Conformal Field Theory

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Chapter 1

Conformal Symmetry

1.1 Introduction

Conformal symmetry is, in essence, the symmetry of **shapes without scales**. In Euclidean geometry, this is often described as "angle-preserving" symmetry. While this definition is correct in a purely spatial setting, it becomes less illuminating once we step into relativistic physics, where time is on equal footing with space. The notion of an "angle" between two events separated in time is not geometrically well-defined in the same way, so we need a more general formulation.

A more robust way to think about conformal transformations is that they are transformations which preserve the metric **up to a local rescaling**. That is, they preserve the light-cone structure of spacetime, and hence the causal relations, but may change distances by position-dependent factors:

$$g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x), \quad \Omega(x) > 0.$$

Here, $\Omega(x)$ is the local scaling factor, and $\Omega(x) = 1$ corresponds to an isometry. This definition works in both Euclidean and Lorentzian settings, and it makes clear why conformal transformations generalize ordinary Poincaré symmetries: they allow stretching of spacetime while preserving angles and null directions.

In flat space, an important special case is the global scaling transformation:

$$x^\mu \;\longrightarrow\; \lambda\, x^\mu,$$

which is not a mere relabeling of coordinates (i.e., not just a diffeomorphism), but a genuine change in the geometry. Under such a transformation, the ratios of lengths along a direction are preserved, and therefore angles are unchanged. This property underlies the term "conformal."

The study of **Conformal Field Theory (CFT)** leverages this symmetry in a very different way from conventional Quantum Field Theory (QFT). In QFT, one typically begins with a Lagrangian and derives correlation functions from the equations of motion and perturbation theory. In CFT, by contrast, the symmetry itself is so constraining that it often determines the form of correlation functions without reference to a specific Lagrangian. This leads naturally to the **conformal bootstrap** program, in which the consistency conditions of conformal symmetry, unitarity, and the operator product expansion are used to solve the theory.

Before we can use these powerful tools, it is important to distinguish conformal transformations from two related but conceptually different ideas: **Weyl rescalings** and **diffeomorphisms**. We will examine these one by one in the next section.

General coordinate invariance (diffeomorphism)

Classical field theories can possess a variety of symmetries. One symmetry we will assume here is **general coordinate invariance**. Using the action principle, this symmetry can be used to show that the energy–momentum tensor is conserved.

In general, the energy–momentum tensor is defined through the variation of the action S under changes in the space–time metric:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}\,.$$

By definition,

$$\delta S = \frac{1}{2}\int d^dx\,\sqrt{g}\,T^{\mu\nu}\,\delta g_{\mu\nu}\,.$$

If the theory is invariant under general coordinate transformations, one can show that

$$(T^{\mu\nu})_{;\nu} = 0\,,$$

where, as usual in general relativity, "$;\nu$" denotes the covariant derivative. In flat coordinates, this condition reduces to

$$\partial_\nu T^{\mu\nu} = 0\,.$$

Weyl invariance

In addition to general coordinate invariance, many field theories possess another powerful symmetry: **Weyl invariance**. While diffeomorphism invariance constrains how the metric responds to arbitrary coordinate changes, Weyl invariance instead concerns how the theory behaves under local rescalings of the metric. Under a Weyl transformation, the metric changes as

$$g_{\mu\nu}(x) \to \Omega(x)\, g_{\mu\nu}(x)\,,$$

or, in infinitesimal form,

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \omega(x)\, g_{\mu\nu}(x)\,.$$

The condition for the action to remain invariant under such a transformation can be expressed in terms of the energy–momentum tensor. Substituting $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}(x)$ into the earlier definition, we find

$$\delta S = \frac{1}{2} \int d^d x\, \sqrt{g}\, T^\mu_{\ \mu}\, \omega(x)\,.$$

Since this must hold for arbitrary functions $\omega(x)$, we conclude that the condition for Weyl invariance is

$$T^\mu_{\ \mu} = 0\,.$$

Thus, just as diffeomorphism invariance implies the covariant conservation of the energy–momentum tensor, Weyl invariance implies that the energy-momentum tensor must be traceless.

Conformal invariance

1.1.1 Conformal Transformations

A *conformal transformation* can be defined as a coordinate transformation that acts on the metric as a Weyl transformation. Consider a general coordinate transformation

$$x \to\ x'\,, \quad x^\mu = f^\mu(x')\,.$$

The metric then transforms as

$$g_{\mu\nu}(x) \to\ g'_{\mu\nu}(x') = \frac{\partial f^\rho}{\partial x'^\mu}\,\frac{\partial f^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}\big(f(x')\big)\,.$$

We now require that the transformed metric be proportional to the original one. Rotations and translations clearly satisfy this condition: they leave the metric unchanged and hence preserve all inner products

$$v \cdot w \ \equiv\ v^\mu g_{\mu\nu} w^\nu\,.$$

They are therefore part of the conformal group. More generally, any coordinate transformation satisfying the above proportionality preserves all angles,

$$\frac{v \cdot w}{\sqrt{v^2\, w^2}}\,,$$

which is the origin of the term "conformal." Later in this chapter we will determine all such transformations explicitly.

If a field theory has a conserved and traceless energy–momentum tensor, it is invariant under both general coordinate transformations and Weyl transformations. Let the action be

$$S \ =\ \int d^d x\, \mathcal{L}\big(\partial_x,\, g_{\mu\nu}(x),\, \phi(x)\big)\,.$$

Here, ϕ denotes any matter field, while the metric $g_{\mu\nu}$ is written separately due to its special rôle. We have also explicitly indicated spacetime derivatives in the Lagrangian.

General coordinate invariance implies

$$S \;=\; S' \;\equiv\; \int d^d x'\, \mathcal{L}\big(\partial_{x'},\, g'_{\mu\nu}(x'),\, \phi'(x')\big),$$

where $g'_{\mu\nu}$ is as given above, and the transformation of ϕ depends on its spin. For a tensor field of rank n one has

$$\phi'_{\mu_1\ldots\mu_n}(x') = \left|\frac{\partial f}{\partial x'}\right|^{\frac{\Delta}{d}} \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}}\, \phi_{\nu_1\ldots\nu_n}\big(f(x')\big), \tag{1.1}$$

where Δ is the scaling dimension of the field. Fields that transform according to Eq. (1.1) under conformal transformations are called *conformal fields*, or equivalently, *primary fields*.

In particular, for a scalar $\phi(x)$ we have simply $\phi'(x') = \phi(f(x'))$. For the derivative of a scalar:

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \;\frac{\partial}{\partial x'^\mu}\phi'(x') = \frac{\partial f^\nu}{\partial x'^\mu}\frac{\partial}{\partial f^\nu}\phi(f(x'))\,,$$

which transforms as a vector. (However, note that nth-order *ordinary* derivatives do not transform as rank-n tensors; this holds only for covariant derivatives.)

If the coordinate transformation $x \to x'$ is of the above type, we can **use Weyl invariance of the action to bring the metric back to its original form**. This yields

$$S = S'' \equiv \int d^d x'\, \mathcal{L}\big(\partial_{x'}, g_{\mu\nu}(f(x')), \phi'(x')\big) = \int d^d x'\, \mathcal{L}\big(\partial_{x'}, g_{\mu\nu}(x), \phi'(x')\big)\,.$$

This is the statement of *conformal symmetry* of the action. We should note that in some cases, an isometry of one metric can act as a conformal transformation for a different metric. In such situations, Weyl rescaling is not needed, since the transformation already preserves the metric under consideration. This happens, for example, in de Sitter (dS) and anti–de Sitter (AdS) spacetimes, where certain isometries correspond to conformal transformations of the induced boundary metric.

If we begin with a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, the background metric remains unchanged under such transformations.[1] This allows us to define conformal transformations for theories in flat space that are not coupled to gravity. We may then ignore general coordinate invariance and start with an action in which no dynamical metric appears.

In this flat-space setting, conformal invariance means that the action is unchanged when we integrate the same Lagrangian (or any scalar physical quantity) written in terms of the transformed fields $\phi'(x')$ over the new coordinates x'.

In $d = 2$ dimensions, this is not really a restriction. A general 2D metric has three independent components: $g_{11}(x)$, $g_{22}(x)$, and $g_{12}(x) = g_{21}(x)$. A general coordinate transformation provides two functions $f^1(x)$ and $f^2(x)$ that can be used to set $g_{12}(x) = 0$ and $g_{11}(x) = \pm g_{22}(x)$ (depending on the signature). The metric can then be written in the form $g(x)\,\eta_{\mu\nu}$, which is called *conformal gauge*. A Weyl transformation can remove the remaining factor $g(x)$, bringing the metric to the form $\eta_{\mu\nu}$.

In more than two dimensions, this procedure does not work in general, so restricting to flat space truly limits us to non-gravitational theories. Even in two dimensions, conformal gauge can be chosen only locally in general, meaning CFT can be applied in coordinate patches, but extra data may be needed to describe the theory globally.

1.2 Infinitesimal Conformal Transformation

The fundamental essence of conformal transformations resides in their infinitesimal form, which serves as a crucial tool for investigating how fields transform under these symmetries. It plays a pivotal role in defining the generator of the conformal group and, subsequently, constraining the set of possible correlators that are compatible with conformal symmetry. Any infinitesimal transformation can be expressed as:

$$x'^\mu = x^\mu + \underset{\substack{\uparrow\\ \text{infinitesimal}}}{\epsilon}{}^{\mu}(x)$$

[1]In perturbation theory, we often describe physics on a perturbed manifold as that of tensor fields living on a background manifold. In this case, the conformal transformation is performed on the background metric, and the change in the perturbation is again dictated by the Killing equation. We first apply the coordinate transformation to the full metric, and then perform a Weyl rescaling to restore the background metric to its original form.

and subsequently,

$$x^\mu = x'^\mu - \epsilon^\mu(x)$$

therefore, the metric transforms like:

$$\begin{aligned}
g'_{\mu\nu} &= \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}}_{\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha(x)} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\
&= \left[\delta^\alpha_\mu - \frac{\partial \epsilon^\alpha(x)}{\partial x'^\mu}\right]\left[\delta^\beta_\nu - \frac{\partial \epsilon^\beta(x)}{\partial x'^\nu}\right] g_{\alpha\beta} \\
&= \delta^\alpha_\mu \delta^\beta_\nu g_{\alpha\beta} - \delta^\beta_\nu \partial_\mu \epsilon^\alpha(x) g_{\alpha\beta} - \delta^\alpha_\mu \partial_\nu \epsilon^\beta(x) g_{\alpha\beta} + \mathcal{O}(\epsilon^2) \\
\Omega(x) g_{\mu\nu} &= g_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu
\end{aligned}$$

In the third step, we used chain rule on $\epsilon^\alpha(x)$ and ignored $\mathcal{O}((\partial\epsilon)^2)$ terms. From the last line, it is reasonable to expect that:

$$\begin{aligned}
g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &= [1 + f(x)] g_{\mu\nu} \\
\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) &= f(x) g_{\mu\nu} \qquad (1.2)
\end{aligned}$$

Contracting Indicies

$$\begin{aligned}
\partial^\mu \epsilon_\mu(x) + \partial^\mu \epsilon_\mu(x) &= f(x) \delta^\mu_\mu \\
2(\partial \cdot \epsilon) &= \underbrace{d}_{\text{dimension of spacetime}}\, f(x) \\
f(x) &= \frac{2}{d} \frac{\partial \epsilon_\mu(x)}{\partial x_\mu}
\end{aligned}$$

Substituting back in (1.2)

$$\boxed{\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \frac{2}{d}(\partial \cdot \epsilon)\ g_{\mu\nu}} \qquad (1.3)$$

Up until now, we have no made any crude assumption. However before we proceed, we will assume that the metric is Euclidean. Now, we operate on both sides by ∂^ν

$$\begin{aligned}
\frac{\partial}{\partial x'_\nu}\left[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)\right] &= \frac{\partial}{\partial x'_\nu}\left(\frac{2}{d}\partial \cdot \epsilon(x)\ \overset{\text{assuming flat metric}}{g_{\mu\nu}}\right) \\
\partial_\mu \underbrace{\partial^\nu \epsilon_\nu}_{\partial\cdot\epsilon} + \underbrace{\partial^\nu \partial_\nu}_{\Box} \epsilon_\mu &= \frac{2}{d} g_{\mu\nu} \partial^\nu \partial \cdot \epsilon \\
\partial_\mu(\partial \cdot \epsilon) + \Box \epsilon &= \frac{2}{d} \partial_\mu (\partial \cdot \epsilon)
\end{aligned}$$

Operating by ∂_ν

$$\begin{aligned}
\partial_\nu[\partial_\mu(\partial \cdot \epsilon) + \Box \epsilon] &= \partial_\nu\left[\frac{2}{d}\partial_\mu(\partial \cdot \epsilon)\right] \\
\left(1 - \frac{2}{d}\right)\partial_\nu \partial_\mu (\partial \cdot \epsilon) + \Box(\partial_\nu \epsilon_\mu) &= 0 \qquad (1.4)
\end{aligned}$$

under relabeling $\mu \leftrightarrow \nu$

$$\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu (\partial \cdot \epsilon) + \Box(\partial_\mu \epsilon_\nu) = 0 \qquad (1.5)$$

adding (1.4) and (1.5)

$$\begin{aligned}
2\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu (\partial \cdot \epsilon) + \Box\underbrace{[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)]}_{\frac{2}{d}(\partial\cdot\epsilon) g_{\mu\nu}} &= 0 \\
\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu (\partial \cdot \epsilon) + \frac{1}{d}\Box(\partial \cdot \epsilon) g_{\mu\nu} &= 0
\end{aligned}$$

$$[g_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu]\,(\partial\cdot\epsilon) = 0 \tag{1.6}$$

Contracting the indicies

$$\begin{aligned}[d\Box + (d-2)\Box](\partial\cdot\epsilon) &= 0\\ 2(d-1)\Box(\partial\cdot\epsilon) &= 0\end{aligned}$$

hence,

$$\boxed{(d-1)\Box(\partial\cdot\epsilon) = 0} \tag{1.7}$$

if $d = 1 \implies$ any $\epsilon^\mu(x)$ satisfies (1.7), therefore, is conformal transformation. It is interesting to note that any 1D QFT is conformal field theory, but for our purpose it's not very useful. We will be concerned with $d \neq 1$ for the rest of this notes unless stated otherwise. Consider the action of ∂_α on (1.3) and then cyclic relabeling of incidies as $\alpha \to \mu \to \nu$

$$\partial_\alpha[\partial_\mu\epsilon_\nu(x) + \partial_\nu\epsilon_\mu(x)] = \frac{2}{d}\partial_\alpha g_{\mu\nu}(\partial\cdot\epsilon) \tag{1.8}$$

$$\partial_\mu[\partial_\nu\epsilon_\alpha(x) + \partial_\alpha\epsilon_\nu(x)] = \frac{2}{d}\partial_\mu g_{\nu\alpha}(\partial\cdot\epsilon) \tag{1.9}$$

$$\partial_\nu[\partial_\alpha\epsilon_\mu(x) + \partial_\mu\epsilon_\alpha(x)] = \frac{2}{d}\partial_\nu g_{\alpha\mu}(\partial\cdot\epsilon) \tag{1.10}$$

Adding the first two equation and subtracting from the last, we get [(1.8) + (1.9) − (1.10)]:

$$\cancel{2}\partial_\alpha\partial_\mu\epsilon_\nu = \frac{\cancel{2}}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon)$$

$$\partial_\alpha\partial_\mu\epsilon_\nu = \frac{1}{d}[g_{\mu\nu}\partial_\alpha + g_{\nu\alpha}\partial_\mu - g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon) \tag{1.11}$$

Referring to eqn 1.7.11 of "Ideas and Methods of Supersymmetry and Supergravity" by Sergio M. Kuzenko, we find that the 3rd order derivation of $\epsilon^\mu(x)$ vanishes. Therefore, the most general conformal transformation is of the type:

$$x'^\mu = x^\mu + \underbrace{a^\mu + b^\mu{}_\nu x^\nu + c^\mu{}_{\nu\alpha}x^\nu x^\alpha}_{\epsilon^\mu}$$

Where, $a^\mu, b^\mu{}_\nu$ and $c^\mu{}_{\nu\alpha}$ are parameters relevant to their transformation. The goal here is simple:

- First find the relevant transformations
- Then based on the transformation rule, find the generators.

For $\epsilon^\mu = a^\mu$:

$$\begin{aligned}x'^\mu &= x^\mu + a^\mu\\ &= x^\mu + \delta^\mu_\nu a^\nu\\ &= x^\mu + (\partial_\nu x^\mu)a^\nu\\ &= [1 + ia^\nu(-i\partial_\nu)]x^\mu\end{aligned}$$

Thus, the generator of translation is $P_\mu - i\partial_\mu$[2]. For $\epsilon^\mu = b^\mu{}_\alpha x^\alpha$, we refer to (1.3)

$$\begin{aligned}\partial_\mu\epsilon_\nu(x) + \partial_\nu\epsilon_\mu(x) &= \frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu}\\ \partial_\mu(b_{\nu\alpha}x^\alpha) + \partial_\nu(b_{\mu\alpha}x^\alpha) &= \frac{2}{d}(\partial^\mu b_{\mu\alpha}x^\alpha)g_{\mu\nu}\\ b_{\nu\alpha}\delta^\alpha_\mu + b_{\mu\alpha}\delta^\alpha_\nu &= \frac{2}{d}(b_{\mu\alpha}g^{\alpha\mu})g_{\mu\nu}\\ b_{\nu\mu} + b_{\mu\nu} &= \frac{2}{d}b^\alpha{}_\alpha g_{\mu\nu}\\ \frac{b_{\nu\mu} + b_{\mu\nu}}{2} &= \frac{1}{d}b^\alpha{}_\alpha g_{\mu\nu}\end{aligned}$$

now,

$$b_{\mu\nu} = \frac{b_{\mu\nu} - b_{\nu\mu}}{2} + \frac{b_{\mu\nu} + b_{\nu\mu}}{2}$$

[2] if we use $[1 - a^\nu(\partial_\nu)]x^\mu$ as the definition, then $P_\mu = i\partial_\mu$ would be the generator

$$= M_{\mu\nu} + \lambda g_{\mu\nu}$$

If $b_{\mu\nu} = \lambda g_{\mu\nu} (M_{\mu\nu} = 0)$

$$\begin{aligned}
x'^{\mu} &= x_{\mu} + b^{\mu}{}_{\nu} x^{\nu} \\
&= x^{\mu} + \lambda g^{\mu\alpha} \underbrace{g_{\alpha\nu} x^{\nu}}_{x_\alpha} \\
&= x^{\mu} + \lambda x^{\mu} \\
&= x^{\mu} + \lambda x^{\nu} \delta^{\mu}_{\nu} \\
&= x^{\mu} + \lambda x^{\nu} (\partial_{\nu} x^{\mu}) \\
&= x^{\mu} + i\lambda x^{\nu} (-i\partial_{\nu} x^{\mu}) \\
&= (1 + i\lambda(-ix^{\nu}\partial_{\nu}))x^{\mu}
\end{aligned}$$

Thus, the generator of dilatation is $D = -ix^{\mu}\partial_{\mu}$. For $b_{\mu\nu} = M_{\mu\nu}(\lambda = 0)$.

$$\begin{aligned}
x'^{\mu} &= x^{\mu} + M^{\mu}{}_{\nu} x^{\nu} \\
&= x^{\mu} + M^{\alpha}{}_{\nu} \delta^{\mu}_{\alpha} x^{\nu} \\
&= x^{\mu} + M^{\alpha}{}_{\nu} (\partial_{\alpha} x^{\mu}) x^{\nu} \\
&= x^{\mu} + M_{\alpha\nu} (\partial^{\alpha} x^{\mu}) x^{\nu} \\
&= x^{\mu} + \frac{M_{\alpha\nu} - M_{\nu\alpha}}{2} (\partial^{\alpha} x^{\mu}) x^{\nu} \qquad \text{relabeling } \nu \leftrightarrow \alpha \\
&= x^{\mu} + \frac{1}{2} M_{\alpha\nu} (\partial^{\alpha} x^{\mu}) x^{\nu} - \frac{1}{2} M_{\nu\alpha} (\partial^{\alpha} x^{\mu}) x^{\nu} \\
&= x^{\mu} + \frac{1}{2} M_{\alpha\nu} (\partial^{\alpha} x^{\mu}) x^{\nu} - \frac{1}{2} M_{\alpha\nu} (\partial^{\nu} x^{\mu}) x^{\alpha} \\
&= x^{\mu} + \frac{1}{2} M_{\alpha\nu} (x^{\nu}\partial^{\alpha} - x^{\alpha}\partial^{\nu}) x^{\mu} \\
&= x^{\mu} + \frac{i}{2} M_{\alpha\nu} \left\{ -i(x^{\nu}\partial^{\alpha} - x^{\alpha}\partial^{\nu}) \right\} x^{\mu} \\
&= x^{\mu} + \frac{i}{2} M_{\alpha\nu} \{ \underbrace{i(x^{\alpha}\partial^{\nu} - x^{\nu}\partial^{\alpha})}_{L^{\alpha\nu}} \} x^{\mu}
\end{aligned}$$

Thus, the generator of rotation is $L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$. Now, the last part $\epsilon^{\mu} = c^{\mu}{}_{\nu\alpha} x^{\nu} x^{\alpha} = c^{\mu}{}_{\alpha\nu} x^{\nu} x^{\alpha}$, we refer to (1.11):

$$\begin{aligned}
\partial_{\alpha}\partial_{\mu}\epsilon_{\nu} &= \frac{1}{d}[g_{\mu\nu}\partial_{\alpha} + g_{\nu\alpha}\partial_{\mu} - g_{\alpha\mu}\partial_{\nu}](\partial \cdot \epsilon) \\
\partial_{\alpha}\partial_{\mu}(c_{\nu\sigma\beta} x^{\sigma} x^{\beta}) &= \frac{1}{d}[g_{\mu\nu}\partial_{\alpha} + g_{\nu\alpha}\partial_{\mu} - g_{\alpha\mu}\partial_{\nu}]\partial^{\mu}(c_{\mu\sigma\beta} x^{\sigma} x^{\beta}) \\
c_{\nu\sigma\beta}\partial_{\alpha}(\delta^{\sigma}_{\mu} x^{\beta} + x^{\sigma}\delta^{\beta}_{\mu}) &= \frac{1}{d}[g_{\mu\nu}\partial_{\alpha} + g_{\nu\alpha}\partial_{\mu} - g_{\alpha\mu}\partial_{\nu}]c_{\mu\sigma\beta}(g^{\sigma\mu} x^{\beta} + x^{\sigma} g^{\beta\mu}) \\
c_{\nu\sigma\beta}(\delta^{\sigma}_{\mu}\delta^{\beta}_{\alpha} + \delta^{\sigma}_{\alpha}\delta^{\beta}_{\mu}) &= \frac{1}{d}[g_{\mu\nu}\partial_{\alpha} + g_{\nu\alpha}\partial_{\mu} - g_{\alpha\mu}\partial_{\nu}](c^{\sigma}{}_{\sigma\beta} x^{\beta} + c^{\beta}{}_{\sigma\beta} x^{\sigma}) \\
2c_{\nu\mu\alpha} &= \frac{2}{d}[g_{\mu\nu}\partial_{\alpha} + g_{\nu\alpha}\partial_{\mu} - g_{\alpha\mu}\partial_{\nu}]c^{\sigma}{}_{\sigma\beta} x^{\beta} \\
c_{\nu\mu\alpha} &= \underbrace{\frac{1}{d}c^{\sigma}{}_{\sigma\beta}}_{b_\beta}[g_{\mu\nu}\delta^{\beta}_{\alpha} + g_{\nu\alpha}\delta^{\beta}_{\mu} - g_{\alpha\mu}\delta^{\beta}_{\nu}] \\
&= g_{\nu\mu} b_{\alpha} + g_{\nu\alpha} b_{\mu} - g_{\mu\alpha} b_{\nu}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\epsilon_{\mu} &= c_{\mu\alpha\beta} x^{\alpha\beta} \\
&= (g_{\mu\alpha} b_{\beta} + g_{\mu\beta} b_{\alpha} - g_{\alpha\beta} b_{\mu}) x^{\alpha} x^{\beta} \\
&= x_{\mu}(b \cdot x) + x_{\mu}(b \cdot x) - b_{\mu}(x \cdot x) \\
&= 2x_{\mu}(b \cdot x) - x^2 b_{\mu}
\end{aligned}$$

Hence, the Special Conformal Transformation looks like:

$$x'^{\mu} = x^{\mu} + 2x^{\mu}(b \cdot x) - x^2 b^{\mu}$$

$$
\begin{aligned}
&= x^\mu + 2(b\cdot x)x^\nu\delta^\mu_\nu - x^2 b^\nu \delta^\mu_\nu \\
&= x^\mu + 2(b\cdot x)x^\nu\partial_\nu x^\mu - x^2 b^\nu \partial_\nu x^\mu \\
&= [1 + 2(b\cdot x)x^\nu\partial_\nu - x^2 b^\nu\partial_\nu]x^\mu \\
&= [1 + \{2b^\alpha x_\alpha x^\nu \partial_\nu - x^2 b^\alpha \partial_\alpha\}]x^\mu \\
&= [1 + ib^\alpha\{\underbrace{-i(2x_\alpha x^\nu\partial_\nu - x^2\partial_\alpha)}_{K_\alpha}\}]x^\mu
\end{aligned}
$$

Hence, the generator for Special Conformal Transformations (SCT) takes the form $K^\mu = -i(2x^\mu x\cdot\partial - x^2\partial^\mu)$. We will now list all the **infinitesimal** transformations and their generators we found in this section.

1. Translation

$$x'^\mu = x^\mu + a^\mu \qquad\qquad P_\mu = -i\partial_\mu \tag{1.12}$$

2. Rotation

$$x'^\mu = x^\mu + M^\mu{}_\nu x^\nu \qquad\qquad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{1.13}$$

3. Dilatation

$$x'^\mu = (1+\lambda)x^\mu \qquad\qquad D = -ix^\mu\partial_\mu \tag{1.14}$$

4. Special Conformal Transformation

$$x'^\mu = x^\mu + 2x^\mu(b\cdot x) - x^2 b^\mu \qquad\qquad K^\mu = -i(2x^\mu x\cdot\partial - x^2\partial^\mu) \tag{1.15}$$

In the above listed transfomations, the parameters $a^\mu, M^\mu{}_\nu, \lambda$ and b^μ are all infinitesimal.

1.3 Finite Conformal Transformation

In the previous section, we considered the infinitesimal conformal transformation, however in this section we will consider the finite conformal transformation.

1. Translation

$$x'^\mu = x^\mu + \underset{\uparrow\text{finite vector}}{a^\mu} = e^{ia^\nu P_\nu}x^\mu$$

2. Dilatation

$$x'^\mu = \left(1 + \frac{\lambda}{N}\right)x^\mu$$

In order to achieve the finite dilatation, we use the infinitesimal transformation recursively by dividing the finite λ into infinitely many λ/N pieces and then transforming

$$
\begin{aligned}
x'^\mu &= \left(1+\frac{\lambda''}{N}\right)\underbrace{\left(1+\frac{\lambda'}{N}\right)\left(1+\frac{\lambda}{N}\right)x^\mu}_{x''^\mu} \\
&= \lim_{N\to\infty}\left(1+\frac{\lambda}{N}\right)^N x^\mu \\
&= e^\lambda x^\mu = e^{i\lambda D}x^\mu
\end{aligned}
$$

3. Rotation

$$
\begin{aligned}
x'^\mu &= \left[1 + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right]^\mu{}_\nu x^\nu \\
&= \left[e^{\frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}}\right]^\mu{}_\nu x^\nu = \Lambda^\mu{}_\nu x^\nu
\end{aligned}
$$

4. The special conformal transformation

$$x'^\mu = x^\mu + 2x^\mu(b\cdot x) - x^2 b^\mu$$

let $b^\mu = \overset{\text{infinitesimal parameter, i.e. } t \text{ is small.}}{t}\, e^\mu$

$$x'^\mu(t) \equiv x^\mu(t) = x^\mu + 2t(e\cdot x)x^\mu - x^2 t e^\mu$$

To find the finite form of the transformation we have to recursively apply the above equation multiple times (Lie Algebra sence). The usual way is to integrate the infinitesimal form. The other way, and since we know that the transformations satisfy the conformal Killing equation, is to find the integral curve of the corresponding conformal Killing vector field as they are equivalent (Differential Geometry sense). Consider the $t-$derivative of the above[3].

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = 2(e\cdot x)x^\mu - x^2 e^\mu \tag{1.16}$$

defining $y^\mu(t) = \frac{x^\mu(t)}{x^2(t)}$

$$\begin{aligned}
\dot{y}^\mu(t) &\overset{\text{quotient rule}}{=} \frac{x^2\dot{x}^\mu - 2(\dot{x}\cdot x)x^\mu}{(x^2)^2} \\
&= \frac{x^2[2(e\cdot x)x^\mu - x^2 e^\mu] - 2[2(e\cdot x)x^\nu - x^2 e^\nu]x_\nu x^\mu}{x^4} \\
&= \frac{x^2[2(e\cdot x)x^\mu - x^2 e^\mu] - 2[2(e\cdot x)x^2 - x^2(e\cdot x)]x^\mu}{x^4} \\
&= \frac{x^2[\cancel{2(e\cdot x)x^\mu} - x^2 e^\mu] - \cancel{2(e\cdot x)x^2 x^\mu}}{x^4} \\
\dot{y}^\mu(t) &= -e^\mu
\end{aligned}$$

Solving the above differential equation

$$\begin{aligned}
y^\mu(t) &= y^\mu(0) - te^\mu \\
\frac{x^\mu(t)}{x^2(t)} &= \frac{x^\mu(0)}{x^2(0)} - te^\mu
\end{aligned}$$

going back to the old notation $x'^\mu \equiv x^\mu(t)$

$$\begin{aligned}
\frac{x'^\mu}{x'^2} &= \frac{x^\mu}{x^2} - te^\mu \\
&= \frac{x^\mu}{x^2} - b^\mu
\end{aligned} \tag{1.17}$$

Squaring both sides

$$\begin{aligned}
\left(\frac{x'^\mu}{x'^2}\right)^2 \equiv \frac{x'^\mu}{x'^2}\frac{x'_\mu}{x'^2} &= \left(\frac{x^\mu}{x^2} - b^\mu\right)^2 \\
\frac{x'^2}{x'^4} &= \left(\frac{x^\mu}{x^2}\right)^2 + b^2 - \frac{2(x\cdot b)}{x^2} \\
\frac{1}{x'^2} &= \frac{1 + b^2x^2 - 2(x\cdot b)}{x^2} \\
x'^2 &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned} \tag{1.18}$$

referring to (1.17)

$$x'^\mu = x'^2\left[\frac{x^\mu}{x^2} - b^\mu\right]$$

[3] when we consider the differential equation, we are no longer thinking of it as transformation but rather flow along a trajectory parameterized by t. This part was taken from pg 16 of "Four point function in momentum spaces and topological terms in gravity"

and substituting (1.18)

$$
\begin{aligned}
x'^{\mu} &= \frac{x^2}{1-2(x\cdot b)+b^2x^2}\left[\frac{x^\mu}{x^2}-b^\mu\right] \\
&= \frac{x^\mu - b^\mu x^2}{1-2(x\cdot b)+b^2x^2}
\end{aligned}
$$

Above procedure also suggests that, finite SCT could be described as a sequence of inversion→translation→inversion. Where inversion is defined as:

$$I(x^\mu) = \frac{x^\mu}{x^2}$$

First we note that the inversion is a global conformal transformation and since it is undefined at origin, it does not have an infinitesimal part i.e. we can not expect inversion to be obtained by exponentiating an element from the conformal Lie algebra. It is not the connected element of conformal group and in embedding space formalism, inversion is related to parity. Another interesting point to note is that there is no parameter associated with the transformation here such as λ for dilatation or b^μ for SCT. Lastly, it is also closely related to the stereographic projection. To show this let us study the stereographic projection of sphere onto a plane. Consider $x \in \mathbb{R}^n$, and define stereographic projection from the **north pole** of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as:

$$X^i = \frac{2x^i}{1+|x|^2}, \quad X^{n+1} = \frac{|x|^2-1}{1+|x|^2}$$

where x^i are coordinates on the projected plane and X^i are the coordiantes on the sphere in embedding space. Now project this point on the sphere back to $\mathbb{R}^n$ via stereographic projection from the **south pole**:

$$x'^i = \frac{X^i}{1+X^{n+1}}$$

Substituting:

$$x'^i = \frac{\frac{2x^i}{1+|x|^2}}{1+\frac{|x|^2-1}{1+|x|^2}} = \frac{2x^i}{(1+|x|^2)+(|x|^2-1)} = \frac{2x^i}{2|x|^2} = \frac{x^i}{|x|^2}$$

Hence, the composition gives:

$$x^i \mapsto \frac{x^i}{|x|^2}$$

which is the **inversion** in the unit sphere. Even though this inversion does not have a killing vector associated with it, but it is reasonable to look for the killing vector associated with stereographic projection. In general, we note that these two transformation would have the following form:

$$x'^\mu = \Omega(x)x^\mu$$

If $\partial_\mu\partial_\nu(\frac{1}{\Omega}) \propto g_{\mu\nu}$. The killing vector associated with it will have the form:

$$K^A{}_\mu = \frac{1}{\Omega^2}\frac{\partial x'^A}{\partial x^\mu}$$

Coming back to special conformal transformation which was the topic at hand, we now look at how they scale the metric tensor.

$$\frac{\partial x'^\mu}{\partial x^\nu} = \left\{\frac{\delta^\mu_\nu - 2b^\mu x_\nu}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\nu + 2b^2x_\nu)}{\Lambda^2}\right\}$$

$$
\begin{aligned}
g_{\alpha\beta}(x) = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}g'_{\mu\nu}(x')\bigg|_{x'=x'(x)} &= \left\{\frac{\delta^\mu_\alpha - 2b^\mu x_\alpha}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\alpha + 2b^2x_\alpha)}{\Lambda^2}\right\} \\
&\times \left\{\frac{\delta^\nu_\beta - 2b^\nu x_\beta}{\Lambda} - \frac{(x^\nu - b^\nu x^2)(-2b_\beta + 2b^2x_\beta)}{\Lambda^2}\right\} g'_{\mu\nu}(x') \\
&= \left\{\frac{\delta^\mu_\alpha - 2b^\mu x_\alpha}{\Lambda} - \frac{(x^\mu - b^\mu x^2)(-2b_\alpha + 2b^2x'_\alpha)}{\Lambda^2}\right\} \\
&\times \left\{\frac{g'_{\mu\beta} - 2b_\mu x_\beta}{\Lambda} - \frac{(x_\mu - b_\mu x^2)(-2b_\beta + 2b^2x_\beta)}{\Lambda^2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{g'_{\alpha\beta} - 2b_\beta x_\alpha - 2b_\alpha x_\beta + 4b^2 x_\alpha x_\beta}{\Lambda^2} - \frac{(x_\beta - b_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&\quad + \frac{(2(b\cdot x)x_\beta - 2b^2 x_\beta x^2)(-2b_\alpha + 2b^2 x_\alpha)}{\Lambda^3} \\
&\quad - \frac{(x_\alpha - 2b_\alpha x^2)(-2b_\alpha + 2b^2 x_\beta)}{\Lambda^3} \\
&\quad + \frac{(2(b\cdot x)x_\alpha - 2b^2 x^2 x_\alpha)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^3} \\
&\quad + \frac{(x^\mu - b^\mu x^2)(x_\mu - b_\mu x^2)(-2b_\alpha + 2b^2 x_\alpha)(-2b_\beta + 2b^2 x_\beta)}{\Lambda^4} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{(-2b_\beta x_\alpha - 2_\alpha x_\beta + 4b^2 x_\alpha x_\beta)(1 - 2b\cdot x + b^2 x^2)}{\Lambda^3} \\
&\quad + \frac{1}{\Lambda^3}\left\{2b_\alpha x_\mu - 2b_\alpha b_\beta x^2 - 2b^2 x_\alpha x_\beta + 2b^2 x^2 x_\alpha b_\beta \right.\\
&\quad\quad \left. -4(b\cdot x)b_\alpha x_\beta + 4b^2 x^2 b_\alpha x_\beta + 4b^2 (b\cdot x) x_\alpha x_\beta - 4b^\mu x^2 x_\alpha x_\beta\right\} + (\alpha \leftrightarrow \beta) \\
&\quad + \frac{\left\{x^2 - 2(b\cdot x)x^2 + b^2 x^4\right\}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\}}{\Lambda^4} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3} \\
&\quad \times \big(-\cancel{2b_\beta x_\alpha} - \cancel{2b_\alpha x_\beta} + \cancel{4b^2 x_\alpha x_\beta} + \cancel{4(b\cdot x)b_\alpha x_\beta} - \cancel{8b^2 (b\cdot x) x_\alpha x_\beta} \\
&\quad\quad -\cancel{2b^2 x^2 b_\beta x_\alpha} - \cancel{2b^2 x^2 b_\alpha x_\beta} + 4b^\mu x^2 x_\alpha x_\beta + \cancel{2b_\alpha x_\beta} + \cancel{2b_\beta x_\alpha} \\
&\quad\quad -4b_\alpha b_\beta x^2 - \cancel{4b^2 x_\alpha x_\beta} + \cancel{2b^2 x^2 x_\alpha b_\beta} + \cancel{2b^2 x^2 x_\beta x_\alpha} - \cancel{4bxb_\alpha x_\beta} \\
&\quad\quad -\cancel{4bxb_\beta x_\alpha} + 4b^2 x^2 b_\alpha x_\beta + 4b^2 x^2 b_\beta x_\alpha + \cancel{8b^2 b^2 x_\alpha x_\beta} - 8b^4 x^2 x_\alpha x_\beta\big) \\
&\quad + x^2 \frac{\Lambda}{\Lambda^4}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3}\left\{-4b_\alpha b_\beta x^2 + 4b^2 x^2 b_\alpha x_\beta + 4b^2 x^2 b_\beta x_\alpha - 4b^4 x^2 x_\alpha x_\beta\right\} \\
&\quad\quad + x^2 \frac{1}{\Lambda^3}\left\{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\right\}
\end{aligned}
$$

$$
g'_{\alpha\beta}(x') = \Lambda^2 g_{\alpha\beta}(x)
$$

Jacobian of the Transformation

The following part is taken from "Conformal Field Theory Primer in $D \geq 3$" by Andrew Evans, pg 36:

$$
\begin{aligned}
\text{Translation:} &\quad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1 \\
\text{Rotation:} &\quad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1 \\
\text{Dilataion:} &\quad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \lambda^{-d} \\
\text{Inversion:} &\quad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \left(\frac{1}{\tilde{x}^2}\right)^d
\end{aligned}
$$

Since the rest are easier to show, we will only focus on showing the last part:

$$
\begin{aligned}
\frac{\partial x^\mu}{\partial \tilde{x}^\nu} &= \frac{1}{\tilde{x}^2}\left[\delta^\mu_\nu - 2\frac{\tilde{x}^\mu \tilde{x}_\nu}{\tilde{x}^2}\right] \\
\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) &= \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}}\frac{\partial x^{\mu_2}}{\partial \tilde{x}^{\nu}_2}\cdots\frac{\partial x^{\mu_d}}{\partial \tilde{x}^{\nu}_d} \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\left[\delta^{\mu_1}_{\nu_1} - 2\frac{\tilde{x}^{\mu_1}\tilde{x}_{\nu_1}}{\tilde{x}^2}\right]\left[\delta^{\mu_2}_{\nu_2} - 2\frac{\tilde{x}^{\mu_2}\tilde{x}_{\nu_2}}{\tilde{x}^2}\right]\cdots\left[\delta^{\mu_d}_{\nu_d} - 2\frac{\tilde{x}^{\mu_d}\tilde{x}_{\nu_d}}{\tilde{x}^2}\right] \\
&= \left(\frac{1}{\tilde{x}^2}\right)^d \frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{i=1}^{d}\delta^{\mu_i}_{\nu_i} - 2\sum_{j=1}^{d}\frac{1}{d!}\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\frac{\tilde{x}^{\mu_j}\tilde{x}_{\nu_j}}{\tilde{x}^2}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} + 0
\end{aligned}
$$

Now we use the identity

$$\epsilon_{i_1\ldots i_k\, i_{k+1}\ldots i_n}\epsilon^{i_1\ldots i_k\, j_{k+1}\ldots j_n} = \delta^{i_1\ldots i_k\, j_{k+1}\ldots j_n}_{i_1\ldots i_k\, i_{k+1}\ldots i_n} = k!\, \delta^{j_{k+1}\ldots j_n}_{i_{k+1}\ldots i_n}$$

which in our case becomes

$$\epsilon_{\mu_1\mu_2\mu_2\ldots\mu_d}\epsilon^{\nu_1\nu_2\nu_2\ldots\nu_d}\prod_{\substack{i=1\\ i\neq j}}^{d}\delta^{\mu_i}_{\nu_i} = (d-1)!\delta^{\mu_j}_{\nu_j}$$

Hence

$$\begin{aligned}\det\left(\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right) &= \left(\frac{1}{\tilde{x}^2}\right)^d\left(\frac{d! - 2\sum_{j=1}^{d}(d-1)!}{d!}\right)\\ &= \left(\frac{1}{\tilde{x}^2}\right)^d\left(1 - \frac{2d(d-1)!}{d!}\right) = -\left(\frac{1}{\tilde{x}^2}\right)^d\end{aligned}$$

How distances transform

Under translation

$$x'^\mu = x^\mu + a^\mu$$

So,

$$\begin{aligned}x'^\mu_a - x'^\mu_b &= x^\mu_a + a^\mu - x^\mu_b - a^\mu\\ &= x^\mu_a - x^\mu_b\end{aligned}$$

Thus, the distances are invariant under translation:

$$|x'_a - x'_b| = |x^\mu_a - x^\mu_b|$$

Under dilatation

$$x'^\mu = (1+\lambda)x^\mu$$

So,

$$\begin{aligned}x'^\mu_a - x'^\mu_b &= (1+\lambda)x^\mu_a - (1+\lambda)x^\mu_b\\ &= (1+\lambda)(x^\mu_a - x^\mu_b)\end{aligned}$$

We find that the distances between two point scales under dilatation, therefore the natural quantity which is invariant under both translation and dilatation is

$$\begin{aligned}\frac{\left|x'^\mu_a - x'^\mu_b\right|}{\left|x'^\mu_c - x'^\mu_d\right|} &= \frac{\cancel{1+\lambda}}{\cancel{1+\lambda}}\frac{|x^\mu_a - x^\mu_b|}{|x^\mu_c - x^\mu_d|}\\ &= \frac{|x^\mu_a - x^\mu_b|}{|x^\mu_c - x^\mu_d|}\end{aligned}$$

Under special conformal transformation

$$\begin{aligned}x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2x^2}\\ &= \frac{x^\mu - b^\mu x^2}{\Lambda^2(x)}\end{aligned}$$

So,

$$\begin{aligned}x'^\mu_a &= \frac{x^\mu_a - b^\mu x^2_a}{\Lambda^2(x_a)}\\ x'^\mu_b &= \frac{x^\mu_b - b^\mu x^2_b}{\Lambda^2(x_b)}\end{aligned}$$

and,

$$x_a'^\mu - x_b'^\mu = \frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}$$

squaring both sides

$$\begin{aligned}
(x_a'^\mu - x_b'^\mu)^2 &= \left(\frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}\right)^2 \\
&= \frac{x_a^2 + b^2(x_a^2)^2 - 2x_a^2(x_a\cdot b)}{\Lambda^4(x_a)} + \frac{x_b^2 + b^2(x_b^2)^2 - 2x_b^2(x_b\cdot b)}{\Lambda^4(x_b)} \\
&\quad - \frac{2}{\Lambda^2(x_a)\Lambda^2(x_b)}\left[x_a\cdot x_b - x_b^2(x_a\cdot b) - x_a^2(b\cdot x_b) + b^2 x_a^2 x_b^2\right] \\
&= x_a^2\left[\frac{1-2(x_a\cdot b)}{\Lambda^4(x_a)} + \frac{\overbrace{2(b\cdot x_b) - b^2 x_b^2}^{1-\Lambda^2(x_b)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1-2(x_b\cdot b)}{\Lambda^4(x_b)} + \frac{\overbrace{2(b\cdot x_a) - b^2 x_a^2}^{1-\Lambda^2(x_a)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2\left[\frac{1-2(x_a\cdot b) - \Lambda^2(x_a)}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1-2(x_b\cdot b) - \Lambda^2(x_b)}{\Lambda^4(x_b)}\right. \\
&\quad \left. + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2\left[\frac{-b^2x_a^2}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{-b^2x_b^2}{\Lambda^4(x_b)} + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a\cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= \frac{(x_a - x_b)^2}{\Lambda^2(x_a)\Lambda^2(x_b)}
\end{aligned}$$

Thus, we find that the ratio of distances are not invariant under SCT.

$$\frac{\left|x_a'^\mu - x_b'^\mu\right|}{|x_a - x_b|} = \frac{1}{\Lambda(x_a)\Lambda(x_b)}$$

where $\Lambda(x_a) = \sqrt{1 - 2(x_a\cdot b) + b^2 x_a^2}$ We can however, construct another quantity which is invariant under SCT.

$$\begin{aligned}
\frac{|x_a' - x_b'|}{|x_b' - x_d'|}\frac{|x_d' - x_c'|}{|x_c' - x_a'|} &= \frac{\frac{|x_a - x_b|}{\Lambda(x_a)\Lambda(x_b)}}{\frac{|x_b - x_d|}{\Lambda(x_b)\Lambda(x_d)}}\frac{\frac{|x_d - x_c|}{\Lambda(x_d)\Lambda(x_c)}}{\frac{|x_c - x_a|}{\Lambda(x_c)\Lambda(x_a)}} \\
&= \frac{|x_a - x_b|}{|x_b - x_d|}\frac{|x_d - x_c|}{|x_c - x_a|}
\end{aligned}$$

Such expressions are called, anharmonic ratios or cross-ratios.

