Isomorphism in Union-Closed Sets

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History

In 1979, Péter Frankl proposed a famous conjecture about finite union-closed families. He stated that in every such family, there exists an element that appears in at least half of the sets. Despite significant efforts, the problem has remained unsolved for more than four decades.

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Union-Closed Conjecture (Frankl, 1979)

Let $\mathcal{K} \subseteq 2^{[n]}$ be a union-closed family of sets. Then there exists an element $i \in \bigcup \mathcal{K}$ such that: $|\mathcal{K}| \leq 2|\mathcal{K}^i|$, where

$$\mathcal{K}^i = \{A \in \mathcal{K} \mid i \in A\}.$$

Examples of Union-Closed Families

Example 1: Small Family

$$\mathcal{K} = \{\varnothing, \{1\}, \{1, 2\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

Here, elements **1** and **2** satisfy the conjecture.

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Example 1: Small Family

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Here, elements **1** and **2** satisfy the conjecture.

Example 2: Another Small Family

$$\mathcal{K} = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$$

In this case, both **a** and **b** satisfy the conjecture.

Isomorphism

? Suppose we have two isomorphic union-closed families. Do the elements have the same frequency counts in both families?

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Definition (1.2)

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be union-closed families of sets. A bijective mapping $h : \mathcal{K}_1 \to \mathcal{K}_2$ is called a **Isomorphism** if, for all $A_1, A_2 \in \mathcal{K}_1$, the following property holds:

$$h(A_1 \cup A_2) = h(A_1) \cup h(A_2).$$

- ✗ In an isomorphic union-closed family, does each element appear in the same number of sets?

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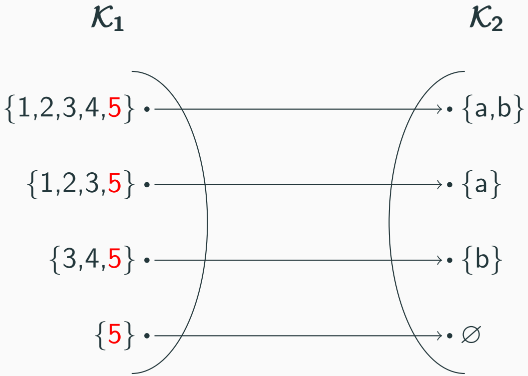

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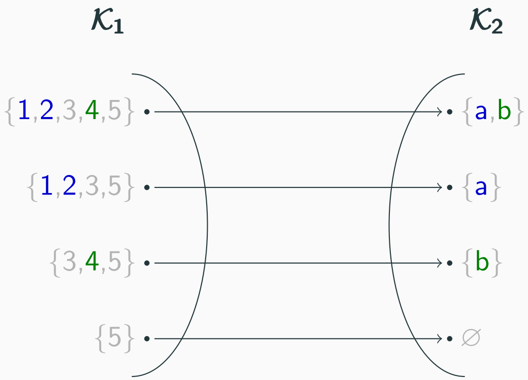
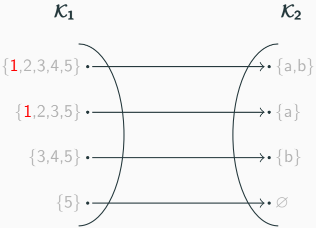
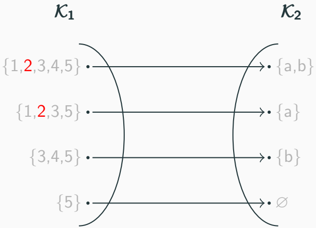



$\mathcal{K}_1$
$\mathcal{K}_2$
{1,2,3,4,5}
{1,2,3,5}
{3,4,5}
{5}
{a,b}
{a}
{b}
∅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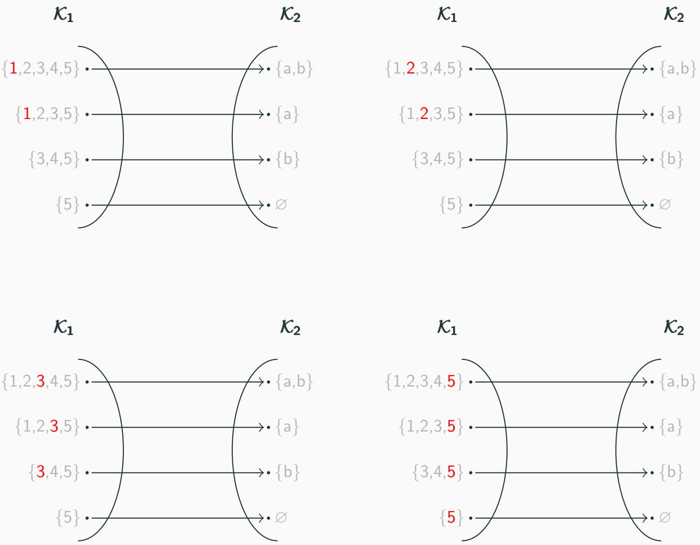
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Definition (1.3)

Let $\mathcal{K} \subseteq 2^{[n]}$. An element $z \in \bigcup \mathcal{K}$ is called **redundant**, if removing z from every set in $\mathcal{K}$ does not change the size of the collection. Specifically, we define:

$$\mathcal{K}^{\setminus z} = \{X \setminus \{z\} \mid X \in \mathcal{K}\},$$

where $|\mathcal{K}| = |\mathcal{K}^{\setminus z}|$. The collection $\mathcal{K}^{\setminus z}$ is called the **reduced collection**.

