformations. Let the action be

$$S \ =\ \int d^d x\, \mathcal{L}\big(\partial_x,\, g_{\mu\nu}(x),\, \phi(x)\big)\,.$$

Here, ϕ denotes any matter field, while the metric $g_{\mu\nu}$ is written separately due to its special rôle. We have also explicitly indicated spacetime derivatives in the Lagrangian.

General coordinate invariance implies

$$S \;=\; S' \;\equiv\; \int d^dx'\,\mathcal{L}\big(\partial_{x'},\, g'_{\mu\nu}(x'),\, \phi'(x')\big),$$

where $g'_{\mu\nu}$ is as given above, and the transformation of ϕ depends on its spin. For a tensor field of rank n one has

$$\phi'_{\mu_1\ldots\mu_n}(x') = \left|\frac{\partial f}{\partial x'}\right|^{\frac{\Delta}{d}} \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}}\, \phi_{\nu_1\ldots\nu_n}\big(f(x')\big), \tag{1.1}$$

where Δ is the scaling dimension of the field. Fields that transform according to Eq. (1.1) under conformal transformations are called *conformal fields*, or equivalently, *primary fields*.

In particular, for a scalar $\phi(x)$ we have simply $\phi'(x') = \phi(f(x'))$. For the derivative of a scalar:

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial x'^\mu}\phi'(x') = \frac{\partial f^\nu}{\partial x'^\mu}\frac{\partial}{\partial f^\nu}\phi(f(x'))\,,$$

which transforms as a vector. (However, note that nth-order *ordinary* derivatives do not transform as rank-n tensors; this holds only for covariant derivatives.)

If the coordinate transformation $x \to x'$ is of the above type, we can **use Weyl invariance of the action to bring the metric back to its original form**. This yields

$$S = S'' \equiv \int d^dx'\,\mathcal{L}\big(\partial_{x'}, g_{\mu\nu}(f(x')), \phi'(x')\big) = \int d^dx'\,\mathcal{L}\big(\partial_{x'}, g_{\mu\nu}(x), \phi'(x')\big)\,.$$

This is the statement of *conformal symmetry* of the action. We should note that in some cases, an isometry of one metric can act as a conformal transformation for a different metric. In such situations, Weyl rescaling is not needed, since the transformation already preserves the metric under consideration. This happens, for example, in de Sitter (dS) and anti–de Sitter (AdS) spacetimes, where certain isometries correspond to conformal transformations of the induced boundary metric.

If we begin with a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, the background metric remains unchanged under such transformations.[1] This allows us to define conformal transformations for theories in flat space that are not coupled to gravity. We may then ignore general coordinate invariance and start with an action in which no dynamical metric appears.

In this flat-space setting, conformal invariance means that the action is unchanged when we integrate the same Lagrangian (or any scalar physical quantity) written in terms of the transformed fields $\phi'(x')$ over the new coordinates x'.

In $d = 2$ dimensions, this is not really a restriction. A general 2D metric has three independent components: $g_{11}(x)$, $g_{22}(x)$, and $g_{12}(x) = g_{21}(x)$. A general coordinate transformation provides two functions $f^1(x)$ and $f^2(x)$ that can be used to set $g_{12}(x) = 0$ and $g_{11}(x) = \pm g_{22}(x)$ (depending on the signature). The metric can then be written in the form $g(x)\,\eta_{\mu\nu}$, which is called *conformal gauge*. A Weyl transformation can remove the remaining factor $g(x)$, bringing the metric to the form $\eta_{\mu\nu}$.

In more than two dimensions, this procedure does not work in general, so restricting to flat space truly limits us to non-gravitational theories. Even in two dimensions, conformal gauge can be chosen only locally in general, meaning CFT can be applied in coordinate patches, but extra data may be needed to describe the theory globally.

1.2 Infinitesimal Conformal Transformation

The fundamental essence of conformal transformations resides in their infinitesimal form, which serves as a crucial tool for investigating how fields transform under these symmetries. It plays a pivotal role in defining the generator of the conformal group and, subsequently, constraining the set of possible correlators that are compatible with conformal symmetry. Any infinitesimal transformation can be expressed as:

$$x'^\mu = x^\mu + \underset{\uparrow\,\text{infinitesimal}}{\epsilon}{}^{\mu}(x)$$

[1] In perturbation theory, we often describe physics on a perturbed manifold as that of tensor fields living on a background manifold. In this case, the conformal transformation is performed on the background metric, and the change in the perturbation is again dictated by the Killing equation. We first apply the coordinate transformation to the full metric, and then perform a Weyl rescaling to restore the background metric to its original form.

and subsequently,

$$x^\mu = x'^\mu - \epsilon^\mu(x)$$

therefore, the metric transforms like:

$$\begin{aligned}
g'_{\mu\nu} &= \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}}_{\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha(x)} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\
&= \left[\delta^\alpha_\mu - \frac{\partial \epsilon^\alpha(x)}{\partial x'^\mu}\right]\left[\delta^\beta_\nu - \frac{\partial \epsilon^\beta(x)}{\partial x'^\nu}\right] g_{\alpha\beta} \\
&= \delta^\alpha_\mu \delta^\beta_\nu g_{\alpha\beta} - \delta^\beta_\nu \partial_\mu \epsilon^\alpha(x) g_{\alpha\beta} - \delta^\alpha_\mu \partial_\nu \epsilon^\beta(x) g_{\alpha\beta} + \mathcal{O}(\epsilon^2) \\
\Omega(x) g_{\mu\nu} &= g_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu
\end{aligned}$$

In the third step, we used chain rule on $\epsilon^\alpha(x)$ and ignored $\mathcal{O}((\partial\epsilon)^2)$ terms. From the last line, it is reasonable to expect that:

$$\begin{aligned}
g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &= [1 + f(x)] g_{\mu\nu} \\
\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) &= f(x) g_{\mu\nu} \qquad (1.2)
\end{aligned}$$

Contracting Indicies

$$\begin{aligned}
\partial^\mu \epsilon_\mu(x) + \partial^\mu \epsilon_\mu(x) &= f(x) \delta^\mu_\mu \\
2(\partial \cdot \epsilon) &= \underset{\uparrow \text{dimension of spacetime}}{d}\ f(x) \\
f(x) &= \frac{2}{d} \frac{\partial \epsilon_\mu(x)}{\partial x_\mu}
\end{aligned}$$

Substituing back in (1.2)

$$\boxed{\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \frac{2}{d}(\partial \cdot \epsilon)\ g_{\mu\nu}} \qquad (1.3)$$

Up until now, we have no made any crude assumption. However before we proceed, we will assume that the metric is Euclidean. Now, we operate on both sides by ∂^ν

$$\begin{aligned}
\frac{\partial}{\partial x'_\nu}\left[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)\right] &= \frac{\partial}{\partial x'_\nu}\left(\frac{2}{d} \partial \cdot \epsilon(x)\ \overset{\text{assuming flat metric}\downarrow}{g_{\mu\nu}}\right) \\
\partial_\mu \underbrace{\partial^\nu \epsilon_\nu}_{\partial \cdot \epsilon} + \underbrace{\partial^\nu \partial_\nu}_{\Box} \epsilon_\mu &= \frac{2}{d} g_{\mu\nu} \partial^\nu \partial \cdot \epsilon \\
\partial_\mu(\partial \cdot \epsilon) + \Box \epsilon &= \frac{2}{d} \partial_\mu (\partial \cdot \epsilon)
\end{aligned}$$

Operating by ∂_ν

$$\begin{aligned}
\partial_\nu[\partial_\mu(\partial \cdot \epsilon) + \Box \epsilon] &= \partial_\nu \left[\frac{2}{d} \partial_\mu (\partial \cdot \epsilon)\right] \\
\left(1 - \frac{2}{d}\right) \partial_\nu \partial_\mu (\partial \cdot \epsilon) + \Box(\partial_\nu \epsilon_\mu) &= 0 \qquad (1.4)
\end{aligned}$$

under relabeling $\mu \leftrightarrow \nu$

$$\left(1 - \frac{2}{d}\right) \partial_\mu \partial_\nu (\partial \cdot \epsilon) + \Box(\partial_\mu \epsilon_\nu) = 0 \qquad (1.5)$$

adding (1.4) and (1.5)

$$\begin{aligned}
2\left(1 - \frac{2}{d}\right) \partial_\mu \partial_\nu (\partial \cdot \epsilon) + \Box \underbrace{[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)]}_{\frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}} &= 0 \\
\left(1 - \frac{2}{d}\right) \partial_\mu \partial_\nu (\partial \cdot \epsilon) + \frac{1}{d} \Box (\partial \cdot \epsilon) g_{\mu\nu} &= 0
\end{aligned}$$

$$[g_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu]\,(\partial\cdot\epsilon) = 0 \tag{1.6}$$

Contracting the indicies

$$\begin{aligned}[][d\Box + (d-2)\Box](\partial\cdot\epsilon) &= 0\\ 2(d-1)\Box(\partial\cdot\epsilon) &= 0\end{aligned}$$

hence,

$$\boxed{(d-1)\Box(\partial\cdot\epsilon) = 0} \tag{1.7}$$

if $d = 1 \implies$ any $\epsilon^\mu(x)$ satisfies (1.7), therefore, is conformal transformation. It is interesting to note that any 1D QFT is conformal field theory, but for our purpose it's not very useful. We will be concerned with $d \neq 1$ for the rest of this notes unless stated otherwise. Consider the action of ∂_α on (1.3) and then cyclic relabeling of incidies as $\alpha \to \mu \to \nu$

$$\partial_\alpha[\partial_\mu\epsilon_\nu(x) + \partial_\nu\epsilon_\mu(x)] = \frac{2}{d}\partial_\alpha g_{\mu\nu}(\partial\cdot\epsilon) \tag{1.8}$$

$$\partial_\mu[\partial_\nu\epsilon_\alpha(x) + \partial_\alpha\epsilon_\nu(x)] = \frac{2}{d}\partial_\mu g_{\nu\alpha}(\partial\cdot\epsilon) \tag{1.9}$$

$$\partial_\nu[\partial_\alpha\epsilon_\mu(x) + \partial_\mu\epsilon_\alpha(x)] = \frac{2}{d}\partial_\nu g_{\alpha\mu}(\partial\cdot\epsilon) \tag{1.10}$$

Adding the first two equation and subtracting from the last, we get [(1.8) + (1.9) − (1.10)]:

$$\cancel{2}\partial_\alpha\partial_\mu\epsilon_\nu = \frac{\cancel{2}}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon)$$

$$\partial_\alpha\partial_\mu\epsilon_\nu = \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon) \tag{1.11}$$

Referring to eqn 1.7.11 of "Ideas and Methods of Supersymmetry and Supergravity" by Sergio M. Kuzenko, we find that the 3rd order derivation of $\epsilon^\mu(x)$ vanishes. Therefore, the most general conformal transformation is of the type:

$$x'^\mu = x^\mu + \underbrace{a^\mu + b^\mu{}_\nu x^\nu + c^\mu{}_{\nu\alpha}x^\nu x^\alpha}_{\epsilon^\mu}$$

Where, $a^\mu, b^\mu{}_\nu$ and $c^\mu{}_{\nu\alpha}$ are parameters relevant to their transformation. The goal here is simple:

- First find the relevant transformations
- Then based on the transformation rule, find the generators.

For $\epsilon^\mu = a^\mu$:

$$\begin{aligned}x'^\mu &= x^\mu + a^\mu\\ &= x^\mu + \delta^\mu_\nu a^\nu\\ &= x^\mu + (\partial_\nu x^\mu)a^\nu\\ &= [1 + ia^\nu(-i\partial_\nu)]x^\mu\end{aligned}$$

Thus, the generator of translation is $P_\mu - i\partial_\mu$[2]. For $\epsilon^\mu = b^\mu{}_\alpha x^\alpha$, we refer to (1.3)

$$\begin{aligned}\partial_\mu\epsilon_\nu(x) + \partial_\nu\epsilon_\mu(x) &= \frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu}\\ \partial_\mu(b_{\nu\alpha}x^\alpha) + \partial_\nu(b_{\mu\alpha}x^\alpha) &= \frac{2}{d}(\partial^\mu b_{\mu\alpha}x^\alpha)g_{\mu\nu}\\ b_{\nu\alpha}\delta^\alpha_\mu + b_{\mu\alpha}\delta^\alpha_\nu &= \frac{2}{d}(b_{\mu\alpha}g^{\alpha\mu})g_{\mu\nu}\\ b_{\nu\mu} + b_{\mu\nu} &= \frac{2}{d}b^\alpha{}_\alpha g_{\mu\nu}\\ \frac{b_{\nu\mu} + b_{\mu\nu}}{2} &= \frac{1}{d}b^\alpha{}_\alpha g_{\mu\nu}\end{aligned}$$

now,

$$b_{\mu\nu} = \frac{b_{\mu\nu} - b_{\nu\mu}}{2} + \frac{b_{\mu\nu} + b_{\nu\mu}}{2}$$

[2]if we use $[1 - a^\nu(\partial_\nu)]x^\mu$ as the definition, then $P_\mu = i\partial_\mu$ would be the generator

$$= M_{\mu\nu} + \lambda g_{\mu\nu}$$

If $b_{\mu\nu} = \lambda g_{\mu\nu}(M_{\mu\nu} = 0)$

$$\begin{aligned}
x'^{\mu} &= x_{\mu} + b^{\mu}{}_{\nu} x^{\nu} \\
&= x^{\mu} + \lambda g^{\mu\alpha} \underbrace{g_{\alpha\nu} x^{\nu}}_{x_\alpha} \\
&= x^{\mu} + \lambda x^{\mu} \\
&= x^{\mu} + \lambda x^{\nu} \delta^{\mu}_{\nu} \\
&= x^{\mu} + \lambda x^{\nu} (\partial_\nu x^{\mu}) \\
&= x^{\mu} + i\lambda x^{\nu} (-i\partial_\nu x^{\mu}) \\
&= (1 + i\lambda(-ix^{\nu}\partial_\nu))x^{\mu}
\end{aligned}$$

Thus, the generator of dilatation is $D = -ix^{\mu}\partial_\mu$. For $b_{\mu\nu} = M_{\mu\nu}(\lambda = 0)$.

$$\begin{aligned}
x'^{\mu} &= x^{\mu} + M^{\mu}{}_{\nu} x^{\nu} \\
&= x^{\mu} + M^{\alpha}{}_{\nu} \delta^{\mu}_{\alpha} x^{\nu} \\
&= x^{\mu} + M^{\alpha}{}_{\nu} (\partial_\alpha x^{\mu}) x^{\nu} \\
&= x^{\mu} + M_{\alpha\nu} (\partial^{\alpha} x^{\mu}) x^{\nu} \\
&= x^{\mu} + \frac{M_{\alpha\nu} - M_{\nu\alpha}}{2} (\partial^{\alpha} x^{\mu}) x^{\nu} \\
&= x^{\mu} + \frac{1}{2} M_{\alpha\nu} (\partial^{\alpha} x^{\mu}) x^{\nu} - \boxed{\frac{1}{2} M_{\nu\alpha} (\partial^{\alpha} x^{\mu}) x^{\nu}} \quad \text{relabeling } \nu \leftrightarrow \alpha \\
&= x^{\mu} + \frac{1}{2} M_{\alpha\nu} (\partial^{\alpha} x^{\mu}) x^{\nu} - \frac{1}{2} M_{\alpha\nu} (\partial^{\nu} x^{\mu}) x^{\alpha} \\
&= x^{\mu} + \frac{1}{2} M_{\alpha\nu} (x^{\nu}\partial^{\alpha} - x^{\alpha}\partial^{\nu}) x^{\mu} \\
&= x^{\mu} + \frac{i}{2} M_{\alpha\nu} \{-i(x^{\nu}\partial^{\alpha} - x^{\alpha}\partial^{\nu})\} x^{\mu} \\
&= x^{\mu} + \frac{i}{2} M_{\alpha\nu} \{\underbrace{i(x^{\alpha}\partial^{\nu} - x^{\nu}\partial^{\alpha})}_{L^{\alpha\nu}}\} x^{\mu}
\end{aligned}$$

Thus, the generator of rotation is $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$. Now, the last part $\epsilon^{\mu} = c^{\mu}{}_{\nu\alpha} x^{\nu} x^{\alpha} = c^{\mu}{}_{\alpha\nu} x^{\nu} x^{\alpha}$, we refer to (1.11):

$$\begin{aligned}
\partial_\alpha \partial_\mu \epsilon_\nu &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial \cdot \epsilon) \\
\partial_\alpha \partial_\mu (c_{\nu\sigma\beta} x^{\sigma} x^{\beta}) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu]\partial^{\mu}(c_{\mu\sigma\beta} x^{\sigma} x^{\beta}) \\
c_{\nu\sigma\beta}\partial_\alpha(\delta^{\sigma}_{\mu} x^{\beta} + x^{\sigma}\delta^{\beta}_{\mu}) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu] c_{\mu\sigma\beta}(g^{\sigma\mu} x^{\beta} + x^{\sigma} g^{\beta\mu}) \\
c_{\nu\sigma\beta}(\delta^{\sigma}_{\mu}\delta^{\beta}_{\alpha} + \delta^{\sigma}_{\alpha}\delta^{\beta}_{\mu}) &= \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](c^{\sigma}{}_{\sigma\beta} x^{\beta} + c^{\beta}{}_{\sigma\beta} x^{\sigma}) \\
2c_{\nu\mu\alpha} &= \frac{2}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu] c^{\sigma}{}_{\sigma\beta} x^{\beta} \\
c_{\nu\mu\alpha} &= \underbrace{\frac{1}{d} c^{\sigma}{}_{\sigma\beta}}_{b_\beta} [g_{\mu\nu}\delta^{\beta}_{\alpha} + g_{\nu\alpha}\delta^{\beta}_{\mu} - g_{\alpha\mu}\delta^{\beta}_{\nu}] \\
&= g_{\nu\mu} b_\alpha + g_{\nu\alpha} b_\mu - g_{\mu\alpha} b_\nu
\end{aligned}$$

Therefore,

$$\begin{aligned}
\epsilon_\mu &= c_{\mu\alpha\beta} x^{\alpha\beta} \\
&= (g_{\mu\alpha} b_\beta + g_{\mu\beta} b_\alpha - g_{\alpha\beta} b_\mu) x^{\alpha} x^{\beta} \\
&= x_\mu (b \cdot x) + x_\mu (b \cdot x) - b_\mu (x \cdot x) \\
&= 2x_\mu (b \cdot x) - x^2 b_\mu
\end{aligned}$$

Hence, the Special Conformal Transformation looks like:

$$x'^{\mu} = x^{\mu} + 2x^{\mu}(b \cdot x) - x^2 b^{\mu}$$

$$
\begin{aligned}
&= x^\mu + 2(b\cdot x)x^\nu \delta^\mu_\nu - x^2 b^\nu \delta^\mu_\nu \\
&= x^\mu + 2(b\cdot x)x^\nu \partial_\nu x^\mu - x^2 b^\nu \partial_\nu x^\mu \\
&= [1 + 2(b\cdot x)x^\nu \partial_\nu - x^2 b^\nu \partial_\nu] x^\mu \\
&= [1 + \{2b^\alpha x_\alpha x^\nu \partial_\nu - x^2 b^\alpha \partial_\alpha\}] x^\mu \\
&= [1 + ib^\alpha \{\underbrace{-i(2x_\alpha x^\nu \partial_\nu - x^2 \partial_\alpha)}_{K_\alpha}\}] x^\mu
\end{aligned}
$$

Hence, the generator for Special Conformal Transformations (SCT) takes the form $K^\mu = -i(2x^\mu x\cdot\partial - x^2\partial^\mu)$. We will now list all the **infinitesimal** transformations and their generators we found in this section.

1. Translation

$$x'^\mu = x^\mu + a^\mu \qquad\qquad P_\mu = -i\partial_\mu \tag{1.12}$$

2. Rotation

$$x'^\mu = x^\mu + M^\mu{}_\nu x^\nu \qquad\qquad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{1.13}$$

3. Dilatation

$$x'^\mu = (1+\lambda)x^\mu \qquad\qquad D = -ix^\mu\partial_\mu \tag{1.14}$$

4. Special Conformal Transformation

$$x'^\mu = x^\mu + 2x^\mu(b\cdot x) - x^2 b^\mu \qquad\qquad K^\mu = -i(2x^\mu x\cdot\partial - x^2\partial^\mu) \tag{1.15}$$

In the above listed transfomations, the parameters $a^\mu, M^\mu{}_\nu, \lambda$ and b^μ are all infinitesimal.

1.3 Finite Conformal Transformation

In the previous section, we considered the infinitesimal conformal transformation, however in this section we will consider the finite conformal transformation.

1. Translation

$$x'^\mu = x^\mu + \underbrace{a^\mu}_{\text{finite vector}} = e^{ia^\nu P_\nu} x^\mu$$

2. Dilatation

$$x'^\mu = \left(1 + \frac{\lambda}{N}\right) x^\mu$$

   In order to achieve the finite dilatation, we use the infinitesimal transformation recursively by dividing the finite λ into infinitely many λ/N pieces and then transforming

$$
\begin{aligned}
x'^\mu &= \left(1+\frac{\lambda''}{N}\right)\underbrace{\left(1+\frac{\lambda'}{N}\right)\left(1+\frac{\lambda}{N}\right)x^\mu}_{x''^\mu} \\
&= \lim_{N\to\infty}\left(1+\frac{\lambda}{N}\right)^N x^\mu \\
&= e^\lambda x^\mu = e^{i\lambda D} x^\mu
\end{aligned}
$$

3. Rotation

$$
\begin{aligned}
x'^\mu &= \left[1 + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right]^\mu{}_\nu x^\nu \\
&= \left[e^{\frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}}\right]^\mu{}_\nu x^\nu = \Lambda^\mu{}_\nu x^\nu
\end{aligned}
$$

4. The special conformal transformation

$$x'^{\mu} = x^{\mu} + 2x^{\mu}(b\cdot x) - x^2 b^{\mu}$$

let $b^{\mu} = \overset{\text{infinitesimal parameter, i.e. } t \text{ is small.}}{t}\, e^{\mu}$

$$x'^{\mu}(t) \equiv x^{\mu}(t) = x^{\mu} + 2t(e\cdot x)x^{\mu} - x^2 t e^{\mu}$$

To find the finite form of the transformation we have to recursively apply the above equation multiple times (Lie Algebra sence). The usual way is to integrate the infinitesimal form. The other way, and since we know that the transformations satisfy the conformal Killing equation, is to find the integral curve of the corresponding conformal Killing vector field as they are equivalent (Differential Geometry sense). Consider the $t-$derivative of the above[3].

$$\frac{\mathrm{d}x^{\mu}(t)}{\mathrm{d}t} = 2(e\cdot x)x^{\mu} - x^2 e^{\mu} \tag{1.16}$$

defining $y^{\mu}(t) = \frac{x^{\mu}(t)}{x^2(t)}$

$$\begin{aligned}
\dot{y}^{\mu}(t) &\overset{\text{quotient rule}}{=} \frac{x^2\dot{x}^{\mu} - 2(\dot{x}\cdot x)x^{\mu}}{(x^2)^2} \\
&= \frac{x^2[2(e\cdot x)x^{\mu} - x^2e^{\mu}] - 2[2(e\cdot x)x^{\nu} - x^2e^{\nu}]x_{\nu}x^{\mu}}{x^4} \\
&= \frac{x^2[2(e\cdot x)x^{\mu} - x^2e^{\mu}] - 2[2(e\cdot x)x^2 - x^2(e\cdot x)]x^{\mu}}{x^4} \\
&= \frac{x^2[\cancel{2(e\cdot x)x^{\mu}} - x^2e^{\mu}] - \cancel{2(e\cdot x)x^2x^{\mu}}}{x^4} \\
\dot{y}^{\mu}(t) &= -e^{\mu}
\end{aligned}$$

Solving the above differential equation

$$\begin{aligned}
y^{\mu}(t) &= y^{\mu}(0) - te^{\mu} \\
\frac{x^{\mu}(t)}{x^2(t)} &= \frac{x^{\mu}(0)}{x^2(0)} - te^{\mu}
\end{aligned}$$

going back to the old notation $x'^{\mu} \equiv x^{\mu}(t)$

$$\begin{aligned}
\frac{x'^{\mu}}{x'^2} &= \frac{x^{\mu}}{x^2} - te^{\mu} \\
&= \frac{x^{\mu}}{x^2} - b^{\mu}
\end{aligned} \tag{1.17}$$

Squaring both sides

$$\begin{aligned}
\left(\frac{x'^{\mu}}{x'^2}\right)^2 \equiv \frac{x'^{\mu}}{x'^2}\frac{x'_{\mu}}{x'^2} &= \left(\frac{x^{\mu}}{x^2} - b^{\mu}\right)^2 \\
\frac{x'^2}{x'^4} &= \left(\frac{x^{\mu}}{x^2}\right)^2 + b^2 - \frac{2(x\cdot b)}{x^2} \\
\frac{1}{x'^2} &= \frac{1 + b^2x^2 - 2(x\cdot b)}{x^2} \\
x'^2 &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned} \tag{1.18}$$

referring to (1.17)

$$x'^{\mu} = x'^2\left[\frac{x^{\mu}}{x^2} - b^{\mu}\right]$$

[3] when we consider the differential equation, we are no longer thinking of it as transformation but rather flow along a trajectory parameterized by t. This part was taken from pg 16 of "Four point function in momentum spaces and topological terms in gravity"

and substituting (1.18)

$$
\begin{aligned}
x'^{\mu} &= \frac{x^2}{1-2(x\cdot b)+b^2x^2}\left[\frac{x^{\mu}}{x^2}-b^{\mu}\right] \\
&= \frac{x^{\mu}-b^{\mu}x^2}{1-2(x\cdot b)+b^2x^2}
\end{aligned}
$$

Above procedure also suggests that, finite SCT could be described as a sequence of inversion→translation→inversion. Where inversion is defined as:

$$
I(x^{\mu}) = \frac{x^{\mu}}{x^2}
$$

First we note that the inversion is a global conformal transformation and since it is undefined at origin, it does not have an infinitesimal part i.e. we can not expect inversion to be obtained by exponentiating an element from the conformal Lie algebra. It is not the connected element of conformal group and in embedding space formalism, inversion is related to parity. Another interesting point to note is that there is no parameter associated with the transformation here such as λ for dilatation or b^{μ} for SCT. Lastly, it is also closely related to the stereographic projection. To show this let us study the stereographic projection of sphere onto a plane. Consider $x \in \mathbb{R}^n$, and define stereographic projection from the **north pole** of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as:

$$
X^i = \frac{2x^i}{1+|x|^2}, \quad X^{n+1} = \frac{|x|^2-1}{1+|x|^2}
$$

where x^i are coordinates on the projected plane and X^i are the coordiantes on the sphere in embedding space. Now project this point on the sphere back to $\mathbb{R}^n$ via stereographic projection from the **south pole**:

$$
x'^i = \frac{X^i}{1+X^{n+1}}
$$

Substituting:

$$
x'^i = \frac{\frac{2x^i}{1+|x|^2}}{1+\frac{|x|^2-1}{1+|x|^2}} = \frac{2x^i}{(1+|x|^2)+(|x|^2-1)} = \frac{2x^i}{2|x|^2} = \frac{x^i}{|x|^2}
$$

Hence, the composition gives:

$$
x^i \mapsto \frac{x^i}{|x|^2}
$$

which is the **inversion** in the unit sphere. Even though this inversion does not have a killing vector associated with it, but it is reasonable to look for the killing vector associated with stereographic projection. In general, we note that these two transformation would have the following form:

$$
x'^{\mu} = \Omega(x)x^{\mu}
$$

If $\partial_{\mu}\partial_{\nu}(\frac{1}{\Omega}) \propto g_{\mu\nu}$. The killing vector associated with it will have the form:

$$
K^{A}{}_{\mu} = \frac{1}{\Omega^2}\frac{\partial x'^A}{\partial x^{\mu}}
$$

Coming back to special conformal transformation which was the topic at hand, we now look at how they scale the metric tensor.

$$
\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \left\{\frac{\delta^{\mu}_{\nu}-2b^{\mu}x_{\nu}}{\Lambda} - \frac{(x^{\mu}-b^{\mu}x^2)(-2b_{\nu}+2b^2x_{\nu})}{\Lambda^2}\right\}
$$

$$
\begin{aligned}
g_{\alpha\beta}(x) = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}g'_{\mu\nu}(x')\bigg|_{x'=x'(x)} &= \left\{\frac{\delta^{\mu}_{\alpha}-2b^{\mu}x_{\alpha}}{\Lambda} - \frac{(x^{\mu}-b^{\mu}x^2)(-2b_{\alpha}+2b^2x_{\alpha})}{\Lambda^2}\right\} \\
&\times \left\{\frac{\delta^{\nu}_{\beta}-2b^{\nu}x_{\beta}}{\Lambda} - \frac{(x^{\nu}-b^{\nu}x^2)(-2b_{\beta}+2b^2x_{\beta})}{\Lambda^2}\right\} g'_{\mu\nu}(x') \\
&= \left\{\frac{\delta^{\mu}_{\alpha}-2b^{\mu}x_{\alpha}}{\Lambda} - \frac{(x^{\mu}-b^{\mu}x^2)(-2b_{\alpha}+2b^2x'_{\alpha})}{\Lambda^2}\right\} \\
&\times \left\{\frac{g'_{\mu\beta}-2b_{\mu}x_{\beta}}{\Lambda} - \frac{(x_{\mu}-b_{\mu}x^2)(-2b_{\beta}+2b^2x_{\beta})}{\Lambda^2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{g'_{\alpha\beta} - 2b_\beta x_\alpha - 2b_\alpha x_\beta + 4b^2 x_\alpha x_\beta}{\Lambda^2} - \frac{(x_\beta - b_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&\quad + \frac{(2(b\cdot x)x_\beta - 2b^2 x_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&\quad - \frac{(x_\alpha - 2b_\alpha x^2)(-2b_\alpha + 2b^2 x_\beta)}{\Lambda^3} \\
&\quad + \frac{(2(b\cdot x)x_\alpha - 2b^2 x^2 x_\alpha)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^3} \\
&\quad + \frac{(x^\mu - b^\mu x^2)(x_\mu - b_\mu x^2)(-2b_\alpha + 2b^2 x_\alpha)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^4} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{(-2b_\beta x_\alpha - 2_\alpha x_\beta + 4b^2 x_\alpha x_\beta)(1 - 2b\cdot x + b^2 x^2)}{\Lambda^3} \\
&\quad + \frac{1}{\Lambda^3}\left\{2b_\alpha x_\mu - 2b_\alpha b_\beta x^2 - 2b^2 x_\alpha x_\beta + 2b^2 x^2 x_\alpha b_\beta \right. \\
&\qquad \left. -4(b\cdot x)b_\alpha x_\beta + 4b^2 x^2 b_\alpha x_\beta + 4b^2(b\cdot x)x_\alpha x_\beta - 4b^\mu x^2 x_\alpha x_\beta\right\} + (\alpha \leftrightarrow \beta) \\
&\quad + \frac{\left\{x^2 - 2(b\cdot x)x^2 + b^2 x^4\right\}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\}}{\Lambda^4} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3} \\
&\quad \times \Big(\cancel{-2b_\beta x_\alpha} - \cancel{2b_\alpha x_\beta} + \cancel{4b^2 x_\alpha x_\beta} + \cancel{4(b\cdot x)b_\alpha x_\beta} - \cancel{8b^2(b\cdot x)x_\alpha x_\beta} \\
&\qquad -\cancel{2b^2 x^2 b_\beta x_\alpha} - \cancel{2b^2 x^2 b_\alpha x_\beta} + 4b^\mu x^2 x_\alpha x_\beta + \cancel{2b_\alpha x_\beta} + \cancel{2b_\beta x_\alpha} \\
&\qquad -4b_\alpha b_\beta x^2 - \cancel{4b^2 x_\alpha x_\beta} + \cancel{2b^2 x^2 x_\alpha b_\beta} + \cancel{2b^2 x^2 x_\beta x_\alpha} - \cancel{4bxb_\alpha x_\beta} \\
&\qquad -\cancel{4bxb_\beta x_\alpha} + 4b^2 x^2 b_\alpha x_\beta + 4b^2 x^2 b_\beta x_\alpha + \cancel{8b^2 b^2 x_\alpha x_\beta} - 8b^4 x^2 x_\alpha x_\beta\Big) \\
&\quad + x^2 \frac{\Lambda}{\Lambda^4}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3}\left\{-4b_\alpha b_\beta x^2 + 4b^2 x^2 b_\alpha x_\beta + 4b^2 x^2 b_\beta x_\alpha - 4b^4 x^2 x_\alpha x_\beta\right\} \\
&\qquad + x^2 \frac{1}{\Lambda^3}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\}
\end{aligned}
$$

$$
g'_{\alpha\beta}(x') = \Lambda^2 g_{\alpha\beta}(x)
$$

Jacobian of the Transformation

The following part is taken from “Conformal Field Theory Primer in $D \geq 3$” by Andrew Evans, pg 36:

$$
\begin{aligned}
\text{Translation:} \quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1 \\
\text{Rotation:} \quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1 \\
\text{Dilataion:} \quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \lambda^{-d} \\
\text{Inversion:} \quad & \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \left(\frac{1}{\tilde{x}^2}\right)^d
\end{aligned}
$$

Since the rest are easier to show, we will only focus on showing the last part:

$$
\begin{aligned}
\frac{\partial x^\mu}{\partial \tilde{x}^\nu} &= \frac{1}{\tilde{x}^2}\left[\delta^\mu_\nu - 2\frac{\tilde{x}^\mu \tilde{x}_\nu}{\tilde{x}^2}\right] \\
\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) &= \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}}\frac{\partial x^{\mu_2}}{\partial \tilde{x}^\nu_2}\cdots\frac{\partial x^{\mu_d}}{\partial \tilde{x}^\nu_d} \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\left[\delta^{\mu_1}_{\nu_1} - 2\frac{\tilde{x}^{\mu_1}\tilde{x}_{\nu_1}}{\tilde{x}^2}\right]\left[\delta^{\mu_2}_{\nu_2} - 2\frac{\tilde{x}^{\mu_2}\tilde{x}_{\nu_2}}{\tilde{x}^2}\right]\cdots\left[\delta^{\mu_d}_{\nu_d} - 2\frac{\tilde{x}^{\mu_d}\tilde{x}_{\nu_d}}{\tilde{x}^2}\right] \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{i=1}^{d}\delta^{\mu_i}_{\nu_i} - 2\sum_{j=1}^{d}\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\tilde{x}^{\mu_j}\tilde{x}_{\nu_j}}{\tilde{x}^2}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} + 0
\end{aligned}
$$