1.4 Lie Algebra of Generators

$$\begin{aligned}
[P_\mu, P_\nu] &= [-i\partial_\mu, -i\partial_\nu] \\
&= -[\partial_\mu, \partial_\nu] = 0
\end{aligned}$$

Some useful identities

$$\begin{aligned}
[x_\alpha, \partial_\beta]f &= x_\alpha\partial_\beta f - \underbrace{\partial_\beta(x_\alpha f)}_{(\partial_\beta x_\alpha)f + x_\alpha\partial_\beta f} \\
&= x_\alpha\partial_\beta f - x_\alpha\partial_\beta f - (\partial_\beta x_\alpha)f
\end{aligned}$$

$$= -(\partial_\beta x_\alpha) f$$

$$\begin{aligned}
[x_\alpha, \partial_\beta] &= -\partial_\beta x_\alpha = -g_{\beta\alpha}\partial^\mu x_\alpha \\
&= g_{\beta\alpha}
\end{aligned} \tag{1.19}$$

next is,

$$\begin{aligned}
[x^2, \partial_\beta] &= [x^\alpha x_\alpha, \partial_\beta] \\
&= x^\alpha [x_\alpha, \partial_\beta] + [x^\alpha, \partial_\beta] x_\alpha \\
&= -x^\alpha g_{\beta\alpha} - \delta^\alpha_\beta x_\alpha \\
&= -x_\beta - x_\beta \\
&= -2x_\beta
\end{aligned} \tag{1.20}$$

and the last one is,

$$\begin{aligned}
[x_\mu x^\nu, \partial_\beta] &= x_\mu [x^\nu, \partial_\beta] + [x_\mu, \partial_\beta] x^\nu \\
&= -x_\nu \delta^\nu_\beta - x^\nu g_{\beta\alpha}
\end{aligned} \tag{1.21}$$

We will now consider, the lie algebra of different operators one by one.

$$\begin{aligned}
[P_\mu, D] &= [-i\partial_\mu, -ix^\alpha \partial_\alpha] \\
&= -[\partial_\mu, x^\alpha \partial_\alpha] \\
&= -x^\alpha [\partial_\mu, \partial_\alpha] - [\partial_\mu, x^\alpha]\partial_\alpha \\
&= -\delta^\alpha_\mu \partial_\alpha = -\partial_\mu = -i(-i\partial_\mu) \\
&= -iP_\mu
\end{aligned}$$

$$\begin{aligned}
[P_\mu, L_{\alpha\beta}] &= [-i\partial_\mu, -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)] \\
&= -[\partial_\mu, x_\alpha \partial_\beta - x_\beta \partial_\alpha] \\
&= -[\partial_\mu, x_\alpha]\partial_\beta + [\partial_\mu, x_\beta]\partial_\alpha \\
&= g_{\alpha\mu}\partial_\beta - g_{\beta\mu}\partial_\alpha \\
&= i(g_{\alpha\mu} P_\beta - g_{\beta\mu} P_\alpha)
\end{aligned}$$

$$\begin{aligned}
[P_\mu, K_\nu] &= [-i\partial_\mu, -i(2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\nu)] \\
&= -[\partial_\mu, 2x_\nu x^\alpha \partial_\alpha - x^2 \partial_\nu] \\
&= -2x_\nu x^\alpha [\partial_\mu, \partial_\alpha] - 2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + x^2[\partial_\mu, \partial_\nu] + [\partial_\mu, x^2]\partial_\nu \\
&= -2[\partial_\mu, x_\nu x^\alpha]\partial_\alpha + [\partial_\mu, x^2]\partial_\nu \\
&= -2(g_{\mu\nu} x^\alpha + \delta^\alpha_\mu x_\nu)\partial_\alpha + 2x_\mu \partial_\nu \\
&= -2g_{\mu\nu} x^\alpha \partial_\alpha - 2(x_\nu \partial_\mu - x_\mu \partial_\nu) \\
&= -2ig_{\mu\nu} D - 2iL_{\mu\nu} \\
&= -2i(g_{\mu\nu} D - L_{\mu\nu})
\end{aligned}$$

$$\begin{aligned}
[D, K_\mu] &= -[x^\alpha \partial_\alpha, 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu] \\
&= -2[x^\alpha \partial_\alpha, x_\mu x^\beta \partial_\beta] + [x^\alpha \partial_\alpha, x^2 \partial_\mu] \\
&= -2\left\{x^\alpha [\partial_\alpha, x_\mu x^\beta]\partial_\beta + x_\mu x^\beta [x^\alpha, \partial_\beta]\partial_\alpha\right\} \\
&\quad + x^\alpha [\partial_\alpha, x^2]\partial_\mu + x^2 [x^\alpha, \partial_\mu]\partial_\alpha \\
&= -2\left\{x^\alpha \left(g_{\alpha\mu} x^\beta + \delta^\beta_\alpha x_\mu\right)\partial_\beta + x_\mu x^\beta (-\delta^\alpha_\beta)\partial_\alpha\right\} \\
&\quad + 2x^2 \partial_\mu - \cancel{x^\alpha x^2 \partial_\alpha \partial_\mu} + \cancel{x^2 x^\alpha \partial_\alpha \partial_\mu} - x^2 \partial_\mu \\
&= -\cancel{2x_\mu x^\beta \partial_\beta} - 2x^\beta x_\mu \partial_\beta + \cancel{2x_\mu x^\beta \partial_\beta} + x^2 \partial_\mu \\
&= -(2x^\beta x_\mu \partial_\beta - x^2 \partial_\mu) \\
&= -iK_\mu
\end{aligned}$$

$$
\begin{aligned}
[K_\mu, L_{\alpha\beta}] &= [-i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu), i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)] \\
&= [2x_\mu x^\nu\partial_\nu - x^2\partial_\mu, x_\alpha\partial_\beta - x_\beta\partial_\alpha] \\
&= 2[x_\mu x^\nu\partial_\nu, x_\alpha\partial_\beta] - [x^2\partial_\mu, x_\alpha\partial_\beta] + \underbrace{2[x_\mu x^\nu\partial_\nu, x_\beta\partial_\alpha] - [x^2\partial_\mu, x_\beta\partial_\alpha]}_{\alpha\leftrightarrow\beta} \\
&= 2\left\{x_\mu x^\nu[\partial_\nu, x_\alpha]\partial_\beta + x_\alpha[x_\mu x^\nu, \partial_\beta]\partial_\nu\right\} - x^2[\partial_\mu, x_\alpha]\partial_\beta - x_\alpha[x^2, \partial_\beta]\partial_\mu - (\alpha\leftrightarrow\beta) \\
&= \cancel{2x_\mu x^\nu(g_{\nu\alpha})\partial_\beta} - 2x_\alpha\left(g_{\mu\beta}x^\nu + \cancel{\delta^\nu_\beta x_\mu}\right)\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + 2x_\alpha x_\beta\partial_\mu - (\alpha\leftrightarrow\beta) \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + \cancel{2x_\alpha x_\beta\partial_\mu} + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2 g_{\mu\beta}\partial_\alpha - \cancel{2x_\beta x_\alpha\partial_\mu} \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2 g_{\mu\beta}\partial_\alpha \\
&= -g_{\mu\beta}(2x_\alpha x^\nu\partial_\nu - x^2\partial_\alpha) + g_{\mu\alpha}(2x_\beta x^\nu\partial_\nu - x^2\partial_\beta) \\
&= ig_{\mu\alpha}K_\beta - ig_{\mu\beta}K_\alpha = i(g_{\mu\alpha}K_\beta - g_{\mu\beta}K_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
[K_\mu, K_\nu] &= -[2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu, 2x_\nu x^\beta\partial_\beta - x^2\partial_\nu] \\
&= -4[x_\mu x^\alpha\partial_\alpha, x_\nu x^\beta\partial_\beta] + 2[x_\mu x^\alpha\partial_\alpha, x^2\partial_\nu] + 2[x^2\partial_\mu, x_\nu x^\beta\partial_\beta] - [x^2\partial_\mu, x^2\partial_\nu] \\
&= -4x_\nu x^\beta\left[x_\mu x^\alpha, \partial_\beta\right]\partial_\alpha - 4x_\mu x^\alpha\left[\partial_\alpha, x_\nu x^\beta\right]\partial_\beta + 2x_\mu x^\alpha\left[\partial_\alpha, x^2\right]\partial_\nu + 2x^2\left[x_\mu x^\alpha, \partial_\nu\right]\partial_\alpha \\
&\quad + 2x^2\left[\partial_\mu, x_\nu x^\beta\right]\partial_\beta + 2x_\nu x^\beta\left[x^2, \partial_\beta\right]\partial_\mu - x^2[\partial_\mu, x^2]\partial_\nu - x^2[x^2, \partial_\nu]\partial_\mu \\
&= \cancel{4x_\nu x^\beta(g_{\mu\beta}x^\alpha + \delta^\alpha_\beta x_\mu)\partial_\alpha} - \cancel{4x_\mu x^\alpha(g_{\alpha\nu}x^\beta + \delta^\beta_\alpha x_\nu)\partial_\beta} + 4x_\mu x^2\partial_\nu - 2x^2(\cancel{g_{\mu\nu}x^\alpha} + \delta^\alpha_\nu x_\mu)\partial_\alpha \\
&\quad + 2x^2(\cancel{g_{\mu\nu}x^\beta} + \delta^\beta_\mu x_\nu)\partial_\beta - 4x_\nu x^2\partial_\mu - 2x^2 x_\mu\partial_\nu + 2x^2 x_\nu\partial_\mu \\
&= \cancel{4x_\mu x^2\partial_\nu - 2x_\mu x^2\partial_\nu} + \cancel{2x_\nu x^2\partial_\mu - 4x_\nu x^2\partial_\mu} - \cancel{2x^2 x_\mu\partial_\nu} + \cancel{2x^2 x_\nu\partial_\mu} \\
&= 0
\end{aligned}
$$

Next, we will see that Conformal Algebra in d dimensions is isomorphic to the Lie algebra of the Lorentz group in $d+2$ dimensions, any conformal covariant correlator in d dimensions should be obtainable from Lorentz covariant expressions in $d+2$ dimensions via some kind of dimensional reduction procedure. This is essentially the idea behind **Embedding Formalism**. We define the following set of new operators:

$$
\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu} \\
J_{0,\mu} &= \frac{1}{2}\left(P_\mu + K_\mu\right) \\
J_{-1,\mu} &= \frac{1}{2}\left(P_\mu - K_\mu\right) \\
J_{-1,0} &= D
\end{aligned}
$$

with the property that

$$J_{ab} = -J_{ba}$$

where

d is dimension of spacetime

$$a, b \in \{-1, 0, 1, \cdots, d\}$$

These new generators, obey $SO(d+1,1)$ lie algebra:

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{bc}J_{ad}\right) \tag{1.22}$$

In this section, we will explicitly assume the form of flat metric as being euclidean, and given as:

$$g_{\mu\nu} = \eta_{\mu\nu} = (\underbrace{1, 1, \cdots, 1}_{d})$$

Our metric in (1.22) would be given as:

$$\eta_{ab} = (\,-1\,,\ 1\,, \underbrace{1, \cdots, 1}_{\mu,\nu}) \tag{1.23}$$

$\eta_{-1-1} = -1$ $\eta_{00} = 1$

If our original metric was Minkowski, we would have had:

$$\eta_{ab} = (-1, 1, \underbrace{-1, \cdots, 1}_{d})$$

We will now check, if (1.22) holds true:

$$
\begin{aligned}
[J_{\mu\nu}, J_{0,\alpha}] &= \left[L_{\mu\nu}, \frac{1}{2}(P_\alpha + K_\alpha)\right] \\
&= \frac{1}{2}[L_{\mu\nu}, P_\alpha] + \frac{1}{2}[L_{\mu\nu}, K_\alpha] \\
&= -\frac{1}{2}[P_\alpha, L_{\mu\nu}] - \frac{1}{2}[K_\alpha, L_{\mu\nu}] \\
&= -\frac{1}{2}\left(\eta_{\alpha\mu}P_\mu - \eta_{\alpha\nu}P_\mu\right) - \frac{1}{2}(\eta_{\alpha\mu}K_\nu - \eta_{\alpha\nu}K_\nu) \\
&= -\eta_{\alpha\mu}\left[\frac{1}{2}(P_\nu + K_\nu)\right] + \eta_{\alpha,\nu}\left[\frac{1}{2}(P_\mu + K_\mu)\right] \\
&= -i\eta_{\alpha\mu}J_{0,\nu} + i\eta_{\alpha\nu}J_{0,\mu}
\end{aligned}
$$

$$
\begin{aligned}
[J_{0,\mu}, J_{-1,0}] &= \left[\frac{1}{2}(P_\mu + K_\mu), D\right] \\
&= \frac{1}{2}[P_\mu, D] + \frac{1}{2}[K_\mu, D] \\
&= -\frac{1}{2}iP_\mu - \frac{1}{2}(-iK_\mu) = \frac{-i}{2}(P_\mu - K_\mu) = -iJ_{-1,\mu}
\end{aligned}
$$

If we assume that the metric in (1.22) is indeed given by (1.23). Then, the algebra (1.22) holds true. This shows the isomorphism between the conformal group of $d-$dimensional Euclidean space and the $SO(d+1,1)$ group of $d+2$ dimensional Minkowski spacetime with $1/2(d+1)(d+2)$ parameters.

Conformal Generators on the Field

Finite form of conformal transformation ($x' = \Lambda x$)[4]

$$
\begin{aligned}
\Phi'_a(x') &= U(\Lambda)\Phi_a(x)U^{-1}(\Lambda) \\
\Phi'_a(\Lambda x) &= \sum_b \pi_{ab}(\Lambda)\Phi_b(x) \\
&= \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x') \\
&= \pi_{ab}(e^{i\omega_g c_g})\Phi_b(e^{-i\omega_g c_g}x')
\end{aligned}
\tag{1.24}
$$

We have dropped the $\sum$ sign and summation over repeated indices are implied. Infinitesimal form of (1.24):

generator only acting on field

generator which only acts on x'^μ

$$
\begin{aligned}
\Phi'_a(x') &= (1 - i\omega_g\, T_g\,)_{ab}\Phi_b(\Lambda^{-1}x') \\
&= (1 - i\omega_g T_g)_{ab}\underbrace{\Phi_b[(1 - i\omega_g\, c_g\,)x'^\mu]}_{\Phi_b(x') + \{(1-i\omega_g c_g)x'^\mu - x'^\mu\}\partial_\mu\Phi_b(x')} \\
&= (1 - i\omega_g T_g)_{ab}[\Phi_b(x') - i\omega_g c_g x'^\mu \partial_\mu \Phi_b(x')] \\
\Phi'(x') &= \Phi(x') - i\omega_g\left[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}\right]\Phi(x') + \mathcal{O}(\omega_g^2)
\end{aligned}
$$

accounts for the change in argument of field

However, we will not use this approach but rather we will consider the transformations at origin and then translate it to every other point. This approach is based on studying the stabilizer subgroup of the Conformal Symmetry.[5] So, if we study the same at origin:

$$
\begin{aligned}
\Phi'(0) &= \Phi(x') - i\omega_g\left[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}\right]\Phi(x')\Bigg|_{x'=0} \\
&= \Phi(0) - i\omega_g T_g \Phi(0)
\end{aligned}
$$

[4]tobias osborne's lecture notes pg 18

[5]pg 7 of "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications"

using translation operator

$$\begin{aligned}
e^{ix^\lambda P_\lambda}\Phi'(0)e^{-ix^\alpha P_\alpha} &= e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} - e^{ix^\lambda P_\lambda} i\omega_g T_g \Phi(0) e^{-ix^\alpha P_\alpha} \\
\Phi'(x) &= \Phi(x) - e^{ix^\lambda P_\lambda} i\omega_g T_g e^{-ix^\sigma P_\sigma} e^{ix^\beta P_\beta}\Phi(0)e^{-ix^\alpha P_\alpha} \\
&= \Phi(x) - i\omega_g \underbrace{e^{ix^\lambda P_\lambda} T_g e^{-ix^\sigma P_\sigma}}_{\text{we will find these “translated operators” later}} \Phi(x)
\end{aligned}$$

For translation

$$\begin{aligned}
\Phi'(x+a) &= e^{ia^\lambda P_\lambda}\Phi(x)e^{-ia^\alpha P_\alpha} \\
&= e^{ia^\lambda [P_\lambda,\ ]}\Phi(x)
\end{aligned}$$

using (1.26)

$$= e^{a\cdot\partial}\Phi(x)$$

For rotation, at $x'^\mu = 0. \implies x^\mu = 0$

$$\Phi'_a(0) = \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}0) = \pi_{ab}(\Lambda)\Phi_b(0)$$

Now, assuming the generator of rotation $T_g = L_{\mu\nu}$ acts like[6]

$$L_{\mu\nu}\Phi_a(0) = S_{\mu\nu}\Phi_a(0) \tag{1.25}$$

at origin. At any other point, it will behave as:

$$\begin{aligned}
L_{\mu\nu}\Phi_a(x) &= e^{ix^\beta P_\beta} L_{\mu\nu}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\
&= \underbrace{e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma}}_{?} \underbrace{e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}}_{\Phi_a(x)}
\end{aligned}$$

by taking the derivative of second term, we obtain the following commutator

$$\begin{aligned}
\Phi_a(x) &= e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\
\partial_\mu\Phi_a(x) &= (\partial_\mu e^{ix^\lambda P_\lambda})\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)(\partial_\mu e^{-ix^\alpha P_\alpha}) \\
&= iP^\mu e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}(-iP^\mu) \\
&= iP^\mu\Phi_a(x) - i\Phi_a(x)P^\mu \\
&= i[P^\mu, \Phi_a(x)]
\end{aligned} \tag{1.26}$$

We will now derive the form of $e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma}$[7]:

$$\begin{aligned}
e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma} &= L_{\mu\nu} + [L_{\mu\nu}, -ix^\alpha P_\alpha] + \frac{1}{2!}[[L_{\mu\nu}, -ix^\alpha P_\alpha], -ix^\alpha P_\alpha] + \ldots \\
&= L_{\mu\nu} + ix^\alpha \underbrace{[P_\alpha, L_{\mu\nu}]}_{i(g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu)} + \ldots \\
&= L_{\mu\nu} + i^2 x^\alpha (g_{\alpha\mu}P_\nu - g_{\alpha\nu}uP_\mu) \\
&= L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \\
&= L_{\mu\nu} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{we found in section 1.2}}
\end{aligned}$$

we know, at $x' = 0$ we have $L_{\mu\nu} = S_{\mu\nu}$, so for the sake of consistency we get

$$e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^\sigma P_\sigma} = \underbrace{S_{\mu\nu}}_{\text{Spin Operator}} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{transforms the argument of field}}$$

The exponential map of above can be found in any textbook on QFT which describes rotation or Lorentz transformation.[8] If we ignore $S_{\mu\nu}$, then we can see how the last part acts on field:

$$x'^\mu = \left(\delta^\mu_\nu + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right)x^\mu$$

[6] pg 10, paragraph 2 of “Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions”

[7] using BCH lemma $e^A B e^{-A} = e^{[A,\ ]}B$

[8] check eqn 1.141 and 1.150 of “QFT in curved spacetime” by Leonard Parker

$$\begin{aligned}
&= x^\mu + \frac{i}{2}\omega_{\alpha\beta} i\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right)x^\nu \\
\Phi'(x) &= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\Phi(x) \\
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta} i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha g^{\beta\sigma}\partial_\sigma - x^\beta g^{\alpha\sigma}\partial_\sigma)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)x^\sigma\partial_\sigma\Phi(x) \\
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}x\cdot\partial\Phi(x) \\
&\approx \Phi\left(x^\mu - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}x^\nu\right) \\
\Phi'(x') &= \Phi(x)
\end{aligned}$$

For dilatation, at $x'^\mu = 0$, $x'^\mu = (1+\lambda)x^\mu = 0 \implies x^\mu = 0$. We have $\omega_g = \lambda$ and $T_g = D$:

$$D\Phi_a(0) = \tilde{\Delta}\Phi_a(0) \tag{1.27}$$

corresponding commutator (by operating it on eigenstate of dilatation)

$$\begin{aligned}
D\left|\Delta\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle + \Phi_\Delta(0)D\left|0\right\rangle \\
\tilde{\Delta}\left|\Delta\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle + 0 \\
\tilde{\Delta}\Phi_\Delta(0)\left|0\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle
\end{aligned}$$

Applying the same procedure, we consider:

$$\begin{aligned}
e^{ix^\beta P_\beta}De^{-ix^\sigma P_\sigma} &= D + [D, -ix^\beta P_\beta] + \frac{1}{2!}[[D, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= D - ix^\alpha(iP_\alpha) \\
&= D + x^\alpha P_\alpha \\
&= D - ix^\alpha\partial_\alpha
\end{aligned} \tag{1.28}$$

for the sake of consistency at $x' = 0$

$$= \tilde{\Delta} - ix^\alpha\partial_\alpha$$

Now, we consider

$$D\Phi_a(x) = (\tilde{\Delta} - ix^\alpha\partial_\alpha)\Phi_a(x)$$

redefining $\tilde{\Delta} \equiv -i\Delta$, we get

$$D\Phi_a(x) = -i(\Delta + x^\alpha\partial_\alpha)\Phi_a(x)$$

Similarly,[9]

$$\begin{aligned}
[D, \Phi_a(x)] &= De^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma}D \\
&= e^{ix^\lambda P_\lambda}\underbrace{e^{ix^\alpha P_\alpha}De^{-ix^\beta P_\beta}}_{=D+x^\alpha P_\alpha}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)\underbrace{e^{-ix^\sigma P_\sigma}De^{-ix^\alpha P_\alpha}}_{=D+x^\alpha P_\alpha}e^{-ix^\lambda P_\lambda} \\
&= e^{ix^\beta P_\beta}[D + x^\alpha P_\alpha, \Phi_a(0)]e^{-ix^\sigma P_\sigma} \\
&= e^{ix^\beta P_\beta}\underbrace{[D, \Phi_a(0)]}_{\tilde{\Delta}\Phi_a(0)}e^{-ix^\sigma P_\sigma} + e^{ix^\beta P_\beta}\underbrace{[x^\alpha P_\alpha, \Phi_a(0)]}_{=x^\alpha[P_\alpha, \Phi_a(0)]}e^{-ix^\sigma P_\sigma} \\
&= \tilde{\Delta}\Phi_a(x) - ix\cdot\partial\Phi_a(x) \\
&= -i(\Delta + x\cdot\partial)\Phi_a(x)
\end{aligned}$$

Finite Dilatation[10], we consider

$$x' = e^\lambda x = e^{i\lambda D}x = \left(1 + i\frac{\lambda}{N}\overset{Dx^\mu = -ix\cdot\partial x^\mu}{D}\right)\dots\left(1 + i\frac{\lambda}{N}D\right)x$$

[9] from pg 31 of 2309.10107, and x is not an operator here but a number

[10] look up *Lectures Notes For An Introduction to Conformal Field Theory A Course Given By Dr. Tobias Osborne*, pg 19

then at origin, the field transforms (active transformation) as:

$$\begin{aligned}
\Phi'_a(0) &= \left(1 + i\frac{\lambda}{N}D\right)\dots\left(1 + i\frac{\lambda}{N}D\right)\Phi_a(0) \\
&= \left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) \qquad \text{(using } D\Phi(0) = \tilde{\Delta}\Phi\text{)} \\
&= e^{i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{-\lambda\Delta_a}\Phi_a(0)
\end{aligned}$$

In passive transformation

$$\begin{aligned}
\Phi'_a(0) &= \left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) \\
&= e^{-i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{\lambda\Delta}\Phi_a(0)
\end{aligned}$$

For arbitrary point (ignoring the change in argument of field and thus generator c_g):

$$\begin{aligned}
\Phi'_a(x') &= \pi_{ab}(e^{i\lambda D})\Phi_b(x) \\
&= [e^{i\lambda\tilde{\Delta}}]_{ab}\Phi_b(x) \\
\Phi'_a(e^{\lambda}x) &= [e^{-\lambda\Delta}]_{ab}\Phi_b(x) = e^{-\lambda\Delta}\Phi_a(x)
\end{aligned}$$

For SCT, $x'^{\mu} = 0 \implies x^{\mu} = 0$. Hence, we will consider the same equations, but in this context:

$$K_\mu\Phi_a(0) = \kappa_\mu\Phi_a(0)$$

Again, applying the same procedure,

$$\begin{aligned}
e^{ix^\beta P_\beta}K_\mu e^{-ix^\sigma P_\sigma} &= K_\mu + [K_\mu, -ix^\beta P_\beta] + \frac{1}{2!}[[K_\mu, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu - ix^\beta[K_\mu, P_\beta] + \frac{1}{2}[-ix^\beta[K_\mu, P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu + 2x^\beta(g_{\mu\beta}D - L_{\mu\beta}) + \frac{1}{2}[2x^\beta(g_{\mu\beta}D - L_{\mu\beta}), -ix^\alpha P_\alpha] \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} - ix_\mu x^\alpha[D, P_\alpha] + ix^\beta x^\alpha \underbrace{[L_{\mu\beta}, P_\alpha]}_{-i(g_{\alpha\mu}P_\beta - g_{\alpha\beta}P_\mu)} \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + x_\mu x^\alpha P_\alpha + x_\mu x^\beta P_\beta - x_\alpha x^\alpha P_\mu \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + 2x_\mu x^\alpha P_\alpha - x_\alpha x^\alpha P_\mu
\end{aligned}$$

From the generator of dilatation and SCT, we have[11]

$$[D, K_\mu] = -iK_\mu \qquad \text{at } x'^\mu = 0 \implies \qquad [\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu$$

and

$$[D, L_{\mu\nu}] = 0 \qquad \text{at } x'^\mu = 0 \implies \qquad [\tilde{\Delta}, S_{\mu\nu}] = 0$$

For primary fields:

$$K_\mu\Phi_a(0) = 0$$

Since, for primary field $\tilde{\Delta}$ commutes with all other operators which belong to the stability subgroup. By Schur's lemma $\tilde{\Delta} \propto I$, where I is an identity operator. The SCT and momentum generator acts as ladder operator for Dilatation.

$$\begin{aligned}
[D, [P_\mu, \Phi(0)]] = [P_\mu, [D, \Phi(0)]] + [[D, P_\mu], \Phi(0)] = -i(\Delta + 1)[P_\mu, \Phi(0)] \\
[D, [K_\mu, \Phi(0)]] = [K_\mu, [D, \Phi(0)]] + [[D, K_\mu], \Phi(0)] = -i(\Delta - 1)[K_\mu, \Phi(0)]
\end{aligned}$$

[11] same notes, look at eqn 65 to 70 (pg 18-19), all these commutators are for T_g

Finite Conformal Transformation of Fields

We begin by noting that *translation* and *rotation* do not introduce any new thing that we hadn't encountered in QFT, it is only the dilatation which does. Upon exponentiating the infinitesimal dilatation:

$$\begin{aligned}\Phi'(x') &= e^{-i\omega_g\left[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}\right]}\Phi(x') \\ &= e^{-i\omega_g T_g} e^{-i\omega_g c_g x'\cdot\frac{\partial}{\partial x'}}\Phi(x') \\ &= e^{-i\omega_g T_g}\Phi(e^{-i\omega_g c_g}x')\end{aligned}$$

This section is taken from "advanced mathematical methods - conformal field theory" by David Duffins.[12]

$$\begin{aligned}\Phi'_a(x') = U(\Lambda)\Phi_a(x')U^{-1}(\Lambda) &= e^{-i\omega_g T_g}\Phi_a e^{i\omega_g T_g} \\ &= e^{-i\omega_g[T_g,\ ]}\Phi_a(x')\end{aligned}$$

For translation

$$\Phi(x) = e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} = e^{x\partial}\Phi(0)$$

Or,

$$\begin{aligned}\Phi'(x') &= \Phi(x) \\ &= \Phi(x' - a) \\ &= e^{-a\frac{\partial}{\partial x'}}\Phi(x') \\ &= e^{-iaP}\Phi(x')\end{aligned}$$

For dilatation ($x' = e^\lambda x$)

$$\begin{aligned}\Phi'_a(x') &= e^{-i\lambda D}\Phi_a(x') \\ &= e^{-\lambda(\Delta + x'\cdot\partial)}\Phi_a(x') \\ &= e^{-\lambda\Delta}\underbrace{e^{-\lambda x\cdot\partial}\Phi_a(x')}_{\Phi_a[e^{-\lambda}x']} \\ &= e^{-\lambda\Delta}\Phi_a(x)\end{aligned}$$

The last part could be understood as:

$$\begin{aligned}\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right] &= e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a(x) \\ \Phi_a\left[\underbrace{\left(1-\frac{\lambda}{N}\right)\dots\left(1-\frac{\lambda}{N}\right)}_{N\text{ terms}}x\right] &= e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a\left[\underbrace{\left(1-\frac{\lambda}{N}\right)\dots\left(1-\frac{\lambda}{N}\right)}_{N-1\text{ terms}}x\right] \\ \Phi_a(e^{-\lambda}x) &= e^{-\lambda x\cdot\partial}\Phi_a(x)\end{aligned}$$

or, alternatively

$$\begin{aligned}e^{-\lambda x\cdot\partial}\Phi_a(x) &= \left(1-\frac{\lambda}{N}x\cdot\partial\right)^N\Phi_a(x) \\ &= \left(1-\frac{\lambda}{N}x\cdot\partial\right)\dots\underbrace{\left(1-\frac{\lambda}{N}x\cdot\partial\right)\Phi_a(x)}_{\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right]} \\ &= \Phi_a\left[\left(1-\frac{\lambda}{N}\right)^N x\right] \\ &= \Phi_a(e^{-\lambda}x)\end{aligned}$$

[12] Active coordinate transformation is given as: $\Phi(x') = U(\Lambda)\Phi(x)U^{-1}(\Lambda)$ whereas passive transformation is given as $\Phi(x') = U^{-1}(\Lambda)\Phi(x)U(\Lambda)$

Chapter 2

Embedding coordinates for Euclidean Space

Consider the embedding space coordinates

$$X^{-1}, X^0, \underbrace{X^1, X^2, \ldots, X^d}_{X^\mu}$$

we introduce the following null coordinates, $X^M = (X^+, X^-, X^\mu)$, where[1]

$$\left.\begin{aligned} X^+ &= X^{-1} + X^0 \\ X^- &= X^{-1} - X^0 \end{aligned}\right\} X^{-1} = \frac{X^+ + X^-}{2}; \quad X^0 = \frac{X^+ - X^-}{2}$$

with the mostly plus metric in $\mathbb{R}^{d+1,1}$, reads as

$$ds^2 = -(dX^{-1})^2 + (dX^0)^2 + \sum_{\mu=1}^{d} (dX^\mu)^2 = \sum_{\mu=1}^{d} (dX^\mu)^2 - dX^+ dX^-$$

with the metric given as:

$$\eta_{MN} = \begin{pmatrix} 0 & -1/2 & 0 & 0 & \cdots \\ -1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 0 & 1 & \\ & & & & \ddots \end{pmatrix}$$

We can easily show that the generators in $d+2$ dimensional space reduces to d dimensional conformal generators in Euclidean space. We go from (X_{-1}, X_0, X_μ) coordinates to (ρ, η, x_μ) by following coordinate transformation.

$$\begin{aligned} X_{-1} &= \frac{\rho(1 - \eta^2 + \vec{x}^2)}{2} \\ X_0 &= \frac{\rho(1 + \eta^2 - \vec{x}^2)}{2} \\ X_\mu &= \rho x_\mu \end{aligned}$$

We can note that $X'^A = \lambda X^A$ corresponds to the same transformation as given above with $\rho' \equiv \lambda\rho$. We will see that it has a profound implication. For now, let us invert the above mentioned transformation as following:

$$\begin{aligned} \rho &= X_{-1} + X_0 \\ \eta &= \frac{\sqrt{\eta_{MN} X^M X^N}}{X_{-1} + X_0} \leftarrow \rho\eta = \sqrt{\eta_{MN} X^M X^N} \end{aligned}$$

[1] the index with lowest numeric value has the same sign in both $X^\pm$. If we had considered, X^{d+1} rather than X^{-1} then the definition would have been something like

$$X^\pm = X^0 \pm X^{d+1}$$

$$x_\mu = \frac{X_\mu}{X_{-1} + X_0}$$

We can write the change of basis as:

$$\begin{aligned}
\frac{\partial}{\partial X_{-1}} &= \frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\
\frac{\partial}{\partial X_0} &= \frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu} \\
\frac{\partial}{\partial X_\mu} &= \frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}
\end{aligned}$$

Then, the generators for Lorentz transformation in $d+2$ dimensional space transforms as:

$$\begin{aligned}
P_\mu = J_{-1,\mu} + J_{0,\mu} &= (X_{-1}+X_0)\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} - \frac{\partial}{\partial X_{-1}}\right) \\
&= \rho\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - \rho x_\mu\left(\frac{1}{\rho\eta}\frac{\partial}{\partial \eta}\right) \\
&= \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x^\mu}
\end{aligned}$$

$$\begin{aligned}
K_\mu = J_{0,\mu} - J_{-1,\mu} &= (X_0 - X_{-1})\frac{\partial}{\partial X_\mu} - X_\mu\left(\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_{-1}}\right) \\
&= \rho(\eta^2 - \vec{x}^2)\left(\frac{x_\mu}{\rho\eta}\frac{\partial}{\partial \eta} + \frac{1}{\rho}\frac{\partial}{\partial x_\mu}\right) - 2\rho x_\mu\left[\frac{\partial}{\partial \rho} - \left(\frac{\eta^2+\vec{x}^2}{2\rho\eta}\right) - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&= \frac{\eta^2-\vec{x}^2}{\eta}x_\mu\frac{\partial}{\partial \eta} + (\eta^2-\vec{x}^2)\frac{\partial}{\partial x_\mu} - 2\rho x_\mu\frac{\partial}{\partial \rho} + x_\mu\frac{\eta^2+\vec{x}^2}{\eta}\frac{\partial}{\partial \eta} + 2x_\mu(x\cdot\partial) \\
&= 2x_\mu(x\cdot\partial) - \vec{x}^2\partial_\mu + \eta^2\partial_\mu - 2\rho x_\mu\frac{\partial}{\partial \rho} + 2x_\mu\eta\frac{\partial}{\partial \eta}
\end{aligned}$$

$$\begin{aligned}
D = J_{-10} &= X_{-1}\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X^{-1}} = X_{-1}\frac{\partial}{\partial X_0} + X_0\frac{\partial}{\partial X_{-1}} \\
&= \frac{\rho(1-\eta^2+\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} + \frac{1-\eta^2-\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&\quad + \frac{\rho(1+\eta^2-\vec{x}^2)}{2}\left[\frac{\partial}{\partial \rho} - \frac{1+\eta^2+\vec{x}^2}{2\rho\eta}\frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho}\frac{\partial}{\partial x_\mu}\right] \\
&= \rho\frac{\partial}{\partial \rho} - x^\mu\frac{\partial}{\partial x^\mu} - \eta\frac{\partial}{\partial \eta}
\end{aligned}$$

For $\eta = 0$, $\rho =$ constant

$$\begin{aligned}
P_\mu &= \partial_\mu \\
K_\mu &= 2x_\mu(x\cdot\partial) - x^\nu x_\nu\partial_\mu \\
D &= -x^\mu\partial_\mu
\end{aligned}$$

Note that conformal algebra is satisfied by both $\pm P_\mu$ and $\pm K_\mu$. The null cone corresponds to $\eta = 0$ but no condition imposed on ρ. Thus, there's a gauge redundancy: different values of ρ acting as scale factor for the coordinates correspond to the same physical point.

Next important to understand now is how the tensor fields transform under conformal transformation. We can use the embedding space to deduce their transformation law which is often more illuminating than the algebra gymnastics.

2.1 Tensor field under conformal transformation

We begin with the embedding space formalism to derive how tensors transform under conformal transformations. In this approach, physical spacetime coordinates x^μ are understood as projections from a higher-dimensional embedding space with coordinates $X^A \in \mathbb{R}^{d,2}$, where the conformal group $SO(d,2)$ acts linearly. Tensors in physical space are then obtained by pulling back embedding space tensors via this projection. Specifically, a tensor in physical space is related to its embedding space counterpart as follows:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \frac{\partial x^{\mu_1}}{\partial X^{A_1}}\cdots\frac{\partial x^{\mu_n}}{\partial X^{A_n}}\frac{\partial X^{B_1}}{\partial x^{\nu_1}}\cdots\frac{\partial X^{B_m}}{\partial x^{\nu_m}}\, T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)$$

The map from embedding to physical space is given by

$$x^\mu = \frac{X^\mu}{X^+} \equiv \frac{X^\mu}{X^0 + X^{-1}}$$

but due to the projective nature of this construction—i.e., physical points correspond to rays in the embedding space—we are free to rescale $X \sim \lambda X$, which is a manifestation of dilatation symmetry. Under this rescaling, the tensor should satisfy

$$T(\lambda X) = \lambda^{-\Delta} T(X)$$

which defines its conformal weight Δ. Using this, the projected tensor can be written as

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \left(\frac{1}{X^+}\right)^{\Delta}\frac{\partial x^{\mu_1}}{\partial X^{A_1}}\cdots\frac{\partial x^{\mu_n}}{\partial X^{A_n}}\frac{\partial X^{B_1}}{\partial x^{\nu_1}}\cdots\frac{\partial X^{B_m}}{\partial x^{\nu_m}}\, T^{A_1\cdots A_n}_{B_1\cdots B_m}\left(\frac{X}{X^+}\right)$$

This includes both the Jacobian factors from the change of variables and the prefactor from conformal weight.

If we choose the embedding slice $X^+ = 1$, then the projection simplifies significantly. In this gauge, we define the following object:

$$e^\mu_A = X^+ \frac{\partial x^\mu}{\partial X^A}$$

and the projected tensor becomes

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = e^{\mu_1}_{A_1}\cdots e^{\mu_n}_{A_n} e^{B_1}_{\nu_1}\cdots e^{B_m}_{\nu_m} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)$$

Using the explicit form of the projection $x^\mu = X^\mu/X^+$, we compute

$$\frac{\partial x^\mu}{\partial X^A} = \frac{\delta^\mu_A(X^0 + X^{-1}) - X^\mu(\delta^0_A + \delta^{-1}_A)}{(X^0 + X^{-1})^2}$$

and setting $X^+ = 1$, we get

$$e^\mu_A = \delta^\mu_A, \qquad e^0_A = e^{-1}_A = -x^\mu$$

However, if we don't choose the slice $X^+ = 1$, then the projected tensor carries an additional scaling dependence:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x) = \left(\frac{1}{X^+}\right)^{\Delta+n-m} \underbrace{e^{\mu_1}_{A_1}\cdots e^{\mu_n}_{A_n} e^{B_1}_{\nu_1}\cdots e^{B_m}_{\nu_m} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X)}_{\text{depends only on physical point}}$$

This shows that tensors projected from different embedding space sections—i.e., different choices of X^+—differ by a power of X^+. So if two representations x and $\tilde{x}$ correspond to the same physical point but lie on different sections (i.e., with different values of X^+), then the corresponding tensors are related as

$$T(\tilde{x}) = \left(\frac{X^+}{\tilde{X}^+}\right)^{\Delta+n-m} T(x)$$

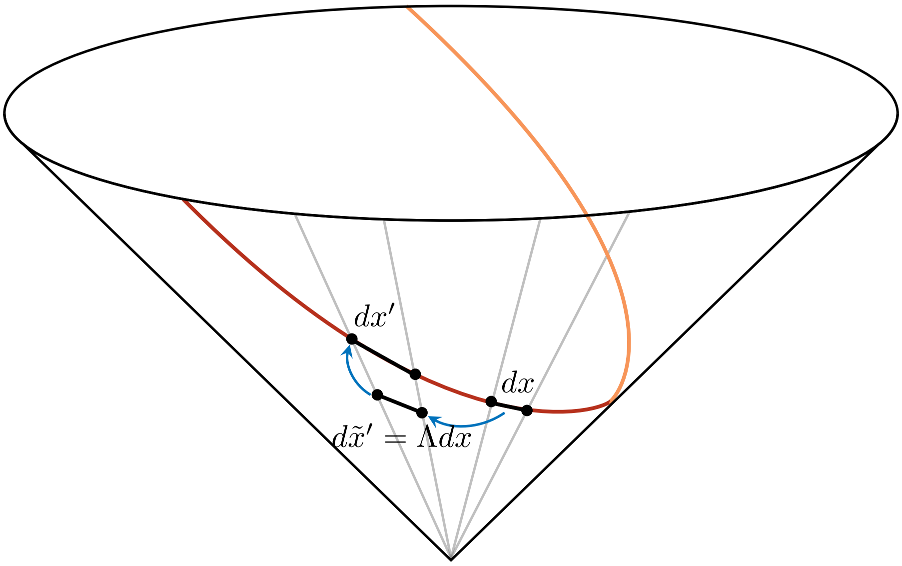


Figure 2.1: Upon Lorentz transformation, the points get mapped to different section however by utilizing the dilatation, we bring it back inside the original Euclidean section.

Next, consider how the tensor transforms under a conformal change of coordinates $x \mapsto x'$ which amounts to applying the corresponding Lorentz transformation $X \mapsto X' = \Lambda X$ in embedding space. Since the Lorentz transformation will map the tensor $T(X)$ to another tensor $T(X')$ living in different section, we will need to use above expression to map it back inside the original section labeled by X^+. The tensor at $\tilde{x}'$ is then:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(\tilde{x}') = \frac{\partial x'^{\mu_1}}{\partial X'^{A_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial X'^{A_n}} \frac{\partial X'^{B_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial X'^{B_m}}{\partial x'^{\nu_m}} T^{A_1\cdots A_n}_{B_1\cdots B_m}(X')$$

Now applying the chain rule, we insert identities:

$$\frac{\partial x'^{\mu}}{\partial X'} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial X'}, \qquad \frac{\partial X'}{\partial x'} = \frac{\partial X'}{\partial x^{\beta}} \cdot \frac{\partial x^{\beta}}{\partial x'}$$

This yields:

$$\underbrace{T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_m}(\tilde{x}')}_{\text{tensor field projected on section } X'^{+}} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) \underbrace{\left[\frac{\partial x^{\alpha_1}}{\partial X^{A_1}} \cdots \frac{\partial x^{\alpha_n}}{\partial X^{A_n}} \frac{\partial X^{B_1}}{\partial x^{\beta_1}} \cdots \frac{\partial X^{B_m}}{\partial x^{\beta_m}}\right] T^{A_1A_2\cdots A_n}_{B_1B_2\cdots B_m}(X)}_{\text{tensor field projected on section } X^{+}}$$

$$\begin{aligned}\left(\frac{X^+}{X'^+}\right)^{\Delta+n-m} T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_m}(x') &= \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x) \\ &= \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right)\left(\frac{X'^+}{X^+}\right)^{\Delta+n-m} T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)\end{aligned}$$

The factor $(X'^+/X^+)^{\Delta+n-m}$ arises because x' and $\tilde{x}'$ are physically the same point, but obtained by projecting from different embedding sections. To express this ratio in terms of the coordinate Jacobian $|\partial x'/\partial x|$, note that from the projection $x^\mu = X^\mu/X^+$ and $x'^\mu = X'^\mu/X'^+$, one finds

$$\left|\frac{\partial x'}{\partial x}\right| = \left|\frac{\partial X}{\partial x} \cdot \frac{\partial x'}{\partial X}\right| = \left|\frac{\partial X}{\partial x} \cdot \frac{\partial x'}{\partial X'} \frac{\partial X'}{\partial X}\right| = \left|\left(\frac{X^+}{X'^+}\right)\frac{\partial X'}{\partial X}\right| = \left|\frac{X^+}{X'^+}\right|^d |\Lambda|$$

and therefore,

$$\left(\frac{X^+}{X'^+}\right)^{\Delta+n-m} = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta+n-m}{d}}$$

Finally, combining everything, the full transformation law for the projected tensor under a conformal coordinate transformation is:

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta+n-m}{d}} \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\right) T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x)$$

This expression makes it manifest that the projected tensor transforms as a tensor under general coordinate transformations, but with an additional *conformal weight* $\Delta + n - m$ that reflects both the homogeneity of the embedding space tensor and the number of upper and lower indices involved in the projection.

2.2 Finding Correlators from Embedding Space

Needless to say, it is significantly easier to construct Lorentz covariant expressions than conformally covariant ones. Therefore, the natural question arises: once we have constructed Lorentz covariant expressions in $d+2$ dimensions, **how do we descend to d dimensions without breaking covariance?**

Since we have already fixed $\eta = 0$ in our derivation of the conformal generators, we now focus on the structure preserved by Lorentz transformations: the null light cone $X^2 = 0$ in embedding space. This cone, defined in $\mathbb{R}^{d+1,1}$ as the space of null rays through the origin, is given by:

$$\begin{aligned} X^2 &= -(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 \\ &= -X^+X^- + \sum_{\mu=1}^{d}(X^\mu)^2 = 0 \end{aligned}$$

Although correlators are initially constructed as Lorentz-invariant functions over the full $d+2$-dimensional ambient space, we now restrict them to the null cone. This constraint effectively reduces the support of such correlators to a $d+1$-dimensional submanifold, since one of the coordinate dependencies—say, X^-—can be eliminated using the condition $X^2 = 0$ (in our case it leads to $\eta = 0$).

Next, we reinterpret embedding space as a fiber bundle over the physical d-dimensional spacetime (where the CFT is defined). Each fiber consists of null lines in the $(d+2)$-dimensional space, and each point in the base space corresponds to an equivalence class of null vectors $X^A \sim \lambda X^A$, for any non-zero λ. This reflects the earlier observation regarding the arbitrariness of ρ: all such rescalings represent the same physical point in d dimensions.

