Isomorphism in Union-Closed Sets

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- The **Union-Closed Sets Conjecture** (Frankl, 1979) states that in any non-empty union-closed family of sets, there exists an element appearing in at least half of the sets.
- The conjecture remains open despite significant efforts.
- We explore **isomorphisms** and their structural impact on union-closed families.

- The conjecture has been studied for over four decades.
- Special cases have been resolved, but a general proof remains elusive.
- Connections to extremal combinatorics and algebraic methods.

Definitions

- A family of sets $\mathcal{K} \subseteq 2^{[n]}$ is **union-closed** if for any $A, B \in \mathcal{K}$, we have $A \cup B \in \mathcal{K}$.
- An **isomorphism** between two union-closed families $\mathcal{K}_1$ and $\mathcal{K}_2$ is a bijection $h : \mathcal{K}_1 \to \mathcal{K}_2$ such that:

$$h(A \cup B) = h(A) \cup h(B), \quad \forall A, B \in \mathcal{K}_1.$$

Example: Union-Closed Family

Consider:

$$\mathcal{K} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

- Every union of sets in $\mathcal{K}$ remains in $\mathcal{K}$.
- Example: $\{a\} \cup \{b\} = \{a, b\} \in \mathcal{K}$.

Main Theorem

Theorem: (Main Result)

- For every isomorphism $h : \mathcal{K}_1 \to \mathcal{K}_2$, there exists a **hyperisomorphism** $H : \bigcup \mathcal{K}_1 \to \bigcup \mathcal{K}_2$ such that:

$$h(A) = \{H(a) \mid a \in A\}, \quad \forall A \in \mathcal{K}_1.$$

Proof Sketch

- Show that any isomorphism h between union-closed families induces a corresponding bijection H on elements.
- Utilize the structure of **minimal elements** to construct H.
- Prove that H preserves inclusion and union properties.

Step 1: Establishing the Structure

- Definition of minimal elements and their role.
- Establish basic properties of pure union-closed families.

Step 2: Constructing the Hyperisomorphism

- Construct H using minimal elements.
- Ensure preservation of key properties.

Step 3: Proving Injectivity and Surjectivity

- Injectivity follows from distinct minimal elements.
- Surjectivity follows from properties of isomorphic families.

Step 4: Conclusion of the Proof

- Verify H satisfies all conditions.
- Summarize key insights from the proof.

Applications of the Result

- Provides a structural approach to union-closed families.
- May lead to new combinatorial methods for tackling the conjecture.
- Opens new research directions in set theory and discrete mathematics.

Open Problems Future Work

- Can hyperisomorphisms be extended to more general set families?
- What are the algorithmic implications of these structures?
- How can this be applied to the broader context of combinatorial optimization?

Conclusion

- We established a connection between **isomorphisms** and **hyperisomorphisms** in union-closed families.
- This contributes to understanding the structural behavior of these families.
- Many open problems remain.

Thank you! Questions?