Introduction to R*

Lecture 2: Atomic Data Types - Homogeneous vectors

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R can be summarized in ${\bf three}$ principles (John M. Chambers, 2016)

- Everything that exists in R is an object.
- Everything that happens in R is a function call.
- Interfaces to other languages are a part of R.

1 R Objects

- An object in R is (internally) represented as a pair: (symbol, value).
- A **symbol** is assigned a **value** by the use of an arrow pointing to the left (<-).
- There are less favored ways:
 - A simple equality sign (=).
 - Using the **assign()** function.

1.1 Examples

• Clean up the global environment i.e. remove all objects from the current R environment.

Recommended!

```
rm(list=ls())
ls()
```

character(0)

• preferred way to assign variables

```
x <- 5.0
x
```

[1] 5

• alternative 1: mainly used to assign default function arguments

```
y = 5.0
y
```

[1] 5

```
mysamplevariance <- function(x, av=0){

n <- length(x)
   if(n>1){
      return(1.0/(n-1)*sum((x-av)^2))
   }
   else{
      stop("ERROR:: Dividing by zero (n==1) || (n==0) ")
   }
}

x <- rnorm(10)
mysamplevariance(x)</pre>
```

[1] 1.303428

```
mysamplevariance(x,mean(x))
```

[1] 1.303176

```
var(x)
[1] 1.303176

• alternative 2: even less used
assign("z", 5.0)
z
[1] 5

• functions are objects
f <- mean
f
function (x, ...)
UseMethod("mean")
<br/>
<br/>
<br/>
cenvironment: namespace:base>
val <- f(1:10)
val
```

[1] 5.5

"Nothing exists except atoms and empty space; everything else is opinion". (Democritos)

2 Atomic Data Types

2.1 The core/atomic data types

- R has the following 6 atomic data types:
 - logical (i.e. boolean)
 - integer
 - double
 - character (i.e. string)
 - complex
 - raw (i.e. byte)

The latter 2 types (i.e. complex and especially raw) are less common.

The **typeof()** function determines the **INTERNAL** storage/type of an R object.

2.1.1 Examples

• boolean/logical values: either TRUE or FALSE

```
x1 <- TRUE
x1
```

[1] TRUE

typeof(x1)

- [1] "logical"
 - integer values $(\in \mathbb{Z})$:

```
x2 <- 3L
x2
```

[1] 3

typeof(x2)

- [1] "integer"
 - double (precision) values:

```
x3 <- 3.14
x3
```

[1] 3.14

typeof(x3)

[1] "double"

• character values/strings

```
x4 <- "Hello world"
x4
```

[1] "Hello world"

typeof(x4)

- [1] "character"
 - complex values ($\in \mathbb{C}$):

```
x5 <- 2.0 + 3i
x5
```

[1] 2+3i

typeof(x5)

[1] "complex"

2.2 Operations on atomic data types

```
• logical operators: ==, !=, &&, ||, !
   • numerical operators: +, -, *, /, ^, ** (same as the caret), but also:
        – integer division: \%/\%
        - modulo operation: %%
        − Note: matrix multiplication will be performed using %*%
   • character/string manipulation:
       - nchar():
       - paste():
       - cat():
       - sprintf():
       - substr():
       - strsplit():
        - Note: Specialized R libraries were developed to manipulate strings e.g. stringr
   • explicit cast/conversion: https://data-flair.training/blogs/r-string-manipulation/
        as.{logical, integer, double, complex, character}()
   • explicit test of the type of a variable:
        is.{logical, integer, double, complex, character}()
2.2.1
       Examples
   • Logical operators:
x <-3
y <-7
(x <= 3) && (y == 7)
[1] TRUE
! (y<7)
[1] TRUE
   • Mathematical operations
2**4
[1] 16
7%%4
[1] 3
7/4
[1] 1.75
7%/%4
[1] 1
```

• String operations

```
s <- "Hello"
nchar(s)
[1] 5
news <- paste(s,"World")</pre>
[1] "Hello World"
sprintf("My new string:%20s\n", news)
[1] "My new string:
                        Hello World\n"
city <- "Witwatersrand"</pre>
substr(city,4,8)
[1] "water"
  • Conversion and testing of types
s <- "Hello World"
is.character(s)
[1] TRUE
s1 <- "-500"
is.character(s1)
[1] TRUE
s2 <- as.double(s1)
is.character(s2)
[1] FALSE
is.double(s2)
[1] TRUE
s3 <- as.complex(s2)
[1] -500+0i
sqrt(s3)
[1] 0+22.36068i
```

2.3 Exercises

- - Calculate $\log_2(10)$ using R's $\log()$ function
 - Perform the inverse operation and check that you get 10 back
- Let z = 3 + 4i
 - Use R's Re(), Im() functions to extract the real and imaginary parts of z.
 - Calculate the modulus of z using R's Mod() function and check whether you the same answer using $\sqrt{\Re(z)^2 + \Im(z)^2}$.
 - whether you the same answer using $\sqrt{\Re(z)^2 + \Im(z)^2}$.

 Calculate the argument of z using R's $\operatorname{Arg}()$ function and check whether you have the same answer using $\operatorname{arctan}\left(\frac{\Im(z)}{\Re(z)}\right)$.

3 Atomic vectors

- An atomic vector is a data structure containing elements of only one atomic data type.
 Therefore, an atomic vector is homogeneous.
- Atomic vectors are stored in a **linear** fashion.
- R does NOT have scalars:
 - An atomic vector of length 1 plays the role of a scalar.
 - Vectors of **length 0** also exist (and they have some use!).
- A **list** is a vector not necessarily of the atomic type.

A list is also known as a **recursive/generic** vector (vide infra).

3.1 Creation of atomic vectors

Atomic vectors can be created in a multiple ways:

- Use of the **vector()** function.
- Use of the **c()** function (**c** stands for concatenate).
- Use of the column operator :
- Use of the **seq()** and **rep()** functions.

The length of a vector can be retrieved using the **length()** function.

