The detunings δ_a, δ_b to each bosonic mode can be chosen as free parameters using the relation

$$\begin{cases} \delta_{(g)} & = (\delta_a - \delta_b)/2 \\ \delta_{(g^s)} & = (\delta_a + \delta_b)/2. \end{cases} \tag{5.16}$$

Finally, considering the pumps $\hat{p}_{(g)}, \hat{p}_{(g^s)}$ in their classical limit allows us to recover Eq. (5.1).

We note that if the detunings are higher than the dissipation rates, the dynamical regime changes, which we want to avoid [61]. Therefore, for all of the simulations in this chapter, we always set the detuning absolute values to be lower than five times the average dissipation rate.

5.1.3 M coupled bosonic modes

The calculations of Section 5.1.2 can be generalized to M coupled bosonic modes $\hat{a}_k$ verifying the commutation relation

$$\left[\hat{a}_j, \hat{a}_k{}^\dagger\right] = \delta^{jk} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}, \tag{5.17}$$

where δ^{jk} is the Kronecker symbol. Three wave mixing allows us to couple the bosonic modes pairwise, with photon conversion at a rate g_{kl} and two-mode squeezing at a rate g^s_{kl} for modes k and l. In the rotating frame, the Hamiltonian of this system writes

$$\boxed{\frac{\hat{H}_{\text{sys}}}{\hbar} = -\sum_{k=1}^{M} \delta_k \hat{a}_k{}^\dagger \hat{a}_k + \sum_{\substack{k,l=1 \\ k<l}}^{M} \left(g_{kl} \hat{a}_k{}^\dagger \hat{a}_l + g^s_{kl} \hat{a}_k{}^\dagger \hat{a}_l{}^\dagger + \text{h.c}\right)} \tag{5.18}$$

where δ_k is the drive detuning of mode k from its resonance frequency. The notation h.c denotes the hermitian conjugate of an expression.

5.2 Driving the bosonic modes with the input-output formalism in the Heisenberg representation

In order to populate the Fock states of the modes, we drive them with nearly resonant drives in the input-output formalism. We will model the evolutions of the operators $\hat{a}_k(t)$ in the Heisenberg picture when the system interacts with a bath of modes, using the quantum Langevin equation. The demonstrations of this section are inspired from

Ref. [158].

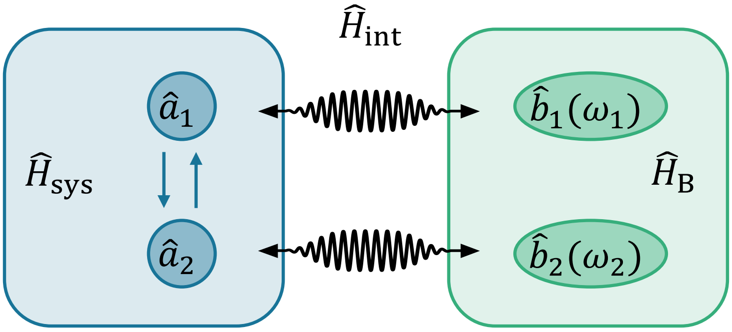

Figure 5.1: Illustration of 2 bosonic modes $\hat{a}_1, \hat{a}_2$ coupled to their respective bath of modes $\hat{b}_1(\omega_1), \hat{b}_2(\omega_2)$. The system and bath are described by their corresponding Hamiltonians $\hat{H}_{\text{sys}}$, $\hat{H}_{\text{B}}$, and their coupling is modeled by the interaction Hamiltonian $\hat{H}_{\text{int}}$.

We suppose a setting illustrated in Fig. 5.1, where each mode in our quantum system is considered to be interacting with a different heat bath k, in the form

$$\begin{aligned} \hat{H} &= \hat{H}_{\text{sys}} + \hat{H}_{\text{B}} + \hat{H}_{\text{int}} \\ \hat{H}_{\text{B}} &= \hbar \sum_{k=1}^{M} \int_{-\infty}^{\infty} d\omega_k \omega_k \hat{b}_k^\dagger(\omega_k) \hat{b}_k(\omega_k) \\ \hat{H}_{\text{int}} &= i\hbar \sum_{k=1}^{M} \int_{-\infty}^{\infty} d\omega_k \gamma_k(\omega_k) \left(\hat{b}_k^\dagger(\omega_k) \hat{c}_k - \hat{c}_k^\dagger \hat{b}_k(\omega_k) \right) \end{aligned} \tag{5.19}$$

