Forward Kinematics of Delta Robot

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List of Symbols

L_i : length of active leg ($\overline{\boldsymbol{B}_i\boldsymbol{A}_i}$)

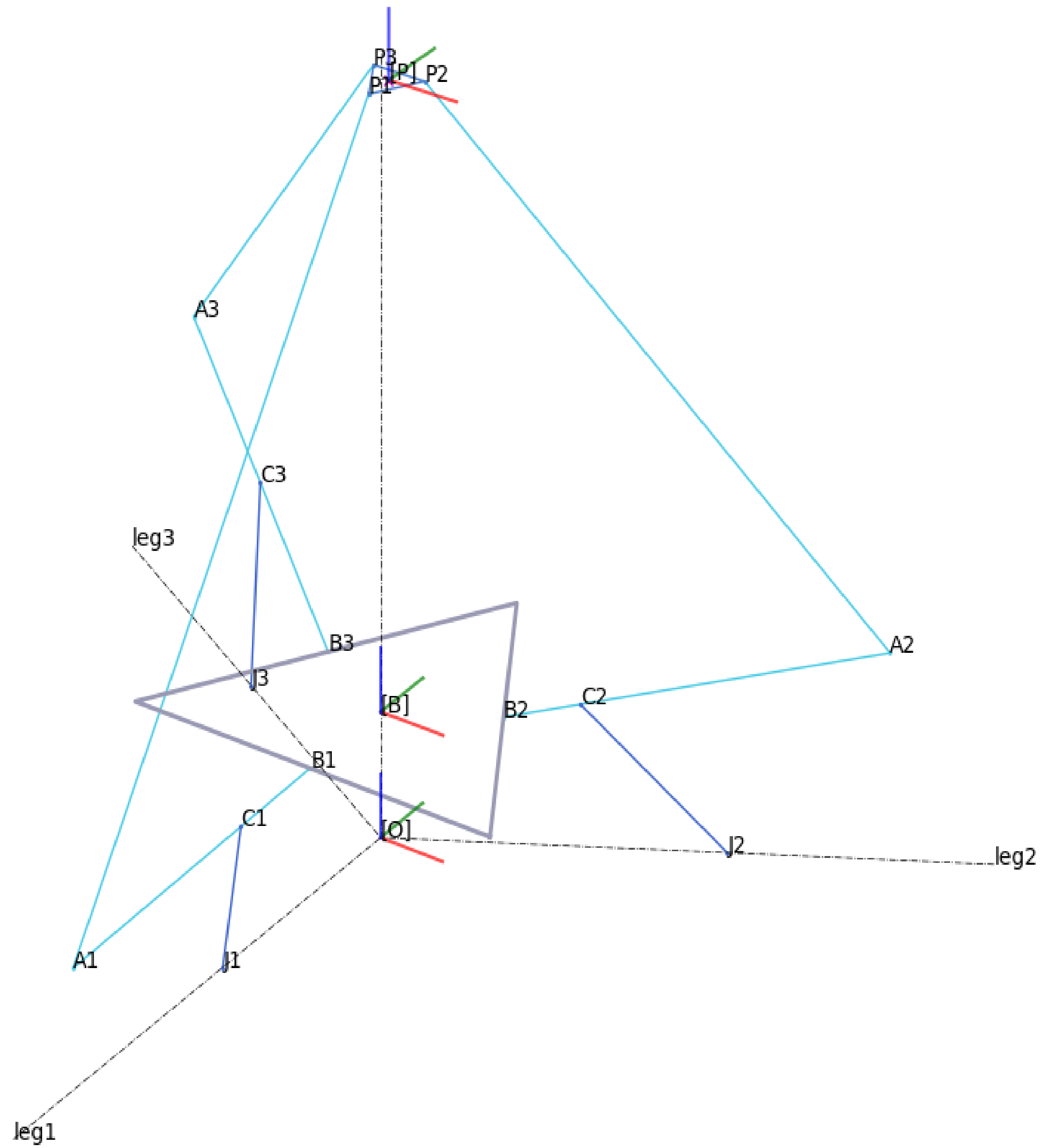
l_i : Length of passive leg ($\overline{\boldsymbol{A}_i\boldsymbol{P}_i}$)

B_i : Hips (Revolute Joint)

A_i : Knees (Universal Joint)

P_i : Ankles (Universal Joint)

$i = 1, 2, 3$



List of Symbols

$[B]$: Base Reference Frame

B_0 : Origin of Base Reference Frame

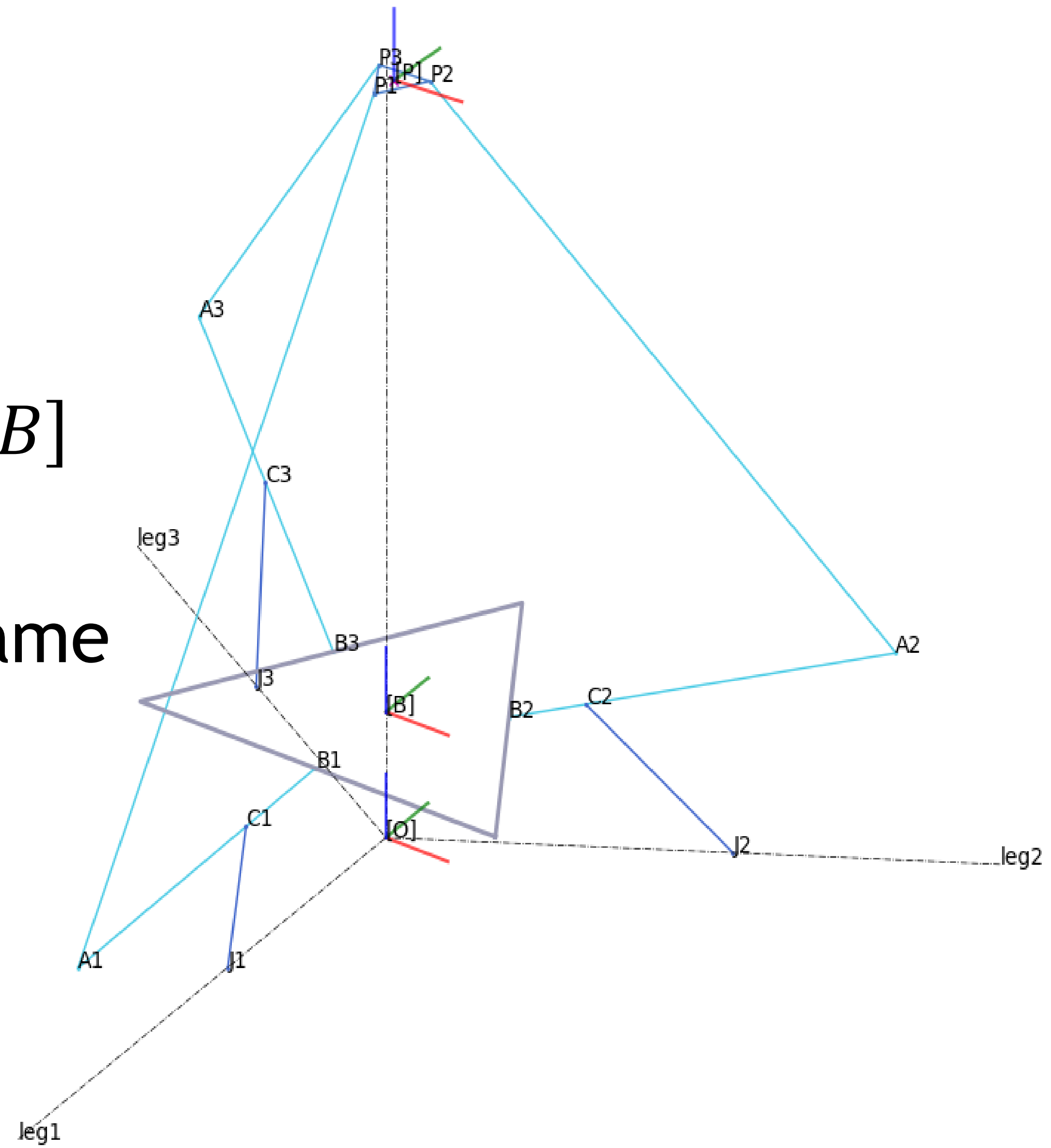
h_B : Z-axis height between $[O]$ and $[B]$