This identification has an important consequence: As mentioned earlier, this introduces gauge redundancy in our description. To eliminate another coordinate, say $X^+ = \rho$, we fix the gauge by selecting a section of the bundle with specific choice of the slice on the embedding space, typically the *Euclidean section*, defined by:

$$X^+ = \rho = f(X^\mu) \equiv f(x^\mu)$$

Although Lorentz transformations may take a null vector outside this section (as they mix time-like and space-like directions), theu can always be brought back by utilising the scaling equivalence $X^A \sim \lambda X^A$. This choice anchors us to physical d-dimensional spacetime, completing the descent from $d+2$ dimensions while maintaining conformal covariance inherited from Lorentz invariance in the higher-dimensional space. With these prescriptions in place we can now identify X^μ with the Euclidean space coordinates x^μ by stripping ρ dependence.

$$X^\mu \equiv x^\mu$$

This leads to definition of X^- based on null condition as:

$$X^- = \frac{\sum_{\mu=1}^{d}(X^\mu)^2}{X^+} = \frac{X^\mu X_\mu}{X^+} = \frac{x^2}{f(X^\mu)}$$

or, equivalently[2]

$$\rho(-\eta^2 + \vec{x}^2) = \frac{\rho^2\vec{x}^2}{\rho} \implies \eta = 0$$

The spacetime interval on this section is given as:

$$ds^2 = dx^2 - dX^+dX^-\Big|_{X^+=f(X^i),X^-=\frac{x^2}{X^+}}$$

This section satisfies two of the following conditions:

- section intersects each of the light rays at some point
- maps each point in d dimensional Euclidean space to a point on the null cone in Embedding space.

We have shown how to get generators of conformal transformation from Lorentz generators by embedding a null cone in ambient space. Let us now analyze how Lorentz Transformation acts on a generic section on that null cone. The Lorentz transformation acting as rotation on the point X^A in the null-cone will move it to another point on the null cone outside the the section $X^B = \Lambda^B{}_A X^A$.

[2] in our notation $x^\mu = \rho\vec{x}^\mu$

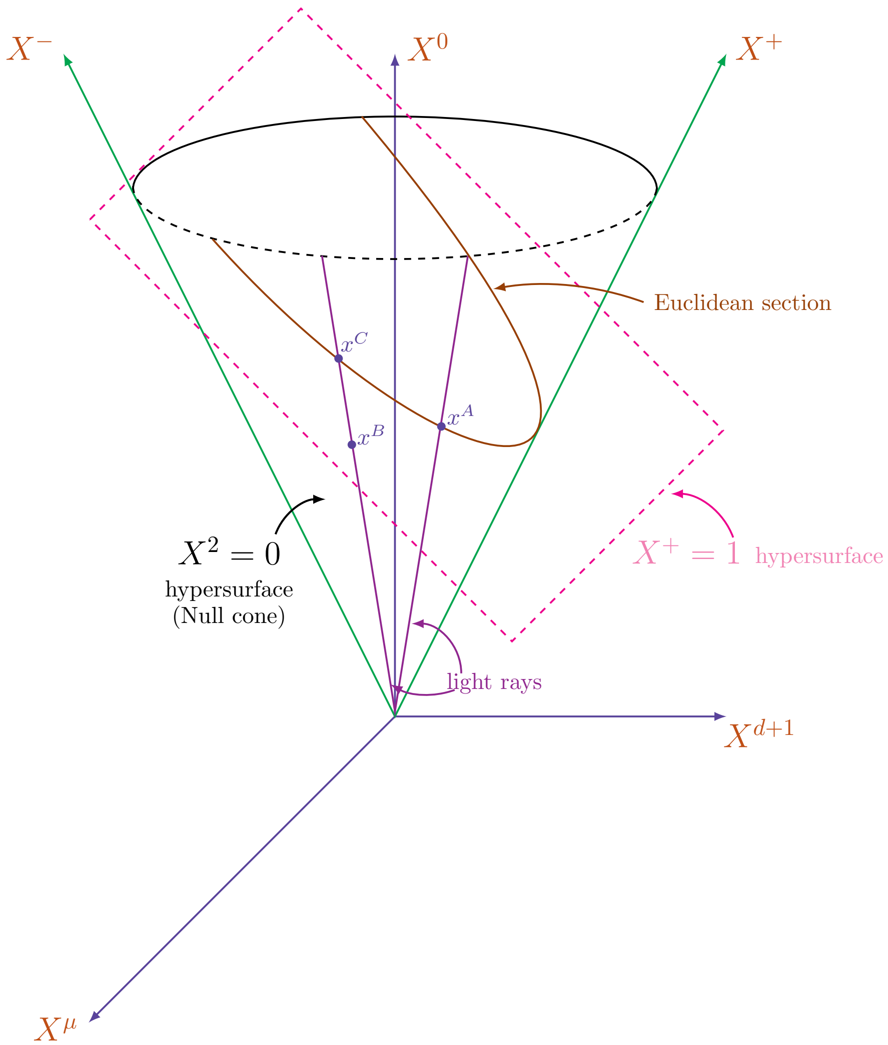


Figure 2.2: The hypersurface perpendicular to X^+ axis cutting at $X^+ = 1$ is shown as a plane and the null hypersurface is shown as the cone. The intersection of these two hypersurfaces describes the Euclidean Section. Dilatations are rotation in the X^0X^{d+1} plane and SCT or momentum generators are rotations in $X^\mu X^{d+1}$ with X^0X^{d+1} plane.

However, suppose via some conformal transformation (dilatation) in d dimensional Euclidean Space, we can move X^B to X^C back into the section. In all these steps, the only thing that changes the metric is the rescaling to get back into the section.

$$\begin{aligned}
ds_B^2 &= dX^M dX_M \\
&= d(\lambda(X)X^M)d(\lambda(X)X_M) \\
&= [\lambda dX^M + X^M(\boldsymbol{\nabla}\lambda \cdot dX)][\lambda dX_M + X_M(\boldsymbol{\nabla}\lambda \cdot dX)] \\
&= \lambda^2 dX^M dX_M + 2\lambda \underbrace{dX^M X_M}_{=0}(\boldsymbol{\nabla}\lambda \cdot dX) + \underbrace{X^M X_M}_{=0}(\boldsymbol{\nabla}\lambda \cdot dX)^2 \\
&= \lambda^2 dX^M dX_M = \lambda^2 ds_C^2
\end{aligned}$$

where we used, $X^2 = 0$ and $X^\mu dX_\mu = 0$ for restricting it to null cone.

Assuming the three conditions we used for simplification applies, the Lorentz Transformation in $d+2$-dimensional spacetime is equivalent to conformal transformation in $d-$dimensional spacetime iff metric in $d-$dimensional space is Euclidean thus, dX_+ in ds^2 has to vanish. It gives us the condition for defining the Euclidean section as $X^+ = \rho = constant$ and thus, for the sake of simplicity, we take it as 1. Thus, we have two conditions which we can use to eliminate two extra degree of freedom.

In the embedding space formalism, choosing an Euclidean section corresponds to picking a specific way to embed the $d-$dimensional space in the $(d+2)-$dimensional space. We define the following map between d dimensional Euclidean Space with conformal symmetry to null cone in $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$

$$(X^+, X^-, X^\mu) \equiv \left(1, x^2, x^\mu\right)$$

Here, we note that choosing a constant value for X^+ would give us a section on the cone on which the induced metric is Euclidean.

2.3 Tensors in Embedding Space

In this section, we will only concern ourselves with traceless and symmetric fields in $\mathbb{R}^d$ and leave the anti-symmetric tensors for future. Consider a symmetric and traceless tensor[3] $O_{M_1 \ldots M_S}$ defined on the cone $X^2 = 0$ in $\mathbb{R}^{d+1,1}$. Under the rescaling $X \to \lambda X$, the tensor transforms as

$$O_{M_1 \ldots M_S}(\lambda X) = \lambda^{-\Delta} O_{M_1 \ldots M_S}(X)$$

i.e. it is a homogeneous function of degree Δ. We expect $O_{M_1 \ldots M_S}$ to get mapped to traceless and symmetric primary field in $\mathbb{R}^d$. Since, each index go from 0 to $d+1$, in $\mathbb{R}^{d+1,1}$ we find that, for $d+2-$dimensional fields other than scalar have 2 more degree of freedom per index than $d-$dimensional fields. In order to remove the extra degree of freedom, we consider the transversality condition.

$$X^{M_1} O_{M_1 \ldots M_S} = 0$$

We define the physical field to be:

$$\phi_{\mu\nu\lambda\ldots}(x) = \frac{\partial X^{M_1}}{\partial x^\mu} \frac{\partial X^{M_2}}{\partial x^\nu} \frac{\partial X^{M_3}}{\partial x^\lambda} \ldots O_{M_1 \ldots M_S}(X) \bigg|_{X = X(x)}$$

Note that, this definition implies a redundancy. Indeed, anything proportional to X^M gives zero since

$$X^2 = 0 \implies X_M \frac{\partial X^M}{\partial x^\mu} = 0$$

Therefore, $O_{M_1 \ldots M_S}(X) \to O_{M_1 \ldots M_S}(X) + X_{M_1} F_{M_2 \ldots M_S}(X)$ gets mapped to the same physical field. This $SO(d+1,1)$ tensor is sometimes referred to as pure gauge in the literature. It is this gauge redundancy that reduces another degree of freedom per index by making it unphysical.

2.4 Examples: Two point and Three point correlator

In the last section we showed how the embedding space formalism put in place could be used to deduce the conformally invariant correlator. In this section we will utilize the formalism and explicitly construct two point and three point function using the formalism developed thus far. From 1.3, we know that the ratios are only invariant under dilatation and translation. Therefore, we seek to construct an invariant out of these ratios and metric tensor which is also invariant under SCT and the exchange of indices $\mu \leftrightarrow \nu$. First we will derive the form of scalar one point correlator. A scalar primary is denoted by $\hat{\mathcal{O}}_\Delta$ (notice the absence of the Lorentz index, which indicates that it is a scalar operator). We want to enforce the invariance of correlator under conformal transformation. For a one-point function, this reduces to

$$\langle \hat{\mathcal{O}}_\Delta(x'^\mu) \rangle = \langle \hat{\tilde{\mathcal{O}}}_\Delta(x'^\mu) \rangle \tag{2.1}$$

This condition must be enforced for all four conformal transformations. We will begin by enforcing translation.

Translation: $\tilde{x}^\mu = x^\mu + a^\mu$

It was given previously that the Jacobian for a translation is one. Therefore, our operator simply does not change under this translation

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \hat{\mathcal{O}}_\Delta(x^\mu)$$

enforcing this in (2.1), we are left with

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu) \rangle$$

[3] symmetric tensors with spin s under $SO(d)$ form irreducible representations that correspond to integer spin particles (bosons). Anti-symmetric tensor fields have interpretation like they correspond to bivector of spinors etc.

where the operators are now the same on both sides of the equation. Notice that a correlation function is just a function. Therefore, this is equivalent to saying

$$f(\tilde{x}^\mu) = f(x^\mu)$$

Since this must be true for every possible translation, this tells us that the function has the same output regardless of what the input is, which means the function must just be some constant. Therefore, by enforcing translation we can conclude that

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu)\rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle = \text{constant} = C$$

We are not yet done. We need to make sure that all four transformations leave the one-point function invariant. Let's see what we can learn when we enforce dilatation.

Dilatation: $\tilde{x}^\mu = \lambda x^\mu$

Applying the Jacobian for dilitation, we see our Primary Scalar Operator transforms as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right|^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta(x^\mu)$$

We want to enforce this in (2.1) and use our results from enforcing translation invariance. This gives us

$$\begin{aligned}\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu)\rangle &= \langle \hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu)\rangle \\ &= \langle \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta(x^\mu)\rangle \\ &= \lambda^{-\Delta}\langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle\end{aligned}$$

We found previously, by enforcing translation, that

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu)\rangle = C$$

which means

$$C = \lambda^{-\Delta} C$$

This equation must be true for arbitrary scale factor λ. Therefore, unless $\Delta = 0$, we can conclude that $C = 0$.

For unitary CFTs, the only $\Delta = 0$ operator is the identity operator. So, with the exception of the identity, all one-point functions must vanish!

$$\boxed{\langle \hat{\mathcal{O}}_\Delta(x^\mu)\rangle = 0 \text{ for } \Delta \neq 0} \tag{2.2}$$

We said that we must impose all four conformal transformations, but the others are trivially satisfied at this point. So we are done with one-point correlators! Again, we'd like to highlight the fact that this is the result for ALL CFTs. You don't need to know anything else about the system, only that it has conformal symmetry.

2.4.1 Two-point Scalar Primary

For two-point functions, we need to enforce

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle \tag{2.3}$$

Again, this must be done for all four conformal transformations. As with the one-point function, we will begin by enforcing translation.

Translation

First, notice that $\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle$ is an object that takes two positions as inputs and gives back a number, so we can just write this as a function of $\tilde{x}_1^\mu$ and $\tilde{x}_2^\mu$

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = f(\tilde{x}_1^\mu, \tilde{x}_2^\mu)$$

We found previously that under translations, scalar primary operators transform as

$$\hat{\tilde{\mathcal{O}}}(\tilde{x}^\mu) = \hat{\mathcal{O}}(x^\mu)$$

Putting this result in the right side of equation (2.3), we find that

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle$$

Notice that the only differences between the left and right side of this equation are the inputs. The function on each side is the same

$$f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) = f(x_1^\mu, x_2^\mu)$$

Under translation, we have $\tilde{x}^\mu = x^\mu + a^\mu \to x^\mu = \tilde{x}^\mu - a^\mu$. If we put this into the previous equation it becomes

$$f(x_1^\mu - a^\mu, x_2^\mu - a^\mu) = f(x_1^\mu, x_2^\mu)\ ,\ \forall a^\mu$$

This must be true regardless of the value of a^μ which means that a^μ must somehow cancel out. This is only satisfied if it is a function of $x_1^\mu - x_2^\mu$ so we have

$$f(x_1^\mu, x_2^\mu) = f(x_1^\mu - x_2^\mu)$$

That is, our function cannot be any function of the two positions. Rather, it can only depend on the displacement between the two positions. Therefore,

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(x_1^\mu - x_2^\mu)$$

Let's now see what we can learn by enforcing rotation.

Rotation $\tilde{x}^\mu = \Lambda^\mu_\nu x^\mu$

The Jacobian for rotation is the same as for translation, 1. Therefore, scalar primary operators transform the same under rotation as they do under translation

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle = \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle$$

Re-expressing this in a more familiar form, as functions, we have

$$f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) = f(x_1^\mu, x_2^\mu)$$

Next, we impose what we found by imposing translational invariance

$$f(\tilde{x}_1^\mu - \tilde{x}_2^\mu) = f(x_1^\mu - x_2^\mu)$$

Expressing the transformed coordinates in terms of our original coordinate system, we find

$$f(\Lambda^\mu_\nu(x_1^\nu - x_2^\nu)) = f(x_1^\mu - x_2^\mu)$$

This tells us that applying a rotation has no effect on the output. This means that the function must depend only on the magnitude of the separation $|x_1^\mu - x_2^\mu|$ (Recall, rotating a vector changes it, but rotating a scalar does nothing). So, from applying translational and rotational invariance, we can conclude that

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = f(|x_1^\mu - x_2^\mu|)$$

We will now continue by enforcing invariance under dilatation.

Dilatation

Recall, under dilatation, scalar primary operators transform as

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \lambda^{-\Delta}\hat{\mathcal{O}}_\Delta(x^\mu)$$

Substituting this into our two-point function condition, eqn. (2.3), we have

$$\begin{aligned}
\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle &= \langle \hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\rangle \\
&= \langle \lambda^{-\Delta_1}\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\lambda^{-\Delta_2}\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle \\
&= \lambda^{-\Delta_1}\lambda^{-\Delta_2}\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle \\
&= \lambda^{-(\Delta_1+\Delta_2)}f(|x_1^\mu - x_2^\mu|)
\end{aligned}$$

where we are able to pull the λ's out of the correlator because they are just scalars. Notice also that we used what we already learned from translational and rotational invariance. This tells us that

$$f(|\tilde{x}_1^\mu - \tilde{x}_2^\mu|) = \lambda^{-(\Delta_1+\Delta_2)} f(|x_1^\mu - x_2^\mu|)$$

We can apply the transformation to the coordinates on the left-hand-side, which gives

$$f(\lambda|x_1^\mu - x_2^\mu|) = \lambda^{-(\Delta_1+\Delta_2)} f(|x_1^\mu - x_2^\mu|)$$

What does this mean? We can consider expanding our function in a power series

$$f(|x_1^\mu - x_2^\mu|) = \sum_n c_n |x_1^\mu - x_2^\mu|^n$$

Substituting this in above gives

$$\sum_n c_n \lambda^n |x_1^\mu - x_2^\mu|^n = \lambda^{-(\Delta_1+\Delta_2)} \sum_n c_n |x_1^\mu - x_2^\mu|^n$$

This is only satisfied for all λ, if all $n = 0$ except $n = -(\Delta_1 + \Delta_2)$. Therefore, after enforcing translation, rotation, and dilatation symmetry we have

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle = C|x_1^\mu - x_2^\mu|^{-(\Delta_1+\Delta_2)} \tag{2.4}$$

where C is some undetermined constant.

Special Conformal Transformation

Enforcing special conformal symmetry directly is a very messy business. Luckily for us, as discussed previously, a special conformal transformation is equivalent to performing an inversion, followed by a translation, followed by another inversion. Since we have already enforced translational invariance, this means it is sufficient to enforce inversion invariance, which is much easier. Recall, an inversion is given by

$$x^\mu = \frac{\tilde{x}^\mu}{\tilde{x}^2}$$

As with the other transformations, we need the Jacobian for inversion in order to see how the operators will transform. This is given by

$$\left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = \frac{1}{\tilde{x}^{2D}}$$

Therefore, under inversion, scalar primary operators transform as

$$\tilde{\hat{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left(\frac{1}{\tilde{x}^{2D}} \right)^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \frac{1}{(\tilde{x}^2)^\Delta} \hat{\mathcal{O}}_\Delta(x^\mu)$$

As usual, we will now go put this into equation (2.3) to enforce the symmetry

$$\begin{aligned}
\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle &= \langle \tilde{\hat{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu) \tilde{\hat{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle \\
&= \left\langle \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \right\rangle \\
&= \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle
\end{aligned}$$

Now, we can use our result from enforcing dilitation to replace $\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle$ on the left and $\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle$ on the right of this equation to get

$$\frac{C}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \frac{C}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}} \tag{2.5}$$

With a bit of algebra, this is equivalent to

$$\frac{(\tilde{x}_1^2)^{\Delta_1} (\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \frac{1}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}} \tag{2.6}$$

In order to put this in a more friendly form, we will use the following identity for inversions. Note: verifying this relationship requires substituting in the inversion transformation and some algebra. The reader is highly encouraged to check it.

$$\frac{\tilde{x}_1^2\tilde{x}_2^2}{(\tilde{x}_1^\mu - \tilde{x}_2^\mu)^2} = \frac{1}{(x_1^\mu - x_2^\mu)^2} \tag{2.7}$$

Using this identity in eqn. (2.6), we find

$$\frac{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1+\Delta_2}} = \left[\frac{\tilde{x}_1^2\tilde{x}_2^2}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^2}\right]^{\frac{\Delta_1+\Delta_2}{2}}$$

This is only satisfied if

$$\Delta_1 = \Delta_2$$

Therefore, we find that the two-point function vanishes, unless the dimensions of the two operators are the same. In summary, the two-point function for scalar primaries in ANY CFT is given by

$$\boxed{\langle\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\rangle = \frac{C\delta_{\Delta_1\Delta_2}}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}}} \tag{2.8}$$

Note that it is standard convention to choose to normalize your operators so that $C = 1$, so you will often see this without the C constant included. We leave it here for complete generality.

2.4.2 Three-point Scalar Primary

For the three-point function, we need to enforce

$$\langle\hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = \langle\hat{\tilde{\mathcal{O}}}_1(x_1^\mu)\hat{\tilde{\mathcal{O}}}_2(x_2^\mu)\hat{\tilde{\mathcal{O}}}_3(x_3^\mu)\rangle \tag{2.9}$$

Enforcing the symmetries for the three-point function follows in a very similar way to the two-point function, so we will not include as much detail. The reader is encouraged to work through any excluded details on their own.

Poincaré

For translations and rotations, the same line of argumentation that was used for two-point functions can be applied. However, instead of two points at our disposal, we have three. Therefore, our function can be a function of the magnitude of the separations between any parings of three points.

$$\langle\hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|)$$

where $|x_{12}^\mu| = |x_1^\mu - x_2^\mu|$, $|x_{23}^\mu| = |x_2^\mu - x_3^\mu|$, and $|x_{31}^\mu| = |x_3^\mu - x_1^\mu|$.

Dilatation

Enforcing dilitation invariance with our three-point function, we have

$$\begin{aligned}\langle\hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(\tilde{x}_3^\mu)\rangle &= \langle\hat{\tilde{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu)\hat{\tilde{\mathcal{O}}}_{\Delta_3}(\tilde{x}_3^\mu)\rangle \\ &= \langle\lambda^{-\Delta_1}\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\lambda^{-\Delta_2}\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\lambda^{-\Delta_3}\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu)\rangle \\ &= \lambda^{-\Delta_1}\lambda^{-\Delta_2}\lambda^{-\Delta_3}\langle\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu)\rangle\end{aligned}$$

Using our results from enforcing Poincaré invariance, this becomes

$$f(|\tilde{x}_{12}^\mu|, |\tilde{x}_{23}^\mu|, |\tilde{x}_{31}^\mu|) = \lambda^{-(\Delta_1+\Delta_2+\Delta_3)}f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) \tag{2.10}$$

Subsituting in the dilitation transformation on the LHS, this is

$$f(\lambda|x_{12}^\mu|, \lambda|x_{23}^\mu|, \lambda|x_{31}^\mu|) = \lambda^{-(\Delta_1+\Delta_2+\Delta_3)}f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) \tag{2.11}$$

As with the two-point function, we can expand our function in a power series.

$$f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) = \sum_{nmp} c_{nmp}|x_{12}^\mu|^n|x_{23}^\mu|^m|x_{31}^\mu|^p \tag{2.12}$$

Substituting this in, we find that all terms must vanish, unless

$$n + m + p = -(\Delta_1 + \Delta_2 + \Delta_3)$$

Therefore, dilitation and Poincaré invariance tell us

$$\langle \hat{\mathcal{O}}_1(x_1^\mu)\hat{\mathcal{O}}_2(x_2^\mu)\hat{\mathcal{O}}_3(x_3^\mu)\rangle = \sum_{nmp=-(\Delta_1+\Delta_2+\Delta_3)} c_{nmp}|x_{12}^\mu|_n|x_{23}^\mu|^m|x_{31}^\mu|^p \tag{2.13}$$

Special Conformal Transformation

Again, to find the effect of imposing special conformal symmetry, we need only to impose inversion symmetry, which is much easier. Although easier, the algebra is still quite nasty and will not be shown here. Ultimately, inversion (therefore special conformal) invariance leads to the additional constraint that all terms vanish, unless

$$\begin{aligned} n &= \Delta_1 + \Delta_2 - \Delta_3 \\ m &= \Delta_1 + \Delta_3 - \Delta_2 \\ p &= \Delta_2 + \Delta_3 - \Delta_1 \end{aligned} \tag{2.14}$$

Therefore, after enforcing all of the conformal symmetries on the 3-point function of scalar primaries, we find

$$\boxed{\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu)\hat{\mathcal{O}}_{\Delta_2}(x_2^\mu)\hat{\mathcal{O}}_{\Delta_3}(x_3^\mu) = \frac{C_{123}}{|x_{12}^\mu|^n|x_{23}^\mu|^m|x_{31}^\mu|^p}} \tag{2.15}$$

where n, m, and p are given by (2.14). We find that, as was the case with the two-point scalar primaries, the spatial dependence of 3-point scalar primaries are completely determined. We are left only with a set of constants C_{123}. It turns out that this set of constants is vitally important to defining any particular conformal field theory and they tell you how much your given operators interact. This set of constants goes by various names including the *3-point coefficients*, the *OPE coefficients*, and the *structure constants*.

2.4.3 Going beyond scalars

Moving on, we next consider the two point corrector of vector field. The ansatz for such a correlator is:[4]

$$\langle J_\mu(x_1)J_\nu(x_2)\rangle = C\underbrace{\frac{1}{|x_1-x_2|^{2\Delta}}}_{\text{same as scalar case}}\left[g_{\mu\nu} + \delta\frac{(x_1-x_2)_\mu(x_1-x_2)_\nu}{(x_1-x_2)^2}\right]$$

The correlation function is invariant under translation, therefore we will consider following redefinition:

$$\begin{aligned} \langle J_\mu(x_1)J_\nu(x_2)\rangle &= \langle J_\mu(x_1-x_2)J_\nu(0)\rangle \\ &= \langle J_\mu(x_{12})J_\nu(0)\rangle = \langle J_\mu(x)J_\nu(0)\rangle \end{aligned}$$

Since SCT is just inversion→translation→inversion, we can this property to our advantage. As the correlation function is already invariant under translations, it suffices to verify its invariance under inversions. If this property holds, then by extension, the correlation function will also be invariant under SCT. The inversion transformation is given as[5]:

$$x'_\mu = \frac{x_\mu}{x^2} \qquad\qquad |x'|^2 = \frac{1}{|x|^2}$$

and

$$\frac{\partial x'_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\frac{x_\nu}{x^2} = \frac{1}{x^2}\left[g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right] = x'^2\underbrace{\left[g_{\mu\nu} - 2\frac{x'_\mu x'_\nu}{x'^2}\right]}_{I_{\mu\nu}}$$

[4]pg 24 of "CFT with boundary and defects" by Herzog

[5]pg 17-18 of "Quantum Gravity and Cosmology based on Conformal Field Theory" and section 4.5 of "A conformal field theory primer in $D \geq 3$" by Andrew Evans

The vector field would transform as

$$\langle J_\mu(x_1')J_\nu(x_2')\rangle = \underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{\Delta/d}\left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{\Delta/d}}_{\substack{\text{this was used to derive}\\ \text{the correlation function for scalar case}}}\overbrace{\frac{\partial x'_\alpha}{\partial x^\mu}\frac{\partial x'_\beta}{\partial x^\nu}}^{\text{without conformal factor}}\langle J^\alpha(x_1)J^\beta(x_2)\rangle$$

we see that

$$\underbrace{\left|\frac{\partial x_\alpha}{\partial x'^\mu}\right|^{\Delta/d}_{x=x_1}}_{|x_1'|^{-2\Delta}}\left|\frac{\partial x_\beta}{\partial x'^\nu}\right|^{\Delta/d}_{x=x_2}\frac{1}{|x_{12}|^{2\Delta}} = |x_1'|^{-2\Delta}|x_2'|^{-2\Delta}\frac{1}{|x_{12}|^{2\Delta}} = \frac{1}{|x_{12}'|^{2\Delta}}$$

where we used

$$\begin{aligned}
|x_{12}'|^2 &= \left(\frac{x_1^\mu}{x_1^2}-\frac{x_2^\mu}{x_2^2}\right)^2\\
&= \frac{|x_1|^2}{x_1^4}+\frac{|x_2|^2}{x_2^4}-2\frac{x_1^\mu}{x_1^2}\frac{x_{2\mu}}{x_2^2}\\
&= \frac{1}{x_1^2}-2\frac{x_1}{x_1^2}\cdot\frac{x_2}{x_2^2}+\frac{1}{x_2^2}\\
&= \frac{(x_1-x_2)^2}{x_1^2x_2^2} = \frac{|x_{12}|^2}{|x_1|^2|x_2|^2} = \frac{|x_{12}|^2}{|x_1'|^{-2}|x_2'|^{-2}}
\end{aligned}$$

Then, we only have to ensure that $g^{\mu\nu}+\delta\frac{x_{12}'^\mu x_{12}'^\nu}{x_{12}'^2}$ is invariant under inversion.[6]

$$\begin{aligned}
g^{\mu\nu}+\delta\frac{(x_{12}')^\mu(x_{12}')^\nu}{(x_{12}')^2} &= \left(\delta^\mu_\alpha-2\frac{x_1^\mu x_{1\alpha}}{x_1^2}\right)\left(\delta^\nu_\beta-2\frac{x_2^\nu x_{2\beta}}{x_2^2}\right)\left[g^{\alpha\beta}+\delta\frac{x_{12}^\alpha x_{12}^\beta}{x_{12}^2}\right]\Bigg|_{x^\mu=\frac{x'^\mu}{|x'|^2}}\\
&= \left(\delta^\mu_\alpha-2\frac{x_1'^\mu x_{1\alpha}'}{|x_1|'^2}\right)\left(\delta^\nu_\beta-2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right)\left[g^{\alpha\beta}+\delta\frac{(x_1'|x_2'|^2-x_2'|x_1'|^2)^\alpha(x_1'|x_2'|^2-x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2}\right]\\
&= \Bigg[g^{\mu\beta}+\delta\frac{(x_1'|x_2'|^2-x_2'|x_1'|^2)^\mu(x_1'|x_2'|^2-x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2}-2\frac{x_1'^\mu x_1'^\beta}{|x_1|'^2}\\
&\quad -2\delta\frac{x_1'^\mu(\cancel{|x_1'|^2}|x_2'|^2-x_1'\cdot x_2'\cancel{|x_1'|^2})(x_1'|x_2'|^2-x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2\cancel{|x_1'|^2}|x_2'|^2}\Bigg]\left(\delta^\nu_\beta-2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right)\\
&= \left[g^{\mu\beta}-\delta\frac{(x_1'|x_2'|^2+x_2'|x_1'|^2-2x_1'(x_1'\cdot x_2'))^\mu(x_1'|x_2'|^2-x_2'|x_1'|^2)^\beta}{x_{12}'^2|x_1'|^2|x_2'|^2}-2\frac{x_1'^\mu x_1'^\beta}{|x_1|'^2}\right]\left(\delta^\nu_\beta-2\frac{x_2'^\nu x_{2\beta}'}{|x_2'|^2}\right)\\
&= \left[g^{\mu\nu}-\delta\frac{(x_1'|x_2'|^2+x_2'|x_1'|^2-2x_1'(x_1'\cdot x_2'))^\mu(x_1'|x_2'|^2-x_2'|x_1'|^2)^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2}-2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2}\right]\\
&\quad -2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2}+2\delta\frac{(x_1'|x_2'|^2+x_2'|x_1'|^2-2x_1'(x_1'\cdot x_2'))^\mu(x_2'\cdot x_1'\cancel{|x_2'|^2}-\cancel{|x_2'|^2}|x_1'|^2)x_2'^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2\cancel{|x_2'|^2}}+4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2}\\
&= g^{\mu\nu}-2\frac{x_1'^\mu x_1'^\nu}{|x_1|'^2}-2\frac{x_2'^\nu x_2'^\mu}{|x_2'|^2}+4\frac{x_1'^\mu x_2'^\nu(x_1'\cdot x_2')}{|x_1|'^2}\\
&\quad -\delta\frac{\{x_1'|x_2'|^2+x_2'|x_1'|^2-2x_1'(x_1'\cdot x_2')\}^\mu\{x_1'|x_2'|^2+x_2'|x_1'|^2-2(x_1'\cdot x_2')x_2'\}^\nu}{x_{12}'^2|x_1'|^2|x_2'|^2}
\end{aligned}$$

[6]the conformal factor is there following eqn 55 of TASI Lectures on the Conformal Bootstrap. The tensor operator under inversion transforms as mentioned in eqn 3.18 of Conformal Field Theory with Boundaries and Defects or eqn 1.55 and 1.60 of EPFL Lectures on Conformal Field Theory in D¿= 3 Dimensions

$$O'^\mu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\frac{\Delta+1}{d}}\frac{\partial x'^\mu}{\partial x^\nu}O^\nu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\Delta/d}I^\mu_\nu(x')O^\nu(x')$$

$$O'_\mu(x') = \left|\frac{\partial x^\alpha}{\partial x'^\beta}\right|^{\frac{\Delta-1}{d}}\frac{\partial x^\nu}{\partial x'^\mu}O_\nu(x')$$

$$
\begin{aligned}
&= g^{\mu\nu} - 2\frac{x_1'^{\mu}x_1'^{\nu}}{|x_1|'^2} - 2\frac{x_2'^{\nu}x_2'^{\mu}}{|x_2'|^2} + 4\frac{x_1'^{\mu}x_2'^{\nu}(x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad - \delta\frac{\{x_2'|x_1'|^2 + |x_1'-x_2'|^2 x_1' - |x_1'|^2 x_1'\}^{\mu}\{x_1'|x_2'|^2 + |x_1'-x_2'|^2 x_2' - |x_2'|^2 x_2'\}^{\nu}}{x_{12}'^2|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - 2\frac{x_1'^{\mu}x_1'^{\nu}}{|x_1|'^2} - 2\frac{x_2'^{\nu}x_2'^{\mu}}{|x_2'|^2} + 4\frac{x_1'^{\mu}x_2'^{\nu}(x_1'\cdot x_2')}{|x_1|'^2} \\
&\quad + \delta\frac{x_{12}'^{\mu}x_{12}'^{\nu}}{x_{12}'^2} - \delta\frac{x_1'^{\mu}}{|x_1'|^2}x_{12}'^{\nu} + \delta\frac{x_2'^{\nu}}{|x_2'|^2}x_{12}'^{\mu} - \delta|x_{12}'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2} \\
&= g^{\mu\nu} - (\delta+2)\frac{x_1'^{\mu}x_1'^{\nu}}{|x_1|'^2} - (\delta+2)\frac{x_2'^{\nu}x_2'^{\mu}}{|x_2'|^2} + 2(\delta+2)\frac{x_1'^{\mu}x_2'^{\nu}(x_1'\cdot x_2')}{|x_1|'^2} + \cancel{\delta\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2}} + \cancel{\delta\frac{x_1'^{\mu}x_2'^{\nu}}{|x_2'|^2}} \\
&\quad - \cancel{\delta|x_1'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2}} - \cancel{\delta|x_2'|^2\frac{x_1'^{\mu}x_2'^{\nu}}{|x_1'|^2|x_2'|^2}} + \delta\frac{x_{12}'^{\mu}x_{12}'^{\nu}}{x_{12}'^2}
\end{aligned}
$$

which implies $\delta = -2$. Hence, the two point function is given as

$$
\langle J_\mu(x)J_\nu(0)\rangle = \frac{C}{|x|^{2\Delta}}\left[g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right]
$$

The embedding space formalism gives the same answer[7]: Considering a tensor field of $SO(d+1,1)$ denoted as $O_{A_1\ldots A_n}(X)$, with the properties

- defined on the null-cone $X^2 = 0$,
- traceless and symmetric,
- homogeneous of degree $-\Delta$ in X, i.e., $O_{A_1\ldots A_n}(\lambda X) = \lambda^{-\Delta}O_{A_1\ldots A_n}(X)$,
- transverse $X^{A_i}O_{A_1\ldots A_n}(X) = 0$, with $i = 1,\ldots,n$

It is clear that those are conditions rendering $O_{A_1\ldots A_n}(X)$ manifestly invariant under $SO(d+1,1)$. In order to find the corresponding tensor in $\mathbb{R}^d$, one has to restrict $O_{A_1\ldots.A_n}(X)$ to the Poincaré section and project the indices as

$$
\langle O^\mu(x_1)O^\nu(x_2)\rangle = \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_1^B}{\partial x_2^\nu}\langle O_A(X_1)O_B(X_2)\rangle
$$

For example, the most general form of the two-point function of two operators with spin-1 and dimension Δ can be derived as:[8]

$$
\left\langle O^A(X_1)O^B(X_2)\right\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta^{AB} + \alpha\frac{X_2^A X_1^B}{X_1\cdot X_2} + \beta\frac{X_1^A X_2^B}{X_1\cdot X_2}\right]
$$

We will drop the last term as it projects to zero anyways.

$$
\left\langle O^A(X_1)O^B(X_2)\right\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta^{AB} + \alpha\frac{X_2^A X_1^B}{X_1\cdot X_2}\right]
$$

According to the transverse condition

$$
X_{A1}\left\langle O^A(X_1)O^B(X_2)\right\rangle = \frac{C_{12}}{(X_1\cdot X_2)^\Delta}[X_1^B + \alpha X_1^B] = 0 \implies \alpha = -1
$$

we now use the projection to find the correlation function in $\mathbb{R}^d$:

$$
\langle O_\mu(x_1)O_\nu(x_2)\rangle = \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\langle O_A(X_1)O_B(X_2)\rangle
$$

[7] section 2.4 of "Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions" or section 5.2.2 of "Conformal Field Theory" by Loriano Bonora

[8] here the terms in bracket is chosen such that they are invariant under the replacement $x \to \lambda x$. We are not using the transformation law for any of them. Under which, even the metric will change to $\eta_{AB} \to \lambda^{-2}\eta_{AB}$.

$$
\begin{aligned}
&= \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\eta_{AB} - \frac{X_{A2}X_{B1}}{X_1\cdot X_2}\right] \\
&= \frac{C_{12}}{(X_1\cdot X_2)^\Delta}\left[\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\eta_{AB} - \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\frac{X_{A2}X_{B1}}{X_1\cdot X_2}\right] \\
&= \frac{C_{12}}{(x_1-x_2)^{2\Delta}}\left[g_{\mu\nu} - 2\frac{(x_1-x_2)_\mu(x_1-x_2)_\nu}{(x_1-x_2)^2}\right]
\end{aligned}
$$

where we used $X^A = (X^a, X^+, X^-) = (x^a, 1, x^2)$, $X_B = \left(x_a, -\frac{1}{2}x^2, -\frac{1}{2}\right)$ and $\eta_{ab} = I_{d\times d}$ with $\eta_{+-} = \eta_{-+} = -1/2$

$$
\begin{aligned}
\frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\eta_{AB} &= g_{\mu\nu} \\
\frac{\partial X_1^A}{\partial x_1^\mu}X_{A2} &= \eta_{ab}\frac{\partial x_1^a}{\partial x_1^\mu}x_2^b - \frac{1}{2}\cancelto{0}{\frac{\partial 1}{\partial x_1^\mu}}x_2^2 - \frac{1}{2}\frac{\partial x_1^2}{\partial x_1^\mu}1 \\
&= \eta_{ab}\delta_\mu^a x_2^b - x_{1\mu} = (x_2-x_1)_\mu = -(x_1-x_2)_\mu \\
\frac{\partial X_2^B}{\partial x_2^\nu}X_{B1} &= \eta_{ab}\frac{\partial x_2^a}{\partial x_2^\mu}x_1^b - \frac{1}{2}\cancelto{0}{\frac{\partial 1}{\partial x_2^\mu}}x_1^2 - \frac{1}{2}\frac{\partial x_2^2}{\partial x_2^\mu}1 = (x_1-x_2)_\nu \\
(X_1-X_2)^A(X_1-X_2)_A &= (x_1-x_2)^a(x_1-x_2)_a - \frac{1}{2}(1-1)(x_1^2-x_2^2) - (x_1^2-x_2^2)(\tfrac{1}{2}-\tfrac{1}{2}) \\
\implies X_1\cdot X_2 &= -\frac{1}{2}(x_1-x_2)^2
\end{aligned}
$$

Next, we boostrap three point correlator. [9] On the null cone we will have

$$
\langle\phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle = \frac{W_M}{(-2X_1\cdot X_2)^{\alpha_{123}}(-2X_1\cdot X_3)^{\alpha_{132}}(-2X_2\cdot X_3)^{\alpha_{231}}}
$$

where the powers α_{ijk} of the scalar factor are determined by the dilatation as in case of scalar operators and the tensor structure W_M equals to

$$
W_M = \frac{(-2X_2\cdot X_3)X_{1M} - (-2X_1\cdot X_3)X_{2M} - (-2X_1\cdot X_2)X_{3M}}{(-2X_1\cdot X_2)^{\frac{1}{2}}(-2X_1\cdot X_3)^{\frac{1}{2}}(-2X_2\cdot X_3)^{\frac{1}{2}}}.
$$

Let us comment a few things on the tensor structure. The relative sign is, as before, fixed by transversality.

$$
\begin{aligned}
(X_1)^M W_M &= 0 \\
(X_2)^M W_M &= 0 \\
(X_3)^M W_M &= 0
\end{aligned}
$$

We drop the term proportional to X_{3M}, since would project to zero anyway. The scaling behavior of correlation function under dilatation is completely determined in the scalar part so the tensor structure have scaling 0 in all variables ($X \to \lambda X \implies W_\mu \to \lambda^0 W_\mu$). Finally, it is immediate to check that the tensor structure is transverse, i.e. $(X_3)_M W_M = 0$. Projecting to physical space as:

$$
\langle\phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle = \frac{\partial X_3^M}{\partial x_3^\mu}\langle\phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle
$$

we find, as explicitly computed before,

$$
\begin{aligned}
&\frac{\partial X_3^M}{\partial x_3^\mu}X_{iM} = (x_i-x_3)_\mu, \quad i = 1,2 \\
&-2X_i\cdot X_j = (x_i-x_j)^2, \quad i = 1,2,3\ (i<j),
\end{aligned}
$$

so that we end up with the tensor structure

$$
W_\mu = \frac{|x_2-x_3|^2(x_1-x_3)_\mu - |x_1-x_3|^2(x_2-x_3)_\mu}{|x_1-x_2||x_1-x_3||x_2-x_3|} = \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}
$$

[9] pg 30 of Masters Thesis on "Spinning Correlators at Finite Temperature" of Oscar Arandes Tejerina

Therefore, the three-point function of two scalars and one spin 1 operators in physical space is given by

$$\begin{aligned}\langle\phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle &= \frac{\frac{|x_{23}|^2(x_{13})_\mu-|x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}\\ &= \frac{|x_{23}|^2(x_{13})_\mu-|x_{13}|^2(x_{23})_\mu}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2+1}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1+1}}\end{aligned}$$

The three-point function of higher-spin operators $J_{\mu_1\ldots\mu_\ell}$ is constructed from the above, analogously as what we did for the two-point functions, since it turns out that W_μ is the only indexed object for three points that is conformal invariant.

2.5 Fermions in Embedding Space

Following is taken from section 3.2 of Lectures on Conformal Field Theories by Hugh Osborn. To discuss spinor fields in the embedding formalism requires extending the usual $d-$dimensional gamma matrices to $d+2$ dimensions. For $d=2n$, we define[10]

$$\begin{aligned}a_0^\pm &= \frac{1}{2}(\pm\gamma^0+\gamma^1)\\ a_1^\pm &= \frac{1}{2}(\gamma^2\pm i\gamma^3)\\ a_2^\pm &= \frac{1}{2}(\gamma^4\pm i\gamma^5)\\ &\vdots\\ a_{\frac{d-2}{2}}^\pm &= \frac{1}{2}(\gamma^{d-2}\pm i\gamma^{d-1})\end{aligned}$$

where gamma matrices satisfy

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$$

One can show that:

$$\begin{aligned}&\{a_i^-,a_j^-\}=\{a_i^+,a_j^+\}=0\\ &\{a_i^-,a_j^+\}=\delta_{ij}\qquad i,j=0,1,2\ldots{}^{d-2}\!/_2.\end{aligned}\tag{2.16}$$

In the literature ${}^{d-2}\!/_2$ is defined as another variable labeled by k, but for the sake for clarity we will keep it explicit. This is the algebra of raising and lowering operators for ${}^{d}\!/_2$ independent two-level systems. We ask how many basis vectors are there (including lowest weight state) which could be formed by operating ${}^{d}\!/_2$ raising a_i^+ on lowest weight state:[11]

$$\sum_{r=0}^{d/2}{}^{d/2}C_r=2^{d/2}$$

It implies that in $d-$dimensions, we have $2^{d/2}\times 2^{d/2}$ dimensional matrix representation for $\gamma-$matrices. We will use the highest weight representation to determine $a_j^\pm$ and then use them to construct γ_μ. From (2.16), we quickly observe that

$$(a_i^-)^2=0=(a_i^+)^2$$

It implies that we can only act a_i or $a_i^\dagger$ once on a state, the second time it acts the state is annihilation. We will build off our intuition from harmonic oscillator (fermionic) and assume that there is a lowest weight state $|\xi\rangle$ such that

$$a_i^-|\xi\rangle=0\quad\text{for all } i$$

Similarly, acting on it once by each $a_i^\dagger$ for all i, we can construct states in the representation. The states can be labeled $s=(s_0,s_1,...,s_{\frac{d-2}{2}})$, where each of the $s_a=\pm\frac{1}{2}$:

$$|\xi^{(s)}\rangle=(a^+_{\frac{d-2}{2}})^{s_{\frac{d-2}{2}}+\frac{1}{2}}\ldots(a_0^+)^{s_0+\frac{1}{2}}|\xi\rangle\tag{2.17}$$

[10] we have abused notation for the sake of avoiding cluttering of indices and $\pm$

[11] we would like to remind ourselves that number of linearly independent basis is defined as the dimension of space.

The lowest weight state $|\xi\rangle$ corresponds to all $s_a = -\frac{1}{2}$. Taking the $|\xi^s\rangle$ as a basis, we derive the matrix elements of γ_μ from the definitions and the anti-commutation relation. Starting with $d = 2$, we have a single two-level system:

$$|\xi^{(\frac{1}{2})}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |\xi^{(-\frac{1}{2})}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

we can construct the raising and lowering operator connecting these two matrices as:

$$a_0^- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \qquad a_0^+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

we find:

$$\gamma^0 = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \qquad \gamma^1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

For $d = 4$, we have 2 independent fermionic oscillator:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

we construct, the following a_i^+ and a_i^- operators for $i = 0, 1$.

$$a_0^+ = \begin{pmatrix}0 & -1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{pmatrix} \qquad a_0^- = \begin{pmatrix}0 & 0 & 0 & 0\\-1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{pmatrix}$$

and

$$a_1^+ = \begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \qquad a_1^- = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}$$

From (2.17) we see that[12]

$$a_0^+\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \qquad a_1^+\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}; \qquad a_0^+a_1^+\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Thus, we conclude that the gamma matrices are gives as:

$$\gamma^0 = \begin{pmatrix}0 & -1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & -1 & 0\end{pmatrix} \qquad \gamma^1 = \begin{pmatrix}0 & -1 & 0 & 0\\-1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix} \qquad \gamma^3 = \begin{pmatrix}0 & 0 & -i & 0\\0 & 0 & 0 & -i\\i & 0 & 0 & 0\\0 & i & 0 & 0\end{pmatrix}$$

The above choice of gamma matrices satisfy the clifford algebra, however the chosen basis is not familiar from QFT textbooks. Given a representation γ^μ in d dimensions, we can construct a representation Γ^μ in $d+2$ dimensions using the prescription,

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix} = -\gamma^\mu \otimes \sigma^3, \quad \mu = 0, \ldots, d-3,$$

[12] $a_0^\pm$ acts like raising and lowering operator in the same oscillator while $a_1^\pm$ changes the oscillator.

$$\Gamma^{d-2} = \mathbb{I} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^1, \quad \Gamma^{d-1} = \mathbb{I} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^2$$

where the σ^i obey

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

The 2×2 matrices that we add act on the index $s_{d-2/2}$, which newly appears in going from $d = 2k$ to $2k + 2$ dimensions. In odd dimensions the first $d - 1$ gamma matrices can be constructed as above, and $\Gamma_d = \pm\Gamma_1\Gamma_2 \dots \Gamma_{d-1}$ completes the gamma matrix algebra. There are two independent representations of the gamma matrix algebra in odd dimensions, differing in the sign of Γ_d. These representations are exchanged by parity, and both representations appear in a parity-conserving theory.