where $\hat{H}_{\text{sys}}$ and $\hat{H}_{\text{B}}$ are respectively the system and bath Hamiltonian, and $\hat{b}_k$ are field operators for the bath k of modes. They verify the commutation relation

$$\left[\hat{b}_k(\omega_k), \hat{b}_k^\dagger(\omega_k') \right] = \delta(\omega_k - \omega_k'), \tag{5.20}$$

where $\delta(t)$ is the Dirac delta function [159]. The operator $\hat{c}_k$ acts on system mode k. The bosonic modes are coupled with strengths $\gamma_k(\omega_k)$ to the bath modes through the interaction Hamiltonian $\hat{H}_{\text{int}}$, which is assumed to be linear in $\hat{b}_k(\omega_k)$ and $\hat{b}_k^\dagger(\omega_k)$. This is a general formulation, so the specific Hamiltonian of our bosonic modes $\hat{H}_{\text{sys}}$ will not affect the following calculations.

We move from the Schrödinger picture where quantum states depend on time to the Heisenberg picture, where quantum states are static and the operators depend on time. The Heisenberg picture operators $\hat{O}(t)$ are related to the Schrödinger picture operators $\hat{O}$ through the transformation

$$\hat{O}(t) = \exp\left(\frac{i}{\hbar}\hat{H}t\right) \hat{O} \exp\left(-\frac{i}{\hbar}\hat{H}t\right). \tag{5.21}$$

Evaluating expectation values of the operators in the Heisenberg picture with respect to the initial quantum state $\hat{\rho}(t=0)$ yields the same result as its expectation value in the Schrödinger picture:

$$\left\langle \hat{O}(t) \right\rangle = \mathrm{Tr}\Big(\hat{\rho}(t=0)\hat{O}(t)\Big) = \mathrm{Tr}\Big(\hat{\rho}(t)\hat{O}\Big). \tag{5.22}$$

As the operators now depend on time, they evolve according to the Heisenberg equations of motion

$$\frac{d\hat{O}}{dt} = -\frac{i}{\hbar}\Big[\hat{O}(t), \hat{H}(t)\Big], \tag{5.23}$$

where the Hamiltonian is unchanged by the Heisenberg picture.

We develop the equations of motion in the Heisenberg picture for the bath of field operators $\hat{b}_k(\omega_k)$ and an arbitrary system operator $\hat{O}$ using Eq. (5.23), and find

$$\frac{d\hat{b}_k(\omega_k)}{dt} = -i\omega_k \hat{b}_k(\omega_k) + \gamma_k(\omega_k)\hat{c}_k \tag{5.24}$$

$$\frac{d\hat{O}}{dt} = -\frac{i}{\hbar}\Big[\hat{O}, \hat{H}_{\text{sys}}\Big] + \sum_{k=1}^{M}\int_{-\infty}^{\infty} d\omega_k \gamma_k(\omega_k)\left(\hat{b}_k^\dagger(\omega_k)\Big[\hat{O}, \hat{c}_k\Big] - \Big[\hat{O}, \hat{c}_k^\dagger\Big]\hat{b}_k(\omega_k)\right). \tag{5.25}$$

Solving Eq. (5.24) gives

$$\hat{b}_k(\omega_k) = e^{-i\omega_k t}\hat{b}_{k,0}(\omega_k) + \gamma_k(\omega_k)\int_0^t e^{-i\omega_k(t-t')}\hat{c}_k(t')dt', \tag{5.26}$$

where $\hat{b}_{k,0}(\omega_k)$ is the initial value of $\hat{b}_k(\omega_k)$ at $t=0$. Substituting Eq. (5.26) into Eq. (5.25) returns the expression

$$\begin{aligned}
\frac{d\hat{O}}{dt} &= -\frac{i}{\hbar}\Big[\hat{O}, \hat{H}_{\text{sys}}\Big] \\
&+ \sum_{k=1}^{M}\int_{-\infty}^{\infty} d\omega_k \gamma_k(\omega_k)\left(e^{i\omega_k t}\hat{b}_{k,0}^\dagger(\omega_k)\Big[\hat{O}, \hat{c}_k\Big] - \Big[\hat{O}, \hat{c}_k^\dagger\Big]e^{-i\omega_k t}\hat{b}_{k,0}(\omega_k)\right) \\
&+ \sum_{k=1}^{M}\int_{-\infty}^{\infty} d\omega_k (\gamma_k(\omega_k))^2 \int_0^t dt' \left(e^{i\omega_k(t-t')}\hat{c}_k^\dagger(t')\Big[\hat{O}, \hat{c}_k\Big] - \Big[\hat{O}, \hat{c}_k^\dagger\Big]e^{-i\omega_k(t-t')}\hat{c}_k(t')\right).
\end{aligned} \tag{5.27}$$

