TEQS - Sheet 1

Fiene Weber
50353222

H.1] a)

$$e^{(A+B)\varepsilon} = \sum_{k=0}^{\infty} \frac{(A+B)^k \varepsilon^k}{k!} = \mathbb{1} + (A+B)\varepsilon + \frac{(A+B)^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)$$

$$= \mathbb{1} + A\varepsilon + B\varepsilon + \frac{1}{2}(A^2 + AB + BA + B^2)\varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

$$e^{A\varepsilon} e^{B\varepsilon} e^{-\frac{1}{2}[A,B]\varepsilon^2 + \mathcal{O}(\varepsilon^3)} = \left(\mathbb{1} + A\varepsilon + \frac{A^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)\right)\left(\mathbb{1} + B\varepsilon + \frac{B^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)\right)$$

$$\cdot\left(\mathbb{1} - \frac{1}{2}[A,B]\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)$$

$$= \mathbb{1} - \frac{1}{2}[A,B]\varepsilon^2 + B\varepsilon + A\varepsilon + AB\varepsilon^2 + \frac{A^2\varepsilon^2}{2} + \frac{B^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)$$

$$= \mathbb{1} + A\varepsilon + B\varepsilon + \frac{AB\varepsilon^2}{2} + \frac{BA\varepsilon^2}{2} + \frac{A^2\varepsilon^2}{2} + \frac{B^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)$$

$$\overset{!}{=} \mathbb{1} + A\varepsilon + B\varepsilon + \frac{1}{2}(A^2 + AB + BA + B^2)\varepsilon^2 + \mathcal{O}(\varepsilon^3) \quad \square$$

b)

$$e^{A\varepsilon} e^{B\varepsilon} = \mathbb{1} + B\varepsilon + \frac{B^2\varepsilon^2}{2} + A\varepsilon + AB\varepsilon^2 + \frac{A^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)$$

$$e^{B\varepsilon} e^{A\varepsilon} e^{[A,B]\varepsilon^2 + \mathcal{O}(\varepsilon^3)} = \left(\mathbb{1} + B\varepsilon + \frac{B^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)\right)$$

$$\cdot\left(\mathbb{1} + A\varepsilon + \frac{A^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)\right)$$

$$\cdot\left(\mathbb{1} + [A,B]\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)$$

$$= \left(\mathbb{1} + A\varepsilon + \frac{A^2\varepsilon^2}{2} + B\varepsilon + BA\varepsilon^2 + \frac{B^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)\right)$$

$$\cdot\left(\mathbb{1} + AB\varepsilon^2 - BA\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)$$

$$= \mathbb{1} + AB\varepsilon^2 - \cancel{BA\varepsilon^2} + A\varepsilon + \frac{A^2\varepsilon^2}{2} + B\varepsilon + \cancel{BA\varepsilon^2} + \frac{B^2\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3)$$

$$= \mathbb{1} + B\varepsilon + A\varepsilon + \frac{B^2\varepsilon^2}{2} + \frac{A^2\varepsilon^2}{2} + AB\varepsilon^2 + \mathcal{O}(\varepsilon^3) \quad \square$$

H.1] c) $\left(e^{\frac{1}{n}A}e^{\frac{1}{n}B}\right)^n \overset{a)}{=} \left(e^{(A+B)\frac{1}{n}}\, e^{\frac{1}{2n^2}[A,B]+\mathcal{O}(\frac{1}{n^3})}\right)^n$

$$= e^{A+B}\, e^{\frac{1}{2n}[A,B]+\mathcal{O}(\frac{1}{n^2})} \xrightarrow[n\to\infty]{} e^{A+B} \qquad \square$$

d) $(b) \Leftrightarrow \left(e^{A}e^{B}\right)^{\varepsilon} = \left(e^{B}e^{A}e^{[A,B]\varepsilon}\right)^{\varepsilon} \quad \Big|\sqrt[\varepsilon]{\ }$