$[P]$: Platform Reference Frame

P_0 : Origin of Platform Reference Frame

$[O]$: Reference Frame

O : Origin of Reference Frame



List of Symbols

C_i : One end of the linear actuator

J_i : Another end of the linear actuator

D_i : Length of linear actuator

t_i : Ratio of C_i $(\overline{\boldsymbol{B}_i\boldsymbol{C}_i}/\overline{\boldsymbol{B}_i\boldsymbol{A}_i})$, $(0 \leq t_i \leq 1)$

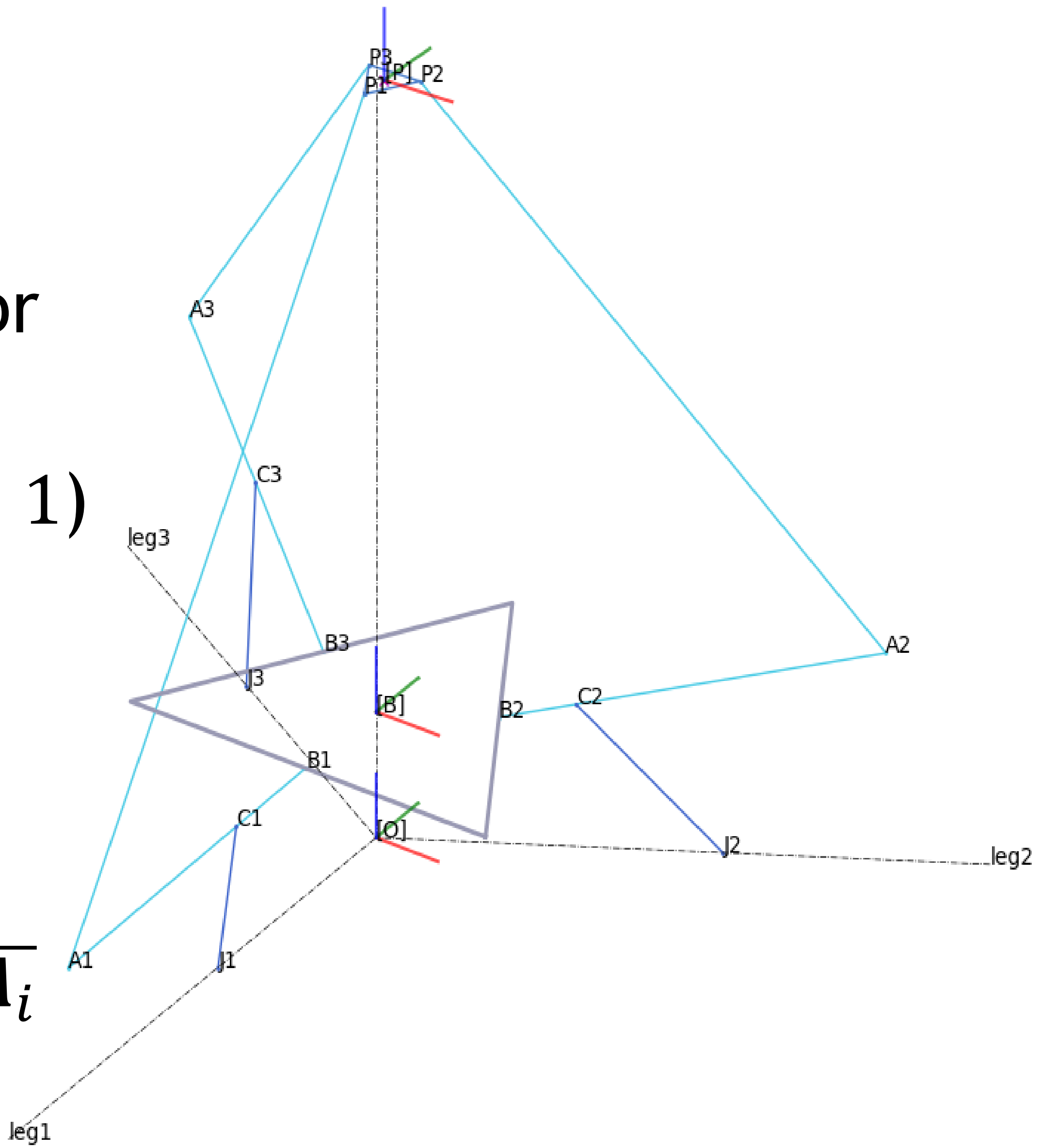
B'_i : xy-plane projection of $\boldsymbol{B}_i$

s_i : Length of $\overline{\boldsymbol{B}'_i\boldsymbol{J}_i}$

θ_i : Angle of Active Leg

= Angle between xy-plane and $\overline{\boldsymbol{B}_i\boldsymbol{A}_i}$

= Angle between $\overline{\boldsymbol{B}_i\boldsymbol{A}_i}$ and $\overline{\boldsymbol{B}'_i\boldsymbol{J}_i}$



List of Symbols

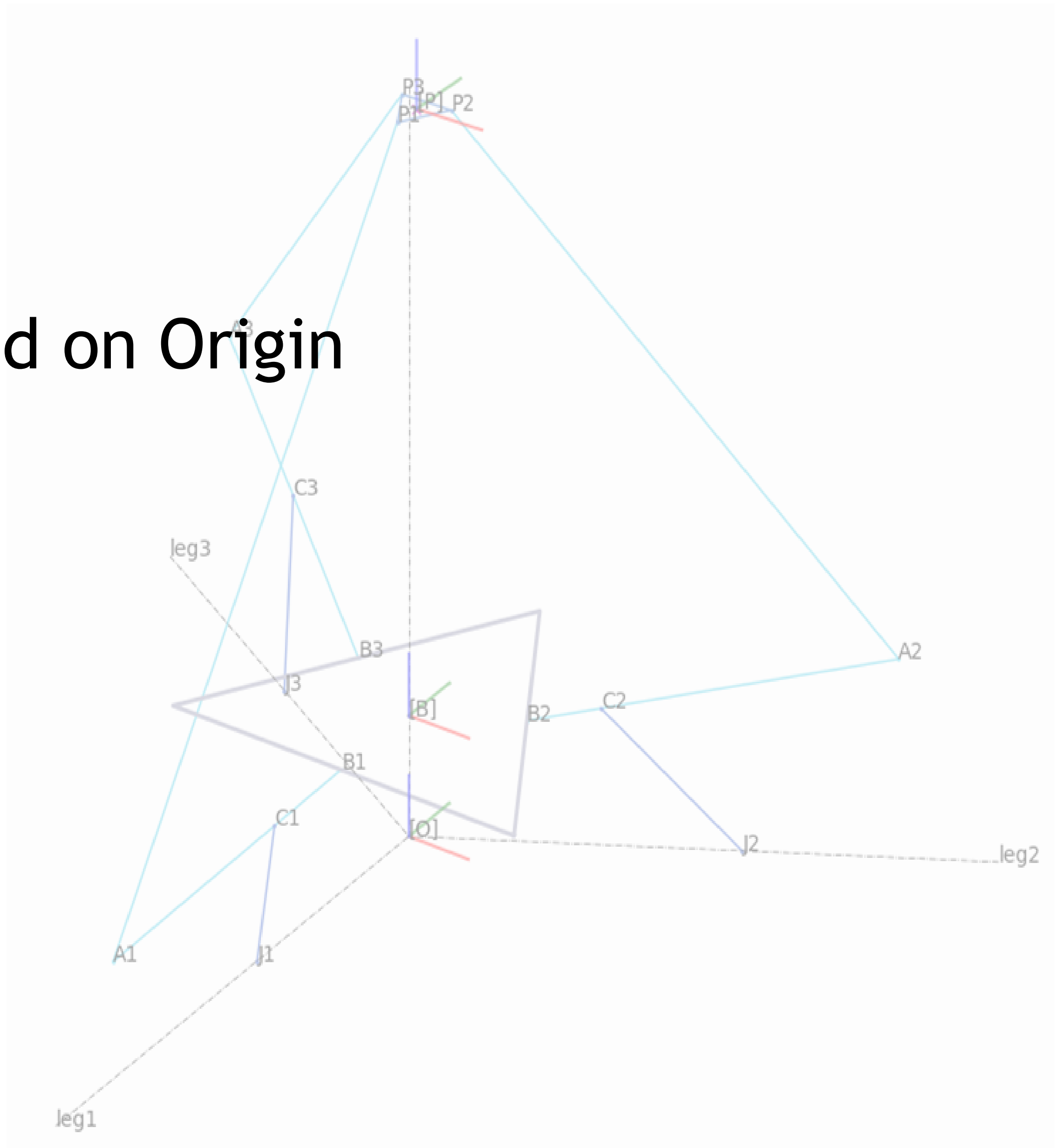
$\boldsymbol{B}, \boldsymbol{A}, \boldsymbol{P}, \boldsymbol{C}$: Bold Alphabet