3.1.1 Examples

```
• use of the vector() function:

x <- vector() # Empty vector (Default:'logical')
x

logical(0)

length(x)

[1] 0

typeof(x)

[1] "logical"

x <- vector(mode="complex", length=4)
x

[1] 0+0i 0+0i 0+0i 0+0i 0+0i

length(x)

[1] 4
x

[1] 0+0i 0+0i 0+0i 0+0i

x[1] <- 4
x

[1] 4+0i 0+0i 0+0i 0+0i 0+0i
```

```
• use of the c() function:
x1 \leftarrow c(3, 2, 5.2, 7)
[1] 3.0 2.0 5.2 7.0
x2 < -c(8, 12, 13)
[1] 8 12 13
x3 \leftarrow c(x2, x1)
x3
[1] 8.0 12.0 13.0 3.0 2.0 5.2 7.0
x4 <- c(FALSE, TRUE, FALSE)
[1] FALSE TRUE FALSE
x5 <- c("Hello", "Salt", "Lake", "City")
x5
[1] "Hello" "Salt" "Lake" "City"
  • use of the column operator:
y1 <- 1:10
y1
[1] 1 2 3 4 5 6 7 8 9 10
y2 <- 5:-5
у2
[1] 5 4 3 2 1 0 -1 -2 -3 -4 -5
y3 <- 2.3:10
уЗ
[1] 2.3 3.3 4.3 5.3 6.3 7.3 8.3 9.3
y4 <- 2.0*7:1
y4
[1] 14 12 10 8 6 4 2
y5 <- 1:7-1
у5
[1] 0 1 2 3 4 5 6
  • seq() and rep() functions
z1 <- seq(from=1, to=15, by=3)</pre>
[1] 1 4 7 10 13
```

```
z2 <- seq(from=-2,to=5,length=4)
z2

[1] -2.0000000  0.3333333  2.6666667  5.0000000

z3 <- rep(c(3,2,4), time=2)
z3

[1] 3 2 4 3 2 4

z4 <- rep(c(3,2,4), each=3)
z4

[1] 3 3 3 2 2 2 4 4 4

z5 <- rep(c(1,7), each=2, time=3)
z5

[1] 1 1 7 7 1 1 7 7 1 1 7 7

length(z5)

[1] 12</pre>
```

3.2 Operations on vectors: element-wise

- All operations on vectors in R happen element by element (cfr. NumPy).
- Vector Recycling:

If 2 vectors of **different** lengths are involved in an operation, the **shortest vector** will be repeated until all elements of the longest vector are matched. A message will be sent to the stdout.

3.2.1 Examples

```
x <- -3:3

x

[1] -3 -2 -1 0 1 2 3

y <- 1:7

y

[1] 1 2 3 4 5 6 7

xy <- x*y

xy

[1] -3 -4 -3 0 5 12 21

xpy <- x^y
xpy

[1] -3 4 -1 0 1 64 2187
```

```
x <- 0:10
y <- 1:2
length(x)

[1] 11
length(y)

[1] 2
x

[1] 0 1 2 3 4 5 6 7 8 9 10
y

[1] 1 2
x+y

Warning in x + y: longer object length is not a multiple of shorter object length

[1] 1 3 3 5 5 7 7 9 9 11 11</pre>
```

3.3 Retrieving elements of vectors

- Indexing: starts at 1 (not 0 like C/C++, Python, Java,) see also: Edsger Dijkstra: Why numbering should start at zero
- Use of vector with indices to extract values.
- Advanced features:
 - use of boolean values to extract values.
 - the membership operator: %in%.
 - the deselect/omit operator: -
 - which(): returns the indices for which the condition is true.
 - **any**()/**all**() functions.
 - * any(): TRUE if at least 1 value is true
 - * all(): TRUE if all values are true

3.3.1 Examples

• Use of a simple index:

```
x <- seq(2,100,by=15)
x[4]
[1] 47
x[1]
[1] 2</pre>
```

• Select several indices at once using vectors:

```
[1] 2 17 32 47 62 77 92
x[3:5]
[1] 32 47 62
x[c(1,3,5,7)]
[1] 2 32 62 92
x[seq(1,7,by=2)]
[1] 2 32 62 92
  • Extraction via booleans (i.e. retain only those values that are equal to TRUE):
X
[1] 2 17 32 47 62 77 92
[1] FALSE FALSE FALSE TRUE TRUE TRUE
x[x>45]
[1] 47 62 77 92
  • Use of the %in% operator:
[1] 2 17 32 47 62 77 92
10 %in% x
[1] FALSE
62 %in% x
[1] TRUE
c(32,33,43) %in% x
[1] TRUE FALSE FALSE
!(c(32,33,43) \%in\% x)
[1] FALSE TRUE TRUE
  • Negate/filter out the elements with negative indices:
[1] 2 17 32 47 62 77 92
```

```
x[-c(2,4,6)]

[1] 2 32 62 92

z <- x[-1] - x[-length(x)]

z

[1] 15 15 15 15 15 15
```

• The which() function returns only those indices of which the condition/expression is true.

```
[1] -0.53910607 -0.46264832  0.78251274 -0.73978303 -0.33846100  0.09327378
[7] -0.14257117 -1.23856432 -1.55542946  0.48506976
which(vecnum>1.0)
integer(0)

• Use of the any()/all() functions.
y <- seq(0,100,by=10)
x

[1] 2 17 32 47 62 77 92
y

[1] 0 10 20 30 40 50 60 70 80 90 100
any(x<y)
Warning in x < y: longer object length is not a multiple of shorter object length
[1] TRUE
all(x[6:7]>y[2:3])
```

3.4 Hash tables

[1] TRUE

Sample 10 numbers from N(0,1)

vecnum <- rnorm(n=10)</pre>

vecnum

A hash table is a data structure which implements an associative array or dictionary. It is an abstract data which maps data to keys.