$$e^{A}e^{B} = e^{B}e^{A}e^{C\mathbb{1}} \qquad \text{FOR } \varepsilon \to 1$$

e) $e^{A}Be^{-A} \overset{A\mapsto\varepsilon A}{\longmapsto} e^{\varepsilon A}Be^{-\varepsilon A} \underset{\substack{\text{AROUND}\\ \varepsilon=0}}{\overset{(TS)}{=}} B + \varepsilon\cdot\left(Ae^{\varepsilon A}Be^{-\varepsilon A} - e^{\varepsilon A}BAe^{-\varepsilon A}\right)\Big|_{\varepsilon=0}$

$$+\varepsilon^2\frac{1}{2}\left(A^2e^{\varepsilon A}Be^{-\varepsilon A} + e^{\varepsilon A}BA^2e^{-\varepsilon A} - Ae^{\varepsilon A}BAe^{-\varepsilon A} - Ae^{\varepsilon A}BAe^{-\varepsilon A}\right)\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^3)$$

$$= B + \varepsilon(AB - BA)$$

$$+\varepsilon^2\frac{1}{2}\underbrace{(A^2B + BA^2 - ABA - ABA)}_{\frac{1}{2}[\varepsilon A,[\varepsilon A,B]]} + \mathcal{O}(\varepsilon^3)$$

REPLACE $\varepsilon A \mapsto A$

$$= B + [A,B] + \frac{1}{2}[A,[A,B]] + \dots$$

$$\overset{!}{=} \sum_{k=0}^{\infty}\frac{1}{k!}[A,B]_k \qquad \square$$

H.2] $H = -J\sum_{\langle i,j\rangle}(c_i^{+}c_j + c_j^{+}c_i)$

a) $\tilde{H}$ MATRIX $\in \mathbb{C}^{3\times3}$ WITH ENTRIES $H_{ij} = \langle i|H|j\rangle$

FOR $i\neq j\neq k\neq i$: $H_{ij} = \langle i|-J(c_i^{+}c_j + c_j^{+}c_i + c_j^{+}c_k + c_k^{+}c_j + c_k^{+}c_i + c_i^{+}c_k)|j\rangle$

$$\overset{!}{=} \langle i|-J(|i\rangle + 0 + 0 + |k\rangle + 0 + 0)$$

$$\overset{!}{=} (\underbrace{\langle i|i\rangle}_{1} + \underbrace{\langle i|k\rangle}_{0})\cdot(-J) = -J$$

$$H_{ii} = \langle i|-J\sum_{\langle i,j\rangle}(c_i^{+}c_j + c_j^{+}c_i)|i\rangle = -J\cdot(\langle i|(0 + |j\rangle + 0 + 0 + |k\rangle + 0) \overset{!}{=} 0$$

$$\to \tilde{H} = \begin{pmatrix} 0 & -J & -J \\ -J & 0 & -J \\ -J & -J & 0 \end{pmatrix}$$

H2) ~~a, form~~ b) $\det(\tilde{H} - \lambda \cdot \mathbb{1}) = 0 \Leftrightarrow 0 = -\lambda(+\lambda^2 - J^2) + J(-J^2)$

$$-J(-J^2)$$

$$= -\lambda^3 + \lambda J^2$$

$$\Rightarrow \lambda_1 = 0$$

$$0 = -\lambda^2 + J^2 \Rightarrow \lambda_{2,3} = \pm J$$

$$\Rightarrow \tilde{H}_{diag} = \begin{pmatrix} J & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -J \end{pmatrix}$$

c) $(\tilde{H} - \lambda \mathbb{1}) \cdot \vec{v} = 0$

$$\lambda_1 \Rightarrow \begin{pmatrix} J & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -J \end{pmatrix} \vec{v}_1 = \begin{pmatrix} J v_1^1 \\ 0 \\ -J v_3^1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \text{ARBITRARY } v_2^1, \; v_1^1 = v_3^1 = 0$$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ v_2^1 \\ 0 \end{pmatrix}$$