Now we use the identity

$$\epsilon_{i_1\ldots i_k\, i_{k+1}\ldots i_n}\epsilon^{i_1\ldots i_k\, j_{k+1}\ldots j_n} = \delta^{i_1\ldots i_k\, j_{k+1}\ldots j_n}_{i_1\ldots i_k\, i_{k+1}\ldots i_n} = k!\, \delta^{j_{k+1}\ldots j_n}_{i_{k+1}\ldots i_n}$$

which in our case becomes

$$\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} = (d-1)!\delta^{\mu_j}_{\nu_j}$$

Hence

$$\begin{aligned}\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) &= \left(\frac{1}{\tilde{x}^2}\right)^d\left(\frac{d! - 2\sum_{j=1}^{d}(d-1)!}{d!}\right)\\ &= \left(\frac{1}{\tilde{x}^2}\right)^d\left(1 - \frac{2d(d-1)!}{d!}\right) = -\left(\frac{1}{\tilde{x}^2}\right)^d\end{aligned}$$

How distances transform

Under translation

$$x'^\mu = x^\mu + a^\mu$$

So,

$$\begin{aligned}x'^\mu_a - x'^\mu_b &= x^\mu_a + a^\mu - x^\mu_b - a^\mu\\ &= x^\mu_a - x^\mu_b\end{aligned}$$

Thus, the distances are invariant under translation:

$$|x'_a - x'_b| = |x^\mu_a - x^\mu_b|$$

Under dilatation

$$x'^\mu = (1+\lambda)x^\mu$$

So,

$$\begin{aligned}x'^\mu_a - x'^\mu_b &= (1+\lambda)x^\mu_a - (1+\lambda)x^\mu_b\\ &= (1+\lambda)(x^\mu_a - x^\mu_b)\end{aligned}$$

We find that the distances between two point scales under dilatation, therefore the natural quantity which is invariant under both translation and dilatation is

$$\begin{aligned}\frac{|x'^\mu_a - x'^\mu_b|}{|x'^\mu_c - x'^\mu_d|} &= \frac{\cancel{1+\lambda}}{\cancel{1+\lambda}}\frac{|x^\mu_a - x^\mu_b|}{|x^\mu_c - x^\mu_d|}\\ &= \frac{|x^\mu_a - x^\mu_b|}{|x^\mu_c - x^\mu_d|}\end{aligned}$$

Under special conformal transformation

$$\begin{aligned}x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2x^2}\\ &= \frac{x^\mu - b^\mu x^2}{\Lambda^2(x)}\end{aligned}$$

So,

$$x'^\mu_a = \frac{x^\mu_a - b^\mu x^2_a}{\Lambda^2(x_a)}$$

$$x'^\mu_b = \frac{x^\mu_b - b^\mu x^2_b}{\Lambda^2(x_b)}$$

and,

$$x'^{\mu}_a - x'^{\mu}_b = \frac{x^\mu_a - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x^\mu_b - b^\mu x_b^2}{\Lambda^2(x_b)}$$

squaring both sides

$$\begin{aligned}
(x'^{\mu}_a - x'^{\mu}_b)^2 &= \left(\frac{x^\mu_a - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x^\mu_b - b^\mu x_b^2}{\Lambda^2(x_b)}\right)^2 \\
&= \frac{x_a^2 + b^2(x_a^2)^2 - 2x_a^2(x_a\cdot b)}{\Lambda^4(x_a)} + \frac{x_b^2 + b^2(x_b^2)^2 - 2x_b^2(x_b\cdot b)}{\Lambda^4(x_b)} \\
&\quad - \frac{2}{\Lambda^2(x_a)\Lambda^2(x_b)}\left[x_a\cdot x_b - x_b^2(x_a\cdot b) - x_a^2(b\cdot x_b) + b^2 x_a^2 x_b^2\right] \\
&= x_a^2\left[\frac{1-2(x_a\cdot b)}{\Lambda^4(x_a)} + \frac{\overbrace{2(b\cdot x_b) - b^2x_b^2}^{1-\Lambda^2(x_b)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1-2(x_b\cdot b)}{\Lambda^4(x_b)} + \frac{\overbrace{2(b\cdot x_a) - b^2x_a^2}^{1-\Lambda^2(x_a)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2\left[\frac{1-2(x_a\cdot b) - \Lambda^2(x_a)}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1-2(x_b\cdot b) - \Lambda^2(x_b)}{\Lambda^4(x_b)}\right. \\
&\quad \left. + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2\left[\frac{-b^2x_a^2}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{-b^2x_b^2}{\Lambda^4(x_b)} + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= \frac{(x_a - x_b)^2}{\Lambda^2(x_a)\Lambda^2(x_b)}
\end{aligned}$$

Thus, we find that the ratio of distances are not invariant under SCT.

$$\frac{\left|x'^{\mu}_a - x'^{\mu}_b\right|}{|x_a - x_b|} = \frac{1}{\Lambda(x_a)\Lambda(x_b)}$$

where $\Lambda(x_a) = \sqrt{1-2(x_a\cdot b) + b^2x_a^2}$ We can however, construct another quantity which is invariant under SCT.

$$\begin{aligned}
\frac{|x'_a - x'_b|}{|x'_b - x'_d|}\frac{|x'_d - x'_c|}{|x'_c - x'_a|} &= \frac{\frac{|x_a - x_b|}{\Lambda(x_a)\Lambda(x_b)}}{\frac{|x_b - x_d|}{\Lambda(x_b)\Lambda(x_d)}}\frac{\frac{|x_d - x_c|}{\Lambda(x_d)\Lambda(x_c)}}{\frac{|x_c - x_a|}{\Lambda(x_c)\Lambda(x_a)}} \\
&= \frac{|x_a - x_b|}{|x_b - x_d|}\frac{|x_d - x_c|}{|x_c - x_a|}
\end{aligned}$$

Such expressions are called, anharmonic ratios or cross-ratios.

1.4 Lie Algebra of Generators

$$\begin{aligned}
[P_\mu, P_\nu] &= [-i\partial_\mu, -i\partial_\nu] \\
&= -[\partial_\mu, \partial_\nu] = 0
\end{aligned}$$

Some useful identities

$$\begin{aligned}
[x_\alpha, \partial_\beta]f &= x_\alpha\partial_\beta f - \underbrace{\partial_\beta(x_\alpha f)}_{(\partial_\beta x_\alpha)f + x_\alpha\partial_\beta f} \\
&= x_\alpha\partial_\beta f - x_\alpha\partial_\beta f - (\partial_\beta x_\alpha)f
\end{aligned}$$

$$= -(\partial_\beta x_\alpha) f$$

$$\begin{aligned}[x_\alpha, \partial_\beta] &= -\partial_\beta x_\alpha = -g_{\beta\alpha}\partial^\mu x_\alpha \\ &= g_{\beta\alpha} \end{aligned} \tag{1.19}$$

next is,

$$\begin{aligned}[x^2, \partial_\beta] &= [x^\alpha x_\alpha, \partial_\beta] \\ &= x^\alpha[x_\alpha, \partial_\beta] + [x^\alpha, \partial_\beta]x_\alpha \\ &= -x^\alpha g_{\beta\alpha} - \delta^\alpha_\beta x_\alpha \\ &= -x_\beta - x_\beta \\ &= -2x_\beta \end{aligned} \tag{1.20}$$

and the last one is,

$$\begin{aligned}[x_\mu x^\nu, \partial_\beta] &= x_\mu[x^\nu, \partial_\beta] + [x_\mu, \partial_\beta]x^\nu \\ &= -x_\nu \delta^\nu_\beta - x^\nu g_{\beta\alpha} \end{aligned} \tag{1.21}$$

We will now consider, the lie algebra of different operators one by one.

$$\begin{aligned}[P_\mu, D] &= [-i\partial_\mu, -ix^\alpha\partial_\alpha] \\ &= -[\partial_\mu, x^\alpha\partial_\alpha] \\ &= -x^\alpha[\partial_\mu, \partial_\alpha] - [\partial_\mu, x^\alpha]\partial_\alpha \\ &= -\delta^\alpha_\mu\partial_\alpha = -\partial_\mu = -i(-i\partial_\mu) \\ &= -iP_\mu \end{aligned}$$

$$\begin{aligned}[P_\mu, L_{\alpha\beta}] &= [-i\partial_\mu, -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\ &= -[\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha] \\ &= -[\partial_\mu, x_\alpha]\partial_\beta + [\partial_\mu, x_\beta]\partial_\alpha \\ &= g_{\alpha\mu}\partial_\beta - g_{\beta\mu}\partial_\alpha \\ &= i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha) \end{aligned}$$

$$\begin{aligned}[P_\mu, K_\nu] &= [-i\partial_\mu, -i(2x_\mu x^\alpha\partial_\alpha - x^2\partial_\nu)] \\ &= -[\partial_\mu, 2x_\nu x^\alpha\partial_\alpha - x^2\partial_\nu] \\ &= -2x_\nu x^\alpha[\partial_\mu, \partial_\alpha] - 2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + x^2[\partial_\mu, \partial_\nu] + [\partial_\mu, x^2]\partial_\nu \\ &= -2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + [\partial_\mu, x^2]\partial_\nu \\ &= -2(g_{\mu\nu}x^\alpha + \delta^\alpha_\mu x_\nu)\partial_\alpha + 2x_\mu\partial_\nu \\ &= -2g_{\mu\nu}x^\alpha\partial_\alpha - 2(x_\nu\partial_\mu - x_\mu\partial_\nu) \\ &= -2ig_{\mu\nu}D - 2iL_{\mu\nu} \\ &= -2i(g_{\mu\nu}D - L_{\mu\nu}) \end{aligned}$$

$$\begin{aligned}[D, K_\mu] &= -[x^\alpha\partial_\alpha, 2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu] \\ &= -2[x^\alpha\partial_\alpha, x_\mu x^\beta\partial_\beta] + [x^\alpha\partial_\alpha, x^2\partial_\mu] \\ &= -2\left\{x^\alpha[\partial_\alpha, x_\mu x^\beta]\partial_\beta + x_\mu x^\beta[x^\alpha, \partial_\beta]\partial_\alpha\right\} \\ &\quad + x^\alpha[\partial_\alpha, x^2]\partial_\mu + x^2[x^\alpha, \partial_\mu]\partial_\alpha \\ &= -2\left\{x^\alpha\left(g_{\alpha\mu}x^\beta + \delta^\beta_\alpha x_\mu\right)\partial_\beta + x_\mu x^\beta(-\delta^\alpha_\beta)\partial_\alpha\right\} \\ &\quad + 2x^2\partial_\mu - \cancel{x^\alpha x^2\partial_\alpha\partial_\mu} + \cancel{x^2 x^\alpha\partial_\alpha\partial_\mu} - x^2\partial_\mu \\ &= -\cancel{2x_\mu x^\beta\partial_\beta} - 2x^\beta x_\mu\partial_\beta + \cancel{2x_\mu x^\beta\partial_\beta} + x^2\partial_\mu \\ &= -(2x^\beta x_\mu\partial_\beta - x^2\partial_\mu) \\ &= -iK_\mu \end{aligned}$$

$$
\begin{aligned}
[K_\mu, L_{\alpha\beta}] &= [-i(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu), i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\
&= [2x_\mu x^\nu \partial_\nu - x^2\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha] \\
&= 2[x_\mu x^\nu\partial_\nu, x_\alpha\partial_\beta] - [x^2\partial_\mu, x_\alpha\partial_\beta] + \underbrace{2[x_\mu x^\nu\partial_\nu, x_\beta\partial_\alpha] - [x^2\partial_\mu, x_\beta\partial_\alpha]}_{\alpha\leftrightarrow\beta} \\
&= 2\left\{x_\mu x^\nu[\partial_\nu, x_\alpha]\partial_\beta + x_\alpha[x_\mu x^\nu, \partial_\beta]\partial_\nu\right\} - x^2[\partial_\mu, x_\alpha]\partial_\beta - x_\alpha[x^2,\partial_\beta]\partial_\mu - (\alpha\leftrightarrow\beta) \\
&= \cancel{2x_\mu x^\nu(g_{\nu\alpha})\partial_\beta} - 2x_\alpha\left(g_{\mu\beta}x^\nu + \cancel{\delta^\nu_\beta x_\mu}\right)\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + 2x_\alpha x_\beta\partial_\mu - (\alpha\leftrightarrow\beta) \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + \cancel{2x_\alpha x_\beta\partial_\mu} + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2 g_{\mu\beta}\partial_\alpha - \cancel{2x_\beta x_\alpha\partial_\mu} \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2 g_{\mu\beta}\partial_\alpha \\
&= -g_{\mu\beta}(2x_\alpha x^\nu\partial_\nu - x^2\partial_\alpha) + g_{\mu\alpha}(2x_\beta x^\nu\partial_\nu - x^2\partial_\beta) \\
&= ig_{\mu\alpha}K_\beta - ig_{\mu\beta}K_\alpha = i(g_{\mu\alpha}K_\beta - g_{\mu\beta}K_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
[K_\mu, K_\nu] &= -[2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu, 2x_\nu x^\beta\partial_\beta - x^2\partial_\nu] \\
&= -4[x_\mu x^\alpha\partial_\alpha, x_\nu x^\beta\partial_\beta] + 2[x_\mu x^\alpha\partial_\alpha, x^2\partial_\nu] + 2[x^2\partial_\mu, x_\nu x^\beta\partial_\beta] - [x^2\partial_\mu, x^2\partial_\nu] \\
&= -4x_\nu x^\beta\left[x_\mu x^\alpha, \partial_\beta\right]\partial_\alpha - 4x_\mu x^\alpha\left[\partial_\alpha, x_\nu x^\beta\right]\partial_\beta + 2x_\mu x^\alpha\left[\partial_\alpha, x^2\right]\partial_\nu + 2x^2\left[x_\mu x^\alpha, \partial_\nu\right]\partial_\alpha \\
&\quad + 2x^2\left[\partial_\mu, x_\nu x^\beta\right]\partial_\beta + 2x_\nu x^\beta\left[x^2, \partial_\beta\right]\partial_\mu - x^2[\partial_\mu, x^2]\partial_\nu - x^2[x^2,\partial_\nu]\partial_\mu \\
&= \cancel{4x_\nu x^\beta(g_{\mu\beta}x^\alpha + \delta^\alpha_\beta x_\mu)\partial_\alpha} - \cancel{4x_\mu x^\alpha(g_{\alpha\nu}x^\beta + \delta^\beta_\alpha x_\nu)\partial_\beta} + 4x_\mu x^2\partial_\nu - 2x^2(\cancel{g_{\mu\nu}x^\alpha} + \delta^\alpha_\nu x_\mu)\partial_\alpha \\
&\quad + 2x^2(\cancel{g_{\mu\nu}x^\beta} + \delta^\beta_\mu x_\nu)\partial_\beta - 4x_\nu x^2\partial_\mu - 2x^2 x_\mu\partial_\nu + 2x^2 x_\nu\partial_\mu \\
&= \cancel{4x_\mu x^2\partial_\nu - 2x_\mu x^2\partial_\nu} + \cancel{2x_\nu x^2\partial_\mu - 4x_\nu x^2\partial_\mu} - \cancel{2x^2 x_\mu\partial_\nu} + \cancel{2x^2 x_\nu\partial_\mu} \\
&= 0
\end{aligned}
$$

Next, we will see that Conformal Algebra in d dimensions is isomorphic to the Lie algebra of the Lorentz group in $d+2$ dimensions, any conformal covariant correlator in d dimensions should be obtainable from Lorentz covariant expressions in $d+2$ dimensions via some kind of dimensional reduction procedure. This is essentially the idea behind **Embedding Formalism**. We define the following set of new operators:

$$
\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu} \\
J_{0,\mu} &= \frac{1}{2}\left(P_\mu + K_\mu\right) \\
J_{-1,\mu} &= \frac{1}{2}\left(P_\mu - K_\mu\right) \\
J_{-1,0} &= D
\end{aligned}
$$

with the property that

$$J_{ab} = -J_{ba}$$

where

$$a, b \in \{-1, 0, 1, \cdots, \overset{\text{d is dimension of spacetime}}{d}\}$$

These new generators, obey $SO(d+1,1)$ lie algebra:

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{bc}J_{ad}\right) \tag{1.22}$$

In this section, we will explicitly assume the form of flat metric as being euclidean, and given as:

$$g_{\mu\nu} = \eta_{\mu\nu} = (\underbrace{1, 1, \cdots, 1}_{d})$$

Our metric in (1.22) would be given as:

$$\eta_{ab} = (\underset{\eta_{-1-1}=-1}{-1}, \underset{\eta_{00}=1}{1}, \underbrace{1, \cdots, 1}_{\mu,\nu}) \tag{1.23}$$

If our original metric was Minkowski, we would have had:

$$\eta_{ab} = (-1, 1, \underbrace{-1, \cdots, 1}_{d})$$

We will now check, if (1.22) holds true:

$$
\begin{aligned}
[J_{\mu\nu}, J_{0,\alpha}] &= \left[L_{\mu\nu}, \frac{1}{2}(P_\alpha + K_\alpha)\right] \\
&= \frac{1}{2}[L_{\mu\nu}, P_\alpha] + \frac{1}{2}[L_{\mu\nu}, K_\alpha] \\
&= -\frac{1}{2}[P_\alpha, L_{\mu\nu}] - \frac{1}{2}[K_\alpha, L_{\mu\nu}] \\
&= -\frac{1}{2}\left(\eta_{\alpha\mu}P_\mu - \eta_{\alpha\nu}P_\mu\right) - \frac{1}{2}(\eta_{\alpha\mu}K_\nu - \eta_{\alpha\nu}K_\nu) \\
&= -\eta_{\alpha\mu}\left[\frac{1}{2}(P_\nu + K_\nu)\right] + \eta_{\alpha,\nu}\left[\frac{1}{2}(P_\mu + K_\mu)\right] \\
&= -i\eta_{\alpha\mu}J_{0,\nu} + i\eta_{\alpha\nu}J_{0,\mu}
\end{aligned}
$$

$$
\begin{aligned}
[J_{0,\mu}, J_{-1,0}] &= \left[\frac{1}{2}(P_\mu + K_\mu), D\right] \\
&= \frac{1}{2}[P_\mu, D] + \frac{1}{2}[K_\mu, D] \\
&= -\frac{1}{2}iP_\mu - \frac{1}{2}(-iK_\mu) = \frac{-i}{2}(P_\mu - K_\mu) = -iJ_{-1,\mu}
\end{aligned}
$$

If we assume that the metric in (1.22) is indeed given by (1.23). Then, the algebra (1.22) holds true. This shows the isomorphism between the conformal group of $d-$dimensional Euclidean space and the $SO(d+1,1)$ group of $d+2$ dimensional Minkowski spacetime with $1/2(d+1)(d+2)$ parameters.

Conformal Generators on the Field

Finite form of conformal transformation $(x' = \Lambda x)$[4]

$$
\begin{aligned}
\Phi'_a(x') &= U(\Lambda)\Phi_a(x)U^{-1}(\Lambda) \\
\Phi'_a(\Lambda x) &= \sum_b \pi_{ab}(\Lambda)\Phi_b(x) \\
&= \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x') \\
&= \pi_{ab}(e^{i\omega_g c_g})\Phi_b(e^{-i\omega_g c_g}x')
\end{aligned}
\tag{1.24}
$$

We have dropped the $\sum$ sign and summation over repeated indices are implied. Infinitesimal form of (1.24):

$$
\begin{aligned}
\Phi'_a(x') &= (1 - i\omega_g \overset{\text{generator only acting on field}}{\overset{\downarrow}{T_g}})_{ab}\Phi_b(\Lambda^{-1}x') \\
&= (1 - i\omega_g T_g)_{ab}\underbrace{\Phi_b[(1 - i\omega_g \overset{\text{generator which only acts on } x'^\mu}{\overset{\downarrow}{c_g}})x'^\mu]}_{\Phi_b(x') + \{(1-i\omega_g c_g)x'^\mu - x'^\mu\}\partial_\mu\Phi_b(x')} \\
&= (1 - i\omega_g T_g)_{ab}[\Phi_b(x') - i\omega_g c_g x'^\mu\partial_\mu\Phi_b(x')] \\
\Phi'(x') &= \Phi(x') - i\omega_g\left[T_g + \underset{\overset{\uparrow}{\text{accounts for the change in argument of field}}}{\boxed{c_g x'^\mu\frac{\partial}{\partial x'^\mu}}}\right]\Phi(x') + \mathcal{O}(\omega_g^2)
\end{aligned}
$$

However, we will not use this approach but rather we will consider the transformations at origin and then translate it to every other point. This approach is based on studying the stabilizer subgroup of the Conformal Symmetry.[5] So, if we study the same at origin:

$$
\begin{aligned}
\Phi'(0) &= \Phi(x') - i\omega_g\left[T_g + c_g x'^\mu\frac{\partial}{\partial x'^\mu}\right]\Phi(x')\bigg|_{x'=0} \\
&= \Phi(0) - i\omega_g T_g\Phi(0)
\end{aligned}
$$

[4] tobias osborne's lecture notes pg 18

[5] pg 7 of "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications"

using translation operator

$$\begin{aligned}
e^{ix^\lambda P_\lambda}\Phi'(0)e^{-ix^\alpha P_\alpha} &= e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} - e^{ix^\lambda P_\lambda} i\omega_g T_g \Phi(0) e^{-ix^\alpha P_\alpha} \\
\Phi'(x) &= \Phi(x) - e^{ix^\lambda P_\lambda} i\omega_g T_g e^{-ix^\sigma P_\sigma} e^{ix^\beta P_\beta}\Phi(0)e^{-ix^\alpha P_\alpha} \\
&= \Phi(x) - i\omega_g \underbrace{e^{ix^\lambda P_\lambda} T_g e^{-ix^\sigma P_\sigma}}_{\text{we will find these “translated operators” later}} \Phi(x)
\end{aligned}$$

For translation

$$\begin{aligned}
\Phi'(x+a) &= e^{ia^\lambda P_\lambda}\Phi(x)e^{-ia^\alpha P_\alpha} \\
&= e^{ia^\lambda [P_\lambda,\ ]}\Phi(x)
\end{aligned}$$

using (1.26)

$$= e^{a\cdot\partial}\Phi(x)$$

For rotation, at $x'^\mu = 0. \implies x^\mu = 0$

$$\Phi'_a(0) = \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}0) = \pi_{ab}(\Lambda)\Phi_b(0)$$

Now, assuming the generator of rotation $T_g = L_{\mu\nu}$ acts like[6]

$$L_{\mu\nu}\Phi_a(0) = S_{\mu\nu}\Phi_a(0) \tag{1.25}$$

at origin. At any other point, it will behave as:

$$\begin{aligned}
L_{\mu\nu}\Phi_a(x) &= e^{ix^\beta P_\beta} L_{\mu\nu}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\
&= \underbrace{e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma}}_{?} \underbrace{e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}}_{\Phi_a(x)}
\end{aligned}$$

by taking the derivative of second term, we obtain the following commutator

$$\begin{aligned}
\Phi_a(x) &= e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\
\partial_\mu \Phi_a(x) &= (\partial_\mu e^{ix^\lambda P_\lambda})\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)(\partial_\mu e^{-ix^\alpha P_\alpha}) \\
&= iP^\mu e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}(-iP^\mu) \\
&= iP^\mu \Phi_a(x) - i\Phi_a(x)P^\mu \\
&= i[P^\mu, \Phi_a(x)]
\end{aligned} \tag{1.26}$$

We will now derive the form of $e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma}$[7]:

$$\begin{aligned}
e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma} &= L_{\mu\nu} + [L_{\mu\nu}, -ix^\alpha P_\alpha] + \frac{1}{2!}[[L_{\mu\nu}, -ix^\alpha P_\alpha], -ix^\alpha P_\alpha] + \dots \\
&= L_{\mu\nu} + ix^\alpha \underbrace{[P_\alpha, L_{\mu\nu}]}_{i(g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu)} + \dots \\
&= L_{\mu\nu} + i^2 x^\alpha (g_{\alpha\mu}P_\nu - g_{\alpha\nu}uP_\mu) \\
&= L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \\
&= L_{\mu\nu} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{we found in section 1.2}}
\end{aligned}$$

we know, at $x' = 0$ we have $L_{\mu\nu} = S_{\mu\nu}$, so for the sake of consistency we get

$$e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma} = \underbrace{S_{\mu\nu}}_{\text{Spin Operator}} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{transforms the argument of field}}$$

The exponential map of above can be found in any textbook on QFT which describes rotation or Lorentz transformation.[8] If we ignore $S_{\mu\nu}$, then we can see how the last part acts on field:

$$x'^\mu = \left(\delta^\mu_\nu + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right)x^\mu$$

[6] pg 10, paragraph 2 of “Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions”

[7] using BCH lemma $e^A B e^{-A} = e^{[A,\ ]}B$

[8] check eqn 1.141 and 1.150 of “QFT in curved spacetime” by Leonard Parker

$$
\begin{aligned}
&= x^\mu + \frac{i}{2}\omega_{\alpha\beta} i \left(x^\alpha \partial^\beta - x^\beta \partial^\alpha\right) x^\nu \\
\Phi'(x) &= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta}\Phi(x) \\
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta} i(x^\alpha \partial^\beta - x^\beta \partial^\alpha)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha g^{\beta\sigma}\partial_\sigma - x^\beta g^{\alpha\sigma}\partial_\sigma)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha \partial^\beta - x^\beta \partial^\alpha)x^\sigma\partial_\sigma\Phi(x) \\
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta} x\cdot\partial\Phi(x) \\
&\approx \Phi\left(x^\mu - \frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta} x^\nu\right) \\
\Phi'(x') &= \Phi(x)
\end{aligned}
$$

For dilatation, at $x'^\mu = 0$, $x'^\mu = (1+\lambda)x^\mu = 0 \implies x^\mu = 0$. We have $\omega_g = \lambda$ and $T_g = D$:

$$
D\Phi_a(0) = \tilde{\Delta}\Phi_a(0) \tag{1.27}
$$

corresponding commutator (by operating it on eigenstate of dilatation)

$$
\begin{aligned}
D\left|\Delta\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle + \Phi_\Delta(0)D\left|0\right\rangle \\
\tilde{\Delta}\left|\Delta\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle + 0 \\
\tilde{\Delta}\Phi_\Delta(0)\left|0\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle
\end{aligned}
$$

Applying the same procedure, we consider:

$$
\begin{aligned}
e^{ix^\beta P_\beta} D e^{-ix^\sigma P_\sigma} &= D + [D, -ix^\beta P_\beta] + \frac{1}{2!}[[D, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= D - ix^\alpha(iP_\alpha) \\
&= D + x^\alpha P_\alpha \\
&= D - ix^\alpha\partial_\alpha
\end{aligned} \tag{1.28}
$$

for the sake of consistency at $x' = 0$

$$
= \tilde{\Delta} - ix^\alpha\partial_\alpha
$$

Now, we consider

$$
D\Phi_a(x) = (\tilde{\Delta} - ix^\alpha\partial_\alpha)\Phi_a(x)
$$

redefining $\tilde{\Delta} \equiv -i\Delta$, we get

$$
D\Phi_a(x) = -i(\Delta + x^\alpha\partial_\alpha)\Phi_a(x)
$$

Similarly,[9]

$$
\begin{aligned}
[D, \Phi_a(x)] &= De^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma}D \\
&= e^{ix^\lambda P_\lambda}\underbrace{e^{ix^\alpha P_\alpha}De^{-ix^\beta P_\beta}}_{=D+x^\alpha P_\alpha}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)\underbrace{e^{-ix^\sigma P_\sigma}De^{-ix^\alpha P_\alpha}}_{=D+x^\alpha P_\alpha}e^{-ix^\lambda P_\lambda} \\
&= e^{ix^\beta P_\beta}[D + x^\alpha P_\alpha, \Phi_a(0)]e^{-ix^\sigma P_\sigma} \\
&= e^{ix^\beta P_\beta}\underbrace{[D, \Phi_a(0)]}_{\tilde{\Delta}\Phi_a(0)}e^{-ix^\sigma P_\sigma} + e^{ix^\beta P_\beta}\underbrace{[x^\alpha P_\alpha, \Phi_a(0)]}_{=x^\alpha[P_\alpha,\Phi_a(0)]}e^{-ix^\sigma P_\sigma} \\
&= \tilde{\Delta}\Phi_a(x) - ix\cdot\partial\Phi_a(x) \\
&= -i(\Delta + x\cdot\partial)\Phi_a(x)
\end{aligned}
$$

Finite Dilatation[10], we consider

$$
x' = e^\lambda x = e^{i\lambda D}x = \left(1 + i\frac{\lambda}{N}\overset{Dx^\mu = -ix\cdot\partial x^\mu}{D}\right)\dots\left(1 + i\frac{\lambda}{N}D\right)x
$$

[9]from pg 31 of 2309.10107, and x is not an operator here but a number

[10]look up *Lectures Notes For An Introduction to Conformal Field Theory A Course Given By Dr. Tobias Osborne*, pg 19

then at origin, the field transforms (active transformation) as:

$$\begin{aligned}
\Phi'_a(0) &= \left(1 + i\frac{\lambda}{N}D\right)\dots\left(1 + i\frac{\lambda}{N}D\right)\Phi_a(0) \\
&= \left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) && \text{(using } D\Phi(0) = \tilde{\Delta}\Phi) \\
&= e^{i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{-\lambda\Delta_a}\Phi_a(0)
\end{aligned}$$

In passive transformation

$$\begin{aligned}
\Phi'_a(0) &= \left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) \\
&= e^{-i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{\lambda\Delta}\Phi_a(0)
\end{aligned}$$

For arbitrary point (ignoring the change in argument of field and thus generator c_g):

$$\begin{aligned}
\Phi'_a(x') &= \pi_{ab}(e^{i\lambda D})\Phi_b(x) \\
&= [e^{i\lambda\tilde{\Delta}}]_{ab}\Phi_b(x) \\
\Phi'_a(e^{\lambda}x) &= [e^{-\lambda\Delta}]_{ab}\Phi_b(x) = e^{-\lambda\Delta}\Phi_a(x)
\end{aligned}$$

For SCT, $x'^{\mu} = 0 \implies x^{\mu} = 0$. Hence, we will consider the same equations, but in this context:

$$K_{\mu}\Phi_a(0) = \kappa_{\mu}\Phi_a(0)$$

Again, applying the same procedure,

$$\begin{aligned}
e^{ix^{\beta}P_{\beta}}K_{\mu}e^{-ix^{\sigma}P_{\sigma}} &= K_{\mu} + [K_{\mu}, -ix^{\beta}P_{\beta}] + \frac{1}{2!}[[K_{\mu}, -ix^{\beta}P_{\beta}], -ix^{\beta}P_{\beta}] + \dots \\
&= K_{\mu} - ix^{\beta}[K_{\mu}, P_{\beta}] + \frac{1}{2}[-ix^{\beta}[K_{\mu}, P_{\beta}], -ix^{\beta}P_{\beta}] + \dots \\
&= K_{\mu} + 2x^{\beta}(g_{\mu\beta}D - L_{\mu\beta}) + \frac{1}{2}[2x^{\beta}(g_{\mu\beta}D - L_{\mu\beta}), -ix^{\alpha}P_{\alpha}] \\
&= K_{\mu} + 2x_{\mu}D - 2x^{\beta}L_{\mu\beta} - ix_{\mu}x^{\alpha}[D, P_{\alpha}] + ix^{\beta}x^{\alpha}\underbrace{[L_{\mu\beta}, P_{\alpha}]}_{-i(g_{\alpha\mu}P_{\beta} - g_{\alpha\beta}P_{\mu})} \\
&= K_{\mu} + 2x_{\mu}D - 2x^{\beta}L_{\mu\beta} + x_{\mu}x^{\alpha}P_{\alpha} + x_{\mu}x^{\beta}P_{\beta} - x_{\alpha}x^{\alpha}P_{\mu} \\
&= K_{\mu} + 2x_{\mu}D - 2x^{\beta}L_{\mu\beta} + 2x_{\mu}x^{\alpha}P_{\alpha} - x_{\alpha}x^{\alpha}P_{\mu}
\end{aligned}$$

From the generator of dilatation and SCT, we have[11]

$$[D, K_{\mu}] = -iK_{\mu} \qquad \text{at } x'^{\mu} = 0 \implies \qquad [\tilde{\Delta}, \kappa_{\mu}] = -i\kappa_{\mu}$$

and

$$[D, L_{\mu\nu}] = 0 \qquad \text{at } x'^{\mu} = 0 \implies \qquad [\tilde{\Delta}, S_{\mu\nu}] = 0$$

For primary fields:

$$K_{\mu}\Phi_a(0) = 0$$

Since, for primary field $\tilde{\Delta}$ commutes with all other operators which belong to the stability subgroup. By Schur's lemma $\tilde{\Delta} \propto I$, where I is an identity operator. The SCT and momentum generator acts as ladder operator for Dilatation.

$$\begin{aligned}
[D, [P_{\mu}, \Phi(0)]] &= [P_{\mu}, [D, \Phi(0)]] + [[D, P_{\mu}], \Phi(0)] = -i(\Delta + 1)[P_{\mu}, \Phi(0)] \\
[D, [K_{\mu}, \Phi(0)]] &= [K_{\mu}, [D, \Phi(0)]] + [[D, K_{\mu}], \Phi(0)] = -i(\Delta - 1)[K_{\mu}, \Phi(0)]
\end{aligned}$$