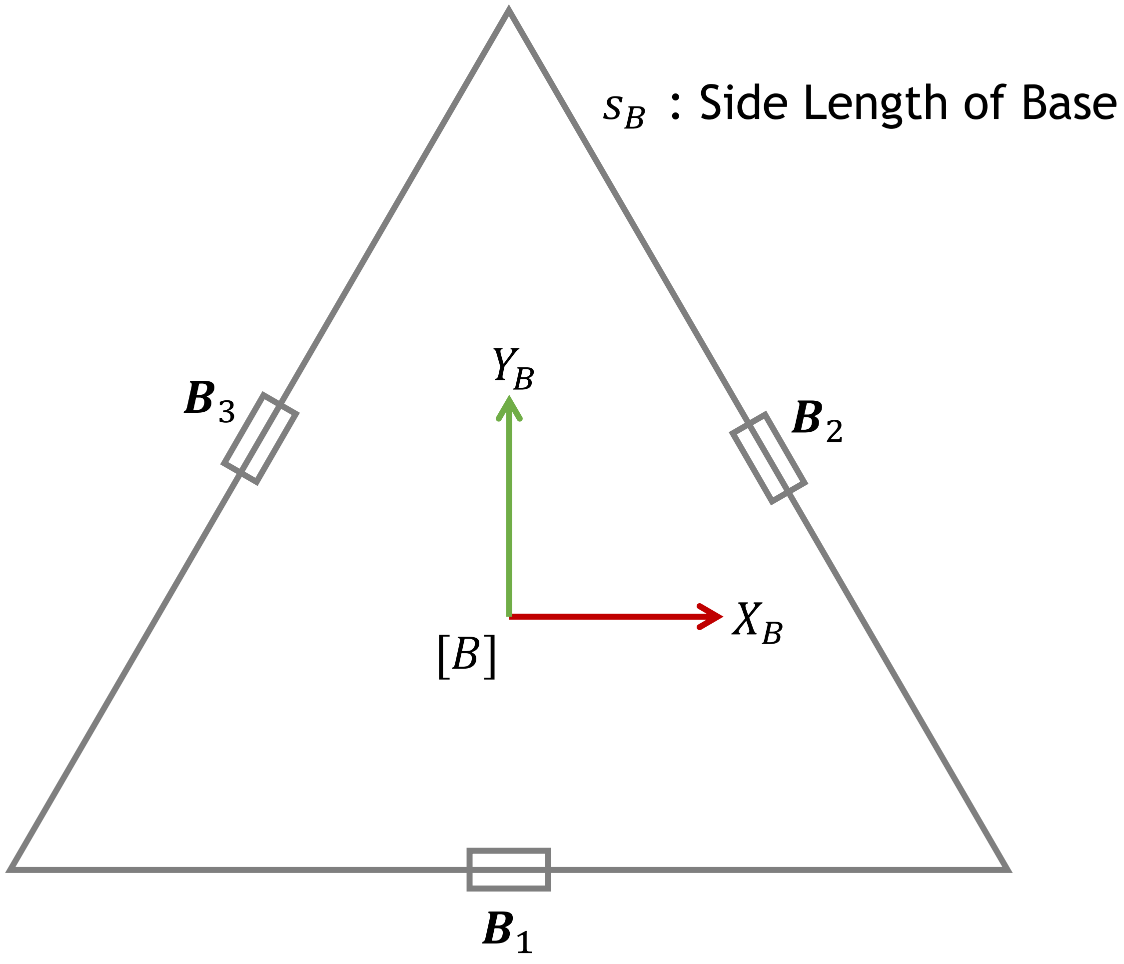
: Row vector or Point based on Origin

${}^{B}\boldsymbol{B}_i$: Row Vector based on $[B]$

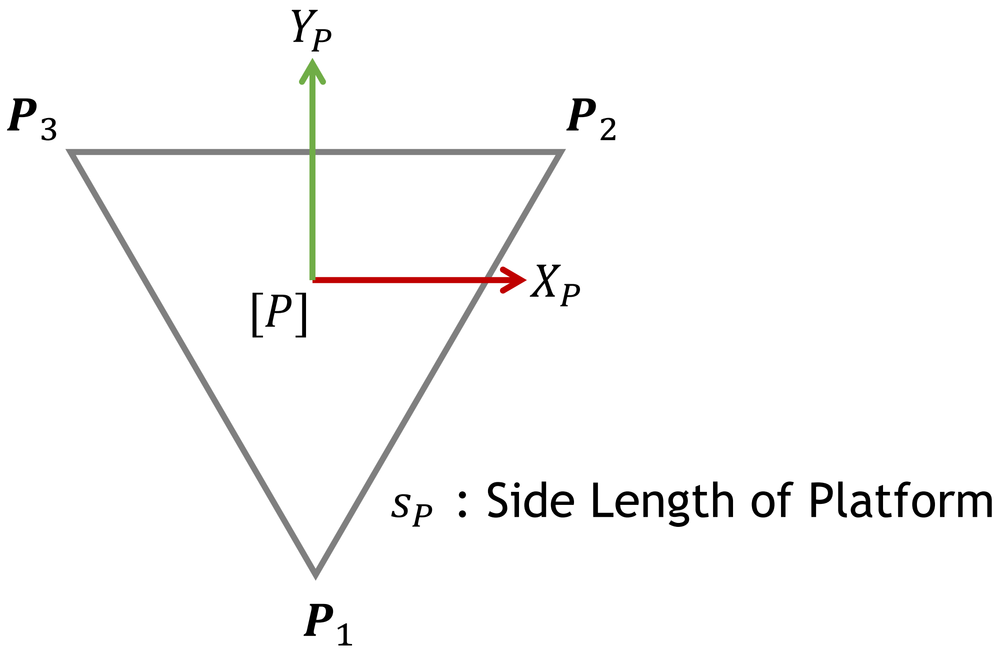
$\underline{B}, \underline{A}, \underline{P}, \underline{C}$: Alphabet with underbar