For visual clarity, we omitted the time argument of operators depending on t, but kept it explicitly for those depending on t'. So far up to Eq. (5.27), the equations are exact. To simplify the integrals in the third line, we introduce the first Markov approximation, which states that the coupling constants are independent of frequency

$$\gamma_k(\omega_k) = \sqrt{\kappa_k/2\pi}, \tag{5.28}$$

where κ_k is the coupling rate of the mode k to its input. So the integrals in ω_k will give Dirac deltas:

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} = 2\pi\delta(t-t'), \tag{5.29}$$

verifying

$$\int_0^t \hat{c}_k(t')\delta(t-t')dt' = \frac{1}{2}\hat{c}_k(t) \tag{5.30}$$

when Eq. (5.29) is obtained as the limit of an integral over a function going smoothly to zero at $\pm\infty$, which happens in our case. We also define the input field operators as

$$\hat{a}_{k,\mathsf{in}}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega_k e^{-i\omega_k t}\hat{b}_{k,0}(\omega_k), \tag{5.31}$$

which satisfy the commutation relation

$$\left[\hat{a}_{j,\mathsf{in}}(t), \hat{a}_{k,\mathsf{in}}{}^{\dagger}(t')\right] = \delta^{jk}\delta(t-t'). \tag{5.32}$$

Using Eqs. (5.28) to (5.32), we derive the quantum Langevin equation governing the time evolution of the operators

$$\begin{aligned}\frac{d\hat{O}}{dt} &= -\frac{i}{\hbar}\left[\hat{O}, \hat{H}_{\mathsf{sys}}\right] \\ &- \sum_{k=1}^{M}\left[\hat{O}, \hat{c}_k^{\dagger}\right]\left(\frac{\kappa_k}{2}\hat{c}_k + \sqrt{\kappa_k}\hat{a}_{k,\mathsf{in}}(t)\right) \\ &- \sum_{k=1}^{M}\left(\frac{\kappa_k}{2}\hat{c}_k^{\dagger} + \sqrt{\kappa_k}\hat{a}_{k,\mathsf{in}}{}^{\dagger}(t)\right)\left[\hat{O}, \hat{c}_k\right].\end{aligned} \tag{5.33}$$

We want to model the dynamics of the field operators in the presence of single-photon dissipation. Single-photon dissipation is modeled by setting $\hat{c}_j \to \hat{a}_j$. By applying the quantum Langevin equation Eq. (5.33) to the field operators $\hat{a}_j$ that describe the single photon loss, we find

$$\boxed{\frac{d\hat{a}_k}{dt} = -\frac{i}{\hbar}\left[\hat{a}_k, \hat{H}_{\mathsf{sys}}\right] - \frac{\kappa_k}{2}\hat{a}_k - \sqrt{\kappa_k}\hat{a}_{k,\mathsf{in}}.} \tag{5.34}$$

We note that the $-\sqrt{\kappa_k}\hat{a}_{k,\mathsf{in}}$ term models nearly resonant driving, and the $-\frac{\kappa_k}{2}\hat{a}_k$ term models single-photon loss. For an isolated system, only the first term of Eq. (5.34) would remain.

In this work, the input modes $\hat{a}_{k,\mathsf{in}}$ will always be coherent states of amplitude $\langle\hat{a}_{k,\mathsf{in}}\rangle \stackrel{\text{def}}{=} \epsilon_k$. We note that this driving formulation is equivalent to the cascaded formalism introduced in Section 4.3.1, with complex drive amplitudes ϵ_k.

5.3 Modeling the quantum state with its Gaussian moments

The coupling Hamiltonian Eq. (5.18) is quadratic in the field operators, and we only model single photon loss, signifying the state of this system will always be Gaussian, if it starts in a Gaussian state [160]. In this section, we will first introduce the tools to define what a Gaussian state is, and how they can be efficiently simulated in polynomial time.