[11] same notes, look at eqn 65 to 70 (pg 18-19), all these commutators are for T_g

Conformal Invariance of Scalar Field

Start from the free, massless scalar action in d dimensions:

$$S[\phi] = \frac{1}{2}\int d^d x\,(\partial_\mu \phi)(\partial^\mu \phi).$$

Under a scale (dilatation) transformation

$$x \mapsto x' = \lambda x, \qquad \lambda > 0,$$

a primary scalar field transforms as

$$\phi(x) \mapsto \phi'(x') = \lambda^{-\Delta}\phi(x),$$

where Δ is the scaling dimension to be determined. The measure scales as[12]

$$d^d x \mapsto d^d x' = \lambda^d d^d x,$$

and the derivative transforms as

$$\partial_\mu \phi(x) \mapsto \partial'_\mu \phi'(x') = \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu\big(\lambda^{-\Delta}\phi(x)\big) = \lambda^{-1}\lambda^{-\Delta}\partial_\mu\phi(x) = \lambda^{-(\Delta+1)}\partial_\mu\phi(x).$$

Hence the kinetic density transforms as

$$(\partial\phi)^2 \mapsto \lambda^{-2(\Delta+1)}(\partial\phi)^2,$$

and the full integrand scales by

$$d^d x\,(\partial\phi)^2 \mapsto \lambda^{d-2(\Delta+1)}\,d^d x\,(\partial\phi)^2.$$

Classical scale invariance of the action requires the exponent of λ to vanish:

$$d - 2(\Delta+1) = 0.$$

Solving for Δ gives

$$\Delta = \frac{d}{2} - 1. \tag{1.29}$$

Finite Conformal Transformation of Fields

We begin by noting that *translation* and *rotation* do not introduce any new thing that we hadn't encountered in QFT, it is only the dilatation which does. Upon exponentiating the infinitesimal dilatation:

$$\begin{aligned}\Phi'(x') &= e^{-i\omega_g\left[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}\right]}\Phi(x')\\ &= e^{-i\omega_g T_g}e^{-i\omega_g c_g x'\cdot\frac{\partial}{\partial x'}}\Phi(x')\\ &= e^{-i\omega_g T_g}\Phi(e^{-i\omega_g c_g}x')\end{aligned}$$

This section is taken from "advanced mathematical methods - conformal field theory" by David Duffins.[13]

$$\begin{aligned}\Phi'_a(x') = U(\Lambda)\Phi_a(x')U^{-1}(\Lambda) &= e^{-i\omega_g T_g}\Phi_a e^{i\omega_g T_g}\\ &= e^{-i\omega_g [T_g,\ \ ]}\Phi_a(x')\end{aligned}$$

For translation

$$\Phi(x) = e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} = e^{x\partial}\Phi(0)$$

Or,

$$\begin{aligned}\Phi'(x') &= \Phi(x)\\ &= \Phi(x'-a)\\ &= e^{-a\frac{\partial}{\partial x'}}\Phi(x')\end{aligned}$$

[12]the metric and metric determinant do not change due to Weyl scaling, resulting in measure being transformation dependent

[13]Active coordinate transformation is given as: $\Phi(x') = U(\Lambda)\Phi(x)U^{-1}(\Lambda)$ whereas passive transformation is given as $\Phi(x') = U^{-1}(\Lambda)\Phi(x)U(\Lambda)$

$$= e^{-iaP}\Phi(x')$$

For dilatation ($x' = e^{\lambda}x$)

$$\begin{aligned}
\Phi'_a(x') &= e^{-i\lambda D}\Phi_a(x') \\
&= e^{-\lambda(\Delta + x'\cdot\partial)}\Phi_a(x') \\
&= e^{-\lambda\Delta}\,\underbrace{e^{-\lambda x\cdot\partial}\Phi_a(x')}_{\Phi_a[e^{-\lambda}x']} \\
&= e^{-\lambda\Delta}\Phi_a(x)
\end{aligned}$$

The last part could be understood as:

$$\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right] = e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a(x)$$

$$\Phi_a\Bigg[\underbrace{\left(1-\frac{\lambda}{N}\right)\dots\left(1-\frac{\lambda}{N}\right)}_{N\text{ terms}}x\Bigg] = e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a\Bigg[\underbrace{\left(1-\frac{\lambda}{N}\right)\dots\left(1-\frac{\lambda}{N}\right)}_{N-1\text{ terms}}x\Bigg]$$

$$\Phi_a(e^{-\lambda}x) = e^{-\lambda x\cdot\partial}\Phi_a(x)$$

or, alternatively

$$\begin{aligned}
e^{-\lambda x\cdot\partial}\Phi_a(x) &= \left(1-\frac{\lambda}{N}x\cdot\partial\right)^N\Phi_a(x) \\
&= \left(1-\frac{\lambda}{N}x\cdot\partial\right)\dots\underbrace{\left(1-\frac{\lambda}{N}x\cdot\partial\right)\Phi_a(x)}_{\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right]} \\
&= \Phi_a\left[\left(1-\frac{\lambda}{N}\right)^N x\right] \\
&= \Phi_a(e^{-\lambda}x)
\end{aligned}$$

Chapter 2

Embedding coordinates for Euclidean Space

We start in the embedding space $\mathbb{R}^{d+1,1}$ with coordinates

$$X^{-1}, X^0, \underbrace{X^1, X^2, \ldots, X^d}_{X^\mu}$$

To simplify the discussion, it is convenient to introduce null coordinates, $X^M = (X^+, X^-, X^\mu)$, defined as[1]

$$\left.\begin{aligned} X^+ &= X^{-1} + X^0 \\ X^- &= X^{-1} - X^0 \end{aligned}\right\} X^{-1} = \frac{X^+ + X^-}{2}; \quad X^0 = \frac{X^+ - X^-}{2}$$

With this choice, the mostly-plus metric of the embedding space becomes

$$ds^2 = -(dX^{-1})^2 + (dX^0)^2 + \sum_{\mu=1}^{d} (dX^\mu)^2 = \sum_{\mu=1}^{d} (dX^\mu)^2 - dX^+ dX^-$$

This can be written in matrix form as

$$\eta_{MN} = \begin{pmatrix} 0 & -1/2 & 0 & 0 & \cdots \\ -1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 0 & 1 & \\ & & & & \ddots \end{pmatrix}$$

The power of the embedding formalism is that the conformal generators in d-dimensional Euclidean space can be realized as Lorentz generators in $d+2$ dimensions. To make this connection explicit, we introduce new coordinates (ρ, η, x_μ) via the transformation

$$\begin{aligned} X_{-1} &= \frac{\rho(1 - \eta^2 + \vec{x}^2)}{2} \\ X_0 &= \frac{\rho(1 + \eta^2 - \vec{x}^2)}{2} \\ X_\mu &= \rho x_\mu \end{aligned}$$

This parametrization makes manifest an important redundancy: scaling all embedding coordinates by an overall factor λ corresponds simply to $\rho \mapsto \lambda\rho$. Thus, points related by rescalings $X'^A = \lambda X^A$ correspond to the same physical point in the conformal compactification. To proceed, we invert the transformation to express (ρ, η, x_μ) in terms of X_A:

[1] the index with lowest numeric value has the same sign in both $X^\pm$. If we had considered, X^{d+1} rather than X^{-1} then the definition would have been something like

$$X^\pm = X^0 \pm X^{d+1}$$

$$\begin{aligned}
\rho &= X_{-1} + X_0 \\
\eta &= \frac{\sqrt{\eta_{MN}X^M X^N}}{X_{-1}+X_0} \leftarrow \rho\eta = \sqrt{\eta_{MN}X^M X^N} \\
x_\mu &= \frac{X_\mu}{X_{-1}+X_0}
\end{aligned}$$

Next, we compute the change of basis for derivatives. Using the chain rule, one finds

$$\begin{aligned}
\frac{\partial}{\partial X_{-1}} &= \frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\
\frac{\partial}{\partial X_0} &= \frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\
\frac{\partial}{\partial X_\mu} &= \frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}
\end{aligned}$$

The Lorentz generators in $d+2$ dimensions are

$$J_{MN} = X_M\partial_N - X_N\partial_M$$

By combining them appropriately, we recover the familiar conformal generators in d dimensions.

- **Translation**

$$\begin{aligned}
P_\mu = J_{-1,\mu} + J_{0,\mu} &= (X_{-1}+X_0)\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} - \frac{\partial}{\partial X_{-1}}\right) \\
&= \rho\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - \rho x_\mu\left(\frac{1}{\rho\eta}\frac{\partial}{\partial \eta}\right) \\
&= \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x^\mu}
\end{aligned}$$

- **Special Conformal Transformation**

$$\begin{aligned}
K_\mu = J_{0,\mu} - J_{-1,\mu} &= (X_0 - X_{-1})\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_{-1}}\right) \\
&= \rho(\eta^2-\vec{x}^2)\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - 2\rho x_\mu\left[\frac{\partial}{\partial \rho} - \left(\frac{\eta^2+\vec{x}^2}{2\rho\eta}\right) - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&= \frac{\eta^2-\vec{x}^2}{\eta}x_\mu\frac{\partial}{\partial \eta} + (\eta^2-\vec{x}^2)\frac{\partial}{\partial x_\mu} - 2\rho x_\mu\frac{\partial}{\partial \rho} + x_\mu\frac{\eta^2+\vec{x}^2}{\eta}\frac{\partial}{\partial \eta} + 2x_\mu(x\cdot\partial) \\
&= 2x_\mu(x\cdot\partial) - \vec{x}^2\partial_\mu + \eta^2\partial_\mu - 2\rho x_\mu\frac{\partial}{\partial \rho} + 2x_\mu\eta\frac{\partial}{\partial \eta}
\end{aligned}$$

- **Dilatation**

$$\begin{aligned}
D = J_{-10} = X_{-1}\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X^{-1}} &= X_{-1}\frac{\partial}{\partial X_0} + X_0\frac{\partial}{\partial X_{-1}} \\
&= \frac{\rho(1-\eta^2+\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&\quad + \frac{\rho(1+\eta^2-\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&= \rho\frac{\partial}{\partial \rho} - x^\mu\frac{\partial}{\partial x^\mu} - \eta\frac{\partial}{\partial \eta}
\end{aligned}$$

If we now restrict to the null cone $\eta = 0$ and fix ρ (using the scaling redundancy), the generators simplify to their standard flat-space form:

$$\begin{aligned}
P_\mu &= \partial_\mu \\
K_\mu &= 2x_\mu(x\cdot\partial) - x^\nu x_\nu\partial_\mu
\end{aligned}$$

$$D = -x^\mu \partial_\mu$$

Note that conformal algebra is satisfied by both $\pm P_\mu$ and $\pm K_\mu$. The null cone corresponds to $\eta = 0$ but no condition imposed on ρ. Thus, there's a gauge redundancy: different values of ρ acting as scale factor for the coordinates correspond to the same physical point.

Next important to understand now is how the tensor fields transform under conformal transformation. We can use the embedding space to deduce their transformation law which is often more illuminating than the algebra gymnastics.

2.1 Tensor field under conformal transformation

The transformation rule for tensor field under conformal transformation is very complicated, therefore we will rely on Embedding formalism to find a simpler form. In this approach, physical spacetime coordinates x^μ are understood as projections from a higher-dimensional embedding space with coordinates $X^A \in \mathbb{R}^{d,2}$, where the conformal group $SO(d,2)$ acts linearly. Tensors in physical space are then obtained by pulling back embedding space tensors via this projection. Specifically, a tensor in physical space is related to its embedding space counterpart as follows:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \frac{\partial x^{\mu_1}}{\partial X^{A_1}} \cdots \frac{\partial x^{\mu_n}}{\partial X^{A_n}} \frac{\partial X^{B_1}}{\partial x^{\nu_1}} \cdots \frac{\partial X^{B_m}}{\partial x^{\nu_m}}\, T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)$$

The map from embedding to physical space is given by

$$x^\mu = \frac{X^\mu}{X^+} \equiv \frac{X^\mu}{X^0 + X^{-1}}$$

but due to the projective nature of this construction—i.e., physical points correspond to rays in the embedding space—we are free to rescale $X \sim \lambda X$, which is a manifestation of dilatation symmetry. Under this rescaling, the tensor should satisfy

$$T(\lambda X) = \lambda^{-\Delta} T(X)$$

which defines its conformal weight Δ. Using this, the projected tensor can be written as

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \left(\frac{1}{X^+}\right)^{\Delta} \frac{\partial x^{\mu_1}}{\partial X^{A_1}} \cdots \frac{\partial x^{\mu_n}}{\partial X^{A_n}} \frac{\partial X^{B_1}}{\partial x^{\nu_1}} \cdots \frac{\partial X^{B_m}}{\partial x^{\nu_m}}\, T^{A_1\cdots A_n}_{B_1\cdots B_m}\left(\frac{X}{X^+}\right)$$

This includes both the Jacobian factors from the change of variables and the prefactor from conformal weight.

If we choose the embedding slice $X^+ = 1$, then the projection simplifies significantly. In this gauge, we define the following object:

$$e^\mu_A = X^+ \frac{\partial x^\mu}{\partial X^A}$$

and the projected tensor becomes

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = e^{\mu_1}_{A_1} \cdots e^{\mu_n}_{A_n} e^{B_1}_{\nu_1} \cdots e^{B_m}_{\nu_m} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)$$

Using the explicit form of the projection $x^\mu = X^\mu / X^+$, we compute

$$\frac{\partial x^\mu}{\partial X^A} = \frac{\delta^\mu_A (X^0 + X^{-1}) - X^\mu(\delta^0_A + \delta^{-1}_A)}{(X^0 + X^{-1})^2}$$

and setting $X^+ = 1$, we get

$$e^\mu_A = \delta^\mu_A, \qquad e^0_A = e^{-1}_A = -x^\mu$$

However, if we don't choose the slice $X^+ = 1$, then the projected tensor carries an additional scaling dependence:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \left(\frac{1}{X^+}\right)^{\Delta+n-m} \underbrace{e^{\mu_1}_{A_1} \cdots e^{\mu_n}_{A_n} e^{B_1}_{\nu_1} \cdots e^{B_m}_{\nu_m} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)}_{\text{depends only on physical point}}$$

This shows that tensors projected from different embedding space sections—i.e., different choices of X^+—differ by a power of X^+. So if two representations x and $\tilde{x}$ correspond to the same physical point but lie on different sections (i.e., with different values of X^+), then the corresponding tensors are related as

$$T(\tilde{x}) = \left(\frac{X^+}{\tilde{X}^+}\right)^{\Delta+n-m} T(x)$$

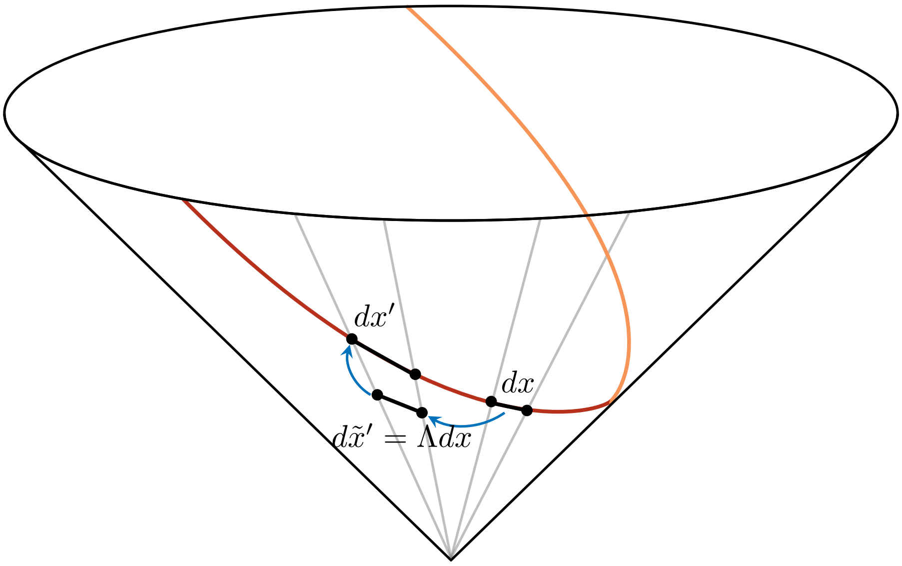


Figure 2.1: Upon Lorentz transformation, the points get mapped to different section however by utilizing the dilatation, we bring it back inside the original Euclidean section.

Next, consider how the tensor transforms under a conformal change of coordinates $x \mapsto x'$ which amounts to applying the corresponding Lorentz transformation $X \mapsto X' = \Lambda X$ in embedding space. Since the Lorentz transformation will map the tensor $T(X)$ to another tensor $T(X')$ living in different section, we will need to use above expression to map it back inside the original section labeled by X^+. The tensor at $\tilde{x}'$ is then:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(\tilde{x}') = \frac{\partial x'^{\mu_1}}{\partial X'^{A_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial X'^{A_n}} \frac{\partial X'^{B_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial X'^{B_m}}{\partial x'^{\nu_m}} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X')$$

Now applying the chain rule, we insert identities:

$$\frac{\partial x'^{\mu}}{\partial X'} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial X'}, \qquad \frac{\partial X'}{\partial x'} = \frac{\partial X'}{\partial x^{\beta}} \cdot \frac{\partial x^{\beta}}{\partial x'}$$

This yields:

$$\underbrace{T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_m}(\tilde{x}')}_{\text{tensor field projected on section } X'^{+}} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right)\underbrace{\left[\frac{\partial x^{\alpha_1}}{\partial X^{A_1}} \cdots \frac{\partial x^{\alpha_n}}{\partial X^{A_n}} \frac{\partial X^{B_1}}{\partial x^{\beta_1}} \cdots \frac{\partial X^{B_m}}{\partial x^{\beta_m}}\right] T^{A_1A_2\cdots A_n}_{B_1B_2\cdots B_m}(X)}_{\text{tensor field projected on section } X^{+}}$$

$$\left(\frac{X^+}{X'^+}\right)^{\Delta+n-m} T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_m}(x') = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)$$
$$= \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right)\left(\frac{X'^+}{X^+}\right)^{\Delta+n-m} T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)$$

The factor $(X'^+/X^+)^{\Delta+n-m}$ arises because x' and $\tilde{x}'$ are physically the same point, but obtained by projecting from different embedding sections. To express this ratio in terms of the coordinate Jacobian $|\partial x'/\partial x|$, note that from the projection $x^\mu = X^\mu/X^+$ and $x'^\mu = X'^\mu/X'^+$, one finds

$$\left|\frac{\partial x'}{\partial x}\right| = \left|\frac{\partial X}{\partial x} \cdot \frac{\partial x'}{\partial X}\right| = \left|\frac{\partial X}{\partial x} \cdot \frac{\partial x'}{\partial X'}\frac{\partial X'}{\partial X}\right| = \left|\left(\frac{X^+}{X'^+}\right)\frac{\partial X'}{\partial X}\right| = \left|\frac{X^+}{X'^+}\right|^d |\Lambda|$$

and therefore,

$$\left(\frac{X^+}{X'^+}\right)^{\Delta+n-m} = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta+n-m}{d}}$$

Finally, combining everything, the full transformation law for the projected tensor under a conformal coordinate transformation is:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta+n-m}{d}} \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)$$

This expression makes it manifest that the projected tensor transforms as a tensor under general coordinate transformations, but with an additional *conformal weight* $\Delta + n - m$ that reflects both the homogeneity of the embedding space tensor and the number of upper and lower indices involved in the projection.

2.2 Finding Correlators from Embedding Space

Needless to say, it is significantly easier to construct Lorentz covariant expressions than conformally covariant ones. Therefore, the natural question arises: once we have constructed Lorentz covariant expressions in $d+2$ dimensions, **how do we descend to d dimensions without breaking covariance?**

Since we have already fixed $\eta = 0$ in our derivation of the conformal generators, we now focus on the structure preserved by Lorentz transformations: the null light cone $X^2 = 0$ in embedding space. This cone, defined in $\mathbb{R}^{d+1,1}$ as the space of null rays through the origin, is given by:

$$
\begin{aligned}
X^2 &= -(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 \\
&= -X^+X^- + \sum_{\mu=1}^{d}(X^\mu)^2 = 0
\end{aligned}
$$

Although correlators are initially constructed as Lorentz-invariant functions over the full $d+2$-dimensional ambient space, we now restrict them to the null cone. This constraint effectively reduces the support of such correlators to a $d+1$-dimensional submanifold, since one of the coordinate dependencies—say, X^-—can be eliminated using the condition $X^2 = 0$ (in our case it leads to $\eta = 0$).

Next, we reinterpret embedding space as a fiber bundle over the physical d-dimensional spacetime (where the CFT is defined). Each fiber consists of null lines in the $(d+2)$-dimensional space, and each point in the base space corresponds to an equivalence class of null vectors $X^A \sim \lambda X^A$, for any non-zero λ. This reflects the earlier observation regarding the arbitrariness of ρ: all such rescalings represent the same physical point in d dimensions.

This identification has an important consequence: As mentioned earlier, this introduces gauge redundancy in our description. To eliminate another coordinate, say $X^+ = \rho$, we fix the gauge by selecting a section of the bundle with specific choice of the slice on the embedding space, typically the *Euclidean section*, defined by:

$$
X^+ = \rho = f(X^\mu) \equiv f(x^\mu)
$$

Although Lorentz transformations may take a null vector outside this section (as they mix time-like and space-like directions), theu can always be brought back by utilising the scaling equivalence $X^A \sim \lambda X^A$. This choice anchors us to physical d-dimensional spacetime, completing the descent from $d+2$ dimensions while maintaining conformal covariance inherited from Lorentz invariance in the higher-dimensional space. With these prescriptions in place we can now identify X^μ with the Euclidean space coordinates x^μ by stripping ρ dependence.

$$
X^\mu \equiv x^\mu
$$

This leads to definition of X^- based on null condition as:

$$
X^- = \frac{\sum_{\mu=1}^{d}(X^\mu)^2}{X^+} = \frac{X^\mu X_\mu}{X^+} = \frac{x^2}{f(X^\mu)}
$$

or, equivalently[2]

$$
\rho(-\eta^2 + \vec{x}^2) = \frac{\rho^2\vec{x}^2}{\rho} \implies \eta = 0
$$

The spacetime interval on this section is given as:

$$
ds^2 = dx^2 - dX^+dX^-\Big|_{X^+=f(X^i),X^-=\frac{x^2}{X^+}}
$$

This section satisfies two of the following conditions:

- section intersects each of the light rays at some point
- maps each point in d dimensional Euclidean space to a point on the null cone in Embedding space.

We have shown how to get generators of conformal transformation from Lorentz generators by embedding a null cone in ambient space. Let us now analyze how Lorentz Transformation acts on a generic section on that null cone. The Lorentz transformation acting as rotation on the point X^A in the null-cone will move it to another point on the null cone outside the the section $X^B = \Lambda^B{}_A X^A$.

[2] in our notation $x^\mu = \rho\vec{x}^\mu$

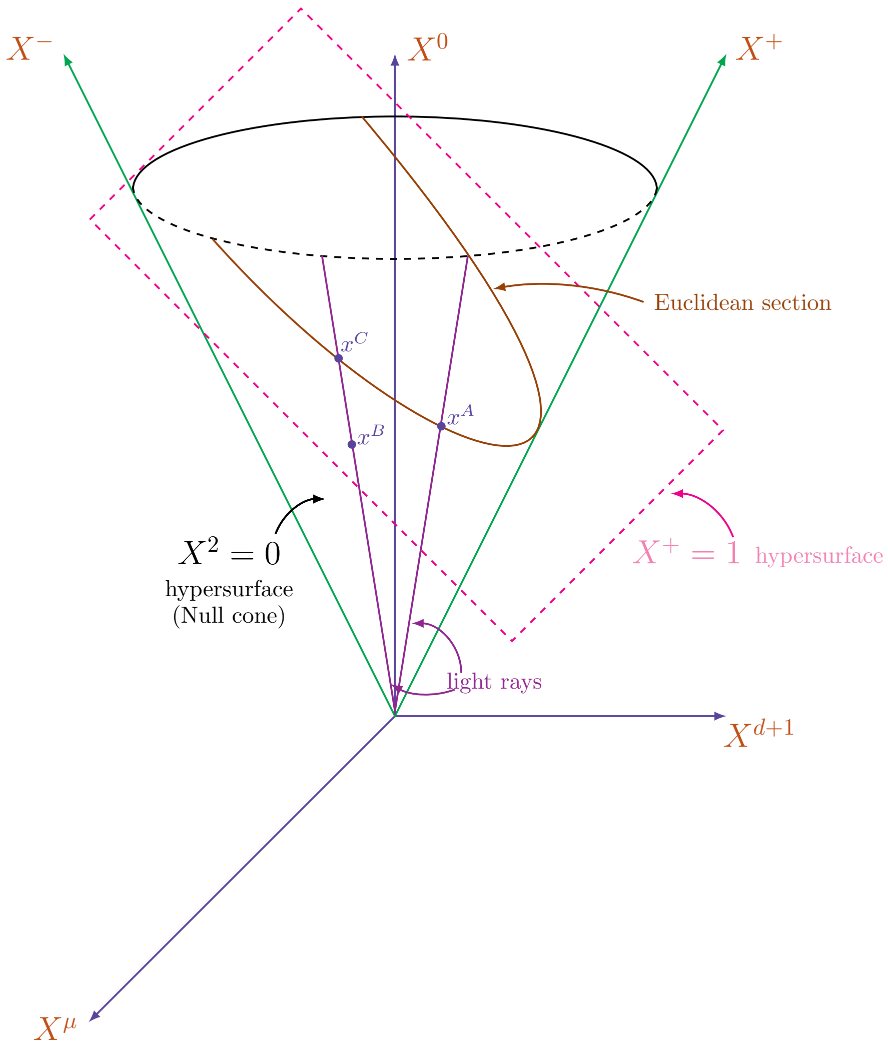


Figure 2.2: The hypersurface perpendicular to X^+ axis cutting at $X^+ = 1$ is shown as a plane and the null hypersurface is shown as the cone. The intersection of these two hypersurfaces describes the Euclidean Section. Dilatations are rotation in the X^0X^{d+1} plane and SCT or momentum generators are rotations in $X^\mu X^{d+1}$ with X^0X^{d+1} plane.

However, suppose via some conformal transformation (dilatation) in d dimensional Euclidean Space, we can move X^B to X^C back into the section. In all these steps, the only thing that changes the metric is the rescaling to get back into the section.

$$\begin{aligned}
ds_B^2 &= dX^M dX_M \\
&= d(\lambda(X)X^M)d(\lambda(X)X_M) \\
&= [\lambda dX^M + X^M(\boldsymbol{\nabla}\lambda \cdot dX)][\lambda dX_M + X_M(\boldsymbol{\nabla}\lambda \cdot dX)] \\
&= \lambda^2 dX^M dX_M + 2\lambda \underbrace{dX^M X_M}_{=0}(\boldsymbol{\nabla}\lambda \cdot dX) + \underbrace{X^M X_M}_{=0}(\boldsymbol{\nabla}\lambda \cdot dX)^2 \\
&= \lambda^2 dX^M dX_M = \lambda^2 ds_C^2
\end{aligned}$$

where we used, $X^2 = 0$ and $X^\mu dX_\mu = 0$ for restricting it to null cone.

Assuming the three conditions we used for simplification applies, the Lorentz Transformation in $d+2$-dimensional spacetime is equivalent to conformal transformation in $d-$dimensional spacetime iff metric in $d-$dimensional space is Euclidean thus, dX_+ in ds^2 has to vanish. It gives us the condition for defining the Euclidean section as $X^+ = \rho = constant$ and thus, for the sake of simplicity, we take it as 1. Thus, we have two conditions which we can use to eliminate two extra degree of freedom.

In the embedding space formalism, choosing an Euclidean section corresponds to picking a specific way to embed the $d-$dimensional space in the $(d+2)-$dimensional space. We define the following map between d dimensional Euclidean Space with conformal symmetry to null cone in $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$

$$(X^+, X^-, X^\mu) \equiv \left(1, x^2, x^\mu\right)$$

Here, we note that choosing a constant value for X^+ would give us a section on the cone on which the induced metric is Euclidean.

2.3 Tensors in Embedding Space

In this section, we will only concern ourselves with traceless and symmetric fields in $\mathbb{R}^d$ and leave the anti-symmetric tensors for future. Consider a symmetric and traceless tensor[3] $O_{M_1\ldots M_S}$ defined on the cone $X^2 = 0$ in $\mathbb{R}^{d+1,1}$. Under the rescaling $X \to \lambda X$, the tensor transforms as

$$O_{M_1\ldots M_S}(\lambda X) = \lambda^{-\Delta} O_{M_1\ldots M_S}(X)$$

i.e. it is a homogeneous function of degree Δ. We expect $O_{M_1\ldots M_S}$ to get mapped to traceless and symmetric primary field in $\mathbb{R}^d$. Since, each index go from 0 to $d+1$, in $\mathbb{R}^{d+1,1}$ we find that, for $d+2-$dimensional fields other than scalar have 2 more degree of freedom per index than $d-$dimensional fields. In order to remove the extra degree of freedom, we consider the transversality condition.

$$X^{M_1} O_{M_1\ldots M_S} = 0$$

We define the physical field to be:

$$\phi_{\mu\nu\lambda\ldots}(x) = \frac{\partial X^{M_1}}{\partial x^\mu}\frac{\partial X^{M_2}}{\partial x^\nu}\frac{\partial X^{M_3}}{\partial x^\lambda}\ldots O_{M_1\ldots M_S}(X)\bigg|_{X=X(x)}$$

Note that, this definition implies a redundancy. Indeed, anything proportional to X^M gives zero since

$$X^2 = 0 \implies X_M \frac{\partial X^M}{\partial x^\mu} = 0$$

Therefore, $O_{M_1\ldots M_S}(X) \to O_{M_1\ldots M_S}(X) + X_{M_1} F_{M_2\ldots M_S}(X)$ gets mapped to the same physical field. This $SO(d+1,1)$ tensor is sometimes referred to as pure gauge in the literature. It is this gauge redundancy that reduces another degree of freedom per index by making it unphysical.

2.4 Examples: Two point and Three point correlator

In the last section we showed how the embedding space formalism put in place could be used to deduce the conformally invariant correlator. In this section we will utilize the formalism and explicitly construct two point and three point function using the formalism developed thus far. From 1.3, we know that the ratios are only invariant under dilatation and translation. Therefore, we seek to construct an invariant out of these ratios and metric tensor which is also invariant under SCT and the exchange of indices $\mu \leftrightarrow \nu$. First we will derive the form of scalar one point correlator. A scalar primary is denoted by $\hat{\mathcal{O}}_\Delta$ (notice the absence of the Lorentz index, which indicates that it is a scalar operator). We want to enforce the invariance of correlator under conformal transformation. For a one-point function, this reduces to

$$\langle \hat{\mathcal{O}}_\Delta(x'^\mu)\rangle = \langle \hat{\tilde{\mathcal{O}}}_\Delta(x'^\mu)\rangle \tag{2.1}$$

This condition must be enforced for all four conformal transformations. We will begin by enforcing translation.

Translation: $\tilde{x}^\mu = x^\mu + a^\mu$

It was given previously that the Jacobian for a translation is one. Therefore, our operator simply does not change under this translation

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \hat{\mathcal{O}}_\Delta(x^\mu)$$

enforcing this in (2.1), we are left with

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu)\rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle$$

[3] symmetric tensors with spin s under $SO(d)$ form irreducible representations that correspond to integer spin particles (bosons). Anti-symmetric tensor fields have interpretation like they correspond to bivector of spinors etc.

where the operators are now the same on both sides of the equation. Notice that a correlation function is just a function. Therefore, this is equivalent to saying

$$f(\tilde{x}^\mu) = f(x^\mu)$$

Since this must be true for every possible translation, this tells us that the function has the same output regardless of what the input is, which means the function must just be some constant. Therefore, by enforcing translation we can conclude that

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu) \rangle = \text{constant} = C$$

We are not yet done. We need to make sure that all four transformations leave the one-point function invariant. Let's see what we can learn when we enforce dilatation.

Dilatation: $\tilde{x}^\mu = \lambda x^\mu$

Applying the Jacobian for dilitation, we see our Primary Scalar Operator transforms as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right|^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \lambda^{-\Delta} \hat{\mathcal{O}}_\Delta(x^\mu)$$

We want to enforce this in (2.1) and use our results from enforcing translation invariance. This gives us

$$\begin{aligned} \langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle &= \langle \hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) \rangle \\ &= \langle \lambda^{-\Delta} \hat{\mathcal{O}}_\Delta(x^\mu) \rangle \\ &= \lambda^{-\Delta} \langle \hat{\mathcal{O}}_\Delta(x^\mu) \rangle \end{aligned}$$

We found previously, by enforcing translation, that

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle = C$$

which means

$$C = \lambda^{-\Delta} C$$

This equation must be true for arbitrary scale factor λ. Therefore, unless $\Delta = 0$, we can conclude that $C = 0$.

For unitary CFTs, the only $\Delta = 0$ operator is the identity operator. So, with the exception of the identity, all one-point functions must vanish!

$$\boxed{\langle \hat{\mathcal{O}}_\Delta(x^\mu) \rangle = 0 \text{ for } \Delta \neq 0} \tag{2.2}$$

We said that we must impose all four conformal transformations, but the others are trivially satisfied at this point. So we are done with one-point correlators! Again, we'd like to highlight the fact that this is the result for ALL CFTs. You don't need to know anything else about the system, only that it has conformal symmetry.

2.4.1 Two-point Scalar Primary

For two-point functions, we need to enforce

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle = \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle \tag{2.3}$$

Again, this must be done for all four conformal transformations. As with the one-point function, we will begin by enforcing translation.