- There are several ways to create one:
 - Map names to an existing vector
 - Add names when creating the vector
- To remove the map, map the names to NULL

3.4.1 Examples

• Creation of 2 independent vectors

```
capitals <- c("Albany", "Providence", "Hartford", "Boston", "Montpelier", "Concord", "Augusta")</pre>
states <- c("NY", "RI", "CT", "MA", "VT", "NH", "ME")
capitals
[1] "Albany"
                  "Providence" "Hartford"
                                             "Boston"
                                                            "Montpelier"
[6] "Concord"
                  "Augusta"
states
[1] "NY" "RI" "CT" "MA" "VT" "NH" "ME"
capitals[3]
[1] "Hartford"
  • Create the hashtable/dictionary
# Method 1
names(capitals) <- states</pre>
capitals
                        RI
                                      CT
                                                    MA
                                                                  VT
                                                                               NH
                                              "Boston" "Montpelier"
    "Albany" "Providence"
                              "Hartford"
                                                                        "Concord"
   "Augusta"
capitals["MA"]
      MA
"Boston"
names(capitals)
[1] "NY" "RI" "CT" "MA" "VT" "NH" "ME"
# Method 2
phonecode <- c("801"="SLC", "206"="Seattle", "307"="Wyoming")</pre>
phonecode
                 206
                           307
      801
    "SLC" "Seattle" "Wyoming"
phonecode["801"]
  801
"SLC"
```

• Dissociate the 2 vectors

```
names(capitals) <- NULL</pre>
capitals
                   "Providence" "Hartford"
[1] "Albany"
                                                 "Boston"
                                                                "Montpelier"
[6] "Concord"
                   "Augusta"
      NA (Not Available values)
3.5
   • NA: stands for 'Not Available'/Missing values and has a length of 1.
     There are in essence 4 versions depending on the type:
        NA (logical - default)
        - NA_integer (integer)
        - NA_real (double precision)
        NA_character (string)
     Under the hood, the version of NA is subjected to coercion:
     logical 
ightarrow integer 
ightarrow double 
ightarrow character
   • some functions e.g. mean() return (by default) NA if
     1 or more instances NA are present in a vector.
   • is.na(): test a vector (element-wise) for NA values.
     Do NOT use:
     x == NA
     but INSTEAD use:
     is.na(x)
3.5.1 Examples
   • Types of NA
x <- NA
typeof(x)
[1] "logical"
# logical NA coerced to double precision NA
x \leftarrow c(3.0, 5.0, NA)
typeof(x[3])
[1] "double"
* Functions on a vector containing NA
mean(x)
[1] NA
mean(x, na.rm=TRUE)
```

[1] 4

```
* Check of the NA availability
```

```
x \leftarrow c(NA, 1, 2, NA)
is.na(x)
```

[1] TRUE FALSE FALSE TRUE

* Functions on a vector containing NA

mean(x)

[1] NA

mean(x, na.rm=TRUE)

[1] 1.5

3.6 NaN and infinities

- NaN (only for numeric types!), and the infinities Inf and -Inf are part of the IEEE 754 floating-point standard.
- To test whether you have:
 - finite numbers: use **is.finite()**
 - infinite numbers: use **is.infinite()**
 - NaNs: use is.nan()
- Further:
 - a NaN will return TRUE when tested by either is.nan() or is.na()
 - a **NA** will return **TRUE** only when tested by **is.na**()

3.6.1 Examples

• Infinities:

```
x <- 5.0/0.0
x
```

[1] Inf

is.finite(x)

[1] FALSE

is.infinite(x)

[1] TRUE

is.nan(x)

[1] FALSE

```
y <- -5.0/0.0
y
```

[1] -Inf

```
is.finite(y)
[1] FALSE
is.infinite(y)
[1] TRUE
is.nan(y)
[1] FALSE
z \leftarrow x + y
[1] NaN
typeof(z)
[1] "double"
is.finite(z)
[1] FALSE
is.infinite(z)
[1] FALSE
is.nan(z)
[1] TRUE
  • is.na() vs. is.nan():
# is.nan
v \leftarrow c(NA, z, 5.0, log(-1.0))
Warning in log(-1): NaNs produced
is.nan(v)
[1] FALSE TRUE FALSE TRUE
# is.na(): also includes NaN!
v \leftarrow c(NA, z, 5.0, log(-1.0))
Warning in log(-1): NaNs produced
is.na(v)
[1] TRUE TRUE FALSE TRUE
```

3.7 Note on logical operators

- &, |, !, xor(): element-wise operators on vectors (cfr. arithmetic operators)
- &&, ||: evaluated from **left** to **right** until result is determined.

3.7.1 Examples

```
• Vector operators (&, |, ! and xor())
```

```
x <- sample(x=1:10, size=10, replace=TRUE)
x</pre>
```

```
[1] 9 6 6 1 7 1 3 9 3 5
```

```
y <- sample(x=1:10, size=10, replace=TRUE)
y
```

[1] 7 5 3 8 10 3 1 2 2 3

```
v1 <- (x<=3)
v1
```

[1] FALSE FALSE TRUE FALSE TRUE FALSE TRUE FALSE

```
v2 <- (y>=7)
v2
```

[1] TRUE FALSE FALSE TRUE TRUE FALSE FALSE FALSE FALSE

v1 & v2

[1] FALSE FALSE FALSE TRUE FALSE FALSE FALSE FALSE FALSE

v1 | v2

[1] TRUE FALSE FALSE TRUE TRUE TRUE TRUE FALSE TRUE FALSE

xor(v1, v2)

[1] TRUE FALSE FALSE TRUE TRUE TRUE FALSE TRUE FALSE

!v1

[1] TRUE TRUE TRUE FALSE TRUE FALSE TRUE FALSE TRUE

```
* && and ||
1 && v1

Warning in 1 && v1: 'length(x) = 10 > 1' in coercion to 'logical(1)'
[1] FALSE
1 & v1
[1] FALSE FALSE TRUE FALSE TRUE TRUE FALSE
1 || v1
```

```
THRESHOLD <- 0.5
val <- runif(1)
if (val>THRESHOLD) {
    sprintf("Value:%4.2f above threshold of %4.2f",val, THRESHOLD)
} else {
    sprintf("Value:%4.2f below threshold of %4.2f",val, THRESHOLD)
}
```

[1] "Value:0.08 below threshold of 0.50"

[1] TRUE

Note: NO lazy-evaluation version of xor()

3.8 Exercises

- Use the **seq()** function to generate the following sequence: 6 13 20 27 34 41 48
- R has the its own inversion function, rev(), e.g,:

```
x <- seq(from=2,to=33,by=3)
x</pre>
```

[1] 2 5 8 11 14 17 20 23 26 29 32

```
y <- rev(x)
```

[1] 32 29 26 23 20 17 14 11 8 5 2

Invert the vector x without invoking the **rev()** function.