We now move onto calculating the correlation function involving spinors in embedding space formalism. To define spinor fields on null cone in embedding space requires that the number of component in $\mathbb{R}^d$ is half the number of components in $\mathbb{R}^{d+1,1}$.

$$\psi(x) \to \Psi(X), \qquad\qquad \bar{\psi}(x) \to \bar{\Psi}(X)$$

which satisfies the following homogeneity condition:

$$\Psi(\lambda X) = \lambda^{-\Delta+\frac{1}{2}}\Psi(X), \qquad \bar{\Psi}(\lambda X) = \lambda^{-\Delta+\frac{1}{2}}\bar{\Psi}(X)$$

The degrees of freedom of $\Psi; \bar{\Psi}$ are reduced to those for $\psi; \bar{\psi}$ by imposing the transversality condition like before:

$$\bar{\Gamma}_A X^A \Psi(X) = 0 \qquad \bar{\Psi}(X)\Gamma_A X^A = 0$$

It introduces the gauge invariance and thus the degrees of freedom are now halved by imposing the equivalence relations

$$\Psi'(X) \sim \Psi' + \bar{\Gamma}_A X^A \zeta(X) \qquad \bar{\Psi}'(X) \sim \bar{\Psi}' + \bar{\zeta}(X)\Gamma_A X^A \tag{2.18}$$

for arbitrary spinor $\zeta(X); \bar{\zeta}(X)$ of appropriate homogeneity. From standard QFT, we are familiar that

$$V_A = \bar{\Psi}\Gamma_A\Psi'$$

transforms like a vector and we also have

$$\begin{aligned} \Psi(X) &= \Gamma_B X^B \Psi'(X), \\ \bar{\Psi}(X) &= \bar{\Psi}'(X)\bar{\Gamma}_B X^B. \end{aligned}$$

Now compute the contraction:

B-th component of coordinate (number)

$$\begin{aligned} V_A = \bar{\Psi}(X)\Gamma_A\Psi'(X) &= \left(\bar{\Psi}'(X)\bar{\Gamma}_B \; X^B\right)\Gamma_A\Psi'(X) \\ &= \bar{\Psi}'(X)\bar{\Gamma}_B X^B \Gamma_A \Psi'(X) \end{aligned}$$

Use the Clifford algebra identity:

$$\begin{aligned} \bar{\Gamma}_A\Gamma_B &= -\bar{\Gamma}_B\Gamma_A + 2\eta_{AB} \\ \Gamma_A\bar{\Gamma}_B &= -\Gamma_B\bar{\Gamma}_A + 2\eta_{AB}, \end{aligned}$$

from (2.18), and above we rewrite:

$$V_A = \bar{\Psi}'\bar{\Gamma}_B X^B \Gamma_A \Psi' = -\bar{\Psi}'\bar{\Gamma}_A\Gamma_B X^B \Psi' + 2X_A\bar{\Psi}'\Psi'$$

The second term, $2X_A\bar{\Psi}'\Psi'$, is proportional to X_A and is hence pure gauge under the equivalence relation:

$$V_A \sim V_A + X_A f(X)$$

so it can be discarded in physical quantities. Therefore, we obtain:

$$\begin{aligned} \bar{\Psi}\Gamma_A\Psi' + X_A f(X) &\sim -\bar{\Psi}'\bar{\Gamma}_A\Gamma_B X^B\Psi' + 2X_A\bar{\Psi}'\Psi' \\ \bar{\Psi}(X)\Gamma_A\Psi'(X) &\sim -\bar{\Psi}'(X)\bar{\Gamma}_A\Psi(X) \end{aligned}$$

whereas,

$$\bar{\Psi}\Psi$$

transforms like a scalar. However, the above is only under (2.18) in odd dimensions. So it does not correspond to a scalar on the projective null cone in even dimensions.

Chapter 3

Anti de Sitter

3.1 Wigner Innou Contraction

We begin by splitting the generators of the conformal algebra of boundary CFT into rotations, boost[1], translations, dilatations, and special conformal transformations (SCTs). To perform the contraction, we first examine the Lorentz generators $J_{\mu\nu}$ (rotations + boost) and the translations P_μ with $\mu = 0, 1, \dots d$.

The guiding principle is that we identify the point in spacetime about which the algebra "blows up," and we rescale only those generators that would otherwise diverge. Specifically, we choose the scaling of the Anti de Sitter coordinates:[2]

$$x^i = \epsilon\, x^i, \qquad x^0 = ct,$$

and take the $\epsilon \to 0$ limit. We are allowed to do this here because the notion of Boost exists in the boundary Minkowski spacetime. Under this scaling, the contracted generators are:

$$\begin{aligned}
J_{ij} &= x_i \partial_j - x_j \partial_i, \\
B_i &= \lim_{\epsilon\to 0} \epsilon J_{0i} = \lim_{\epsilon\to 0} \epsilon\big(x_0 \tfrac{1}{\epsilon}\partial_i - \epsilon x^i \partial_0\big) = -t\,\partial_i, \\
H &= P_0 = \partial_t, \\
P_i &= \partial_i.
\end{aligned}$$

To obtain the **Galilean Conformal Algebra (GCA)**, we perform a non-relativistic contraction of the full conformal algebra $SO(d+1,2)$, the conformal symmetry group of $(d+1)$-dimensional Minkowski space (and equivalently the isometry group of AdS_{d+2}) which sits at the boundary of AdS_{d+2}. The dilatation generator contracts to

$$D = -x \cdot \partial = -x^i \partial_i - t\,\partial_t,$$

while the special conformal generators become

$$\begin{aligned}
K_\mu &= \big[2x_\mu (x\cdot\partial) - (x\cdot x)\partial_\mu\big], \\
K = K_0 &= \Big[-2t(x\cdot\partial) + \big(t^2 - \epsilon^2 |\vec{x}|^2\big)\partial_0\Big] = -2t^2\partial_t - 2t(x^i\partial_i) + \big(t^2 - \epsilon^2|\vec{x}|^2\big)\partial_t \;\to\; -\big[2t(x^i\partial_i) + t^2\partial_t\big], \\
K_i &= \Big[2\epsilon x_i (x\cdot\partial) + \big(t^2 - \epsilon^2|\vec{x}|^2\big)\partial_i\Big] \;\to\; t^2\,\partial_i,
\end{aligned}$$

where in the last step we have taken the $\epsilon \to 0$ limit. The resulting finite-dimensional Galilean Conformal Algebra (GCA) in d spatial dimensions has the generators

$$(J_{ij}, P_i, H, B_i, D, K, K_i),$$

where

- J_{ij} are spatial rotations, $\frac{d(d-1)}{2}$ in number,
- P_i are spatial translations, d in number,
- H is the time translation, 1 generator,

[1] A note that in de Sitter case we will not have Lorentz generator on the boundary because the holography for de Sitter space exists with Euclidean space and as such there's no notion of boost that pre-existed even if we perform the wick rotation.

[2] followed Anatomy of Null contraction

- B_i are Galilean boosts, d in number,
- D is the dilatation, 1 generator,
- K is the temporal special conformal, 1 generator,
- K_i are the spatial special conformals, d in number.

The total number of generators is

$$\frac{d(d-1)}{2}+3d+3=\frac{d^2-d+6d+6}{2}=\frac{d^2+5d+6}{2}=\frac{(d+1+1)(d+1+2)}{2}$$

which is exactly the same as $(d+2)(d+3)/2$ number of generators for $SO(d+1,2)$. **Example in $d=3$:**

$$3+3+1+3+1+1+3=15 \text{ generators}.$$

For the boundary of 4D AdS, which is $2+1$ dimensional Minkowski, the nonrelativistic contraction gives

$$J=1,\quad P_i=2,\quad H=1,\quad B_i=2,\quad D=1,\quad K=1,\quad K_i=2,$$

so that the total is

$$1+2+1+2+1+1+2=10 \text{ generators}.$$

Hence, no generators are truly "lost"; the reduction in number reflects the lower-dimensional boundary rather than an actual loss of symmetry.

Derivation of the Lie algebra

The Wigner Innou contracted generators have the following form:

$$J_{ij}=x_i\partial_j-x_j\partial_i,\quad B_i=-t\partial_i,\quad H=\partial_t,\quad P_i=\partial_i,$$
$$D=-x^k\partial_k-t\partial_t,\quad K=2t(x^i\partial_i)+t^2\partial_t,\quad K_i=t^2\partial_i.$$

which we will use to find the algebra. But first let us outline some useful identities.

$$[D,\partial_t]=\partial_t,\qquad [D,\partial_i]=\partial_i,\qquad [\partial_t,t]=1,\quad [D,t]=-t.$$

$$[J_{ij},J_{kl}]=\delta_{jk}J_{il}-\delta_{ik}J_{jl}-\delta_{jl}J_{ik}+\delta_{il}J_{jk},$$

and for any spatial vector field $V_k=f(t,x)\partial_k$,

$$[J_{ij},V_k]=\delta_{jk}V_i-\delta_{ik}V_j.$$

Now, let us derive the algebra:

$$[P_i,P_j]=0,\qquad [P_i,B_j]=0,\qquad [B_i,B_j]=0,\qquad [K_i,K_j]=0,$$

since ∂_i commute and each spatial generator is proportional to ∂_i. Next,

$$[H,B_i]=[\partial_t,-t\partial_i]=\partial_t(-t)\partial_i=-\partial_i=-P_i,$$
$$[H,P_i]=[\partial_t,\partial_i]=0.$$

$$[D,H]=[-x^k\partial_k-t\partial_t,\partial_t]=-[t\partial_t,\partial_t]=\partial_t=H,$$
$$[D,P_i]=[-x^k\partial_k,\partial_i]=\partial_i=P_i,\qquad [D,B_i]=[-t\partial_t,-t\partial_i]=0,$$
$$[D,K_i]=[-x^k\partial_k-t\partial_t,\,t^2\partial_i]=(-2t^2)\partial_i=-K_i,$$
$$[D,K]=[D,2t(x^i\partial_i)+t^2\partial_t]=2[D,t]D+[D,t^2]\partial_t=2(-t)D+(-2t^2)\partial_t=-2tD-t^2\partial_t=-K.$$

$$[H,K]=[\partial_t,\,2t(x^i\partial_i)+t^2\partial_t]=2[\partial_t,t]x^i\partial_i+2t[\partial_t,x^i\partial_i]+[\partial_t,t^2]\partial_t=2(x^i\partial_i+t\partial_t)=-2D,$$

$$[K,P_i]=[2t(x^i\partial_i)+t^2\partial_t,\partial_i]=2t[(x^i\partial_i),\partial_i]+t^2[\partial_t,\partial_i]=-2t\partial_i=2B_i,$$

$$\begin{aligned}
[K, B_i] &= [2t(x^i\partial_i) + t^2\partial_t, -t\partial_i] = [2t(x^i\partial_i), -t\partial_i] + [t^2\partial_t, -t\partial_i] \\
&= \underbrace{[2t(x^i\partial_i), -t]}_{=0}\partial_i - t[2t(x^i\partial_i), \partial_i] + [t^2\partial_t, -t]\partial_i - t\underbrace{[t^2\partial_t, \partial_i]}_{=0} \\
&= 2t^2[\partial_i, x^j\partial_j] - t^2\partial_i = 2t^2\delta_i^j\partial_j - t^2\partial_i \\
&= t^2\partial_i = K_i,
\end{aligned}$$

$$[K, K_i] = [2t(x^i\partial_i) + t^2\partial_t, t^2\partial_i] = 0.$$

$$\begin{aligned}
[K_i, H] &= [t^2\partial_i, \partial_t] = \partial_t(t^2)\partial_i = 2t\partial_i = 2B_i, \\
[K_i, P_j] &= [t^2\partial_i, \partial_j] = 0, \qquad [K_i, B_j] = 0.
\end{aligned}$$

compactly,

$$\begin{aligned}
[J_{ij}, J_{rs}] &= \delta_{ir}J_{js} - \delta_{is}J_{jr} - \delta_{jr}J_{is} + \delta_{js}J_{ir} \\
[J_{ij}, B_r] &= [J_{ij}, J_{0r}] = B_i\delta_{jr} - B_j\delta_{ir}, \\
[J_{ij}, P_r] &= [J_{ij}, J_{Dr}] = P_i\delta_{jr} - P_j\delta_{ir}, \\
[J_{ij}, H] &= 0, \\
[B_i, B_j] &= 0, \qquad [P_i, P_j] = 0, \qquad [B_i, P_j] = 0, \\
[H, P_i] &= 0, \qquad [H, B_i] = -P_i.
\end{aligned}$$

$$\begin{array}{llll}
[K, K_i] = 0, & [K, B_i] = K_i, & [K, P_i] = 2B_i, & \\
[J_{ij}, K_r] = K_i\delta_{jr} - K_j\delta_{ir}, & [J_{ij}, K] = 0, & [J_{ij}, D] = 0, & \\
[K_i, K_j] = 0, & [K_i, B_j] = 0, & [K_i, P_j] = 0, & [H, K_i] = -2B_i, \\
[D, H] = H, & [D, K_i] = -K_i, & [D, B_i] = 0, & [D, P_i] = P_i, \\
[H, K] = -2D, & [D, K] = -K. & &
\end{array}$$

3.2 Infinite dimensional extension

We have looked at how the finite dimensional GCA is constructed out of the contraction of the relativistic conformal algebra in the previous section. The most interesting feature of the GCA is that it admits a very natural extension to an infinite dimensional algebra of the Virasoro-Kac-Moody type. To see this we denote

$$\begin{array}{lll}
L^{(-1)} = H & L^{(0)} = D & L^{(+1)} = K \\
M_i^{(-1)} = P_i & M_i^{(0)} = B_i & M_i^{+1} = K_i
\end{array}$$

The algebra in the new form takes very simpler and compact form:

$$\begin{aligned}
[L^{-1}, M_i^{-1}] &= [H, P_i] = 0 \\
[L^{-1}, M_i^{0}] &= [H, B_i] = -P_i = (-1-0)M_i^{-1} \\
[L^{-1}, M_i^{+1}] &= [H, K_i] = -2B_i = (-1-1)M_i^{0} \\
[L^{0}, M_i^{-1}] &= [D, P_i] = P_i = (0-(-1))M_i^{-1} \\
[L^{0}, M_i^{0}] &= [D, B_i] = 0 \\
[L^{0}, M_i^{+1}] &= [D, K_i] = -K_i = (0-1)M_i^{+1} \\
[L^{+1}, M_i^{-1}] &= [K, P_i] = 2B_i = (1+1)M_i^{0} \\
[L^{+1}, M_i^{0}] &= [K, B_i] = K_i = (1-0)M_i^{1} \\
[L^{+1}, M_i^{+1}] &= [K, K_i] = 0
\end{aligned}$$

$$[L^0, L^{-1}] = [D, H] = H = (0-(-1))L^{-1},$$
$$[L^0, L^{+1}] = [D, K] = -K = (0-1)L^{+1},$$
$$[L^{-1}, L^{+1}] = [H, K] = [H, K] = -2D = (-1-1)L^0.$$

The finite dimensional GCA that we just derived can now be recast as:

$$\begin{aligned}
[J_{ij}, L^{(n)}] &= 0 \\
[L^{(m)}, M_i^{(n)}] &= (m-n)M_i^{(m+n)} \\
[J_{ij}, M_k^{(m)}] &= M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik} \\
[M_i^{(m)}, M_j^{(n)}] &= 0 \\
[L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)}
\end{aligned}$$

with the indices $m, n = 0, \pm 1$, we have made the $SL(2, R)$ subalgebra of the generators manifest. In fact, we can define the vector fields

$$L^{(n)} = -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t, \qquad M_i^{(n)} = t^{n+1}\partial_i$$

with $n = 0, \pm 1$. These together with J_{ij} are exactly the vector fields which generate GCA. For J_{ij}, we can define

$$J_a^{(n)} = J_{ij}^{(n)} = t^n(x_i\partial_j - x_j\partial_i)$$

Then,

$$J_{ij}^{(m)} = t^m R_{ij}, \qquad R_{ij} := x_i\partial_j - x_j\partial_i, \qquad L^{(n)} = t^n \ell^{(n)}, \qquad \ell^{(n)} := -(n+1)x^k\partial_k - t\partial_t.$$

We use the identity for Lie brackets of rescaled vector fields:

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

with $f = t^m,\ g = t^n,\ X = R_{ij},\ Y = \ell^{(n)}$ we have $X(g) = R_{ij}(t^n) = 0$ (rotations do not act on t). Also

$$\ell^{(n)}(t^m) = -(n+1)x^k\partial_k(t^m) - t\partial_t(t^m) = -mt^m.$$

Since $[R_{ij}, \ell^{(n)}] = 0$ (rotations commute with $x^k\partial_k$ and with ∂_t), the bracket becomes

$$[J_{ij}^{(m)}, L^{(n)}] = t^{m+n}[R_{ij}, \ell^{(n)}] - t^n\big(\ell^{(n)}(t^m)\big)R_{ij} = 0 - t^n(-mt^m)R_{ij} = m\, t^{m+n}R_{ij}.$$

Finally

$$m\, t^{m+n}R_{ij} = m\, J_{ij}^{(m+n)}.$$

$$\boxed{[J_{ij}^{(m)}, L^{(n)}] = m\, J_{ij}^{(m+n)}\ .}$$

Next, we consider

$$J_{ij}^{(m)} = t^m R_{ij}, \qquad R_{ij} := x_i\partial_j - x_j\partial_i, \qquad M_k^{(n)} = t^{n+1}\partial_k.$$

with $f = t^m,\ g = t^{n+1},\ X = R_{ij},\ Y = \partial_k$ we have $X(g) = R_{ij}(t^{n+1}) = 0$ (rotations do not act on t), and $Y(f) = \partial_k(t^m) = 0$. Thus

$$[J_{ij}^{(m)}, M_k^{(n)}] = t^{m+n+1}[R_{ij}, \partial_k].$$

But

$$[R_{ij}, \partial_k] = [\ x_i\partial_j - x_j\partial_i, \partial_k\ ] = \delta_{jk}\partial_i - \delta_{ik}\partial_j.$$

Hence

$$\boxed{[J_{ij}^{(m)}, M_k^{(n)}] = t^{m+n+1}(\delta_{jk}\partial_i - \delta_{ik}\partial_j) = \delta_{jk}M_i^{(m+n)} - \delta_{ik}M_j^{(m+n)}}$$

and lastly,

$$J_{ij}^{(m)} = t^m R_{ij}, \qquad R_{ij} := x_i\partial_j - x_j\partial_i.$$

We want $[J_{ij}^{(m)}, J_{kl}^{(n)}]$. Here $f = t^m,\ g = t^n,\ X = R_{ij},\ Y = R_{kl}$. Since $R_{ij}(t^n) = 0$ and $R_{kl}(t^m) = 0$, the last two terms vanish. Thus

$$[J_{ij}^{(m)}, J_{kl}^{(n)}] = t^{m+n}[R_{ij}, R_{kl}] = t^{(m+n)}f_{abc}R_c = f_{abc}J_c^{(m+n)}.$$

Hence,

$$\boxed{[J_{ij}^{(m)}, J_{kl}^{(n)}] = f_{abc}J_c^{(m+n)}}$$

The index a labels the generators of the spatial rotation group $SO(d)$ and f_{abc} are the corresponding structure constants. We see that the vector fields generate a $SO(d)$ Kac-Moody algebra without any central terms.

Chapter 4

de Sitter

4.1 Introduction

Anti–de Sitter (AdS) and de Sitter (dS) are two of the three maximally symmetric solutions of Einstein's equations with constant curvature. Unlike flat Minkowski space, they feature genuine gravitational effects: the dynamics of spacetime itself influences particle motion. AdS has constant negative curvature, while dS has constant positive curvature. Their isometry algebras can be described linearly by embedding them in higher-dimensional flat (zero-curvature) spaces, where AdS_d is realized as a hyperboloid in $\mathbb{R}^{d-1,2}$, while dS_d sits inside $\mathbb{R}^{d,1}$. In both cases the isometry group is simply the Lorentz group of the embedding space:

$$\mathrm{Isom}(AdS_d) = SO(d-1,2), \qquad \mathrm{Isom}(dS_d) = SO(d,1).$$

To compare with the Poincaré algebra, one splits the generators J_{AB} of the embedding group into "Lorentz" generators $iM_{\mu\nu} = J_{\mu\nu}$ and "translation-like" generators $iP_\mu = \frac{1}{R}J_{\mu d}$, where R is the (A)dS radius and $\mu, \nu = 0, 1 \ldots d-1$. The algebra of the embedding-space isometry generators is independent of the choice of metric signature and can be written as:

$$[J_{AB}, J_{CD}] = \eta_{AC}J_{BD} - \eta_{AD}J_{BC} + \eta_{BD}J_{AC} - \eta_{BC}J_{AD}$$

The commutators then organize as:

- The Lorentz subalgebra:

  $$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho}),$$

  valid for both AdS and dS.

- Lorentz action on "translations":

  $$[M_{\mu\nu}, P_\rho] = -[J_{\mu\nu}, J_{\rho d}] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu).$$

  valid for both AdS and dS.

- Commutator of translations:

  $$[P_\mu, P_\nu] = \frac{1}{R^2}[J_{\mu d}, J_{\nu d}] = \frac{\eta_{dd}}{R^2}J_{\mu\nu} = \eta_{dd}\frac{i}{R^2}M_{\mu\nu},$$

where the **sign distinguishes the cases**: minus for AdS, plus for dS (mostly plus signature).[1] This is the only algebraic difference, and it encodes the curvature sign of the spacetime. The sign flip of a single commutator, while all others remain unchanged, is what ultimately alters the underlying physics. Locally, de Sitter (dS) and Anti-de Sitter (AdS) spaces can be related by analytic continuation, but this correspondence does not extend globally. The obstruction lies in their fundamentally different global structures: AdS has topology $\mathbb{R} \times S^{d-1}$ with a timelike conformal boundary, while dS has topology S^d with spacelike conformal boundaries at future and past infinity. These differences in causal structure, boundary conditions, and topology prevent a global analytic continuation between the two spacetimes. (verify?)

[1] refer to Newton-Hooke Algebras, Non-relativistic Branes and Generalized pp-wave Metrics but flip the metric sign to account for signature convention change.

The flat-space limit is obtained by letting $R \to \infty$. In this contraction, the $[P_\mu, P_\nu]$ commutator vanishes, recovering the familiar Poincaré algebra $ISO(d-1,1)$ with commuting translations. Thus, both AdS and dS reduce to flat Minkowski symmetry in the infinite-radius limit, but neither contains Poincaré as a subgroup at finite radius. The distinct signatures have important consequences for holography. In AdS, the boundary is timelike, and the isometry group $SO(d-1,2)$ acts as the Lorentzian conformal group of a $(d-1)$-dimensional Minkowski boundary, giving the basis for AdS/CFT. In dS, the boundary is spacelike, and the same structure $SO(d,1)$ acts as the Euclidean conformal group of a $(d-1)$-dimensional space, motivating the dS/CFT correspondence. This is a key principle of holography, where the isometry group of the higher-dimensional "bulk" spacetime matches the conformal group of the lower-dimensional boundary which can be seen from the table below.

Geometry Type	**Dimensions**	**Isometry Group**	**Conformal Group**
Euclidean	d	$ISO(d)$	$SO(d+1,1)$
de Sitter	$d+1$	$SO(d+1,1)$	$SO(d+1,2)$
Minkowski	d	$ISO(d-1,1)$	$SO(d,2)$
Anti-de Sitter	$d+1$	$SO(d,2)$	$SO(d,3)$

From a geometric perspective, $\Lambda = 0$ corresponds to flat Minkowski space, $\Lambda > 0$ gives dS as the maximally symmetric spacetime of constant positive curvature, and $\Lambda < 0$ gives AdS with constant negative curvature. For example, dS_4 can be realized as the hyperboloid in $\mathbb{R}^{4,1}$:[2]

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = l^2$$

It describes a time-like hyperboloid with topology $\mathbb{R} \times S^3$, the S^3 arising from the slicing at fixed X^0,

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \underbrace{l^2 + (X^0)^2}_{\text{constant}} > 0 \tag{4.1}$$

The spacetime interval in $\mathbb{R}^{4,1}$ is

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2.$$

Since de Sitter space is embedded in $\mathbb{R}^{(4,1)}$, whose isometry group is $\mathrm{ISO}(4,1)$, the induced metric on de Sitter space must be invariant under a certain subgroup of $\mathrm{ISO}(4,1)$ restricted to the hypersurface. However, the way these ambient-space isometries act on $\mathbb{R}^{4,1}$ differs from how they act on the de Sitter hypersurface $\mathbb{M}^{3,1}$. For this reason, we describe them in terms that reflect their action on de Sitter space itself:

- 3 Rotation + 3 Translation
$$x_i \to a_i + R_{ij}x_j$$
- 1 Dilatation
$$x_\mu \to \lambda x_\mu$$
- 3 Special Conformal Transformation ($\eta \to 0$ or $b^\mu = (0, \vec{b})$)
$$x_i \to \frac{x_i - b_i(-\eta^2 + \vec{x}^2)}{1 - 2\vec{b}\cdot\vec{x} + b^2(-\eta^2 + \vec{x}^2)}$$

We will derive these in the next section. For now, note that four-dimensional de Sitter space has a total of

$$\frac{5(5-1)}{2} = 10$$

independent isometries, forming the group $\mathrm{SO}(4,1)$. This happens because the choice of de Sitter hypersurface breaks the translation invariance of the ambient space.

[2]we define it like that because it helps us realize the de Sitter isometries in linear fashion.

Coordinatizing de Sitter

There are many ways to introduce coordinate system on de Sitter space. We will start by setting a cartesian like coordinate system in ambient space and then introduce $d-2$ dimensional hypersurfaces which will slice the de Sitter space into space+time. The family of curve orthogonal to this hypersurface will become our timelike direction on de Sitter and points on this $d-2$ dimensional hypersurface will be spacelike. This will lead to natural introduction of inducing embedding space coordinate to label the points on de Sitter hypersurface. Let us first define n^a as normal vectors on 3−sphere (S^3). We can parameterize (2.1) as following

$$
\begin{aligned}
(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 &= l^2 + (X^0)^2 \\
l^2 \underbrace{\sum_{a,b=1}^{4} \delta_{ab} n^a n^b}_{1} \cosh\Big(\frac{\tau}{l}\Big) &= l^2 + l^2 \sinh\Big(\frac{\tau}{l}\Big)
\end{aligned}
$$

i.e. we can introduce following global coordinates on de Sitter space as[3]:

$$
\left.\begin{aligned}
X^0 &= l^2 \sinh\big(\tfrac{\tau}{l}\big) \\
X^a &= l^2 n^a \cosh\big(\tfrac{\tau}{l}\big); \qquad a = 1,2,3,4
\end{aligned}\right\} \text{closed slicing}
$$

we also define following notation:

$$
\sum_{a,b=1}^{d-1} \delta_{ab} dn^a dn^b = d\Omega_{d-1}^2
$$

For $\mathbb{S}^3$, the normal vector n^a looks like

$$
\begin{aligned}
n^1 &= \cos\chi \\
n^2 &= \sin\chi\cos\theta \\
n^3 &= \sin\chi\sin\theta\cos\phi \\
n^4 &= \sin\chi\sin\theta\sin\phi
\end{aligned}
$$

where

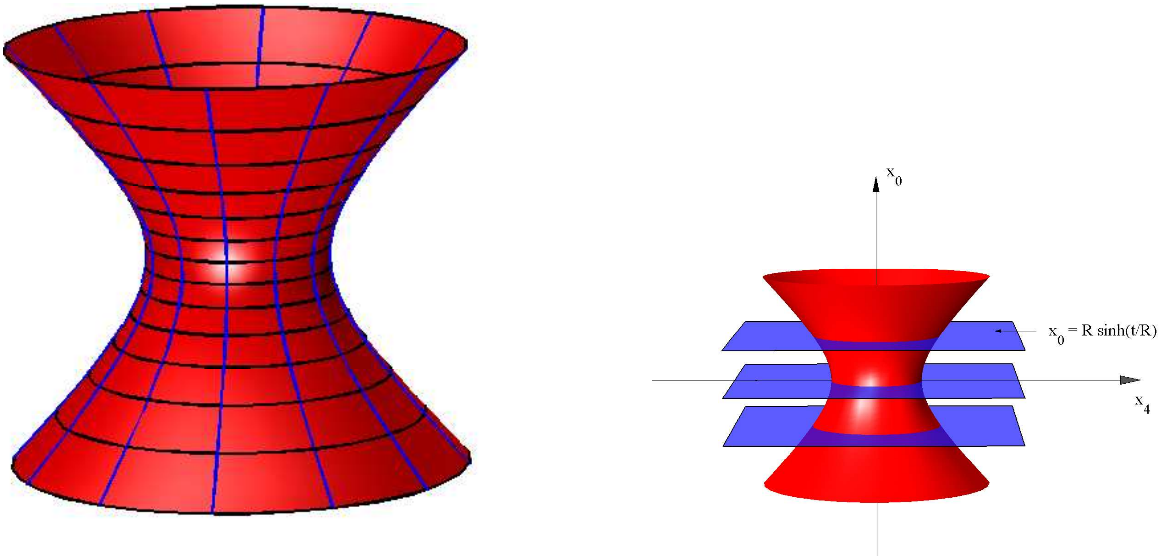
$$
\chi \in [0,\pi], \quad \theta \in [0,\pi], \quad \phi \in [0,2\pi]
$$

We quickly note that the above definition of coordinates gives rise to following induced metric on the hyperboloid.

$$
ds^2 = -d\tau^2 + l^2 \cosh^2\Big(\frac{\tau}{l}\Big) d\Omega_3^2
$$

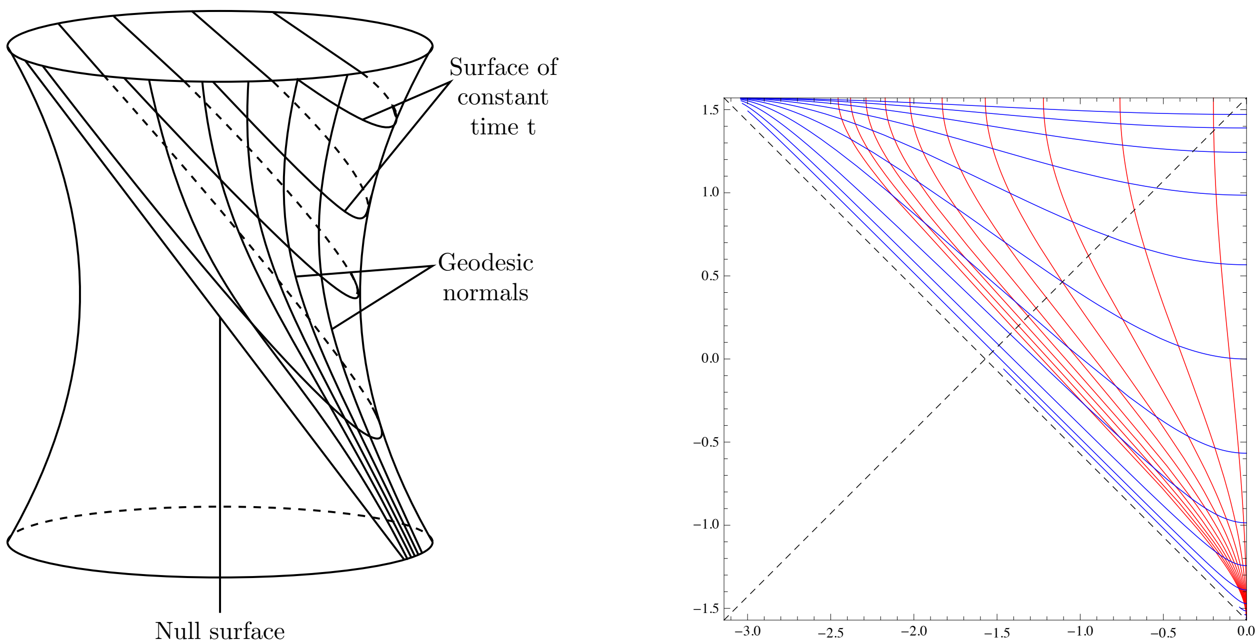
where $d\Omega_3^2$ is metric on S^3. Constant time slices are then compact. For $\tau > 0$, this is the typical picture of a closed Universe whose size is expanding exponentially as time evolves forward. The minimal size of the sphere is at $\tau = 0$, where the radius of the sphere is one (in units of the dS radius). In these coordinates dS_4 looks like a 3−sphere which starts out infinitely large at $\tau = -\infty$, then shrinks to a minimal finite size at $\tau = 0$, then grows again to infinite size as $\tau = \infty$. Note that this metric depends explicitly on the global time; dS does not have a **global** timelike Killing vector.

[3] chapter 4 of "Jerry B. Griffiths, Jirí Podolský - Exact Space-Times in Einstein's General Relativity"



(a) (b)

4.1.1 Flat coordinates



(c) (d)

Figure 4.1: (c) de Sitter geodesic. (d) Flat slicing of de Sitter, drawn on the Penrose diagram. Blue curves are constant-t flat slices, and red curves are the surfaces of constant-r. Intersections of the blue curves with the dashed line are the cross-sections of the cosmological horizon.

These are the coordinates $t, \mathbf{x}$, defined by

$$X^0 = l \sinh\left(\frac{ct}{l}\right) + \frac{1}{2l} x^2 e^{\frac{ct}{l}},$$

$$X^1 = l \cosh\left(\frac{ct}{l}\right) - \frac{1}{2l} x^2 e^{\frac{ct}{l}},$$

$$X^i = x^i e^{\frac{ct}{l}}, \quad i = 2, 3, 4.$$

where $x^2 = \delta_{ab} x^a x^b$. These coordinates do not cover the full de Sitter space, but only the patch

$$X^0 + X^1 = l e^{ct/l} > 0.$$

observable by observer at south pole ($\theta = \pi$) in penrose diagram. What it implies is that the space is coordinatized by a collection of plane in the ambient space with slope -1 and X^d intercept given by $He^{t/H} > 0$. The flat slicing covers only half the Penrose diagram, so this metric by itself is past-geodesically incomplete. Timelike worldlines, unless they are specially chosen to sit at the South pole, will exit the flat slicing in the past, in finite affine time. In these coordinates, the metric reads

$$ds^2 = -c^2dt^2 + e^{2ct/l}\sum_{i=1}^{3} dx_i^2.$$

This metric can be regarded as a FRW cosmology with an exponential function $a(t)$ by replacing $l = \frac{c}{H}$:

$$a(t) = e^{Ht}. \tag{4.2}$$

If we draw a diagonal plane through the embedding diagram, this is the 'upper triangle.' Similar coordinates can be chosen to cover just the 'lower triangle.'

4.1.2 Conformally flat coordinates

There is another coordinate which is conformal to flat Cartesian coordinates and covers more parts of de Sitter space than flat coordinates. Using familiar cartesian-like coordinates $(c\eta, x, y, z)$, the de Sitter hyperboloid is covered by:

$$\begin{aligned} x^0 &= \frac{l^2+s}{2c\eta} \\ x^1 &= \frac{l^2-s}{2c\eta} \\ x^2 &= l\frac{x}{c\eta} \\ x^3 &= l\frac{y}{c\eta} \\ x^4 &= l\frac{z}{c\eta} \end{aligned}$$

where $s = -c^2\eta^2 + x^2 + y^2 + z^2$ with $\eta, x, y, z \in (-\infty, \infty)$. Note that these coordinates still do not cover the full de Sitter space but only the patch

$$x^0 + x^1 \neq 0$$

In these coordinates, the de Sitter metric is

$$ds^2 = \frac{l^2}{c^2\eta^2}(-c^2d\eta^2 + d\vec{x}^2)$$

where η is usually referred to as conformal time. This is usually the preferred frame for the computation of cosmological correlators. In these coordinates the time η is not a Killing vector, and the only manifest symmetries are translations and rotations of the x^i coordinate.

4.1.3 Static coordinates

A very important aspect of dS space is that no single observer has access to the full spacetime. This is clear by just looking at the Penrose diagram. An important set of coordinates are those that describe the region accessible to a single observer. This is the intersection between the region of space that can affect the observer and the region that can be affected by them. In terms of embedding coordiantes, they are given by:

$$\begin{aligned} X^0 &= l\sqrt{1-\frac{r^2}{l^2}}\sinh\left(\frac{ct}{l}\right), \\ X^1 &= l\sqrt{1-\frac{r^2}{l^2}}\cosh\left(\frac{ct}{l}\right), \\ X^a &= rn^a, \quad a = 2,3,4 \end{aligned}$$

where

$$0 \leq r < l$$

They only cover the region

$$x^1 > 0, \quad r^2 < l^2.$$

These are accelerating observers in the ambient space:

$$-(X^0)^2 + (X^1)^2 + (X^a)^2 = (l^2 - \cancel{r^2}) + \cancel{r^2} = \frac{1}{\alpha^2}$$

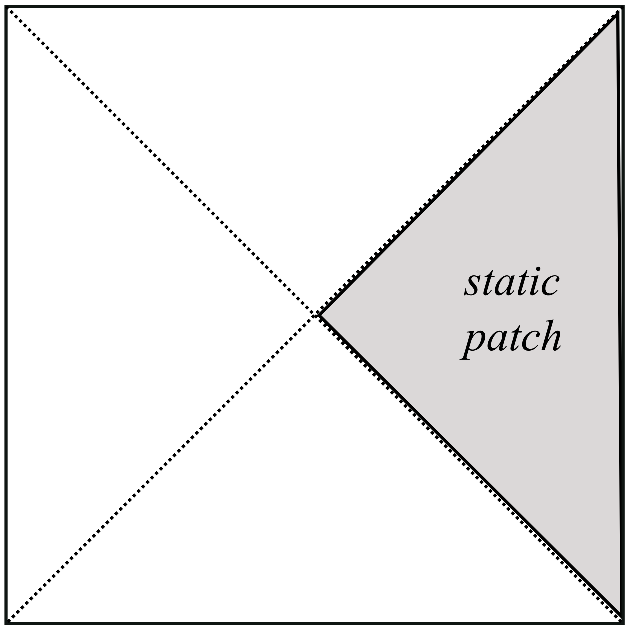


Figure 4.2: Static patch, on the Penrose diagram. This is the causal patch of an observer sitting at the north pole, i.e. $\theta = 0$ in global coordinates, i.e. $r = 0$ in static coordinates. The right edge of the diagram is $r_{\text{static}} = H$; the bifurcate Killing horizon is $r_{\text{static}} =$. The other three patches can also be covered by (independent) static coordinate systems, much like the four regions of the Penrose diagram for Schwarzschild black holes

In these coordinates, the metric reads

$$ds^2 = \left(1 - \frac{r^2}{l^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r^2}{l^2}} - r^2 d\Omega_3^2.$$

Notice the presence of an explicit horizon at $r = l$, which makes manifest the presence of event horizons for observers in de Sitter space.

4.1.4 Conformal Structure

The conformal boundary of de Sitter spacetime is located at $\eta = 0$ and $\eta = \pi$, corresponding to past and future infinity. Unlike in Minkowski space, these infinities, $\mathcal{I}^-$ and $\mathcal{I}^+$, are **spacelike** in de Sitter. To analyze the causal structure, it is convenient to introduce a conformal time coordinate T, defined by

$$\cos T = \frac{1}{\cosh\left(\frac{\tau}{H}\right)}.$$

This maps the global time coordinate τ to a finite range,

$$-\tfrac{\pi}{2} \leq T \leq \tfrac{\pi}{2}.$$

The spatial sections of de Sitter space are 3-spheres. Writing the metric on S^3 as

$$d\theta^2 + \sin^2\theta \, d\Omega_2^2,$$

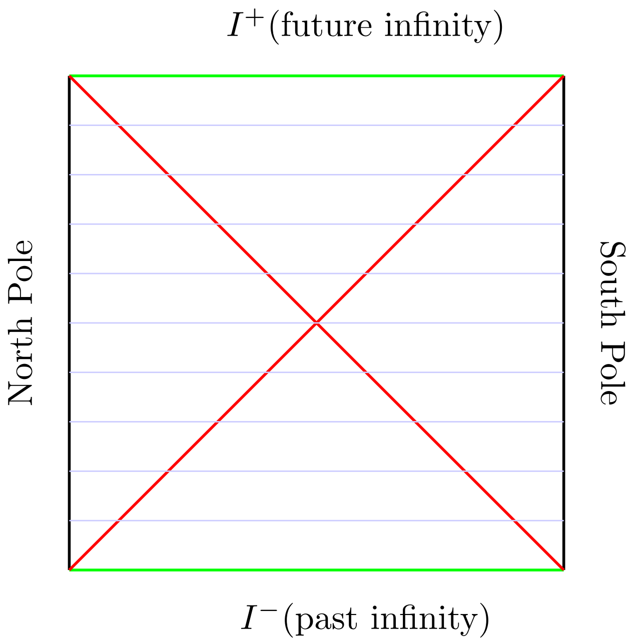
the full de Sitter metric becomes conformal to

$$ds^2 = \frac{1}{\cos^2 T}\Big(- dT^2 + d\theta^2 + \sin^2\theta \, d\Omega_2^2\Big),$$

with $\theta \in [0, \pi]$.

Suppressing the transverse 2-sphere $d\Omega_2$ and including the endpoints at $T = \pm\frac{\pi}{2}$ (past and future infinity), we obtain the Penrose diagram of de Sitter space in the (T, θ) plane. Each horizontal slice of this diagram (shown in blue) corresponds to a 3-sphere of radius

$$R(T) = \frac{1}{\cosh T}.$$



In this representation, each point along a horizontal line stands for a 2-sphere, except at the vertical boundaries $\theta = 0$ and $\theta = \pi$, where the geometry collapses to a single point. These are referred to as the North and South poles of the 3-sphere, and it is often convenient to imagine inertial observers located there.

Finally, the symmetry group of the de Sitter boundary is isomorphic to the conformal group of $\mathbb{R}^3$. This group includes rotations, translations, dilatations, and special conformal transformations. Since the boundary is spacelike, however, there is no notion of Lorentz boosts on the boundary.

4.2 Killing Vectors

We noted at the start of this chapter that de Sitter space is maximally symmetric space with 10 killing vector associated with translation (3), rotation (3), dilatation (1) and SCT (3). There are two ways to find these killing vector

- Either find the coordianate transformation such that metric is independent of one or more coordinates.
- Solve the killing equation in some coordinate system

Since finding all the relevant coordiante transformation is very much dependent on luck, we choose to solve the killing equation in conformal coordinate. The metric for de Sitter space in conformal coordinates is given as:

$$ds^2 = \frac{1}{(H\eta)^2}(-d\eta^2 + d\vec{x}^2) = \frac{1}{(H\eta)^2}\eta_{\mu\nu}dx^\mu dx^\nu$$

The killing equation would be given as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

It is easier to first focus on the case where $\Gamma^\lambda_{\mu\nu} = 0$, so will we start with that.