Translation

First, notice that $\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle$ is an object that takes two positions as inputs and gives back a number, so we can just write this as a function of $\tilde{x}_1^\mu$ and $\tilde{x}_2^\mu$

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle = f(\tilde{x}_1^\mu, \tilde{x}_2^\mu)$$

We found previously that under translations, scalar primary operators transform as

$$\hat{\tilde{\mathcal{O}}}(\tilde{x}^\mu) = \hat{\mathcal{O}}(x^\mu)$$

Putting this result in the right side of equation (2.3), we find that

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle$$

Notice that the only differences between the left and right side of this equation are the inputs. The function on each side is the same

$$f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) = f(x_1^\mu, x_2^\mu)$$

Under translation, we have $\tilde{x}^\mu = x^\mu + a^\mu \to x^\mu = \tilde{x}^\mu - a^\mu$. If we put this into the previous equation it becomes

$$f(x_1^\mu - a^\mu, x_2^\mu - a^\mu) = f(x_1^\mu, x_2^\mu)\ ,\ \forall a^\mu$$

This must be true regardless of the value of a^μ which means that a^μ must somehow cancel out. This is only satisfied if it is a function of $x_1^\mu - x_2^\mu$ so we have

$$f(x_1^\mu, x_2^\mu) = f(x_1^\mu - x_2^\mu)$$

That is, our function cannot be any function of the two positions. Rather, it can only depend on the displacement between the two positions. Therefore,

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(x_1^\mu - x_2^\mu)$$

Let's now see what we can learn by enforcing rotation.

Rotation $\tilde{x}^\mu = \Lambda^\mu_\nu x^\mu$

The Jacobian for rotation is the same as for translation, 1. Therefore, scalar primary operators transform the same under rotation as they do under translation

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle$$

Re-expressing this in a more familiar form, as functions, we have

$$f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) = f(x_1^\mu, x_2^\mu)$$

Next, we impose what we found by imposing translational invariance

$$f(\tilde{x}_1^\mu - \tilde{x}_2^\mu) = f(x_1^\mu - x_2^\mu)$$

Expressing the transformed coordinates in terms of our original coordinate system, we find

$$f(\Lambda^\mu_\nu(x_1^\nu - x_2^\nu)) = f(x_1^\mu - x_2^\mu)$$

This tells us that applying a rotation has no effect on the output. This means that the function must depend only on the magnitude of the separation $|x_1^\mu - x_2^\mu|$ (Recall, rotating a vector changes it, but rotating a scalar does nothing). So, from applying translational and rotational invariance, we can conclude that

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(|x_1^\mu - x_2^\mu|)$$

We will now continue by enforcing invariance under dilatation.

Dilatation

Recall, under dilatation, scalar primary operators transform as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta(x^\mu)$$

Substituting this into our two-point function condition, eqn. (2.3), we have

$$\begin{aligned}\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle &= \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle \\ &= \langle \lambda^{-\Delta_1}\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\lambda^{-\Delta_2}\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle \\ &= \lambda^{-\Delta_1}\lambda^{-\Delta_2}\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle \\ &= \lambda^{-(\Delta_1+\Delta_2)}f(|x_1^\mu - x_2^\mu|)\end{aligned}$$

where we are able to pull the λ's out of the correlator because they are just scalars. Notice also that we used what we already learned from translational and rotational invariance. This tells us that

$$f(|\tilde{x}_1^\mu - \tilde{x}_2^\mu|) = \lambda^{-(\Delta_1+\Delta_2)} f(|x_1^\mu - x_2^\mu|)$$

We can apply the transformation to the coordinates on the left-hand-side, which gives

$$f(\lambda|x_1^\mu - x_2^\mu|) = \lambda^{-(\Delta_1+\Delta_2)} f(|x_1^\mu - x_2^\mu|)$$

What does this mean? We can consider expanding our function in a power series

$$f(|x_1^\mu - x_2^\mu|) = \sum_n c_n |x_1^\mu - x_2^\mu|^n$$

Substituting this in above gives

$$\sum_n c_n \lambda^n |x_1^\mu - x_2^\mu|^n = \lambda^{-(\Delta_1+\Delta_2)} \sum_n c_n |x_1^\mu - x_2^\mu|^n$$

This is only satisfied for all λ, if all $n = 0$ except $n = -(\Delta_1 + \Delta_2)$. Therefore, after enforcing translation, rotation, and dilatation symmetry we have

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle = C|x_1^\mu - x_2^\mu|^{-(\Delta_1+\Delta_2)} \tag{2.4}$$

where C is some undetermined constant.

Special Conformal Transformation

Enforcing special conformal symmetry directly is a very messy business. Luckily for us, as discussed previously, a special conformal transformation is equivalent to performing an inversion, followed by a translation, followed by another inversion. Since we have already enforced translational invariance, this means it is sufficient to enforce inversion invariance, which is much easier. Recall, an inversion is given by

$$x^\mu = \frac{\tilde{x}^\mu}{\tilde{x}^2}$$

As with the other transformations, we need the Jacobian for inversion in order to see how the operators will transform. This is given by

$$\left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = \frac{1}{\tilde{x}^{2D}}$$

Therefore, under inversion, scalar primary operators transform as

$$\tilde{\hat{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left(\frac{1}{\tilde{x}^{2D}}\right)^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \frac{1}{(\tilde{x}^2)^\Delta} \hat{\mathcal{O}}_\Delta(x^\mu)$$

As usual, we will now go put this into equation (2.3) to enforce the symmetry

$$\begin{aligned}
\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle &= \langle \tilde{\hat{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu) \tilde{\hat{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle \\
&= \left\langle \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \right\rangle \\
&= \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle
\end{aligned}$$

Now, we can use our result from enforcing dilitation to replace $\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle$ on the left and $\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle$ on the right of this equation to get

$$\frac{C}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \frac{C}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}} \tag{2.5}$$

With a bit of algebra, this is equivalent to

$$\frac{(\tilde{x}_1^2)^{\Delta_1} (\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \frac{1}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}} \tag{2.6}$$

In order to put this in a more friendly form, we will use the following identity for inversions. Note: verifying this relationship requires substituting in the inversion transformation and some algebra. The reader is highly encouraged to check it.

$$\frac{\tilde{x}_1^2\tilde{x}_2^2}{(\tilde{x}_1^\mu - \tilde{x}_2^\mu)^2} = \frac{1}{(x_1^\mu - x_2^\mu)^2} \tag{2.7}$$

Using this identity in eqn. (2.6), we find

$$\frac{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \left[\frac{\tilde{x}_1^2\tilde{x}_2^2}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^2}\right]^{\frac{\Delta_1+\Delta_2}{2}}$$

This is only satisfied if

$$\Delta_1 = \Delta_2$$

Therefore, we find that the two-point function vanishes, unless the dimensions of the two operators are the same. In summary, the two-point function for scalar primaries in ANY CFT is given by

$$\boxed{\langle\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = \frac{C\delta_{\Delta_1\Delta_2}}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}}} \tag{2.8}$$

Note that it is standard convention to choose to normalize your operators so that $C = 1$, so you will often see this without the C constant included. We leave it here for complete generality.

From Quantum Field Theory

The Euclidean space correlator can be calculated by Fourier Transforming the momentum space propagator

$$G_E^{(0)}(x - x') = \int \frac{d^Dp}{(2\pi)^D}\, G_E^{(0)}(p)\, e^{\,ip_\mu(x_\mu - x'_\mu)}$$

where,

$$G_E^{(0)}(p) = \frac{1}{p^2}$$

Therefore, the correlation function in real (Euclidean) space is the integral

$$G_E^{(0)}(x - x') = \int \frac{d^Dp}{(2\pi)^D}\frac{e^{\,ip_\mu(x_\mu - x'_\mu)}}{p^2}$$

we can simplify the integrand by using Schwinger trick:

$$\frac{1}{p^2} = \frac{1}{2}\int_0^\infty d\alpha\, e^{-\frac{\alpha}{2}p^2}$$

Now, let's use this to re-express our correlator as double integral

$$G_E^{(0)}(x - x') = \frac{1}{2}\int_0^\infty d\alpha \int \frac{d^Dp}{(2\pi)^D}\, e^{-\frac{\alpha}{2}p^2 + ip_\mu(x_\mu - x'_\mu)}$$

The integrand is gaussian and integral can be calculated by considering shift in variable

$$\frac{\alpha}{2}p^2 - ip_\mu(x_\mu - x'_\mu) = \frac{1}{2}\Big(\sqrt{\alpha}p_\mu - i\tfrac{x_\mu - x'_\mu}{\sqrt{\alpha}}\Big)^2 + \frac{1}{2}\Big(\tfrac{x_\mu - x'_\mu}{\sqrt{\alpha}}\Big)^2$$

and by using the Gaussian integral,

$$\int \frac{d^Dp}{(2\pi)^D}\, e^{-\frac{1}{2}\left(\sqrt{\alpha}p_\mu - i\frac{x_\mu - x'_\mu}{\sqrt{\alpha}}\right)^2} = (2\pi\alpha)^{-D/2}$$

We find the correlation function to be,

$$G_E^{(0)}(x - x') = \frac{1}{2(2\pi)^{D/2}}\int_0^\infty d\alpha\, \alpha^{-D/2}\, e^{-\frac{|x-x'|^2}{2\alpha}} = \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{\frac{D}{2}}|x - x'|^{D-2}}$$

This is same as what we found previously by considering only symmetry constraint with $\Delta = \frac{D}{2} - 1$ as found in (1.29).

2.4.2 Three-point Scalar Primary

For the three-point function, we need to enforce

$$\langle \hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = \langle \hat{\tilde{\mathcal{O}}}_1(x_1^\mu)\hat{\tilde{\mathcal{O}}}_2(x_2^\mu)\hat{\tilde{\mathcal{O}}}_3(x_3^\mu)\rangle \tag{2.9}$$

Enforcing the symmetries for the three-point function follows in a very similar way to the two-point function, so we will not include as much detail. The reader is encouraged to work through any excluded details on their own.

Poincaré

For translations and rotations, the same line of argumentation that was used for two-point functions can be applied. However, instead of two points at our disposal, we have three. Therefore, our function can be a function of the magnitude of the separations between any parings of three points.

$$\langle \hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|)$$

where $|x_{12}^\mu| = |x_1^\mu - x_2^\mu|$, $|x_{23}^\mu| = |x_2^\mu - x_3^\mu|$, and $|x_{31}^\mu| = |x_3^\mu - x_1^\mu|$.

Dilatation

Enforcing dilitation invariance with our three-point function, we have

$$\begin{aligned}\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(\tilde{x}_3^\mu)\rangle &= \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_3}(\tilde{x}_3^\mu)\rangle \\ &= \langle \lambda^{-\Delta_1}\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\lambda^{-\Delta_2}\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\lambda^{-\Delta_3}\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu)\rangle \\ &= \lambda^{-\Delta_1}\lambda^{-\Delta_2}\lambda^{-\Delta_3}\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu)\rangle\end{aligned}$$

Using our results from enforcing Poincaré invariance, this becomes

$$f(|\tilde{x}_{12}^\mu|, |\tilde{x}_{23}^\mu|, |\tilde{x}_{31}^\mu|) = \lambda^{-(\Delta_1+\Delta_2+\Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) \tag{2.10}$$

Subsituting in the dilitation transformation on the LHS, this is

$$f(\lambda|x_{12}^\mu|, \lambda|x_{23}^\mu|, \lambda|x_{31}^\mu|) = \lambda^{-(\Delta_1+\Delta_2+\Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) \tag{2.11}$$

As with the two-point function, we can expand our function in a power series.

$$f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) = \sum_{nmp} c_{nmp}|x_{12}^\mu|^n|x_{23}^\mu|^m|x_{31}^\mu|^p \tag{2.12}$$

Substituting this in, we find that all terms must vanish, unless

$$n + m + p = -(\Delta_1 + \Delta_2 + \Delta_3)$$

Therefore, dilitation and Poincaré invariance tell us

$$\langle \hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = \sum_{nmp=-(\Delta_1+\Delta_2+\Delta_3)} c_{nmp}|x_{12}^\mu|_n|x_{23}^\mu|^m|x_{31}^\mu|^p \tag{2.13}$$

Special Conformal Transformation

Again, to find the effect of imposing special conformal symmetry, we need only to impose inversion symmetry, which is much easier. Although easier, the algebra is still quite nasty and will not be shown here. Ultimately, inversion (therefore special conformal) invariance leads to the additional constraint that all terms vanish, unless

$$\begin{aligned} n &= \Delta_1 + \Delta_2 - \Delta_3 \\ m &= \Delta_1 + \Delta_3 - \Delta_2 \\ p &= \Delta_2 + \Delta_3 - \Delta_1 \end{aligned} \tag{2.14}$$

Therefore, after enforcing all of the conformal symmetries on the 3-point function of scalar primaries, we find

$$\boxed{\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu) = \frac{C_{123}}{|x_{12}^\mu|^n |x_{23}^\mu|^m |x_{31}^\mu|^p}} \tag{2.15}$$

where n, m, and p are given by (2.14). We find that, as was the case with the two-point scalar primaries, the spatial dependence of 3-point scalar primaries are completely determined. We are left only with a set of constants C_{123}. It turns out that this set of constants is vitally important to defining any particular conformal field theory and they tell you how much your given operators interact. This set of constants goes by various names including the *3-point coefficients*, the *OPE coefficients*, and the *structure constants*.

2.4.3 Going beyond scalars

Moving on, we next consider the two point corrector of vector field. The ansatz for such a correlator is:[4]

$$\langle J_\mu(x_1) J_\nu(x_2)\rangle = C \underbrace{\frac{1}{|x_1 - x_2|^{2\Delta}}}_{\text{same as scalar case}} \left[g_{\mu\nu} + \delta \frac{(x_1 - x_2)_\mu (x_1 - x_2)_\nu}{(x_1 - x_2)^2}\right]$$

The correlation function is invariant under translation, therefore we will consider following redefinition:

$$\begin{aligned}\langle J_\mu(x_1) J_\nu(x_2)\rangle &= \langle J_\mu(x_1 - x_2) J_\nu(0)\rangle \\ &= \langle J_\mu(x_{12}) J_\nu(0)\rangle = \langle J_\mu(x) J_\nu(0)\rangle\end{aligned}$$

Since SCT is just inversion→translation→inversion, we can this property to our advantage. As the correlation function is already invariant under translations, it suffices to verify its invariance under inversions. If this property holds, then by extension, the correlation function will also be invariant under SCT. The inversion transformation is given as[5]:

$$x'_\mu = \frac{x_\mu}{x^2} \qquad\qquad |x'|^2 = \frac{1}{|x|^2}$$

and

$$\frac{\partial x'_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\frac{x_\nu}{x^2} = \frac{1}{x^2}\left[g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right] = x'^2 \underbrace{\left[g_{\mu\nu} - 2\frac{x'_\mu x'_\nu}{x'^2}\right]}_{I_{\mu\nu}}$$

The vector field would transform as

$$\langle J_\mu(x'_1) J_\nu(x'_2)\rangle = \underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{\Delta/d} \left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{\Delta/d}}_{\substack{\text{this was used to derive} \\ \text{the correlation function for scalar case}}} \overbrace{\frac{\partial x'_\alpha}{\partial x^\mu}\frac{\partial x'_\beta}{\partial x^\nu}}^{\text{without conformal factor}} \left\langle J^\alpha(x_1) J^\beta(x_2)\right\rangle$$

we see that

$$\underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{\Delta/d}_{x=x_1}}_{|x'_1|^{-2\Delta}} \left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{\Delta/d}_{x=x_2} \frac{1}{|x_{12}|^{2\Delta}} = |x'_1|^{-2\Delta}|x'_2|^{-2\Delta}\frac{1}{|x_{12}|^{2\Delta}} = \frac{1}{|x'_{12}|^{2\Delta}}$$

where we used

$$|x'_{12}|^2 = \left(\frac{x_1^\mu}{x_1^2} - \frac{x_2^\mu}{x_2^2}\right)^2$$

[4] pg 24 of "CFT with boundary and defects" by Herzog

[5] pg 17-18 of "Quantum Gravity and Cosmology based on Conformal Field Theory" and section 4.5 of "A conformal field theory primer in $D \geq 3$" by Andrew Evans

$$
\begin{aligned}
&= \frac{|x_1|^2}{x_1^4} + \frac{|x_2|^2}{x_2^4} - 2\frac{x_1^\mu}{x_1^2}\frac{x_{2\mu}}{x_2^2} \\
&= \frac{1}{x_1^2} - 2\frac{x_1}{x_1^2}\cdot\frac{x_2}{x_2^2} + \frac{1}{x_2^2} \\
&= \frac{(x_1-x_2)^2}{x_1^2x_2^2} = \frac{|x_{12}|^2}{|x_1|^2|x_2|^2} = \frac{|x_{12}|^2}{|x_1'|^{-2}|x_2'|^{-2}}
\end{aligned}
$$

Then, we only have to ensure that $g^{\mu\nu} + \delta\frac{x_{12}'^\mu x_{12}'^\nu}{x_{12}'^2}$ is invariant under inversion.[6]

$$
\begin{aligned}
g^{\mu\nu} + \delta\frac{(x_{12}')^\mu (x_{12}')^\nu}{(x_{12}')^2} &= \left(\delta^\mu_\alpha - 2\frac{x_1^\mu x_{1\alpha}}{x_1^2}\right)\left(\delta^\nu_\beta - 2\frac{x_2^\nu x_{2\beta}}{x_2^2}\right)\left[g^{\alpha\beta} + \delta\frac{x_{12}^\alpha x_{12}^\beta}{x_{12}^2}\right]\Bigg|_{x^\mu = \frac{x'^\mu}{|x'|^2}} \\
&= \left(\delta^\mu_\alpha - 2\frac{x_1'^\mu x_{1\alpha}'}{|x_1|'^2}\right)\left(\delta^\nu_\beta - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right)\left[g^{\alpha\beta} + \delta\frac{(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\alpha (x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2}\right] \\
&= \Bigg[g^{\mu\beta} + \delta\frac{(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\mu (x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\beta}{|x_1|'^2} \\
&\quad - 2\delta\frac{x_1'^\mu(\cancel{|x_1'|^2}|x_2'|^2 - x_1'\cdot x_2'\cancel{|x_1'|^2})(x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2\cancel{|x_1'|^2}|x_2'|^2}\Bigg]\left(\delta^\nu_\beta - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right) \\
&= \left[g^{\mu\beta} - \delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu (x_1'|x_2'|^2 - x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\beta}{|x_1|'^2}\right]\left(\delta^\nu_\beta - 2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right) \\
&= \left[g^{\mu\nu} - \delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu (x_1'|x_2'|^2 - x_2'|x_1'|^2)^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2}\right] \\
&\quad - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 2\delta\frac{(x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2'))^\mu (x_2'\cdot x_1'\cancel{|x_2'|^2} - \cancel{|x_2'|^2}|x_1'|^2)x_2'^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2\cancel{|x_2'|^2}} + 4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad - \delta\frac{\{x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2x_1'(x_1'\cdot x_2')\}^\mu\{x_1'|x_2'|^2 + x_2'|x_1'|^2 - 2(x_1'\cdot x_2')x_2'\}^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad - \delta\frac{\{x_2'|x_1'|^2 + |x_1'-x_2'|^2x_1' - |x_1'|^2x_1'\}^\mu\{x_1'|x_2'|^2 + |x_1'-x_2'|^2x_2' - |x_2'|^2x_2'\}^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - 2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad + \delta\frac{x_{12}'^\mu x_{12}'^\nu}{x_{12}'^2} - \delta\frac{x_1'^\mu}{|x_1'|^2}x_{12}'^\nu + \delta\frac{x_2'^\nu}{|x_2'|^2}x_{12}'^\mu - \delta|x_{12}'|^2\frac{x_1'^\mu x_2'^\nu}{|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - (\delta+2)\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2} - (\delta+2)\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2} + 2(\delta+2)\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2} + \cancel{\delta\frac{x_1'^\mu x_2'^\nu}{|x_1'|^2}} + \cancel{\delta\frac{x_1'^\mu x_2'^\nu}{|x_2'|^2}} \\
&\quad - \cancel{\delta|x_1'|^2\frac{x_1'^\mu x_2'^\nu}{|x_1'|^2|x_2'|^2}} - \cancel{\delta|x_2'|^2\frac{x_1'^\mu x_2'^\nu}{|x_1'|^2|x_2'|^2}} + \delta\frac{x_{12}'^\mu x_{12}'^\nu}{x_{12}'^2}
\end{aligned}
$$

which implies $\delta = -2$. Hence, the two point function is given as

[6] the conformal factor is there following eqn 55 of TASI Lectures on the Conformal Bootstrap. The tensor operator under inversion transforms as mentioned in eqn 3.18 of Conformal Field Theory with Boundaries and Defects or eqn 1.55 and 1.60 of EPFL Lectures on Conformal Field Theory in D¿= 3 Dimensions

$$
O'^\mu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\frac{\Delta+1}{d}}\frac{\partial x'^\mu}{\partial x^\nu}O^\nu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\Delta/d} I^\mu_\nu(x')O^\nu(x')
$$

$$
O'_\mu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\frac{\Delta-1}{d}}\frac{\partial x^\nu}{\partial x'^\mu}O_\nu(x')
$$

$$\langle J_\mu(x)J_\nu(0)\rangle = \frac{C}{|x|^{2\Delta}}\left[g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right]$$

The embedding space formalism gives the same answer[7]: Considering a tensor field of $SO(d+1,1)$ denoted as $O_{A_1\ldots A_n}(X)$, with the properties

- defined on the null-cone $X^2 = 0$,
- traceless and symmetric,
- homogeneous of degree $-\Delta$ in X, i.e., $O_{A_1\ldots A_n}(\lambda X) = \lambda^{-\Delta} O_{A_1\ldots A_n}(X)$,
- transverse $X^{A_i} O_{A_1\ldots A_n}(X) = 0$, with $i = 1, \ldots, n$

It is clear that those are conditions rendering $O_{A_1\ldots A_n}(X)$ manifestly invariant under $SO(d+1,1)$. In order to find the corresponding tensor in $\mathbb{R}^d$, one has to restrict $O_{A_1\ldots\ldots A_n}(X)$ to the Poincaré section and project the indices as

$$\langle O^\mu(x_1)O^\nu(x_2)\rangle = \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_1^B}{\partial x_2^\nu}\langle O_A(X_1)O_B(X_2)\rangle$$

For example, the most general form of the two-point function of two operators with spin-1 and dimension Δ can be derived as:[8]

$$\left\langle O^A(X_1)O^B(X_2)\right\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta^{AB} + \alpha\frac{X_2^A X_1^B}{X_1\cdot X_2} + \beta\frac{X_1^A X_2^B}{X_1\cdot X_2}\right]$$

We will drop the last term as it projects to zero anyways.

$$\left\langle O^A(X_1)O^B(X_2)\right\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta^{AB} + \alpha\frac{X_2^A X_1^B}{X_1\cdot X_2}\right]$$

According to the transverse condition

$$X_{A1}\left\langle O^A(X_1)O^B(X_2)\right\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}[X_1^B + \alpha X_1^B] = 0 \implies \alpha = -1$$

we now use the projection to find the correlation function in $\mathbb{R}^d$:

$$\begin{aligned}
\langle O_\mu(x_1)O_\nu(x_2)\rangle &= \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\langle O_A(X_1)O_B(X_2)\rangle \\
&= \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta_{AB} - \frac{X_{A2}X_{B1}}{X_1\cdot X_2}\right] \\
&= \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\eta_{AB} - \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\frac{X_{A2}X_{B1}}{X_1\cdot X_2}\right] \\
&= \frac{C_{12}}{(x_1-x_2)^{2\Delta}}\left[g_{\mu\nu} - 2\frac{(x_1-x_2)_\mu(x_1-x_2)_\nu}{(x_1-x_2)^2}\right]
\end{aligned}$$

where we used $X^A = (X^a, X^+, X^-) = (x^a, 1, x^2)$, $X_B = \left(x_a, -\frac{1}{2}x^2, -\frac{1}{2}\right)$ and $\eta_{ab} = I_{d\times d}$ with $\eta_{+-} = \eta_{-+} = -1/2$

$$\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\eta_{AB} = g_{\mu\nu}$$

$$\frac{\partial X_1^A}{\partial x_1^\mu}X_{A2} = \eta_{ab}\frac{\partial x_1^a}{\partial x_1^\mu}x_2^b - \frac{1}{2}\cancelto{0}{\frac{\partial 1}{\partial x_1^\mu}}x_2^2 - \frac{1}{2}\frac{\partial x_1^2}{\partial x_1^\mu}1$$

[7] section 2.4 of "Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions" or section 5.2.2 of "Conformal Field Theory" by Loriano Bonora

[8] here the terms in bracket is chosen such that they are invariant under the replacement $x \to \lambda x$. We are not using the transformation law for any of them. Under which, even the metric will change to $\eta_{AB} \to \lambda^{-2}\eta_{AB}$.

$$= \eta_{ab}\delta^a_\mu x_2^b - x_{1\mu} = (x_2 - x_1)_\mu = -(x_1 - x_2)_\mu$$

$$\frac{\partial X_2^B}{\partial x_2^\nu} X_{B1} = \eta_{ab}\frac{\partial x_2^a}{\partial x_2^\mu} x_1^b - \frac{1}{2}\cancelto{0}{\frac{\partial 1}{\partial x_2^\mu}} x_1^2 - \frac{1}{2}\frac{\partial x_2^2}{\partial x_2^\mu} 1 = (x_1 - x_2)_\nu$$

$$(X_1 - X_2)^A (X_1 - X_2)_A = (x_1 - x_2)^a (x_1 - x_2)_a - \frac{1}{2}(1-1)(x_1^2 - x_2^2) - (x_1^2 - x_2^2)(\tfrac{1}{2} - \tfrac{1}{2})$$

$$\implies X_1 \cdot X_2 = -\frac{1}{2}(x_1 - x_2)^2$$

Next, we boostrap three point correlator. [9] On the null cone we will have

$$\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle = \frac{W_M}{(-2X_1\cdot X_2)^{\alpha_{123}}(-2X_1\cdot X_3)^{\alpha_{132}}(-2X_2\cdot X_3)^{\alpha_{231}}}$$

where the powers α_{ijk} of the scalar factor are determined by the dilatation as in case of scalar operators and the tensor structure W_M equals to

$$W_M = \frac{(-2X_2\cdot X_3)X_{1M} - (-2X_1\cdot X_3)X_{2M} - (-2X_1\cdot X_2)X_{3M}}{(-2X_1\cdot X_2)^{\frac{1}{2}}(-2X_1\cdot X_3)^{\frac{1}{2}}(-2X_2\cdot X_3)^{\frac{1}{2}}}.$$

Let us comment a few things on the tensor structure. The relative sign is, as before, fixed by transversality.

$$(X_1)^M W_M = 0$$
$$(X_2)^M W_M = 0$$
$$(X_3)^M W_M = 0$$

We drop the term proportional to X_{3M}, since would project to zero anyway. The scaling behavior of correlation function under dilatation is completely determined in the scalar part so the tensor structure have scaling 0 in all variables ($X \to \lambda X \implies W_\mu \to \lambda^0 W_\mu$). Finally, it is immediate to check that the tensor structure is transverse, i.e. $(X_3)_M W_M = 0$. Projecting to physical space as:

$$\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle = \frac{\partial X_3^M}{\partial x_3^\mu}\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle$$

we find, as explicitly computed before,

$$\frac{\partial X_3^M}{\partial x_3^\mu} X_{iM} = (x_i - x_3)_\mu, \quad i = 1,2$$
$$-2X_i\cdot X_j = (x_i - x_j)^2, \quad i = 1,2,3\ (i<j),$$

so that we end up with the tensor structure

$$W_\mu = \frac{|x_2 - x_3|^2(x_1 - x_3)_\mu - |x_1 - x_3|^2(x_2 - x_3)_\mu}{|x_1 - x_2||x_1 - x_3||x_2 - x_3|} = \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}$$

Therefore, the three-point function of two scalars and one spin 1 operators in physical space is given by

$$\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle = \frac{\frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$
$$= \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2+1}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1+1}}$$

The three-point function of higher-spin operators $J_{\mu_1\dots\mu_\ell}$ is constructed from the above, analogously as what we did for the two-point functions, since it turns out that W_μ is the only indexed object for three points that is conformal invariant.

2.5 Fermions in Embedding Space

Following is taken from section 3.2 of Lectures on Conformal Field Theories by Hugh Osborn. To discuss spinor fields in the embedding formalism requires extending the usual $d-$dimensional gamma matrices to $d+2$ dimensions. For $d = 2n$, we define[10]

$$a_0^\pm = \frac{1}{2}(\pm\gamma^0 + \gamma^1)$$

[9] pg 30 of Masters Thesis on "Spinning Correlators at Finite Temperature" of Oscar Arandes Tejerina

[10] we have abused notation for the sake of avoiding cluttering of indices and $\pm$

$$\begin{aligned} a_1^{\pm} &= \frac{1}{2}(\gamma^2 \pm i\gamma^3) \\ a_2^{\pm} &= \frac{1}{2}(\gamma^4 \pm i\gamma^5) \\ &\vdots \\ a_{\frac{d-2}{2}}^{\pm} &= \frac{1}{2}(\gamma^{d-2} \pm i\gamma^{d-1}) \end{aligned}$$

where gamma matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

One can show that:

$$\begin{aligned} \{a_i^-, a_j^-\} &= \{a_i^+, a_j^+\} = 0 \\ \{a_i^-, a_j^+\} &= \delta_{ij} \qquad i,j = 0,1,2\ldots {}^{d-2}/_2. \end{aligned} \tag{2.16}$$

In the literature ${}^{d-2}/_2$ is defined as another variable labeled by k, but for the sake for clarity we will keep it explicit. This is the algebra of raising and lowering operators for ${}^{d}/_2$ independent two-level systems. We ask how many basis vectors are there (including lowest weight state) which could be formed by operating ${}^{d}/_2$ raising a_i^+ on lowest weight state:[11]

$$\sum_{r=0}^{d/2} {}^{d/2}C_r = 2^{d/2}$$

It implies that in $d-$dimensions, we have $2^{d/2} \times 2^{d/2}$ dimensional matrix representation for $\gamma-$matrices. We will use the highest weight representation to determine $a_j^{\pm}$ and then use them to construct γ_μ. From (2.16), we quickly observe that

$$(a_i^-)^2 = 0 = (a_i^+)^2$$

It implies that we can only act a_i or $a_i^\dagger$ once on a state, the second time it acts the state is annihilation. We will build off our intuition from harmonic oscillator (fermionic) and assume that there is a lowest weight state $|\xi\rangle$ such that

$$a_i^- |\xi\rangle = 0 \quad \text{for all } i$$

Similarly, acting on it once by each $a_i^\dagger$ for all i, we can construct states in the representation. The states can be labeled $s = (s_0, s_1, ..., s_{\frac{d-2}{2}})$, where each of the $s_a = \pm\frac{1}{2}$:

$$|\xi^{(s)}\rangle = (a_{\frac{d-2}{2}}^+)^{s_{\frac{d-2}{2}}+\frac{1}{2}} \ldots (a_0^+)^{s_0+\frac{1}{2}} |\xi\rangle \tag{2.17}$$

The lowest weight state $|\xi\rangle$ corresponds to all $s_a = -\frac{1}{2}$. Taking the $|\xi^s\rangle$ as a basis, we derive the matrix elements of γ_μ from the definitions and the anti-commutation relation. Starting with $d = 2$, we have a single two-level system:

$$|\xi^{(\frac{1}{2})}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad |\xi^{(-\frac{1}{2})}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we can construct the raising and lowering operator connecting these two matrices as:

$$a_0^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad\qquad a_0^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we find:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad\qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $d = 4$, we have 2 independent fermionic oscillator:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

[11] we would like to remind ourselves that number of linearly independent basis is defined as the dimension of space.

we construct, the following a_i^+ and a_i^- operators for $i = 0, 1$.

$$a_0^+ = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_0^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$a_1^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_1^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

From (2.17) we see that[12]

$$a_0^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \qquad a_1^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \qquad a_0^+ a_1^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, we conclude that the gamma matrices are gives as:

$$\gamma^0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

The above choice of gamma matrices satisfy the clifford algebra, however the chosen basis is not familiar from QFT textbooks. Given a representation γ^μ in d dimensions, we can construct a representation Γ^μ in $d+2$ dimensions using the prescription,

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\gamma^\mu \otimes \sigma^3, \quad \mu = 0, \ldots, d-3,$$

$$\Gamma^{d-2} = \mathbb{I} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^1, \quad \Gamma^{d-1} = \mathbb{I} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^2$$

where the σ^i obey

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

The 2×2 matrices that we add act on the index $s_{d-2/2}$, which newly appears in going from $d = 2k$ to $2k+2$ dimensions. In odd dimensions the first $d-1$ gamma matrices can be constructed as above, and $\Gamma_d = \pm \Gamma_1 \Gamma_2 \ldots \Gamma_{d-1}$ completes the gamma matrix algebra. There are two independent representations of the gamma matrix algebra in odd dimensions, differing in the sign of Γ_d. These representations are exchanged by parity, and both representations appear in a parity-conserving theory.

We now move onto calculating the correlation function involving spinors in embedding space formalism. To define spinor fields on null cone in embedding space requires that the number of component in $\mathbb{R}^d$ is half the number of components in $\mathbb{R}^{d+1,1}$.

$$\psi(x) \to \Psi(X), \qquad \bar{\psi}(x) \to \bar{\Psi}(X)$$

which satisfies the following homogeneity condition:

$$\Psi(\lambda X) = \lambda^{-\Delta+\frac{1}{2}} \Psi(X), \qquad \bar{\Psi}(\lambda X) = \lambda^{-\Delta+\frac{1}{2}} \bar{\Psi}(X)$$

The degrees of freedom of $\Psi; \bar{\Psi}$ are reduced to those for $\psi; \bar{\psi}$ by imposing the transversality condition like before:

$$\bar{\Gamma}_A X^A \Psi(X) = 0 \qquad \bar{\Psi}(X) \Gamma_A X^A = 0$$

[12] $a_0^\pm$ acts like raising and lowering operator in the same oscillator while $a_1^\pm$ changes the oscillator.