: Matrix





Base Reference Frame

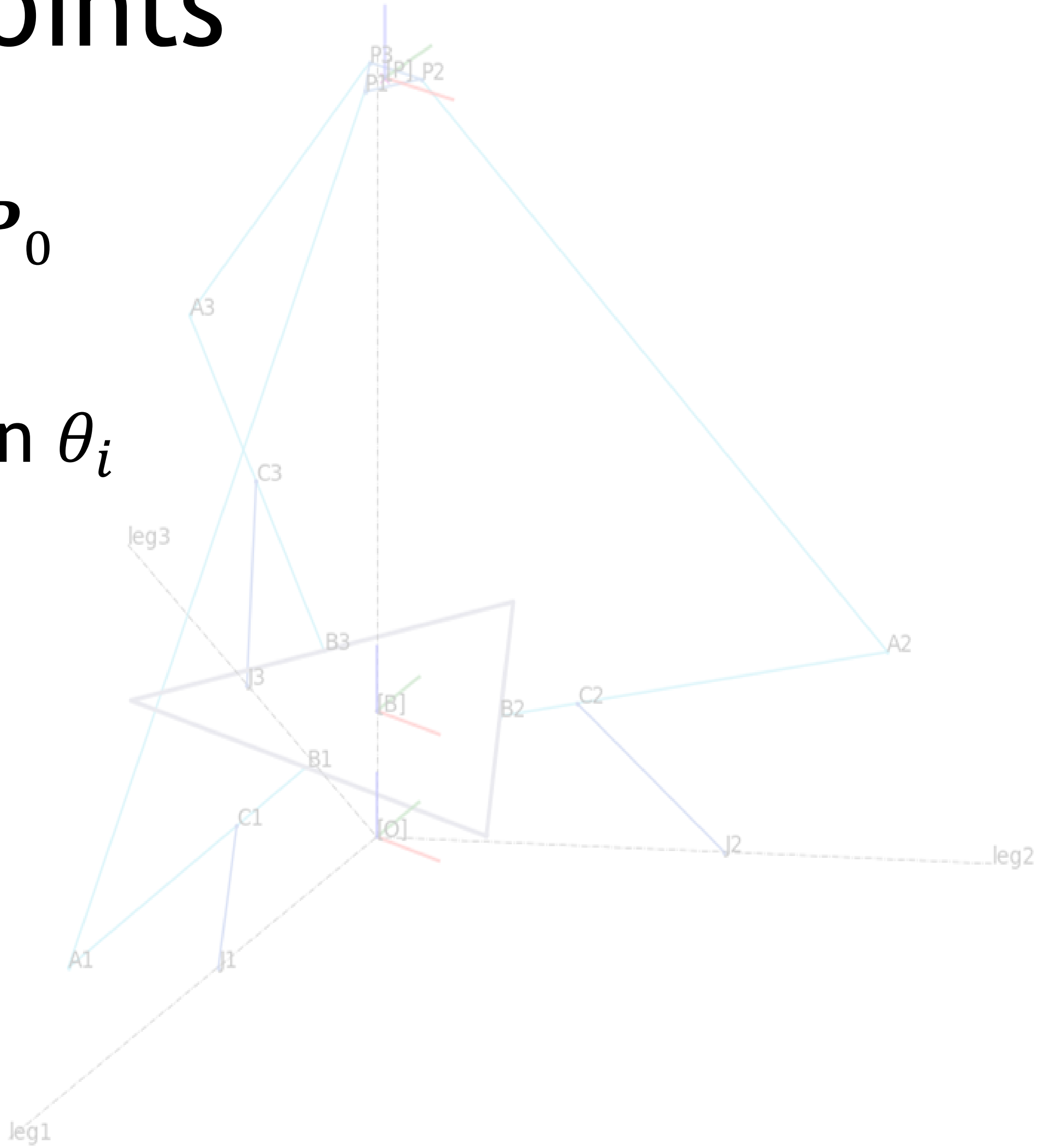


Platform Reference Frame

Forward Kinematics - Points

Forward Kinematics : Find ${}^{B}\boldsymbol{P}_0$ or ${}^{O}\boldsymbol{P}_0$

1. Find Point Coordinates depends on θ_i
2. Find ${}^{B}\boldsymbol{P}_0$ depends on θ_i
3. Find θ_i depends on D_i

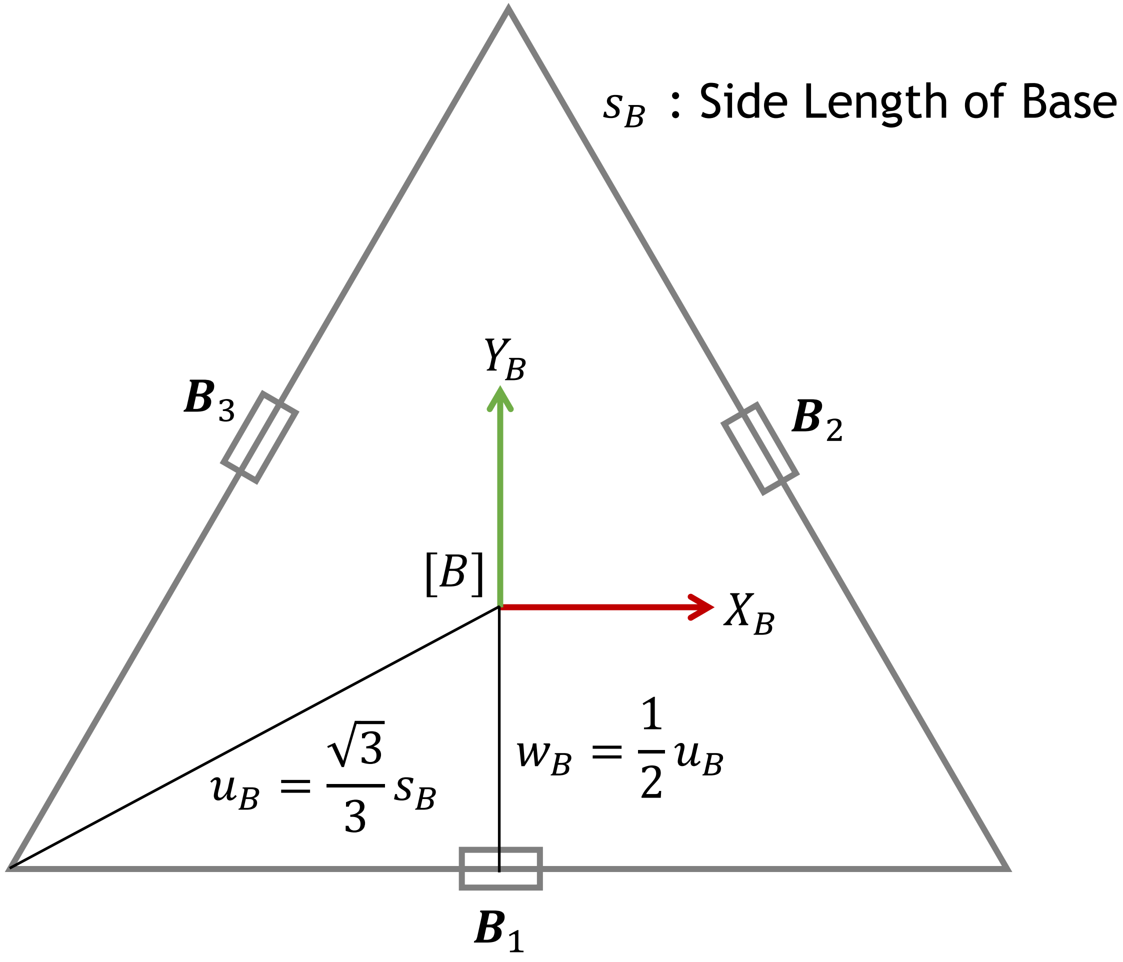


FWD Kinematics - Points

$$ {}^{B}\boldsymbol{B}_1 = \begin{bmatrix} 0 & -w_B & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{6}s_B & 0 \end{bmatrix} $$

$$ {}^{B}\boldsymbol{B}_1 = \begin{bmatrix} w_B \cos 30° & w_B \sin 30° & 0 \end{bmatrix} = \begin{bmatrix} \frac{s_B}{4} & \frac{\sqrt{3}}{12}s_B & 0 \end{bmatrix} $$

$$ {}^{B}\boldsymbol{B}_1 = \begin{bmatrix} w_B \cos 30° & -w_B \sin 30° & 0 \end{bmatrix} = \begin{bmatrix} \frac{s_B}{4} & -\frac{\sqrt{3}}{12}s_B & 0 \end{bmatrix} $$