4.2.1 Translation

The first thing to note is that metric is already independent of x^i, therefore it has 3 killing vector associated with translation. We can show that they are $\xi^\mu = \delta^\mu_i$, and

$$\xi_\mu = g_{\mu\nu}\xi^\nu = g_{\mu\nu}\delta^\nu_i = g_{\mu i}$$

From killing equation:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = \xi_{\mu,\nu} + \xi_{\nu,\mu} - 2\Gamma^\rho_{\mu\nu}\xi_\rho$$

Substituting $\xi_\rho = g_{\rho i}$, we get:

$$g_{\mu i,\nu} + g_{\nu i,\mu} - g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})g_{\rho i}$$

Now, using $g_{\rho i}g^{\rho\sigma} = \delta^\sigma_i$, the Killing condition simplifies to:

$$g_{\mu i,\nu} + g_{\nu i,\mu} - (g_{i\mu,\nu} + g_{i\nu,\mu} - g_{\mu\nu,i}) = 0$$

Hence,

$$\xi^x = (0,1,0,0) \qquad \xi^y = (0,0,1,0) \qquad \xi^z = (0,0,0,1)$$

or more compactly

$$\boxed{\xi = \xi^i\partial_i = \partial_i}$$

To generate translation from them, we just exponentiate the generators:

$$e^{a\partial_x}x = x + a\partial_x x + 0 = x + a$$

4.2.2 Rotation

Next, we solve for $\mu \neq \nu$ case where $\Gamma^\lambda_{\mu\nu} = 0$:

$$\begin{aligned}
\partial_x\xi_y + \partial_y\xi_x &= 0\\
\partial_x\xi_z + \partial_z\xi_x &= 0\\
\partial_y\xi_z + \partial_z\xi_y &= 0
\end{aligned}$$

Since $\partial_i g^{\mu\nu} = 0$, we can rewrite them as

$$\begin{aligned}
\partial_x\xi^y + \partial_y\xi^x &= 0\\
\partial_x\xi^z + \partial_z\xi^x &= 0\\
\partial_y\xi^z + \partial_z\xi^y &= 0
\end{aligned}$$

These are same as killing vector for flat space rotation.

$$\xi = \epsilon^{(i)jk}x_j\partial_k$$

we can just exponentiate them and find rotation. We will do it for $i = 3$ which describes the rotation about z-axis

$$\xi = (0,-y,x,0) \equiv -y\partial_x + x\partial_y$$

The relevant transformation is:

$$\begin{aligned}
e^{\theta\xi}x &= \sum_{n=0}^{\infty}\frac{1}{n!}\theta^n\left(-y\partial_x + x\partial_y\right)^n x\\
&= x + \theta\left(-y\partial_x + x\partial_y\right)x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 x + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 x\ldots\\
&= x - \theta y - \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)y - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)x\ldots\\
&= \left(1 - \frac{\theta^2}{2!} + \ldots\right)x - \left(\theta - \frac{\theta^3}{3!} + \ldots\right)y\\
&= \cos(\theta)x - \sin(\theta)y
\end{aligned}$$

Similarly,

$$e^{\theta\xi}y = \sum_{n=0}^{\infty}\frac{1}{n!}\theta^n\left(-y\partial_x + x\partial_y\right)^n y$$

$$
\begin{aligned}
&= y + \theta\left(-y\partial_x + x\partial_y\right) y + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 y + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 y \dots \\
&= y + \theta x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right) x - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right) y \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \dots\right) y + \left(\theta - \frac{\theta^3}{3!} + \dots\right) x \\
&= \cos(\theta) y + \sin(\theta) x
\end{aligned}
$$

Represented in matrix form:

$$
\begin{bmatrix} \eta' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta \\ x \\ y \\ z \end{bmatrix}
$$

4.2.3 Dilatation

Now to proceed further we need to write down the non vanishing connection terms:

$$
\begin{aligned}
\Gamma^x_{\eta x} &= -\frac{1}{\eta} - \frac{H'}{H} & \Gamma^y_{\eta y} &= -\frac{1}{\eta} - \frac{H'}{H} \\
\Gamma^z_{\eta z} &= -\frac{1}{\eta} - \frac{H'}{H} & \Gamma^\eta_{xx} &= -\frac{1}{\eta} - \frac{H'}{H} \\
\Gamma^\eta_{yy} &= -\frac{1}{\eta} - \frac{H'}{H} & \Gamma^\eta_{zz} &= -\frac{1}{\eta} - \frac{H'}{H} \\
\Gamma^\eta_{\eta\eta} &= -\frac{1}{\eta} - \frac{H'}{H}
\end{aligned}
$$

More compactly, we can express them as:

$$
\Gamma^i_{\eta i} = \Gamma^\eta_{\mu\mu} = -\frac{1}{\eta} + \epsilon H a(\eta) = -\frac{1}{\eta}\left(1 + \frac{\epsilon}{1-\epsilon}\right) = -\frac{1}{\eta(1-\epsilon)} \approx -\frac{1}{\eta} - \frac{\epsilon}{\eta}
$$

Since, $\epsilon = 0$ for de Sitter space, we can drop those terms. Just a quick clarification—the conformal Killing vectors of quasi-de Sitter space match those of de Sitter space only at leading order in slow-roll. These should not be confused with the isometry-generating Killing vectors, which strictly preserve the metric, unlike conformal Killing vectors that preserve it only up to a conformal factor. The killing vectors being studied here are isometry generating killing vectors for de Sitter space not conformal killing vector. They just happen to be conformal killing vector of $\mathbb{R}^3$ and hence the association of name dilatation. The killing equation for such can be given as:

$$
\begin{aligned}
\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu &= 0 \\
\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\Gamma^\lambda_{\mu\nu}\xi_\lambda &= 0
\end{aligned}
$$

for $\mu = \nu$, we have following four expressions:

$$
\begin{aligned}
\partial_x \xi_x &= -\frac{1}{\eta}\xi_\eta & \partial_y \xi_y &= -\frac{1}{\eta}\xi_\eta \\
\partial_z \xi_z &= -\frac{1}{\eta}\xi_\eta & \partial_\eta \xi_\eta &= -\frac{1}{\eta}\xi_\eta
\end{aligned}
$$

We can solve the last equation as:

$$
\frac{d\xi_\eta}{\xi_\eta} = -\frac{d\eta}{\eta} \quad \Rightarrow \quad \ln\xi_\eta = -\ln\eta + \ln f(x,y,z)
$$

which leads to

$$
\boxed{\xi_\eta(\eta, x, y, z) = \frac{f(x,y,z)}{\eta}}
$$

Now we determine $f(x,y,z)$:

$$
\xi_\eta = \frac{f(x,y,z)}{\eta} \Rightarrow \partial_\eta \xi_\eta = -\frac{1}{\eta}\xi_\eta = -\frac{f(x,y,z)}{\eta^2}
$$

So,

$$\partial_x \xi_x = -\frac{f(x,y,z)}{\eta^2}; \qquad \partial_y \xi_y = -\frac{f(x,y,z)}{\eta^2}; \qquad \partial_z \xi_z = -\frac{f(x,y,z)}{\eta^2}$$

For now let us focus on the specific case where,

$$f(x,y,z) = \text{const} = A \Rightarrow \xi_\eta = \frac{A}{\eta}$$

We will determine the specific form of $f(x,y,z)$ later. Then:

$$\partial_x \xi_x = -\frac{A}{\eta^2} \Rightarrow \xi_x = -\frac{A}{\eta^2}x + \phi_1(y,z,\eta)$$

$$\partial_y \xi_y = -\frac{A}{\eta^2} \Rightarrow \xi_y = -\frac{A}{\eta^2}y + \phi_2(x,z,\eta)$$

$$\partial_z \xi_z = -\frac{A}{\eta^2} \Rightarrow \xi_z = -\frac{A}{\eta^2}z + \phi_3(x,y,\eta)$$

For now lets set $\phi_i = 0$ which describes rotation and fix coefficient A such that in the component form:

$$\boxed{\xi^\mu = g^{\mu\nu}\xi_\nu = (\eta, x, y, z)}$$

is the killing vector associated with dilatation. With the basis, it looks like:

$$D = \xi^\mu \partial_\mu = x^\mu \partial_\mu$$

Exponentiating them

$$\begin{aligned}
e^{\lambda D} x = e^{\lambda x \cdot \partial_x} x &= \lim_{N\to\infty} \left(1 + \frac{\lambda}{N} x \cdot \partial\right)^N x \\
&= \lim_{N\to\infty} \underbrace{\left(1 + \frac{\lambda}{N} x \cdot \partial\right) \dots \left(1 + \frac{\lambda}{N} x \cdot \partial\right)}_{N \text{ terms}} x \\
&= \lim_{N\to\infty} \underbrace{\left(1 + \frac{\lambda}{N} x \cdot \partial\right) \dots \left(1 + \frac{\lambda}{N} x \cdot \partial\right)}_{N-1 \text{ terms}} \left(x + \frac{\lambda}{N} x\right) \\
&= \lim_{N\to\infty} \left(1 + \frac{\lambda}{N}\right) \underbrace{\left(1 + \frac{\lambda}{N} x \cdot \partial\right) \dots \left(1 + \frac{\lambda}{N} x \cdot \partial\right)}_{N-1 \text{ terms}} x \\
&= e^\lambda x
\end{aligned}$$

or, more simply we can utilize $x \cdot \partial_x x = x$ and then

$$\begin{aligned}
e^{\lambda D} x &= \sum_{n=0}^{\infty} \frac{(\lambda x \cdot \partial_x)^n}{n!} x \\
&= 1 + \frac{\lambda}{1!} x \cdot \partial_x x + \frac{\lambda^2}{2!} (x \cdot \partial_x)(x \cdot \partial_x x) + \dots \\
&= \left(1 + \lambda + \frac{\lambda^2}{2!} + \dots\right) x = e^\lambda x
\end{aligned}$$

4.2.4 Special conformal transformation

The remaining killing equation which lead to SCT are as follows:

$$\partial_i \xi_\eta + \partial_\eta \xi_i + \frac{2}{\eta}\xi_i = 0$$

Going back we found the solution to

$$\partial_\eta\, \xi_\eta \;=\; -\frac{1}{\eta}\, \xi_\eta,$$

as

$$\xi_\eta(\eta,x,y,z) \;=\; \frac{f(x,y,z)}{\eta} \quad \text{for some function } f(x,y,z).$$

This time we won't assume it to just be a constant. Therefore the killing equation to be solved becomes:

$$\frac{1}{\eta}\,\partial_i f(x,y,z) \;+\; \partial_\eta\,\xi_i \;+\; \frac{2}{\eta}\,\xi_i = 0,$$

which is a first-order linear ordinary differential equation in η for each spatial component ξ_i. The homogeneous part $\partial_\eta\xi_i + (2/\eta)\xi_i = 0$ integrates immediately to $\xi_i^{(\mathrm{hom})}(\eta) = C_i(x,y,z)/\eta^2$. A particular solution of the inhomogeneous equation can be found by considering power series in η, defined as $\xi_i = A(x,y,z)\eta^\alpha$

$$A\alpha\eta^{\alpha-1} + 2A\eta^{\alpha-1} = -\eta^{-1}\partial_i f(x,y,z)$$
$$(\alpha+2)A\eta^{\alpha-1} = -\eta^{-1}\partial_i f(x,y,z) \implies \alpha = 0, A = -\frac{1}{2}\partial_i f(x,y,z)$$

Therefore, $\xi_i^{(\mathrm{part})} = -\frac{1}{2}\,\partial_i f(x,y,z)$. Hence the full solution is

$$\xi_\eta = \frac{f(x,y,z)}{\eta}$$
$$\xi_i(\eta,x,y,z) = \frac{C_i(x,y,z)}{\eta^2} \;-\; \frac{1}{2}\,\partial_i f(x,y,z)$$

Meanwhile, from the $i \neq j$ Killing equations:

$$\partial_i\xi_j + \partial_j\xi_i = \frac{1}{\eta^2}(\partial_i C_j + \partial_j C_i) - \partial_i\partial_j f = 0$$

we infer $\partial_i\partial_j f = 0$ and $\partial_i C_j + \partial_j C_i = 0$. From $\partial_i\partial_j f = 0$ one concludes f is at most linear in (x,y,z), so

$$f(x,y,z) = a_x\,x + a_y\,y + a_z\,z + A$$

with constant coefficients a_i, A. There are further constraints on C_i as:

$$\partial_i\xi_i = \frac{1}{\eta^2}\partial_i C_i(x,y,z) - \frac{1}{2}\partial_i\partial_i f(x,y,z) = -\frac{f(x,y,z)}{\eta^2}$$
$$\partial_i C_i(x,y,z) = -f(x,y,z)$$

Since we have already studied dilatation, where we studied the effect of $A \neq 0$, therefore let us set A to zero,

$$C_i = -\int f(x,y,z)dx_i = -(a\cdot x)x_i + \frac{1}{2}a_i(x^i)^2 + g(x_j \forall j \neq i)$$

Then

$$\partial_i C_j + \partial_j C_i = 0 \implies C_i = -\left(\sum_{j=1}^{3} a_j x^j\right)x_i + \frac{1}{2}a_i\left(\sum_{j=1}^{3} x^j x^j\right)$$

Let choose $(a_x,a_y,a_z) = -\frac{2}{H^2}(b_x,b_y,b_z)$ so ξ_i appears without fraction. Then

$$\xi_\eta(\eta,x,y,z) = -\frac{2b_i x^i}{H^2\eta} \qquad \xi_i(\eta,x,y,z) = \frac{2(b\cdot x)x_i - b_i\vec{x}^2}{H^2\eta^2} - \frac{1}{2H^2}\partial_i\left(-2b_j x^j\right) = \frac{2(b\cdot x)x_i - b_i\vec{x}^2}{H^2\eta^2} + \frac{b_i}{H^2}$$

Raising indices gives[4]

$$\xi^\eta = 2\,\eta\left(b_i x^i\right), \qquad \xi^i = (\eta^2 - \vec{x}^2)b^i + 2(b\cdot x)x^i$$

Since $x^\mu = (\eta, x^i)$ and $b^\mu = (0, b^i)$, this coincides exactly with

$$\xi^\mu = 2(b\cdot x)x^\mu - b^\mu\left(-\eta^2 + \vec{x}^2\right), \quad b^0 = 0$$

These are the generators of the isometry of de Sitter space not the conformal transformations of de Sitter space. We should be careful in regard to not interpret these killing vectors as conformal killing vectors. Since b^μ is spacelike with $b^0 = 0$ it does not generate SCT for $\mathbb{R}^4$ but for $\mathbb{R}^3$ on $\eta = 0$ hypersurface. We can see that these generators of de Sitter isometry has similar form as the expressions in section 1.12

[4]indices of b_i and x_i are raised and lowered by $\eta_{\mu\nu}$

- Translation
$$P_i = \partial_i$$
- Rotation
$$L_{ij} = x_i\partial_j - x_j\partial_i$$
- Dilatation
$$D = x^\mu \partial_\mu$$
- SCT
$$K^i = [2(b^j x_j)x^i - b^i(\eta_{kl}x^k x^l)]\partial_i$$

4.3 Ambient space

The isometries of de Sitter space are highly non-linear, which makes them difficult to study directly in intrinsic coordinates. A more convenient approach is to view them from the perspective of the higher-dimensional flat embedding space. The special relativity of higher dimension will apppear as general relativity on de Sitter space. The generators of $D+1$ dimensional lorentz transformations in such embedding space are given as:

$$J_{MN} = X_M\partial_N - X_N\partial_M$$

where $M, N = 0, 1, 2, \dots D$. There are several ways to coordinatize the embedded de Sitter space, for this section we choose to work in flat slicing. The coordinate transformation between the two given as:

$$\left.\begin{aligned} X^0 &= \tfrac{\rho}{2(-\eta)}(1-s) \\ X^D &= \tfrac{\rho}{2(-\eta)}(1+s) \end{aligned}\right\} \implies \rho = -\eta(X^D + X^0)$$
$$X^i = \frac{\rho}{-\eta}x^i \implies x^i = \frac{-\eta}{\rho}X^i = \frac{X^i}{X^0 + X^D}$$

where

$$s = \eta^2 - \delta_{ij}x^i x^j$$

and $i = 1, 2, \dots D-1$. We can note from below that ρ =constant hypersurface are hyperbolic in nature.

$$g_{AB}X^A X^B = -(X^0)^2 + (X^D)^2 + \delta_{ij}X^i X^j = \frac{\rho^2 s^2}{\eta^2} + \frac{\rho^2}{\eta^2}\delta_{ij}x^i x^j = \rho^2$$

Inverting above relation,

$$\begin{aligned} \rho &= \sqrt{g_{AB}X^A X^B} \implies \frac{\partial\rho}{\partial X^0} = -\frac{X^0}{\rho} = \frac{s-1}{2(-\eta)} \\ \eta &= -\frac{\rho}{X^0 + X^D} \implies \frac{\partial\eta}{\partial X^0} = \frac{X_0 X_D + X_D^2 + X^i X_i}{(X^D + X^0)^2\sqrt{g_{AB}X^A X^B}} = \frac{\frac{1+s}{2} + \delta_{ij}x^i x^j}{\rho} \\ x^i &= \frac{X^i}{X^0 + X^D} \end{aligned}$$

we can now use these to express the derivatives in the two coordinate system in the following manner:

$$\begin{aligned} \frac{\partial}{\partial X^0} &= \frac{1}{2(-\eta)}(-1 + \eta^2 - \delta_{ij}x^i x^j)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(1 + \eta^2 + \delta_{ij}x^i x^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i \\ \frac{\partial}{\partial X^D} &= \frac{1}{2(-\eta)}(1 + \eta^2 - \delta_{ij}x^i x^j)\frac{\partial}{\partial\rho} + \frac{1}{2\rho}(-1 + \eta^2 + \delta_{ij}x^i x^j)\frac{\partial}{\partial\eta} + \frac{\eta}{\rho}x^i\partial_i \\ \frac{\partial}{\partial X^i} &= \frac{x_i}{-\eta}\frac{\partial}{\partial\rho} - \frac{x_i}{\rho}\frac{\partial}{\partial\eta} - \frac{\eta}{\rho}\partial_i \end{aligned}$$

Metric in this coordinate

$$ds^2 = -(dX^0)^2 + (dX^D)^2 + \delta_{ij}dX^i dX^j = d\rho^2 + \rho^2\frac{\left(-d\eta^2 + \delta_{ij}dx^i dx^j\right)}{\eta^2}$$

Then

$$P_i = J_{Di} - J_{0i} = X_D\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^D} - X_0\frac{\partial}{\partial X^i} + X_i\frac{\partial}{\partial X^0}$$

$$
\begin{aligned}
&= (X_D - X_0)\frac{\partial}{\partial X^i} - X_i\left(\frac{\partial}{\partial X^D} - \frac{\partial}{\partial X^0}\right) \\
&= (X^D + X^0)\frac{\partial}{\partial X^i} - X^i\left(\frac{\partial}{\partial X^D} - \frac{\partial}{\partial X^0}\right) \\
&= \frac{\rho}{-\eta}\left(\frac{x^i}{-\eta}\frac{\partial}{\partial \rho} - \frac{x^i}{\rho}\frac{\partial}{\partial \eta} - \frac{\eta}{\rho}\partial_i\right) - \frac{\rho}{-\eta}x^i\left(\frac{1}{-\eta}\frac{\partial}{\partial \rho} - \frac{1}{\rho}\frac{\partial}{\partial \eta}\right) \\
&= \partial_i
\end{aligned}
$$

since $i = 1, 2, \dots D-1$, there are $D-1$ momenta and SCT generator.

$$
\begin{aligned}
K_i = J_{Di} + J_{0i} &= X_D\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^D} + X_0\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^0} \\
&= (X_D + X_0)\frac{\partial}{\partial X^i} - X_i\left(\frac{\partial}{\partial X^D} + \frac{\partial}{\partial X^0}\right) \\
&= (X^D - X^0)\frac{\partial}{\partial X^i} - X^i\left(\frac{\partial}{\partial X^D} + \frac{\partial}{\partial X^0}\right) \\
&= \frac{\rho s}{-\eta}\left(\cancel{\frac{x^i}{-\eta}\frac{\partial}{\partial \rho}} - \frac{x^i}{\rho}\frac{\partial}{\partial \eta} - \frac{\eta}{\rho}\partial_i\right) - \frac{\rho}{-\eta}x^i\left(\cancel{\frac{s}{-\eta}\frac{\partial}{\partial \rho}} + \frac{\eta^2 + \delta_{ij}x^i x^j}{\rho}\frac{\partial}{\partial \eta} + \frac{2\eta}{\rho}x^j\partial_j\right) \\
&= \cancel{\frac{x^i s}{\eta}\frac{\partial}{\partial \eta}} + s\partial_i + \frac{2\eta^2 - \cancel{s}}{\eta}x_i\frac{\partial}{\partial \eta} + 2x_i(x^j\partial_j) \\
&= (\eta^2 - \delta_{jk}x^j x^k)\partial_i + 2x_i\eta\partial\eta + 2x_i(x^j\partial_j) \\
&= 2x_i(x\cdot\partial) - (-\eta^2 + \vec{x}^2)\partial_i
\end{aligned}
$$

$$
\begin{aligned}
D = J_{D0} &= X_D\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X^D} = X_D\frac{\partial}{\partial X^0} + X^0\frac{\partial}{\partial X^D} \\
&= \frac{\rho}{2(-\eta)}(1+s)\left[\frac{1}{2(-\eta)}(-1+s)\frac{\partial}{\partial \rho} + \frac{1}{2\rho}(1 + \eta^2 + \delta_{ij}x^i x^j)\frac{\partial}{\partial \eta} + \frac{\eta}{\rho}x^i\partial_i\right] \\
&\quad + \frac{\rho}{2(-\eta)}(1-s)\left[\frac{1}{2(-\eta)}(1+s)\frac{\partial}{\partial \rho} + \frac{1}{2\rho}(-1 + \eta^2 + \delta_{ij}x^i x^j)\frac{\partial}{\partial \eta} + \frac{\eta}{\rho}x^i\partial_i\right] \\
&= -\eta\frac{\partial}{\partial \eta} - x^i\partial_i = -x\cdot\partial
\end{aligned}
$$

This exercise showed us that there is a more transparent way to visualize the isometries of de Sitter space: by embedding it into a higher-dimensional ambient space where the symmetry algebra acts linearly. However, one important question remains: **what happens to the de Sitter radius?**

In particular, if we take the flat-space limit $l \to \infty$, how do the de Sitter generators reduce to the Poincaré generators, and which symmetries morph into which? To study this question more carefully, we will next examine how the de Sitter isometries can be mapped via a stereographic projection onto Minkowski space, which itself can be embedded in the same ambient space. This approach makes it straightforward to perform the Inönü–Wigner contraction and explicitly see how the symmetry algebra of de Sitter reduces to that of flat spacetime.

Stereographic Projection of de Sitter Space

So far, the generators we wrote down have no explicit dependence on the de Sitter radius l. To introduce this dependence, we work in stereographic coordinates, where de Sitter space is projected conformally onto Minkowski space. We begin by recalling that de Sitter space can be realized as a hyperboloid embedded in $(d+1)$-dimensional Minkowski space:

$$-X_0^2 + X_1^2 + \cdots + X_{d-1}^2 + X_d^2 = l^2.$$

The normal to this surface could be given as:

$$n_A = \frac{\partial}{\partial x^A}\left(X^B X_B - l^2\right) = 2X_A$$

Any tensor field K which is tangent to this hypersurface has to satisfy the following transversality condition.

$$X^A K_A = 0$$

Now, to perform the projection, let us consider the separation vector starting from $(0, -l)$ and ending at (x^μ, l) in $\mathbb{R}^{d,1}$. The components of the separation vector between the points could be given as:

$$\begin{aligned} X^A(\Omega(x)) &= (0,0,\dots,-l) + \Omega(x)[(x^\mu, l) - (0,0,\dots,-l)] \\ &= (0,0,\dots,-l) + \Omega(x)(x^\mu, 2l) \\ &= (\Omega(x)x^\mu, 2\Omega(x)l - l) \end{aligned}$$

The points on this line eventually passes through the hyperboloid $X_A X^A = l^2$, so it leads to the following condition over $\Omega(x)$:

$$\begin{aligned} g_{\mu\nu}\Omega(x)^2 x^\mu x^\nu + l^2(2\Omega(x) - 1)^2 &= l^2 \\ \Omega(x)^2 x^2 + l^2(\cancel{1} - 4\Omega(x) + 4\Omega(x)^2) &= \cancel{l^2} \\ \Omega(x)^2 x^2 - 4\Omega(x) l^2 + 4\Omega(x)^2 l^2 &= 0 \\ \Omega(x)^2(x^2 + 4l^2) - 4\Omega(x)l^2 &= 0 \\ \Omega(x)[\Omega(x)(x^2 + 4l^2) - 4l^2] = 0 \implies \Omega(x) &= \frac{4l^2}{x^2 + 4l^2} \end{aligned}$$

Resulting in the following relationship between the points over de Sitter hypersurface and plane at $X^d = l$.

$$\begin{aligned} X^\mu &= \Omega(x)x^\mu \\ X^d &= 2\Omega(x)l - l = l\left(2\frac{1}{1+\frac{x^2}{4l^2}} - 1\right) = l\left(\frac{1 - \frac{x^2}{4l^2}}{1 + \frac{x^2}{4l^2}}\right) \end{aligned}$$

Using this we define the stereographic projection by the following conformal transformation:[5]

$$\begin{aligned} X^\mu &\equiv r^\mu = \Omega(x)x^\mu \\ X^4 &\equiv r^4 = l\Omega(x)\left(1 - \frac{x^2}{4l^2}\right) \\ \Omega(x) &= \frac{1}{1 + \frac{x^2}{4l^2}} \end{aligned}$$

Here, r^a is the cartesian coordinate on de Sitter space and x^a is the coordinate on projected minkowski space. We can note that $r_A r^A = l^2$ is imposed on de Sitter coordinates but no such condition is imposed on $x^2 = \eta_{\mu\nu}x^\mu x^\nu$. The inverse of transformation could be found by the considering that each hyperbolic hypersurface gets mapped onto their respective plane sitting the $X^d = l$:

$$\begin{aligned} x^\mu &= \left(1 + \frac{l - X^4}{1 + X^4}\right) X^\mu = \left(\frac{2}{1 + \frac{X^4}{\sqrt{X^\mu X_\mu + (X^4)^2}}}\right) X^\mu = \frac{1}{\Omega(x)} X^\mu \\ l &= \sqrt{X^\mu X_\mu + (X^4)^2} \end{aligned}$$

We will now define the following object:[6]

$$\begin{aligned} K^\mu_A &= \frac{1}{\Omega(x)^2}\frac{\partial r_A}{\partial x_\mu} = \frac{\delta^\mu_A}{\Omega} - r_A \frac{\partial}{\partial x_\mu}\left(\frac{1}{\Omega(x)}\right) \\ &= \left(1 + \frac{x^2}{4l^2}\right)^2 \frac{\partial r_A}{\partial x_\mu} \end{aligned}$$

whose explicit form can be derived by using:

$$\frac{\partial r_\nu}{\partial x_\mu} = \Omega(x)\delta^\mu_\nu - x_\nu \Omega(x)^2 \frac{x^\mu}{2l^2} \qquad\qquad \frac{\partial r_4}{\partial x^\mu} = -\Omega(x)^2 \frac{x_\mu}{l}$$

This object has certain properties which as we will see, would make it a versatile tool at our disposal. But rather than using this straightforward method, we will derive their explicit form in somewhat lengthy but illuminating way which will reveal their true nature. These K^μ_A are found to satisfy the conformal killing equation

$$\frac{\partial}{\partial x^\mu} K^A{}_\nu + \frac{\partial}{\partial x^\nu} K^A{}_\mu = \delta^A_\mu \partial_\nu \left(\frac{1}{\Omega}\right) - \delta^A_\nu \partial_\mu \left(\frac{1}{\Omega}\right) - x^A \partial_\mu \partial_\nu \left(\frac{1}{\Omega}\right)$$

[5] pg 369 of Group Theoretical Concepts and Methods in Elementary Particle Physics by Feza Gürsey

[6] this insight was provided in A unified construction of Skyrme-type non-linear sigma models via the higher dimensional Landau models

$$
\begin{aligned}
&+ \delta^A_\nu \partial_\mu \left(\frac{1}{\Omega}\right) - \delta^A_\mu \partial_\nu \left(\frac{1}{\Omega}\right) - x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) \\
&= -2x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) = 2f(x) g_{\mu\nu}
\end{aligned}
$$

and

$$
\begin{aligned}
f(x) g_{\mu\nu} &= -x^A \partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) \\
f(x) g^{\mu\nu} g_{\mu\nu} &= -x^A \partial_\mu \partial^\mu \left(\frac{1}{\Omega}\right) = \partial_\mu K^{A\mu} \\
f(x) &= \frac{\partial_\mu K^{A\mu}}{d}
\end{aligned}
$$

So, ${K^A}_\mu$ could be interpreted as killing vector as long as $\partial_\nu \partial_\mu \left(\frac{1}{\Omega}\right) \propto g_{\mu\nu}$. There is additional property that ${K^A}_\mu$ satisfies, namely transversality condition:[7]

$$
r^A {K_A}^\mu \Big|_{r_A r^A = l^2} = 0 \implies x^\nu {K_\nu}^\mu + l \left(1 - \frac{x^2}{4l^2}\right) {K_4}^\mu = 0
$$

Since we can write the general form of conformal killing vectors as following,

$$
K^\mu_a = t^\mu_a + \epsilon_a x^\mu + \omega^{\mu\nu}_a x_\nu + \lambda^\mu_a x^2 - 2\lambda^\sigma_a x_\sigma x^\mu
$$

Inserting above in transverality condition leads to:

$$
x^\nu (t^\mu_\nu + \epsilon_\nu x^\mu + \omega^{\mu\sigma}_\nu x_\sigma + \lambda^\mu_\nu x^2 - 2\lambda^\sigma_\nu x_\sigma x^\mu) + l \left(1 - \frac{x^\nu x_\nu}{4l^2}\right) (t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) = 0
$$

Or,

$$
\begin{aligned}
x^\nu t^\mu_\nu + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + \lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu) + l(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) \\
- \frac{x^\nu x_\nu}{4l}(t^\mu_4 + \epsilon_4 x^\mu + \omega^{\mu\nu}_4 x_\nu + \lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu) = 0
\end{aligned}
$$

It can be conveniently written as:

$$
\begin{aligned}
l t^\mu_4 + (x^\nu t^\mu_\nu + l\epsilon_4 x^\mu + l\omega^{\mu\nu}_4 x_\nu) + \left[-\frac{x^2}{4l} t^\mu_4 + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + l\lambda^\mu_4 x^2 - 2l\lambda^\sigma_4 x_\sigma x^\mu\right] \\
+ \left[\lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu - \frac{1}{4l}\epsilon_4 x^2 x^\mu - \frac{1}{4l}\omega^{\mu\nu}_4 x^2 x_\nu\right] + \left[-\frac{x^4}{4l}\lambda^\mu_4 + \frac{2x^2 x_\sigma x^\mu}{4l}\lambda^\sigma_4\right] = 0
\end{aligned}
$$

Let us set coefficient of x^ν to zero, order by order.

$$
\begin{aligned}
t^\mu_4 &= 0 \\
x^\nu t^\mu_\nu + l\epsilon_4 x^\mu + l\omega^{\mu\nu}_4 x_\nu &= 0 \\
-\frac{x^2}{4l} t^\mu_4 + \epsilon_\nu x^\nu x^\mu + \omega^{\mu\sigma}_\nu x^\nu x_\sigma + l\lambda^\mu_4 x^2 - 2l\lambda^\sigma_4 x_\sigma x^\mu &= 0 \\
\lambda^\mu_\nu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^\mu - \frac{1}{4l}\epsilon_4 x^2 x^\mu - \frac{1}{4l}\omega^{\mu\nu}_4 x^2 x_\nu &= 0 \\
-\frac{x^4}{4l}\lambda^\mu_4 + \frac{2x^2 x_\sigma x^\mu}{4l}\lambda^\sigma_4 &= 0
\end{aligned}
$$

contracting the last four equations with x_μ:

$$
t^\mu_\nu x_\mu x^\nu + l\epsilon_4 x^2 = 0
$$

[7] following the procedure outlined in section 2 of Gauge Theories on de Sitter space and Killing Vectors

$$\begin{aligned}
\epsilon_\nu x^\nu x^2 + l\lambda^\mu_4 x_\mu x^2 - 2l\lambda^\sigma_4 x_\sigma x^2 &= 0 \\
\lambda^\mu_\nu x_\mu x^\nu x^2 - 2\lambda^\sigma_\nu x^\nu x_\sigma x^2 - \frac{1}{4l}\epsilon_4 x^4 &= 0 \\
x^4 \lambda^\mu_4 &= 0
\end{aligned}$$

Hence,

$$\begin{aligned}
t^\mu_\nu x_\mu x^\nu + l\epsilon_4 x^2 &= 0 \\
\epsilon_\nu x^\nu x^2 &= 0 \\
-\lambda^\sigma_\nu x^\nu x_\sigma x^2 - \frac{1}{4l}\epsilon_4 x^4 &= 0
\end{aligned}$$

Since t generates translation, we have $t^\mu_\nu = \delta^\mu_\nu$:

$$\begin{aligned}
\epsilon_4 &= -\frac{1}{l} \\
\lambda_{\mu\nu} &= \frac{1}{4l^2} g_{\mu\nu}
\end{aligned}$$

Substituting them in Killing vector, we get:

$$\begin{aligned}
K^\mu_\nu &= \left(1 + \frac{x^2}{4l^2}\right)\delta^\mu_\nu - \frac{x_\nu x^\mu}{2l^2} \\
&= \delta^\mu_\nu + \frac{1}{4l^2}(x^2\delta^\mu_\nu - 2x_\nu x^\mu) \\
K^\mu_4 &= -\frac{x^\mu}{l}
\end{aligned}$$

These satisfy few properties:

$$\begin{aligned}
K^\mu_A K^{A\nu} &= \left[\delta^\mu_\alpha + \frac{1}{4l^2}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)\right]\left[g^{\alpha\nu} + \frac{1}{4l^2}(x^2\, g^{\alpha\nu} - 2x^\nu x^\alpha)\right] + \frac{x^\mu x^\nu}{l^2} \\
&= g^{\mu\nu} + \frac{1}{4l^2}(x^2 g^{\mu\nu} - \cancel{2x^\nu x^\mu}) + \frac{1}{4l^2}(x^2 g^{\mu\nu} - \cancel{2x^\nu x^\mu}) + \frac{1}{16l^4}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)(x^2\, g^{\alpha\nu} - 2x^\nu x^\alpha) + \cancel{\frac{x^\mu x^\nu}{l^2}} \\
&= g^{\mu\nu} + \frac{x^2}{2l^2} g^{\mu\nu} + \frac{1}{16l^4}(x^4 g^{\mu\beta} - 2x^2 x^\mu x^\nu - 2x^2 x^\mu x^\nu + 4x^\alpha x_\alpha x^\mu x^\nu) \\
&= \left(1 + \frac{x^2}{4l^2}\right)^2 g^{\mu\nu} = \frac{1}{\Omega(x)^2} g^{\mu\nu} \\
K^\mu_B K_{C\mu} &= \left[\delta^\mu_\alpha + \frac{1}{4l^2}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)\right]\left[g_{\beta\mu} + \frac{1}{4l^2}(x^2\, g_{\beta\mu} - 2x_\mu x_\beta)\right]\delta^\alpha_B\delta^\beta_C + \frac{x^\mu x_\mu}{l^2}\delta^4_B\delta^4_C \\
&\quad - \frac{x_\mu}{l}\left[\delta^\mu_\alpha + \frac{1}{4l^2}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)\right]\delta^4_B\delta^\alpha_C - \left[\delta^\mu_\alpha + \frac{1}{4l^2}(x^2\delta^\mu_\alpha - 2x_\alpha x^\mu)\right]\frac{x_\mu}{l}\delta^\alpha_B\delta^4_C \\
&= \left(1 + \frac{x^2}{4l^2}\right)^2\left(g_{\alpha\beta} - \frac{r_\alpha r_\beta}{l^2}\right)\delta^\alpha_B\delta^\beta_C + \left(1 + \frac{x^2}{4l^2}\right)^2\left(1 - \frac{r_4^2}{l^2}\right)\delta^4_B\delta^4_C - \frac{1}{l}\left[x_\alpha - \frac{x^2 x_\alpha}{4l^2}\right](\delta^4_B\delta^\alpha_C + \delta^\alpha_B\delta^4_C) \\
&= \left(1 + \frac{x^2}{4l^2}\right)^2\left(g_{BC} - \frac{r_B r_C}{l^2}\right)
\end{aligned}$$

It is precisely this property which makes it possible to use these killing vector as projection operator[8]

$$\begin{aligned}
A_\mu &= \frac{\partial r^B}{\partial x^\mu}\hat{A}_B = \Omega(x)^2 K^B{}_\mu \hat{A}_B \\
K^\mu_C A_\mu &= \Omega(x)^2 K^B{}_\mu K_C{}^\mu \hat{A}_B = \left(\delta^B_C - \frac{r^B r_C}{l^2}\right)\hat{A}_B \\
\hat{A}_B &= K^\mu_B A_\mu \qquad \text{(where we used } r^B\hat{A}_B = 0\text{)}
\end{aligned}$$

[8]hat tensors are on de Sitter and without hat ones are on Minkowski

It makes sense because $\frac{\partial r^A}{\partial x^\mu}$ forms a $d \times (d-1)$ matrix and as such does not have $\frac{\partial x^\nu}{\partial r^A}$ as its inverse. However, all hope is not lost, the inverse could be found but the procedure is slightly different and given as following:

$$\begin{aligned}
\frac{\partial r^A}{\partial x^\mu}\frac{\partial r_A}{\partial x_\nu} &= \Omega^4(x)K^A_\mu K^\nu_A = \Omega^2(x)\delta^\nu_\mu \\
\frac{\partial r^A}{\partial x^\mu}\frac{1}{\Omega(x)^2}\frac{\partial r_A}{\partial x_\nu} &= \delta^\nu_\mu \\
\left(\frac{\partial r^A}{\partial x^\mu}\right)^{-1} &= \frac{1}{\Omega(x)^2}\frac{\partial r_A}{\partial x_\nu} = K^\nu_A
\end{aligned}$$

We can define the derivative along de Sitter hyperspace independent of choice of slicing by projecting the derivatives of ambient space and killing off the remaining components of it:

$$\begin{aligned}
\hat{\partial}_A &= \frac{\partial}{\partial r^A} - \frac{r_A r_B}{l^2}\frac{\partial}{\partial r_B} \implies r^A\hat{\partial}_A = 0 \\
&\equiv K^\mu_A \frac{\partial}{\partial x^\mu}
\end{aligned}$$

using $r_A r^A = l^2$ and $\eta_{AB} = (-,+,+\dots)$

$$\begin{aligned}
J_{AB} &= r_A\frac{\partial}{\partial r^B} - r_B\frac{\partial}{\partial r^A} \\
&= r_A\left(\hat{\partial}_B + \frac{r_B r_C}{l^2}\frac{\partial}{\partial r_C}\right) - r_B\left(\hat{\partial}_A + \frac{r_A r_C}{l^2}\frac{\partial}{\partial r_C}\right) \\
&= (r_A\hat{\partial}_B - r_B\hat{\partial}_A) \\
&= (r_A K^\mu_B - r_B K^\mu_A)\partial_\mu
\end{aligned}$$

in a more compact form:

$$r^A J_{AB} = l^2 K^\mu_B \partial_\mu$$

Then, for $A, B \neq 4$

$$\begin{aligned}
L_{ab} &= \Omega(x)\left[x_a\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_b - \frac{x_a x_b x^\mu}{4l^2} - x_b\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_a + \frac{x_a x_b x^\mu}{4l^2}\right]\partial_\mu \\
&= (x_a\delta^\mu_b - x_b\delta^\mu_a)\partial_\mu
\end{aligned}$$

These are spatial rotation on the hypersurface getting mapped to spatial rotation in the projected Minkowski space. Boosts in the ambient space and projected Minkowski plane could be given as

$$K_i = J_{0i}$$

Now we calculate the de Sitter Momenta

$$\begin{aligned}
lP_\nu = J_{4\nu} &= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left\{\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu - \frac{x_\nu x^\mu}{2l^2}\right\} + \frac{x_\nu x^\mu}{l}\right]\partial_\mu \\
&= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu - \frac{x_\nu x^\mu}{2l} + \frac{x^2}{4l^2}\frac{x_\nu x^\mu}{2l} + \frac{x_\nu x^\mu}{l}\right]\partial_\mu \\
&= \Omega(x)\left[l\left(1-\frac{x^2}{4l^2}\right)\left(1+\frac{x^2}{4l^2}\right)\delta^\mu_\nu + \frac{x^2}{4l^2}\frac{x_\nu x^\mu}{2l} + \frac{x_\nu x^\mu}{2l}\right]\partial_\mu \\
&= \left[l\left(1-\frac{x^2}{4l^2}\right)\delta^\mu_\nu + \frac{x_\nu x^\mu}{2l}\right]\partial_\mu \\
&= l\partial_\nu + \frac{1}{4l}\left[-x^2\delta^\mu_\nu + 2x_\nu x^\mu\right]\partial_\mu
\end{aligned}$$

The zeroth component of this momenta (which was dilatation on the hypersurface) becomes our new Hamiltonian on the projected Minkowski plane:

$$H = \frac{J_{40}}{l}$$

The algebra on the projected Minkowski plane is:

$$[J_{\mu\nu}, J_{\lambda\rho}] = \eta_{\mu\lambda}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\lambda} + \eta_{\nu\rho}J_{\mu\lambda} - \eta_{\nu\lambda}J_{\mu\rho}$$

$$[J_{\mu\nu}, P_\lambda] = \frac{1}{l}[J_{\mu\nu}, J_{4\lambda}] = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu$$
$$[P_\mu, P_\nu] = \frac{1}{c^2 l^2}[J_{4\mu}, J_{4\nu}] = \frac{1}{c^2 l^2}J_{\mu\nu}$$

We are now in a position to answer the question of how the de Sitter generators morph in the flat-space limit. As $l \to \infty$ the de Sitter momenta begin to commute and we recover Lorentz algebra. In this limit, the mixed generators behave as follows:

- $J_{D\mu}$ reduces to the generators of Minkowski translations ($D = 4$),
- J_{0i} reduces to the generators of Minkowski boosts ($D - 1 = 3$).
- J_{ab} stays the generator of spatial rotation ($\frac{(D-1)(D-2)}{2} = 3$).

That counts to $4 + 3 + 3 = 10$ generators and this is how they get mapped onto Minkowski plane. Physically, this corresponds to taking the cosmological constant $\Lambda \to 0$, which flattens the spacetime. This connection becomes evident when we recall the relation between the scalar curvature of de Sitter space, the cosmological constant, and the de Sitter radius:

$$R = \frac{d(d-1)}{l^2} = \frac{2d}{d-2}\Lambda = d(d-1)\left(\frac{H}{c}\right)^2$$

4.4 Wigner Innou Contraction

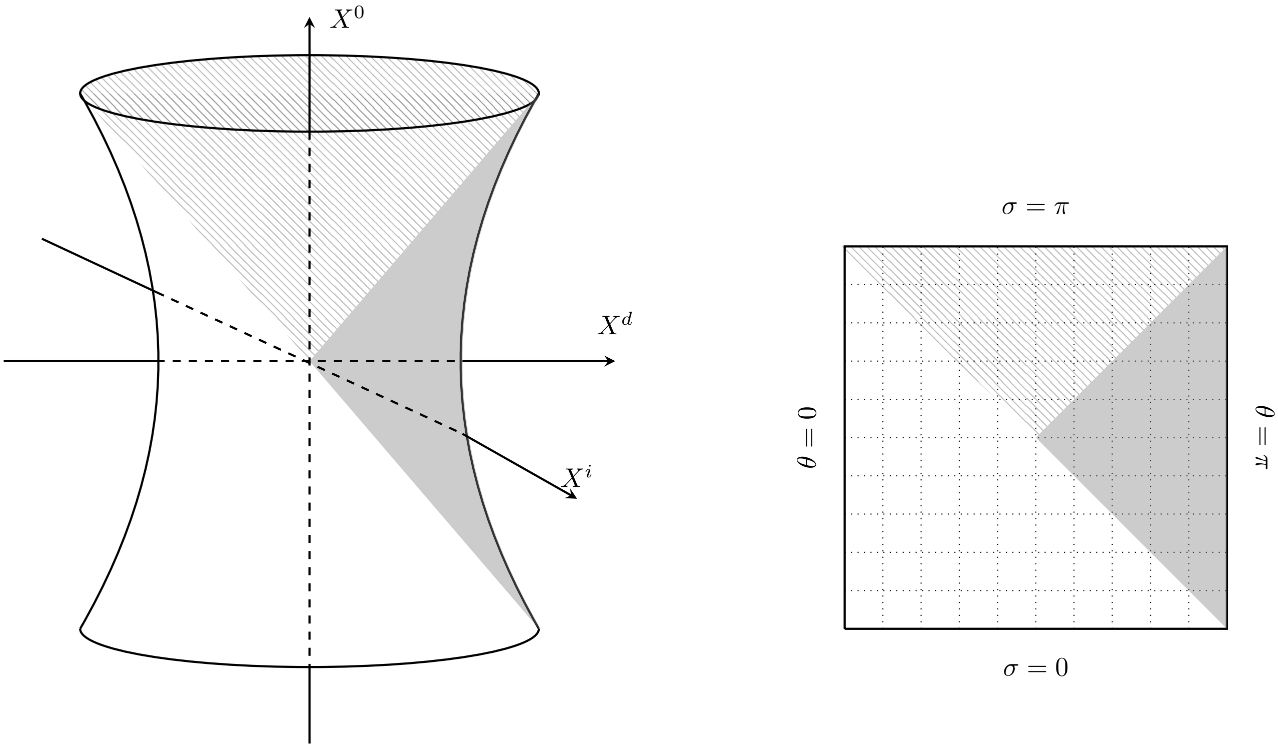


Figure 4.3: To take the $l \to \infty$ limit, imagine placing a plane at $X^d = l$. The plane will be coordinatized by X^μ and the rotation in $X^\mu X^d$ plane will appear as translation. That's why we split the generators of the ambient space into 6 lorentz generator + 4 translation. Taking the limit basically implies that we are confining ourselves to origin of the plane near the $X^d = l$ plane.