- Create the following R vector using **only** the **seq()** and **rep()** functions: -8 -8 -8 -8 0 8 8 8 16 16 16 16 16
- Create the following vector (do **not** use **c()**!): -512 -216 -64 -8 0 8 64 216 512 1000
- Generate a random vector of integers using the following code:

```
x <- sample(x=0:1000, size=100, replace=TRUE)
```

- Invoke the above code to generate the vector x
- Find if there are any integers in the vector **x** which can be divided by 4 and 6
- Find those numbers and their corresponding indices in the vector x.
- The Taylor series for $\ln(1+x)$ is converging when |x|<1 and is given by:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$$

Calculate the sum of the first 5, 10, 15 terms in the above expression to approximate ln(1.2). Compare with R's value i.e.: log(1.2).

• The logarithmic return in finance is defined as:

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

- Generate a financial time series using the following R code:

thecasino <- abs(rcauchy(1000))+1.E-6

- Calculate the logarithmic return for the financial time series thecasino.
 Your adjusted time series will be 1 element shorter in length than the original one.
 Compare your result with diff(log(thecasino)).
- Monte-Carlo approximation of π

Let S1 be the square spanned by the following 4 vertices: $\{(0,0),(0,1),(1,0),(1,1)\}$. Let S2 be the first quadrant of the unit-circle $\mathcal{C}: x^2 + y^2 = 1$.

The ratio ρ defined as:

$$\rho := \frac{\text{Area S2}}{\text{Area S1}} = \frac{\# \text{Points in S2}}{\# \text{Points in S1}}$$

allows us to estimate $\frac{\pi}{4}$ numerically.

Therefore:

- Sample 100000 independent x-coordinates from Unif.
- Sample 100000 independent y-coordinates from Unif.
- Calculate an approximate value for π using the Monte-Carlo approach.

Note: The uniform distribution [0,1) (Unif) can be sampled using runif().

• A family has installed a device to monitor their daily energy consumption (in kWh). When a measurement fails or is unavailable NA is recorded.

You can invoke the following code to generate the measurements generated by the device.

```
dailyusage <- 30.0 + runif(365, min=0, max=5.0)
dailyusage[sample(1:365, sample(1:50,1), replace=FALSE)] <- NA</pre>
```

- How many measurements failed?
- What is the average daily energy consumption (based on the non-failed) measurements?

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out" (Emil Artin)

4 Matrices & Arrays

Matrices and arrays are **homogeneous atomic vectors** with an **extra** attribute: dimension

By default, the elements are stored in a **column-major** fashion. (cfr. **Fortran**). However, we can store the elements in **row-major** order (cfr. **C**) as well.

4.1 Creation of matrices

Matrices can be created in different ways:

- use of the **matrix()** function
- use of rbind()/cbind()
- set the attributes() of a vector
- special functions like e.g. diag()

4.1.1 Examples

• use of the **matrix()** function:

The **matrix()** function creates a matrix based on a vector.

By default, the elements are stored in a column-major fashion.

The use of the flag byrow=TRUE will store the data in a row-major fashion.

```
A <- matrix(data=1:10, nrow=2) # Column-major (like Fortran)
     [,1] [,2] [,3] [,4] [,5]
                       7
[1,]
        1
             3
                  5
[2,]
                  6
                            10
B <- matrix(data=c(2,3,893,0.17), nrow=2, ncol=2)
     [,1]
            [,2]
[1,]
        2 893.00
[2,]
            0.17
```

```
C <- matrix(data=1:10, nrow=2, byrow=TRUE) # Row-major (like C, C++)

[,1] [,2] [,3] [,4] [,5]

[1,] 1 2 3 4 5

[2,] 6 7 8 9 10
```

• use of the **rbind()/cbind()** functions:

```
rbind(): Bind several vectors (as rows) into a matrix.
cbind(): Bind several vectors (as columns) into a matrix.
```

```
A <- rbind(1:10,11:20)
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
[1,]
              2
                   3
                        4
                              5
                                   6
                                         7
                                              8
                                                    9
                                                         10
[2,]
       11
             12
                  13
                       14
                             15
                                  16
                                        17
                                                   19
                                                         20
typeof(A)
[1] "integer"
class(A)
[1] "matrix" "array"
```

```
B <- cbind(1:5,6:10,11:15)
B
```

```
[,1] [,2] [,3]
[1,]
        1
              6
                   11
[2,]
         2
              7
                   12
[3,]
        3
              8
                   13
[4,]
                   14
[5,]
         5
             10
                   15
class(B)
```

- [1] "matrix" "array"
 - modifying the dim attribute.

The **fundamental** difference between an R vector and matrix is the presence (in the case of matrices) of a non NULL dim attribute.

We change easily convert a vector into a matrix by setting the dimensions of the vector:

- through the dim() function.
- through the attr() function.

The inverse can be done as well by setting the dim attribute of matrix to NULL.