$$\lambda_2 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -J & 0 \\ 0 & 0 & -2J \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 0 \\ -J v_2^2 \\ -2J v_3^2 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} v_1^2 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 \Rightarrow \begin{pmatrix} 2J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_3 = \begin{pmatrix} 2J v_1^3 \\ J v_2^3 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ v_3^3 \end{pmatrix}$$

$$\Rightarrow \tilde{H}_{diag} = \left(J\vec{v}_2 \;\; 0 \cdot \vec{v}_1 \;\; -J\vec{v}_3 \right)$$

d) $U(t) = e^{-iHt} = \mathbb{1} - itH - \frac{t^2 H^2}{2} + \dots$

$H \stackrel{!}{=} \tilde{H}$

$$= \left(\vec{v}_2 \middle| \vec{v}_1 \middle| \vec{v}_3 \right) - it \left(J\vec{v}_2 \middle| \vec{0} \middle| -J\vec{v}_3 \right) - \frac{t^2}{2} \left(J^2 \vec{v}_2 \middle| \vec{0} \middle| J^2 \vec{v}_3 \right)$$

$$= \left(\vec{v}_2 - itJ\vec{v}_2 - \frac{t^2}{2}\vec{v}_2 + \dots \middle| \vec{v}_1 \middle| \vec{v}_3 \left(1 + itJ - \frac{t^2}{2}J^2 + \dots\right) \right)$$

H.3] a) $L=2$: $\{ |-\tfrac{1}{2},-\tfrac{1}{2}\rangle, |-\tfrac{1}{2},\tfrac{1}{2}\rangle, |\tfrac{1}{2},-\tfrac{1}{2}\rangle, |\tfrac{1}{2},\tfrac{1}{2}\rangle \}$

$L=3$: $\{ |-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle, |-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\rangle, |-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\rangle,$
$|\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle, |-\tfrac{1}{2},+\tfrac{1}{2},\tfrac{1}{2}\rangle, |\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\rangle,$
$|+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2}\rangle, |\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\rangle \}$

b) $S_j^i = \underbrace{\mathbb{1}\otimes\dots\otimes S_j^i\otimes\dots\otimes\mathbb{1}}_{L-1}$, $i\in\{x,y,z\}$, $j\in\{0,\cancel{1},1,\dots,L-1\}$

(ACTS ON j'th PARTICLE)

SPIN UP := $\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{2}$ SPIN DOWN := $\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}$ (SPINORS

DENOTE BASIS STATES AS $|m^s_1, m^s_2\rangle$ FOR $L=2$, $|m^s_1, m^s_2, m^s_3\rangle$ FOR $L=3$, RESPECTIVELY

$L=2$: $S_0^x = S_0^x\otimes\mathbb{1}_2 = \frac{1}{2}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$

$S_1^x = \mathbb{1}_2\otimes S_1^x = \frac{1}{2}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$ ($S_0^x = S_2^x$ FOR PERIODIC BOUNDARY CONDITIONS)

AND SIMILAR FOR S_j^y & S_j^z

$H = J\,(S_0^x S_1^x + S_0^y S_1^y + S_0^z S_1^z + S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z)$

REWRITE BASIS: $\left\{ \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix} \right\}$

$\Rightarrow \langle +\tfrac{1}{2},+\tfrac{1}{2}|H|+\tfrac{1}{2},+\tfrac{1}{2}\rangle = H_{11} = \langle +\tfrac{1}{2},+\tfrac{1}{2}|\,J\Big[S_0^x\begin{pmatrix}1\\0\end{pmatrix}\otimes S_1^x\begin{pmatrix}1\\0\end{pmatrix}$
$+ S_0^y\begin{pmatrix}1\\0\end{pmatrix} S_1^y\begin{pmatrix}1\\0\end{pmatrix} + S_0^z\begin{pmatrix}1\\0\end{pmatrix} S_1^z\begin{pmatrix}1\\0\end{pmatrix} + \underbrace{S_1^x\begin{pmatrix}1\\0\end{pmatrix} S_0^x\begin{pmatrix}1\\0\end{pmatrix} + S_1^y\begin{pmatrix}1\\0\end{pmatrix} S_2^y\begin{pmatrix}1\\0\end{pmatrix} + S_1^z\begin{pmatrix}1\\0\end{pmatrix} S_0^z\begin{pmatrix}1\\0\end{pmatrix}}_{\text{ONLY IF PER. BOUND. COND.}} \Big]$