It introduces the gauge invariance and thus the degrees of freedom are now halved by imposing the equivalence relations

$$\Psi'(X) \sim \Psi' + \bar{\Gamma}_A X^A \zeta(X) \qquad \bar{\Psi}'(X) \sim \bar{\Psi}' + \bar{\zeta}(X)\Gamma_A X^A \tag{2.18}$$

for arbitrary spinor $\zeta(X); \bar{\zeta}(X)$ of appropriate homogeneity. From standard QFT, we are familiar that

$$V_A = \bar{\Psi}\Gamma_A\Psi'$$

transforms like a vector and we also have

$$\begin{aligned} \Psi(X) &= \Gamma_B X^B \Psi'(X), \\ \bar{\Psi}(X) &= \bar{\Psi}'(X)\bar{\Gamma}_B X^B. \end{aligned}$$

Now compute the contraction:

$$\begin{aligned} V_A = \bar{\Psi}(X)\Gamma_A\Psi'(X) &= \left(\bar{\Psi}'(X)\bar{\Gamma}_B \overset{\text{B-th component of coordinate (number)}}{X^B}\right)\Gamma_A\Psi'(X) \\ &= \bar{\Psi}'(X)\bar{\Gamma}_B X^B \Gamma_A \Psi'(X) \end{aligned}$$

Use the Clifford algebra identity:

$$\begin{aligned} \bar{\Gamma}_A\Gamma_B &= -\bar{\Gamma}_B\Gamma_A + 2\eta_{AB} \\ \Gamma_A\bar{\Gamma}_B &= -\Gamma_B\bar{\Gamma}_A + 2\eta_{AB}, \end{aligned}$$

from (2.18), and above we rewrite:

$$V_A = \bar{\Psi}'\bar{\Gamma}_B X^B \Gamma_A \Psi' = -\bar{\Psi}'\bar{\Gamma}_A\Gamma_B X^B \Psi' + 2X_A\bar{\Psi}'\Psi'$$

The second term, $2X_A\bar{\Psi}'\Psi'$, is proportional to X_A and is hence pure gauge under the equivalence relation:

$$V_A \sim V_A + X_A f(X)$$

so it can be discarded in physical quantities. Therefore, we obtain:

$$\begin{aligned} \bar{\Psi}\Gamma_A\Psi' + X_A f(X) &\sim -\bar{\Psi}'\bar{\Gamma}_A\Gamma_B X^B\Psi' + 2X_A\bar{\Psi}'\Psi' \\ \bar{\Psi}(X)\Gamma_A\Psi'(X) &\sim -\bar{\Psi}'(X)\bar{\Gamma}_A\Psi(X) \end{aligned}$$

whereas,

$$\bar{\Psi}\Psi$$

transforms like a scalar. However, the above is only under (2.18) in odd dimensions. So it does not correspond to a scalar on the projective null cone in even dimensions.

Chapter 3

2D CFT

Conformal invariance takes a new meaning in two dimensions. As already apparent in previous chapter, the case $d = 2$ requires special attention. Indeed, there exists in two dimensions an infinite variety of coordinate transformations that, although not everywhere well-defined, are locally conformal: they are holomorphic mappings from the complex plane (or part of it) onto itself. Among this infinite set of mappings one must distinguish the 6-parameter global conformal group, made of one-to-one mappings of the complex plane into itself. The analysis of the previous chapter still holds when considering these transformations only. However, a local field theory should be sensitive to local symmetries, even if the related transformations are not globally defined. It is local conformal invariance that enables exact solutions of two-dimensional conformal field theories.

3.1 Conformal Group in Two Dimensions

We begin by considering the flat two-dimensional Euclidean space with coordinates x^0 and x^1. A conformal transformation in this space takes the form

$$x^\mu \to x^\mu + \epsilon^\mu(x)$$

where the conformal Killing equation imposes the condition

$$\begin{aligned}\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &= \frac{2}{d} g_{\mu\nu} \partial^\rho \epsilon_\rho \\ \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &= g_{\mu\nu}(\partial_0 \epsilon_0 + \partial_1 \epsilon_1)\end{aligned}$$

Evaluating these relations component by component, we obtain:

- For $\mu = \nu = 0$

$$2\partial_0 \epsilon_0 = \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \implies \partial_0 \epsilon_0 = \partial_1 \epsilon_1$$

- For $\mu = \nu = 1$

$$2\partial_1 \epsilon_1 = \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \implies \partial_1 \epsilon_1 = \partial_0 \epsilon_0$$

- For $\mu = 0, \nu = 1$

$$\partial_0 \epsilon_1 + \partial_1 \epsilon_0 = 0 \implies \partial_0 \epsilon_1 = -\partial_1 \epsilon_0$$

Thus only two independent constraints remain:

$$\boxed{\begin{aligned}\partial_0 \epsilon_0 &= \partial_1 \epsilon_1 \\ \partial_0 \epsilon_1 &= -\partial_1 \epsilon_0\end{aligned}}$$

To make further progress, it is convenient to introduce complex coordinates $(z_1, z_2) \equiv (z, \bar{z})$ by embedding the $\mathbb{R}^2$ plane in $\mathbb{C}^2$. This embedding allows us to treat the two real directions as components of a single complex variable on the hypersurface $z_1^* = z_2$, which greatly simplifies the form of the constraints.

$$\left.\begin{aligned} z &= x^0 + ix^1 \\ \bar{z} &= x^0 - ix^1 \end{aligned}\right\} \begin{aligned} x^0 &= \frac{z+\bar{z}}{2} \\ x^1 &= \frac{z-\bar{z}}{2i} \end{aligned}$$

Now let us see how the transformation acts in these coordinates. Under $x^\mu \to x^\mu + \epsilon^\mu$, we find

$$z \to x^0 + \epsilon^0 + i(x^1 + \epsilon^1) = (x^0 + ix^1) + (\epsilon^0 + i\epsilon^1) = z + \epsilon(z, \bar{z})$$

and similarly,

$$\bar{z} \to \bar{z} + \bar{\epsilon}(z, \bar{z})$$

where we define $\bar{\epsilon} = \epsilon^0 - i\epsilon^1$. To rewrite the constraints, we must also translate derivatives into complex coordinates. Using the chain rule,

$$\begin{aligned} \partial_0 &= \frac{\partial}{\partial x^0} = \frac{\partial z}{\partial x^0}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x^0}\frac{\partial}{\partial \bar{z}} = \partial + \bar{\partial} \\ \partial_1 &= \frac{\partial}{\partial x^1} = \frac{\partial z}{\partial x^1}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x^1}\frac{\partial}{\partial \bar{z}} = i(\partial - \bar{\partial}) \end{aligned}$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$ are the Wirtinger derivatives:

$$\partial = \frac{\partial}{\partial z} = \frac{\partial x^0}{\partial z}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial z}\frac{\partial}{\partial x^1} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} - i\frac{\partial}{\partial x^1}\right) \tag{3.1}$$

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{\partial x^0}{\partial \bar{z}}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial}{\partial x^1} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x^1}\right) \tag{3.2}$$

We also note that

$$\begin{aligned} \epsilon_0 &= \frac{\epsilon + \bar{\epsilon}}{2} \\ \epsilon_1 &= \frac{\epsilon - \bar{\epsilon}}{2i} \end{aligned}$$

Substituting into the first constraint $\partial_0\epsilon_0 = \partial_1\epsilon_1$, we obtain.

$$\begin{aligned} \partial_0\epsilon_0 &= \partial_1\epsilon_1 \\ (\partial + \bar{\partial})\left(\frac{\epsilon + \bar{\epsilon}}{2}\right) &= i(\partial - \bar{\partial})\left(\frac{\epsilon - \bar{\epsilon}}{2i}\right) \\ \partial\epsilon + \partial\bar{\epsilon} + \bar{\partial}\epsilon + \bar{\partial}\bar{\epsilon} &= \partial\epsilon - \partial\bar{\epsilon} - \bar{\partial}\epsilon + \bar{\partial}\bar{\epsilon} \\ \bar{\partial}\epsilon &= -\partial\bar{\epsilon} \end{aligned} \tag{3.3}$$

From the second constraint $\partial_0\epsilon_1 = -\partial_1\epsilon_0$, we similarly find

$$\begin{aligned} \partial_0\epsilon_1 &= -\partial_1\epsilon_0 \\ (\partial + \bar{\partial})\left(\frac{\epsilon - \bar{\epsilon}}{2i}\right) &= -i(\partial - \bar{\partial})\left(\frac{\epsilon + \bar{\epsilon}}{2}\right) \\ \partial\epsilon - \partial\bar{\epsilon} + \bar{\partial}\epsilon - \bar{\partial}\bar{\epsilon} &= \partial\epsilon + \partial\bar{\epsilon} - \bar{\partial}\epsilon - \bar{\partial}\bar{\epsilon} \\ \bar{\partial}\epsilon &= \partial\bar{\epsilon} \end{aligned} \tag{3.4}$$

Combining both (3.3) and (3.4), we conclude

$$\begin{aligned} \partial\bar{\epsilon} = 0 &\implies \frac{\partial\bar{\epsilon}}{\partial z} = 0 \\ \bar{\partial}\epsilon = 0 &\implies \frac{\partial\epsilon}{\partial \bar{z}} = 0 \end{aligned}$$

This shows that $\epsilon(z)$ is holomorphic while $\bar{\epsilon}(\bar{z})$ is antiholomorphic. Therefore, the conformal transformation factorizes neatly as

$$\begin{aligned} z &\to z + \epsilon(z) \\ \bar{z} &\to \bar{z} + \bar{\epsilon}(\bar{z}) \end{aligned}$$

or equivalently

$$z \to f(z)$$

with

$$\frac{\partial f}{\partial \bar{z}} = 0$$

i.e. $f(z)$ is analytic or holomorphic. Since $\epsilon(z)$ is holormorphic, it admits a Laurent expansion.

$$\begin{aligned} z' &= z + \epsilon(z) \\ &= z + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \\ &= z + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \partial z \\ &= \left(1 + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \partial\right) z \end{aligned}$$

It is now easy to read off the generators of conformal transformation in 2d from above. We define,

$$r_n = z^n \partial$$

Similarly,

$$\begin{aligned} \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) \\ &= \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \\ &= \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \bar{\partial} \bar{z} \\ &= \left(1 + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \bar{\partial}\right) \bar{z} \end{aligned}$$

and hence

$$\bar{r}_n = \bar{z}^n \bar{\partial}$$

To determine the structure of the conformal algebra, we now compute the commutator between two such generators. A straightforward calculation shows that the r_n satisfy the following commutation relation:

$$\begin{aligned} [r_m, r_n] f(z, \bar{z}) &= [z^m \partial, z^n \partial] f(z, \bar{z}) \\ &= z^m \partial(z^n \partial f) - z^n \partial(z^m \partial f) \\ &= z^{m+n} \partial^2 f + z^m (\partial z^n) \partial f - z^{m+n} \partial^2 f - z^n (\partial z^m) \partial f \\ &= (n - m) z^{m+n-1} \partial f \\ &= (n - m) r_{m+n-1} f \end{aligned}$$

Thus,

$$\boxed{[r_m, r_n] = (n - m) r_{m+n-1}}$$

Following the same steps, we can also find

$$\boxed{[\bar{r}_m, \bar{r}_n] = (n - m) \bar{r}_{m+n-1}}$$

and

$$\boxed{[r_m, \bar{r}_n] = 0}$$

It is conventional to redefine the generators as $r_n = -l_{n-1}$. With this shift, the algebra becomes

$$\begin{aligned} [r_m, r_n] &= (n - m) r_{m+n-1} \\ [-l_{m-1}, -l_{n-1}] &= -(n - m) l_{m+n-2} \end{aligned}$$

and then shifting $m \to m + 1$ and $n \to n + 1$

$$[l_m, l_n] = (m - n) l_{m+n}$$

similarly,

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}$$
$$[l_m, \bar{l}_n] = 0$$

where

$$l_n = -r_{n+1} = -z^{n+1}\partial$$
$$\bar{l}_n = -\bar{r}_{n+1} = -\bar{z}^{n+1}\bar{\partial}$$

This is recognized as **Witt algebra**. They generate the conformal transformation in 2d Euclidean plane. To find the subset of generators l_n which generate global conformal transformation, we note that z^n has two singularity, one at $z = 0$ for $n < 0$ and another at $z = \infty$ for $n > 0$. Therefore, we impose the restriction that the representations of Witt algebra does not blow up in the limit $z \to 0$ or $z \to \infty$. We start with $z \to 0$ limit, $l_n = -z^{n+1}\partial$ needs to converge, implying:

$$n + 1 \geq 0 \implies n \geq -1$$

Since $z \to \infty$ does not appear as a pole, we consider the conformal mapping $w = \frac{1}{z}$ which brings the infinity at origin. Then,

$$\begin{aligned}\partial = \frac{\partial}{\partial z} &= \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \\ &= -\frac{1}{z^2}\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}\end{aligned}$$

Then,

$$\begin{aligned}l_n &= -z^{n+1}\partial = w^{-n-1}w^2\partial_w \\ &= w^{-n+1}\partial_w\end{aligned}$$

Then,

$$1 - n \geq 0 \implies n \leq 1$$

Therefore, l_{-1}, l_0 and l_1 generate global conformal transformation on the 2d Euclidean space.

3.2 Global Conformal Transformation

Last section was devoted to finding the generators of global conformal transformation. In this part, we look at the infinitesimal transformation generated by l_{-1} , l_0 and l_1 and then find their finite counterpart. For $n = -1$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_{-1}$ looks like:

$$l_{-1} = z^{-1+1}\partial = \partial$$
$$\bar{l}_{-1} = \bar{z}^{-1+1}\bar{\partial} = \bar{\partial}$$

So, this generates translation. For $n = 0$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_0 z$ looks like:

$$l_0 = -z^{0+1}\partial = -z\partial$$
$$\bar{l}_0 = -\bar{z}\bar{\partial}$$

To interpret how the works in complex plane, we will use the polar coordinates $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$ where $r = \sqrt{z\bar{z}}$ and $\theta = \frac{1}{2i}\ln\left(\frac{z}{\bar{z}}\right)$

$$\begin{aligned}\partial = \frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} \\ &= \frac{\bar{z}}{2\sqrt{z\bar{z}}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar{z}}{z}\frac{\partial(z/\bar{z})}{\partial z}\frac{\partial}{\partial \theta} \\ &= \frac{\bar{z}}{2r}\frac{\partial}{\partial r} + \frac{1}{2iz}\frac{\partial}{\partial \theta}\end{aligned}$$

$$= \frac{e^{-i\theta}}{2}\frac{\partial}{\partial r} + \frac{e^{-i\theta}}{2ir}\frac{\partial}{\partial\theta}$$

and,

$$\begin{aligned}
\bar\partial = \frac{\partial}{\partial\bar z} &= \frac{\partial r}{\partial\bar z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial\bar z}\frac{\partial}{\partial\theta} \\
&= \frac{z}{2\sqrt{z\bar z}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar z}{z}\frac{\partial(z/\bar z)}{\partial\bar z}\frac{\partial}{\partial\theta} \\
&= \frac{z}{2\sqrt{z\bar z}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar z}{z}\left(-\frac{z}{\bar z^2}\right)\frac{\partial}{\partial\theta} \\
&= \frac{z}{2r}\frac{\partial}{\partial r} - \frac{1}{2i\bar z}\frac{\partial}{\partial\theta} \\
&= \frac{e^{i\theta}}{2}\frac{\partial}{\partial r} - \frac{e^{i\theta}}{2ir}\frac{\partial}{\partial\theta}
\end{aligned}$$

we can now recast or generators in polar coordinates as following:

$$\begin{aligned}
l_0 = -z\partial &= -re^{i\theta}\left[\frac{e^{-i\theta}}{2}\frac{\partial}{\partial r} + \frac{e^{-i\theta}}{2ir}\frac{\partial}{\partial\theta}\right] \\
&= -\frac{r}{2}\left[\partial_r - \frac{i}{r}\partial_\theta\right]
\end{aligned}$$

$$\begin{aligned}
\bar l_0 &= -\bar z\bar\partial \\
&= -re^{-i\theta}\left[\frac{e^{i\theta}}{2}\frac{\partial}{\partial r} - \frac{e^{i\theta}}{2ir}\frac{\partial}{\partial\theta}\right] \\
&= -\frac{r}{2}\left[\partial_r + \frac{i}{r}\partial_\theta\right]
\end{aligned}$$

It is now clear that the following combination

$$l_0 + \bar l_0 = -r\partial_r$$

is the generator of dilatation and

$$i(l_0 - \bar l_0) = \partial_\theta$$

is the generator of rotation. For $n = -1$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_1 z^2$ looks like:

$$l_1 = -z^2\partial$$

These are nothing but Mobius transformation.

$$z' = \frac{az+b}{cz+d}$$

For translation we have $a = 1, c = 0$ and $d = 1$ and we have $ad - bc = 1$. For dilatation and rotation, we have

$$\begin{aligned}
z' &= z - \epsilon_0 z \\
&= (1-\epsilon_0)z \\
&= \frac{\frac{1-\epsilon_0}{\sqrt{1-\epsilon_0}}z + 0}{0z + \frac{1}{\sqrt{1+\epsilon_0}}} = \frac{\sqrt{1+\epsilon_0}z + 0}{0z + \frac{1}{\sqrt{1+\epsilon_0}}}
\end{aligned}$$

with $a = \sqrt{1+\epsilon_0}, b = 0, c = 0$ and $d = \frac{1}{\sqrt{1+\epsilon_0}}$ so that $ad - bc = 1$. For SCT, we have

$$\begin{aligned}
z' &= z + \epsilon_1 z^2 \\
&= z(1-\epsilon_1 z)^{-1}
\end{aligned}$$

$$= \frac{z}{1-\epsilon_1 z} = \frac{z+0}{-\epsilon_1 z+1}$$

Since ϵ_n is in general a complex number. These are the set of 2×2 complex matrices with unit determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{with } ad-bc=1$$

These matrices form the group $SL(2,\mathbb{C})$. But under the mapping $a\to -a$, $b\to -b$, $c\to -c$ and $d\to -d$, the generate the same transformation. Therefore, the conformal transformation in 2d forms the group $SL(2,\mathbb{C})/\mathbb{Z}_2$. In the Lorentzian case the group is replaced by $SL(2,R)\times SL(2,R)=SO(2,2)$ where one factor of $SL(2,R)$ pertains to left-movers and the other to right-movers.

3.3 Virasoro Algebra

In some cases, the regularization scheme used to quantize a classical theory does not preserve the original symmetry. The classical generators, which we obtain by solving the Killing equations, satisfy a certain algebra—for example, the Witt algebra. However, in the quantum system, the requirement of normal ordering to render observables finite modifies the algebraic structure. In the next chapter, we will learn how to construct the corresponding quantum operators systematically using the operator formalism. As a result, these quantum operators no longer satisfy the Witt algebra but instead form the Virasoro algebra. This is unavoidable; it's a reflection of the fact that the classical symmetry is realized projectively in the quantum theory.

The Virasoro algebra is the **central extension** of Witt algebra. To centrally extend the algebra, a new generator—called the center—is introduced, which commutes with all other generators. To distinguish from the generators of Witt algebra, we use the notation L_n to denote the generators of Virasoro algebra. The algebra is then given as:

$$[L_n,L_m]=(n-m)L_{n+m}+\underbrace{c\rho(n,m)}_{\text{complex number}}$$

This algebra contains a finite-dimensional subalgebra generated by $L_{0,\pm1}$,$\bar{L}_{0,\pm1}$. These are the generators that are well defined all over the complex plane and form the global conformal group in two dimensions just as we saw in Witt algebra. The rest of the generators are local. The object $\rho(n,m)$ which we have added to centrally extend the algebra has certain properties: it is a number which depends on n and m, it commutes with all L_n, it satisfies recursion relation and it vanishes for certain n and m. To see this, let us first note that

$$\begin{aligned}
[L_n,L_m] &= (n-m)L_{n+m}+c\rho(n,m)\\
[L_m,L_n] &= -(n-m)L_{n+m}+c\rho(m,n)
\end{aligned}$$

adding above expressions

$$\begin{aligned}
[L_n,L_m]+[L_m,L_n] &= (n-m)L_{n+m}+c\rho(n,m)-(n-m)L_{n+m}+c\rho(m,n)\\
\rho(n,m)+\rho(m,n) &= 0
\end{aligned}$$

We have shown that $\rho(n,m)$ is anti-symmetric under the exchange of $n\leftrightarrow m$. Now let us set $m=0$,

$$\begin{aligned}
[L_n^{\text{old}},L_0] &= nL_n^{\text{old}}+c\rho(n,0)\\
&= n\left[L_n^{\text{old}}+\frac{c}{n}\rho(n,0)\right]
\end{aligned}$$

Redefining the generators for $n\neq 0$ as:

$$L_n^{\text{new}} = L_n^{\text{old}}+\frac{c}{n}\rho(n,0)$$

and

$$L_0^{\text{new}} = L_0^{\text{old}}$$

The algebra of the translated generators should not changed:

$$\begin{aligned}
[L_n^{\text{old}},L_0] &= [L_n^{\text{new}}-\frac{c}{n}\rho(n,0),L_0]\\
[L_n^{\text{new}},L_0]-\left[\frac{c}{n}\rho(n,0),L_0\right] &= nL_n^{\text{old}}+c\rho(n,0)\\
[L_n^{\text{new}},L_0] &= nL_n^{\text{new}}
\end{aligned}$$

This proves that $\rho(n,0)=0$ for $n\neq 0$. For $n=1, m=-1$ we have:

$$\begin{aligned}[L_1^{\text{old}}, L_{-1}^{\text{old}}] &= 2L_0^{\text{old}} + c\rho(1,-1)\\ &= 2\left[L_0^{\text{old}} + \frac{c}{2}\rho(1,-1)\right]\\ &= 2L_0^{\text{new}}\end{aligned}$$

This, proves that $\rho(1,-1)=0$, to find other n and m values for which $\rho(n,m)$ vanishes, we will utilize the jacobi identity.

$$[L_n,[L_m,L_r]] + [L_r,[L_n,L_m]] + [L_m,[L_r,L_n]] = 0$$

For $r=0$

$$\begin{aligned}[L_n, mL_m] + [L_0,(n-m)L_{n+m}] + [L_m,-nL_n] &= 0\\ m\left[(n-m)L_{n+m} + c\rho(n,m)\right] + (n-m)\left[-(n+m)L_{n+m}\right] - n\left[(m-n)L_{m+n} + c\rho(m,n)\right] &= 0\\ \underbrace{\left[m(n-m)-(n-m)(n+m)-n(m-n)\right]}_{=0} L_{m+n} - c(m+n)\rho(m,n) &= 0\end{aligned}$$

Hence,

$$(m+n)\rho(m,n)=0$$

It means, $\rho(m,n)$ can only be non zero. if $m+n=0$. So, $\rho(m,n)=\rho(n,-n)\delta_{m+n,0}$. Next to find the recursion relation, we utilize,

$$\begin{aligned}[L_n,[L_1,L_{-1-n}]] + [L_{-1-n},[L_n,L_1]] + [L_1,[L_{-1-n},L_n]] &= 0\\ (n+2)[L_n,L_{-n}] + (n-1)[L_{-1-n},L_{n+1}] - (2n+1)[L_1,L_{-1}] &= 0\\ (n+2)\left[2nL_0 + c\rho(n,-n)\right] + (n-1)[-2(n+1)L_0 + c\rho(-1-n,1+n)] - (2n+1)2L_0 &= 0\\ \underbrace{\left[2n(n+2)-2(n-1)(n+1)-2(2n+1)\right]}_{=0} L_0 + c[(n+2)\rho(n,-n)-(n-1)\rho(n+1,-n-1)] &= 0\end{aligned}$$

Hence the recursion relation,

$$\rho(n+1,-n-1) = \frac{n+2}{n-1}\rho(n,-n)$$

under $n\to n+1$,

$$\rho(n,-n) = \frac{n+1}{n-2}\rho(n-1,-n+1)$$

We can see that for $n=1$

$$\rho(1,-1) = \frac{2}{-1}\rho(0,0) = 0$$

For $n=2$

$$\rho(2,-2) = \frac{3}{0}\rho(1,-1) = \text{indeterminate}$$

So, we only need to fix $\rho(2,-2)$ and all the other $\rho(n,m)$ gets fixed from that via this recursion relation.

$$\begin{aligned}\rho(n,-n) &= \frac{n+1}{n-2}\rho(n-1,-n+1)\\ &= \frac{n+1}{n-2}\cdots\frac{4}{1}\rho(2,-2)\\ &= \frac{1}{3!}\frac{(n+1)!}{(n-2)!}\rho(2,-2) = {}^nC_3 \quad \rho(2,-2)\end{aligned}$$

For free boson we normally like to set $c=1$ which forces the normalize of $\rho(2,-2)={}^1\!/_2$ since $c\times\rho(n,m) = \xi c\times\frac{\rho(n,m)}{\xi}$ needs to stay constant.

$$[L_n,L_m] = (n-m)L_{n+m} + {}^nC_3\rho(2,-2)$$

$$
\begin{aligned}
&= (n-m)L_{n+m} + c\frac{(n+1)!}{6(n-2)!}\frac{1}{2} \\
&= (n-m)L_{n+m} + c\frac{(n+1)!}{12(n-2)!}
\end{aligned}
$$

A short remark about the central charge that sometimes c and $\bar{c}$ aren't complex conjugate of each other which allows us to interpret z and $\bar{z}$ as being independent variables even though they are related to each other via complex conjugation operation.

3.4 How Fields transform under $SL(2,\mathbb{C})$

There are four types of fields we will be concerned about in 2d CFT. They are chiral fields that only depend on z ($\bar{h}=0$) and anti-chiral fields which only depend on $\bar{z}$ ($h=0$). It is also common to use the terminology holomorphic and anti-holomorphic in order to distinguish between chiral and anti-chiral quantities. There is a special kind of field which has the same transformation law for both Global conformal transformation as well as local conformal transformaion, we call them Primary fields. Under conformal transformation $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, these Primary fields transform as (passive):

$$
\phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(z,\bar{z})\Big|_{z=f(z),\bar{z}=\bar{f}(\bar{z})} = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z}))
$$

where $(h,\bar{h})$ are conformal dimensions/weight of the field $\phi(z,\bar{z})$. Sometimes this transformation law (active) is also expressed as:

$$
\phi'(z,\bar{z}) = \left(\frac{d}{dz}f^{-1}(z)\right)^h \left(\frac{d}{d\bar{z}}\overline{f^{-1}}(\bar{z})\right)^{\bar{h}} \phi\big(f^{-1}(z),\overline{f^{-1}}(\bar{z})\big)
$$

Define $\quad \zeta = f^{-1}(z), \qquad \bar{\zeta} = \overline{f^{-1}}(\bar{z}).$

$$
\phi'(z,\bar{z}) = \left(\frac{d\zeta}{dz}\right)^h \left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{\bar{h}} \phi(\zeta,\bar{\zeta}) \qquad \text{with } \zeta = f^{-1}(z),\ \bar{\zeta} = \overline{f^{-1}}(\bar{z}).
$$

Now write the derivative of the inverse in terms of the derivative of f :

$$
\frac{d\zeta}{dz}\Big|_{\zeta=f^{-1}(z)} = \frac{1}{\frac{d}{d\zeta}f(\zeta)\Big|_{\zeta=f^{-1}(z)}}, \qquad \frac{d\bar{\zeta}}{d\bar{z}}\Big|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})} = \frac{1}{\frac{d}{d\bar{\zeta}}\overline{f(\zeta)}\Big|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})}}.
$$

$$
\therefore \quad \phi'(z,\bar{z}) = \left(\frac{1}{\frac{d}{d\zeta}f(\zeta)\big|_{\zeta=f^{-1}(z)}}\right)^h \left(\frac{1}{\frac{d}{d\bar{\zeta}}\overline{f(\zeta)}\big|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})}}\right)^{\bar{h}} \phi\big(f^{-1}(z),\overline{f^{-1}}(\bar{z})\big).
$$

Now replace the left-hand argument $f^{-1}(z)$ by z (relabel $\zeta \mapsto z,\ \bar{\zeta} \mapsto \bar{z}$):

$$
\boxed{\phi'(z,\bar{z}) = \left(\frac{d}{dz}f(z)\Big|_{z=f^{-1}(z)}\right)^{-h} \left(\frac{d}{d\bar{z}}\overline{f(z)}\Big|_{\bar{z}=\overline{f^{-1}}(\bar{z})}\right)^{-\bar{h}} \phi\big(f^{-1}(z),\overline{f^{-1}}(\bar{z})\big)\ .}
$$

These are active transformation because the RHS has the form $\pi_{ab}\phi_b(\Lambda^{-1}x)$. To see how the transformation works, we consider the following example, $z' = \lambda z$

$$
\begin{aligned}
\phi'(z,\bar{z}) &= \left(\frac{\partial \lambda z}{\partial z}\right)^h \left(\frac{\partial \bar{\lambda z}}{\partial \bar{z}}\right)^{\bar{h}} \phi(\lambda z,\lambda\bar{z}) \\
&= \lambda^h \lambda^{\bar{h}} \phi(\lambda z,\lambda\bar{z}) \\
\phi(z',\bar{z}') &= \lambda^{-h}\lambda^{-\bar{h}}\phi'(z,\bar{z})
\end{aligned}
$$

under infinitesimal conformal transformation $z' = f(z) = z + \epsilon$,

$$
\begin{aligned}
\phi'(z,\bar{z}) &= [1+\partial\epsilon]^{h}[1+\bar{\partial}\bar{\epsilon}]^{\bar{h}}\phi(z+\epsilon,\bar{z}+\bar{\epsilon}) \\
&= [1+\partial\epsilon]^{h}[1+\bar{\partial}\bar{\epsilon}]^{\bar{h}}[\phi(z,\bar{z})+\epsilon\partial\phi(z,\bar{z})+\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})+\mathcal{O}(\epsilon^2)] \\
&= \phi(z,\bar{z})+\epsilon\partial\phi(z,\bar{z})+\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})+h\phi(z,\bar{z})\partial\epsilon+\bar{h}\phi(z,\bar{z})\bar{\partial}\bar{\epsilon}+\mathcal{O}(\epsilon^2) \\
&= [1+(\epsilon\partial+h\partial\epsilon)+(\bar{\epsilon}\bar{\partial}+\bar{h}\bar{\partial}\bar{\epsilon})]\phi(z,\bar{z})
\end{aligned}
$$

So, the variation $\delta\phi(z,\bar{z})$ can be given as:

$$
\sum_{n=-\infty}^{\infty}\epsilon_n[L_n,\phi]+\bar{\epsilon}_n[\bar{L}_n,\phi]=\delta\phi(z,\bar{z})=[(\epsilon\partial+h\partial\epsilon)+(\bar{\epsilon}\bar{\partial}+\bar{h}\bar{\partial}\bar{\epsilon})]\phi(z,\bar{z})
$$

In the active point of view,

$$
\begin{aligned}
\phi'(z,\bar{z}) &= [1+\partial\epsilon]^{-h}[1+\bar{\partial}\bar{\epsilon}]^{\bar{-}h}\phi(z-\epsilon,\bar{z}-\bar{\epsilon}) \\
&= [1+\partial\epsilon]^{-h}[1+\bar{\partial}\bar{\epsilon}]^{-\bar{h}}[\phi(z,\bar{z})-\epsilon\partial\phi(z,\bar{z})-\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})+\mathcal{O}(\epsilon^2)] \\
&= \phi(z,\bar{z})-\epsilon\partial\phi(z,\bar{z})-\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})-h\phi(z,\bar{z})\partial\epsilon-\bar{h}\phi(z,\bar{z})\bar{\partial}\bar{\epsilon}+\mathcal{O}(\epsilon^2) \\
&= [1-(\epsilon\partial+h\partial\epsilon)-(\bar{\epsilon}\bar{\partial}+\bar{h}\bar{\partial}\bar{\epsilon})]\phi(z,\bar{z})
\end{aligned}
$$

The other kind is **quasi-primary field**, they only transform as primary field under Global conformal transformation.