5.3.1 Gaussian states

A Gaussian state is defined by its Gaussian distribution in the phase space. To express the two moments of this distribution in systems of M bosonic modes, we first introduce the quadrature operators

$$\hat{x}_k \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(\hat{a}_k + \hat{a}_k{}^\dagger), \qquad \hat{p}_k \stackrel{\text{def}}{=} \frac{1}{i\sqrt{2}}(\hat{a}_k - \hat{a}_k{}^\dagger). \tag{5.35}$$

Let $\hat{R} \stackrel{\text{def}}{=} (\hat{x}_1, \ldots, \hat{x}_M, \hat{p}_1, \ldots, \hat{p}_M)^T$ be the column vector containing the quadratures of the modes, we can define the first and second moments of the quadratures as

$$\boldsymbol{\alpha}^R \stackrel{\text{def}}{=} \left\langle \hat{R} \right\rangle \tag{5.36a}$$

$$\boldsymbol{\sigma}^R_{kl} \stackrel{\text{def}}{=} \frac{1}{2} \left\langle \hat{R}_k \hat{R}_l^\dagger + \hat{R}_l^\dagger \hat{R}_k \right\rangle - \boldsymbol{\alpha}^R_k (\boldsymbol{\alpha}^R_l)^*. \tag{5.36b}$$

We label $\boldsymbol{\alpha}^R \in \mathbb{C}^{2M\times 1}$ the quadrature displacement vector and $\boldsymbol{\sigma}^R \in \mathbb{C}^{2M\times 2M}$ the quadrature covariance matrix.

The quantum quasi-probability distribution of a M mode mixed state $\hat{\rho}$ in the phase space can be represented using the Wigner function [161]

$$W(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{(2\pi)^M} \int_{\mathbb{R}^M} \left\langle \boldsymbol{x} + \frac{1}{2}\boldsymbol{q} \middle| \hat{\rho} \middle| \boldsymbol{x} - \frac{1}{2}\boldsymbol{q} \right\rangle e^{i\boldsymbol{p}\cdot\boldsymbol{q}} d^M \boldsymbol{q}, \tag{5.37}$$

where $\boldsymbol{x}$ and $\boldsymbol{p}$ denote M-dimensional real vectors. We denote $|x_k\rangle$ an eigenstate of the position quadrature operator $\hat{x}_k$ with the eigenvalue x_k, and $|\boldsymbol{x}\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_M\rangle$.

The Wigner function can be reformulated as $W(\boldsymbol{r})$, for $\boldsymbol{r} = (x_1, \ldots, x_M, p_1, \ldots, p_M)$, such that $W(\boldsymbol{r}) = W(\boldsymbol{x}, \boldsymbol{p})$. For Gaussian states, the Wigner function is a Gaussian

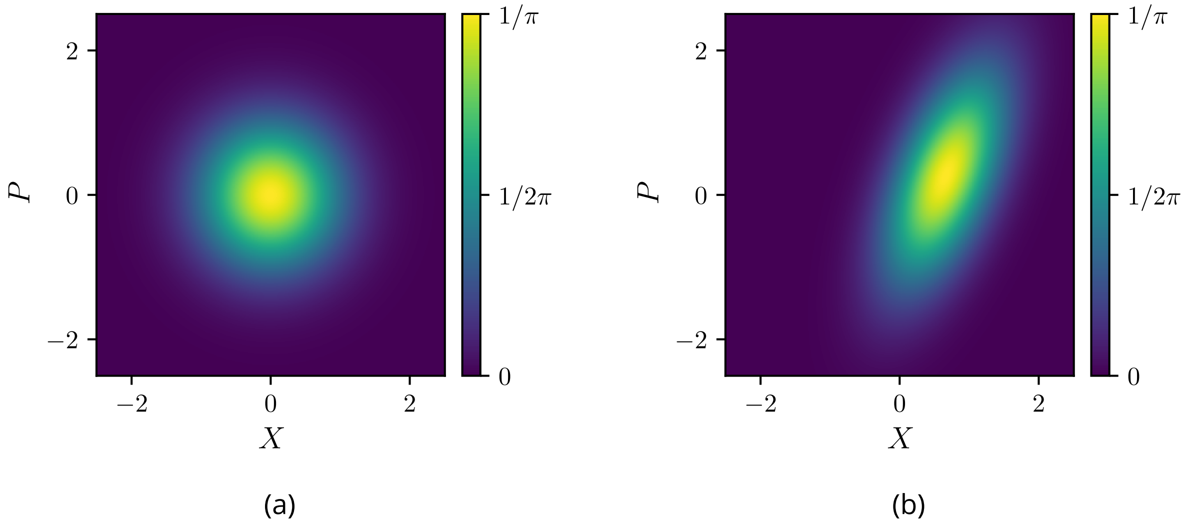

Figure 5.2: Wigner function distribution of two different single mode Gaussian states in the phase space. (a) Vacuum state. (b) Gaussian state with $\boldsymbol{\alpha}^R = (0.66, 0.26)^T$ and $\boldsymbol{\sigma}^R = \begin{pmatrix} 0.35 & 0.34 \\ 0.34 & 1.03 \end{pmatrix}$.