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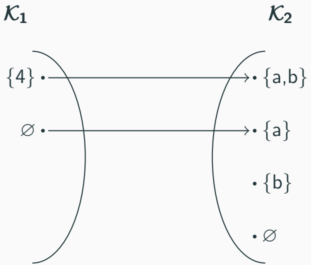
Definition (1.4)

A collection $\mathcal{K} \subseteq 2^{[n]}$ is called **pure** if it does not have any **redundant** element.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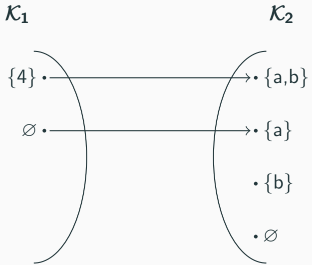
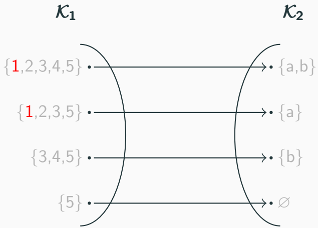
$\{4\}$
$\varnothing$
$\{a,b\}$
$\{a\}$
$\{b\}$
$\varnothing$


$\mathcal{K}_1$
$\mathcal{K}_2$
{4}
∅
{a,b}
{a}
{b}
∅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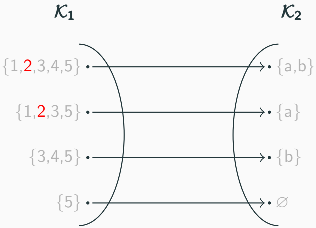
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Algorithm Redundancy Removal

```
while redundant elements exist do
    Identify all redundant elements
    Remove one redundant element
end while
```

Algorithm Redundancy Removal

1: **while** redundant elements exist **do**
2: Identify all redundant elements
3: Remove one redundant element
4: **end while**

Purified Collection

The **Purified Collection** of $\mathcal{K}$, denoted by $\mathcal{K}^*$, is constructed by iteratively removing all redundant elements from $\mathcal{K}$.

Pure Collection

1. $\mathcal{K}^*$ is not well defined at all.
2. The number of elements in different version of $\mathcal{K}^*$ might be different.

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Corollary (1.3)

For any union-closed family $\mathcal{K}$, there exists an isomorphism $h : \mathcal{K} \to \mathcal{K}^*$.

Corollary (2.1)

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two pure union-closed families of sets. If there exists an isomorphism $h : \mathcal{K}_1 \to \mathcal{K}_2$, then:

$$\left|\bigcup \mathcal{K}_1\right| = \left|\bigcup \mathcal{K}_2\right|.$$

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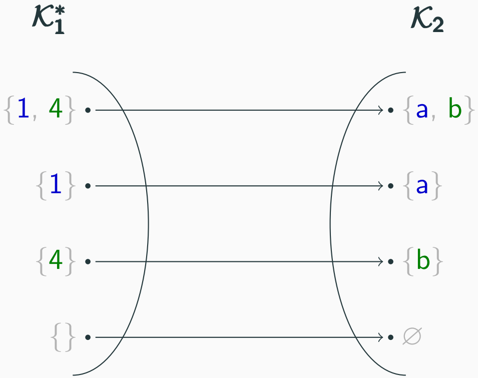
- The Frankl conjecture is true iff it's true for **pure Union-closed family**.
- Without loss of generality we can suppose our families are **pure**.

✓ In an isomorphic **pure union-closed family**, does each element appear in the same number of sets?



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Theorem (Cardinality)

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be pure union-closed families of sets. If there exists an isomorphism $h : \mathcal{K}_1 \to \mathcal{K}_2$, then for every $A \in \mathcal{K}_1$, we have:

$$|A| = |h(A)|.$$

Corollary (2.2)

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be pure union-closed families of sets. If there exists an isomorphism $h : \mathcal{K}_1 \to \mathcal{K}_2$, then for any $A, B \in \mathcal{K}_1$, the following properties hold:

1. $|A \cup B| = |h(A) \cup h(B)|$.
2. $|A \cap B| = |h(A) \cap h(B)|$.
3. $|A \setminus B| = |h(A) \setminus h(B)|$.
4. $|A^c| = |h(A)^c|$,

where $A^c = \bigcup \mathcal{K}_1 \setminus A$ *and* $h(A)^c = \bigcup \mathcal{K}_2 \setminus h(A)$.

Definition (3.1)

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two union-closed families of sets. A bijective mapping $H : \bigcup \mathcal{K}_1 \to \bigcup \mathcal{K}_2$ is called a **hyperisomorphism** if the induced mapping $h : \mathcal{K}_1 \to \mathcal{K}_2$, defined by $h(A) = H(A)$, is an isomorphism.

Hyperisomorphism

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Lemma (3.4)

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two pure union-closed families of sets, and $h : \mathcal{K}_1 \to \mathcal{K}_2$ be an isomorphism between them. Then, for each $i \in \bigcup \mathcal{K}_1$, there exists a unique $j \in \bigcup \mathcal{K}_2$ such that:

$$\mathcal{K}_2^j = h(\mathcal{K}_1^i),$$

where $h(\mathcal{K}_1^i) = \{h(A) \mid A \in \mathcal{K}_1^i\}$.

Theorem (3.1)

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be pure union-closed families of sets. For every isomorphism $h : \mathcal{K}_1 \to \mathcal{K}_2$, there exists a **hyperisomorphism** *$H : \bigcup \mathcal{K}_1 \to \bigcup \mathcal{K}_2$ such that:*

$$h(A) = \{H(a) \mid a \in A\} \quad \text{for all } A \in \mathcal{K}_1.$$

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Thank You!

Any Questions?

- Email: m.moghadas11235@gmail.com
- Paper available on ArXiv.