4.1.2 Examples

```
A <- 1:10
typeof(A)

[1] "integer"
class(A)

[1] "integer"
dim(A)
```

NULL

```
# Matrix
B <- matrix(1:10,nrow=2,ncol=5,byrow=TRUE)</pre>
typeof(B)
[1] "integer"
class(B)
[1] "matrix" "array"
dim(B)
[1] 2 5
# Vector
A <- 1:10
[1] 1 2 3 4 5 6 7 8 9 10
dim(A)
NULL
typeof(A)
[1] "integer"
class(A)
[1] "integer"
\# OPTION I: Using the dim function transform a vector into a matrix
dim(A) \leftarrow c(2,5)
     [,1] [,2] [,3] [,4] [,5]
[1,] 1 3 5 7 9
[2,]
       2 4 6 8 10
dim(A)
[1] 2 5
typeof(A)
[1] "integer"
class(A)
[1] "matrix" "array"
```

```
\# Converting the matrix back to a vector
dim(A) <- NULL</pre>
dim(A)
NULL
typeof(A)
[1] "integer"
class(A)
[1] "integer"
# Option II: More general way
# Convert vector into a matrix
A <- 1:8
[1] 1 2 3 4 5 6 7 8
class(A)
[1] "integer"
attr(A,'dim') <- c(2,4)
     [,1] [,2] [,3] [,4]
[1,]
       1 3 5
[2,]
        2
              4
                   6
class(A)
[1] "matrix" "array"
# Convert matrix into a vector.
attr(A, 'dim') <- NULL</pre>
[1] 1 2 3 4 5 6 7 8
class(A)
[1] "integer"
4.2
      Operations on matrices
  • Operations like *,/, + happpen element-wise.
   • There are also more specialized functions:
       - the mean over rows and columns (rowMeans(), colMeans())
       - linear algebra functions (\%*\%, \mathbf{t}(), \ldots)
```

4.2.1 Examples

[1,] 1 3 5 [2,] 2 4 6

• Operations (by **default: element-by-element**): A <- matrix(1:10, nrow=2) B <- matrix(seq(10, 100, by=10), nrow=2)</pre> [,1] [,2] [,3] [,4] [,5] [1,] 1 3 5 7 9 [2,] 2 4 6 8 10 [,1] [,2] [,3] [,4] [,5] [1,] 10 30 50 70 90 [2,] 20 40 60 80 100 A*B [,1] [,2] [,3] [,4] [,5] [1,] 10 90 250 490 810 [2,] 40 160 360 640 1000 C <- matrix(rep(2,10), nrow=2)</pre> C**A[,1] [,2] [,3] [,4] [,5] [1,] 2 8 32 128 512 [2,] 4 16 64 256 1024 • Calculate row and column means : # Means of rows and columns [,1] [,2] [,3] [,4] [,5] [1,] 1 3 5 7 9 [2,] 2 4 6 8 10 rowMeans(A) [1] 5 6 colMeans(A) [1] 1.5 3.5 5.5 7.5 9.5 • Matrix multiplication (%*%) : A <- matrix(1:6, nrow=2) [,1] [,2] [,3]

```
B <- matrix(seq(10,120,by=10), nrow=3)</pre>
     [,1] [,2] [,3] [,4]
[1,]
       10
             40
                   70 100
[2,]
                   80
                      110
[3,]
       30
             60
                   90 120
C <- A%*%B
     [,1] [,2] [,3] [,4]
[1,] 220 490 760 1030
[2,] 280 640 1000 1360
dim(C)
[1] 2 4
   • Linear algebra routines
Some of the more common ones in R:
   • solve(): invert a square matrix
   • diag()
       - extracts the diagonal of a matrix when a matrix is provided.
       - creates a diagonal matrix when a vector is provided.
   • eigen(): calculates the eigenvalues and eigenvectors of a matrix
   • det() : calculates the determinant of a matrix.
   • t(): calculates the transpose<sup>1</sup> of a matrix.
# Invert matrix A
A \leftarrow matrix(c(1, 3, 2, 4), ncol = 2, byrow = T)
Ainv <- solve(A)
Ainv %*% A
     [,1] [,2]
[1,]
        1
              0
[2,]
        0
# Create a diagonal matrix
C \leftarrow diag(c(1,4,7))
     [,1] [,2] [,3]
[1,]
              0
                    0
         1
[2,]
                    0
              0
                    7
[3,]
# Extract the diagonal elements
```

D <- matrix(1:8,nrow=4)</pre>

¹Can also be used for dataframes (see later)

```
[1,]
[2,]
[3,]
        3
             7
[4,]
diag(D)
[1] 1 6
# Calculate eigenvalues and eigenvectors of A
r <- eigen(A)
eigen() decomposition
$values
[1] 5.3722813 -0.3722813
$vectors
                      [,2]
           [,1]
[1,] -0.5657675 -0.9093767
[2,] -0.8245648 0.4159736
# Eigenvalues
r$values
[1] 5.3722813 -0.3722813
# Matrix with eigenvectors
r$vectors
           [,1]
                      [,2]
[1,] -0.5657675 -0.9093767
[2,] -0.8245648 0.4159736
# Diagonal Matrix (Similarity Transformation)
solve(r$vectors) %*% A %*% r$vectors
              [,1]
                          [,2]
[1,] 5.372281e+00 0.0000000
[2,] -3.330669e-16 -0.3722813
Note that under the hood R calls BLAS and LAPACK.
# Find the version used of BLAS and LAPACK
La_library()
[1] "/usr/lib/x86_64-linux-gnu/lapack/liblapack.so.3.7.1"
extSoftVersion()["BLAS"]
                                              BLAS
"/usr/lib/x86_64-linux-gnu/blas/libblas.so.3.7.1"
```

4.3 Retrieving elements/subsetting

[,1] [,2]

Matrices (and arrays) can be subsetted in different ways:

- use an **index** for each dimension, where the dimensions are comma-separated
 - If an index for a dimension is omitted:
 consider all dimensions (may lead to reduction of the dimension)
 - but you can use drop=FALSE to prevent dimensionality reduction.
- use another **vector** (can be either linear or a vector for each dimension)
- by using another matrix.