distribution defined by the quadrature displacement vector and covariance matrix

$$W(\boldsymbol{r}) = \frac{1}{(2\pi)^M \sqrt{\det(\boldsymbol{\sigma}^R)}} \exp\left(-\frac{1}{2} \left(\boldsymbol{\alpha}^R - \boldsymbol{r}\right)^T \left(\boldsymbol{\sigma}^R\right)^{-1} \left(\boldsymbol{\alpha}^R - \boldsymbol{r}\right) \right). \tag{5.38}$$

A notable Gaussian state is the vacuum $|\mathbf{0}\rangle \stackrel{\text{def}}{=} |0\rangle_1 \otimes ... \otimes |0\rangle_M$, with moments

$$\begin{cases} \boldsymbol{\alpha}^R & = (0, \ldots, 0) \\ \boldsymbol{\sigma}^R & = \boldsymbol{\sigma}_0 \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\mathbb{1}_M}{2} & \mathbb{0}_M \\ \mathbb{0}_M & \frac{\mathbb{1}_M}{2} \end{pmatrix} \end{cases}. \tag{5.39}$$

$\mathbb{1}_M$ and $\mathbb{0}_M$ denote the unitary and zero matrices of size $M \times M$. The Wigner function of a single mode in a vacuum state is shown in Fig. 5.2a. The Wigner function distribution of a single mode Gaussian state with $\boldsymbol{\alpha}^R = (0.66, 0.26)^T$ and $\boldsymbol{\sigma}^R = \begin{pmatrix} 0.35 & 0.34 \\ 0.34 & 1.03 \end{pmatrix}$ is depicted in the phase space in Fig. 5.2b, to illustrate its Gaussian shape.

For the remainder of this manuscript, it will be more convenient to represent Gaussian states using their field operator displacement and covariance matrix, instead of those of the field quadratures. Let $\hat{A} \stackrel{\text{def}}{=} \left(\hat{a}_1, \ldots, \hat{a}_M, \hat{a}_1{}^\dagger, \ldots, \hat{a}_M{}^\dagger\right)^T$ be the column vector containing the field operators of M bosonic modes. The displacement vector and covariance matrix are

$$\boldsymbol{\alpha} \stackrel{\text{def}}{=} \left\langle \hat{A} \right\rangle \tag{5.40a}$$

$$\boldsymbol{\sigma}_{kl} \stackrel{\text{def}}{=} \frac{1}{2} \left\langle \hat{A}_k \hat{A}_l^{\dagger} + \hat{A}_l^{\dagger} \hat{A}_k \right\rangle - \boldsymbol{\alpha}_k \boldsymbol{\alpha}_l^{*}. \tag{5.40b}$$

The field operator moments can be derived from the quadrature moments

$$\boldsymbol{\alpha} = \gamma \boldsymbol{\alpha}^{R} \tag{5.41a}$$

$$\boldsymbol{\sigma} = \gamma \boldsymbol{\sigma}^{R} \gamma^{\dagger}, \tag{5.41b}$$

where γ is the change of frame matrix

$$\gamma \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \mathbb{1}_M & i\mathbb{1}_M \\ \hline \mathbb{1}_M & -i\mathbb{1}_M \end{array} \right), \quad \gamma^{-1} = \gamma^{\dagger} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \mathbb{1}_M & \mathbb{1}_M \\ \hline -i\mathbb{1}_M & i\mathbb{1}_M \end{array} \right). \tag{5.42}$$

Moving to the Heisenberg picture allows to define the field operator moments with respect to time

$$\boldsymbol{\alpha}(t) \stackrel{\text{def}}{=} \left\langle \hat{A}(t) \right\rangle \tag{5.43a}$$

$$\boldsymbol{\sigma}_{kl}(t, t') \stackrel{\text{def}}{=} \frac{1}{2} \left\langle \hat{A}_k(t) \hat{A}_l^{\dagger}(t') + \hat{A}_l^{\dagger}(t') \hat{A}_k(t) \right\rangle - \boldsymbol{\alpha}_k(t) \boldsymbol{\alpha}_l^{*}(t'). \tag{5.43b}$$

For visual clarity, we will simplify the notation of $\boldsymbol{\sigma}(t, t')$ to $\boldsymbol{\sigma}(t)$ if $t = t'$.