$$H_{11} = \underbrace{\left(\cancel{\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}} \right)}_{\langle \frac{1}{2}, \frac{1}{2} |} \frac{J}{4} \Bigg[\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\i\end{pmatrix} \otimes \begin{pmatrix}0\\i\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

$$\underbrace{+ \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\i\end{pmatrix} \otimes \begin{pmatrix}0\\i\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}}_{\text{ONLY IF PER. BOUND. COND.}} \Bigg]$$

$$= \frac{J}{4} \Big[1 + \underbrace{1}_{\text{ONLY IF PER. BOUND. COND.}} \Big]$$

$$H_{44} = \langle -\tfrac{1}{2}, -\tfrac{1}{2} | H | -\tfrac{1}{2}, -\tfrac{1}{2} \rangle = \langle -\tfrac{1}{2}, -\tfrac{1}{2} | \frac{J}{4} \Bigg[\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-i\\0\end{pmatrix} \otimes \begin{pmatrix}-i\\0\end{pmatrix} + \overbrace{\begin{pmatrix}0\\-1\end{pmatrix} \otimes \begin{pmatrix}0\\-1\end{pmatrix}}^{(-1)^2 \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}}$$

$$\underbrace{+ \text{———} \;\; '' \;\; \text{———}}_{\text{IFF PER. B. C.}} \Bigg]$$

$$= \frac{J}{4} \Big[+1 + \underbrace{1}_{\text{IFF P.B.C.}} \Big]$$

$$H_{22} = \langle \tfrac{1}{2}, -\tfrac{1}{2} | H | \tfrac{1}{2}, -\tfrac{1}{2} \rangle = \langle \tfrac{1}{2}, -\tfrac{1}{2} | \frac{J}{4} \Big[S_0^x \begin{pmatrix}1\\0\end{pmatrix} S_1^x \begin{pmatrix}0\\1\end{pmatrix} + S_0^y \begin{pmatrix}1\\0\end{pmatrix} \otimes S_1^y \begin{pmatrix}0\\1\end{pmatrix} + S_0^z \begin{pmatrix}1\\0\end{pmatrix} S_1^z \begin{pmatrix}0\\1\end{pmatrix}$$

$$\underbrace{+ S_1^x \begin{pmatrix}0\\1\end{pmatrix} S_0^x \begin{pmatrix}1\\0\end{pmatrix} + S_1^y \begin{pmatrix}0\\1\end{pmatrix} S_0^y \begin{pmatrix}1\\0\end{pmatrix} + S_1^z \begin{pmatrix}0\\1\end{pmatrix} S_0^z \begin{pmatrix}1\\0\end{pmatrix}}_{\text{IFF P.B.C.}} \Big]$$

$$= \langle \tfrac{1}{2}, -\tfrac{1}{2} | \frac{J}{4} \Bigg[\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-i\\0\end{pmatrix} \otimes \begin{pmatrix}0\\i\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\-1\end{pmatrix}$$

$$\underbrace{+ \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-i\\0\end{pmatrix} \otimes \begin{pmatrix}0\\i\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}}_{\text{IFF P.B.C}} \Bigg]$$