4.3.1 Examples

• Use of indices:

```
A <- matrix(1:30, nrow=6, ncol=5)
     [,1] [,2] [,3] [,4] [,5]
[1,]
             7
        1
                  13
                       19
                             25
[2,]
        2
             8
                  14
                        20
                             26
[3,]
             9
                       21
                             27
                  15
[4,]
        4
             10
                 16
                       22
                             28
[5,]
             11
                  17
                       23
                             29
[6,]
        6
             12
                  18
                       24
                             30
```

```
A[3,4]
```

[1] 21

A[6,2]

[1] 12

```
x1 <- A[2,]
x1
```

```
[1] 2 8 14 20 26
dim(x1)
```

NULL

```
x2 <- A[,3]
x2
```

```
[1] 13 14 15 16 17 18
dim(x2)
```

NULL

The flag drop=FALSE can be used to prevent dimensionality reduction

```
y1 <- A[2,,drop=FALSE]
y1</pre>
```

```
[,1] [,2] [,3] [,4] [,5]
[1,] 2 8 14 20 26
dim(y1)
[1] 1 5
y2 <- A[,3,drop=FALSE]
  [,1]
[1,] 13
[2,] 14
[3,] 15
[4,] 16
[5,] 17
[6,] 18
dim(y2)
[1] 6 1
• Use of vector(s):
  [,1] [,2] [,3] [,4] [,5]
[1,] 1 7 13 19 25
[2,] 2 8 14 20 26
[3,] 3 9 15 21 27
[4,] 4 10 16 22 28
[5,] 5 11 17 23 29
[6,] 6 12 18 24 30
x1 \leftarrow A[2:4,]
  [,1] [,2] [,3] [,4] [,5]
[1,] 2 8 14 20 26
[2,] 3 9 15 21 27
[3,] 4 10 16 22 28
dim(x1)
[1] 3 5
x2 \leftarrow A[,1:3]
x2
 [,1] [,2] [,3]
[1,] 1 7 13
[2,] 2 8 14 [3,] 3 9 15
```

```
[4,] 4 10 16
[5,] 5 11 17
[6,] 6 12 18
```

dim(x2)

[1] 6 3

Using a vector for EACH dimension

A[c(1,3),c(2,4)]

[1,] [,2] [1,] 7 19 [2,] 9 21

Using 1 vector => Linear index

A[c(1,3,8,10)]

[1] 1 3 8 10

A[c(TRUE, FALSE, TRUE, TRUE, FALSE, TRUE), c(2,3)]

[,1] [,2] [1,] 7 13

[2,] 9 15

[3,] 10 16

[4,] 12 18

A[c(TRUE, FALSE, TRUE, TRUE, FALSE, TRUE),]

[,1] [,2] [,3] [,4] [,5] [1,] 1 7 13 19 25 [2,] 3 9 15 21 27 [3,] 4 10 16 22 28 [4,] 6 12 18 24 30

Use of a linear index

A[c(TRUE, FALSE, TRUE, TRUE, FALSE, TRUE)]

- [1] 1 3 4 6 7 9 10 12 13 15 16 18 19 21 22 24 25 27 28 30
 - Use of a matrix:

Α

[1,1] [,2] [,3] [,4] [,5] [1,] 1 7 13 19 25 [2,] 2 8 14 20 26

```
[3,]
              9
                   15
                         21
                               27
[4,]
         4
              10
                   16
                         22
                               28
[5,]
         5
              11
                   17
                         23
                               29
[6,]
         6
              12
                         24
                    18
                               30
```

[1] 2 27 10 30

4.4 Hash tables/dictionaries

We can also use hashes for matrices. We can select one or both dimensions. To create hashes, for: - rows: use rownames - columns: use colnames

To remove the hash, use the **NULL** (like for vectors).

4.4.1 Examples

```
A1 < - c(0)
              , 5471.52, 5091.57, 5392.82,
        5416.45, 4584.33, 4904.83, 3851.73)
A2 < -c(5471.52,
                        0, 1315.28, 927.35,
        1505.11, 944.40, 1157.42, 1945.42)
A3 \leftarrow c(5091.57, 1315.28,
                                  0, 2166.00,
        2724.01, 1571.76, 293.52, 1240.77)
A4 \leftarrow c(5392.82, 927.35, 2166.00,
         577.85, 973.23, 1947.28, 2422.32)
A5 \leftarrow c(5416.45, 1505.11, 2724.01, 577.85,
               0, 1366.63, 2490.97, 2838.62)
A6 \leftarrow c(4584.33, 944.40, 1571.76, 973.23,
                        0, 1290.15, 1474.26)
        1366.63,
A7 <- c(4904.83, 1157.42, 293.52, 1947.28,
        2490.97, 1290.15,
                                  0, 1064.41)
A8 \leftarrow c(3851.73, 1945.42, 1240.77, 2422.32,
        2838.62, 1474.26, 1064.41,
```

```
dist <- rbind(A1,A2,A3,A4,A5,A6,A7,A8)
dist
                      [,3]
                              [,4]
                                       [,5]
                                               [,6]
      [,1]
              [,2]
                                                       [,7]
                                                               [,8]
      0.00\ 5471.52\ 5091.57\ 5392.82\ 5416.45\ 4584.33\ 4904.83\ 3851.73
A1
A2 5471.52
              0.00 1315.28 927.35 1505.11 944.40 1157.42 1945.42
A3 5091.57 1315.28
                      0.00 2166.00 2724.01 1571.76 293.52 1240.77
A4 5392.82 927.35 2166.00
                              0.00 577.85 973.23 1947.28 2422.32
A5 5416.45 1505.11 2724.01 577.85
                                       0.00 1366.63 2490.97 2838.62
A6 4584.33 944.40 1571.76 973.23 1366.63
                                              0.00 1290.15 1474.26
```

```
A7 4904.83 1157.42 293.52 1947.28 2490.97 1290.15 0.00 1064.41
A8 3851.73 1945.42 1240.77 2422.32 2838.62 1474.26 1064.41 0.00
```

```
# Adding hashes to both rows and columns
cities <- c("Anchorage","Atlanta","Austin","Baltimore","Boston", "Chicago", "Dallas","Denver")
rownames(dist) <- cities
colnames(dist) <- cities
dist</pre>
```

```
Anchorage Atlanta Austin Baltimore Boston Chicago Dallas Denver
Anchorage
              0.00 5471.52 5091.57 5392.82 5416.45 4584.33 4904.83 3851.73
Atlanta
           5471.52
                     0.00 1315.28 927.35 1505.11 944.40 1157.42 1945.42
Austin
           5091.57 1315.28
                             0.00 2166.00 2724.01 1571.76 293.52 1240.77
Baltimore 5392.82 927.35 2166.00
                                      0.00 577.85 973.23 1947.28 2422.32
          5416.45 1505.11 2724.01 577.85
                                              0.00 1366.63 2490.97 2838.62
Boston
          4584.33 944.40 1571.76 973.23 1366.63
                                                     0.00 1290.15 1474.26
Chicago
Dallas
          4904.83 1157.42 293.52 1947.28 2490.97 1290.15
                                                             0.00 1064.41
Denver
          3851.73 1945.42 1240.77 2422.32 2838.62 1474.26 1064.41
                                                                    0.00
```

```
dist["Chicago", "Denver"]
[1] 1474.26
dist["Austin", "Boston"]
```