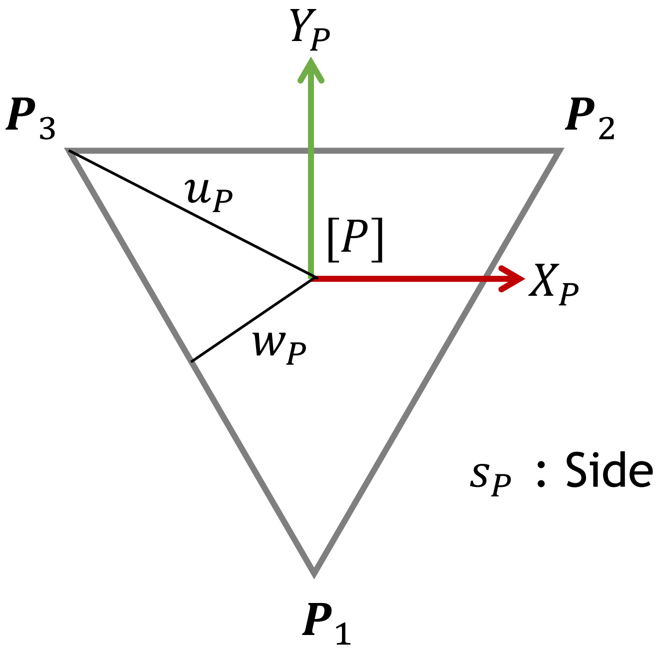
Base Reference Frame

FWD Kinematics - Points

$$^{P}\boldsymbol{P}_1 = [0 \quad -u_P \quad 0] = \begin{bmatrix} 0 & -\dfrac{\sqrt{3}}{3}s_P & 0 \end{bmatrix}$$

$$^{P}\boldsymbol{P}_2 = [u_P \cos 30° \quad u_P \sin 30° \quad 0] = \begin{bmatrix} \dfrac{s_P}{2} & \dfrac{\sqrt{3}}{6}s_B & 0 \end{bmatrix}$$

$$^{P}\boldsymbol{P}_3 = [u_P \cos 30° \quad -u_P \sin 30° \quad 0] = \begin{bmatrix} \dfrac{s_B}{2} & -\dfrac{\sqrt{3}}{6}s_B & 0 \end{bmatrix}$$



s_P : Side Length of Platform

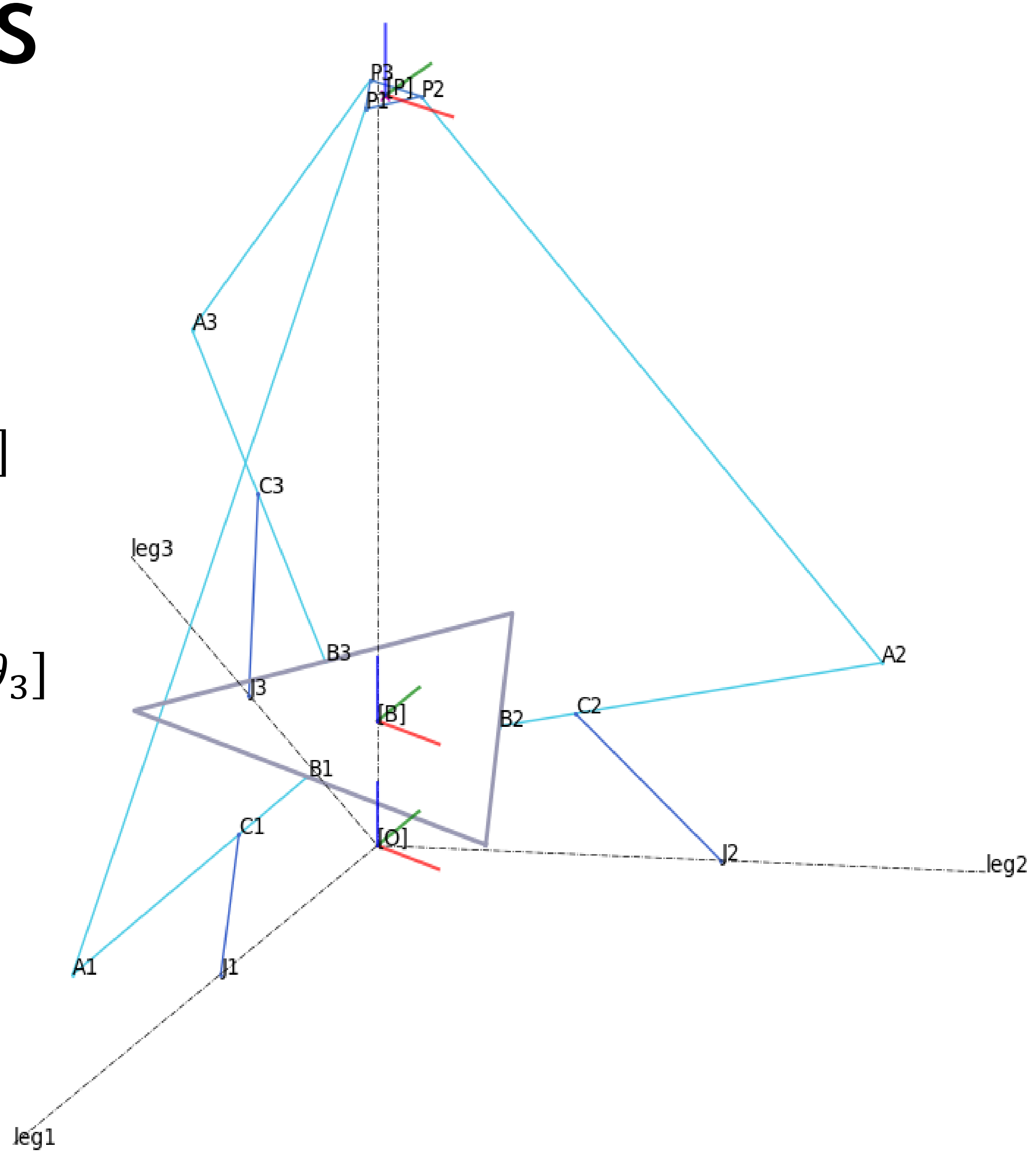
Platform Reference Frame

FWD Kinematics - Points

$$^{B}\boldsymbol{A}_1 = [0 \quad -L_1\cos\theta_1 \quad L_1\sin\theta_1]$$

$$^{B}\boldsymbol{A}_2 = [L_2\cos\theta_2\cos 30° \quad L_2\cos\theta_2\sin 30° \quad L_2\sin\theta_2]$$