3.5 Energy Momentum Tensor

Usually, a Field Theory is defined in terms of a Lagrangian action from which one can derive various objects and properties of the theory. In particular, the energy–momentum tensor can be deduced from the variation of the action with respect to the metric and so it encodes the behaviour of the theory under infinitesimal transformations $g_{\mu\nu}\to g_{\mu\nu}+\delta g_{\mu\nu}$.

$$
\begin{aligned}
S &= \int d^Dx\mathcal{L} \\
\delta S &= \int d^Dx\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\delta g^{\mu\nu}=\int d^DxT_{\mu\nu}\delta g^{\mu\nu}
\end{aligned}
$$

Under $x^\mu\to x^\mu+\epsilon^\mu(x)$, $\delta g^{\mu\nu}=\partial^\mu\epsilon^\nu+\partial^\nu\epsilon^\mu$.

$$
\begin{aligned}
\delta S &= \int d^DxT_{\mu\nu}(\partial^\mu\epsilon^\nu+\partial^\nu\epsilon^\mu) \\
&= \int d^DxT_{\mu\nu}\frac{2}{D}(\partial^\rho\epsilon_\rho)g^{\mu\nu}=\frac{2}{D}\int d^DxT^\mu_\mu(\partial^\rho\epsilon_\rho)=0 \implies T^\mu_\mu=0
\end{aligned}
$$

This tell us that for the conformal invarience to hold, the stress energy tensor has to be traceless.

$$
\begin{aligned}
\delta S &= \int d^DxT_{\mu\nu}(\partial^\mu\epsilon^\nu+\partial^\nu\epsilon^\mu) \\
&= 2\int d^DxT_{\mu\nu}\partial^\mu\epsilon^\nu \\
&= 2\int d^Dx\partial^\mu(T_{\mu\nu}\epsilon^\nu)-2\int d^Dx(\partial^\mu T_{\mu\nu})\epsilon^\nu=0 \implies \partial^\mu T_{\mu\nu}=0
\end{aligned}
$$

Hence, we conclude that stress energy tensor is canonically conserved as well as traceless in the presence of conformal symmetry. Next we see how this condition gets translated in complex coordinates. The goal is to find T_{zz}, $T_{z\bar{z}}$, $T_{\bar{z}z}$ and $T_{\bar{z}\bar{z}}$.

$$
\begin{aligned}
T_{zz} &= \frac{\partial x^0}{\partial z}\frac{\partial x^0}{\partial z}T_{00}+2\frac{\partial x^0}{\partial z}\frac{\partial x^1}{\partial z}T_{01}+\frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial z}T_{11} \\
&= \frac{1}{4}T_{00}+2\frac{1}{2}\frac{1}{2i}T_{01}-\frac{1}{4}T_{11} \\
&= \frac{1}{4}[T_{00}-2iT_{01}-T_{11}]
\end{aligned}
$$

$$
\begin{aligned}
T_{z\bar{z}} &= \frac{\partial x^0}{\partial z}\frac{\partial x^0}{\partial \bar{z}}T_{00} + \frac{\partial x^0}{\partial z}\frac{\partial x^1}{\partial \bar{z}}T_{01} + \frac{\partial x^1}{\partial z}\frac{\partial x^0}{\partial \bar{z}}T_{10} + \frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial \bar{z}}T_{11} \\
&= \frac{1}{2}\frac{1}{2}T_{00} + \frac{1}{2}\frac{-1}{2i}T_{01} + \frac{1}{2i}\frac{1}{2}T_{10} + \frac{1}{2i}\frac{-1}{2i}T_{11} \\
&= \frac{1}{4}[T_{00} + T_{11}] = \frac{1}{4}\delta^{\mu\nu}T_{\mu\nu} = \frac{1}{4}T^{\mu}_{\mu} = 0
\end{aligned}
$$

$$
\begin{aligned}
T_{\bar{z}\bar{z}} &= \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^0}{\partial \bar{z}}T_{00} + 2\frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^1}{\partial \bar{z}}T_{01} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial x^1}{\partial \bar{z}}T_{11} \\
&= \frac{1}{4}T_{00} + 2\frac{1}{2}\frac{-1}{2i}T_{01} - \frac{1}{4}T_{11} \\
&= \frac{1}{4}[T_{00} + 2iT_{01} - T_{11}]
\end{aligned}
$$

Next we investigate the form of $\partial^{\mu}T_{\mu\nu}$ in complex coordinates.

$$
\begin{aligned}
\partial_0 T_{00} + \partial_1 T_{10} = 0 \\
\partial_0 T_{01} + \partial_1 T_{11} = 0
\end{aligned}
\tag{3.5}
$$

We can now calculate $\partial_{\bar{z}}T_{zz}$:

$$\bar{\partial}T_{zz} = \frac{1}{4}(\bar{\partial}T_{00} - 2i\bar{\partial}T_{01} - \bar{\partial}T_{11})$$

using (3.1) and $T^{\mu}_{\mu} = T_{00} + T_{11} = 0$

$$
\begin{aligned}
&= \frac{1}{8}[(\partial_0 + i\partial_1)T_{00} - 2i(\partial_0 + i\partial_1)T_{01} - (\partial_0 + i\partial_1)T_{11}] \\
&= \frac{1}{8}[(\partial_0 T_{00} + 2\partial_1 T_{01} - \partial_0 T_{11}) + i(\partial_1 T_{00} - 2\partial_0 T_{01} - \partial_1 T_{11})] \\
&= \frac{1}{8}[2(\partial_0 T_{00} + \partial_1 T_{01}) - 2i(\partial_1 T_{11} + \partial_0 T_{01})] = 0
\end{aligned}
$$

In the last step we used (3.5). Similarly one can show

$$\partial T_{\bar{z}\bar{z}} = 0$$

Thus, T_{zz} is the chiral field we discussed earlier and $T_{\bar{z}\bar{z}}$ is the anti-chiral field. So in complex coordinates, the stress energy tensor looks like:

$$T_{\mu\nu}(z,\bar{z}) = \begin{bmatrix} 0 & T(z) \\ \bar{T}(\bar{z}) & 0 \end{bmatrix}$$

3.6 Ward Identities

The consequence of a symmetry of the action and measure on correlation functions may also be expressed via the so-called Ward identities. An infinitesimal transformation may be written in terms of the generator as:

$$\phi'(x) = \phi(x) - i\omega_a G_a \phi(x)$$

where ω_a is a collection of infinitesimal, constant parameters. We will consider the above variation of fields in the correlation function. The action is not invariant under such local transformation and its variation is given by:

$$\delta S = \int d^D x (\partial_\mu T^{\mu\nu})\epsilon_\nu$$

where j^{μ}_{a} is the conserved current. The correlation function can be given as:

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \frac{1}{Z}\int D\phi\phi(x_1)\phi(x_2)\ldots\phi(x_n)e^{-S[\phi]}$$

We use the invariance of correlators under the transformation to argue,

$$
\begin{aligned}
\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle &= \langle \phi'(x_1)\phi'(x_2)\ldots\phi'(x_n)\rangle \\
&= \frac{1}{Z}\int D\phi'\phi'(x_1)\phi'(x_2)\ldots\phi'(x_n)e^{-S[\phi']} = \frac{1}{Z}\int D\phi'\phi'(x_1)\phi'(x_2)\ldots\phi'(x_n)e^{-S[\phi]-\delta S[\phi]}
\end{aligned}
$$

$$= \frac{1}{Z}\int D\phi'[\phi(x_1)\phi(x_2)\dots\phi(x_n)+\delta(\phi(x_1)\phi(x_2)\dots\phi(x_n))]e^{-S[\phi]-\int d^Dx\partial_\mu T^{\mu\nu}\epsilon_\nu(x)}$$
$$= \frac{1}{Z}\int D\phi[\phi(x_1)\phi(x_2)\dots\phi(x_n)+\delta(\phi(x_1)\phi(x_2)\dots\phi(x_n))]e^{-S[\phi]}\left(1-\int d^Dx\partial_\mu T^{\mu\nu}\epsilon_\nu(x)+\dots\right)$$

When expanded to first order in $\omega_a(x)$, the above yields

$$\langle\delta(\phi(x_1)\phi(x_2)\dots\phi(x_n))\rangle = \int dx\,\langle\partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle\,\epsilon_\nu(x)$$
$$-i\sum_{i=1}^{N}[\phi(x_1)\dots G_i^\mu\phi(x_i)\dots\phi(x_n)]\epsilon_\mu(x_i) = \int dx\,\langle\partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle\,\epsilon_\nu(x)$$
$$-i\int d^Dx\epsilon_\mu(x)\delta^D(x-x_i)\sum_{i=1}^{N}[\phi(x_1)\dots G_i^\mu\phi(x_i)\dots\phi(x_n)] = \int dx\,\langle\partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle\,\epsilon_\nu(x)$$

Hence,

$$\boxed{\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}(x)\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots G_i^\mu\phi(x_i)\dots\phi(x_n)\rangle}$$

Translation

The generator of Translation $P^\mu = -i\partial^\mu$ is given as:

$$\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}(x)\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = -\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots\partial_i^\mu\phi(x_i)\dots\phi(x_n)\rangle$$

Rotation

The generator of Rotation is $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) + S^{\mu\nu}$ and the associated conserved current is $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$.

$$\frac{\partial}{\partial x^\mu}\langle j^{\mu\nu\rho}(x)\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots J_i^{\nu\rho}\phi(x_i)\dots\phi(x_n)\rangle$$
$$\frac{\partial}{\partial x^\mu}\langle(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots[i(x^\nu\partial^\rho - x^\rho\partial^\nu)+S^{\nu\rho}]_i\phi(x_i)\dots\phi(x_n)\rangle$$
$$\cancel{x^\rho\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}\phi(x_1)\dots\phi(x_n)\rangle} + \delta_\mu^\rho\langle T^{\mu\nu}\phi(x_1)\dots\phi(x_n)\rangle - \cancel{x^\nu\frac{\partial}{\partial x^\mu}\langle T^{\mu\rho}\phi(x_1)\dots\phi(x_n)\rangle} - \delta_\mu^\nu\langle T^{\mu\rho}\phi(x_1)\dots\phi(x_n)\rangle$$
$$= -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots[\cancel{i(x^\nu\partial^\rho} - \cancel{x^\rho\partial^\nu})+S^{\nu\rho}]_i\phi(x_i)\dots\phi(x_n)\rangle$$
$$\langle(T^{\rho\nu} - T^{\nu\rho})\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\,\langle\phi(x_1)\dots S_i^{\nu\rho}\phi(x_i)\dots\phi(x_n)\rangle$$

Dilatation

The generator for Dilatation is $D = x\cdot\partial$ and the associated conserved current is $j^\mu = T^\mu_\nu x^\nu$.

$$S[\phi] = \int d^2x\,\mathcal{L}(\phi^A,\partial_\mu\phi^A)$$
$$x'^\mu = (1+\varepsilon)x^\mu, \qquad \delta x^\mu = \varepsilon x^\mu, \qquad \bar{\delta}\phi^A = -\varepsilon\Delta_A\phi^A$$
$$\delta_{\text{tot}}\phi^A = \bar{\delta}\phi^A + \delta x^\nu\partial_\nu\phi^A$$

From Noether Theorem

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\bar{\delta}\phi^A - T^\mu{}_\nu\,\delta x^\nu$$

For dilatation, (ε factor suppressed)

$$j_D^\mu = -\sum_A \Delta_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\phi^A - x_\nu T^{\mu\nu}$$

Let's define $V^\mu \equiv -\sum_A \Delta_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\phi^A$, then

$$j_D^\mu = V^\mu - x_\nu T^{\mu\nu}$$

$$\begin{aligned}
\partial_\mu j_D^\mu &= \partial_\mu V^\mu - \partial_\mu(x_\nu T^{\mu\nu}) \\
&= \partial_\mu V^\mu - T^\mu{}_\mu - x_\nu \partial_\mu T^{\mu\nu} \\
&= \partial_\mu V^\mu + T^\mu{}_\mu \quad (\partial_\mu T^{\mu\nu} = 0)
\end{aligned}$$

$$\partial_\mu j_D^\mu = 0 \iff T^\mu{}_\mu + \partial_\mu V^\mu = 0$$

The related Ward identity becomes:

$$\partial_\mu \langle T_\nu^\mu x^\nu \phi(x_1)\dots\phi(x_n)\rangle = -\sum_i \delta(x-x_i)\left[x_i^\nu \frac{\partial}{\partial x_i^\nu}\langle\phi(x_1)\dots\phi(x_n)\rangle + \Delta_i \langle\phi(x_1)\dots\phi(x_n)\rangle\right]$$

which becomes

$$\left\langle T_\mu^\mu \phi(x_1)\dots\phi(x_n)\right\rangle = -\sum_i \delta(x-x_i)\Delta_i \langle\phi(x_1)\dots\phi(x_n)\rangle$$

3.6.1 In complex coordinates

We wish to rewrite these identities in terms of complex coordinates and complex components. But, we will first derive the relevant identities for doing so.

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

The metric in Euclidean space could be given as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In complex coordinates, it becomes:

$$\begin{aligned}
g^{\alpha\beta} &= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}
\end{aligned}$$

The anti-symmetric levi civita tensor in cartesian coordinates has the following form:

$$\epsilon^{\mu\nu}_{\text{cartesian}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

using the tensorial transformation law for the above coordinate transformation,

$$\begin{aligned}
\epsilon^{\alpha\beta} &= J^\alpha_\mu \epsilon^{\mu\nu}_{\text{cartesian}} J^\beta_\nu \\
&= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}
\end{aligned}$$

Lowering the indices:

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{-i}{2} & 0 \end{pmatrix}$$

For the delta functions, we remind ourselves that it is not defined as:

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$$

But by the expression

$$\int_M d^2x\delta(x)f(z) = f(0)$$

Any function which satisfies this property is a valid representation of delta function. The ∞ of $\delta(x)$ at $x = 0$ in the above mentioned representation is defined by the integral $\int_{-\infty}^{\infty}\delta(x)dx = 1$. For more detail refer to section 7.2 of the textbook **Green's function** by *G.F. Roach*. It just so happens to be, that

$$\delta(x) = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z}$$

satisfies the required property. This identity is justified as follows. We consider a vector F^μ whose divergence is integrated within a region M of the complex plane bounded by the contour ∂M. Gauss's theorem may be applied:

$$\int_M d^2x\,\partial_\mu F^\mu = \int_{\partial M} d\xi_\mu\,F^\mu \tag{3.6}$$

where $d\xi_\mu$ is an outward-directed differential of circumference, orthogonal to the boundary ∂M of the domain of integration. It is more convenient to use a counterclockwise differential ds^ρ, parallel to the contour ∂M: $d\xi_\mu = \epsilon_{\mu\rho}ds^\rho$. In terms of complex coordinates, the above surface integral is nothing but a contour integral, where the (anti)holomorphic component of ds^ρ is dz ($d\bar{z}$):

$$\begin{aligned}\int_M d^2x\,\partial_\mu F^\mu &= \int_{\partial M} ds^\sigma\epsilon_{\sigma\mu}F^\mu &(3.7)\\ &= \int_{\partial M}\left(dz\,\epsilon_{z\bar{z}}F^{\bar{z}} + d\bar{z}\,\epsilon_{\bar{z}z}F^z\right)\\ &= \frac{i}{2}\int_{\partial M}\left(-dz\,F^{\bar{z}} + d\bar{z}\,F^z\right) &(3.8)\end{aligned}$$

Here the contour ∂M circles counterclockwise. If $F^{\bar{z}}$ (F^z) is holomorphic (antiholomorphic), then Cauchy's theorem may be applied; otherwise the contour ∂M must stay fixed. We consider then a holomorphic function $f(z)$ and check the correctness of the first representation in Eq. (5.33) by integrating it against $f(z)$ within a neighborhood M of the origin:

$$\begin{aligned}\int_M d^2x\,\delta(x)f(z) &= \frac{1}{\pi}\int_M d^2x\,f(z)\,\partial_{\bar{z}}\frac{1}{z}\\ &= \frac{1}{\pi}\int_M d^2x\,\partial_{\bar{z}}\left(\frac{f(z)}{z}\right)\\ &= \frac{1}{2\pi i}\int_{\partial M} dz\,\frac{f(z)}{z}\\ &= f(0) &(3.9)\end{aligned}$$

In the second equation we have used the assumption that $f(z)$ is analytic within M, and in the third equation we have used the form (3.7) of Gauss's theorem with $F^{\bar{z}} = f(z)/\pi z$ and $F^z = 0$, and in the last equation we used Cauchy's residue theorem. Since the original ward identity was covariant, we now only need the following object to write the Ward identities in complex coordinates:

$$\begin{aligned}\partial_\mu\langle T^\mu_z\phi(z_1)\dots\phi(z_n)\rangle &= g^{\alpha\beta}\partial_\alpha\langle T_{\beta z}\phi(z_1)\dots\phi(z_n)\rangle = 2\partial_z\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle + 2\partial_{\bar{z}}\langle T_{zz}\phi(z_1)\dots\phi(z_n)\rangle\\ \partial_\mu\langle T^\mu_{\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle &= g^{\alpha\beta}\partial_\alpha\langle T_{\beta\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle = 2\partial_z\langle T_{\bar{z}\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\partial_{\bar{z}}\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle\\ \langle T^\mu_\mu\phi(z_1)\dots\phi(z_n)\rangle &= g^{\alpha\beta}\langle T_{\alpha\beta}\phi(z_1)\dots\phi(z_n)\rangle = 2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle\\ \epsilon_{\mu\nu}\langle T^{\mu\nu}\phi(z_1)\dots\phi(z_n)\rangle &= \epsilon^{\alpha\beta}\langle T_{\alpha\beta}\phi(z_1)\dots\phi(z_n)\rangle = -2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle\end{aligned}$$

The Ward identities are then explicitly written as:

$$2\pi\partial_z\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}\phi(z_1)\dots\phi(z_n)\rangle = -\sum_i\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle\phi(z_1)\dots\phi(w_i)\dots\phi(z_n)\rangle$$

$$2\pi\partial_z\langle T_{\bar{z}\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle = -\sum_i\partial_z\frac{1}{\bar{z}-\bar{z}_i}\partial_{\bar{z}_i}\langle\phi(z_1)\dots\phi(w_i)\dots\phi(z_n)\rangle$$

$$2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle = -\sum_{i=1}^{N}\delta(z-w_i)\Delta_i\langle\phi(z_1)\dots\dots\phi(z_n)\rangle$$
$$-2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle = -\sum_{i=1}^{N}\delta(z-w_i)s_i\langle\phi(z_1)\dots\dots\phi(z_n)\rangle$$

Adding and subtracting the last two expressions,

$$2\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle = -\sum_{i=1}^{N}\delta(z-w_i)\frac{\Delta_i+s_i}{2}\langle\phi(z_1)\dots\dots\phi(z_n)\rangle$$
$$2\pi\langle T_{\bar{z}z}\phi(z_1)\dots\phi(z_n)\rangle = -\sum_{i}\partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle\phi(z_1)\dots\phi(z_n)\rangle \tag{3.10}$$

$$2\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle = -\sum_{i=1}^{N}\delta(z-w_i)\frac{\Delta_i-s_i}{2}\langle\phi(z_1)\dots\dots\phi(z_n)\rangle$$
$$2\pi\langle T_{z\bar{z}}\phi(z_1)\dots\phi(z_n)\rangle = -\sum_{i}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle\phi(z_1)\dots\phi(z_n)\rangle \tag{3.11}$$

Inserting these relations in the first two Ward identities:

$$\partial_{\bar{z}}\left\{\langle T(z,\bar{z})\phi(z_1)\dots\phi(z_n)\rangle - \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle\phi(z_1)\dots\phi(z_n)\rangle + \frac{h_i}{(z-w_i)^2}\langle\phi(z_1)\dots\phi(z_n)\rangle\right]\right\} = 0 \tag{3.12}$$

$$\partial_{z}\left\{\langle \bar{T}(z,\bar{z})\phi(z_1)\dots\phi(z_n)\rangle - \sum_{i=1}^{n}\left[\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle\phi(z_1)\dots\phi(z_n)\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}\langle\phi(z_1)\dots\phi(z_n)\rangle\right]\right\} = 0 \tag{3.13}$$

where we have introduced a renormalized energy-momentum tensor

$$T = -2\pi T_{zz}, \qquad \bar{T} = -2\pi T_{\bar{z}\bar{z}}. \tag{3.14}$$

Thus the expressions between braces in (3.12) and ((3.13)) are respectively holomorphic and antiholomorphic; we may write

$$\langle T(z)\phi(z_1)\dots\phi(z_n)\rangle = \sum_{i=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle\phi(z_1)\dots\phi(z_n)\rangle + \frac{h_i}{(z-w_i)^2}\langle\phi(z_1)\dots\phi(z_n)\rangle\right\} + \text{reg.} \tag{3.15}$$

where "reg." stands for a holomorphic function of z, regular at $z = w_i$. There is a similar expression for the antiholomorphic counterpart. The Ward identity shows that the correlator of the field $T(z)$ with primary fields $\phi(w_i, \bar{w}_i)$ becomes singular as z approaches the points w_i. The OPE of the energy-momentum tensor with primary fields is written simply by removing the brackets $\langle\dots\rangle$, it being understood that OPE is meaningful only within correlation functions.

3.7 Free Fields and Operator Product Expansion

The operator product expansion, or OPE, is the representation of a product of operators (at positions z and w, respectively) by a sum of terms, each being a single operator, well-defined as $z \to w$, multiplied by a c-number function of $z-w$, possibly diverging as $z \to w$, and which embodies the infinite fluctuations as the two positions tend toward each other. For a single primary field ϕ of conformal dimension h and $\bar{h}$, we have the OPE from Ward identity as:

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w})$$
$$T(\bar{z})\phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{z})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) \tag{3.16}$$

whenever appearing in OPEs, the symbol $\sim$ will mean equality modulo expressions regular as $z \to w$. Of course, the OPE contains also an infinite number of regular terms which, for the Energy Momentum tensor, can not be obtained from the conformal ward identity. In general, we would write the OPE of two fields $A(z)$ and $B(w)$ as

$$A(z)B(w) = \sum_{n=-\infty}^{\infty} \frac{\{AB\}_n(w)}{(z-w)^n}$$

where the composite $\{AB\}_n(w)$ are non singular at $z = w$. For instance, $\{T\phi\}_1 = \partial_w \phi(w)$. We stress that so far, quantities appearing in (3.16) are not operators but simply fields occuring within correlation functions. We will now proceed with specific examples, in order to familiarize ourselves with basic techniques and with simple but important systems.

The Free Boson

From the point of view of the canonical or path integral formalism, the simplest conformal field theory is that of a free massless boson ϕ, with the following action:

$$S = \frac{1}{2}g \int d^2x \partial_\mu \phi \partial^\mu \phi$$

where g is some normalization parameter that we leave unspecified at the moment. The two-point function, or propagator can be found by comparing with

$$S = \frac{1}{2} \int \mathrm{d}^d x \mathrm{d}^d y \, \phi(x) A(x,y) \phi(y)$$

we have

$$A(x,y) = -g\delta^{(2)}(x-y)\Box \,, \tag{3.17}$$

We can calculate the two-point function $K(x,y) \equiv \langle \varphi(x_1)\varphi(x_2)\rangle = A^{-1}$ by solving the following equation:

$$\begin{aligned}
\int d^2u A(x,u) K(u,y) &= \delta^{(2)}(x-y) \\
-\int d^2u g\delta^{(2)}(x-u)\Box K(u,y) &= \delta^{(2)}(x-y) \\
-g\Box K(x,y) &= \delta^{(2)}(x-y) \,,
\end{aligned}$$

Because of rotation and translation invariance, the propagator $K(x,y)$ should depend only on the distance separating the two points. Thus, we can write $K(x,y) \equiv K(\rho)$ with $\rho = |x-y|$, and integrate over x within a disk of radius ρ around y. We find

$$\begin{aligned}
1 &= 2\pi g \int_0^r \mathrm{d}\rho\rho \Big(-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho K'(\rho)) \Big) \\
&= 2\pi g(-rK'(r))
\end{aligned}$$

The solution of the two-point function for massless free boson can be obtained up to an additive constant,

$$\langle \phi(x)\phi(y)\rangle = -\frac{1}{4\pi g} \ln(x-y)^2 + \text{const} \tag{3.18}$$

In terms of complex coordinates, this is

$$\langle \phi(z,\bar z)\phi(w,\bar w)\rangle = -\frac{1}{4\pi g}\{\ln(z-w) + \ln(\bar z - \bar w)\} + \text{const}$$

The holomorphic and anti-holomorphic components can be separated by taking the derivatives $\partial\phi$ and $\bar\partial\phi$:

$$\begin{aligned}
\langle \partial_z\phi(z,\bar z)\partial_w\phi(w,\bar w)\rangle &= -\frac{1}{4\pi g}\partial_w\partial_z\{\ln(z-w)\} \\
&= -\frac{1}{4\pi g}\partial_w \frac{1}{z-w} = -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \\
\langle \partial_{\bar z}\phi(z,\bar z)\partial_{\bar w}\phi(w,\bar w)\rangle &= -\frac{1}{4\pi g}\partial_{\bar w}\partial_{\bar z}\{\ln(\bar z-\bar w)\} \\
&= -\frac{1}{4\pi g}\partial_{\bar w} \frac{1}{\bar z-\bar w} = -\frac{1}{4\pi g}\frac{1}{(\bar z-\bar w)^2}
\end{aligned}$$

In the following, we will focus on holomorphic field $\partial\phi \equiv \partial_z\phi$.

$$\begin{aligned}
T_{\mu\nu} &= g\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right] \\
&= g\partial_\mu\phi\partial_\nu\phi - \frac{g}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \\
&= g\partial_\mu\phi\partial_\nu\phi - \frac{g}{2}\eta_{\mu\nu}\times 2\eta^{z\bar{z}}\partial_z\phi\,\underbrace{\partial_{\bar{z}}\phi}_{=0} \\
&= g\partial_\mu\phi\partial_\nu\phi
\end{aligned}$$

Then,

$$T(z) = -2\pi T_{zz} = -2\pi g:\partial\phi\partial\phi: \qquad \text{and} \qquad \bar{T}(\bar{z}) = 0$$

Like all composite field, the energy momentum tensor has to be normal ordered, in order to ensure the vanishing of its vacuum expectation value. More explicitly, the exact meaning of above expression is

$$T(z) = -2\pi g \lim_{w\to z}\left[\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle\right]$$

The OPE of $T(z)$ with $\partial\phi$ may be calculated from Wick's theorem:[1]

$$\begin{aligned}
T(z)\partial\phi(w) &= -2\pi g:\partial\phi(z)\partial\phi(z):\partial\phi(w) \\
&\sim -2\pi g:\partial\phi(z)\partial\phi(z):\partial\phi(w) - 2\pi g:\partial\phi(z)\partial\phi(z):\partial\phi(w) \\
&\sim \frac{\partial\phi(z)}{(z-w)^2}
\end{aligned}$$

By expanding $\phi(z)$ around w, we arrive at the OPE

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)}$$

This shows that $\partial\phi$ is a primary field with conformal dimension $h = 1$. Wick's theorem also allows us to calculate the OPE of energy-momentum tensor with itself:

$$\begin{aligned}
T(z)T(w) &= 4\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\sim 4\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): + 4\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\quad + 4\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): + 4\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\quad + 4\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): + 4\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\sim 8\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): + 16\pi^2g^2:\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g:\partial\phi(z)\partial\phi(w):}{(z-w)^2}
\end{aligned}$$

Expanding $\phi(z)$ around w, we arrive at the OPE

$$\begin{aligned}
&\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g:\partial\phi(w)\partial\phi(w):}{(z-w)^2} - \frac{4\pi g:\partial^2\phi(w)\partial\phi(w):}{(z-w)} \\
&\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}
\end{aligned}$$

We immediately see that the energy momentum tensor is not strictly a primary field, because of the anomalous term $1/2(z-w)^4$ which does not appear in (3.16).

Free Fermion

Now we consider another simple model: free fermion. In two dimensions, the action of a free Majorana fermion is

$$S = \frac{1}{2}g\int \mathrm{d}^2x\Psi^\dagger\gamma^0\gamma^\mu\partial_\mu\Psi\,, \tag{3.19}$$

[1] Since we are Wick contracting the normal ordered product of operators, the only term that it will give rise to are cross-contractions, since the wick contraction of operators which are already normal ordered vanishes.

where the gamma matrices γ^μ satisfy the so-called **Clifford algebra**:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,, \tag{3.20}$$

and we impose the **Majorana condition** ($\Psi^* = \Psi$) to the fermionic field to remove a half of the degrees of freedom. In the Euclidean space $\eta^{\mu\nu} = \mathrm{diag}(1,1)$, we take a basis of Dirac matrices as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.21}$$

and therefore,

$$\begin{aligned}
\gamma^0\gamma^\mu\partial_\mu &= \gamma^0(\gamma^0\partial_0 + \gamma^1\partial_1) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial + i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\bar{\partial}\right] \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \partial_0 - i\partial_1 \\ \partial_0 + i\partial_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2\partial_z \\ 2\partial_{\bar{z}} & 0 \end{pmatrix} \\
&= 2\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}
\end{aligned}$$

Using this basis, we can express the action as

$$S = g\int \mathrm{d}^2x(\overline{\psi}\partial\overline{\psi} + \psi\bar{\partial}\psi)\,, \tag{3.22}$$

where we write the two-component spinor Ψ as $(\psi, \overline{\psi})$. Since the equations of motion are $\partial\overline{\psi} = 0$ and $\overline{\partial}\psi = 0$, $\psi(z)$ and $\overline{\psi}(\overline{z})$ holomorphic and antiholomorphic field, respectively.

Now let us calculate the two-point function as in the free fermion

$$K_{ij} = \langle\Psi_i(x)\Psi_j(y)\rangle \quad (i,j = 1,2)\,. \tag{3.23}$$

The action can be expressed by

$$S = \frac{1}{2}\int \mathrm{d}^2x\mathrm{d}^2y\Psi_i(x)A_{ij}(x,y)\Psi_j(y)\,, \tag{3.24}$$

where the kernel is

$$A_{ij}(x,y) = g\delta(x-y)(\gamma^0\gamma^\mu)_{ij}\partial_\mu\,. \tag{3.25}$$

Recalling that propagator K_{ij} is the inverse of A_{ij}. Therefore, we can write the equation for K as

$$\begin{gathered}
\int d^2uA(x,u)K(u,y) = \delta^{(2)}(x-y) \\
g\int d^2u\delta^{(2)}(x-u)(\gamma^0\gamma^\mu)_{ij}\frac{\partial}{\partial x^\mu}K(u,y) = \delta^{(2)}(x-y)\delta_{ij} \\
g(\gamma^0\gamma^\mu)_{ik}\frac{\partial}{\partial x^\mu}K_{kj}(x,y) = \delta(x-y)\delta_{ij}\,.
\end{gathered}$$

In terms of complex coordinates, this becomes

$$2g\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}\begin{pmatrix} \langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle & \langle\psi(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle \\ \langle\overline{\psi}(z,\overline{z})\psi(w,\overline{w})) & \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle \end{pmatrix} = \frac{1}{\pi}\begin{pmatrix} \overline{\partial}\frac{1}{z-w} & 0 \\ 0 & \partial\frac{1}{\overline{z}-\overline{w}} \end{pmatrix}, \tag{3.26}$$

where we have used the complex form of the δ-function:

$$\delta(x) = \frac{1}{\pi}\overline{\partial}\frac{1}{z} = \frac{1}{\pi}\partial\frac{1}{\overline{z}}\,.$$

Therefore, we obtain the two-point functions for the fermionic fields

$$\begin{aligned}
\langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle &= \frac{1}{2\pi g}\frac{1}{z-w}\,, \\
\langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle &= \frac{1}{2\pi g}\frac{1}{\overline{z}-\overline{w}}\,,
\end{aligned}$$

$$\langle \psi(z,\bar{z})\overline{\psi}(w,\overline{w})\rangle = 0$$

These, after differentiation, imply

$$\begin{aligned}
\langle \partial_z \psi(z,\bar{z})\psi(w,\bar{w})\rangle &= -\frac{1}{2\pi g}\frac{1}{(z-w)^2} \\
\langle \partial_z \psi(z,\bar{z})\partial_w \psi(w,\bar{w})\rangle &= -\frac{1}{\pi g}\frac{1}{(z-w)^3}
\end{aligned}$$

Thus the OPE of two holomorphic fields can be written as:

$$\psi(z)\psi(w) \sim \frac{1}{2\pi g}\frac{1}{z-w} \tag{3.27}$$

In order to see whether the fermion field is a primary field or not, we can calculate its OPE with the energy-momentum tensor. By using (**??**) in complex-coordinate form, we can calculate all the components of the energy-momentum tensor.

$$\begin{aligned}
T^{\bar{z}\bar{z}} &= \frac{\partial \mathcal{L}}{\partial(\partial_{\bar{z}}\Phi)}\partial^{\bar{z}}\Phi = g^{\bar{z}z}\frac{\partial \mathcal{L}}{\partial(\bar{\partial}\Phi)}\partial\Phi = 2\frac{\partial \mathcal{L}}{\partial(\bar{\partial}\Phi)}\partial\Phi = 2g\psi\partial\psi\,, \\
T^{zz} &= \frac{\partial \mathcal{L}}{\partial(\partial_{z}\Phi)}\partial^{z}\Phi = g^{z\bar{z}}\frac{\partial \mathcal{L}}{\partial(\partial\Phi)}\overline{\partial}\Phi = 2\frac{\partial \mathcal{L}}{\partial(\partial\Phi)}\overline{\partial}\Phi = 2g\overline{\psi\partial\psi}\,, \\
T^{z\bar{z}} &= \frac{\partial \mathcal{L}}{\partial(\partial_{z}\Phi)}\partial^{\bar{z}}\Phi - g^{z\bar{z}}\mathcal{L} = 2\frac{\partial \mathcal{L}}{\partial(\partial\Phi)}\partial\Phi - 2\mathcal{L} = -2g\psi\overline{\partial}\psi\,.
\end{aligned}$$

The traceless condition $T^{z\bar{z}}$ is preserved when taking into account the equation of motion, as we have discussed. The holomorphic part is defined as:

$$T(z) = -\pi g : \psi(z)\partial\psi(z) : \,. \tag{3.28}$$

The normal-ordering product can be written in an equivalent way for the free field as follow:

$$: \psi\partial\psi : (z) = \lim_{w\to z}\left(\psi(z)\partial\psi(w) - \langle\psi(z)\partial\psi(w)\rangle\right), \tag{3.29}$$

which is the same expression as in bosonic field theory. Then we can calculate the OPE between the fermion field and energy-momentum tensor directly.

$$\begin{aligned}
T(z)\psi(w) &= -\pi g : \psi(z)\partial\psi(z) : \psi(w) \\
&= -\pi g : \psi(z)\partial\psi(z) : \psi(w) - \pi g : \psi(z)\partial\psi(z) : \psi(w) \\
&\sim \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + \frac{1}{2}\frac{\partial\psi(z)}{(z-w)} \\
&\sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}
\end{aligned}$$

In contracting $\psi(z)$ with $\psi(w)$ we have carried $\psi(w)$ over $\partial\psi(z)$, thus introducing a $(-)$ sign by Pauli's principle. The OPE immediately tells us that in the free fermion model, ψ is a primary field with conformal dimension 1/2. TT OPE in this model can also be obtained directly by calculation.

$$T(z)T(w) \sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,. \tag{3.30}$$

The Ghost System

In string theory applications, there appears another simple system, with the following action:

$$S = \frac{1}{2}g\int d^2x b_{\mu\nu}\partial^\mu c^\nu$$

where $b_{\mu\nu}$ is a traceless symmetric tensor, and where both c^μ and $b_{\mu\nu}$ are fermions (anti-commuting fields). These fields are called ghosts because they are not fundamental dynamical fields, but rather represent a jacobian arising from a change of variables in some functional integrals. More precisely, they are known as *reparametrization ghosts*. The role of these ghost fields is to cancel the unphysical gauge degrees of freedom.