[1] 2724.01

4.5 Arrays

Say something about arrays.

4.6 Exercises

• Create the following matrix A, given by:

```
[,1] [,2] [,3]
                       [,4]
                                          [,6]
        3
              9
                                 243
                                           729
[1,]
                   27
                         81
        5
[2,]
             25
                 125
                        625
                                3125
                                         15625
[3,]
        7
                 343
                       2401
                               16807
                                        117649
             49
[4,]
       11
            121 1331 14641
                              161051
                                       1771561
[5,]
       13
            169 2197 28561
                              371293
                                       4826809
[6,]
       17
            289 4913 83521 1419857 24137569
```

- 1. get element 343
- 2. get the elements 25, 625, 2197 and 4826809 (all at once).
- 3. get the fourth row as a vector.
- 4. get the fourth row as a matrix.
- 5. get columns 2 and 3 (at the same time).
- 6. get everything except rows 2 and 4.
- 7. the diagonal of matrix A.
- Linear regression:

- Step 1:

Create a synthetic data set by executing the following R code:

```
x <- seq(from=0, to=20.0, by=0.25)
a <- 2.0
b <- 1.5
c <- 0.5
y <- a + b*x + c*x^2 + rnorm(length(x))</pre>
```

- Step 2:

Our goal is to use the following linear model, i.e.:

$$Y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \epsilon_i$$

or in matrix form:

$$Y = X \beta + \epsilon \tag{1}$$

to fit the previously generated data set.

In Eq.(1), we have:

- * Y: a $n \times 1$ column vector.
- * X: a $n \times 3$ matrix.
- * β : a 3 × 1 column vector.
- * ϵ is: a $n \times 1$ column vector and $\sim N(0, \sigma^2)$

An estimate for β ($\hat{\beta}$) can be found (using Least-Squares, MLE see e.g. (Seber & Lee, 2012)) and has the following form:

$$\widehat{\beta} = (\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}\mathbf{Y} \tag{2}$$

where:

the column vector \mathbf{Y} is given by:

$$\mathbf{Y} := \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

and the matrix \mathbf{X}^2 takes the following form:

$$\mathbf{X} := \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix}$$

Calculate $\hat{\beta}$ using Eq.(2).

An estimate for the residuals $(\hat{\epsilon})$ is given by:

$$\widehat{\epsilon} = \mathbf{Y} - \mathbf{X}\,\widehat{\boldsymbol{\beta}} \tag{3}$$

Calculate $\hat{\epsilon}$ using Eq.(3).

- Step 3:

You can check your results using the following ${\tt R}$ code.

```
myquadfit <- lm(y ~ x + I(x^2))
cat(sprintf("The estimates for beta::\n"))
cat(myquadfit$coefficients)
cat(sprintf("The residuals::\n"))
cat(myquadfit$residuals)</pre>
```

 $^{^2{\}rm This}$ is a known as a Vandermonde matrix.

5 Special Data Types (Factors and Date/Time types)

Every R object has attributes (i.e. properties or metadata). They can be classified as:

- intrinisic properties e.g. length()
- external properties (to be set by the user)

- factor : **factor()** (see next section)

5.1 Attributes

```
can be get/retrieved using attributes().
can be set:

individually using attr()
in generally using structure()

some attributes can (also) be set/unset with special functions:

names: names()
dimension: dim()
comment: comment()
time series: tsp()
```

5.1.1 Examples

• 1 attribute:

```
x <- 1:5
x

[1] 1 2 3 4 5
attr(x, 'prop1') <- "hello"
attributes(x)

$prop1
[1] "hello"
x

[1] 1 2 3 4 5
attr(,"prop1")
[1] "hello"</pre>
```

```
attr(x, 'prop1') <- NULL
attributes(x)

NULL
x</pre>
```

[1] 1 2 3 4 5

```
• more than 1 attribute:
  y <- 1:8
  У
  [1] 1 2 3 4 5 6 7 8
  y <- structure(y, dim=c(2,4), tag="trial")
      [,1] [,2] [,3] [,4]
  [1,] 1 3 5 7
  [2,] 2 4 6 8
  attr(,"tag")
  [1] "trial"
  attributes(y)
  $dim
  [1] 2 4
  $tag
  [1] "trial"
  typeof(y)
  [1] "integer"
  class(y)
  [1] "matrix" "array"
  # Remove BOTH attributes
  y <- structure(y, dim=NULL, tag=NULL)</pre>
  [1] 1 2 3 4 5 6 7 8
  attributes(y)
  NULL
  typeof(y)
  [1] "integer"
  class(y)
  [1] "integer"
• names()
  # Set the names attribute
  capitals <- c("Salt Lake City", "Carson City", "Boise", "Santa Fe")</pre>
```

```
names(capitals) <- c("UT", "NV", "ID", "NM")</pre>
  capitals
                UT
                                 NV
                                                  ID
                                                                   NM
  "Salt Lake City"
                      "Carson City"
                                            "Boise"
                                                           "Santa Fe"
  attributes(capitals)
  $names
  [1] "UT" "NV" "ID" "NM"
  # Remove the names attribute
  names(capitals) <- NULL</pre>
  capitals
  [1] "Salt Lake City" "Carson City"
                                        "Boise"
                                                         "Santa Fe"
• dim()
  x <- 1:12
   [1] 1 2 3 4 5 6 7 8 9 10 11 12
  typeof(x)
  [1] "integer"
  class(x)
  [1] "integer"
  # Set the dimension attribute
  dim(x) \leftarrow c(3,4)
      [,1] [,2] [,3] [,4]
  [1,] 1 4 7
  [2,]
         2
            5 8
                      11
  [3,]
                       12
  typeof(x)
  [1] "integer"
  class(x)
  [1] "matrix" "array"
```

```
# Remove the dimension attribute
dim(x) <- NULL
x

[1] 1 2 3 4 5 6 7 8 9 10 11 12
typeof(x)

[1] "integer"
class(x)

[1] "integer"

comment()
x <- structure(1:6, comment="My vector")
typeof(x)

[1] "integer"
class(x)

[1] "integer"
comment(x)

[1] "My vector"</pre>
```