Method 1

We will first use the stereographic projection to see how the Newton Hook limit could be achieved. Under Wigner Innou contraction the rotations about these planes will appear as translation and they are generated by Momentum operator.

$$
\begin{aligned}
P_\mu &= \frac{J_{4\mu}}{l} \\
L_{ab} &= J_{ab} = x_a\partial_b - x_b\partial_a \\
-K_a = L_{a0} &= \frac{J_{a0}}{c}
\end{aligned}
$$

where $a = 1,2,3$ and $\mu = 0,1,2,3$ with mostly plus metric signature:

$$[J_{AB}, J_{CD}] = \eta_{AC}J_{BD} - \eta_{AD}J_{BC} + \eta_{BD}J_{AC} - \eta_{BC}J_{AD}$$

which leads to

$$
\begin{aligned}
[J_{\mu\nu}, J_{\lambda\rho}] &= \eta_{\mu\lambda}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\lambda} + \eta_{\nu\rho}J_{\mu\lambda} - \eta_{\nu\lambda}J_{\mu\rho} \\
[J_{\mu\nu}, P_\lambda] &= \frac{1}{l}[J_{\mu\nu}, J_{4\lambda}] = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu \\
[P_\mu, P_\nu] &= \frac{1}{c^2l^2}[J_{4\mu}, J_{4\nu}] = \frac{1}{c^2l^2}J_{\mu\nu}
\end{aligned}
$$

Then, the algebra becomes:[9]

$$
\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{ad}L_{be} + \delta_{be}L_{ad} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0} \\
[L_{0b}, L_{0e}] &= \frac{1}{c^2}[J_{0b}, J_{0e}] = \eta_{00}\frac{1}{c^2}J_{be}, \\
[L_{ab}, P_d] &= \frac{1}{l}[J_{ab}, J_{4d}] = \delta_{bd}\frac{J_{a4}}{l} - \delta_{ad}\frac{J_{b4}}{l} = \delta_{ad}P_b - \delta_{bd}P_a \\
[L_{a0}, P_b] &= \frac{1}{cl}[J_{a0}, J_{4b}] = -\frac{1}{cl}\delta_{ab}J_{04} = \frac{1}{c}\delta_{ab}P_0, \\
[L_{a0}, P_0] &= \frac{1}{cl}[J_{a0}, J_{40}] = \eta_{00}\frac{1}{cl}J_{a4} = \frac{1}{c}P_a, \\
[L_{ab}, P_0] &= \frac{1}{l}[J_{ab}, J_{40}] = 0, \\
[P_a, P_b] &= \frac{1}{l^2}[J_{4a}, J_{4b}] = \frac{\eta_{44}}{l^2}J_{ab} = \frac{1}{l^2}L_{ab}, \\
[P_a, P_0] &= \frac{1}{l^2}[J_{4a}, J_{40}] = \frac{\eta_{44}}{l^2}L_{a0} = -\frac{c}{l^2}K_a \\
[P_0, P_0] &= \frac{1}{l^2}[J_{40}, J_{40}] = 0.
\end{aligned}
$$

We find that unless we scale P_0 by c. The resulting algebra will be ill defined as $[L_{a0}, P_0] \to 0$ and it will not match with Galilean group where $[L_{a0}, P_0] = P_a$.

$$
\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{be}L_{ad} - \delta_{ad}L_{be} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0} \\
[L_{0b}, L_{0e}] &= -\frac{1}{c^2}L_{be}, \\
[L_{ab}, P_d] &= \delta_{ad}P_b - \delta_{bd}P_a \\
[L_{a0}, P_b] &= \frac{1}{c^2}\delta_{ab}P_0, \\
[L_{a0}, P_0] &= P_a, \\
[L_{ab}, P_0] &= 0, \\
[P_a, P_b] &= \frac{1}{\tau^2c^2}L_{ab}, \\
[P_a, P_0] &= \frac{1}{\tau^2}L_{a0}, \\
[P_0, P_0] &= 0.
\end{aligned}
$$

[9]This form of the analysis was suggested in section 2.4.2 of de Sitter Relativity: Foundation and some physical implications

where $\tau = \frac{l}{c}$ is kept constant during the process of taking the limit $l \to \infty$ and $c \to \infty$:

$$\begin{aligned}
[L_{ab}, L_{de}] &= \delta_{be}L_{ad} - \delta_{ad}L_{be} - \delta_{bd}L_{ae} - \delta_{ae}L_{bd} \\
[L_{ab}, L_{d0}] &= \delta_{ad}L_{b0} - \delta_{bd}L_{a0} \\
[L_{0b}, L_{0e}] &= 0 \\
[L_{ab}, P_d] &= \delta_{ad}P_b - \delta_{bd}P_a \\
[L_{a0}, P_b] &= 0, \\
[L_{a0}, P_0] &= P_a, \\
[L_{ab}, P_0] &= 0, \\
[P_a, P_b] &= 0 \\
[P_a, P_0] &= -\frac{1}{\tau^2}K_a, \\
[P_0, P_0] &= 0.
\end{aligned}$$

which is same as

$$\begin{array}{lll}
[J_i, J_j] = \epsilon_{ijk}J_k, & [J_i, P_j] = \epsilon_{ijk}P_k, & [J_i, K_j] = \epsilon_{ijk}K_k, \\
[P_i, P_j] = 0, & [K_i, K_j] = 0, & [K_i, P_j] = 0, \\
[H, J_i] = 0, & [H, P_i] = \dfrac{c^2\Lambda}{3}K_i, & [H, K_i] = P_i,
\end{array}$$

for

$$J_i = \frac{1}{2}\epsilon_{ijk}J_{jk} \qquad\qquad P_0 = H$$

Method 2

Alternatively, we can derive the contraction by considering the Lie algebra of the Killing vector fields in the ambient space in the form

$$\begin{array}{lll}
[J_i, J_j] = \epsilon_{ijk}J_k, & [J_i, P_j] = \epsilon_{ijk}P_k, & [J_i, K_j] = \epsilon_{ijk}K_k, \\
[P_i, P_j] = \epsilon_{ijk}J_k, & [K_i, K_j] = -\epsilon_{ijk}J_k, & [K_i, P_j] = -\delta_{ij}H, \\
[H, J_i] = 0, & [H, P_i] = K_i, & [H, K_i] = P_i,
\end{array}$$

Under the Wigner-Innou contraction to be considered here, we only have to examine Lie brackets which doesn't involve J_i. Therefore, we only have to worry about the following 5 commutators:

$$\begin{array}{lll}
[P_i, P_j] = \epsilon_{ijk}J_k, & [K_i, K_j] = -\epsilon_{ijk}J_k, & [K_i, P_j] = -\delta_{ij}H, \\
[H, P_i] = K_i, & [H, K_i] = P_i, &
\end{array}$$

We will consider the following rescaling as suggested in section 2 of covariant formulation of newton-hooke particle and its canonical analysis[10]. The ambient space operators are scaled as following:

$$\tilde{P}_i = \frac{P_i}{cl} = \frac{J_{4i}}{cl} \qquad \tilde{P}_0 \equiv \tilde{H} = \frac{H}{l} = \frac{J_{40}}{l} \qquad \tilde{K}_i = \frac{K_i}{c} = \frac{J_{0i}}{c}$$

With this rescaling, the algebra becomes:

$$\begin{array}{lll}
[J_i, J_j] = \epsilon_{ijk}J_k, & [J_i, \tilde{P}_j] = \epsilon_{ijk}\tilde{P}_k, & [J_i, \tilde{K}_j] = \epsilon_{ijk}\tilde{K}_k, \\
[\tilde{P}_i, \tilde{P}_j] = \dfrac{1}{c^2l^2}\epsilon_{ijk}J_k & [\tilde{K}_i, \tilde{K}_j] = -\dfrac{1}{c^2}\epsilon_{ijk}J_k & [\tilde{K}_i, \tilde{P}_j] = -\dfrac{1}{c^2}\delta_{ij}\tilde{H} \\
[\tilde{H}, J_i] = 0, & [\tilde{H}, \tilde{P}_i] = \dfrac{1}{l^2}K_i, & [\tilde{H}, \tilde{K}_i] = \tilde{P}_i,
\end{array}$$

Now define the parameter $l = \frac{c^2\Lambda}{3}$ so that in the limit $c \to \infty$ and $\Lambda \to 0$ with $c^2\Lambda$ fixed, the commutators contract to

$$[J_i, J_j] = \epsilon_{ijk}J_k, \qquad [J_i, P_j] = \epsilon_{ijk}P_k, \qquad [J_i, K_j] = \epsilon_{ijk}K_k,$$

[10]Boosts are scaled down by c for Galilean contraction.

$$[P_i, P_j] = 0, \qquad [K_i, K_j] = 0, \qquad [K_i, P_j] = 0,$$
$$[H, J_i] = 0, \qquad [H, P_i] = \frac{c^2\Lambda}{3} K_i, \qquad [H, K_i] = P_i,$$

It is often convenient to consider the central extension of Newton Hooke' group by redefining $\tilde{H} = \frac{H}{l} - mc^2$. The only commutator this modifies is

$$[\tilde{K}_i, \tilde{P}_j] = \lim_{c\to\infty} \frac{1}{c^2 l}[K_i, P_j] = -\lim_{c\to\infty} \frac{1}{c^2}\delta_{ij}\frac{H}{l} = -\lim_{c\to\infty}\left(m + \frac{\tilde{H}}{c^2}\right)\delta_{ij} = -m\delta_{ij}$$

where m appearing as central charge is the mass of the particle.

Method 3

We embed d-dimensional spacetime into a $(d+1)$-dimensional space with coordinates $X^A = (X^0, X^i, X^D)$, $i = 1, \ldots, d-1$, and define the Lorentz generators

$$J_{AB} = X_A\partial_B - X_B\partial_A.$$

Translation generators

Introduce a large parameter l and set $X^D = l$, $X^\mu = x^\mu$. Then

$$J_{D\mu} = X_D\partial_\mu - X_\mu\partial_D = l\partial_\mu - x_\mu\partial_D.$$

Divide by l and take $l \to \infty$:

$$\frac{1}{l}J_{D\mu} = \partial_\mu - \frac{x_\mu\partial_D}{l} \xrightarrow{l\to\infty} \partial_\mu.$$

Thus the generators corresponding to translations after contraction are

$$P_\mu = \partial_\mu.$$

Spatial rotations

The spatial rotations remain unchanged:

$$J_{ij} = x_i\partial_j - x_j\partial_i.$$

Galilean boosts

Set $X^0 = ct$, $X^i = x^i$, then

$$J_{0i} = X_0\partial_i - X_i\partial_0 = ct\partial_i - x_i\frac{1}{c}\partial_t.$$

Divide by c and take $c \to \infty$:

$$B_i = \lim_{c\to\infty}\frac{1}{c}J_{0i} = t\partial_i.$$

Dilatation

The dilatation generator in embedding space is

$$D = J_{D0} = X_D\partial_0 - X_0\partial_D = l\partial_0 - ct\partial_D.$$

Divide by l and take $l \to \infty$:

$$D \xrightarrow{l\to\infty} \partial_0.$$

Note that the contraction here took place in the ambient space, not on the hypersurface, so we did not concern ourselves with

$$\begin{aligned}
P_i &= J_{Di} - J_{0i} \\
&= \frac{1}{l}\left[X_D\partial_i - X_i\partial_D\right] - \frac{1}{c}\left[X_0\partial_i - X_i\partial_0\right] \\
&= \frac{1}{l}\left[l\partial_i - x_i\partial_D\right] - \frac{1}{c}\left[ct\partial_i + x_i\frac{1}{c}\partial_t\right] \\
&= (1-t)\partial_i
\end{aligned}$$

and

$$\begin{aligned}
K_i &= J_{Di} + J_{0i} \\
&= \frac{1}{l}\left[X_D\partial_i - X_i\partial_D\right] + \frac{1}{c}\left[X_0\partial_i - X_i\partial_0\right] \\
&= \frac{1}{l}\left[l\partial_i - x_i\partial_D\right] + \frac{1}{c}\left[ct\partial_i + x_i\frac{1}{c}\partial_t\right] \\
&= (1+t)\partial_i
\end{aligned}$$

because these combination do not survive in the traditional sense. Now, to find the algebra, we compute the commutators of the above representations with each other, since the algebra is independent of the choice of representation and the point in spacetime at which it is evaluated.

4.5 Action

We are given the action of a relativistic point particle in a de Sitter background,

$$S = -mc\int d\tau\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

In the static patch, the metric components take simpler familiar form:

$$g_{00} = -f(r), \quad g_{rr} = \frac{1}{f(r)}, \quad f(r) = 1 - \Lambda r^2, \quad r^2 = x^i x^i$$

Then the action reduces to:

$$S = -mc\int d\tau\sqrt{f(r)\dot{t}\dot{t} - \frac{1}{f(r)}\dot{r}^2}$$

$$S = \frac{m}{2}\int d\tau\left(\frac{\dot{x}^i\dot{x}^i}{\dot{t}} + \frac{\epsilon x^i x^i\dot{t}}{R^2}\right) - mc^2\int d\tau\dot{t}$$

Chapter 5

Ward Identities in de Sitter space

The infinitesimal conformal transformation which respects the de Sitter isometry will look like

$$\begin{aligned}
\text{dilation:} \quad & \eta \to \eta(1+\lambda), & \mathbf{x} &\to \mathbf{x}(1+\lambda), \\
\text{SCT:} \quad & \eta \to \eta(1-2\mathbf{b}\cdot\mathbf{x}), & \mathbf{x} &\to \mathbf{x} - 2(\mathbf{b}\cdot\mathbf{x})\mathbf{x} + (\mathbf{x}^2-\eta^2)\mathbf{b},
\end{aligned}$$

We consider the following quantity:

$$\begin{aligned}
e^{i\lambda D}\langle \ldots \rangle = \langle \ldots \rangle &\implies D\langle \ldots \rangle = 0 \\
e^{i\vec{b}\cdot\vec{K}}\langle \ldots \rangle = \langle \ldots \rangle &\implies \vec{b}\cdot\vec{K}\langle \ldots \rangle \equiv \mathbf{b}\cdot\mathbf{K}\ \langle \ldots \rangle\ = 0
\end{aligned}$$

function, not an operator

Since, at late time ($\eta \to 0$) we will be decomposing our fields like:[1]

$$\Phi = \sum_{\{\Delta\}} \eta^{\Delta} O_{\Delta}(\vec{x})$$

where Δ is the scaling dimension of O_Δ. We deduce how the generators act on the boundary operator O_Δ by using the fact that scaling dimension of Φ is zero in de sitter space. Under $x \to x' \equiv \lambda x$, $\partial'_\mu = \lambda^{-1}\partial_\mu$ and $\phi'(x') = \lambda^{-\Delta}\phi(x)$:[2]

$$\begin{aligned}
S &= \int d^4x' \frac{(\partial'_0\phi')^2 - (\nabla'\phi')^2}{(H\eta')^2} \\
&= \lambda^4 \int d^4x \frac{\lambda^{-2(1+\Delta)}[(\partial_0\phi)^2 - (\nabla\phi)^2]}{\lambda^2 (H\eta)^2} \implies \Delta = 0
\end{aligned}$$

If we don't wish to use the weyl scaling, just the coordinate transformation then:

$$\begin{aligned}
S &= \int d^4x' \sqrt{g'} g'^{\mu\nu} \partial'_\mu\phi\partial'_\nu\phi = \int d^4x' \frac{(\partial'_0\phi')^2 - (\nabla'\phi')^2}{(H\eta')^2} \\
&= \int d^4x \sqrt{g} g'^{\mu\nu} \partial'_\mu\phi\partial'_\nu\phi = \int d^4x \lambda^2 \frac{\lambda^{-2(1+\Delta)}[(\partial_0\phi)^2 - (\nabla\phi)^2]}{(H\eta)^2} \implies \Delta = 0
\end{aligned}$$

Let us now come back to dilatation and study how it acts on O:

$$\begin{aligned}
D\Phi &= -x\cdot\partial\Phi \\
&= -x\cdot\partial \sum_{\{\Delta\}} \eta^{\Delta} O_\Delta \\
&= \sum_{\{\Delta\}} (-\eta\partial_\eta - \vec{x}\partial_{\vec{x}})\eta^{\Delta} O_\Delta \\
&= \sum_{\{\Delta\}} \eta^{\Delta} \underbrace{(-\Delta - \vec{x}\partial_{\vec{x}})}_{D} O_\Delta
\end{aligned}$$

[1] $\eta = \lim_{t\to\infty} -\frac{e^{-Ht}}{H} = 0$

[2] note that we used the weyl scaling to remove the λ^2 factor from metric under change of coordinate. So this basically amounts to substituting the coordinate transformation in the metric.

$$
\begin{aligned}
&= D\sum_{\{\Delta\}}\eta^{\Delta}O_{\Delta} \\
b\cdot K\Phi &= b^{\mu}[-2x_{\mu}\eta\partial_{\eta} - 2x_{\mu}\vec{x}\cdot\partial_{\vec{x}} - (\eta^2 - |\vec{x}|^2)\partial_{\mu}]\Phi \\
&= [-2(\vec{b}\cdot\vec{x})\eta\partial_{\eta} - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} - (\eta^2 - |\vec{x}|^2)\vec{b}\cdot\partial_{\vec{x}}]\Phi \\
&= \sum_{\{\Delta\}}\underbrace{[-2(\vec{b}\cdot\vec{x})\Delta - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} - (\eta^2 - |\vec{x}|^2)\vec{b}\cdot\partial_{\vec{x}}]}_{b\cdot K}\eta^{\Delta}O_{\Delta}
\end{aligned}
$$

in the limit $\eta\to 0$

$$
= [-2(\vec{b}\cdot\vec{x})\Delta - 2(\vec{b}\cdot\vec{x})\vec{x}\cdot\partial_{\vec{x}} + |\vec{x}|^2\vec{b}\cdot\partial_{\vec{x}}]\Phi
$$

In momentum space, the above operators take the following form:

$$
\begin{aligned}
D &: (3-\eta\partial_{\eta}) + k^i\partial_{k_i}, \\
\mathbf{b}\cdot\mathbf{K} &: (3-\eta\partial_{\eta})\,2b^i\partial_{k_i} - \mathbf{b}\cdot\mathbf{k}\partial_{k_i}\partial_{k_i} + 2k^i\partial_{k_i}b^j\partial_{k_j}.
\end{aligned}
$$

or, simply

$$
\begin{aligned}
D &: (3-\Delta) + k^i\partial_{k_i}, \\
\mathbf{b}\cdot\mathbf{K} &: (3-\Delta)\,2b^i\partial_{k_i} - \mathbf{b}\cdot\mathbf{k}\partial_{k_i}\partial_{k_i} + 2k^i\partial_{k_i}b^j\partial_{k_j}.
\end{aligned}
$$

There are two ways to derive this expression, we will discuss both of them. The first is based on using Fourier Transform:

$$
f(\vec{x}) = \int d^3x e^{i\vec{x}\cdot\vec{k}}f(\vec{k})
$$

In our case

$$
\begin{aligned}
D\underbrace{\int d^3k e^{i\vec{x}\cdot\vec{k}}O_{\Delta}(\vec{k})}_{O_{\Delta}(\vec{x})} &= -\left(\Delta + x^j\frac{\partial}{\partial x^j}\right)\int d^3k e^{i\vec{x}\cdot\vec{k}}O_{\Delta}(\vec{k}) \\
&= \int d^3k\left[-\Delta - x^j\frac{\partial}{\partial x^j}\right]e^{i\vec{x}\cdot\vec{k}}O_{\Delta}(\vec{k}) \\
&= \int d^3k[-\Delta e^{i\vec{x}\cdot\vec{k}} - x^j e^{i\vec{x}\cdot\vec{k}}(ik_j)]O_{\Delta}(\vec{k})
\end{aligned}
$$

we have to get rid of x^j, so we consider:

$$
\begin{aligned}
&= \int d^3k\left[-\Delta e^{i\vec{x}\cdot\vec{k}} - \left(-i\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right)ik_j\right]O_{\Delta}(\vec{k}) \\
&= \int d^3k\left[-\Delta e^{i\vec{x}\cdot\vec{k}} - \left(\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right)k_j\right]O_{\Delta}(\vec{k})
\end{aligned}
$$

integrating by parts the second term

$$
\begin{aligned}
&= \int d^3k e^{i\vec{x}\cdot\vec{k}}\left[-\Delta + \left(\frac{\partial}{\partial k_j}\right)k_j\right]O_{\Delta}(\vec{p}) \\
&= \int d^3k e^{i\vec{x}\cdot\vec{k}}\left(-\Delta + \cancelto{3}{\frac{\partial k_j}{\partial k_j}} + k_j\frac{\partial}{\partial k_j}\right)O_{\Delta}(k)).
\end{aligned}
$$

Thus the action of the dilatation generator in momentum space is

$$
DO_{\Delta}(\vec{k}) = \left(3 - \Delta + k^j\frac{\partial}{\partial k^j}\right)O_{\Delta}(\vec{k})
$$

The second way is to replace the following in the expression in coordinate space (it can be seen operating the corresponding derivative operator on $e^{ix^{\mu}k_{\mu}}$ and the corresponding integration by parts).

$$
x_{\mu}\to -i\frac{\partial}{\partial k^{\mu}}
$$

$$\frac{\partial}{\partial x_\mu} \to -ik^\mu$$

Substituting in

$$DO_\Delta(\vec{x}) = -\left(\Delta + x^j \frac{\partial}{\partial x^j}\right) O_\Delta(\vec{x})$$

we get

$$DO_\Delta(\vec{k}) = -\left[\Delta - i\frac{\partial}{\partial k^j}\left(-ik^j\right)\right] O_\Delta(\vec{k}) = -\left(\Delta - 3 - k^j \frac{\partial}{\partial k^j}\right) O_\Delta(\vec{k})$$

5.1 Conformal Ward Identity in momentum space

The following is taken from "Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex". We start by assuming that the generator of conformal transformation annihilates the correlation function.

$$\begin{aligned} D\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\ldots O_n(\vec{x}_n)\right\rangle &= 0 \\ e^{ix_n\cdot P} D\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\ldots O_n(\vec{x}_n)\right\rangle e^{-ix_n\cdot P} &= 0 \\ \underbrace{e^{ix_n\cdot P} D e^{-ix_n\cdot P}}_{D+\vec{x}_n\cdot\partial_{\vec{x}_n}} \underbrace{e^{ix_n\cdot P}\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\ldots O_n(\vec{x}_n)\right\rangle e^{-ix_n\cdot P}}_{\left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\ldots O_n(0)\right\rangle} &= 0 \end{aligned}$$

now, in Fourier space:

$$\begin{aligned} \left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\ldots O_n(0)\right\rangle &= \int d^dk_1\ldots d^dk_n e^{\sum_{j=1}^{n-1} ik_j\cdot(x_j-x_n)+0} \\ &\quad\times \delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\ldots O_n(\vec{k}_n)\right\rangle \\ &= \int d^dk_1\ldots d^dk_n e^{\sum_{j=1}^{n-1} ik_j\cdot x_j - ix_n\cdot(\sum_{j=1}^{n-1}k_j)} \\ &\quad\times \delta^d\left(k_n + \sum_j^{n-1} k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\ldots O_n(\vec{k}_n)\right\rangle \\ &\equiv \int d^dk_1\ldots d^dk_n e^{\sum_{j=1}^{n} ik_j\cdot x_j} \\ &\quad\times \delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\ldots O_n(\vec{k}_n)\right\rangle \end{aligned}$$

From first and third equality, we observe that the replacement we need to make are:

$$(x_j - x_n)_\mu \to -i\frac{\partial}{\partial k_j^\mu}$$

$$\frac{\partial}{\partial x_\mu} \to -ik^\mu$$

Thus,

$$\begin{aligned} (\underbrace{D+\vec{x}_n\cdot\partial_{\vec{x}_n}}_{-(\sum_{j=1}^n \Delta_j + x_j\cdot\partial_{x_j}) + x_n\partial_{x_n}})\left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\ldots O_n(0)\right\rangle &= 0 \\ -\left[\sum_j^n \Delta_j + \sum_j^{n-1}(x_j-x_n)\cdot\partial_{x_j}\right]\left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\ldots O_n(0)\right\rangle &= 0 \\ -\Bigg[\underbrace{\sum_j^n \Delta_j + \sum_j^{n-1}\left(-i\frac{\partial}{\partial k_j^\mu}\right)\cdot(-ik_j^\mu)}_{-[\Delta-(n-1)d-\sum_{j=1}^{n-1}k_j\cdot\partial_{k_j}]}\Bigg]\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\ldots O_n(\vec{k}_n)\right\rangle \delta^d\left(\sum_j^n k_j\right) &= 0 \end{aligned}$$

The alternate way to do the same is by explicitly doing it. We then will have to use

$$\int dx f(x)\partial_x \delta(x-a) = -\int dx \partial_x f(x)\delta(x-a) = -\partial_x f(a)$$

Consider the following integral:

$$\begin{aligned} I_{\alpha\beta} &= \int d^d k \left[\frac{\partial}{\partial k^\alpha}\delta^d(k^\mu)\right] k_\beta \\ &= -\int d^d k \delta^d(k^\mu) \underbrace{\frac{\partial k_\beta}{\partial k^\alpha}}_{g_{\alpha\beta}} = -g_{\alpha\beta} \end{aligned} \tag{5.1}$$

However, we also know that:

$$\begin{aligned} \int d^d k \delta^d(k^\mu) &= 1 \\ \int d^d k \delta^d(k^\mu) \frac{k^2}{k^2} &= 1 \\ \int d^d k \delta^d(k^\mu) \frac{g^{\alpha\beta} k_\alpha k_\beta}{k^2} &= 1 \\ \int d^d k \frac{\delta^d(k^\mu)}{k^2} k_\alpha k_\beta &= \frac{1}{d} g_{\alpha\beta} \end{aligned} \tag{5.2}$$

from (3.1) and (3.2), we get

$$\frac{\partial}{\partial k^\alpha}\delta^d(k^\mu) = -\frac{d}{k^2} k_\alpha \delta^d(k^\mu)$$

using the above, we derive:

$$k^\alpha \frac{\partial}{\partial k^\alpha}\delta^3(\vec{k}) = -\frac{3}{k^2} k^\alpha k_\alpha \delta^3(\vec{k}) = -3\delta^3(\vec{k})$$

Then, we have:

$$\begin{aligned} &-\sum_{j=1}^{n}\left(\Delta_j + x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right)\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle \\ &= -\sum_{j=1}^{3}\left(\Delta_j + x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right)\int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{ik_l x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\ &= -\int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{ik_l x_l}\left(\displaystyle\sum_{j=1}^{n}\Delta_j - \underbrace{\sum_{j=1}^{n} d}_{nd} - \sum_{j=1}^{n} k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle + \delta'_{\text{term}} \end{aligned}$$

where

$$\delta'_{\text{term}} = \sum_{j=1}^{n}\int \prod_{l=1}^{n} d^d k_l \left[k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\delta^d(\textstyle\sum_{i=1}^n \vec{k}_i)\right] e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle$$

defining $P = \sum_{i=1}^{n}\vec{k}_i$

$$\begin{aligned} &= \sum_{j=1}^{n}\int \prod_{l=1}^{n} d^d k_l \left[k_j^\alpha \frac{\partial}{\partial P^\alpha}\delta^d(P)\right] e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\ &= \int \prod_{l=1}^{n} d^d k_l \left[P^\alpha \frac{\partial}{\partial P^\alpha}\delta^d(P)\right] e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\ &= -d\int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{ik_l x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \end{aligned}$$

Thus, finally

$$
\begin{aligned}
-\sum_{j=1}^{n}\left(x_j^{\alpha}\frac{\partial}{\partial x_j^{\alpha}}+\Delta_j\right)\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle = -\int\prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^{n} k_i)e^{ik_l x_l}& \\
\left(\sum_{j=1}^{n}\Delta_j-(n-1)d-\sum_{j=1}^{n}k_j^{\alpha}\frac{\partial}{\partial k_j^{\alpha}}\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle&
\end{aligned}
\tag{5.3}
$$

Before we proceed, there is another identity we'd like to derive which would be very helpful. [3]

$$
\begin{aligned}
\partial_\alpha\partial_\beta\delta^d(k^\mu) &= \partial_\alpha\left[-\frac{d}{k^2}\delta^d(k^\mu)k_\beta\right] \\
&= -d\left(\frac{\partial}{\partial k^\alpha}k^{-2}\right)\delta^d(k^\mu)k_\beta-\frac{d}{k^2}[\partial_\alpha\delta^d(k)]k_\beta-\frac{d}{k^2}\delta^d(k^\mu)\partial_\alpha k_\beta \\
&= \frac{2d}{k^3}\frac{k_\alpha}{k}\delta^d(k^\mu)k_\beta+\frac{d^2}{k^4}\delta^d(k^\mu)k_\alpha k_\beta-\frac{d}{k^2}\delta^d(k^\mu)g_{\alpha\beta} \\
\\
&= \frac{d(d+2)}{k^4}\delta^d(k^\mu)k_\alpha k_\beta-\frac{d}{k^2}\delta^d(k^\mu)g_{\alpha\beta}
\end{aligned}
$$

We will now discuss the same for SCT. We perform the same substitution

$$
\begin{aligned}
-iK &= -2x_\mu\Delta-2x_\mu\underbrace{\vec{x}\cdot\partial_{\vec{x}}}_{-i\frac{\partial}{\partial k^\alpha}(-ik^\alpha)}+|\vec{x}|^2\partial_\mu \\
&= 2i\Delta\frac{\partial}{\partial k^\mu}-2i\frac{\partial^2}{\partial k^\mu\partial k^\alpha}k^\alpha+i\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}k_\mu \\
&= 2i\Delta\frac{\partial}{\partial k^\mu}-\underbrace{2i\frac{\partial}{\partial k^\mu}\left(d+k^\alpha\frac{\partial}{\partial k^\alpha}\right)}_{2i\left(d\frac{\partial}{\partial k^\mu}+\frac{\partial}{\partial k^\mu}+k^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha}\right)}+\underbrace{i\frac{\partial}{\partial k^\alpha}\left(\delta_\mu^\alpha+k_\mu\frac{\partial}{\partial k_\alpha}\right)}_{2i\frac{\partial}{\partial k^\mu}+ik_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}} \\
&= 2i\Delta\frac{\partial}{\partial k^\mu}-2id\frac{\partial}{\partial k^\mu}-2ik^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha}+ik_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha} \\
&= i\left[2(\Delta-d)\frac{\partial}{\partial k^\mu}-2k^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha}+k_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}\right]
\end{aligned}
$$

Then,

$$
\begin{aligned}
K^\mu\delta^d(\textstyle\sum_{i=1}^{n}p_i^k) &= \sum_{j=1}^{n}\left[k_j^\mu\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\alpha}}-2k_j^\alpha\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\mu}}+2(\Delta_j-d)\frac{\partial}{\partial k_{j\mu}}\right]\delta^d(\overset{P^\mu=\sum_{i=1}^{n}k_i^\mu}{\overset{\downarrow}{\textstyle\sum_{i=1}^{n}k_i^\mu}}) \\
&= \sum_{j=1}^{n}\left[k_j^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha}-2k_j^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu}+2(\Delta_j-d)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \left[\left(\sum_{j=1}^{n}k_j^\mu\right)\frac{\partial^2}{\partial P^\alpha\partial P_\alpha}-2\left(\sum_{j=1}^{n}k_j^\alpha\right)\frac{\partial^2}{\partial P^\alpha\partial P_\mu}+2\sum_{j=1}^{n}(\Delta_j-d)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \left[P^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha}-2P^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu}+2(\Delta-nd)\frac{\partial}{\partial P_\mu}\right]\delta^d(P) \\
&= \frac{2d}{P^2}\delta^d(P)P^\mu-2\frac{d^2+d}{P^2}\delta^d(P)P^\mu+\frac{2d(\Delta-nd)}{P^2}\delta^d(P)P^\mu \\
&= 2d\left[nd-d-\Delta\right]\frac{\delta^d(P)P^\mu}{P^2}
\end{aligned}
$$

[3]In the third line, we have used

$$
\begin{aligned}
\frac{\partial k}{\partial k^\alpha} &= \frac{\partial\sqrt{k^\alpha k_\alpha}}{\partial k^\alpha} \\
&= \frac{1}{2\sqrt{k^\alpha k_\alpha}}\left[\frac{\partial k^\alpha}{\partial k^\alpha}k_\alpha+k_\alpha\frac{\partial k^\alpha}{\partial k^\alpha}\right]=\frac{k^\alpha}{k}
\end{aligned}
$$

$$= -2[(n-1)d - \Delta]\frac{\partial \delta^d(P)}{\partial P^\mu}$$

In the fourth line, we used $\sum_{j=1}^n \Delta_j = \Delta$. Now, when K operates on the correlation function, it produces three kinds of terms,

- All operators in K acting purely on $\langle O_1(p_1)\dots O_n(p_n)\rangle$
- All operators in K acting purely on $\delta^d(\sum_{i=1}^n p_i)$
- Operators acting on both $\langle O_1(p_1)\dots O_n(p_n)\rangle$ and $\delta^d(\sum_{i=1}^n p_i)$

We will consider the action of

$$k_j^\mu \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\alpha}} - 2k_j^\alpha \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\mu}}$$

then, they will operate like:

$$2\frac{\partial \delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\alpha}\underbrace{\left[k_j^\mu \frac{\partial}{\partial k_{j\alpha}} - k_j^\alpha \frac{\partial}{\partial k_{j\mu}}\right]}_{iL_{\mu\alpha}}\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle - 2\frac{\partial \delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\mu} k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

First part vanishes due to rotational invariance. Therefore the extra terms would be:

$$\delta'_{\text{terms}} = -2\frac{\partial \delta^d(P)}{\partial P}\left[(n-1)d - \Delta + \sum_{j=1}^n k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\right]\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

The above vanishes from (3.3). Therefore the SCT ward identity is given as:

$$K_\mu \langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle = -\left[2(\Delta - d)\frac{\partial}{\partial k^\mu} - 2k^\alpha \frac{\partial^2}{\partial k^\mu \partial k^\alpha} + k_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha}\right]\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle \tag{5.4}$$

Chapter 6

Maldacena's Consistency condition

It is expected that the de Sitter isometries are broken during inflation and the Hubble factor is no exactly constant during the evolution. In this chapter, we aim to study how the breaking of de Sitter isometries result in further conditions imposed over correlators. First we will use the pullback map to bring the metric over perturbed manifold to a tensor field over unperturbed manifold. Next, we will consider the Lie derivative of the pullback metric to study how the resulting tensor field changes along the integral curve of the generators of isometries of unpertubed metric:

$$\delta g_{\mu\nu} = \mathcal{L}_{\xi} g_{\mu\nu}$$

In order to match it with the literature, we can subtract the unperturbed metric from this rank 2 tensor and consider upto first order term.

Chapter 7

2D CFT

In the first chapter we learnt that conformal field in 2 dimensions are very special. In this chapter we will look at why? We will start by defining the two coordinate labels for the flat 2d euclidean space x^0 and x^1. The condition for the conformal transformation in such space is given as:

$$x^\mu \to x^\mu + \epsilon^\mu(x)$$

with

$$\begin{aligned}
\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &= \frac{2}{d} g_{\mu\nu} \partial^\rho \epsilon_\rho \\
\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &= g_{\mu\nu}(\partial_0 \epsilon_0 + \partial_1 \epsilon_1)
\end{aligned}$$

for $\mu = \nu = 0$

$$2\partial_0 \epsilon_0 = \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \implies \partial_0 \epsilon_0 = \partial_1 \epsilon_1$$

for $\mu = \nu = 1$

$$2\partial_1 \epsilon_1 = \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \implies \partial_1 \epsilon_1 = \partial_0 \epsilon_0$$

for $\mu = 0, \nu = 1$

$$\partial_0 \epsilon_1 + \partial_1 \epsilon_0 = 0 \implies \partial_0 \epsilon_1 = -\partial_1 \epsilon_0$$

So, in conclusion we only have two constraints imposed over ϵ^μ

$$\boxed{\begin{aligned} \partial_0 \epsilon_0 &= \partial_1 \epsilon_1 \\ \partial_0 \epsilon_1 &= -\partial_1 \epsilon_0 \end{aligned}}$$

Now, let us embed our flat 2d surface inside 4 dimensional space, where the coordinates are given by:

$$\left.\begin{aligned} z &= x^0 + ix^1 \\ \bar{z} &= x^0 - ix^1 \end{aligned}\right\} \begin{aligned} x^0 &= \frac{z+\bar{z}}{2} \\ x^1 &= \frac{z-\bar{z}}{2i} \end{aligned}$$

We will see the advantage of embedding our flat euclidean space soon, let us first consider how the conformal transformation $x^\mu \to x^\mu + \epsilon^\mu$ acts in the new coordinate:

$$z \to x^0 + \epsilon^0 + i(x^1 + \epsilon^1) = (x^0 + ix^1) + (\epsilon^0 + i\epsilon^1) = z + \epsilon(z, \bar{z})$$

and accordingly,

$$\bar{z} \to \bar{z} + \bar{\epsilon}(z, \bar{z})$$

with $\bar{\epsilon} = \epsilon^0 - i\epsilon^1$. Here we have used the notation

$$\begin{aligned}
\partial_0 &= \frac{\partial}{\partial x^0} = \frac{\partial z}{\partial x^0}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x^0}\frac{\partial}{\partial \bar{z}} = \partial + \bar{\partial} \\
\partial_1 &= \frac{\partial}{\partial x^1} = \frac{\partial z}{\partial x^1}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x^1}\frac{\partial}{\partial \bar{z}} = i(\partial - \bar{\partial})
\end{aligned}$$

and then, accordingly we define the Wirtinger derivative:

$$\partial = \frac{\partial}{\partial z} = \frac{\partial x^0}{\partial z}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial z}\frac{\partial}{\partial x^1} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} - i\frac{\partial}{\partial x^1}\right) \tag{7.1}$$

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{\partial x^0}{\partial \bar{z}}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial}{\partial x^1} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x^1}\right) \tag{7.2}$$

and,

$$\epsilon_0 = \frac{\epsilon + \bar{\epsilon}}{2}$$
$$\epsilon_1 = \frac{\epsilon - \bar{\epsilon}}{2i}$$

Having derived how ϵ_μ, x_μ and ∂_μ looks like in the new coordiante, we now use these relation to re-express the constraint over ϵ_μ to ϵ:

$$\begin{aligned}
\partial_0\epsilon_0 &= \partial_1\epsilon_1 \\
(\partial + \bar{\partial})\left(\frac{\epsilon + \bar{\epsilon}}{2}\right) &= i(\partial - \bar{\partial})\left(\frac{\epsilon - \bar{\epsilon}}{2i}\right) \\
\partial\epsilon + \partial\bar{\epsilon} + \bar{\partial}\epsilon + \bar{\partial}\bar{\epsilon} &= \partial\epsilon - \partial\bar{\epsilon} - \bar{\partial}\epsilon + \bar{\partial}\bar{\epsilon} \\
\bar{\partial}\epsilon &= -\partial\bar{\epsilon}
\end{aligned} \tag{7.3}$$

and,

$$\begin{aligned}
\partial_0\epsilon_1 &= -\partial_1\epsilon_0 \\
(\partial + \bar{\partial})\left(\frac{\epsilon - \bar{\epsilon}}{2i}\right) &= -i(\partial - \bar{\partial})\left(\frac{\epsilon + \bar{\epsilon}}{2}\right) \\
\partial\epsilon - \partial\bar{\epsilon} + \bar{\partial}\epsilon - \bar{\partial}\bar{\epsilon} &= \partial\epsilon + \partial\bar{\epsilon} - \bar{\partial}\epsilon - \bar{\partial}\bar{\epsilon} \\
\bar{\partial}\epsilon &= \partial\bar{\epsilon}
\end{aligned} \tag{7.4}$$

From (7.3) and (7.4), we conclude

$$\partial\bar{\epsilon} = 0 \implies \frac{\partial\bar{\epsilon}}{\partial z} = 0$$
$$\bar{\partial}\epsilon = 0 \implies \frac{\partial\epsilon}{\partial \bar{z}} = 0$$

Hence, the conformal transformation could be expressed in these coordinates as:

$$z \to z + \epsilon(z)$$
$$\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$$

or

$$z \to f(z)$$

Then,

$$\frac{\partial f}{\partial \bar{z}} = 0$$

i.e. $f(z)$ is analytic or holomorphic. Since $\epsilon(z)$ is an analytic function in z, it means we can perform a series expansion.

$$\begin{aligned}
z' &= z + \epsilon(z) \\
&= z + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \\
&= z + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \partial z \\
&= \left(1 + \sum_{n\in\mathbb{Z}} \epsilon_n z^n \partial\right) z
\end{aligned}$$

It is now easy to read off the generators of conformal transformation in 2d from above. We define,

$$r_n = z^n\partial$$

Similarly,

$$\begin{aligned}\bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) \\ &= \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \\ &= \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \bar{\partial}\bar{z} \\ &= \left(1 + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \bar{\partial}\right)\bar{z}\end{aligned}$$

and hence

$$\bar{r}_n = \bar{z}^n \bar{\partial}$$

We know that the algebra of the generators are independent of the representation. Therefore, the next step is to find the algebra of r_n and r_m in this representation.

$$\begin{aligned}[r_m, r_n] f(z,\bar{z}) &= [z^m\partial, z^n\partial] f(z,\bar{z}) \\ &= z^m\partial(z^n\partial f) - z^n\partial(z^m\partial f) \\ &= z^{m+n}\partial^2 f + z^m(\partial z^n)\partial f - z^{m+n}\partial^2 f - z^n(\partial z^m)\partial f \\ &= (n-m) z^{m+n-1}\partial f \\ &= (n-m) r_{m+n-1} f\end{aligned}$$

Thus,

$$\boxed{[r_m, r_n] = (n-m) r_{m+n-1}}$$

Following the same steps, we can also find

$$\boxed{[\bar{r}_m, \bar{r}_n] = (n-m) \bar{r}_{m+n-1}}$$

and

$$\boxed{[r_m, \bar{r}_n] = 0}$$

We can simplify this by redefining the generator as $r_n = -l_{n-1}$. Then

$$\begin{aligned}[r_m, r_n] &= (n-m) r_{m+n-1} \\ [-l_{m-1}, -l_{n-1}] &= -(n-m) l_{m+n-2}\end{aligned}$$

and then shifting $m \to m+1$ and $n \to n+1$

$$[l_m, l_n] = (m-n) l_{m+n}$$

with

$$\begin{aligned}l_n &= -r_{n+1} = -z^{n+1}\partial \\ \bar{l}_n &= -\bar{r}_{n+1} = -\bar{z}^{n+1}\bar{\partial}\end{aligned}$$

and

$$\begin{aligned}[\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n} \\ [l_m, \bar{l}_n] &= 0\end{aligned}$$

This is **Witt algebra**. The generate the conformal transformation in 2d Euclidean plane. To find the subset of generators l_n which generate global conformal transformation, we note that z^n has two singularity, one at $z = 0$ for $n < 0$ and another at $z = \infty$ for $n > 0$. Therefore, we impose the restriction that the representations of Witt algebra does not blow up in the limit $z \to 0$ or $z \to \infty$. We start with $z \to 0$ limit, $l_n = -z^{n+1}\partial$ needs to converge:

$$n + 1 \geq 0 \implies n \geq -1$$

Since $z \to \infty$ does not appear as a pole, we consider the conformal mapping $w = \frac{1}{z}$ which brings the infinity at origin. Then,

$$\partial = \frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w}$$

$$= -\frac{1}{z^2}\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}$$

Then,

$$\begin{aligned} l_n &= -z^{n+1}\partial = w^{-n-1}w^2\partial_w \\ &= w^{-n+1}\partial_w \end{aligned}$$

Then,

$$1 - n \geq 0 \implies n \leq 1$$

Therefore, l_{-1}, l_0 and l_1 generate global conformal transformation on the 2d Euclidean space.

7.1 Global Conformal Transformation

In the last section, we found the generators of global conformal transformation. In this part, we look at the infinitesimal transformation generated by l_{-1} , l_0 and l_1 and then find their finite counterpart. For $n = -1$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_{-1}$ looks like:

$$\begin{aligned} l_{-1} &= z^{-1+1}\partial = \partial \\ \bar{l}_{-1} &= \bar{z}^{-1+1}\bar{\partial} = \bar{\partial} \end{aligned}$$

So, this generates translation. For $n = 0$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_0 z$ looks like:

$$\begin{aligned} l_0 &= -z^{0+1}\partial = -z\partial \\ \bar{l}_0 &= -\bar{z}\bar{\partial} \end{aligned}$$

to interpret how the work in complex plane, we will use the polar coordinates $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$ where $r = \sqrt{z\bar{z}}$ and $\theta = \frac{1}{2i}\ln\left(\frac{z}{\bar{z}}\right)$

$$\begin{aligned} \partial = \frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} \\ &= \frac{\bar{z}}{2\sqrt{z\bar{z}}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar{z}}{z}\frac{\partial(z/\bar{z})}{\partial z}\frac{\partial}{\partial\theta} \\ &= \frac{\bar{z}}{2r}\frac{\partial}{\partial r} + \frac{1}{2iz}\frac{\partial}{\partial\theta} \\ &= \frac{e^{-i\theta}}{2}\frac{\partial}{\partial r} + \frac{e^{-i\theta}}{2ir}\frac{\partial}{\partial\theta} \end{aligned}$$

and,

$$\begin{aligned} \bar{\partial} = \frac{\partial}{\partial\bar{z}} &= \frac{\partial r}{\partial\bar{z}}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial\bar{z}}\frac{\partial}{\partial\theta} \\ &= \frac{z}{2\sqrt{z\bar{z}}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar{z}}{z}\frac{\partial(z/\bar{z})}{\partial\bar{z}}\frac{\partial}{\partial\theta} \\ &= \frac{z}{2\sqrt{z\bar{z}}}\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\bar{z}}{z}\left(-\frac{z}{\bar{z}^2}\right)\frac{\partial}{\partial\theta} \\ &= \frac{z}{2r}\frac{\partial}{\partial r} - \frac{1}{2i\bar{z}}\frac{\partial}{\partial\theta} \\ &= \frac{e^{i\theta}}{2}\frac{\partial}{\partial r} - \frac{e^{i\theta}}{2ir}\frac{\partial}{\partial\theta} \end{aligned}$$

$$\begin{aligned} l_0 = -z\partial &= -re^{i\theta}\left[\frac{e^{-i\theta}}{2}\frac{\partial}{\partial r} + \frac{e^{-i\theta}}{2ir}\frac{\partial}{\partial\theta}\right] \\ &= -\frac{r}{2}\left[\partial_r - \frac{i}{r}\partial_\theta\right] \end{aligned}$$

$$\begin{aligned}
\bar{l}_0 &= -\bar{z}\bar{\partial} \\
&= -re^{-i\theta}\left[\frac{e^{i\theta}}{2}\frac{\partial}{\partial r} - \frac{e^{i\theta}}{2ir}\frac{\partial}{\partial\theta}\right] \\
&= -\frac{r}{2}\left[\partial_r + \frac{i}{r}\partial_\theta\right]
\end{aligned}$$

So,

$$l_0 + \bar{l}_0 = -r\partial_r$$

is the generator of dilatation and

$$i(l_0 - \bar{l}_0) = \partial_\theta$$

is the generator of rotation. For $n = -1$, the generators associated with the infinitesimal transformation $z \to z + \epsilon_1 z^2$ looks like:

$$l_1 = -z^2\partial$$

These are nothing but Mobius transformation.

$$z' = \frac{az+b}{cz+d}$$

For translation we have $a = 1, c = 0$ and $d = 1$ and we have $ad - bc = 1$. For dilatation and rotation, we have

$$\begin{aligned}
z' &= z - \epsilon_0 z \\
&= (1-\epsilon_0)z \\
&= \frac{\frac{1-\epsilon_0}{\sqrt{1-\epsilon_0}}z + 0}{0z + \frac{1}{\sqrt{1+\epsilon_0}}} = \frac{\sqrt{1+\epsilon_0}z + 0}{0z + \frac{1}{\sqrt{1+\epsilon_0}}}
\end{aligned}$$

with $a = \sqrt{1+\epsilon_0}, b = 0, c = 0$ and $d = \frac{1}{\sqrt{1+\epsilon_0}}$ so that $ad - bc = 1$. For SCT, we have

$$\begin{aligned}
z' &= z + \epsilon_1 z^2 \\
&= z(1-\epsilon_1 z)^{-1} \\
&= \frac{z}{1-\epsilon_1 z} = \frac{z+0}{-\epsilon_1 z + 1}
\end{aligned}$$

Since ϵ_n is in general a complex number. These are the set of 2×2 complex matrices with unit determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{with } ad - bc = 1$$

These matrices form the group $SL(2,\mathbb{C})$. But under the mapping $a \to -a$, $b \to -b$, $c \to -c$ and $d \to -d$, the generate the same transformation. Therefore, the conformal transformation in 2d forms the group $SL(2,\mathbb{C})/\mathbb{Z}_2$. In the Lorentzian case the group is replaced by $SL(2,R) \times SL(2,R) = SO(2,2)$ where one factor of $SL(2,R)$ pertains to left-movers and the other to right-movers.