5.3.2 Evolution of the Gaussian moments

Since Gaussian states are fully described by their polynomially-sized displacement and covariance matrix, it is more computationally efficient to compute their dynamics using the Gaussian variables in the Heisenberg picture, rather than simulating the exponentially-sized density matrix in the Schrödinger picture. In this section we will use the quantum Langevin equation Eq. (5.34) to derive the time evolution equations for the displacement and covariance matrix of a Gaussian mode.

We start by developing the unitary evolution term induced by the coupling Hamiltonian $\hat{H}_{\text{sys}}$ using the commutation property $[a, bc] = [a, b]c + b[a, c]$ and Eq. (5.17), which gives

$$
\begin{aligned}
-\frac{i}{\hbar}\left[\hat{a}_k, \hat{H}_{\text{sys}}\right] &= i\sum_{j=1}^{M}\delta_j\left[\hat{a}_k, \hat{a}_j{}^\dagger\hat{a}_j\right] \\
&- i\sum_{\substack{i,j=1\\ i<j}}^{M}\left(g_{ij}\left[\hat{a}_k, \hat{a}_i{}^\dagger\hat{a}_j\right] + g_{ij}{}^*\left[\hat{a}_k, \hat{a}_i\hat{a}_j{}^\dagger\right] + g^s_{ij}\left[\hat{a}_k, \hat{a}_i{}^\dagger\hat{a}_j{}^\dagger\right] + g^s_{ij}{}^*[\hat{a}_k, \hat{a}_i\hat{a}_j]\right) \\
&= i\delta_k\hat{a}_k - i\sum_{j=1}^{M}\left(\gamma_k(g_{kj} + g_{jk}{}^*)\hat{a}_j + (g^s_{kj} + g^s_{jk})\hat{a}_j{}^\dagger\right).
\end{aligned}
\tag{5.44}
$$

For the quadratic system Hamiltonian $\hat{H}_{\text{sys}}$ of Eq. (5.18), the Langevin equation becomes

$$
\frac{d\hat{a}_k}{dt} = i\delta_k\hat{a}_k - i\sum_{j=1}^{M}\left((g_{kj} + g_{jk}{}^*)\hat{a}_j + (g^s_{kj} + g^s_{jk})\hat{a}_j{}^\dagger\right) - \frac{\kappa_k}{2}\hat{a}_k - \sqrt{\kappa_k}\hat{a}_{k,\text{in}}. \tag{5.45}
$$

Since $\hat{a}_k$ and $\hat{a}_{k,\text{in}}$ only evolve linearly with this equation, we notice that the system of differential equations defined by Eq. (5.45) is closed, and thus they are exactly solvable. We describe the evolution of a Gaussian state using the coupling matrix [80]

$$
L \stackrel{\text{def}}{=} \frac{1}{i\hbar}\begin{pmatrix} G & G^s \\ -G^{s\dagger} & -G^T \end{pmatrix}, \tag{5.46}
$$

where the matrix elements are

$$
(G)_{k,l} \stackrel{\text{def}}{=} \hbar\times\begin{cases} -\delta_k & \text{if } k = l \\ g_{kl} & \text{if } k < l \\ g_{kl}{}^* & \text{if } k > l \end{cases}, \quad (G^s)_{k,l} \stackrel{\text{def}}{=} \hbar\times\begin{cases} 0 & \text{if } k = l \\ g^s_{kl} & \text{otherwise.} \end{cases} \tag{5.47}
$$

The vectorized Langevin equation for the entire system becomes

$$
\frac{d\hat{A}}{dt} = L\hat{A} - \frac{K}{2}\hat{A} - \sqrt{K}\hat{A}_{\text{in}}, \tag{5.48}
$$

where

$$
\begin{cases} K & \stackrel{\text{def}}{=} \text{diag}(\kappa_1, \ldots, \kappa_M, \kappa_1, \ldots, \kappa_M) \\ \hat{A}_{\text{in}} & \stackrel{\text{def}}{=} \left(\hat{a}_{1,\text{in}}, \ldots, \hat{a}_{M,\text{in}}, \hat{a}_{1,\text{in}}{}^\dagger, \ldots, \hat{a}_{M,\text{in}}{}^\dagger\right)^T. \end{cases} \tag{5.49}
$$