The equation of motion are

$$\partial^\alpha b_{\alpha\mu} = 0 \qquad \text{and} \qquad \partial^\alpha c^\beta + \partial^\beta c^\alpha = 0$$

In holomorphic form we write $c = c^z$ and $\bar{c} = c^{\bar{z}}$. The only nonzero components of the traceless symmetric tensor $b_{\mu\nu}$ are $b = b_{zz}$ and $\bar{b} = b_{\bar{z}\bar{z}}$. The equations of motion are then

$$\begin{array}{lll} \bar{\partial} b = 0 & \partial \bar{b} = 0 & \\ \bar{\partial} c = 0 & \partial \bar{c} = 0 & \partial c = -\bar{\partial}\bar{c} \end{array}$$

The propagator is calculated in the usual way, by writing the action as:

$$S = \frac{1}{2}\int d^2x d^2y b_{\mu\nu}(x) A^{\mu\nu}_{\alpha} c^{\alpha}(y)$$

$$A^{\mu\nu}_{\alpha} = \frac{1}{2} g \delta^{\nu}_{\alpha} \delta(x-y)\partial^{\mu}$$

where we must consider (μ,ν) as a single composite index, symmetric under the exchange of μ and ν. The factor of $\frac{1}{2}$ in front of $A^{\mu\nu}_{\alpha}$ compensates the double counting of each pair (μ,ν) in the sum, which should be avoided since $b^{\mu\nu}$ is the same degree of freedom as $b^{\nu\mu}$. Again the propagator is $K = A^{-1}$, satisfying

$$\frac{1}{2} g \delta^{\mu}_{\alpha} \partial^{\nu} K^{\beta}_{\mu\nu}(x,y) = \delta(x-y)\delta_{\alpha\beta}$$

or, in complex representation

$$g \partial_{\bar{z}} K^{\beta}_{zz} = \frac{1}{\pi}\partial_{\bar{z}} \frac{1}{z-w}\delta_{\beta z}$$

which implies

$$\langle b(z) c(w)\rangle = K^{z}_{zz}(z,w) = \frac{1}{\pi g}\frac{1}{z-w}$$

In OPE form, this is

$$\boxed{b(z)c(w) \sim \frac{1}{\pi g}\frac{1}{z-w}}$$

from which we immediately derive

$$\begin{aligned} \langle c(z) b(w)\rangle &= -\langle b(w) c(z)\rangle = \frac{1}{\pi g}\frac{1}{z-w} \\ \langle b(z)\partial_w c(w)\rangle &= -\langle \partial_z c(z) b(w)\rangle = \partial_w\left(\frac{1}{\pi g}\frac{1}{z-w}\right) = \frac{1}{\pi g}\frac{1}{(z-w)^2} \\ \langle \partial_z b(z) c(w)\rangle &= -\frac{1}{\pi g}\frac{1}{(z-w)^2} \\ \langle \partial_z b(z)\partial_w c(w)\rangle &= -\frac{2}{\pi g}\frac{1}{(z-w)^3} \end{aligned}$$

The canonical energy-momentum tensor for this system is

$$T^{\mu\nu}_{B} = \frac{1}{2} g\left[b^{\mu\alpha}\partial^{\nu} c_{\alpha} - \eta^{\mu\nu} b^{\alpha\beta}\partial_{\alpha} c_{\beta}\right]$$

The Belinfante tensor is

$$T^{\mu\nu}_{B} = \frac{1}{2} g\left[b^{\mu\alpha}\partial^{\nu} c_{\alpha} + b^{\nu\alpha}\partial^{\mu} c_{\alpha} + \partial_{\alpha} b^{\mu\nu} c^{\alpha} - \eta^{\mu\nu} b^{\alpha\beta}\partial_{\alpha} c_{\beta}\right]$$

The normal ordered holomorphic component is obtained from the above by setting $\mu = \nu = 1$, that is, by considering $T^{\bar{z}\bar{z}} = 4T_{zz}$:

$$T(z) = \pi g : (2\partial c\, b + c\partial b) :$$

The OPE for this stress energy tensor with c is again calculated using Wick's theorem:

$$\begin{aligned} T(z)c(w) &= \pi g : (2\partial c\, b + c\partial b) : c(w) \\ &= \pi g : 2\partial c\, b : c(w) + \pi g : c\partial b : c(w) \\ &\sim \frac{2\partial_z c(z)}{z-w} - \frac{c(z)}{(z-w)^2} \\ &\sim \frac{2\partial_w c(w)}{z-w} - \frac{c(w) + (z-w)\partial_w c(w)}{(z-w)^2} \end{aligned}$$

$$\sim \frac{\partial_w c(w)}{z-w} - \frac{c(w)}{(z-w)^2}$$

$$\begin{aligned}
T(z)b(w) &= \pi g : (2\partial c\, b + c\partial b) : b(w) \\
&= -\pi g : 2b\partial c : b(w) - \pi g : \partial b\, c : b(w) \\
&\sim \frac{2b(z)}{(z-w)^2} - \frac{\partial_z b(z)}{(z-w)} \\
&\sim \frac{2b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)}
\end{aligned}$$

One must be careful because b and c are anticommuting so that interchange of fields flips the sign: one should anticommute the fields being paired until they are next to each other before doing the Wick contraction. The OPE of T with itself, may contain more terms, which add up to following:

$$\begin{aligned}
T(z)T(w) &= \pi^2 g^2 : (2\partial c(z)\, b(z) + c(z)\partial b(z)) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) : \\
&= \pi^2 g^2 : 2\partial c(z)\, b(z) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) : + \pi^2 g^2 : c(z)\partial b(z) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) : \\
&= 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) : + 2\pi^2 g : \partial c(z)\, b(z) :: c(w)\partial b(w) : + 2\pi^2 g^2 : c(z)\partial b(z) :: \partial c(w)\, b(w) : \\
&\quad + \pi^2 g^2 : c(z)\partial b(z) :: c(w)\partial b(w) : \\
&= 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) : + 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) : + 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) : \\
&\quad + 2\pi^2 g : \partial c(z)\, b(z) :: c(w)\partial b(w) : + 2\pi^2 g : \partial c(z)\, b(z) :: c(w)\partial b(w) : + 2\pi^2 g : \partial c(z)\, b(z) :: c(w)\partial b(w) : \\
&\quad + 2\pi^2 g^2 : c(z)\partial b(z) :: \partial c(w)\, b(w) : + 2\pi^2 g^2 : c(z)\partial b(z) :: \partial c(w)\, b(w) : + 2\pi^2 g^2 : c(z)\partial b(z) :: \partial c(w)\, b(w) : \\
&\quad + \pi^2 g^2 : c(z)\partial b(z) :: c(w)\partial b(w) : + \pi^2 g^2 : c(z)\partial b(z) :: c(w)\partial b(w) : + \pi^2 g^2 : c(z)\partial b(z) :: c(w)\partial b(w) :
\end{aligned}$$

The end result is

$$\begin{aligned}
T(z)T(w) = {}& \frac{-4}{(z-w)^4} + \frac{4\pi g : \partial c(z) b(w) :}{(z-w)^2} - \frac{4\pi g : b(z)\partial c(w) :}{(z-w)^2} - \frac{4}{(z-w)^4} + \frac{2\pi g : \partial c(z)\partial b(w) :}{z-w} - \frac{4\pi g : b(z) c(w) :}{(z-w)^3} \\
& - \frac{4}{(z-w)^4} - \frac{4\pi g : c(z) b(w) :}{(z-w)^3} + \frac{2\pi g : \partial b(z)\partial c(w) :}{z-w} - \frac{1}{(z-w)^4} - \frac{\pi g : c(z)\partial b(w) :}{(z-w)^2} + \frac{\pi g : \partial b(z) c(w) :}{(z-w)^2} + \ldots
\end{aligned}$$

After some Taylor expansions to turn $f(z)$ functions into $f(w)$ functions, together with a little collecting of terms, this can be written as,

$$T(z)T(w) = \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

3.8 Central Charge

The specific models treated in the last section lead us naturally to the following general OPE of the energy-momentum tensor.

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

where the constant c, not to be confused with ghost field c_μ, depends on the specific model under study: it is equal to 1 for the free boson, $1/2$ for the free fermion, -26 for the reparametrization ghosts, and -2 for the simple ghost system. This model dependent constant term is called the *central charge*. Except for this anomalous term, the OPE simply means that T is a quasi-primary field with conformal dimension $h = 2$.

The central charge may not be determined from symmetry considerations: its value is determined by the short-distance behavior of the theory. For free fields, as seen in the previous section, it is determined by applying Wick's theorem on the normal-ordered energy-momentum tensor. When two decoupled systems (e.g., two free fields) are put together, the energy-momentum tensor of the total system is simply the sum of the energy-momentum tensors associated with each part, and the associated central charge is simply the sum of the central charges of the parts. Thus, the central charge is somehow an extensive measure of the number of degrees of freedom of the system.

Transformation of the Energy-Momentum Tensor

The departure of OPE from the general form (3.16) means that the energy-momentum tensor does not exactly transform like a primary field of dimension 2, contrary to what we expect classically. This happens because the normal ordering is not invariant under conformal transformation. According to conformal ward identity

$$\begin{aligned}\delta_\epsilon T(w) &= -\frac{1}{2\pi i}\oint_C dz\epsilon(z)T(z)T(w)\\ &= -\frac{1}{2\pi i}\oint_C dz\epsilon(z)\left[\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{(z-w)}+\text{reg.}\right]\\ &= -\frac{c/2}{3!}\partial_w^3\epsilon(w)-2T(w)\partial_w\epsilon(w)-\epsilon(w)\partial_w T(w)\end{aligned}$$

In the third equation, we used Cauchy's integral formula for derivatives. The "exponentiation" of this infinitesimal variation to a finite transformation $z \to w(z)$ is:

$$T'(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left[T(z)-\frac{c}{12}\{w;z\}\right] \tag{3.31}$$

where we have introduced the Schwarzian derivative:

$$\{w;z\} = \frac{\frac{\mathrm{d}^3w}{\mathrm{d}z^3}}{\frac{\mathrm{d}w}{\mathrm{d}z}}-\frac{3}{2}\left(\frac{\frac{\mathrm{d}^2w}{\mathrm{d}z^2}}{\frac{\mathrm{d}w}{\mathrm{d}z}}\right)^2$$

Instead of giving the long and technical proof of the last statement, we shall derive the above for free boson system. We write the free boson energy-momentum tensor as

$$T(z) = -2\pi g\lim_{\delta\to 0} :\partial\phi\partial\phi: = -2\pi g\lim_{\delta\to 0}\left[\partial\phi\left(z+\frac{1}{2}\delta\right)\partial\phi\left(z-\frac{1}{2}\delta\right)+\frac{1}{4\pi\delta^2}\right]$$

Consider the transformation $z \to w(z)$. Since ϕ has conformal dimension zero $\partial\phi$ transforms as

$$\partial_z\phi = \frac{\partial w}{\partial z}\frac{\partial\phi'(z)}{\partial w} = w^{(1)}\partial_w\phi'(w)$$

(here we denote the n-th derivative of w by $w^{(n)}$ in order to lighten the notation). Hence $T(z)$ transforms as:

$$T(z) = -2\pi g\lim_{\delta\to 0}\left[w^{(1)}\left(z+\frac{1}{2}\delta\right)w^{(1)}\left(z-\frac{1}{2}\delta\right)\partial_w\phi'\left(w\left(z-\frac{1}{2}\delta\right)\right)\partial_w\phi'\left(w\left(z+\frac{1}{2}\delta\right)\right)+\frac{1}{4\pi\delta^2}\right] \tag{3.32}$$

we will use the following to simplify the above:

$$\begin{aligned}w(z+\delta/2) &\simeq w(z)+\frac{\delta}{2}\partial_z w(z)+\frac{1}{2!}\left(\frac{\delta}{2}\right)^2\partial_z^2 w(z)+\frac{1}{3!}\left(\frac{\delta}{2}\right)^3\partial_z^3 w(z)+\dots\\ \partial_z w(z+\delta/2) &\simeq \partial_z w(z)+\frac{\delta}{2}\partial_z^2 w(z)+\frac{1}{2!}\left(\frac{\delta}{2}\right)^2\partial_z^3 w(z)+\dots\\ \big[w(z+\delta/2)-w(z-\delta/2)\big]^2 &= \big(\partial_z w(z)\big)^2\delta^2+\frac{1}{12}\big(\partial_z^3 w\,\partial_z w(z)\big)\delta^4+\mathcal{O}(\delta^6),\\ &= (w^{(1)}\delta)^2\left[1+\frac{1}{12}\frac{w^{(3)}}{w^{(1)}}\delta^2+\dots\right]\end{aligned}$$

so the inverse is then,

$$\frac{1}{[w(z+\delta/2)-w(z-\delta/2)]^2} = \frac{1}{\delta^2}\frac{1}{(\partial_z w(z))^2}-\frac{1}{12}\frac{\partial_z^3 w(z)}{(\partial_z w(z))^3}+\mathcal{O}(\delta^2) \tag{3.33}$$

$$\begin{aligned}\partial_z w(z+\delta/2)\partial_z w(z-\delta/2) &= \left[\partial_z w(z)+\frac{\delta}{2}\partial_z^2 w(z)+\frac{1}{2!}\left(\frac{\delta}{2}\right)^2\partial_z^3 w(z)+\dots\right]\left[\partial_z w(z)-\frac{\delta}{2}\partial_z^2 w(z)+\frac{1}{2!}\left(\frac{\delta}{2}\right)^2\partial_z^3 w(z)+\dots\right]\\ &= (w^{(1)})^2-\left(\frac{\delta}{2}\right)^2(w^{(2)})^2+\left(\frac{\delta}{2}\right)^2 w^{(1)}w^{(3)}+\dots\end{aligned} \tag{3.34}$$

Then,

$$
\begin{aligned}
T(z) &= \lim_{\delta\to 0}\left[w^{(1)}\left(z+\frac{1}{2}\delta\right)w^{(1)}\left(z-\frac{1}{2}\delta\right)\left\{-2\pi g : \partial_w\phi'(w)\partial_w\phi'(w) : + \frac{1}{2\left[w\left(z+\frac{\delta}{2}\right)-w\left(z-\frac{\delta}{2}\right)\right]^2}\right\} - \frac{1}{2\delta^2}\right] \\
&= (w^{(1)}(z))^2 T'(w) + \lim_{\delta\to 0}\left[\frac{w^{(1)}\left(z+\frac{\delta}{2}\right)w^{(1)}\left(z-\frac{\delta}{2}\right)}{2\left[w\left(z+\frac{\delta}{2}\right)-w\left(z+\frac{\delta}{2}\right)\right]^2} - \frac{1}{2\delta^2}\right] \\
&= (w^{(1)}(z))^2 T'(w) \\
&\quad + \lim_{\delta\to 0}\left[\left\{(w^{(1)})^2 - \left(\frac{\delta}{2}\right)^2 (w^{(2)})^2 + \left(\frac{\delta}{2}\right)^2 w^{(1)}w^{(3)} + \dots\right\}\left\{\frac{1}{2\delta^2}\frac{1}{(w^{(1)})^2} - \frac{1}{24}\frac{w^{(3)}}{(w^{(1)})^3} + \dots\right\} - \frac{1}{2\delta^2}\right] \\
&= (w^{(1)}(z))^2 T'(w) + \lim_{\delta\to 0}\left[\cancel{\frac{1}{2\delta^2}} - \frac{1}{8}\left(\frac{w^{(2)}}{w^{(1)}}\right)^2 - \frac{1}{24}\frac{w^{(3)}}{w^{(1)}} + \frac{1}{8}\frac{w^{(3)}}{w^{(1)}} + \dots - \cancel{\frac{1}{2\delta^2}}\right] \\
&= (w^{(1)}(z))^2 T'(w) + \frac{1}{12}\left[\frac{w^{(3)}}{w^{(1)}} - \frac{3}{2}\left(\frac{w^{(2)}}{w^{(1)}}\right)^2\right]
\end{aligned}
$$

Here we use (3.33) and (3.34) in the second step for simplication. There is another way to look at this derivation which sheds light on the origin of schwarzian derivative term. Let us focus on how the propagator transforms under conformal transformation:

$$
\begin{aligned}
\langle\phi(w(z_1))\phi(w(z_2))\rangle &= -\ln\left|w(z_1)-w(z_2)\right|^2 \\
&= -\ln\left|z_{12}\frac{w(z_1)-w(z_2)}{z_1-z_2}\right|^2 \\
&= -\ln|z_{12}|^2 - \ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right|^2 \\
&= \langle\phi(z_1)\phi(z_2)\rangle - \ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right|^2
\end{aligned}
$$

Let us write $w_1 \equiv w(z_1)$ and $w_2 \equiv w(z_2)$. One can show that for $|z_{12}| = |z_1 - z_2|$ small:

$$
\begin{aligned}
\partial_{z_1}\partial_{z_2}\ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right| &= \partial_{z_1}\partial_{z_2}\ln|w(z_1)-w(z_2)|^2 - \partial_{z_1}\partial_{z_2}\ln|z_{12}|^2 \\
&= \partial_{z_1}\left[\frac{-w^{(1)}(z_2)}{w(z_1)-w(z_2)}\right] - \frac{1}{z_{12}^2} \\
&= \frac{w^{(1)}(z_2)w^{(1)}(z_1)}{(w(z_1)-w(z_2))^2} - \frac{1}{z_{12}^2} \\
&= \frac{2}{12}\left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2}w_2} - \frac{3}{2}\left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2}w_2}\right)^2\right] + \mathcal{O}(z_{12}).
\end{aligned}
$$

Since only the $z_{12} \to 0$ limit is of interest we can drop all terms on the right-hand side that vanish in this limit. Substituting the result of this into the above we learn that:

$$
\boxed{\lim_{z_1\to z_2}\partial_{z_1}\partial_{z_2}\langle\phi(w(z_1))\phi(w(z_2))\rangle = \lim_{z_1\to z_2}\partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle - \frac{2}{12}\left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2}w_2} - \frac{3}{2}\left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2}w_2}\right)^2\right]}
$$

The non-invariance of the vacuum and thus correlator is really what opens up the possibility of a trace anomaly $\langle U^{-1}T(w)U\rangle \equiv \langle T'(w)\rangle \neq 0$ and it is not at all coincidental. The conformal transformations which do change the vacuum are those that have non-vanishing Schwarzian derivative, and thus an extra inhomogeneous central charge term. Using this we can derive the transformation law for energy-momentum tensor:

$$
\begin{aligned}
T^{(w)}(z_2) &\equiv : \lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_w \\
&= \lim_{z_1\to z_2}\Big[-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)-\partial_{z_1}\partial_{z_2}\left\langle\phi(w(z_1))\phi(w(z_2))\right\rangle\Big)\Big] \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)-\partial_{z_1}\partial_{z_2}\left\langle\phi(z_1)\phi(z_2)\right\rangle+\frac{2}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2}-\frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big]\Big)\Big\} \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)-\partial_{z_1}\partial_{z_2}\left\langle\phi(z_1)\phi(z_2)\right\rangle\Big)\Big\}-\frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2}-\frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big] \\
&=: \lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_z -\frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2}-\frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big] \\
&= T^{(z)}(z_2)-\frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2}-\frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big]
\end{aligned}
\tag{3.35}
$$

where we noted in the last two lines that:

$$
\begin{aligned}
T^{(z)}(z_2) &\equiv : \lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_z \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)-\partial_{z_1}\partial_{z_2}\left\langle\phi(z_1)\phi(z_2)\right\rangle\Big)\Big\}
\end{aligned}
\tag{3.36}
$$

as shown above. So we learn that a finite holomorphic change in normal ordering, $z \to w(z)$, with fixed coordinates, z_2, of the energy-momentum tensor is given by:

$$
\boxed{T^{(w)}(z_2) = T^{(z)}(z_2)-\frac{1}{12}\left[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2}-\frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\right]}
\tag{3.37}
$$

Chapter 4

Operator Formalism

In the previous chapter, conformal symmetry was seen to impose constraints on correlation functions in the form of Ward identities. These identities were conveniently expressed using operator product expansions between the energy–momentum tensor and local fields, but the OPEs were understood only as a shorthand for singularities inside correlators. Nothing required a Hilbert space or an operator formalism: in principle, everything could have been computed directly in the path integral by evaluating Green's functions and extracting their short-distance behavior. Up to this point, all we really needed was a way to compute the two-point correlator, whether by solving the Schwinger–Dyson equations or by brute-force path integration. The OPEs then followed from the explicit Green's function.

From here onward the viewpoint changes. Instead of relying on an explicit propagator, we will systematically construct an operator formalism in which OPEs can be obtained directly from symmetry and representation theory. This is a qualitatively new approach: even when the Green's function is not accessible by solving the equations of motion or performing the path integral, the operator method still allows us to determine the structure of OPEs and, in turn, recover the correlators themselves.

The Hilbert space is what makes this possible. Once a Hilbert space is defined, local fields are no longer only insertions in correlation functions — they correspond to states, and operators act as maps on this space. This representation turns the OPE into an actual operator identity, rather than a mnemonic for propagator singularities. The OPE then expresses how the action of one operator near another decomposes into the basis of states, giving us algebraic control that was absent in the purely path-integral description.

To set this up, we must specify a notion of "time," since operators evolve with respect to it. In Euclidean space, the natural choice is to take the radial coordinate as time, leading to radial quantization. States are then defined on concentric circles, time evolution is dilation, and contour integrals of the stress tensor implement the Virasoro algebra. Within this framework, commutators appear as contour manipulations, and the OPE becomes a universal computational tool, independent of explicit Green's functions.

4.1 Radial quantization

In the operator formalism one must first distinguish a time direction from a space direction. In Minkowski spacetime this choice is canonical, but in Euclidean space it is arbitrary. A Hilbert space by itself is just a complete inner product space; what makes it physically meaningful is the specification of a **time direction**, which singles out a Hamiltonian as the generator of time translations. The Hamiltonian defines a **vacuum state** (its lowest-energy eigenstate) and organizes the remaining states as excitations built on top of it.

This structure underlies any quantum theory: begin with a vacuum, generate excitations, and classify states by the Hamiltonian spectrum. In two-dimensional conformal field theory we can choose a different notion of "time": the **radial direction** from the origin. To motivate this further, consider $\mathbb{R}^d$ in spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega_{d-1} = r^2 \left[\frac{dr^2}{r^2} + d\Omega_{d-1}\right]$$

Now let $t = \log(r)$ so that

$$ds^2 = e^{2t}[dt^2 + d\Omega_{d-1}]$$

which is conformally related to the metric on $\mathbb{R} \times S^{d-1}$. Now if we consider a CFT on $\mathbb{R}^d$, the theory should be invariant under rescaling of metric, so studying that theory on $\mathbb{R}^d$ should be equivalent to studying the theory on $\mathbb{R} \times S^{d-1}$.

From a Minkowski space point of view (in particular in the context of string theory), we will initially define our theory on an infinite space-time cylinder, with time t going from $-\infty$ to $+\infty$ along the "flat" direction of the cylinder, and space being compactified with x going from 0 to L, and the point $(0,t)$ and (L,t) being identified. If we continue to Euclidian space, the cylinder is described by a single complex coordinate

$$w = t + ix, \qquad x \sim x + L,$$

with Hamiltonian $H = -i\partial_t$. We can then consider the conformal map of this cylinder to the plane via

$$z = e^{\frac{2\pi}{L}w}.$$

Here $t \to -\infty$ maps to $z = 0$, $t \to +\infty$ to $z = \infty$, and the compact x direction gives angular periodicity. A very interesting feature of this map is that it takes circles of constant radius in $\mathbb{R}^2$ to constant t slices on $\mathbb{R} \times S$. As a consequence, the dilatation operator on $\mathbb{R}^2$ which maps circles onto circles with different radius, corresponds to time translation on $\mathbb{R} \times S$, so it behave as Hamiltonian.

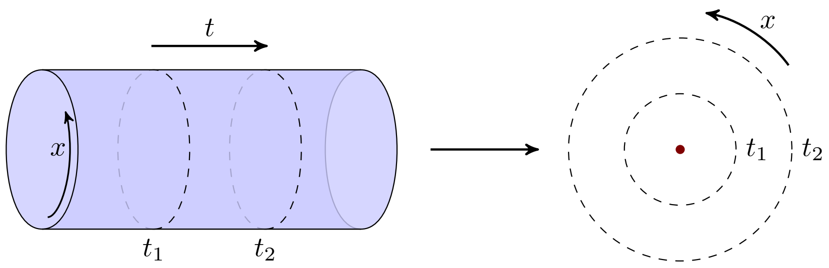


Figure 4.1: Conformal map from the cylinder to the complex plane.

Since periodicity in x is reflected in the periodicity of w and $\bar{w}$. All operators on cylinder admit the following expansion:

$$\phi(w,\bar{w}) = \sum_{m,n\in\mathbb{Z}} \phi_{n,m} e^{-mw} e^{-n\bar{w}} \tag{4.1}$$

where $\phi_{n,m}$ do not have any $(w,\bar{w})$ dependence. Consider the transformation of primary field living on cylinder to $z-$plane under the conformal mapping $z = e^w$:

$$\begin{aligned}
\Phi(z,\bar{z}) &= (\partial_z w)^h (\partial_{\bar{z}}\bar{w})^{\bar{h}} \Phi(w,\bar{w}) \\
&= z^{-h}\bar{z}^{-\bar{h}} \sum_{m,n\in\mathbb{Z}} \phi_{n,m} e^{-mw} e^{-n\bar{w}} \\
&= z^{-h}\bar{z}^{-\bar{h}} \sum_{m,n\in\mathbb{Z}} \phi_{m,m} z^{-n}\bar{z}^{-m} \\
&= \sum_{m,n} \phi_{n,m} z^{-n-h}\bar{z}^{-m-\bar{h}}
\end{aligned}$$

In free-field theories the vacuum is annihilated by positive-frequency modes; in interacting theories we assume asymptotic fields behave freely, e.g.

$$\phi_{\text{in}} \propto \lim_{t\to-\infty} \phi(x,t).$$

In radial quantization, this translates into

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle\,.$$

Expanding the operator in modes, we have

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \sum_{m,n} \phi_{n,m}\, z^{-n-h}\,\bar{z}^{-m-\bar{h}}\,|0\rangle\,.$$

For this expression to be non-singular as $z \to 0$, the modes with negative powers of z and $\bar{z}$ must annihilate the vacuum. This requires

$$n + h > 0 \;\Rightarrow\; n > -h, \qquad m + \bar{h} > 0 \;\Rightarrow\; m > -\bar{h},$$

so that

$$\phi_{n,m}\left|0\right\rangle = 0 \qquad \text{for } n > -h \text{ and } m > -\bar{h}.$$

Thus, only the non-negative powers contribute. For $n + h < 0$ and $m + \bar{h} < 0$, we have

$$\lim_{z,\bar{z}\to 0} z^{-n-h}\bar{z}^{-m-\bar{h}}\left|0\right\rangle = 0$$

The only non-trivial contribution comes from the modes with

$$n + h = 0, \qquad m + \bar{h} = 0,$$

yielding

$$\left|\phi_{\text{in}}\right\rangle = \phi_{-h,-\bar{h}}\left|0\right\rangle.$$

Once the Hilbert space is tied to a vacuum and a Hamiltonian (here, the dilatation operator), operators can be classified by scaling dimensions, excitations arranged systematically, and operator product expansions formulated as exact operator identities—rather than being inferred indirectly from correlation functions.

The Hermitian product

On this Hilbert space we must also define a bilinear product, which we do indirectly by defining an asymptotic "out" state, together with the action of Hermitian conjugation on conformal fields.

$$\left\langle\phi_{\text{out}}\right| = \left|\phi_{\text{in}}\right\rangle^\dagger = \left(\lim_{z,\bar{z}\to 0}\phi(z,\bar{z})\left|0\right\rangle\right)^\dagger$$

In Minkowski space, Hermitian conjugation does not affect the space-time coordinates. Things are different in Euclidian space, since the Euclidian time $\tau = it$ must be reversed ($\tau \to -\tau$) upon Hermitian conjugation if t is to be left unchanged. In radial quantization this corresponds to the mapping:

$$\begin{aligned} w \equiv e^{\tau+ix} &\to e^{-\tau+ix} \\ &= e^{-(\tau-ix)} = \frac{1}{\overline{e^{\tau+ix}}} = \frac{1}{\bar{z}} \end{aligned}$$

Since ϕ is a primary field,

$$\begin{aligned} \phi(z,\bar{z}) &= \left(\frac{\partial w}{\partial z}\right)^h\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}}\phi(w,\bar{w}) \\ &= \left(-\frac{1}{z^2}\right)^h\left(-\frac{1}{\bar{z}^2}\right)^{\bar{h}}\phi(w,\bar{w}) \\ &= (-1)^h(w^{2h})(-1)^{-\bar{h}}(\bar{w}^{2\bar{h}})\phi(w,\bar{w}) \\ &= (-1)^{h+\bar{h}}w^{2h}\bar{w}^{2\bar{h}}\phi(w,\bar{w}) \\ \phi(w,\bar{w}) &= (-1)^{h+\bar{h}}w^{-2h}\bar{w}^{-2\bar{h}}\phi(z,\bar{z}) \end{aligned}$$

Since $\left|\phi_{\text{in}}\right\rangle$ were defined using $\lim\limits_{z\to 0}$, so $\lim\limits_{w\to 0}$ could be used to define $\left|\phi_{\text{out}}\right\rangle$ state:

$$\left\langle\phi_{\text{out}}\right| = \lim_{z,\bar{z}\to\infty}\left\langle 0\right|\phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) == \lim_{w,\bar{w}\to 0}(-1)^{h+\bar{h}}w^{-2h}\bar{w}^{-2\bar{h}}\left\langle 0\right|\phi\left(\frac{1}{w},\frac{1}{\bar{w}}\right)$$

relabelling $w = z$,

$$\left\langle\phi_{\text{out}}\right| = \lim_{z,\bar{z}\to 0}(-1)^{h+\bar{h}}z^{-2h}\bar{z}^{-2\bar{h}}\left\langle 0\right|\phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right)$$

The factor $(-1)^{h+\bar{h}}$ is coming due to spin, if we ignore that:

$$\left\langle\phi_{\text{out}}\right| = \lim_{z,\bar{z}\to 0}z^{-2h}\bar{z}^{-2\bar{h}}\left\langle 0\right|\phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) = \lim_{z,\bar{z}\to 0}\left\langle 0\right|\phi(z,\bar{z})^\dagger$$

Hence, we conclude

$$\phi(z,\bar{z})^\dagger = z^{-2h}\bar{z}^{-2\bar{h}}\phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) \tag{4.2}$$

Or we could alternatively do Fourier expansion on the radial plane, the adjoint property then reads

$$\begin{aligned}\Phi(z,\bar{z})^\dagger &= \bar{z}^{-2h} z^{-2\bar{h}} \Phi(1/\bar{z}, 1/z) \\ &= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n} \phi_{n,m} \left(\frac{1}{\bar{z}}\right)^{-n-h} \left(\frac{1}{z}\right)^{-m-\bar{h}} \\ &= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n} \phi_{n,m} (\bar{z})^{n+h} (z)^{m+\bar{h}} \\ &= \sum_{m,n} \phi_{n,m} z^{m-\bar{h}} \bar{z}^{n-h}\end{aligned}$$

$$\sum_{m,n} \phi^\dagger_{n,m} \overline{(z^{-n-h}\bar{z}^{-m-\bar{h}})} = \sum_{m,n} \phi_{-n,-m} z^{-m-\bar{h}} \bar{z}^{-n-h} \implies \boxed{\phi^\dagger_{n,m} = \phi_{-n,-m}}$$

The inner product is well defined as well:

$$\begin{aligned}\langle \phi_{\text{out}} | \phi_{\text{in}} \rangle &= \lim_{z,\bar{z},w,\overline{w}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0 | \phi(1/\bar{z}, 1/z) \phi(w,\overline{w}) | 0 \rangle \\ &= \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0 | \phi(\bar{\xi}, \xi) \phi(0,0) | 0 \rangle \\ &= \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \frac{C_{12}}{\bar{\xi}^{2h} \xi^{2\bar{h}}} = C_{12}\end{aligned}$$

Unless the prefactors in (4.2) were missing the limit would have been ill defined.