7.2 Virasoro Algebra

In some cases, the regularization scheme used to quantize a classical theory does not preserve the original symmetry. The classical generators, which we obtain by solving the Killing equations, satisfy a certain algebra—for example, the Witt algebra. However, in the quantum system, the requirement of normal ordering to render observables finite modifies the algebraic structure. In the next chapter, we will learn how to construct the corresponding quantum operators systematically using the operator formalism. As a result, these quantum operators no longer satisfy the Witt algebra but instead form the Virasoro algebra. This is unavoidable; it's a reflection of the fact that the classical symmetry is realized projectively in the quantum theory.

The Virasoro algebra is the **central extension** of Witt algebra. To centrally extend the algebra, a new generator—called the center—is introduced, which commutes with all other generators. To distinguish from the generators of Witt algebra, we use the notation L_n to denote the generators of Virasoro algebra. The algebra is then given as:

$$[L_n, L_m] = (n-m)L_{n+m} + \underbrace{c\rho(n,m)}_{\text{complex number}}$$

This algebra contains a finite-dimensional subalgebra generated by $L_{0,\pm1}$,$\bar{L}_{0,\pm1}$. These are the generators that are well defined all over the complex plane and form the global conformal group in two dimensions just as we saw in Witt algebra. The rest of the generators are local. The object $\rho(n,m)$ which we have added to centrally extend the algebra has certain properties: it is a number which depends on n and m, it commutes with all L_n, it satisfies recursion relation and it vanishes for certain n and m. To see this, let us first note that

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + c\rho(n,m) \\
[L_m, L_n] &= -(n-m)L_{n+m} + c\rho(m,n)
\end{aligned}$$

adding above expressions

$$\begin{aligned}
[L_n, L_m] + [L_m, L_n] &= (n-m)L_{n+m} + c\rho(n,m) - (n-m)L_{n+m} + c\rho(m,n) \\
\rho(n,m) + \rho(m,n) &= 0
\end{aligned}$$

We have shown that $\rho(n,m)$ is anti-symmetric under the exchange of $n \leftrightarrow m$. Now let us set $m = 0$,

$$\begin{aligned}
[L_n^{\text{old}}, L_0] &= nL_n^{\text{old}} + c\rho(n,0) \\
&= n\left[L_n^{\text{old}} + \frac{c}{n}\rho(n,0)\right]
\end{aligned}$$

Redefining the generators for $n \neq 0$ as:

$$L_n^{\text{new}} = L_n^{\text{old}} + \frac{c}{n}\rho(n,0)$$

and

$$L_0^{\text{new}} = L_0^{\text{old}}$$

The algebra of the translated generators should not changed:

$$\begin{aligned}
[L_n^{\text{old}}, L_0] &= [L_n^{\text{new}} - \frac{c}{n}\rho(n,0), L_0] \\
[L_n^{\text{new}}, L_0] - \left[\frac{c}{n}\rho(n,0), L_0\right] &= nL_n^{\text{old}} + c\rho(n,0) \\
[L_n^{\text{new}}, L_0] &= nL_n^{\text{new}}
\end{aligned}$$

This proves that $\rho(n,0) = 0$ for $n \neq 0$. For $n = 1, m = -1$ we have:

$$\begin{aligned}
[L_1^{\text{old}}, L_{-1}^{\text{old}}] &= 2L_0^{\text{old}} + c\rho(1,-1) \\
&= 2\left[L_0^{\text{old}} + \frac{c}{2}\rho(1,-1)\right] \\
&= 2L_0^{\text{new}}
\end{aligned}$$

This, proves that $\rho(1,-1) = 0$, to find other n and m values for which $\rho(n,m)$ vanishes, we will utilize the jacobi identity.

$$[L_n, [L_m, L_r]] + [L_r, [L_n, L_m]] + [L_m, [L_r, L_n]] = 0$$

For $r = 0$

$$\begin{aligned}
[L_n, mL_m] + [L_0, (n-m)L_{n+m}] + [L_m, -nL_n] &= 0 \\
m\left[(n-m)L_{n+m} + c\rho(n,m)\right] + (n-m)\left[-(n+m)L_{n+m}\right] - n\left[(m-n)L_{m+n} + c\rho(m,n)\right] &= 0 \\
\underbrace{\left[m(n-m) - (n-m)(n+m) - n(m-n)\right]}_{=0} L_{m+n} - c(m+n)\rho(m,n) &= 0
\end{aligned}$$

Hence,

$$(m+n)\rho(m,n) = 0$$

It means, $\rho(m,n)$ can only be non zero. if $m+n = 0$. So, $\rho(m,n) = \rho(n,-n)\delta_{m+n,0}$. Next to find the recursion relation, we utilize,

$$[L_n, [L_1, L_{-1-n}]] + [L_{-1-n}, [L_n, L_1]] + [L_1, [L_{-1-n}, L_n]] = 0$$

$$\begin{aligned}(n+2)[L_n,L_{-n}]+(n-1)[L_{-1-n},L_{n+1}]-(2n+1)[L_1,L_{-1}]=0\\(n+2)\left[2nL_0+c\rho(n,-n)\right]+(n-1)[-2(n+1)L_0+c\rho(-1-n,1+n)]-(2n+1)2L_0=0\\\underbrace{[2n(n+2)-2(n-1)(n+1)-2(2n+1)]}_{=0}L_0+c[(n+2)\rho(n,-n)-(n-1)\rho(n+1,-n-1)]=0\end{aligned}$$

Hence the recursion relation,

$$\rho(n+1,-n-1)=\frac{n+2}{n-1}\rho(n,-n)$$

under $n\to n+1$,

$$\rho(n,-n)=\frac{n+1}{n-2}\rho(n-1,-n+1)$$

We can see that for $n=1$

$$\rho(1,-1)=\frac{2}{-1}\rho(0,0)=0$$

For $n=2$

$$\rho(2,-2)=\frac{3}{0}\rho(1,-1)=\text{indeterminate}$$

So, we only need to fix $\rho(2,-2)$ and all the other $\rho(n,m)$ gets fixed from that via this recursion relation.

$$\begin{aligned}\rho(n,-n)&=\frac{n+1}{n-2}\rho(n-1,-n+1)\\&=\frac{n+1}{n-2}\cdots\frac{4}{1}\rho(2,-2)\\&=\frac{1}{3!}\frac{(n+1)!}{(n-2)!}\rho(2,-2)={}^nC_3\quad\rho(2,-2)\end{aligned}$$

For free boson we normally like to set $c=1$ which forces the normalize of $\rho(2,-2)=1/2$ since $c\times\rho(n,m)=\xi c\times\frac{\rho(n,m)}{\xi}$ needs to stay constant.

$$\begin{aligned}[L_n,L_m]&=(n-m)L_{n+m}+{}^nC_3\rho(2,-2)\\&=(n-m)L_{n+m}+c\frac{(n+1)!}{6(n-2)!}\frac{1}{2}\\&=(n-m)L_{n+m}+c\frac{(n+1)!}{12(n-2)!}\end{aligned}$$

A short remark about the central charge that sometimes c and $\bar{c}$ aren't complex conjugate of each other which allows us to interpret z and $\bar{z}$ as being independent variables even though they are related to each other via complex conjugation operation.

7.3 How Fields transform under $SL(2,\mathbb{C})$

There are four types of fields we will be concerned about in 2d CFT. They are chiral fields that only depend on z ($\bar{h}=0$) and anti-chiral fields which only depend on $\bar{z}$ ($h=0$). It is also common to use the terminology holomorphic and anti-holomorphic in order to distinguish between chiral and anti-chiral quantities. There is a special kind of field which has the same transformation law for both Global conformal transformation as well as local conformal transformaion, we call them Primary fields. Under conformal transformation $z\to f(z)$ and $\bar{z}\to\bar{f}(\bar{z})$, these Primary fields transform as (passive):

$$\phi'(z,\bar{z})=\left(\frac{\partial f}{\partial z}\right)^h\left(\frac{\partial\bar{f}}{\partial\bar{z}}\right)^{\bar{h}}\phi(z,\bar{z})\Big|_{z=f(z),\bar{z}=\bar{f}(\bar{z})}=\left(\frac{\partial f}{\partial z}\right)^h\left(\frac{\partial\bar{f}}{\partial\bar{z}}\right)^{\bar{h}}\phi(f(z),\bar{f}(\bar{z}))$$

where $(h,\bar{h})$ are conformal dimensions/weight of the field $\phi(z,\bar{z})$. Sometimes this transformation law (active) is also expressed as:

$$\phi'(z,\bar{z}) = \left(\frac{d}{dz}f^{-1}(z)\right)^h \left(\frac{d}{d\bar{z}}\overline{f^{-1}}(\bar{z})\right)^{\bar{h}} \phi\big(f^{-1}(z), \overline{f^{-1}}(\bar{z})\big)$$

Define $\zeta = f^{-1}(z), \qquad \bar{\zeta} = \overline{f^{-1}}(\bar{z})$.

$$\phi'(z,\bar{z}) = \left(\frac{d\zeta}{dz}\right)^h \left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{\bar{h}} \phi(\zeta,\bar{\zeta}) \qquad \text{with } \zeta = f^{-1}(z),\ \bar{\zeta} = \overline{f^{-1}}(\bar{z}).$$

Now write the derivative of the inverse in terms of the derivative of f :

$$\left.\frac{d\zeta}{dz}\right|_{\zeta=f^{-1}(z)} = \frac{1}{\left.\frac{d}{d\zeta}f(\zeta)\right|_{\zeta=f^{-1}(z)}}, \qquad \left.\frac{d\bar{\zeta}}{d\bar{z}}\right|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})} = \frac{1}{\left.\frac{d}{d\bar{\zeta}}\overline{f(\zeta)}\right|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})}}.$$

$$\therefore \quad \phi'(z,\bar{z}) = \left(\frac{1}{\frac{d}{d\zeta}f(\zeta)|_{\zeta=f^{-1}(z)}}\right)^h \left(\frac{1}{\frac{d}{d\bar{\zeta}}\overline{f(\zeta)}|_{\bar{\zeta}=\overline{f^{-1}}(\bar{z})}}\right)^{\bar{h}} \phi\big(f^{-1}(z), \overline{f^{-1}}(\bar{z})\big).$$

Now replace the left-hand argument $f^{-1}(z)$ by z (relabel $\zeta \mapsto z,\ \bar{\zeta} \mapsto \bar{z}$):

$$\boxed{\phi'(z,\bar{z}) = \left(\left.\frac{d}{dz}f(z)\right|_{z=f^{-1}(z)}\right)^{-h} \left(\left.\frac{d}{d\bar{z}}\overline{f(z)}\right|_{\bar{z}=\overline{f^{-1}}(\bar{z})}\right)^{-\bar{h}} \phi\big(f^{-1}(z), \overline{f^{-1}}(\bar{z})\big)\ .}$$

These are active transformation because the RHS has the form $\pi_{ab}\phi_b(\Lambda^{-1}x)$. To see how the transformation works, we consider the following example, $z' = \lambda z$

$$\begin{aligned}
\phi'(z,\bar{z}) &= \left(\frac{\partial \lambda z}{\partial z}\right)^h \left(\frac{\partial \bar{\lambda z}}{\partial \bar{z}}\right)^{\bar{h}} \phi(\lambda z, \lambda\bar{z}) \\
&= \lambda^h \lambda^{\bar{h}} \phi(\lambda z, \lambda \bar{z}) \\
\phi(z', \bar{z}') &= \lambda^{-h}\lambda^{-\bar{h}}\phi'(z,\bar{z})
\end{aligned}$$

under infinitesimal conformal transformation $z' = f(z) = z + \epsilon$,

$$\begin{aligned}
\phi'(z,\bar{z}) &= [1+\partial\epsilon]^h[1+\bar{\partial}\bar{\epsilon}]^{\bar{h}}\phi(z+\epsilon, \bar{z}+\bar{\epsilon}) \\
&= [1+\partial\epsilon]^h[1+\bar{\partial}\bar{\epsilon}]^{\bar{h}}[\phi(z,\bar{z}) + \epsilon\partial\phi(z,\bar{z}) + \bar{\epsilon}\bar{\partial}\phi(z,\bar{z}) + \mathcal{O}\big(\epsilon^2\big)] \\
&= \phi(z,\bar{z}) + \epsilon\partial\phi(z,\bar{z}) + \bar{\epsilon}\bar{\partial}\phi(z,\bar{z}) + h\phi(z,\bar{z})\partial\epsilon + \bar{h}\phi(z,\bar{z})\bar{\partial}\bar{\epsilon} + \mathcal{O}\big(\epsilon^2\big) \\
&= [1 + (\epsilon\partial + h\partial\epsilon) + (\bar{\epsilon}\bar{\partial} + \bar{h}\bar{\partial}\bar{\epsilon})]\phi(z,\bar{z})
\end{aligned}$$

So, the variation $\delta\phi(z,\bar{z})$ can be given as:

$$\sum_{n=-\infty}^{\infty} \epsilon_n[L_n,\phi] + \bar{\epsilon}_n[\bar{L}_n,\phi] = \delta\phi(z,\bar{z}) = [(\epsilon\partial + h\partial\epsilon) + (\bar{\epsilon}\bar{\partial} + \bar{h}\bar{\partial}\bar{\epsilon})]\phi(z,\bar{z})$$

In the active point of view,

$$\begin{aligned}
\phi'(z,\bar{z}) &= [1+\partial\epsilon]^{-h}[1+\bar{\partial}\bar{\epsilon}]^{\bar{-h}}\phi(z-\epsilon, \bar{z}-\bar{\epsilon}) \\
&= [1+\partial\epsilon]^{-h}[1+\bar{\partial}\bar{\epsilon}]^{-\bar{h}}[\phi(z,\bar{z}) - \epsilon\partial\phi(z,\bar{z}) - \bar{\epsilon}\bar{\partial}\phi(z,\bar{z}) + \mathcal{O}\big(\epsilon^2\big)] \\
&= \phi(z,\bar{z}) - \epsilon\partial\phi(z,\bar{z}) - \bar{\epsilon}\bar{\partial}\phi(z,\bar{z}) - h\phi(z,\bar{z})\partial\epsilon - \bar{h}\phi(z,\bar{z})\bar{\partial}\bar{\epsilon} + \mathcal{O}\big(\epsilon^2\big) \\
&= [1 - (\epsilon\partial + h\partial\epsilon) - (\bar{\epsilon}\bar{\partial} + \bar{h}\bar{\partial}\bar{\epsilon})]\phi(z,\bar{z})
\end{aligned}$$

The other kind is **quasi-primary field**, they only transform as primary field under Global conformal transformation.

7.4 Energy Momentum Tensor

Usually, a Field Theory is defined in terms of a Lagrangian action from which one can derive various objects and properties of the theory. In particular, the energy–momentum tensor can be deduced from the variation of the action with respect to the metric and so it encodes the behaviour of the theory under infinitesimal transformations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$.

$$\begin{aligned} S &= \int d^D x \mathcal{L} \\ \delta S &= \int d^D x \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \int d^D x T_{\mu\nu} \delta g^{\mu\nu} \end{aligned}$$

Under $x^\mu \to x^\mu + \epsilon^\mu(x)$, $\delta g^{\mu\nu} = \partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu$.

$$\begin{aligned} \delta S &= \int d^D x T_{\mu\nu} (\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) \\ &= \int d^D x T_{\mu\nu} \frac{2}{D} (\partial^\rho \epsilon_\rho) g^{\mu\nu} = \frac{2}{D} \int d^D x T^\mu_\mu (\partial^\rho \epsilon_\rho) = 0 \implies T^\mu_\mu = 0 \end{aligned}$$

This tell us that for the conformal invarience to hold, the stress energy tensor has to be traceless.

$$\begin{aligned} \delta S &= \int d^D x T_{\mu\nu} (\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) \\ &= 2 \int d^D x T_{\mu\nu} \partial^\mu \epsilon^\nu \\ &= 2 \int d^D x \partial^\mu (T_{\mu\nu} \epsilon^\nu) - 2 \int d^D x (\partial^\mu T_{\mu\nu}) \epsilon^\nu = 0 \implies \partial^\mu T_{\mu\nu} = 0 \end{aligned}$$

Hence, we conclude that stress energy tensor is canonically conserved as well as traceless in the presence of conformal symmetry. Next we see how this condition gets translated in complex coordinates. The goal is to find T_{zz}, $T_{z\bar{z}}$, $T_{\bar{z}z}$ and $T_{\bar{z}\bar{z}}$.

$$\begin{aligned} T_{zz} &= \frac{\partial x^0}{\partial z} \frac{\partial x^0}{\partial z} T_{00} + 2 \frac{\partial x^0}{\partial z} \frac{\partial x^1}{\partial z} T_{01} + \frac{\partial x^1}{\partial z} \frac{\partial x^1}{\partial z} T_{11} \\ &= \frac{1}{4} T_{00} + 2 \frac{1}{2} \frac{1}{2i} T_{01} - \frac{1}{4} T_{11} \\ &= \frac{1}{4} [T_{00} - 2i T_{01} - T_{11}] \end{aligned}$$

$$\begin{aligned} T_{z\bar{z}} &= \frac{\partial x^0}{\partial z} \frac{\partial x^0}{\partial \bar{z}} T_{00} + \frac{\partial x^0}{\partial z} \frac{\partial x^1}{\partial \bar{z}} T_{01} + \frac{\partial x^1}{\partial z} \frac{\partial x^0}{\partial \bar{z}} T_{10} + \frac{\partial x^1}{\partial z} \frac{\partial x^1}{\partial \bar{z}} T_{11} \\ &= \frac{1}{2} \frac{1}{2} T_{00} + \frac{1}{2} \frac{-1}{2i} T_{01} + \frac{1}{2i} \frac{1}{2} T_{10} + \frac{1}{2i} \frac{-1}{2i} T_{11} \\ &= \frac{1}{4} [T_{00} + T_{11}] = \frac{1}{4} \delta^{\mu\nu} T_{\mu\nu} = \frac{1}{4} T^\mu_\mu = 0 \end{aligned}$$

$$\begin{aligned} T_{\bar{z}\bar{z}} &= \frac{\partial x^0}{\partial \bar{z}} \frac{\partial x^0}{\partial \bar{z}} T_{00} + 2 \frac{\partial x^0}{\partial \bar{z}} \frac{\partial x^1}{\partial \bar{z}} T_{01} + \frac{\partial x^1}{\partial \bar{z}} \frac{\partial x^1}{\partial \bar{z}} T_{11} \\ &= \frac{1}{4} T_{00} + 2 \frac{1}{2} \frac{-1}{2i} T_{01} - \frac{1}{4} T_{11} \\ &= \frac{1}{4} [T_{00} + 2i T_{01} - T_{11}] \end{aligned}$$

Next we investigate the form of $\partial^\mu T_{\mu\nu}$ in complex coordinates.

$$\begin{aligned} \partial_0 T_{00} + \partial_1 T_{10} = 0 \\ \partial_0 T_{01} + \partial_1 T_{11} = 0 \end{aligned} \tag{7.5}$$

We can now calculate $\partial_{\bar{z}} T_{zz}$:

$$\bar{\partial} T_{zz} = \frac{1}{4} (\bar{\partial} T_{00} - 2i \bar{\partial} T_{01} - \bar{\partial} T_{11})$$

using (7.1) and $T^{\mu}_{\mu} = T_{00} + T_{11} = 0$

$$\begin{aligned}
&= \frac{1}{8}[(\partial_0 + i\partial_1)T_{00} - 2i(\partial_0 + i\partial_1)T_{01} - (\partial_0 + i\partial_1)T_{11}] \\
&= \frac{1}{8}[(\partial_0 T_{00} + 2\partial_1 T_{01} - \partial_0 T_{11}) + i(\partial_1 T_{00} - 2\partial_0 T_{01} - \partial_1 T_{11})] \\
&= \frac{1}{8}[2(\partial_0 T_{00} + \partial_1 T_{01}) - 2i(\partial_1 T_{11} + \partial_0 T_{01})] = 0
\end{aligned}$$

In the last step we used (7.5). Similarly one can show

$$\partial T_{\bar{z}\bar{z}} = 0$$

Thus, T_{zz} is the chiral field we discussed earlier and $T_{\bar{z}\bar{z}}$ is the anti-chiral field. So in complex coordinates, the stress energy tensor looks like:

$$T_{\mu\nu}(z, \bar{z}) = \begin{bmatrix} 0 & T(z) \\ \bar{T}(\bar{z}) & 0 \end{bmatrix}$$

7.5 Ward Identities

The consequence of a symmetry of the action and measure on correlation functions may also be expressed via the so-called Ward identities. An infinitesimal transformation may be written in terms of the generator as:

$$\phi'(x) = \phi(x) - i\omega_a G_a \phi(x)$$

where ω_a is a collection of infinitesimal, constant parameters. We will consider the above variation of fields in the correlation function. The action is not invariant under such local transformation and its variation is given by:

$$\delta S = \int d^D x (\partial_\mu T^{\mu\nu})\epsilon_\nu$$

where j^{μ}_{a} is the conserved current. The correlation function can be given as:

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \frac{1}{Z}\int D\phi\phi(x_1)\phi(x_2)\ldots\phi(x_n)e^{-S[\phi]}$$

We use the invariance of correlators under the transformation to argue,

$$\begin{aligned}
\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle &= \langle \phi'(x_1)\phi'(x_2)\ldots\phi'(x_n)\rangle \\
&= \frac{1}{Z}\int D\phi'\phi'(x_1)\phi'(x_2)\ldots\phi'(x_n)e^{-S[\phi']} = \frac{1}{Z}\int D\phi'\phi'(x_1)\phi'(x_2)\ldots\phi'(x_n)e^{-S[\phi]-\delta S[\phi]} \\
&= \frac{1}{Z}\int D\phi'[\phi(x_1)\phi(x_2)\ldots\phi(x_n) + \delta(\phi(x_1)\phi(x_2)\ldots\phi(x_n))]e^{-S[\phi]-\int d^D x\partial_\mu T^{\mu\nu}\epsilon_\nu(x)} \\
&= \frac{1}{Z}\int D\phi[\phi(x_1)\phi(x_2)\ldots\phi(x_n) + \delta(\phi(x_1)\phi(x_2)\ldots\phi(x_n))]e^{-S[\phi]}\left(1 - \int d^D x\partial_\mu T^{\mu\nu}\epsilon_\nu(x) + \ldots\right)
\end{aligned}$$

When expanded to first order in $\omega_a(x)$, the above yields

$$\begin{aligned}
\langle \delta(\phi(x_1)\phi(x_2)\ldots\phi(x_n))\rangle &= \int dx \langle \partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle \epsilon_\nu(x) \\
-i\sum_{i=1}^{N}[\phi(x_1)\ldots G^{\mu}_{i}\phi(x_i)\ldots\phi(x_n)]\epsilon_\mu(x_i) &= \int dx \langle \partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle \epsilon_\nu(x) \\
-i\int d^D x\epsilon_\mu(x)\delta^D(x - x_i)\sum_{i=1}^{N}[\phi(x_1)\ldots G^{\mu}_{i}\phi(x_i)\ldots\phi(x_n)] &= \int dx \langle \partial_\mu T^{\mu\nu}\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle \epsilon_\nu(x)
\end{aligned}$$

Hence,

$$\boxed{\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}(x)\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x - x_i)\langle \phi(x_1)\ldots G^{\mu}_{i}\phi(x_i)\ldots\phi(x_n)\rangle}$$

Translation

The generator of Translation $P^\mu = -i\partial^\mu$ is given as:

$$\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}(x)\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = -\sum_{i=1}^{N}\delta(x-x_i)\langle\phi(x_1)\dots\partial_i^\mu\phi(x_i)\dots\phi(x_n)\rangle$$

Rotation

The generator of Rotation is $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) + S^{\mu\nu}$ and the associated conserved current is $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$.

$$\frac{\partial}{\partial x^\mu}\langle j^{\mu\nu\rho}(x)\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\langle\phi(x_1)\dots J_i^{\nu\rho}\phi(x_i)\dots\phi(x_n)\rangle$$

$$\frac{\partial}{\partial x^\mu}\langle(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\langle\phi(x_1)\dots[i(x^\nu\partial^\rho - x^\rho\partial^\nu) + S^{\nu\rho}]_i\phi(x_i)\dots\phi(x_n)\rangle$$

$$\cancel{x^\rho\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}\phi(x_1)\dots\phi(x_n)\rangle} + \delta_\mu^\rho\langle T^{\mu\nu}\phi(x_1)\dots\phi(x_n)\rangle - \cancel{x^\nu\frac{\partial}{\partial x^\mu}\langle T^{\mu\rho}\phi(x_1)\dots\phi(x_n)\rangle} - \delta_\mu^\nu\langle T^{\mu\rho}\phi(x_1)\dots\phi(x_n)\rangle$$
$$= -i\sum_{i=1}^{N}\delta(x-x_i)\langle\phi(x_1)\dots[\cancel{i(x^\nu\partial^\rho} - \cancel{x^\rho\partial^\nu}) + S^{\nu\rho}]_i\phi(x_i)\dots\phi(x_n)\rangle$$

$$\langle(T^{\rho\nu} - T^{\nu\rho})\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{i=1}^{N}\delta(x-x_i)\langle\phi(x_1)\dots S_i^{\nu\rho}\phi(x_i)\dots\phi(x_n)\rangle$$

Dilatation

The generator for Dilatation is $D = x\cdot\partial$ and the associated conserved current is $j^\mu = T^\mu_\nu x^\nu$.

$$S[\phi] = \int d^2x\,\mathcal{L}(\phi^A, \partial_\mu\phi^A)$$
$$x'^\mu = (1+\varepsilon)x^\mu, \qquad \delta x^\mu = \varepsilon x^\mu, \qquad \bar{\delta}\phi^A = -\varepsilon\Delta_A\phi^A$$
$$\delta_{\text{tot}}\phi^A = \bar{\delta}\phi^A + \delta x^\nu\partial_\nu\phi^A$$

From Noether Theorem

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\bar{\delta}\phi^A - T^\mu{}_\nu\,\delta x^\nu$$

For dilatation, (ε factor suppressed)

$$j_D^\mu = -\sum_A \Delta_A\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\phi^A - x_\nu T^{\mu\nu}$$

Let's define $V^\mu \equiv -\sum_A\Delta_A\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\phi^A$, then

$$j_D^\mu = V^\mu - x_\nu T^{\mu\nu}$$

$$\begin{aligned}\partial_\mu j_D^\mu &= \partial_\mu V^\mu - \partial_\mu(x_\nu T^{\mu\nu})\\ &= \partial_\mu V^\mu - T^\mu{}_\mu - x_\nu\partial_\mu T^{\mu\nu}\\ &= \partial_\mu V^\mu + T^\mu{}_\mu \quad (\partial_\mu T^{\mu\nu} = 0)\end{aligned}$$

$$\partial_\mu j_D^\mu = 0 \iff T^\mu{}_\mu + \partial_\mu V^\mu = 0$$

The related Ward identity becomes:

$$\partial_\mu\langle T^\mu_\nu x^\nu\phi(x_1)\dots\phi(x_n)\rangle = -\sum_i\delta(x-x_i)\left[x_i^\nu\frac{\partial}{\partial x_i^\nu}\langle\phi(x_1)\dots\phi(x_n)\rangle + \Delta_i\langle\phi(x_1)\dots\phi(x_n)\rangle\right]$$

which becomes

$$\langle T^\mu_\mu\phi(x_1)\dots\phi(x_n)\rangle = -\sum_i\delta(x-x_i)\Delta_i\langle\phi(x_1)\dots\phi(x_n)\rangle$$

7.5.1 In complex coordinates

We wish to rewrite these identities in terms of complex coordinates and complex components. But, we will first derive the relevant identities for doing so.

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

The metric in Euclidean space could be given as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In complex coordinates, it becomes:

$$\begin{aligned} g^{\alpha\beta} &= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \end{aligned}$$

The anti-symmetric levi civita tensor in cartesian coordinates has the following form:

$$\epsilon^{\mu\nu}_{\text{cartesian}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

using the tensorial transformation law for the above coordinate transformation,

$$\begin{aligned} \epsilon^{\alpha\beta} &= J^{\alpha}_{\mu} \epsilon^{\mu\nu}_{\text{cartesian}} J^{\beta}_{\nu} \\ &= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \end{aligned}$$

Lowering the indices:

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{-i}{2} & 0 \end{pmatrix}$$

For the delta functions, we remind ourselves that it is not defined as:

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$$

But by the expression

$$\int_M d^2x \delta(x) f(z) = f(0)$$

Any function which satisfies this property is a valid representation of delta function. The ∞ of $\delta(x)$ at $x = 0$ in the above mentioned representation is defined by the integral $\int_{-\infty}^{\infty} \delta(x)dx = 1$. For more detail refer to section 7.2 of the textbook **Green's function** by *G.F. Roach*. It just so happens to be, that

$$\delta(x) = \frac{1}{\pi}\partial_z \frac{1}{\bar{z}} = \frac{1}{\pi}\partial_{\bar{z}} \frac{1}{z}$$

satisfies the required property. This identity is justified as follows. We consider a vector F^μ whose divergence is integrated within a region M of the complex plane bounded by the contour ∂M. Gauss's theorem may be applied:

$$\int_M d^2x\, \partial_\mu F^\mu = \int_{\partial M} d\xi_\mu\, F^\mu \tag{7.6}$$

where $d\xi_\mu$ is an outward-directed differential of circumference, orthogonal to the boundary ∂M of the domain of integration. It is more convenient to use a counterclockwise differential ds^ρ, parallel to the contour ∂M: $d\xi_\mu = \epsilon_{\mu\rho} ds^\rho$. In terms of complex coordinates, the above surface integral is nothing but a contour integral, where the (anti)holomorphic component of ds^ρ is dz ($d\bar{z}$):

$$\int_M d^2x\, \partial_\mu F^\mu = \int_{\partial M} ds^\sigma \epsilon_{\sigma\mu} F^\mu \tag{7.7}$$

$$
\begin{aligned}
&= \int_{\partial M} \left(dz\, \epsilon_{z\bar{z}} F^{\bar{z}} + d\bar{z}\, \epsilon_{\bar{z}z} F^{z}\right) \\
&= \frac{i}{2} \int_{\partial M} \left(-dz\, F^{\bar{z}} + d\bar{z}\, F^{z}\right)
\end{aligned}
\tag{7.8}
$$

Here the contour ∂M circles counterclockwise. If $F^{\bar{z}}$ (F^z) is holomorphic (antiholomorphic), then Cauchy's theorem may be applied; otherwise the contour ∂M must stay fixed. We consider then a holomorphic function $f(z)$ and check the correctness of the first representation in Eq. (5.33) by integrating it against $f(z)$ within a neighborhood M of the origin:

$$
\begin{aligned}
\int_M d^2x\, \delta(x) f(z) &= \frac{1}{\pi} \int_M d^2x\, f(z)\, \partial_{\bar{z}} \frac{1}{z} \\
&= \frac{1}{\pi} \int_M d^2x\, \partial_{\bar{z}} \left(\frac{f(z)}{z}\right) \\
&= \frac{1}{2\pi i} \int_{\partial M} dz\, \frac{f(z)}{z} \\
&= f(0)
\end{aligned}
\tag{7.9}
$$

In the second equation we have used the assumption that $f(z)$ is analytic within M, and in the third equation we have used the form (7.7) of Gauss's theorem with $F^{\bar{z}} = f(z)/\pi z$ and $F^z = 0$, and in the last equation we used Cauchy's residue theorem. Since the original ward identity was covariant, we now only need the following object to write the Ward identities in complex coordinates:

$$
\begin{aligned}
\partial_\mu \langle T^\mu_z \phi(z_1)\ldots\phi(z_n)\rangle &= g^{\alpha\beta}\partial_\alpha \langle T_{\beta z}\phi(z_1)\ldots\phi(z_n)\rangle = 2\partial_z \langle T_{\bar{z}z}\phi(z_1)\ldots\phi(z_n)\rangle + 2\partial_{\bar{z}} \langle T_{zz}\phi(z_1)\ldots\phi(z_n)\rangle \\
\partial_\mu \langle T^\mu_{\bar{z}} \phi(z_1)\ldots\phi(z_n)\rangle &= g^{\alpha\beta}\partial_\alpha \langle T_{\beta \bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle = 2\partial_z \langle T_{\bar{z}\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle + 2\partial_{\bar{z}} \langle T_{z\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle \\
\left\langle T^\mu_\mu \phi(z_1)\ldots\phi(z_n)\right\rangle &= g^{\alpha\beta} \langle T_{\alpha\beta}\phi(z_1)\ldots\phi(z_n)\rangle = 2\langle T_{z\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\ldots\phi(z_n)\rangle \\
\epsilon_{\mu\nu} \langle T^{\mu\nu} \phi(z_1)\ldots\phi(z_n)\rangle &= \epsilon^{\alpha\beta} \langle T_{\alpha\beta}\phi(z_1)\ldots\phi(z_n)\rangle = -2\langle T_{z\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\ldots\phi(z_n)\rangle
\end{aligned}
$$

The Ward identities are then explicitly written as:

$$
\begin{aligned}
2\pi\partial_z \langle T_{\bar{z}z}\phi(z_1)\ldots\phi(z_n)\rangle + 2\pi\partial_{\bar{z}} \langle T_{zz}\phi(z_1)\ldots\phi(z_n)\rangle &= -\sum_i \partial_{\bar{z}} \frac{1}{z-w_i} \partial_{w_i} \langle \phi(z_1)\ldots\phi(w_i)\ldots\phi(z_n)\rangle \\
2\pi\partial_z \langle T_{\bar{z}\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle + 2\pi\partial_{\bar{z}} \langle T_{z\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle &= -\sum_i \partial_{z} \frac{1}{\bar{z}-\bar{z}_i} \partial_{\bar{z}_i} \langle \phi(z_1)\ldots\phi(w_i)\ldots\phi(z_n)\rangle \\
2\langle T_{z\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\ldots\phi(z_n)\rangle &= -\sum_{i=1}^N \delta(z-w_i)\Delta_i \langle \phi(z_1)\ldots\ldots\phi(z_n)\rangle \\
-2\langle T_{z\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle + 2\langle T_{\bar{z}z}\phi(z_1)\ldots\phi(z_n)\rangle &= -\sum_{i=1}^N \delta(z-w_i) s_i \langle \phi(z_1)\ldots\ldots\phi(z_n)\rangle
\end{aligned}
$$

Adding and subtracting the last two expressions,

$$
\begin{aligned}
2\langle T_{\bar{z}z}\phi(z_1)\ldots\phi(z_n)\rangle &= -\sum_{i=1}^N \delta(z-w_i)\frac{\Delta_i + s_i}{2} \langle \phi(z_1)\ldots\ldots\phi(z_n)\rangle \\
2\pi\langle T_{\bar{z}z}\phi(z_1)\ldots\phi(z_n)\rangle &= -\sum_i \partial_{\bar{z}} \frac{1}{z-w_i} h_i \langle \phi(z_1)\ldots\phi(z_n)\rangle
\end{aligned}
\tag{7.10}
$$

$$
\begin{aligned}
2\langle T_{z\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle &= -\sum_{i=1}^N \delta(z-w_i)\frac{\Delta_i - s_i}{2} \langle \phi(z_1)\ldots\ldots\phi(z_n)\rangle \\
2\pi\langle T_{z\bar{z}}\phi(z_1)\ldots\phi(z_n)\rangle &= -\sum_i \partial_{z} \frac{1}{\bar{z}-\bar{w}_i} \bar{h}_i \langle \phi(z_1)\ldots\phi(z_n)\rangle
\end{aligned}
\tag{7.11}
$$

Inserting these relations in the first two Ward identities:

$$
\partial_{\bar{z}} \left\{ \langle T(z,\bar{z})\phi(z_1)\ldots\phi(z_n)\rangle - \sum_{i=1}^n \left[\frac{1}{z-w_i}\partial_{w_i}\langle \phi(z_1)\ldots\phi(z_n)\rangle + \frac{h_i}{(z-w_i)^2}\langle \phi(z_1)\ldots\phi(z_n)\rangle \right] \right\} = 0 \tag{7.12}
$$

$$\partial_z\left\{\langle\bar{T}(z,\bar{z})\phi(z_1)\dots\phi(z_n)\rangle-\sum_{i=1}^{n}\left[\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle\phi(z_1)\dots\phi(z_n)\rangle+\frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}\langle\phi(z_1)\dots\phi(z_n)\rangle\right]\right\}=0 \tag{7.13}$$

where we have introduced a renormalized energy-momentum tensor

$$T=-2\pi T_{zz}, \qquad \bar{T}=-2\pi T_{\bar{z}\bar{z}}. \tag{7.14}$$

Thus the expressions between braces in (7.12) and ((7.13)) are respectively holomorphic and antiholomorphic; we may write

$$\langle T(z)\phi(z_1)\dots\phi(z_n)\rangle=\sum_{i=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle\phi(z_1)\dots\phi(z_n)\rangle+\frac{h_i}{(z-w_i)^2}\langle\phi(z_1)\dots\phi(z_n)\rangle\right\}+\text{reg.} \tag{7.15}$$

where "reg." stands for a holomorphic function of z, regular at $z=w_i$. There is a similar expression for the antiholomorphic counterpart. The Ward identity shows that the correlator of the field $T(z)$ with primary fields $\phi(w_i,\bar{w}_i)$ becomes singular as z approaches the points w_i. The OPE of the energy-momentum tensor with primary fields is written simply by removing the brackets $\langle\dots\rangle$, it being understood that OPE is meaningful only within correlation functions.

7.6 Free Fields and Operator Product Expansion

The operator product expansion, or OPE, is the representation of a product of operators (at positions z and w, respectively) by a sum of terms, each being a single operator, well-defined as $z\to w$, multiplied by a c-number function of $z-w$, possibly diverging as $z\to w$, and which embodies the infinite fluctuations as the two positions tend toward each other. For a single primary field ϕ of conformal dimension h and $\bar{h}$, we have the OPE from Ward identity as:

$$\begin{aligned}
T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w})+\frac{1}{z-w}\partial_w\phi(w,\bar{w})\\
T(\bar{z})\phi(w,\bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{z})^2}\phi(w,\bar{w})+\frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w})
\end{aligned} \tag{7.16}$$

whenever appearing in OPEs, the symbol $\sim$ will mean equality modulo expressions regular as $z\to w$. Of course, the OPE contains also an infinite number of regular terms which, for the Energy Momentum tensor, can not be obtained from the conformal ward identity. In general, we would write the OPE of two fields $A(z)$ and $B(w)$ as

$$A(z)B(w)=\sum_{n=-\infty}^{\infty}\frac{\{AB\}_n(w)}{(z-w)^n}$$

where the composite $\{AB\}_n(w)$ are non singular at $z=w$. For instance, $\{T\phi\}_1=\partial_w\phi(w)$. We stress that so far, quantities appearing in (7.16) are not operators but simply fields occuring within correlation functions. We will now proceed with specific examples, in order to familiarize ourselves with basic techniques and with simple but important systems.

The Free Boson

From the point of view of the canonical or path integral formalism, the simplest conformal field theory is that of a free massless boson ϕ, with the following action:

$$S=\frac{1}{2}g\int d^2x\partial_\mu\phi\partial^\mu\phi$$

where g is some normalization parameter that we leave unspecified at the moment. The two-point function, or propagator can be found by comparing with

$$S=\frac{1}{2}\int \mathrm{d}^dx\mathrm{d}^dy\,\phi(x)A(x,y)\phi(y)$$

we have

$$A(x,y)=-g\delta^{(2)}(x-y)\Box\,, \tag{7.17}$$

We can calculate the two-point function $K(x,y)\equiv\langle\varphi(x_1)\varphi(x_2)\rangle=A^{-1}$ by solving the following equation:

$$\int d^2uA(x,u)K(u,y)=\delta^{(2)}(x-y)$$

$$-\int d^2ug\delta^{(2)}(x-u)\Box K(u,y)=\delta^{(2)}(x-y)$$
$$-g\Box K(x,y)=\delta^{(2)}(x-y)\,,$$

Because of rotation and translation invariance, the propagator $K(x,y)$ should depend only on the distance separating the two points. Thus, we can write $K(x,y)\equiv K(\rho)$ with $\rho=|x-y|$, and integrate over x within a disk of radius ρ around y. We find

$$\begin{aligned}1&=2\pi g\int_0^r \mathrm{d}\rho\rho\Big(-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho K'(\rho))\Big)\\&=2\pi g(-rK'(r))\end{aligned}$$

The solution of the two-point function for massless free boson can be obtained up to an additive constant,

$$\langle\phi(x)\phi(y)\rangle=-\frac{1}{4\pi g}\ln(x-y)^2+\text{const}$$

In terms of complex coordinates, this is

$$\langle\phi(z,\bar z)\phi(w,\bar w)\rangle=-\frac{1}{4\pi g}\{\ln(z-w)+\ln(\bar z-\bar w)\}+\text{const}$$

The holomorphic and anti-holomorphic components can be separated by taking the derivatives $\partial\phi$ and $\bar\partial\phi$:

$$\begin{aligned}\langle\partial_z\phi(z,\bar z)\partial_w\phi(w,\bar w)\rangle&=-\frac{1}{4\pi g}\partial_w\partial_z\{\ln(z-w)\}\\&=-\frac{1}{4\pi g}\partial_w\frac{1}{z-w}=-\frac{1}{4\pi g}\frac{1}{(z-w)^2}\\\langle\partial_{\bar z}\phi(z,\bar z)\partial_{\bar w}\phi(w,\bar w)\rangle&=-\frac{1}{4\pi g}\partial_{\bar w}\partial_{\bar z}\{\ln(\bar z-\bar w)\}\\&=-\frac{1}{4\pi g}\partial_{\bar w}\frac{1}{\bar z-\bar w}=-\frac{1}{4\pi g}\frac{1}{(\bar z-\bar w)^2}\end{aligned}$$

In the following, we will focus on holomorphic field $\partial\phi\equiv\partial_z\phi$.

$$\begin{aligned}T_{\mu\nu}&=g\left[\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right]\\&=g\partial_\mu\phi\partial_\nu\phi-\frac{g}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\\&=g\partial_\mu\phi\partial_\nu\phi-\frac{g}{2}\eta_{\mu\nu}\times2\eta^{z\bar z}\partial_z\phi\underbrace{\partial_{\bar z}\phi}_{=0}\\&=g\partial_\mu\phi\partial_\nu\phi\end{aligned}$$

Then,

$$T(z)=-2\pi T_{zz}=-2\pi g:\partial\phi\partial\phi:\qquad\text{and}\qquad\bar T(\bar z)=0$$

Like all composite field, the energy momentum tensor has to be normal ordered, in order to ensure the vanishing of its vacuum expectation value. More explicitly, the exact meaning of above expression is

$$T(z)=-2\pi g\lim_{w\to z}[\partial\phi(z)\partial\phi(w)-\langle\partial\phi(z)\partial\phi(w)\rangle]$$

The OPE of $T(z)$ with $\partial\phi$ may be calculated from Wick's theorem:[1]

$$\begin{aligned}T(z)\partial\phi(w)&=-2\pi g:\partial\phi(z)\partial\phi(z):\partial\phi(w)\\&\sim-2\pi g:\partial\phi(z)\contraction{}{\partial\phi(z):}{}{\partial\phi(w)}\partial\phi(z):\partial\phi(w)-2\pi g:\contraction{}{\partial\phi(z)\partial\phi(z):}{}{\partial\phi(w)}\partial\phi(z)\partial\phi(z):\partial\phi(w)\\&\sim\frac{\partial\phi(z)}{(z-w)^2}\end{aligned}$$

[1]Since we are Wick contracting the normal ordered product of operators, the only term that it will give rise to are cross-contractions, since the wick contraction of operators which are already normal ordered vanishes.

By expanding $\phi(z)$ around w, we arrive at the OPE

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)}$$

This shows that $\partial\phi$ is a primary field with conformal dimension $h = 1$. Wick's theorem also allows us to calculate the OPE of energy-momentum tensor with itself:

$$\begin{aligned}
T(z)T(w) &= 4\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\sim 4\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \; + 4\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\quad + 4\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \; + 4\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\quad + 4\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \; + 4\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\sim 8\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \; + 16\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w): \\
&\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g :\partial\phi(z)\partial\phi(w):}{(z-w)^2}
\end{aligned}$$

Expanding $\phi(z)$ around w, we arrive at the OPE

$$\begin{aligned}
&\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g :\partial\phi(w)\partial\phi(w):}{(z-w)^2} - \frac{4\pi g :\partial^2\phi(w)\partial\phi(w):}{(z-w)} \\
&\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}
\end{aligned}$$

We immediately see that the energy momentum tensor is not strictly a primary field, because of the anomalous term $1/2(z-w)^4$ which does not appear in (7.16).