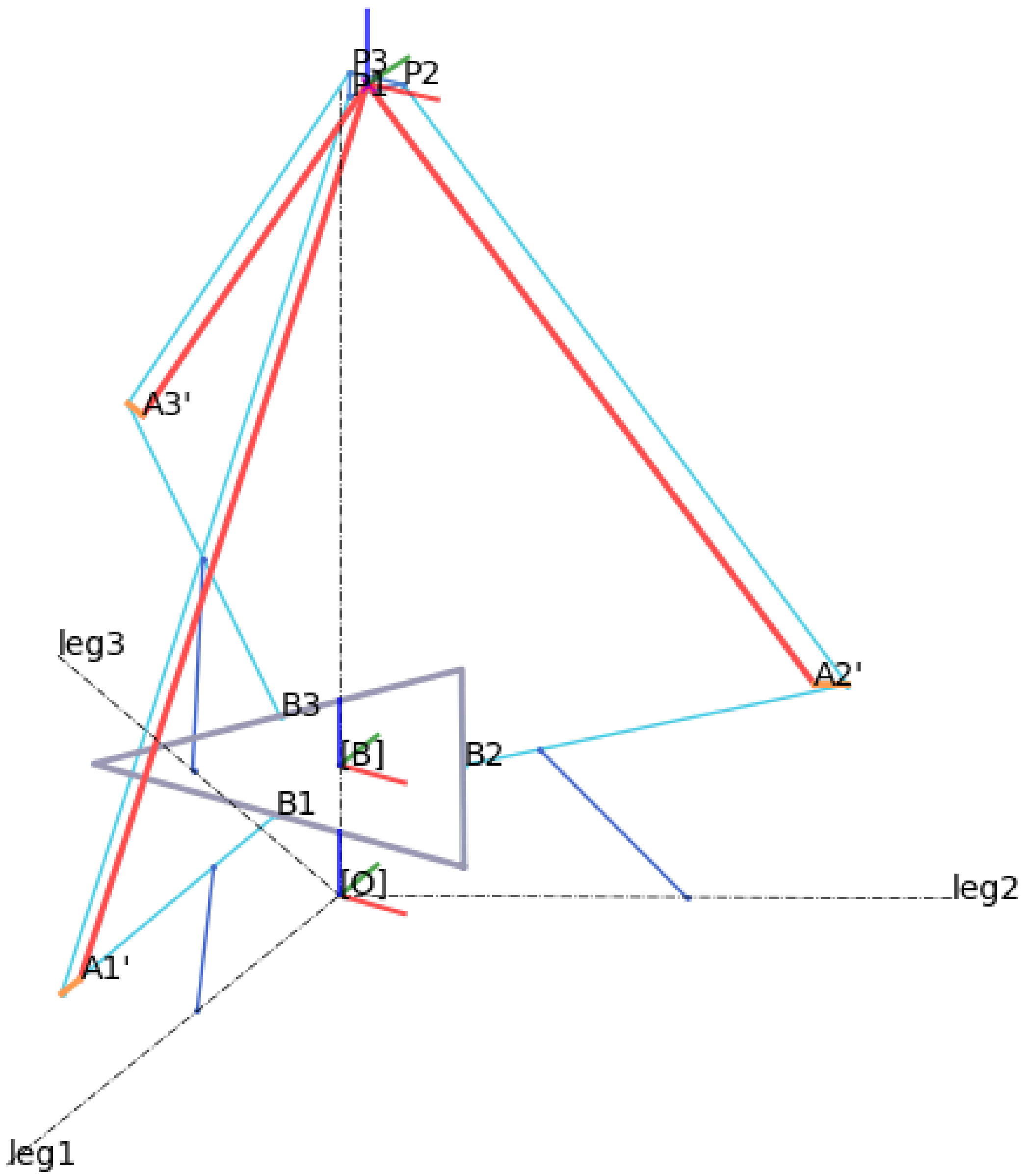
$$^{B}\boldsymbol{A}_3 = [-L_3\cos\theta_3\cos 30° \quad L_3\cos\theta_3\sin 30° \quad L_3\sin\theta_3]$$



FWD Kinematics - Points

Let ${}^{B}\boldsymbol{P'}_i$ is the point
such that ${}^{B}\boldsymbol{P}_i$ moves by $-{}^{P}\boldsymbol{P}_i$

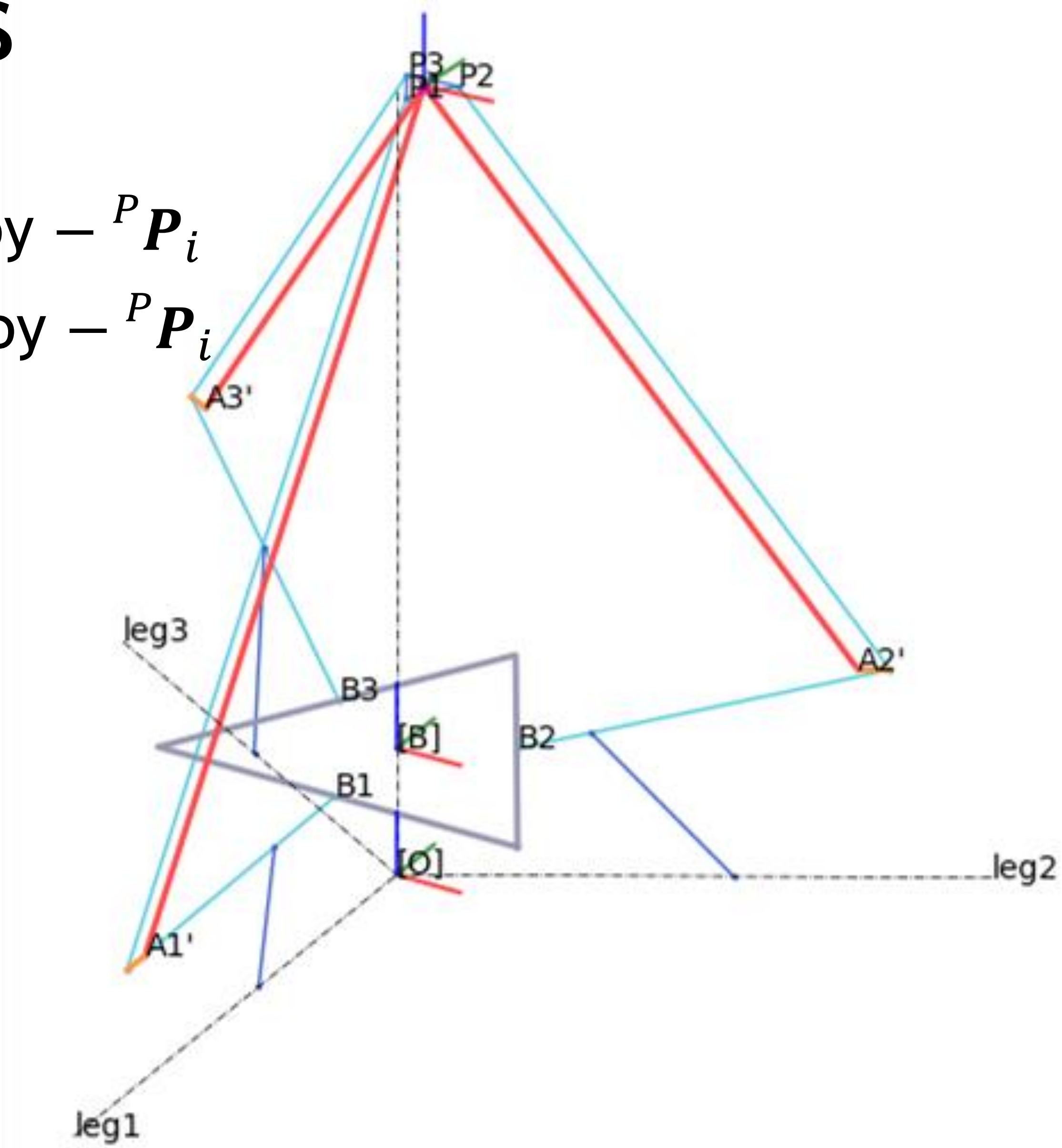
Then all ${}^{B}\boldsymbol{P'}_i$ gather at a point ${}^{B}\boldsymbol{P}_0$



FWD Kinematics - Points

Let ${}^{B}\boldsymbol{P'}_i$ is the point such that ${}^{B}\boldsymbol{P}_i$ moves by $-{}^{P}\boldsymbol{P}_i$
And ${}^{B}\boldsymbol{A'}_i$ is the point such that ${}^{B}\boldsymbol{A}_i$ moves by $-{}^{P}\boldsymbol{P}_i$