The general solution to this first-order linear differential equation is

$$\hat{A}(t) = F(t)\hat{A}(t=0) - \int_0^t F(t-\tau)\sqrt{K}\hat{A}_{\mathsf{in}}(\tau)d\tau, \tag{5.50}$$

where we have introduced the propagator matrix

$$F(t) \stackrel{\text{def}}{=} \exp(F't), \tag{5.51}$$

with

$$F' \stackrel{\text{def}}{=} L - \frac{K}{2}. \tag{5.52}$$

Time evolution of the displacement

Using Eq. (5.48) and the definition of the field operator displacement $\boldsymbol{\alpha}(t)$ from Eq. (5.43a), we obtain

$$\boxed{\boldsymbol{\alpha}(t) = F(t)\boldsymbol{\alpha}(t=0) - \int_0^t F(t-\tau)\sqrt{K}\boldsymbol{\alpha}_{\mathsf{in}}(\tau)d\tau,} \tag{5.53}$$

where $\boldsymbol{\alpha}_{\mathsf{in}}(t) \stackrel{\text{def}}{=} \left\langle \hat{A}_{\mathsf{in}}(t) \right\rangle = (\epsilon_1(t), \ldots, \epsilon_M(t), {\epsilon_1}^*(t), \ldots, {\epsilon_M}^*(t))^T$ is the displacement vector of the input modes.

Time evolution of the covariance matrix

The calculation of $\boldsymbol{\sigma}(t)$ is lengthier. First we develop the different terms of Eq. (5.43b)

$$\begin{aligned}
\boldsymbol{\alpha}_i(t)\boldsymbol{\alpha}_j(t) &= \sum_{k,l=1}^{2M} F_{ik}(t){F_{jl}}^*(t)\boldsymbol{\alpha}_k(t=0)\boldsymbol{\alpha}_l(t=0) \\
&+ \sum_{k,l=1}^{2M} \int_0^t \int_0^t F_{ik}(t-\tau){F_{jl}}^*(t-\tau')\sqrt{K_k K_l}\boldsymbol{\alpha}_{\mathsf{in},k}(\tau){\boldsymbol{\alpha}_{\mathsf{in},l}}^*(\tau')d\tau d\tau' \\
&- \sum_{k,l=1}^{2M} F_{ik}(t)\boldsymbol{\alpha}_k(t=0)\int_0^t {F_{jl}}^*(t-\tau)\sqrt{K_l}{\boldsymbol{\alpha}_{\mathsf{in},l}}^*(\tau)d\tau \\
&- \sum_{k,l=1}^{2M} {F_{jl}}^*(t){\boldsymbol{\alpha}_l}^*(t=0)\int_0^t F_{ik}(t-\tau)\sqrt{K_k}\boldsymbol{\alpha}_{\mathsf{in},k}(\tau)d\tau
\end{aligned} \tag{5.54}$$

$$
\begin{aligned}
\left\langle \hat{A}_i(t)\hat{A}_j(t)\right\rangle = &\sum_{k,l=1}^{2M} F_{ik}(t)F_{jl}{}^*(t)\left\langle \hat{A}_k(t=0)\hat{A}_l{}^\dagger(t=0)\right\rangle \\
&+\sum_{k,l=1}^{2M}\int_0^t\int_0^t F_{ik}(t-\tau)F_{jl}{}^*(t-\tau')\sqrt{K_kK_l}\left\langle \hat{A}_{\text{in},k}(\tau)\hat{A}_{\text{in},l}{}^\dagger(\tau')\right\rangle d\tau d\tau' \\
&-\sum_{k,l=1}^{2M} F_{ik}(t)\boldsymbol{\alpha}_k(t=0)\int_0^t F_{jl}{}^*(t-\tau)\sqrt{K_l}\boldsymbol{\alpha}_{\text{in},l}{}^*(\tau)d\tau \\
&-\sum_{k,l=1}^{2M} F_{jl}{}^*(t)\boldsymbol{\alpha}_l{}^*(t=0)\int_0^t F_{ik}(t-\tau)\sqrt{K_k}\boldsymbol{\alpha}_{\text{in},k}(\tau)d\tau.
\end{aligned}
\tag{5.55}
$$

We observe that the last two terms in Eq. (5.54) and Eq. (5.55) will cancel out. So the expression for the covariance matrix is

$$
\boldsymbol{\sigma}_{ij}(t) = \sum_{k,l=1}^{2M} F_{ik}(t)F_{jl}{}^*(t)\boldsymbol{\sigma}_{kl}(t=0)+\sum_{k,l=1}^{2M}\sqrt{K_kK_l}\int_0^t\int_0^t F_{ik}(t-\tau)F_{jl}{}^*(t-\tau')\boldsymbol{\sigma}_{\text{in},kl}(\tau,\tau')d\tau d\tau',
\tag{5.56}
$$

where $\boldsymbol{\sigma}_{\text{in}}$ is the covariance matrix of the input modes. In the setting where input modes are in coherent states without temporal correlations, their covariance matrix verifies

$$
\boldsymbol{\sigma}_{\text{in}}(t,t') = \delta(t-t')\boldsymbol{\sigma}_0.
\tag{5.57}
$$