Free Fermion

Now we consider another simple model: free fermion. In two dimensions, the action of a free Majorana fermion is

$$S = \frac{1}{2}g\int \mathrm{d}^2x\Psi^\dagger\gamma^0\gamma^\mu\partial_\mu\Psi\,, \tag{7.18}$$

where the gamma matrices γ^μ satisfy the so-called **Clifford algebra**:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,, \tag{7.19}$$

and we impose the **Majorana condition** ($\Psi^* = \Psi$) to the fermionic field to remove a half of the degrees of freedom. In the Euclidean space $\eta^{\mu\nu} = \mathrm{diag}(1,1)$, we take a basis of Dirac matrices as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.20}$$

and therefore,

$$\begin{aligned}
\gamma^0\gamma^\mu\partial_\mu &= \gamma^0(\gamma^0\partial_0 + \gamma^1\partial_1) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial + i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\bar\partial\right] \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \partial_0 - i\partial_1 \\ \partial_0 + i\partial_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2\partial_z \\ 2\partial_{\bar z} & 0 \end{pmatrix} \\
&= 2\begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix}
\end{aligned}$$

Using this basis, we can express the action as

$$S = g\int \mathrm{d}^2x(\overline{\psi}\partial\overline{\psi} + \psi\overline{\partial}\psi)\,, \tag{7.21}$$

where we write the two-component spinor Ψ as $(\psi, \overline{\psi})$. Since the equations of motion are $\partial\overline{\psi} = 0$ and $\overline{\partial}\psi = 0$, $\psi(z)$ and $\overline{\psi}(\overline{z})$ holomorphic and antiholomorphic field, respectively.

Now let us calculate the two-point function as in the free fermion

$$K_{ij} = \langle \Psi_i(x)\Psi_j(y)\rangle \quad (i, j = 1, 2) \,. \tag{7.22}$$

The action can be expressed by

$$S = \frac{1}{2}\int \mathrm{d}^2x\mathrm{d}^2y \Psi_i(x) A_{ij}(x, y)\Psi_j(y) \,, \tag{7.23}$$

where the kernel is

$$A_{ij}(x, y) = g\delta(x - y)(\gamma^0\gamma^\mu)_{ij}\partial_\mu \,. \tag{7.24}$$

Recalling that propagator K_{ij} is the inverse of A_{ij}. Therefore, we can write the equation for K as

$$\begin{aligned} \int d^2u A(x, u)K(u, y) &= \delta^{(2)}(x - y) \\ g\int d^2u \delta^{(2)}(x - u)(\gamma^0\gamma^\mu)_{ij}\frac{\partial}{\partial x^\mu} K(u, y) &= \delta^{(2)}(x - y)\delta_{ij} \\ g(\gamma^0\gamma^\mu)_{ik}\frac{\partial}{\partial x^\mu} K_{kj}(x, y) &= \delta(x - y)\delta_{ij} \,. \end{aligned}$$

In terms of complex coordinates, this becomes

$$2g\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}\begin{pmatrix} \langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle & \langle\psi(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle \\ \langle\overline{\psi}(z,\overline{z})\psi(w,\overline{w})) & \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle \end{pmatrix} = \frac{1}{\pi}\begin{pmatrix} \overline{\partial}\frac{1}{z-w} & 0 \\ 0 & \partial\frac{1}{\overline{z}-\overline{w}} \end{pmatrix} , \tag{7.25}$$

where we have used the complex form of the δ-function:

$$\delta(x) = \frac{1}{\pi}\overline{\partial}\frac{1}{z} = \frac{1}{\pi}\partial\frac{1}{\overline{z}} \,.$$

Therefore, we obtain the two-point functions for the fermionic fields

$$\begin{aligned} \langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle &= \frac{1}{2\pi g}\frac{1}{z - w} \,, \\ \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle &= \frac{1}{2\pi g}\frac{1}{\overline{z} - \overline{w}} \,, \\ \langle\psi(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle &= 0 \end{aligned}$$

These, after differentiation, imply

$$\begin{aligned} \langle\partial_z\psi(z,\bar{z})\psi(w,\bar{w})\rangle &= -\frac{1}{2\pi g}\frac{1}{(z - w)^2} \\ \langle\partial_z\psi(z,\bar{z})\partial_w\psi(w,\bar{w})\rangle &= -\frac{1}{\pi g}\frac{1}{(z - w)^3} \end{aligned}$$

Thus the OPE of two holomorphic fields can be written as:

$$\psi(z)\psi(w) \sim \frac{1}{2\pi g}\frac{1}{z - w} \tag{7.26}$$

In order to see whether the fermion field is a primary field or not, we can calculate its OPE with the energy-momentum tensor. By using (**??**) in complex-coordinate form, we can calculate all the components of the energy-momentum tensor.

$$\begin{aligned} T^{\overline{z}\overline{z}} &= \frac{\partial\mathcal{L}}{\partial(\partial_{\bar{z}}\Phi)}\partial^{\bar{z}}\Phi = g^{\bar{z}z}\frac{\partial\mathcal{L}}{\partial(\overline{\partial}\Phi)}\partial\Phi = 2\frac{\partial\mathcal{L}}{\partial(\overline{\partial}\Phi)}\partial\Phi = 2g\psi\partial\psi \,, \\ T^{zz} &= \frac{\partial\mathcal{L}}{\partial(\partial_z\Phi)}\partial^z\Phi = g^{z\bar{z}}\frac{\partial\mathcal{L}}{\partial(\partial\Phi)}\overline{\partial}\Phi = 2\frac{\partial\mathcal{L}}{\partial(\partial\Phi)}\overline{\partial}\Phi = 2g\overline{\psi\partial\psi} \,, \\ T^{z\overline{z}} &= \frac{\partial\mathcal{L}}{\partial(\partial_z\Phi)}\partial^{\bar{z}}\Phi - g^{z\overline{z}}\mathcal{L} = 2\frac{\partial\mathcal{L}}{\partial(\partial\Phi)}\partial\Phi - 2\mathcal{L} = -2g\psi\overline{\partial}\psi \,. \end{aligned}$$

The traceless condition $T^{z\overline{z}}$ is preserved when taking into account the equation of motion, as we have discussed. The holomorphic part is defined as:

$$T(z) = -\pi g : \psi(z)\partial\psi(z) : \,. \tag{7.27}$$

The normal-ordering product can be written in an equivalent way for the free field as follow:

$$: \psi\partial\psi : (z) = \lim_{w\to z} \left(\psi(z)\partial\psi(w) - \langle\psi(z)\partial\psi(w)\rangle\right), \tag{7.28}$$

which is the same expression as in bosonic field theory. Then we can calculate the OPE between the fermion field and energy-momentum tensor directly.

$$\begin{aligned} T(z)\psi(w) &= -\pi g : \psi(z)\partial\psi(z) : \psi(w) \\ &= -\pi g : \psi(z)\partial\psi(z) : \psi(w) - \pi g : \psi(z)\partial\psi(z) : \psi(w) \\ &\sim \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + \frac{1}{2}\frac{\partial\psi(z)}{(z-w)} \\ &\sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w} \end{aligned}$$

In contracting $\psi(z)$ with $\psi(w)$ we have carried $\psi(w)$ over $\partial\psi(z)$, thus introducing a $(-)$ sign by Pauli's principle. The OPE immediately tells us that in the free fermion model, ψ is a primary field with conformal dimension 1/2. TT OPE in this model can also be obtained directly by calculation.

$$T(z)T(w) \sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,. \tag{7.29}$$

The Ghost System

In string theory applications, there appears another simple system, with the following action:

$$S = \frac{1}{2}g\int d^2x b_{\mu\nu}\partial^\mu c^\nu$$

where $b_{\mu\nu}$ is a traceless symmetric tensor, and where both c^μ and $b_{\mu\nu}$ are fermions (anti-commuting fields). These fields are called ghosts because they are not fundamental dynamical fields, but rather represent a jacobian arising from a change of variables in some functional integrals. More precisely, they are known as *reparametrization ghosts*. The role of these ghost fields is to cancel the unphysical gauge degrees of freedom.

The equation of motion are

$$\partial^\alpha b_{\alpha\mu} = 0 \qquad \text{and} \qquad \partial^\alpha c^\beta + \partial^\beta c^\alpha = 0$$

In holomorphic form we write $c = c^z$ and $\bar{c} = c^{\bar{z}}$. The only nonzero components of the traceless symmetric tensor $b_{\mu\nu}$ are $b = b_{zz}$ and $\bar{b} = b_{\bar{z}\bar{z}}$. The equations of motion are then

$$\begin{array}{lll} \bar\partial b = 0 & \partial\bar{b} = 0 & \\ \bar\partial c = 0 & \partial\bar{c} = 0 & \partial c = -\bar\partial\bar{c} \end{array}$$

The propagator is calculated in the usual way, by writing the action as:

$$S = \frac{1}{2}\int d^2xd^2y b_{\mu\nu}(x)A^{\mu\nu}_\alpha c^\alpha(y)$$

$$A^{\mu\nu}_\alpha = \frac{1}{2}g\delta^\nu_\alpha\delta(x-y)\partial^\mu$$

where we must consider (μ,ν) as a single composite index, symmetric under the exchange of μ and ν. The factor of $\frac{1}{2}$ in front of $A^{\mu\nu}_\alpha$ compensates the double counting of each pair (μ,ν) in the sum, which should be avoided since $b^{\mu\nu}$ is the same degree of freedom as $b^{\nu\mu}$. Again the propagator is $K = A^{-1}$, satisfying

$$\frac{1}{2}g\delta^\mu_\alpha\partial^\nu K^\beta_{\mu\nu}(x,y) = \delta(x-y)\delta_{\alpha\beta}$$

or, in complex representation

$$g\partial_{\bar{z}}K^\beta_{zz} = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z-w}\delta_{\beta z}$$

which implies

$$\langle b(z)c(w)\rangle = K^z_{zz}(z,w) = \frac{1}{\pi g}\frac{1}{z-w}$$

In OPE form, this is

$$\boxed{b(z)c(w) \sim \frac{1}{\pi g}\frac{1}{z-w}}$$

from which we immediately derive

$$\begin{aligned}
\langle c(z)b(w)\rangle = -\langle b(w)c(z)\rangle &= \frac{1}{\pi g}\frac{1}{z-w} \\
\langle b(z)\partial_w c(w)\rangle = -\langle \partial_z c(z)b(w)\rangle = \partial_w\left(\frac{1}{\pi g}\frac{1}{z-w}\right) &= \frac{1}{\pi g}\frac{1}{(z-w)^2} \\
\langle \partial_z b(z)c(w)\rangle &= -\frac{1}{\pi g}\frac{1}{(z-w)^2} \\
\langle \partial_z b(z)\partial_w c(w)\rangle &= -\frac{2}{\pi g}\frac{1}{(z-w)^3}
\end{aligned}$$

The canonical energy-momentum tensor for this system is

$$T_B^{\mu\nu} = \frac{1}{2}g\left[b^{\mu\alpha}\partial^\nu c_\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right]$$

The Belinfante tensor is

$$T_B^{\mu\nu} = \frac{1}{2}g\left[b^{\mu\alpha}\partial^\nu c_\alpha + b^{\nu\alpha}\partial^\mu c_\alpha + \partial_\alpha b^{\mu\nu}c^\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right]$$

The normal ordered holomorphic component is obtained from the above by setting $\mu = \nu = 1$, that is, by considering $T^{\bar{z}\bar{z}} = 4T_{zz}$:

$$T(z) = \pi g : (2\partial c\, b + c\partial b) :$$

The OPE for this stress energy tensor with c is again calculated using Wick's theorem:

$$\begin{aligned}
T(z)c(w) &= \pi g : (2\partial c\, b + c\partial b) : c(w) \\
&= \pi g : 2\partial c\, b : c(w) + \pi g : c\partial b : c(w) \\
&\sim \frac{2\partial_z c(z)}{z-w} - \frac{c(z)}{(z-w)^2} \\
&\sim \frac{2\partial_w c(w)}{z-w} - \frac{c(w) + (z-w)\partial_w c(w)}{(z-w)^2} \\
&\sim \frac{\partial_w c(w)}{z-w} - \frac{c(w)}{(z-w)^2}
\end{aligned}$$

$$\begin{aligned}
T(z)b(w) &= \pi g : (2\partial c\, b + c\partial b) : b(w) \\
&= -\pi g : 2b\partial c : b(w) - \pi g : \partial b\, c : b(w) \\
&\sim \frac{2b(z)}{(z-w)^2} - \frac{\partial_z b(z)}{(z-w)} \\
&\sim \frac{2b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)}
\end{aligned}$$

One must be careful because b and c are anticommuting so that interchange of fields flips the sign: one should anticommute the fields being paired until they are next to each other before doing the Wick contraction. The OPE of T with itself, may contain more terms, which add up to following:

$$\begin{aligned}
T(z)T(w) &= \pi^2 g^2 : (2\partial c(z)\, b(z) + c(z)\partial b(z)) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) : \\
&= \pi^2 g^2 : 2\partial c(z)\, b(z) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) : + \pi^2 g^2 : c(z)\partial b(z) :: (2\partial c(w)\, b(w) + c(w)\partial b(w)) : \\
&= 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) : + 2\pi^2 g : \partial c(z)\, b(z) :: c(w)\partial b(w) : + 2\pi^2 g^2 : c(z)\partial b(z) :: \partial c(w)\, b(w) : \\
&\quad + \pi^2 g^2 : c(z)\partial b(z) :: c(w)\partial b(w) : \\
&= 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) : + 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) : + 4\pi^2 g^2 : \partial c(z)\, b(z) :: \partial c(w)\, b(w) :
\end{aligned}$$

$$+ 2\pi^2 g : \partial c(z)\, b(z) :: c(w) \partial b(w) : + 2\pi^2 g : \partial c(z)\, b(z) :: c(w) \partial b(w) : + 2\pi^2 g : \partial c(z)\, b(z) :: c(w) \partial b(w) :$$
$$+ 2\pi^2 g^2 : c(z) \partial b(z) :: \partial c(w)\, b(w) : + 2\pi^2 g^2 : c(z) \partial b(z) :: \partial c(w)\, b(w) : + 2\pi^2 g^2 : c(z) \partial b(z) :: \partial c(w)\, b(w) :$$
$$+ \pi^2 g^2 : c(z) \partial b(z) :: c(w) \partial b(w) : + \pi^2 g^2 : c(z) \partial b(z) :: c(w) \partial b(w) : + \pi^2 g^2 : c(z) \partial b(z) :: c(w) \partial b(w) :$$

The end result is

$$T(z)T(w) = \frac{-4}{(z-w)^4} + \frac{4\pi g : \partial c(z) b(w) :}{(z-w)^2} - \frac{4\pi g : b(z) \partial c(w) :}{(z-w)^2} - \frac{4}{(z-w)^4} + \frac{2\pi g : \partial c(z) \partial b(w) :}{z-w} - \frac{4\pi g : b(z) c(w) :}{(z-w)^3}$$
$$- \frac{4}{(z-w)^4} - \frac{4\pi g : c(z) b(w) :}{(z-w)^3} + \frac{2\pi g : \partial b(z) \partial c(w) :}{z-w} - \frac{1}{(z-w)^4} - \frac{\pi g : c(z) \partial b(w) :}{(z-w)^2} + \frac{\pi g : \partial b(z) c(w) :}{(z-w)^2} + \dots$$

After some Taylor expansions to turn $f(z)$ functions into $f(w)$ functions, together with a little collecting of terms, this can be written as,

$$T(z)T(w) = \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

7.7 Central Charge

The specific models treated in the last section lead us naturally to the following general OPE of the energy-momentum tensor.

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

where the constant c, not to be confused with ghost field c_μ, depends on the specific model under study: it is equal to 1 for the free boson, 1/2 for the free fermion, -26 for the reparametrization ghosts, and -2 for the simple ghost system. This model dependent constant term is called the *central charge*. Except for this anomalous term, the OPE simply means that T is a quasi-primary field with conformal dimension $h = 2$.

The central charge may not be determined from symmetry considerations: its value is determined by the short-distance behavior of the theory. For free fields, as seen in the previous section, it is determined by applying Wick's theorem on the normal-ordered energy-momentum tensor. When two decoupled systems (e.g., two free fields) are put together, the energy-momentum tensor of the total system is simply the sum of the energy-momentum tensors associated with each part, and the associated central charge is simply the sum of the central charges of the parts. Thus, the central charge is somehow an extensive measure of the number of degrees of freedom of the system.

Transformation of the Energy-Momentum Tensor

The departure of OPE from the general form (7.16) means that the energy-momentum tensor does not exactly transform like a primary field of dimension 2, contrary to what we expect classically. This happens because the normal ordering is not invariant under conformal transformation. According to conformal ward identity

$$\begin{aligned}
\delta_\epsilon T(w) &= -\frac{1}{2\pi i} \oint_C dz \epsilon(z) T(z) T(w) \\
&= -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.} \right] \\
&= -\frac{c/2}{3!} \partial_w^3 \epsilon(w) - 2T(w) \partial_w \epsilon(w) - \epsilon(w) \partial_w T(w)
\end{aligned}$$

In the third equation, we used Cauchy's integral formula for derivatives. The "exponentiation" of this infinitesimal variation to a finite transformation $z \to w(z)$ is:

$$T'(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z} \right)^{-2} \left[T(z) - \frac{c}{12} \{w; z\} \right]$$

where we have introduced the Schwarzian derivative:

$$\{w; z\} = \frac{\frac{\mathrm{d}^3 w}{\mathrm{d}z^3}}{\frac{\mathrm{d}w}{\mathrm{d}z}} - \frac{3}{2} \left(\frac{\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}}{\frac{\mathrm{d}w}{\mathrm{d}z}} \right)^2$$

Instead of giving the long and technical proof of the last statement, we shall derive the above for free boson system. We write the free boson energy-momentum tensor as

$$T(z) = -2\pi g \lim_{\delta\to 0} :\partial\phi\partial\phi: = -2\pi g \lim_{\delta\to 0}\left[\partial\phi\left(z+\frac{1}{2}\delta\right)\partial\phi\left(z-\frac{1}{2}\delta\right)+\frac{1}{4\pi\delta^2}\right]$$

Consider the transformation $z \to w(z)$. Since ϕ has conformal dimension zero $\partial\phi$ transforms as

$$\partial_z\phi = \frac{\partial w}{\partial z}\frac{\partial\phi'(z)}{\partial w} = w^{(1)}\partial_w\phi'(w)$$

(here we denote the n-th derivative of w by $w^{(n)}$ in order to lighten the notation). Hence $T(z)$ transforms as:

$$T(z) = -2\pi g \lim_{\delta\to 0}\left[w^{(1)}\left(z+\frac{1}{2}\delta\right)w^{(1)}\left(z-\frac{1}{2}\delta\right)\partial_w\phi'\left(w\left(z-\frac{1}{2}\delta\right)\right)\partial_w\phi'\left(w\left(z+\frac{1}{2}\delta\right)\right)+\frac{1}{4\pi\delta^2}\right]$$

we will use the following to simplify the above:

$$\begin{aligned}
w(z+\delta/2) &\simeq w(z) + \frac{\delta}{2}\partial_z w(z) + \frac{1}{2!}\left(\frac{\delta}{2}\right)^2\partial_z^2 w(z) + \frac{1}{3!}\left(\frac{\delta}{2}\right)^3\partial_z^3 w(z) + \dots \\
\partial_z w(z+\delta/2) &\simeq \partial_z w(z) + \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!}\left(\frac{\delta}{2}\right)^2\partial_z^3 w(z) + \dots \\
\big[w(z+\delta/2) - w(z-\delta/2)\big]^2 &= \big(\partial_z w(z)\big)^2\delta^2 + \frac{1}{12}\big(\partial_z^3 w\,\partial_z w(z)\big)\delta^4 + \mathcal{O}(\delta^6), \\
&= (w^{(1)}\delta)^2\left[1+\frac{1}{12}\frac{w^{(3)}}{w^{(1)}}\delta^2+\dots\right]
\end{aligned}$$

so the inverse is then,

$$\frac{1}{[w(z+\delta/2)-w(z-\delta/2)]^2} = \frac{1}{\delta^2}\frac{1}{(\partial_z w(z))^2} - \frac{1}{12}\frac{\partial_z^3 w(z)}{(\partial_z w(z))^3} + \mathcal{O}(\delta^2) \tag{7.30}$$

$$\begin{aligned}
\partial_z w(z+\delta/2)\partial_z w(z-\delta/2) &= \left[\partial_z w(z) + \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!}\left(\frac{\delta}{2}\right)^2\partial_z^3 w(z) + \dots\right]\left[\partial_z w(z) - \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!}\left(\frac{\delta}{2}\right)^2\partial_z^3 w(z) + \dots\right] \\
&= (w^{(1)})^2 - \left(\frac{\delta}{2}\right)^2(w^{(2)})^2 + \left(\frac{\delta}{2}\right)^2 w^{(1)}w^{(3)} + \dots
\end{aligned} \tag{7.31}$$

Then,

$$\begin{aligned}
T(z) &= \lim_{\delta\to 0}\left[w^{(1)}\left(z+\frac{1}{2}\delta\right)w^{(1)}\left(z-\frac{1}{2}\delta\right)\left\{-2\pi g :\partial_w\phi'(w)\partial_w\phi'(w): + \frac{1}{2\left[w\left(z+\frac{\delta}{2}\right)-w\left(z-\frac{\delta}{2}\right)\right]^2}\right\} - \frac{1}{2\delta^2}\right] \\
&= (w^{(1)}(z))^2 T'(w) + \lim_{\delta\to 0}\left[\frac{w^{(1)}\left(z+\frac{\delta}{2}\right)w^{(1)}\left(z-\frac{\delta}{2}\right)}{2\left[w\left(z+\frac{\delta}{2}\right)-w\left(z+\frac{\delta}{2}\right)\right]^2} - \frac{1}{2\delta^2}\right] \\
&= (w^{(1)}(z))^2 T'(w) \\
&\quad + \lim_{\delta\to 0}\left[\left\{(w^{(1)})^2 - \left(\frac{\delta}{2}\right)^2(w^{(2)})^2 + \left(\frac{\delta}{2}\right)^2 w^{(1)}w^{(3)} + \dots\right\}\left\{\frac{1}{2\delta^2}\frac{1}{(w^{(1)})^2} - \frac{1}{24}\frac{w^{(3)}}{(w^{(1)})^3} + \dots\right\} - \frac{1}{2\delta^2}\right] \\
&= (w^{(1)}(z))^2 T'(w) + \lim_{\delta\to 0}\left[\cancel{\frac{1}{2\delta^2}} - \frac{1}{8}\left(\frac{w^{(2)}}{w^{(1)}}\right)^2 - \frac{1}{24}\frac{w^{(3)}}{w^{(1)}} + \frac{1}{8}\frac{w^{(3)}}{w^{(1)}} + \dots - \cancel{\frac{1}{2\delta^2}}\right] \\
&= (w^{(1)}(z))^2 T'(w) + \frac{1}{12}\left[\frac{w^{(3)}}{w^{(1)}} - \frac{3}{2}\left(\frac{w^{(2)}}{w^{(1)}}\right)^2\right]
\end{aligned}$$

Here we use (7.30) and (7.31) in the second step for simplication. There is another way to look at this derivation which sheds light on the origin of schwarzian derivative term. Let us focus on how the propagator transforms under conformal transformation:

$$\begin{aligned}
\langle\phi(w(z_1))\phi(w(z_2))\rangle &= -\ln\big|w(z_1)-w(z_2)\big|^2 \\
&= -\ln\big|z_{12}\frac{w(z_1)-w(z_2)}{z_1-z_2}\big|^2
\end{aligned}$$

$$
\begin{aligned}
&= -\ln|z_{12}|^2 - \ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right|^2 \\
&= \langle\phi(z_1)\phi(z_2)\rangle - \ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right|^2
\end{aligned}
$$

Let us write $w_1 \equiv w(z_1)$ and $w_2 \equiv w(z_2)$. One can show that for $|z_{12}| = |z_1 - z_2|$ small:

$$
\begin{aligned}
\partial_{z_1}\partial_{z_2}\ln\left|\frac{w(z_1)-w(z_2)}{z_{12}}\right| &= \partial_{z_1}\partial_{z_2}\ln|w(z_1)-w(z_2)|^2 - \partial_{z_1}\partial_{z_2}\ln|z_{12}|^2 \\
&= \partial_{z_1}\left[\frac{-w^{(1)}(z_2)}{w(z_1)-w(z_2)}\right] - \frac{1}{z_{12}^2} \\
&= \frac{w^{(1)}(z_2)w^{(1)}(z_1)}{(w(z_1)-w(z_2))^2} - \frac{1}{z_{12}^2} \\
&= \frac{2}{12}\left[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\left(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\right)^2\right] + \mathcal{O}(z_{12}).
\end{aligned}
$$

Since only the $z_{12}\to 0$ limit is of interest we can drop all terms on the right-hand side that vanish in this limit. Substituting the result of this into the above we learn that:

$$
\boxed{\lim_{z_1\to z_2}\partial_{z_1}\partial_{z_2}\langle\phi(w(z_1))\phi(w(z_2))\rangle = \lim_{z_1\to z_2}\partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle - \frac{2}{12}\left[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\left(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\right)^2\right]}
$$

The non-invariance of the vacuum and thus correlator is really what opens up the possibility of a trace anomaly $\langle U^{-1}T(w)U\rangle \equiv \langle T'(w)\rangle \neq 0$ and it is not at all coincidental. The conformal transformations which do change the vacuum are those that have non-vanishing Schwarzian derivative, and thus an extra inhomogeneous central charge term. Using this we can derive the transformation law for energy-momentum tensor:

$$
\begin{aligned}
T^{(w)}(z_2) &\equiv :\lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_w \\
&= \lim_{z_1\to z_2}\Big[-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) - \partial_{z_1}\partial_{z_2}\langle\phi(w(z_1))\phi(w(z_2))\rangle\Big)\Big] \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) - \partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle + \frac{2}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big]\Big)\Big\} \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) - \partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle\Big)\Big\} - \frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big] \\
&=: \lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_z - \frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big] \\
&= T^{(z)}(z_2) - \frac{1}{12}\Big[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\Big(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\Big)^2\Big]
\end{aligned}
\tag{7.32}
$$

where we noted in the last two lines that:

$$
\begin{aligned}
T^{(z)}(z_2) &\equiv :\lim_{z_1\to z_2}\Big[-\frac{1}{2}\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2)\Big]:_z \\
&= \lim_{z_1\to z_2}\Big\{-\frac{1}{2}\Big(\partial_{z_1}\phi(z_1)\partial_{z_2}\phi(z_2) - \partial_{z_1}\partial_{z_2}\langle\phi(z_1)\phi(z_2)\rangle\Big)\Big\}
\end{aligned}
\tag{7.33}
$$

as shown above. So we learn that a finite holomorphic change in normal ordering, $z \to w(z)$, with fixed coordinates, z_2, of the energy-momentum tensor is given by:

$$
\boxed{T^{(w)}(z_2) = T^{(z)}(z_2) - \frac{1}{12}\left[\frac{\partial^3_{z_2}w_2}{\partial_{z_2}w_2} - \frac{3}{2}\left(\frac{\partial^2_{z_2}w_2}{\partial_{z_2}w_2}\right)^2\right]}
$$

Chapter 8

Operator Formalism

In the previous chapter, conformal symmetry was seen to impose constraints on correlation functions in the form of Ward identities. These identities were conveniently expressed using operator product expansions between the energy–momentum tensor and local fields, but the OPEs were understood only as a shorthand for singularities inside correlators. Nothing required a Hilbert space or an operator formalism: in principle, everything could have been computed directly in the path integral by evaluating Green's functions and extracting their short-distance behavior. Up to this point, all we really needed was a way to compute the two-point correlator, whether by solving the Schwinger–Dyson equations or by brute-force path integration. The OPEs then followed from the explicit Green's function.

From here onward the viewpoint changes. Instead of relying on an explicit propagator, we will systematically construct an operator formalism in which OPEs can be obtained directly from symmetry and representation theory. This is a qualitatively new approach: even when the Green's function is not accessible by solving the equations of motion or performing the path integral, the operator method still allows us to determine the structure of OPEs and, in turn, recover the correlators themselves.

The Hilbert space is what makes this possible. Once a Hilbert space is defined, local fields are no longer only insertions in correlation functions — they correspond to states, and operators act as maps on this space. This representation turns the OPE into an actual operator identity, rather than a mnemonic for propagator singularities. The OPE then expresses how the action of one operator near another decomposes into the basis of states, giving us algebraic control that was absent in the purely path-integral description.

To set this up, we must specify a notion of "time," since operators evolve with respect to it. In Euclidean space, the natural choice is to take the radial coordinate as time, leading to radial quantization. States are then defined on concentric circles, time evolution is dilation, and contour integrals of the stress tensor implement the Virasoro algebra. Within this framework, commutators appear as contour manipulations, and the OPE becomes a universal computational tool, independent of explicit Green's functions.

8.1 Radial quantization

In the operator formalism one must first distinguish a time direction from a space direction. In Minkowski spacetime this choice is canonical, but in Euclidean space it is arbitrary. A Hilbert space by itself is just a complete inner product space; what makes it physically meaningful is the specification of a **time direction**, which singles out a Hamiltonian as the generator of time translations. The Hamiltonian defines a **vacuum state** (its lowest-energy eigenstate) and organizes the remaining states as excitations built on top of it.

This structure underlies any quantum theory: begin with a vacuum, generate excitations, and classify states by the Hamiltonian spectrum. In two-dimensional conformal field theory we can choose a different notion of "time": the **radial direction** from the origin. In order to make this choice more natural from a Minkowski space point of view (in particular in the context of string theory), we may initially define our theory on an infinite space-time cylinder, with time t going from $-\infty$ to $+\infty$ along the "flat" direction of the cylinder, and space being compactified with x going from 0 to L, and the point $(0,t)$ and (L,t) being identified. If we continue to Euclidian space, the cylinder is described by a single complex coordinate

$$w = t + ix, \qquad x \sim x + L,$$

with Hamiltonian $H = -i\partial_t$. We can then consider the conformal map of this cylinder to the plane via

$$z = e^{\frac{2\pi}{L} w}.$$

Here $t \to -\infty$ maps to $z = 0$, $t \to +\infty$ to $z = \infty$, and the compact x direction gives angular periodicity. Under this mapping, translations in t become dilatations in z, so the cylinder Hamiltonian is identified with the dilatation operator D on the plane and the equal-time slices on the cylinder become circles on the plane.

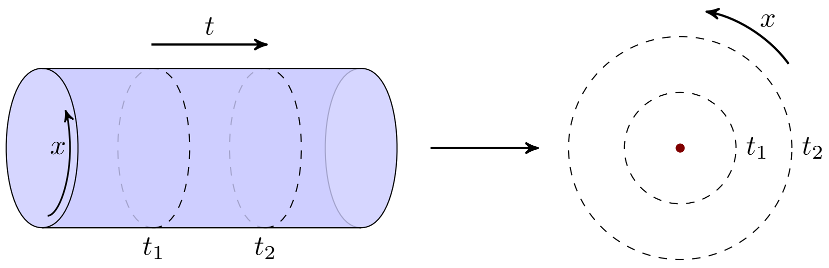


Figure 8.1: Conformal map from the cylinder to the complex plane.

Since periodicity in x is reflected in the periodicity of w and $\bar{w}$. All operators on cylinder admit the following expansion:

$$\phi(w, \bar{w}) = \sum_{m,n \in \mathbb{Z}} \phi_{n,m} e^{-mw} e^{-n\bar{w}} \tag{8.1}$$

where $\phi_{n,m}$ do not have any $(w, \bar{w})$ dependence. Consider the transformation of primary field living on cylinder to $z-$plane under the conformal mapping $z = e^w$:

$$\begin{aligned}
\Phi(z, \bar{z}) &= (\partial_z w)^h (\partial_{\bar{z}} \bar{w})^{\bar{h}} \Phi(w, \bar{w}) \\
&= z^{-h} \bar{z}^{-\bar{h}} \sum_{m,n \in \mathbb{Z}} \phi_{n,m} e^{-mw} e^{-n\bar{w}} \\
&= z^{-h} \bar{z}^{-\bar{h}} \sum_{m,n \in \mathbb{Z}} \phi_{m,m} z^{-n} \bar{z}^{-m} \\
&= \sum_{m,n} \phi_{n,m} z^{-n-h} \bar{z}^{-m-\bar{h}}
\end{aligned}$$

In free-field theories the vacuum is annihilated by positive-frequency modes; in interacting theories we assume asymptotic fields behave freely, e.g.

$$\phi_{\text{in}} \propto \lim_{t \to -\infty} \phi(x, t).$$

In radial quantization, this translates into

$$|\phi_{\text{in}}\rangle = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z}) |0\rangle .$$

Expanding the operator in modes, we have

$$|\phi_{\text{in}}\rangle = \lim_{z, \bar{z} \to 0} \sum_{m,n} \phi_{n,m} \, z^{-n-h} \, \bar{z}^{-m-\bar{h}} \, |0\rangle .$$

For this expression to be non-singular as $z \to 0$, the modes with negative powers of z and $\bar{z}$ must annihilate the vacuum. This requires

$$n + h > 0 \quad \Rightarrow \quad n > -h, \qquad m + \bar{h} > 0 \quad \Rightarrow \quad m > -\bar{h},$$

so that

$$\phi_{n,m} |0\rangle = 0 \qquad \text{for } n > -h \text{ and } m > -\bar{h}.$$

Thus, only the non-negative powers contribute. For $n + h < 0$ and $m + \bar{h} < 0$, we have

$$\lim_{z, \bar{z} \to 0} z^{-n-h} \bar{z}^{-m-\bar{h}} |0\rangle = 0$$

The only non-trivial contribution comes from the modes with

$$n + h = 0, \qquad m + \bar{h} = 0,$$

yielding

$$|\phi_{\text{in}}\rangle = \phi_{-h,-\bar{h}} |0\rangle .$$

Once the Hilbert space is tied to a vacuum and a Hamiltonian (here, the dilatation operator), operators can be classified by scaling dimensions, excitations arranged systematically, and operator product expansions formulated as exact operator identities—rather than being inferred indirectly from correlation functions.

The Hermitian product

On this Hilbert space we must also define a bilinear product, which we do indirectly by defining an asymptotic "out" state, together with the action of Hermitian conjugation on conformal fields.

$$\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^{\dagger} = \left(\lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle\right)^{\dagger}$$

In Minkowski space, Hermitian conjugation does not affect the space-time coordinates. Things are different in Euclidian space, since the Euclidian time $\tau = it$ must be reversed ($\tau \to -\tau$) upon Hermitian conjugation if t is to be left unchanged. In radial quantization this corresponds to the mapping:

$$\begin{aligned} w \equiv e^{\tau+ix} &\to e^{-\tau+ix} \\ &= e^{-(\tau-ix)} = \frac{1}{\overline{e^{\tau+ix}}} = \frac{1}{\bar{z}} \end{aligned}$$

Since ϕ is a primary field,

$$\begin{aligned} \phi(z,\bar{z}) &= \left(\frac{\partial w}{\partial z}\right)^{h}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \phi(w,\bar{w}) \\ &= \left(-\frac{1}{z^2}\right)^{h}\left(-\frac{1}{\bar{z}^2}\right)^{\bar{h}} \phi(w,\bar{w}) \\ &= (-1)^{h}(w^{2h})(-1)^{-\bar{h}}(\bar{w}^{2\bar{h}})\phi(w,\bar{w}) \\ &= (-1)^{h+\bar{h}} w^{2h}\bar{w}^{2\bar{h}}\phi(w,\bar{w}) \\ \phi(w,\bar{w}) &= (-1)^{h+\bar{h}} w^{-2h}\bar{w}^{-2\bar{h}}\phi(z,\bar{z}) \end{aligned}$$

Since $|\phi_{\text{in}}\rangle$ were defined using $\lim\limits_{z\to 0}$, so $\lim\limits_{w\to 0}$ could be used to define $|\phi_{\text{out}}\rangle$ state:

$$\langle\phi_{\text{out}}| = \lim_{z,\bar{z}\to\infty} \langle 0|\,\phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) == \lim_{w,\bar{w}\to 0}(-1)^{h+\bar{h}} w^{-2h}\bar{w}^{-2\bar{h}} \langle 0|\,\phi\left(\frac{1}{w},\frac{1}{\bar{w}}\right)$$

relabelling $w = z$,

$$\langle\phi_{\text{out}}| = \lim_{z,\bar{z}\to 0}(-1)^{h+\bar{h}} z^{-2h}\bar{z}^{-2\bar{h}} \langle 0|\,\phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right)$$

The factor $(-1)^{h+\bar{h}}$ is coming due to spin, if we ignore that:

$$\langle\phi_{\text{out}}| = \lim_{z,\bar{z}\to 0} z^{-2h}\bar{z}^{-2\bar{h}} \langle 0|\,\phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) = \lim_{z,\bar{z}\to 0} \langle 0|\,\phi(z,\bar{z})^{\dagger}$$

Hence, we conclude

$$\phi(z,\bar{z})^{\dagger} = z^{-2h}\bar{z}^{-2\bar{h}}\phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right) \tag{8.2}$$

Or we could alternatively do Fourier expansion on the radial plane, the adjoint property then reads

$$\begin{aligned} \Phi(z,\bar{z})^{\dagger} &= \bar{z}^{-2h} z^{-2\bar{h}} \Phi(1/\bar{z}, 1/z) \\ &= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n} \phi_{n,m} \left(\frac{1}{\bar{z}}\right)^{-n-h}\left(\frac{1}{z}\right)^{-m-\bar{h}} \\ &= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n} \phi_{n,m} (\bar{z})^{n+h}(z)^{m+\bar{h}} \\ &= \sum_{m,n} \phi_{n,m} z^{m-\bar{h}}\bar{z}^{n-h} \end{aligned}$$

$$\sum_{m,n}\phi^{\dagger}_{n,m}\overline{(z^{-n-h}\overline{z}^{-m-\bar{h}})} = \sum_{m,n}\phi_{-n,-m}z^{-m-\bar{h}}\overline{z}^{-n-h} \quad\Longrightarrow\quad \boxed{\phi^{\dagger}_{n,m} = \phi_{-n,-m}}$$

The inner product is well defined as well:

$$\begin{aligned}
\langle\phi_{\text{out}}|\phi_{\text{in}}\rangle &= \lim_{z,\overline{z},w,\overline{w}\to 0} \overline{z}^{-2h}z^{-2\overline{h}}\langle 0|\,\phi(1/\overline{z},1/z)\phi(w,\overline{w})\,|0\rangle \\
&= \lim_{\xi,\overline{\xi}\to\infty} \overline{\xi}^{2h}\xi^{2\overline{h}}\langle 0|\,\phi(\overline{\xi},\xi)\phi(0,0)\,|0\rangle \\
&= \lim_{\xi,\overline{\xi}\to\infty} \overline{\xi}^{2h}\xi^{2\overline{h}}\frac{C_{12}}{\overline{\xi}^{2h}\xi^{2\overline{h}}} = C_{12}
\end{aligned}$$

Unless the prefactors in (8.2) were missing the limit would have been ill defined.

8.1.1 Radial Ordering

Within radial quantization, the time ordering that appears in the definition of correlation functions becomes a radial ordering,

$$\mathcal{R}(\phi_1(z)\phi_2(w)) = \begin{cases}\phi_1(z)\phi_2(w) & |z| > |w|\,, \\ \phi_2(w)\phi_1(z) & |z| < |w|\,.\end{cases} \tag{8.3}$$

As usual, we will always omit the radial-ordered operator in the correlation function as well as in the OPE expansion. One consequence after specifying time direction is that we can relate OPE to commutation relations. For this, let us consider the contour integral around w for two holomorphic fields $a(z)$ and $b(w)$. If the contour is not radially ordered over the whole path, we can decompose it into contributions that are radially ordered:

$$\oint_w dz\, a(z)b(w) \;=\; \underbrace{\oint_{|z|>|w|} dz\, a(z)b(w)}_{\text{radial ordering: } z \text{ outside } w} \;-\; \underbrace{\oint_{|z|<|w|} dz\, b(w)a(z)}_{\text{radial ordering: } z \text{ inside } w} \;=\; [A,\, b(w)]\,, \tag{8.4}$$

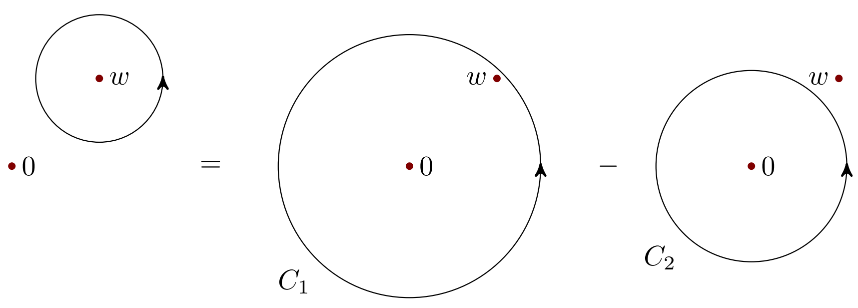
where the operator A is the contour integral of $a(z)$ at a fixed time

$$A = \oint \mathrm{d}z\, a(z)\,. \tag{8.5}$$

Here we take the contours C_1 and C_2 at fixed-radius $|w|+\epsilon$ and $|w|-\epsilon$ with a small positive number ϵ as illustrated in Figure 8.2 . Then, with $B = \oint \mathrm{d}z\, b(z)$, we can generalize the relation (**??**) to

$$[A,B] = \oint_0 \mathrm{d}w[A,b(w)] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z a(z)b(w)\,. \tag{8.6}$$

Figure 8.2: Subtraction of contours



8.2 Virasoro Algebra

8.2.1 Conformal Generators

The ward identity

$$\delta_\epsilon\langle\phi_1\ldots\phi_n\rangle = \int_M d^2x\partial_\mu\langle T^{\mu\nu}\epsilon_\nu\phi_1\ldots\phi_n\rangle$$

could be expressed in complex coordinates using (7.7):

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\left<\phi_1\dots\phi_n\right> &= \frac{i}{2}\int_C\left[-dz\left<T^{\bar{z}\bar{z}}\epsilon_{\bar{z}}\phi_1\dots\phi_n\right>+d\bar{z}\left<T^{zz}\epsilon_z\phi_1\dots\phi_n\right>\right]\\ &= -\frac{1}{2\pi i}\int_C dz\left<T(z)\epsilon\phi_1\dots\phi_n\right>+\frac{1}{2\pi i}\int_C d\bar{z}\left<\bar{T}(\bar{z})\bar{\epsilon}\phi_1\dots\phi_n\right>\end{aligned}$$

where we used,

$$\begin{aligned}T &= -2\pi T_{zz} = -2\pi g_{z\mu}g_{z\nu}T^{\mu\nu} = -2\pi\frac{1}{4}T^{\bar{z}\bar{z}} = -\frac{\pi}{2}T^{\bar{z}\bar{z}}\\ \bar{T} &= -2\pi T_{\bar{z}\bar{z}} = -2\pi g_{\bar{z}\mu}g_{\bar{z}\nu}T^{\mu\nu} = -2\pi\frac{1}{4}T^{zz} = -\frac{\pi}{2}T^{zz}\end{aligned}$$

and

$$\begin{aligned}\epsilon &= \epsilon^z = g^{z\mu}\epsilon_\mu = 2\epsilon_{\bar{z}}\\ \bar{\epsilon} &= \epsilon^{\bar{z}} = g^{\bar{z}\mu}\epsilon_\mu = 2\epsilon_z\end{aligned}$$

If we apply, (8.4) and (8.6) to the conformal ward identity. Let $\epsilon(z)$ be holomorphic component of an infinitesimal conformal change of coordinates. We then define the conformal charge

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z) \tag{8.7}$$

with the help of (8.4), the conformal ward identity translates into

$$\delta_\epsilon\phi = -[Q_\epsilon,\phi]$$

which means that the operator Q_ϵ is the generator of conformal transformation. We may expand energy-momentum tensor according to (8.1):

$$\begin{aligned}T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2}L_n & L_n &= \frac{1}{2\pi i}\oint dz z^{n+1}T(z)\\ \bar{T}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n & \bar{L}_n &= \frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+1}\bar{T}(\bar{z})\end{aligned}$$

we may also expand the infinitesimal conformal change $\epsilon(z)$ as:

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n$$

Then the expression, (8.7) becomes:

$$\begin{aligned}Q_\epsilon &= \frac{1}{2\pi i}\oint dz\left(\sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n\right)T(z)\\ &= \sum_{n\in\mathbb{Z}}\epsilon_n\frac{1}{2\pi i}\oint dz z^{n+1}T(z)\\ &= \sum_{n\in\mathbb{Z}}\epsilon_n L_n\end{aligned}$$

The mode operators L_n and L_m of the energy-momentum tensor are the generators of the local conformal transformations on the Hilbert space, exactly like l_n and l_m of Witt algebra. The next part is to find the algebra obeyed by L_n and L_m which we described in previous chapter as the central extension of Witt algebra. We will now prove that here:

$$\begin{aligned}[L_m,L_n] &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}z^{m+1}w^{n+1}[T(z),T(w)]\\ &= \oint\frac{dw}{2\pi i}w^{n+1}\oint_{C(w)}\frac{dz}{2\pi i}z^{m+1}\mathcal{R}\big(T(z)T(w)\big)\\ &= \oint\frac{dw}{2\pi i}w^{n+1}\oint_{C(w)}\frac{dz}{2\pi i}z^{m+1}\left[\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_w T(w)}{z-w}\right]\end{aligned}$$

$$
\begin{aligned}
&= \oint \frac{dw}{2\pi i}\, w^{n+1} \left[(m+1)m(m-1)w^{m-2}\frac{c}{2\cdot 3!} + 2(m+1)w^m T(w) + w^{m+1}\partial_w T(w) \right] \\
&= \oint \frac{dw}{2\pi i} \left[\frac{c}{12}(m^3-m)\, w^{m+n-1} + 2(m+1)w^{m+n+1}T(w) + w^{m+n+2}\partial_w T(w) \right] \\
&= \oint \frac{dw}{2\pi i} \left[\frac{c}{12}(m^3-m)\, w^{m+n-1} + 2(m+1)w^{m+n+1}T(w) + \partial_w \{w^{m+n+2}T(w)\} - \partial_w w^{m+n+2}T(w) \right] \\
&= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + 2(m+1)L_{m+n} + 0 - \oint \frac{dw}{2\pi i}\,(m+n+2)T(w)w^{m+n+1} \\
&= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + 2(m+1)L_{m+n} - (m+n+2)L_{m+n} \\
&= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}.
\end{aligned}
$$

Collectively,

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n} \\
[L_m, \bar{L}_n] &= 0 \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}
\end{aligned}
$$