4.1.1 Radial Ordering

Within radial quantization, the time ordering that appears in the definition of correlation functions becomes a radial ordering,

$$\mathcal{R}(\phi_1(z)\phi_2(w)) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w| \,, \\ \phi_2(w)\phi_1(z) & |z| < |w| \,. \end{cases} \tag{4.3}$$

As usual, we will always omit the radial-ordered operator in the correlation function as well as in the OPE expansion. One consequence after specifying time direction is that we can relate OPE to commutation relations. For this, let us consider the contour integral around w for two holomorphic fields $a(z)$ and $b(w)$. If the contour is not radially ordered over the whole path, we can decompose it into contributions that are radially ordered:

$$\oint_w dz\, a(z)b(w) \;=\; \underbrace{\oint_{|z|>|w|} dz\, a(z)b(w)}_{\text{radial ordering: } z \text{ outside } w} \;-\; \underbrace{\oint_{|z|<|w|} dz\, b(w)a(z)}_{\text{radial ordering: } z \text{ inside } w} \;=\; [A,\, b(w)]\,, \tag{4.4}$$

where the operator A is the contour integral of $a(z)$ at a fixed time

$$A = \oint \mathrm{d}z\, a(z)\,. \tag{4.5}$$

Here we take the contours C_1 and C_2 at fixed-radius $|w| + \epsilon$ and $|w| - \epsilon$ with a small positive number ϵ as illustrated in Figure 4.2 . Then, with $B = \oint \mathrm{d}z\, b(z)$, we can generalize the relation (**??**) to

$$[A, B] = \oint_0 \mathrm{d}w [A, b(w)] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z a(z) b(w)\,. \tag{4.6}$$

4.2 Virasoro Algebra

4.2.1 Conformal Generators

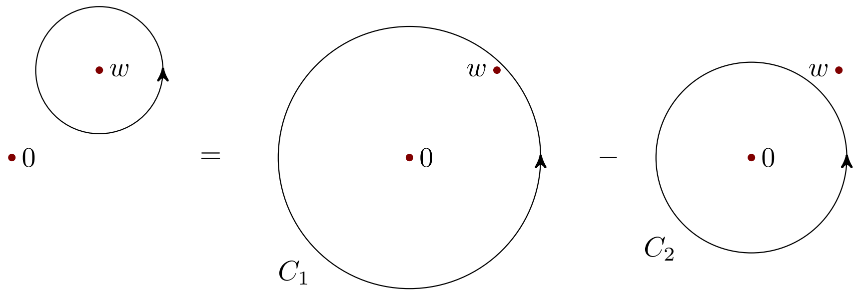
The ward identity

$$\delta_\epsilon \langle \phi_1 \dots \phi_n \rangle = \int_M d^2x \partial_\mu \langle T^{\mu\nu} \epsilon_\nu \phi_1 \dots \phi_n \rangle$$

could be expressed in complex coordinates using (3.7):

$$\delta_{\epsilon,\bar{\epsilon}} \langle \phi_1 \dots \phi_n \rangle = \frac{i}{2} \int_C \left[-dz \left\langle T^{\bar{z}\bar{z}} \epsilon_{\bar{z}} \phi_1 \dots \phi_n \right\rangle + d\bar{z} \left\langle T^{zz} \epsilon_z \phi_1 \dots \phi_n \right\rangle \right]$$

Figure 4.2: Subtraction of contours



$$= -\frac{1}{2\pi i}\int_C dz\,\langle T(z)\epsilon\phi_1\ldots\phi_n\rangle + \frac{1}{2\pi i}\int_C d\bar{z}\,\langle \bar{T}(\bar{z})\bar{\epsilon}\phi_1\ldots\phi_n\rangle$$

where we used,

$$\begin{aligned}T &= -2\pi T_{zz} = -2\pi g_{z\mu}g_{z\nu}T^{\mu\nu} = -2\pi\frac{1}{4}T^{\bar{z}\bar{z}} = -\frac{\pi}{2}T^{\bar{z}\bar{z}}\\ \bar{T} &= -2\pi T_{\bar{z}\bar{z}} = -2\pi g_{\bar{z}\mu}g_{\bar{z}\nu}T^{\mu\nu} = -2\pi\frac{1}{4}T^{zz} = -\frac{\pi}{2}T^{zz}\end{aligned}$$

and

$$\begin{aligned}\epsilon &= \epsilon^z = g^{z\mu}\epsilon_\mu = 2\epsilon_{\bar{z}}\\ \bar{\epsilon} &= \epsilon^{\bar{z}} = g^{\bar{z}\mu}\epsilon_\mu = 2\epsilon_z\end{aligned}$$

If we apply, (4.4) and (4.6) to the conformal ward identity. Let $\epsilon(z)$ be holomorphic component of an infinitesimal conformal change of coordinates. We then define the conformal charge

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z) \tag{4.7}$$

with the help of (4.4), the conformal ward identity translates into

$$\delta_\epsilon\phi = -[Q_\epsilon,\phi]$$

which means that the operator Q_ϵ is the generator of conformal transformation. We may expand energy-momentum tensor according to (4.1):

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \qquad L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T(z)$$

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n \qquad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+1}\bar{T}(\bar{z})$$

we may also expand the infinitesimal conformal change $\epsilon(z)$ as:

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n$$

Then the expression, (4.7) becomes:

$$\begin{aligned}Q_\epsilon &= \frac{1}{2\pi i}\oint dz\left(\sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n\right)T(z)\\ &= \sum_{n\in\mathbb{Z}}\epsilon_n\frac{1}{2\pi i}\oint dz z^{n+1}T(z)\\ &= \sum_{n\in\mathbb{Z}}\epsilon_n L_n\end{aligned}$$

The mode operators L_n and L_m of the energy-momentum tensor are the generators of the local conformal transformations on the Hilbert space, exactly like l_n and l_m of Witt algebra. The next part is to find the algebra obeyed by L_n and L_m which we described in previous chapter as the central extension of Witt algebra. We will now prove that here:

$$\begin{aligned}
[L_m, L_n] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\, z^{m+1} w^{n+1} [T(z), T(w)] \\
&= \oint \frac{dw}{2\pi i}\, w^{n+1} \oint_{C(w)} \frac{dz}{2\pi i}\, z^{m+1} \mathcal{R}\big(T(z)T(w)\big) \\
&= \oint \frac{dw}{2\pi i}\, w^{n+1} \oint_{C(w)} \frac{dz}{2\pi i}\, z^{m+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}\right] \\
&= \oint \frac{dw}{2\pi i}\, w^{n+1} \left[(m+1)m(m-1)w^{m-2}\frac{c}{2\cdot 3!} + 2(m+1)w^m T(w) + w^{m+1}\partial_w T(w)\right] \\
&= \oint \frac{dw}{2\pi i} \left[\frac{c}{12}(m^3-m)\, w^{m+n-1} + 2(m+1)w^{m+n+1}T(w) + w^{m+n+2}\partial_w T(w)\right] \\
&= \oint \frac{dw}{2\pi i} \left[\frac{c}{12}(m^3-m)\, w^{m+n-1} + 2(m+1)w^{m+n+1}T(w) + \partial_w\{w^{m+n+2}T(w)\} - \partial_w w^{m+n+2} T(w)\right] \\
&= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + 2(m+1)L_{m+n} + 0 - \oint \frac{dw}{2\pi i}\,(m+n+2)T(w)w^{m+n+1} \\
&= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + 2(m+1)L_{m+n} - (m+n+2)L_{m+n} \\
&= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}.
\end{aligned}$$

Collectively,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n} \\
[L_m, \bar{L}_n] &= 0 \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}
\end{aligned}$$

4.3 The Hilbert Space

The vacuum state $|0\rangle$ in CFT must be invariant under global conformal transformation. This means the generator of transformation

$$\frac{az+b}{cz+d}$$

which are $L_{\pm 1}$ and L_0 must annihilate the vacuum. This, in turn is recovered from the condition that $T(z)\,|0\rangle$ and $\bar{T}(\bar{z})\,|0\rangle$ are well defined as $z, \bar{z} \to 0$,

$$\begin{aligned}
\lim_{z\to 0} T(z)\,|0\rangle &= \lim_{z\to 0} \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\,|0\rangle \\
\lim_{\bar{z}\to 0} \bar{T}(\bar{z})\,|0\rangle &= \lim_{\bar{z}\to 0} \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n\,|0\rangle
\end{aligned}$$

Here, $z^{-(n+2)}$ diverges at $z=0$ if $n+2>0$, i.e. if $n>-2$, or equivalently $n \geq -1$. Which implies, we need

$$\begin{aligned}
L_n\,|0\rangle &= 0 \\
\bar{L}_n\,|0\rangle &= 0
\end{aligned} \qquad \text{(for } n \geq -1\text{)}$$

This includes a subcondition regarding invariance of $|0\rangle$ under global conformal group. It also implies the vanishing of the vacuum expectation value of the energy-momentum tensor:[1]

$$\langle 0|\,T(z)\,|0\rangle = \left(\sum_{n=-\infty}^{-1} z^{-n-2}\,\langle 0|\,L_n\right)|0\rangle + \sum_{n=0}^{\infty} z^{-n-2}\,\langle 0|\,(L_n\,|0\rangle) = 0$$

[1]If the vacuum is not invariant under local conformal transformation i.e. if vacuum is not annihilated by L_n for $n > 1$, then we can not ensure the vanishing of vev of energy-momentum tensor.

$$\langle 0|\bar{T}(\bar{z})|0\rangle = \left(\sum_{n=-\infty}^{-1} z^{-n-2}\langle 0|\bar{L}_n\right)|0\rangle + \sum_{n=0}^{\infty} z^{-n-2}\langle 0|\left(\bar{L}_n|0\rangle\right) = 0$$

Primary fields, when acting on vacuum, create asymptotic states, eigenstates of the Hamiltonian. A simple demonstration follows from the OPE (3.16) between $T(z)$ and a primary field $\phi(z,\bar{z})$ of dimension $(h,\bar{h})$, translated into operator language:

$$\begin{aligned}
[L_n, \phi(w,\bar{w})] &= \frac{1}{2\pi i}\oint_w dz\, z^{n+1} T(z)\phi(w,\bar{w}) \\
&= \frac{1}{2\pi i}\oint_w dz\, z^{n+1}\left[\frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{z-w} + \text{reg.}\right] \\
&= h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \qquad (n \geq -1)
\end{aligned}$$

The antiholomorphic counterpart of this relation is

$$[\bar{L}_n, \phi(w,\bar{w})] = \bar{h}(n+1)\bar{w}^n\phi(w,\bar{w}) + \bar{w}^{n+1}\bar{\partial}\phi(w,\bar{w}) \qquad (n \geq -1)$$

After applying these relations to the asymptotic state

$$|h,\bar{h}\rangle \equiv \phi(0,0)|0\rangle,$$

we conclude that

$$\begin{aligned}
L_0\left|h,\bar{h}\right\rangle &= L_0\phi(0,0)|0\rangle \\
&= \phi(0,0)L_0|0\rangle + [L_0, \phi(0,0)]|0\rangle \\
&= 0 + h\phi(0,0)|0\rangle + 0 = h\left|h,\bar{h}\right\rangle
\end{aligned}$$

compactly,

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle, \qquad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle.$$

Thus $|h,\bar{h}\rangle$ is an eigenstate of the Hamiltonian. Likewise, we have

$$L_n|h,\bar{h}\rangle = 0, \qquad \bar{L}_n|h,\bar{h}\rangle = 0, \qquad \text{if } n > 0.$$

Excited states above the asymptotic state $\left|h,\bar{h}\right\rangle$ may be obtained by applying ladder operators. The generators $L_{-m}(m>0)$ increases the eigenstate of L_0 a.k.a conformal weight by virtue of virasoro algebra

$$[L_0, L_{-m}] = (0+m)L_{0-m} = mL_{-m}$$

This means the excited states may be obtained by successive applications of these operators on the asymptotic state $|h\rangle$:

$$L_{-k_1}L_{-k_2}\dots L_{-k_n}|0\rangle \qquad (1 \geq k_1 \geq \dots \geq k_n)$$

This state is the eigenstate of L_0 with eigenvalue

$$h' = h + k_1 + k_2 + \dots + k_n = h + N$$

and these are called descendants of the asymptotic state $|h\rangle$ and the integer N is called the level of the descendant.

4.4 Free Boson

This section gives a detailed account of the canonical quantization of the free boson on the cylinder. The mode expansions are obtained, after imposing the appropriate boundary conditions. The mapping from the cylinder to the complex plane is used to define the conformal generators and, in particular, the vacuum energies. Free-field theories are of special importance not only because they can be solved explicitly, but also because they are the building blocks of more complicated models, or can be shown to be equivalent to interesting statistical models.

Mode Expansion

We let $\varphi(x,t)$ be a free Bose field defined on a cylinder of circumference L:

$$\varphi(x+L,t) \equiv \varphi(x,t)$$

This field may be Fourier expanded as follows:

$$\varphi(x,t) = \sum_n e^{2\pi inx/L}\,\varphi_n(t),$$

with Fourier coefficients

$$\varphi_n(t) = \frac{1}{L}\int_0^L dx\, e^{-2\pi inx/L}\,\varphi(x,t).$$

Next, we try to find the Fourier coefficients of kinetic part.

$$\begin{aligned}
\partial_t\varphi(x,t) &= \sum_n e^{2\pi inx/L}\,\dot\varphi_n(t),\\
(\partial_t\varphi)^2 &= \sum_{n,m} e^{2\pi i(n+m)x/L}\,\dot\varphi_n\dot\varphi_m.
\end{aligned}$$

Only the $m=-n$ term survives the integral over x, so

$$\boxed{\int_0^L dx\,(\partial_t\varphi)^2 = L\sum_n \dot\varphi_n\dot\varphi_{-n}} \tag{4.8}$$

Similarly,

$$\begin{aligned}
\partial_x\varphi(x,t) &= \sum_n \frac{2\pi in}{L} e^{2\pi inx/L}\,\varphi_n(t),\\
(\partial_x\varphi)^2 &= \sum_{n,m} \frac{2\pi in}{L}\frac{2\pi im}{L} e^{2\pi i(n+m)x/L}\,\varphi_n\varphi_m.
\end{aligned}$$

$$\boxed{\int_0^L dx\,(\partial_x\varphi)^2 = L\sum_n \left(\frac{2\pi n}{L}\right)^2 \varphi_n\varphi_{-n}} \tag{4.9}$$

Putting them in the free field Lagrangian

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}g\int_0^L dx\,\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right]\\
&= \frac{gL}{2}\sum_n \left(\dot\varphi_n\dot\varphi_{-n} - \left(\frac{2\pi n}{L}\right)^2 \varphi_n\varphi_{-n}\right).
\end{aligned}$$

The momentum conjugate to φ_n is

$$\pi_n = \frac{\partial\mathcal{L}}{\partial\dot\varphi_n} = gL\,\dot\varphi_{-n} \qquad\qquad [\varphi_n,\pi_m] = i\delta_{nm}.$$

The Hamiltonian is

$$\begin{aligned}
H &= \sum_n \pi_n\dot\varphi_n - \mathcal{L}\\
&= \frac{1}{2gL}\sum_n \left(\pi_n\pi_{-n} + (2\pi ng)^2\,\varphi_n\varphi_{-n}\right).
\end{aligned}$$

Thus the system reduces to a sum of decoupled harmonic oscillators with frequencies $\omega_n = \frac{2\pi|n|}{L}$. The usual procedure is to define creation and annihilation operators $\tilde a_n$ and $\tilde a_n^\dagger$:

$$\tilde a_n = \frac{1}{\sqrt{4\pi g|n|}}(2\pi g|n|\phi_n + i\pi_{-n}) \tag{4.10}$$

such that $\left[\tilde{a}_n, \tilde{a}^\dagger_m\right] = \delta_{n,m}$ and $[\tilde{a}_n, \tilde{a}_m] = 0$. This ofcourse does not work for zero mode ϕ_0. Instead we shall use the following operators:

$$a_n = \begin{cases} -i\sqrt{n}\,\tilde{a}_n & (n>0) \\ i\sqrt{-n}\,\tilde{a}^\dagger_{-n} & (n<0) \end{cases}, \qquad \bar{a}_n = \begin{cases} -i\sqrt{n}\tilde{a}_{-n} & (n>0) \\ i\sqrt{-n}\tilde{a}^\dagger_n & (n<0) \end{cases}$$

and we treat the zero mode ϕ_0 separately. The associated commutation relations are

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_n, \tilde{a}_m] = 0, \quad [\tilde{a}_n, \tilde{a}_m] = n\delta_{n+m,0} \tag{4.11}$$

The Hamiltonian is then expressible as

$$H = \frac{1}{2gL}\sum_n \left(\pi_n\pi_{-n} + (2\pi gn)^2\varphi_n\varphi_{-n}\right) = \frac{\pi_0^2}{2gL} + \frac{(2\pi g\,0)^2}{2gL}\phi_0^2 + \frac{1}{2gL}\sum_{n\neq 0}\left(\pi_n\pi_{-n} + (2\pi gn)^2\varphi_n\varphi_{-n}\right) \tag{4.12}$$

We start with the oscillator operators:

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi gn}}(2\pi gn\,\varphi_n + i\pi_{-n}), \quad \tilde{a}^\dagger_n = \frac{1}{\sqrt{4\pi gn}}(2\pi gn\,\varphi_{-n} - i\pi_n).$$

$$\tilde{a}_{-n} = \frac{1}{\sqrt{4\pi gn}}(2\pi gn\,\varphi_{-n} + i\pi_n), \quad \tilde{a}^\dagger_{-n} = \frac{1}{\sqrt{4\pi gn}}(2\pi gn\,\varphi_n - i\pi_{-n}).$$

$$4\pi gn\,\tilde{a}^\dagger_n\tilde{a}_n = (2\pi gn)^2\varphi_{-n}\varphi_n + \pi_n\pi_{-n} + i[\text{cross terms}],$$

$$4\pi gn\,\tilde{a}^\dagger_{-n}\tilde{a}_{-n} = (2\pi gn)^2\varphi_n\varphi_{-n} + \pi_{-n}\pi_n - i[\text{same cross terms}],$$

adding above equations,

$$\overset{2}{\cancel{4}}\pi gn(\tilde{a}^\dagger_n\tilde{a}_n + \tilde{a}^\dagger_{-n}\tilde{a}_{-n}) = \cancel{2}[(2\pi gn)^2\varphi_n\varphi_{-n} + \pi_n\pi_{-n}] \implies 2\pi gn(\tilde{a}^\dagger_n\tilde{a}_n + \tilde{a}^\dagger_{-n}\tilde{a}_{-n}) = \pi_n\pi_{-n} + (2\pi gn)^2\varphi_n\varphi_{-n}$$

Substituting it in (4.12),

$$H = \frac{1}{2gL}\pi_0^2 + \frac{\pi}{L}\sum_{n\neq 0} n(\tilde{a}^\dagger_n\tilde{a}_n + \tilde{a}^\dagger_{-n}\tilde{a}_{-n}).$$

$$a_n = -i\sqrt{n}\tilde{a}_n, \quad a_{-n} = a^\dagger_n = i\sqrt{n}\tilde{a}^\dagger_n, \qquad \bar{a}_n = -i\sqrt{n}\tilde{a}_{-n}, \quad \bar{a}_{-n} = \bar{a}^\dagger_n = i\sqrt{n}\tilde{a}^\dagger_{-n} \qquad (n>0).$$

So

$$n\tilde{a}^\dagger_n\tilde{a}_n = (i\sqrt{n}\tilde{a}^\dagger_n)(-i\sqrt{n}\tilde{a}_n) = a_{-n}a_n, \quad n\tilde{a}^\dagger_{-n}\tilde{a}_{-n} = (i\sqrt{n}\tilde{a}^\dagger_{-n})(-i\sqrt{n}\tilde{a}_{-n}) = \tilde{a}_{-n}\tilde{a}_n.$$

$$\boxed{H = \frac{1}{2gL}\pi_0^2 + \frac{\pi}{L}\sum_{n\neq 0}(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n)}. \tag{4.13}$$

The commutation relation (4.11), leads to

$$\begin{aligned} [H, a_{-m}] &= \frac{\pi}{L}\sum_{n\neq 0}[a_{-n}a_n,\, a_{-m}] \\ &= \frac{\pi}{L}\sum_{n\neq 0}\Big(a_{-n}[a_n, a_{-m}] + [a_{-n}, a_{-m}]a_n\Big) \\ &= \frac{\pi}{L}\sum_{n\neq 0}\Big(a_{-n}\,n\,\delta_{n-m,0} + (-n)\,\delta_{-n-m,0}\,a_n\Big) \\ &= \frac{2\pi}{L}m\,a_{-m} = -i\dot{a} \end{aligned}$$

which means for $a_{-m}(m>0)$, when applied to an eigenstate of H of energy E, produces another eigenstate with energy $E + 2m\pi/L$. Since the fourier modes are

$$\varphi_n = \frac{i}{n\sqrt{4\pi g}}(a_n - \bar{a}_{-n})$$

The mode expansion at $t = 0$ may be written as:

$$\varphi(x) = \varphi_0 + \sum_{n\neq 0} \varphi_n e^{\frac{2\pi i n x}{L}} = \varphi_0 + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n} (a_n - \tilde{a}_{-n}) e^{2\pi i n x/L}$$

The time evolution of the operators ϕ_0, a_n and $\bar{a}_n$ in the Heisenberg picture follow from the above Hamiltonian:

$$[H, \phi_0] = \frac{1}{2gL}[\pi_0^2, \phi_0] = -\frac{\pi_0[\phi_0, \pi_0] + [\phi_0, \pi_0]\pi_0}{2gL} = -\frac{i}{gL}\pi_0 \qquad \text{(using } [\varphi_0, \pi_0] = i\text{)}$$

Hence,

$$\begin{aligned} \varphi_0(t) &= \varphi_0(0) + \frac{1}{gL}\pi_0 t, \\ a_n(t) &= a_n(0)\, e^{-2\pi i n t/L}, \\ \tilde{a}_n(t) &= \tilde{a}_n(0)\, e^{-2\pi i n t/L} \end{aligned}$$

In terms of constant operators, the mode expansion of the field at arbitrary time is then,

$$\varphi_{\text{cyl}} = \varphi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n} \left(a_n e^{-2\pi i n(t-x)/L} + \overline{a}_n e^{-2\pi i n(t+x)/L} \right) \tag{4.14}$$

where 'cyl' means the field defined on the cylinder.

Relation between Cylinder and Plane

Now, we move to Euclidean space-time ($t = -i\tau$) by taking $w = \tau - ix$ and $\overline{w} = \tau + ix$. The mode expansion of boson field on the cylinder now is then

$$\varphi_{\text{cyl}}(w, \overline{w}) = \varphi_0 - i\frac{1}{2gL}\pi_0(w + \overline{w}) + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n} \left(a_n e^{-2\pi n w/L} + \overline{a}_n e^{-2\pi n \overline{w})/L} \right), \tag{4.15}$$

with now $w \sim w + iL$. Using a conformal transformation as in Figure 4.1, we map all the operators from the cylinder to the complex plane:

$$z = e^{2\pi w/L}, \qquad \overline{z} = e^{2\pi \overline{w}/L}$$

we finally obtain the expansion by simply replacing w by z.

$$\varphi_{\text{pl}}(z, \overline{z}) = \varphi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\overline{z}) + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n}(a_n z^{-n} + \overline{a}_n \overline{z}^{-n}). \tag{4.16}$$

The propagator could be calculated by assuming $|z| > |z'|$ and considering the product

$$\begin{aligned} \varphi_{\text{pl}}(w, \bar{w})\, \varphi_{\text{pl}}(w', \bar{w}') \;=\; & \Big[\varphi_0 - i\frac{1}{4\pi g}\pi_0 \ln|z|^2 + i\frac{1}{\sqrt{4\pi g}} \sum_{m\neq 0} \frac{1}{m}(a_m z^{-m} + \overline{a}_m \overline{z}^{-m})\Big] \\ & \times \Big[\varphi_0 - i\frac{1}{4\pi g}\pi_0 \ln|z'|^2 + i\frac{1}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n}(a_n z'^{-n} + \overline{a}_n \overline{z}'^{-n})\Big]. \end{aligned}$$

We will move annihilation operators ($a_k, \overline{a}_k$ with $k > 0$) and π_0 to the right to reach normal ordering; non-vanishing c-number contributions come from $[\varphi_0, \pi_0] = i$ and $[a_m, a_n] = m\delta_{m+n,0}$, $[\overline{a}_m, \overline{a}_n] = m\delta_{m+n,0}$. Let's first focus on these terms

$$\begin{aligned} &\Big(\varphi_0 - i\frac{1}{4\pi g}\pi_0 \ln|z|^2\Big)\Big(\varphi_0 - i\frac{1}{4\pi g}\pi_0 \ln|z'|^2\Big) \\ &= \varphi_0^2 - i\frac{1}{4\pi g}\big(\varphi_0\pi_0 \ln|z'|^2 + \pi_0\varphi_0 \ln|z|^2\big) - \Big(\frac{1}{4\pi g}\Big)^2 \pi_0^2 \ln|z|^2 \ln|z'|^2 \\ &= \varphi_0^2 - i\frac{1}{4\pi g}\big(\varphi_0\pi_0 \ln|z'|^2 + (\varphi_0\pi_0 - i) \ln|z|^2\big) - \Big(\frac{1}{4\pi g}\Big)^2 \pi_0^2 \ln|z|^2 \ln|z'|^2 \\ &= \varphi_0^2 - i\frac{1}{4\pi g}\varphi_0\pi_0\big(\ln|z'|^2 + \ln|z|^2\big) - \frac{1}{4\pi g}\ln|z|^2 - \Big(\frac{1}{4\pi g}\Big)^2 \pi_0^2 \ln|z|^2 \ln|z'|^2 \\ &= \underbrace{\varphi_0^2 - i\frac{1}{4\pi g}\varphi_0\pi_0\big(\ln|z'|^2 + \ln|z|^2\big) - \Big(\frac{1}{4\pi g}\Big)^2 \pi_0^2 \ln|z|^2 \ln|z'|^2}_{\text{normal ordered}} - \frac{1}{4\pi g}\ln|z|^2 \end{aligned}$$

$$=:\Big(\varphi_0 - i\frac{1}{4\pi g}\pi_0 \ln|z|^2\Big)\Big(\varphi_0 - i\frac{1}{4\pi g}\pi_0 \ln|z'|^2\Big): \;-\; \frac{1}{4\pi g}\ln|z|^2,$$

The zero-mode reordering produces the c-number $-\frac{1}{4\pi g}\ln|z|^2$. Next, we'll consider holomorphic oscillator pieces

$$i\frac{1}{\sqrt{4\pi g}}\sum_{m\neq 0}\frac{1}{m}a_m z^{-m} \quad \text{and} \quad i\frac{1}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}a_n z'^{-n}.$$

We can split the sum in two parts,

$$\sum_{m\neq 0}\frac{1}{m}a_m z^{-m} = \sum_{m=-\infty}^{-1}\frac{1}{m}a_m z^{-m} + \sum_{m=1}^{\infty}\frac{1}{m}a_m z^{-m}.$$

The only non-normal-ordered combination producing a c-number is the product of the $m>0$ part from the first bracket with the $n<0$ part from the second bracket. That term (including prefactors) is

$$\Big(i\frac{1}{\sqrt{4\pi g}}\Big)^2\Big(\sum_{m=1}^{\infty}\frac{1}{m}a_m z^{-m}\Big)\Big(\sum_{n=-\infty}^{-1}\frac{1}{n}a_n z'^{-n}\Big) = -\frac{1}{4\pi g}\sum_{m=1}^{\infty}\sum_{n=-\infty}^{-1}\frac{1}{mn}\,a_m a_n\, z^{-m}z'^{-n}.$$

Using $a_m a_n =: a_n a_m : +[a_m, a_n] =: a_n a_m : +m\delta_{m+n,0}$. The only contributing commutator in the sum over n occurs when $n=-m$. Extracting that:

$$-\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m(-m)}\,m\,z^{-m}z'^{m} = -\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{-1}{m}\Big(\frac{z'}{z}\Big)^m = +\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{z'}{z}\Big)^m.$$

Thus the holomorphic oscillators give the c-number

$$\boxed{\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{z'}{z}\Big)^m}.$$

By identical steps for $\overline{a}_m$ modes we obtain

$$\boxed{\frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{\bar z'}{\bar z}\Big)^m}.$$

Collecting the zero-mode, holomorphic and anti-holomorphic c-number contributions:

$$-\frac{1}{4\pi g}\ln|z|^2 \;+\; \frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{z'}{z}\Big)^m \;+\; \frac{1}{4\pi g}\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{\bar z'}{\bar z}\Big)^m.$$

We can use $-\ln(1-x)=\sum_{m=1}^{\infty}x^m/m$ to simplify the series expansion (valid for $|z'|<|z|$):

$$\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{z'}{z}\Big)^m = -\ln\Big(1-\frac{z'}{z}\Big), \qquad \sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{\bar z'}{\bar z}\Big)^m = -\ln\Big(1-\frac{\bar z'}{\bar z}\Big).$$

So the total c-number becomes

$$-\frac{1}{4\pi g}\ln|z|^2 \;-\; \frac{1}{4\pi g}\ln\Big|1-\frac{z'}{z}\Big|^2 = -\frac{1}{4\pi g}\ln\Big|z\Big(1-\frac{z'}{z}\Big)\Big|^2 = -\frac{1}{4\pi g}\ln|z-z'|^2.$$

Therefore

$$\boxed{\varphi_{\rm pl}(w,\bar w)\varphi_{\rm pl}(w',\bar w') =: \varphi_{\rm pl}(w,\bar w)\varphi_{\rm pl}(w',\bar w') : -\frac{1}{4\pi g}\,\ln|z-z'|^2}$$

Since we know that the normal ordered product vanishes inside correlator, the propagator could be simply read off as:

$$\boxed{\langle\varphi_{\rm pl}(w,\bar w)\,\varphi_{\rm pl}(w',\bar w')\rangle = -\,\frac{1}{4\pi g}\,\ln|z-z'|^2}$$

This is the same as (3.18). From now on, we will drop the subscript "pl." We know that φ is **not** itself a primary field, but its derivatives $\partial\phi$ and $\bar{\partial}\bar{\phi}$ are. Now we concentrate on the holomorphic field $\partial\varphi$

$$i\partial\varphi(z) = \frac{1}{4\pi g}\frac{\pi_0}{z} + \frac{1}{4\pi g}\sum_{n\neq 0} a_n z^{-n-1} = \frac{1}{4\pi g}a_0 z^{0-1} + \frac{1}{4\pi g}\sum_{n\neq 0} a_n z^{-n-1} .$$

We may write the zero modes by a_0 and $\overline{a}_0$:

$$a_0 \equiv \overline{a}_0 \equiv \pi_0$$

Now the commutation relations of (4.11) can be extended to include the zero mode operator without changing the form of the algebra. The mode expansion of $\partial\varphi$ is consistent with the expansion of a primary field with $h = 1$ then becomes:

$$i\partial\varphi(z) = \sum_n a_n z^{-n-1} . \tag{4.17}$$

Often the normalization of $g = \frac{1}{4\pi}$ is used in this kind of work. Using this mode expansion, we can also compute the two-point function of $\partial\varphi$. For ($|z| > |w|$):

$$\langle \varphi(z)\partial\varphi(w)\rangle = \sum_{m,n\neq 0} \frac{1}{n}\langle a_n a_m\rangle z^{-n} w^{-m-1} .$$

According to the commutation relation (4.11) and the fact that a_m and a_{-m} are annihilation and creation operators respectively, it follows that[2]

$$\langle \varphi(z)\partial\varphi(w)\rangle = \sum_{n>0}\sum_{m\neq 0} \frac{n\delta_{n,-m} z^{-n} w^{-m-1}}{n} = \frac{1}{w}\sum_{n>0} ({}^{w}\!/\!_{z})^n = \frac{1}{w}\frac{w/z}{1-w/z} = \frac{1}{z-w} . \tag{4.18}$$

Its differentiation with respect to z provides the two-point function.

$$\langle \partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{(z-w)^2} . \tag{4.19}$$

The holomorphic energy-momentum tensor is given by

$$T(z) = -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) := \frac{1}{2}\sum_{n,m\in\mathbb{Z}} z^{-n-m-2} : a_n a_m : , \tag{4.20}$$

which implies

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_{n-m} a_m : \qquad (n \neq 0)$$
$$L_0 = \sum_{n>0} a_{-n} a_n + \frac{1}{2}a_0^2 .$$

The Hamiltonian (4.13) now can be written as,

$$H = \frac{2\pi}{L}(L_0 + \overline{L}_0) . \tag{4.21}$$

This confirms the role of L_0 and $\overline{L}_0$ as a Hamiltonian. Since we place the free boson on a cylinder of size L, energy is proportional to $\frac{2\pi}{L}$, which is called the **finite-size scaling**. The mode operator a_m plays a similar role to L_m with respect to L_0, because of the commutation relation $[L_0, a_{-m}] = m a_{-m}$. Therefore its effect on the conformal dimension is the same as that of L_m.

By definition, the normal ordering prescription implies that $\langle T(z)\rangle = 0$.[3] We can always define the normal ordering product by subtracting all the singular terms from the OPE. In fact the general expectation value for energy-momentum tensor (3.32) can be written as

$$\langle T(z)\rangle = -\frac{1}{2}\lim_{\epsilon\to 0}\left(\langle \partial\varphi(z+\epsilon)\partial\varphi(z)\rangle + \frac{1}{\epsilon^2}\right) . \qquad (\text{with } g = \tfrac{1}{4\pi})$$

[2] $\langle a_n a_m\rangle = \langle 0|\, a_m a_n \,|0\rangle + n\delta_{n,-m}\langle 0|0\rangle = n\delta_{n,-m}$ for $n > 0$

[3] However, as we will see in the next section, this is not the case for the fermionic theory with anti-periodic boundary condition.

We will see that this relation is useful when we consider the anti-periodic boundary condition later. By plugging in to the two-point function, we see that $\langle T(z)\rangle = 0$, which implies that $(L_0)_{\rm pl}$ vanishes on the vacuum. We now map the theory back to a cylinder $z \to w = \frac{L}{2\pi}\ln z$, by using (3.37):

$$T_{\rm cyl}(w) = \left(\frac{{\rm d}w}{{\rm d}z}\right)^{-2}\left[T_{\rm pl}(z) - \frac{1}{12}\{w,z\}\right] = \left(\frac{L}{2\pi z}\right)^{-2}\left[T_{\rm pl}(z) - \frac{1}{12}\frac{1}{2z^2}\right] = \left(\frac{2\pi}{L}\right)^{2}\left[T_{\rm pl}(z)z^2 - \frac{1}{24}\right] \tag{4.22}$$

where we use the central charge $c = 1$ for the free boson. Taking the expectation value on both sides, we have

$$\langle T_{\rm cyl}\rangle = -\frac{1}{24}\left(\frac{2\pi}{L}\right)^2 . \tag{4.23}$$

This implies

$$L_{0,\rm cyl} = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n - \frac{1}{24} . \tag{4.24}$$

The Hamiltonian is now written as

$$H = \frac{2\pi}{L}(L_{0,\rm cyl} + \overline{L}_{0,\rm cyl}) . \tag{4.25}$$

Actually, in the general case, Hamiltonian of a theory defined on a cylinder with central charge c can be written as

$$H = \frac{2\pi}{L}(L_{0,\rm pl} + \overline{L}_{0,\rm pl} - \frac{c}{12}) . \tag{4.26}$$

Thus, we can infer that the central charge c shows up as the vacuum energy of a theory on a cylinder, and this is one instance of finite-size effects .

4.5 Vertex Operator

Since canonical scaling dimension $(h, \bar{h})$ of the bosonic field φ vanishes, it is possible to construct an infinite variety of local fields related to φ without introducing a scale, namely the so called vertex operators. These operators become relevant in string theory because their action on absolute vacuum creates the conformal vacuum state of the theory. They are indeed primary fields and defined as:

$$\mathcal{V}_\alpha =: e^{i\alpha\varphi(z,\bar{z})} :$$

The normal ordering has the following meaning, in terms of the operators appearing in the mode expansion