Then ${}^{B}\boldsymbol{P}_0$ is the intersection of 3 Spheres
with each center is ${}^{B}\boldsymbol{A'}_i$ and radius is l_i



FWD Kinematics – Intersection of Spheres

In this section, bold alphabet means column vector

Let $\boldsymbol{c_j} = \{x_j, y_j, z_j\}^T$ is center of the sphere with a radius of R_j,
and $\mathbf{x} = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}^{\mathrm{T}}$ is intersection of spheres. $(j = 1, 2, 3)$

Then $\mathbf{x}$ satisfying the nonlinear equation

$$\left\|\mathbf{x} - \boldsymbol{c_j}\right\|^2 = R_j^2$$

or equivalently,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{x}^T\boldsymbol{c_j} + \boldsymbol{c_j}^T\boldsymbol{c_j} = R_j^2$$

FWD Kinematics – Intersection of Spheres

At equation

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{x}^T\boldsymbol{c_j} + \boldsymbol{c_j^T}\boldsymbol{c_j} = R_j^2$$

Let $r = \mathbf{x}^T\mathbf{x}$, and $b_j = \boldsymbol{c_j^T}\boldsymbol{c_j} - R_j^2$ then equation can rewrite as

$$\boldsymbol{c_j^{\mathrm{T}}}\mathbf{x} = \frac{r + b_j}{2}$$

FWD Kinematics – Intersection of Spheres

In matrix form, equation

$$\boldsymbol{c}_{\boldsymbol{j}}^{\mathrm{T}}\mathbf{x} = \frac{r + b_j}{2}$$

become

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \frac{r}{2} + \frac{1}{2} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\underline{C}^T\mathbf{x} = \frac{r\mathbf{e} + \mathbf{b}}{2}$$

$$\underline{C} = [\boldsymbol{c}_{\mathbf{1}} \quad \boldsymbol{c}_{\mathbf{2}} \quad \boldsymbol{c}_{\mathbf{3}}], \qquad \mathbf{e} = \{1 \quad 1 \quad 1\}^T, \qquad \mathbf{b} = \{b_1 \quad b_2 \quad b_3\}^T$$

FWD Kinematics – Intersection of Spheres

Therefore

$$\mathbf{x} = \underline{C}^{-T}\left(\frac{r\mathbf{e} + \mathbf{b}}{2}\right)$$

$$\mathbf{x} = \frac{r\mathbf{u} + \mathbf{v}}{2}$$

$$\mathbf{u} = \underline{C}^{-T}\mathbf{e}, \qquad \mathbf{v} = \underline{C}^{-T}\mathbf{b}$$

FWD Kinematics – Intersection of Spheres

Since, $r = \mathbf{x}^T \mathbf{x}$

$$r = \mathbf{x}^T \mathbf{x} = \frac{1}{4}(r\mathbf{u} + \mathbf{v})^{\mathrm{T}}(r\mathbf{u} + \mathbf{v})$$

Therefore

$$\left(\mathbf{u}^{\mathrm{T}}\mathbf{u}\right)r^2 + \left(2\mathbf{u}^{\mathrm{T}}\mathbf{v} - 4\right)r + \mathbf{v}^{\mathrm{T}}\mathbf{v} = 0$$

Which is a quadratic equation in the scalar r.

$$r = \frac{-\left(\mathbf{u}^{\mathrm{T}}\mathbf{v} - 2\right) + \sigma\sqrt{(\mathbf{u}^{\mathrm{T}}\mathbf{v} - 2)^2 - (\mathbf{u}^{\mathrm{T}}\mathbf{u})(\mathbf{v}^{\mathrm{T}}\mathbf{v})}}{\mathbf{u}^{\mathrm{T}}\mathbf{u}}, \qquad \sigma = \pm 1$$

FWD Kinematics - ${}^{B}\boldsymbol{P}_0$

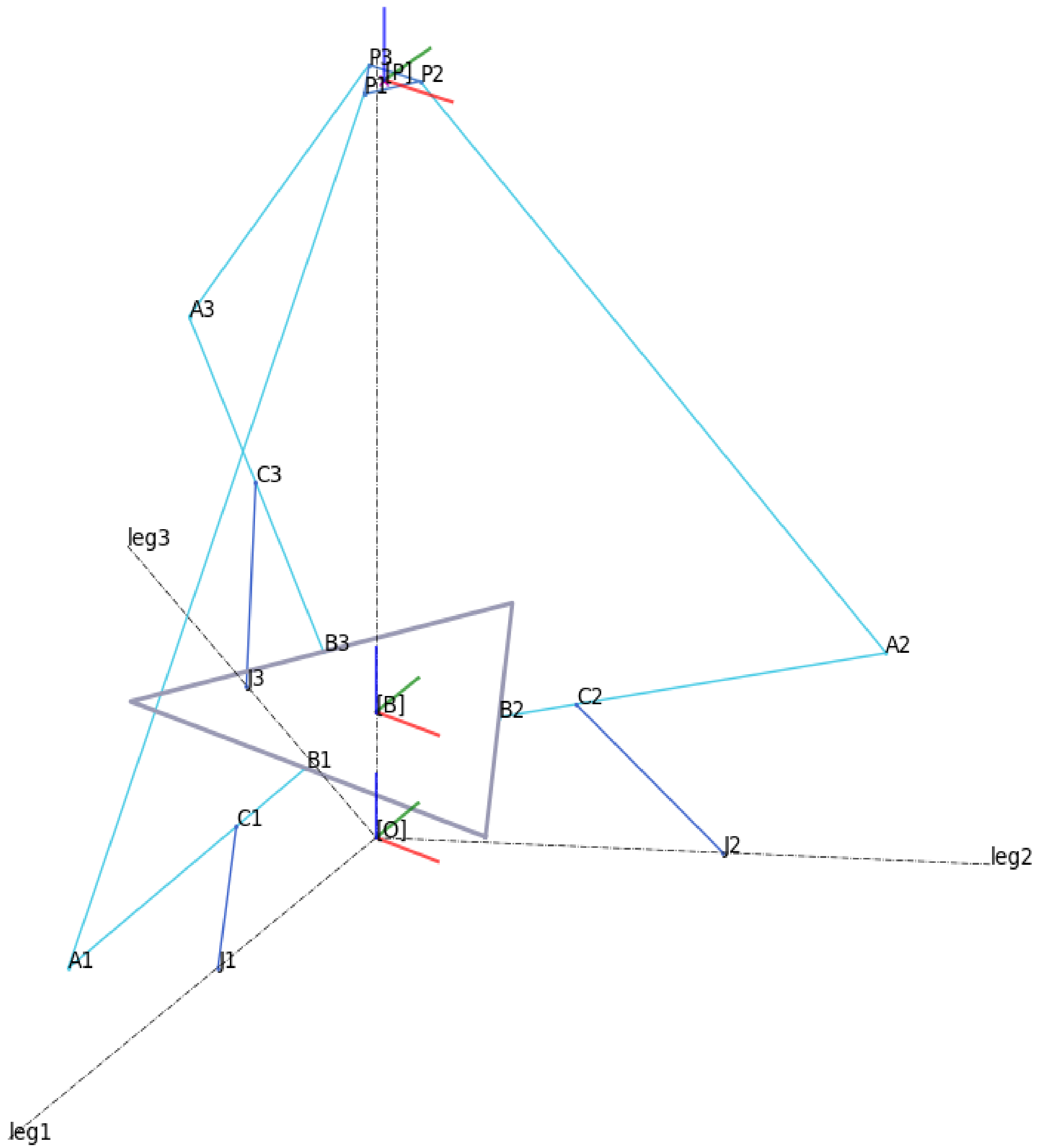
$$ {}^{B}\boldsymbol{P}_0 = \mathbf{x} = \frac{r\mathbf{u} + \mathbf{v}}{2} $$

