MODULE TWO PROBLEM SET

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Directions: Type your solutions into this document and be sure to show all steps for arriving at your solution. Just giving a final number may not receive full credit.

PROBLEM 1

Part 1. Indicate whether the argument is valid or invalid. For valid arguments, prove that the argument is valid using a truth table. For invalid arguments, give truth values for the variables showing that the argument is not valid.

(1)

$$\boxed{\begin{array}{c}(p \wedge q) \rightarrow r \\ \\ \therefore (p \vee q) \rightarrow r\end{array}}$$

p	q	r	(p ∧ q)⟶ r	(p ∨ q)⟶ r
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

proving the argument is **valid**

Part 2. Converse and inverse errors are typical forms of invalid arguments. Prove that each argument is invalid by giving truth values for the variables showing that the argument is invalid. You may find it easier to find the truth values by constructing a truth table.

(a) Converse error

$$\boxed{\begin{array}{c}p \rightarrow q \\ q \\ \\ \therefore p\end{array}}$$

p	q	p ⟶ q
T	T	T
T	F	F
F	T	T
F	F	T

Truth values:
p = T
q = p⟶ q = F

proving it is **invalid**

(b) Inverse error

$p \to q$
$\neg p$
$\therefore \neg q$

p	q	p ⟶ q	¬ p	¬ q
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Truth values:
p = ¬ q = T
q = p⟶ q = ¬ p = F
proving it is **invalid**

Part 3. Which of the following arguments are invalid and which are valid? Prove your answer by replacing each proposition with a variable to obtain the form of the argument. Then prove that the form is valid or invalid.

(a)

The patient has high blood pressure or diabetes or both.
The patient has diabetes or high cholesterol or both.
∴ The patient has high blood pressure or high cholesterol.

P(x) : x has high blood pressure
D(x) : x has diabetes
C(x) : x has high cholesterol

$\exists x(P(x) \vee D(x))$
$\exists x(D(x) \vee C(x)$

$\exists x(P(x) \vee C(x))$

1. $\exists x(P(x) \vee D(x))$: Hypothesis

2. $\exists x(D(x) \vee C(x)$: Hypothesis
3. $P(c) \vee D(c)$: Existential Instantiation 1
4. $D(c) \vee C(c)$: Existential Instantiation 2
5. $P(c) \vee C(c)$: Hypothetical Syllogism 3,4
6. $\exists(P(x) \vee C(x))$: Existential Generalization 5

The argument is **Valid** since they use or and is proven above.

PROBLEM 2

Part 1. Which of the following arguments are valid? Explain your reasoning.

(a) I have a student in my class who is getting an A. Therefore, John, a student in my class, is getting an A.

Invalid
It states a student in the class is getting an A. Not every student in the class is getting an A. Nothing is proved to show that John is the(or one of the) student(s) that is receiving an A.

(b) Every Girl Scout who sells at least 30 boxes of cookies will get a prize. Suzy, a Girl Scout, got a prize. Therefore, Suzy sold at least 30 boxes of cookies.

Valid
In order to get the prize a girl scout must sell 30 boxes of cookies. If Suzy did not sell 30 boxes she would not have received the prize. Since Suzy did get a prize she had to of sold 30 boxes of cookies.

Part 2. Determine whether each argument is valid. If the argument is valid, give a proof using the laws of logic. If the argument is invalid, give values for the predicates P and Q over the domain a, b that demonstrate the argument is invalid.

(a)

$$\boxed{\begin{array}{c} \exists x\,(P(x) \;\wedge\; Q(x)) \\ \\ \therefore \exists x\, Q(x) \;\wedge\; \exists x\, P(x) \end{array}}$$

1. $\exists x(P(x) \wedge Q(x))$ Hypothesis
2. $P(c) \wedge Q(c)$ Existential Instantiation 1
3. $P(c)$ Simplifaction 2
4. $Q(c)$ Simplifaction 2
5. $\exists x P(x)$ Exisential Generalzation 3
6. $\exists x Q(x)$ Exisential Generalzation 4
7. $\exists x Q(x) \wedge \exists x P(x)$ Conjunction 5, 6

(b)

$$\boxed{\begin{array}{c} \forall x\,(P(x) \;\vee\; Q(x)) \\ \\ \therefore \forall x\, Q(x) \;\vee\; \forall x\, P(x) \end{array}}$$

1. $\forall x(P(x) \vee Q(x))$ Hypothesis
2. $P(c) \vee Q(c)$ Universal Instantiation 1

Not Valid cannot prove any further.

Problem 3

Prove the following using a direct proof. Your proof should be expressed in complete English sentences.