In this regime, the Eq. (5.56) simplifies to

$$
\begin{aligned}
\boldsymbol{\sigma}_{ij}(t) &= \sum_{k,l=1}^{2M} F_{ik}(t)F_{jl}{}^*(t)\boldsymbol{\sigma}_{kl}(t=0)+\sum_{k,l=1}^{2M}\sqrt{K_kK_l}\int_0^t F_{ik}(t-\tau)F_{jl}{}^*(t-\tau)\boldsymbol{\sigma}_{0,kl}d\tau \\
&= \sum_{k,l=1}^{2M} F_{ik}(t)F_{jl}{}^*(t)\boldsymbol{\sigma}_{kl}(t=0)+\frac{1}{2}\sum_{k=1}^{2M}K_k\int_0^t F_{ik}(t-\tau)F_{jk}{}^*(t-\tau)d\tau.
\end{aligned}
\tag{5.58}
$$

Finally, we get the expression

$$
\boxed{\boldsymbol{\sigma}(t) = F(t)\boldsymbol{\sigma}(t=0)F^\dagger(t)+\boldsymbol{\sigma}_0\int_0^t F(t-\tau)KF^\dagger(t-\tau)d\tau.}
\tag{5.59}
$$

5.3.3 Computation of the displacement and covariance matrix via diagonalization

To simulate the bosonic modes when they are in Gaussian states, we can compute the displacement $\boldsymbol{\alpha}(t)$ and covariance matrix $\boldsymbol{\sigma}(t)$. Their computation is possible by integrating Eqs. (5.53) and (5.59), but this method is time-consuming. Instead, diagonalizing the propagator matrix F allows for their efficient computation, as we will detail below.

The input modes are set to be in coherent states of constant values $\boldsymbol{\alpha}_{\text{in}}$, and we assume the propagator matrix generator F' is diagonalizable as

$$\Lambda = U^{-1}F'U = \text{diag}\left(\lambda_1, \ldots, \lambda_{2M}\right), \tag{5.60}$$

where U is an invertible matrix, and λ_k are eigenvalues of F'. The propagator matrix can be diagonalized using the exponential of a diagonal matrix

$$\begin{aligned} F(t) &= \exp(F't) \\ &= Ue^{t\Lambda}U^{-1}. \end{aligned} \tag{5.61}$$

The integral of $\boldsymbol{\alpha}(t)$ in Eq. (5.53) becomes

$$\begin{aligned} \boldsymbol{\alpha}(t) &= F(t)\boldsymbol{\alpha}(t=0) + \sqrt{K}U\left(\int_0^t e^{\Lambda(t-\tau)}d\tau\right)U^{-1}\boldsymbol{\alpha}_{\text{in}} \\ &= F(t)\boldsymbol{\alpha}(t=0) + \sqrt{K}UI_1U^{-1}\boldsymbol{\alpha}_{\text{in}}, \end{aligned} \tag{5.62}$$

where $I_1 = \Lambda^{-1}\left(e^{\Lambda t} - \mathbb{1}_{2M}\right)$. To compute the covariance matrix, we introduce the matrices P and I_2 such that

$$P = U^{-1}K(U^{-1})^\dagger \tag{5.63}$$

$$(I_2)_{i,j} = (P)_{i,j}\frac{\exp((\lambda_i + {\lambda_j}^*)\,t) - 1}{\lambda_i + {\lambda_j}^*}. \tag{5.64}$$

Finally, we find

$$\boldsymbol{\sigma}(t) = F(t)\boldsymbol{\sigma}(t=0)F(t)^\dagger + \boldsymbol{\sigma}_0 UI_2U^\dagger. \tag{5.65}$$

Simulating the evolution of Gaussian states using Eqs. (5.62) and (5.65) in the Heisenberg picture is much more efficient than solving the Lindblad master equation of their density matrices in the Schrödinger representation, because their size scales exponentially with the number of modes. In the calculations considered above, the most resource intensive operation is matrix diagonalization, which is of time complexity $\mathcal{O}(M^3)$ using the QR algorithm [162]. In my simulations, this was my method for computing the displacement and covariance matrix.