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Exercise 3

H1)

a) $$D_T^2 z = \{\{z,T\},T\} = \frac{\partial}{\partial q}\left(\frac{\partial z}{\partial q}\frac{\partial T}{\partial p}\right)\frac{\partial T}{\partial p} - \underbrace{\overbrace{\frac{\partial T}{\partial q}}^{=0}\frac{\partial}{\partial p}\left(\frac{\partial z}{\partial q}\frac{\partial T}{\partial p}\right)}_{=0}$$

$$= \underbrace{\frac{\partial z^2}{\partial^2 q}}_{=0}\left(\frac{\partial T}{\partial p}\right)^2 + \frac{\partial z}{\partial q}\underbrace{\frac{\partial^2 T}{\partial q \partial p}}_{=0}\frac{\partial T}{\partial p} = 0$$

$$\boxed{\begin{aligned}\{z,T\} &= \frac{\partial z}{\partial q}\frac{\partial T}{\partial p} - \underbrace{\frac{\partial T}{\partial q}}_{=0}\frac{\partial z}{\partial p}\\ &= \frac{\partial z}{\partial q}\frac{\partial T}{\partial p}\end{aligned}}$$

$$\boxed{z = \begin{pmatrix} q \\ p \end{pmatrix} \Rightarrow \begin{aligned}\frac{\partial^2 z}{\partial q^2} &= \vec{0}\\ \frac{\partial^2 z}{\partial p^2} &= \vec{0}\end{aligned}}$$

like for T

$$\boxed{\{z,V\} = -\frac{\partial V}{\partial q}\frac{\partial z}{\partial p}}$$

In the same way:

$$D_V^2 z = \{\{z,V\},V\} = \frac{\partial}{\partial q}\left(-\frac{\partial V}{\partial q}\frac{\partial z}{\partial p}\right)\underbrace{\frac{\partial V}{\partial p}}_{=0} - \frac{\partial V}{\partial q}\frac{\partial}{\partial p}\left(-\frac{\partial V}{\partial q}\frac{\partial z}{\partial p}\right)$$

$$= \frac{\partial V}{\partial q}\left(\underbrace{\frac{\partial^2 V}{\partial p \partial q}}_{=0}\frac{\partial z}{\partial p} + \frac{\partial V}{\partial q}\underbrace{\frac{\partial^2 z}{\partial p^2}}_{=0}\right) = 0$$

b) $$T(p) = \frac{p^2}{2m}$$

commutator acting on a function

$$[D_V,[D_V,[D_V,D_T]]]z = D_V[D_V,[D_V,D_T]]z - [D_V,[D_V,D_T]]D_V z$$

$$= D_V^2[D_V,D_T]z - D_V[D_V,D_T]D_V z - D_V[D_V,D_T]D_V z$$

$$+ [D_V,D_T]\underbrace{D_V^2 z}_{=0}$$

$$= D_V^3 D_T z - D_V^2 D_T D_V z - 2\left(D_V^2 D_T D_V z - D_V D_T \underbrace{D_V^2 z}_{=0}\right)$$

$$= \underbrace{D_V^3 D_T z}_{=0} - \underbrace{3 D_V^2 D_T D_V z}_{=0} = 0$$

Explaination ↓

- $D_T z = \{z, T\} = \frac{\partial z}{\partial q}\frac{\partial T}{\partial p} = \begin{pmatrix} p/m \\ 0 \end{pmatrix} \Rightarrow D_V^3 \begin{pmatrix} p/m \\ 0 \end{pmatrix} = D_V \underbrace{D_V^2 \begin{pmatrix} p/m \\ 0 \end{pmatrix}}_{=0} = 0$, $\frac{\partial^2}{\partial p^2}\begin{pmatrix} p/m \\ 0 \end{pmatrix} = 0$

- $D_V z = \{z, V\} = -\frac{\partial V}{\partial q}\frac{\partial z}{\partial p} = -\frac{\partial V}{\partial q}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow D_T D_V z = \left\{ \begin{pmatrix} 0 \\ -\frac{\partial V}{\partial q} \end{pmatrix}, T \right\} = \frac{\partial}{\partial q}\begin{pmatrix} 0 \\ -\frac{\partial V}{\partial q} \end{pmatrix} \frac{\partial T}{\partial p} = \begin{pmatrix} 0 \\ -\frac{\partial^2 V}{\partial q^2}\frac{p}{m} \end{pmatrix}$$

$$\text{and } D_V^2 D_T D_V z = \underbrace{D_V^2 \begin{pmatrix} 0 \\ -\frac{\partial^2 V}{\partial q^2} p/m \end{pmatrix}}_{=0} = 0 \quad , \quad \frac{\partial^2}{\partial p^2}\begin{pmatrix} 0 \\ -\frac{\partial^2 V}{\partial q^2} p/m \end{pmatrix} = 0$$

thus $[D_V, [D_V, [D_V, D_T]]] = 0$

H3)

a) $H(0) = -J \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z \Rightarrow$ two possible ground states $|\Psi_0\rangle = |\uparrow\uparrow\uparrow \dots \uparrow\rangle$, $|\Phi_0\rangle = |\downarrow\downarrow\downarrow \dots \downarrow\rangle$

Eigenstates of σ^z

$\Rightarrow \langle H(0) \rangle = E = -JN$ the ground state energy

b) Linear interpolation $H(t) = (1 - \frac{t}{T}) H(0) + \frac{t}{T} H(T)$

$$= (1 - \frac{t}{T})\left[-J \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z\right] + \frac{t}{T}\left[-J \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - h_f \sum_{i=1}^{N} \sigma_i^x\right]$$

$$= -J \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - \frac{t}{T} h_f \sum_{i=1}^{N} \sigma_i^x \quad , t \in [0, T]$$

H4)

a) $i \frac{d}{dt} |\psi(T)\rangle = H |\psi(\tau)\rangle = -\frac{d}{d\tau} |\psi(\tau)\rangle$ for $t = i\tau$

$$\Rightarrow \boxed{\frac{d}{d\tau} |\psi(\tau)\rangle = -H |\psi(\tau)\rangle}$$

b) $|\psi(\tau)\rangle = e^{-H\tau} \underbrace{|\psi(0)\rangle}_{\text{ground state}} = \sum_n c_n e^{-E_n \tau} |E_n\rangle$; for τ large, E_0 (the smallest eigenvalue) dominates

for $H(h)|E_n\rangle = E_n|E_n\rangle$

c) The exponential function $e^{-E_n \tau}$ decays rapidly $\Rightarrow$ the norm $\langle \psi(\tau) | \psi(\tau) \rangle$ decreases exponentially with higher τ. Periodic renormalizing $\rightarrow$ keeps representation of state accurate