If a, b, and c are integers such that b is a multiple of a^3 and c is a multiple of b^2, then c is a multiple of a^6.

Proof:
Let a,b,and c be integers
$b = na^3$ for some integer n
$c = mb^2$ for some integer m
Since c is a multiple of b^2
$c = m(na^3)^2$
$c = (mna^6)$
Since m and n are both integers c is a multiple of a^6

PROBLEM 4

Prove the following using a direct proof:

The sum of the squares of 4 consecutive integers is an even integer. Proof:
Let k, k+1, k+2, and k+3 be integers
We can sum them together
M = (k)+(k+1)+(k+2)+(k+3)
Which can also be written as
M = 4k + (1+2+3)
It can then be simplified as
M = 4k + 6
If 2 gets factored out
M = 2 + (2k+3)
n is an even number such that:
$n = 2 * j, j \in$ Z

PROBLEM 5

Prove the following using a proof by contrapositive:

Let x be a rational number. Prove that if xy is irrational, then y is irrational.

Let x and y be real numbers
Assume x and x+y are rational
Suppose that y is irrational
If there are Integers j,k,m,n
k and n do not equal 0 such that
$x + y = \frac{j}{k} and x = \frac{m}{n}$
we substitute the x
$\frac{m}{n} + y = \frac{j}{k} or y = \frac{jn-km}{kn}$
jn - km and kn are both integers and kn does not equal 0
Hence proves if xy is irrational then y is irrational

Problem 6

Prove the following using a proof by contradiction:

The average of four real numbers is greater than or equal to at least one of the numbers.

Let the four real numbers be a,b,c,and d
For integer; $= \frac{a+b+c+d}{4}$
Assume $f < a, f < b, f < c, f < d$
It can also written as $f < \frac{a+b+c+d}{4}$ Because $f = \frac{a+b+c+d}{4}$ it can be plugged in as $f < f$
Since f can not be less than f
The average of 4 real numbers is greater than or equal to at least one of the numbers.

Problem 7

Let $q = \dfrac{a}{b}$ and $r = \dfrac{c}{d}$ be two rational numbers written in lowest terms. Let $s = q + r$ and $s = \dfrac{e}{f}$ be written in lowest terms. Assume that s is not 0.

Prove or disprove the following two statements.

a. If b and d are odd, then f is odd.

Because $q = \frac{a}{b}$ and is an odd rational number q also equals $\frac{a}{2k+1}$
Because $r = \frac{c}{d}$ and is an odd rational number r also equals $\frac{c}{2n+1}$
Since s is equal to q + r it stands to reason s also equals $\frac{a}{b} + \frac{c}{d}$
or $s = \frac{ad}{bd} + \frac{cb}{bd}$
Since $s = \frac{e}{f}$
$(\frac{ad}{bd} + \frac{cb}{bd}) = \frac{e}{f}$
Hence if b and d are odd then f is odd

b. If b and d are even, then f is even.

Because $q = \frac{a}{b}$ and is an even rational number q also equals $\frac{a}{2k}$
Because $r = \frac{c}{d}$ and is an even rational number r also equals $\frac{c}{2n}$
Since s is equal to q + r it stands to reason s also equals $\frac{a}{b} + \frac{c}{d}$
or $s = \frac{ad}{bd} + \frac{cb}{bd}$
Since $s = \frac{e}{f}$
$(\frac{ad}{bd} + \frac{cb}{bd}) = \frac{e}{f}$
Hence if b and d are even then f is even

Problem 8

Define $P(n)$ to be the assertion that:

$$\sum_{j=1}^{n} j^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that $P(3)$ is true.

$\displaystyle\sum_{j=1}^{3} j^2 = 14 = \frac{3(3+1)(2(3)+1)}{6}$
P(3) is true

(b) Express $P(k)$.

$\displaystyle\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2(k)+1)}{6}$

(c) Express $P(k+1)$.

$\displaystyle\sum_{j=1}^{k+1} j^2 = \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$

(d) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the base case?

base case being j = 1
$1^2 = \frac{1(1+1)(2(1)+1)}{6}$

(e) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the inductive step?

base case must be proven first
base case being j = 1

$1^2 = \frac{1(1+1)(2(1)+1)}{6}$
because base case is true

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

If base case is

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

true
then P(k + 1) is true

$$\sum_{j=1}^{k+1} j^2 = \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$$

and P(k) is true

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2(k)+1)}{6}$$

(g) Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Since all have been proven true

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

P(k + 1)

$$\sum_{j=1}^{k+1} j^2 = \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$$

P(k)

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2(k)+1)}{6}$$

then P(n)

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \text{ must be true}$$