$$ {}^{B}\boldsymbol{A}'_i = {}^{B}\boldsymbol{A}_i - {}^{P}\boldsymbol{P}_i, \qquad \underline{C}^{\mathrm{T}} = \begin{Bmatrix} {}^{B}\boldsymbol{A}'_1 \\ {}^{B}\boldsymbol{A}'_2 \\ {}^{B}\boldsymbol{A}'_3 \end{Bmatrix}, \qquad \mathbf{u} = \underline{C}^{-T}\mathbf{e}, \qquad \mathbf{v} = \underline{C}^{-T}\mathbf{b} $$

$$ \mathbf{e} = \{1 \quad 1 \quad 1\}^T, \qquad \mathbf{b} = \{b_1 \quad b_2 \quad b_3\}^T, \qquad b_i = {}^{B}\boldsymbol{A}'^{\,T}_i \, {}^{B}\boldsymbol{A}'_i - l_i^2 $$

$$ r = \frac{-\left(\mathbf{u}^{\mathrm{T}}\mathbf{v} - 2\right) + \sigma\sqrt{(\mathbf{u}^{\mathrm{T}}\mathbf{v} - 2)^2 - (\mathbf{u}^{\mathrm{T}}\mathbf{u})(\mathbf{v}^{\mathrm{T}}\mathbf{v})}}{\mathbf{u}^{\mathrm{T}}\mathbf{u}}, \qquad \sigma = \pm 1 $$

FWD Kinematics - ${}^{B}\boldsymbol{P}_0$

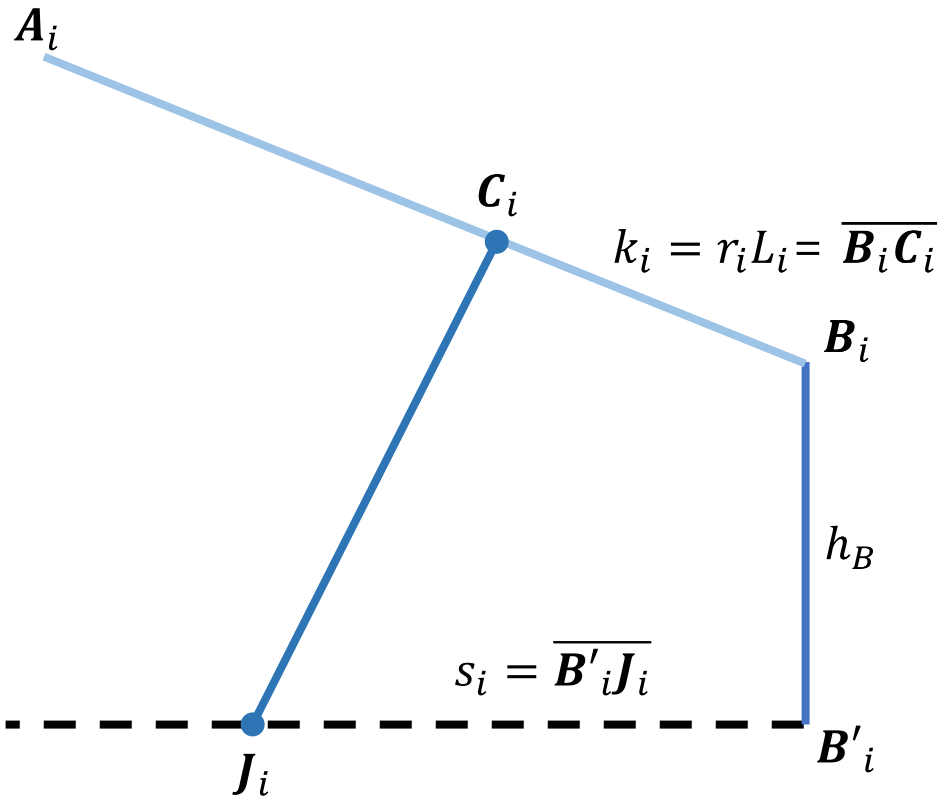


FWD Kinematics - $\theta_i(D_i)$

$[B]$ is equilvalent to moving $[O]$ by h_B following Z-axis

$$\boldsymbol{C}_i = t_i\boldsymbol{A}_i + (1 - t_i)\boldsymbol{B}_i$$

$$\boldsymbol{J}_i = \boldsymbol{B}_i' + \frac{\boldsymbol{B}_i'}{\|\boldsymbol{B}_i'\|} s_i$$



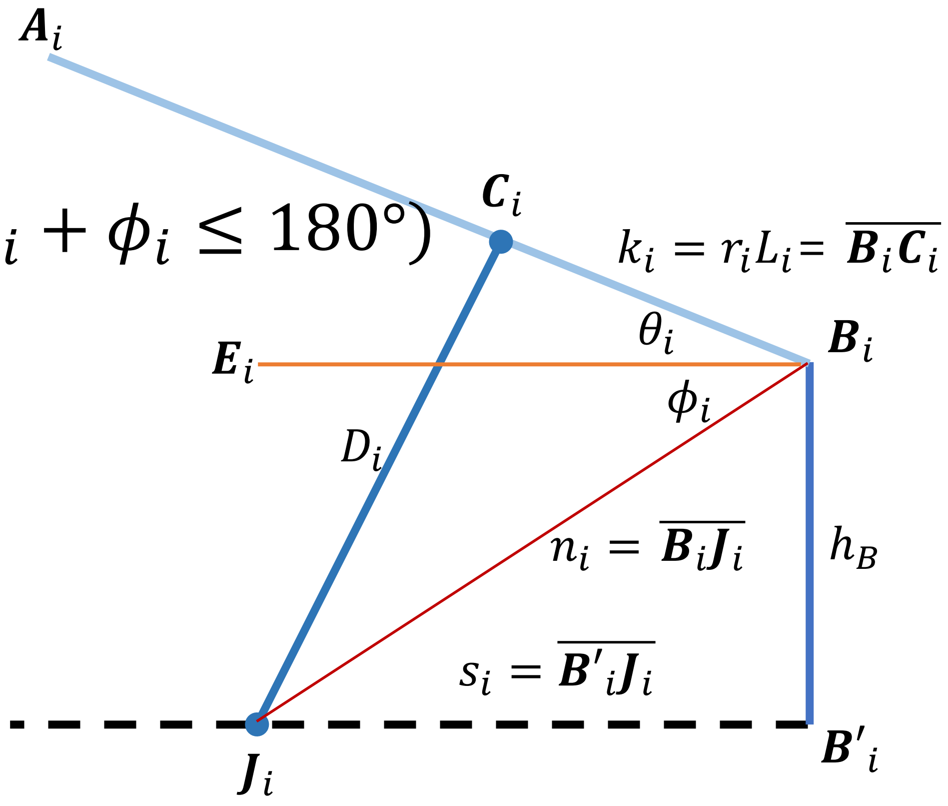
FWD Kinematics - $\theta_i(D_i)$

$$\phi_i = \angle \boldsymbol{E}_i \boldsymbol{B}_i \boldsymbol{J}_i = \cos^{-1}\left(\frac{s_i}{n_i}\right), \qquad (0 < \phi_i \leq 90°)$$

2nd Law of Cosine

$$D_i^2 = k_i^2 + n_i^2 - 2n_i k_i \cos(\theta_i + \phi_i), \qquad (0 \leq \theta_i + \phi_i \leq 180°)$$

$$\theta_i = \cos^{-1}\left(\frac{D_i^2 - k_i^2 - n_i^2}{2n_i k_i}\right) - \cos^{-1}\left(\frac{s_i}{n_i}\right)$$



Reference

[1] Robert L. Williams II, “The Delta Parallel Robot : Kinematics Solutions”, 2016

[2] I.D. Coope, “Reliable computation of the points of intersection of n spheres in n-dimension”, 2000