5.2 Factor variables (Categorical variables)

- Factor variables (factors, categorical variables) are discrete variables (i.e not continuous). The factors bear labels (levels) which are mapped into integers.
- Therefore, factors are stored as integer vector with 2 attributes:

```
class= "factor"levels: a vector with the "labels".
```

- By default (unordered) the labels are mapped alphabetically to the integers. We can impose our own ordering between integers and labels (levels).
- Useful functions:

```
levels(): provides the levels of a factor
table(): returns the counts of each level
is.factor(): tests whether a variable is a factor variable
is.ordered(): tests whether a variable is an ordered factor variable
```

5.2.1 Examples

• Creation of an unordered factor

```
[17] VeryLow
  Levels: High Low Medium VeryHigh VeryLow
  # by default: the levels are stored ALPHABETICALLY (i.e. unordered)
  levels(myfac.temp.data)
  [1] "High"
                                        "VeryHigh" "VeryLow"
                 "Low"
                             "Medium"
  table(myfac.temp.data)
  myfac.temp.data
      High
                Low Medium VeryHigh VeryLow
         2
                5
                           2
                                     6
  is.factor(myfac.temp.data)
  [1] TRUE
  is.ordered(myfac.temp.data)
  [1] FALSE
• Creation of an ordered factor
  # Creation of an unordered factor
  temp.data <- c("High", "Low", "VeryHigh", "Low", "VeryLow", "Medium",</pre>
                 "VeryHigh", "VeryHigh", "Low", "Low", "Medium", "VeryHigh",
                 "VeryHigh", "VeryHigh", "Low", "High", "VeryLow")
  myfac2.temp.data <- factor(temp.data, ordered=TRUE,</pre>
                              levels=c("VeryLow","Low","Medium","High","VeryHigh"))
 myfac2.temp.data
   [1] High
                                            VeryLow Medium
                                                               VeryHigh VeryHigh
                Low
                         VeryHigh Low
   [9] Low
                         Medium
                                  VeryHigh VeryHigh VeryHigh Low
                Low
                                                                        High
  [17] VeryLow
  Levels: VeryLow < Low < Medium < High < VeryHigh
  # The ordering is NOW imposed
  levels(myfac2.temp.data)
  [1] "VeryLow" "Low"
                             "Medium"
                                        "High"
                                                    "VeryHigh"
  table(myfac2.temp.data)
  myfac2.temp.data
                                  High VeryHigh
  VeryLow
                Low
                      Medium
```

VeryLow Medium VeryHigh VeryHigh

VeryHigh VeryHigh VeryHigh Low

[1] High

[9] Low

Low

Low

VeryHigh Low

Medium

2 5 2 2 6

```
is.factor(myfac2.temp.data)
  [1] TRUE
  is.ordered(myfac2.temp.data)
  [1] TRUE
  # Stripping a factor to the essentials: integer vector
  attributes(myfac2.temp.data)
  $levels
  [1] "VeryLow" "Low"
                                                     "VeryHigh"
                              "Medium"
                                         "High"
  $class
  [1] "ordered" "factor"
  class(myfac2.temp.data) <- NULL</pre>
  levels(myfac2.temp.data) <- NULL</pre>
  myfac2.temp.data
   [1] 4 2 5 2 1 3 5 5 2 2 3 5 5 5 2 4 1
   Dates and times in R.
• Date class:
    - represents calendar dates
    - built on top of doubles with class attribute 'Date'
    - 0 : Jan 1. 1970 (Unix Epoch time)
    - as.Date(): method to cast string to a Date
• POSIXct and POSIXIt : date and time
```

5.3.1 Examples

• Date

5.3

```
today <- Sys.Date()
today</pre>
```

- POSIXIt: stored as bluelist with elements for seconds, minutes, hours, day, month, year, etc.

- **POSIXct**: stores date/time values as the #seconds since Jan. 1, 1970

• lubridate: a very useful package for dates and times:

[1] "2022-11-16"

```
# Attributes of Date
class(today)
[1] "Date"
attributes(today)
$class
[1] "Date"
unclass(today)
[1] 19312
d0 <- structure(0, class='Date')</pre>
[1] "1970-01-01"
class(d0)
[1] "Date"
typeof(d0)
[1] "double"
# Convert a string into a Date
d1 <- as.Date("2022-01-01")</pre>
d1
[1] "2022-01-01"
class(d1)
[1] "Date"
typeof(d1)
[1] "double"
```

• POSIXct

```
# Convert a string into a POSIXct object
now_ct <- as.POSIXct("2018-08-01 22:00", tzone="MST")
now_ct

[1] "2018-08-01 22:00:00 MDT"

attributes(now_ct)

$class
[1] "POSIXct" "POSIXt"

$tzone
[1] ""
typeof(now_ct)

[1] "double"

# Removal of the attributes
attr(now_ct, "tzone") <- NULL
unclass(now_ct)

[1] 1533182400</pre>
```

Bibliography

Seber G.A.F. & Lee A.J. (2012). Linear Regression Analysis. Wiley Series in Probability and Statistics. Wiley.