$$= \frac{J}{4} \Big[-1 + \underbrace{1}_{\text{IFF P.B.C.}} \Big]$$

$$H_{33} = \langle -\tfrac{1}{2},\tfrac{1}{2} | H | -\tfrac{1}{2},\tfrac{1}{2} \rangle = \langle -\tfrac{1}{2},\tfrac{1}{2} | \tfrac{J}{4} \Big[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\i\end{pmatrix}\otimes\begin{pmatrix}-i\\0\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}$$

$$+ \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\i\end{pmatrix}\otimes\begin{pmatrix}-i\\0\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\-1\end{pmatrix} \Big]$$

$$= \frac{J}{4}\Big[-1 \underbrace{+1}_{\text{IFF P.B.C.}} \Big]$$

$$\cancel{\Rightarrow H = \frac{J}{4}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}} \quad \text{~~WITH OPEN BOUND. COND.~~}$$

$$\cancel{H = \frac{J}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} \quad \text{~~WITH PER. BOUND. COND.~~}$$

$$\cancel{L=3: \quad H = J\left(S_0^x S_1^x + S_0^y S_1^y + S_0^z S_1^z + S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z + S_2^x S_3^x + S_2^y S_3^y + S_2^z S_3^z\right)}$$

$$\cancel{H_{max} = \langle \tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2} | H | \tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2} \rangle}$$

$$H_{12} = \langle +- | H | ++ \rangle = \frac{J}{4}\Big[0+0+0+0+0+0\Big] = 0$$

$$H_{13} = \langle -+ | H | ++ \rangle = \frac{J}{4}\Big[0+0+0+0 \ldots\Big] = 0$$

$$H_{14} = \langle -- | H | ++ \rangle = \frac{J}{4}\Big[1 - 1 + 0 + \underbrace{1 - 1 + 0}_{\text{IFF P.B.C.}}\Big] = 0$$

$$H_{42} = H_{43} = 0 = H_{41}$$

$$H_{21} = \langle ++ | H | +- \rangle = \frac{J}{4}\Big[0+0+0+0+0+0\Big] = 0$$

$$= H_{31} = H_{24} = H_{34}$$

$$H_{23} = \langle -+ | H | +- \rangle = \frac{J}{4}\Big[1+1+0+\underbrace{0+0-1}_{\text{IFF P.B.C.}}\Big] = \begin{cases} 2, & \text{OPEN BOUND.} \\ 1, & \text{PER. BOUND.} \end{cases}$$

$$H_{32} = \langle +- | H | -+ \rangle = \frac{J}{4}\Big[1+1+0+0+0-1\Big] = \begin{cases} 2, & \text{OP. BOUND.} \\ 1, & \text{PER. BOUND.} \end{cases}$$

$$\Rightarrow \quad H_2^{open} = \frac{J}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ OPEN BOUNDARIES

$$H_2^{per} = \frac{J}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ PER. BOUND. COND.

FOR $L=3$ ONLY OPEN BOUNDARIES:

$$H_{op} = J\left(S_0^x S_1^x + S_0^y S_1^y + S_0^z S_1^z + S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z \right)$$

$$\cancel{+ S_2^x S_3^x + S_2^y S_3^y + S_2^z S_3^z}$$

DOESN'T CONTRIBUTE

$$\rightarrow \tilde{H} = H_3: \quad \text{Matrix} \in \mathbb{R}^{8\times 8}$$

$$\begin{pmatrix} \langle +++|H|+++\rangle & \langle ++-|H|+++\rangle & \langle +-+|H|+++\rangle & \cdots & \langle ---|H|+++\rangle \\ \langle +++|H|++-\rangle & \langle ++-|H|++-\rangle & & & \\ \langle +++|H|+-+\rangle & & \langle +-+|H|+-+\rangle & & \\ \langle +++|H|-++\rangle & & & \langle -++|H|-++\rangle & \\ -''-|+--\rangle & & & & \ddots \\ -''-|--+\rangle & & & & \\ -''-|-+-\rangle & & & & \\ \langle +++|H|---\rangle & & & & \langle ---|H|---\rangle \end{pmatrix}$$

$$= \frac{J}{4}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & -2 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}$$