Problem 1

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 5 & 8 \\ 1 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 4 \\ -1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 3 & -4 \end{pmatrix} x(t) + 7u(t)$$

(a) compute the analytical expressions of state and output responses, with initial conditions $x(0) = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ and null input $u(t) = 0$;

(b) repeat the exercise with a step signal of amplitude 4 as input.

$$\dot{x}(t) = A x(t) + B u(t)$$

$$A = \begin{pmatrix} 5 & 8 \\ 1 & 3 \end{pmatrix} \; ; \; B = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

a) $\mathcal{L}\{\dot{x}(t)\} \Rightarrow sX(s) - x(0) = AX(s) + BU(s)$

$\Rightarrow sX(s) - AX(s) = BU(s) + x(0)$

$\Rightarrow X(s)[sI - A] = x(0) + BU(s)$

$\Rightarrow X(s) = (sI - A)^{-1}(x(0) + BU(s))$

$u(t) = 0 \Leftrightarrow U(s) = 0$

after performing the matlab operations:

res1: $\begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \end{bmatrix}$ pol1: $\begin{bmatrix} 7 \\ 7 \\ 1 \\ 1 \end{bmatrix}$ $X_1(s) = \frac{-2}{s-7} + \frac{5}{s-1} \Rightarrow x_1(t) = (-2e^{7t} + 5e^{t})$

res2: $\begin{bmatrix} -0.5 \\ 0 \\ -2.5 \\ 0 \end{bmatrix}$ pol2: $\begin{bmatrix} 7 \\ 7 \\ 1 \\ 1 \end{bmatrix}$ $X_2(s) = \frac{-0.5}{s-7} - \frac{2.5}{s-1} \Rightarrow x_2(t) = (-0.5e^{7t} - 2.5e^{t})$

$$x(t) = \begin{pmatrix} -2e^{7t} + 5e^{t} \\ -0.5e^{7t} - 2.5e^{t} \end{pmatrix} \varepsilon(t)$$

to ensure it starts after $t=0$

finding $y(t)$:

$\mathcal{L}\{y(t)\} \Rightarrow Y(s) = (3 \quad -4)\, X(s)$

after performing the matlab operations;

$y = \dfrac{21\,(s-8.143)}{(s-7)(s-1)} \Rightarrow res = \begin{bmatrix} -4 \\ 25 \end{bmatrix} \quad pol = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$Y(s) = \dfrac{-4}{s-7} + \dfrac{25}{s-1} \Rightarrow$

$y(t) = (-4e^{7t} + 25e^{t})\,\varepsilon(t)$

↳ to ensure it starts after $t=0$

6) instead of having $u(t)=0$; now we have $u(t) = 4\,\varepsilon(t)$

hence $U(s) = \dfrac{4}{s}$

$res1 = \begin{bmatrix} -1.24 \\ 15.67 \\ -11.43 \end{bmatrix} \quad pol1: \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \quad x_1(t) = -1.24e^{7t} + 15.67e^{t} - 11.43$

$res2 = \begin{bmatrix} -0.31 \\ -7.83 \\ 5.14 \end{bmatrix} \quad pol2: \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \quad x_2(t) = -0.31e^{7t} - 7.83e^{t} + 5.14$

$resy = \begin{bmatrix} -2.48 \\ 78.33 \\ -26.86 \end{bmatrix} \quad poly: \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \quad y(t) = -2.48e^{7t} + 78.33e^{t} - 26.86$

Problem 2

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 27 & 0 \\ -23 & 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

compute the analytical expressions of state and output responses, with initial conditions $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and input $u(t) = \begin{pmatrix} 0 \\ \delta(t) \end{pmatrix}$.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 27 & 0 \\ -23 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\dot{x}(t) = \overbrace{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}^{A} x(t) + \underbrace{\overbrace{\begin{pmatrix} 27 & 0 \\ -23 & 1 \end{pmatrix}}^{B} \begin{pmatrix} 0 \\ \delta(t) \end{pmatrix}}_{\delta(t) \xleftrightarrow{\mathcal{L}} 1} \Bigg\} \; U = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\mathcal{L}\{\}$

$$sX(s) - x(0) = AX(s) + BU$$

$$X(s)[sI - A] = BU + x(0)$$
$$X(s) = (sI - A)^{-1}(BU + x(0))$$

after making the matlab calculations:

$$\text{res } X1: \begin{bmatrix} -0.58j \\ +0.58j \end{bmatrix} \qquad \text{pol } X1 = \begin{bmatrix} -0.5 + 0.87j \\ -0.5 - 0.87j \end{bmatrix}$$

$$\text{res } X2: \begin{bmatrix} 0.5 + 0.29j \\ 0.5 - 0.29j \end{bmatrix} \qquad \text{pol } X2 = \begin{bmatrix} -0.5 + 0.87j \\ -0.5 - 0.87j \end{bmatrix}$$

$$\text{res}\,y = \begin{bmatrix} -0.58j \\ 0.58j \end{bmatrix} \qquad \text{pol}\,y = \begin{bmatrix} -0.5 & +0.87j \\ -0.5 & -0.87j \end{bmatrix}$$

Since our findings are in imaginary domain we have to apply Euler's formula

$$\mathcal{L}^{-1}\left(\frac{R}{s-\sigma_0-j\omega_0} + \frac{R^*}{s-\sigma_0+j\omega_0}\right) = |R|\, e^{\sigma_0 t}\, 2\cos(\arg(R) + \omega_0 t)\,\varepsilon(t)$$

$$x_1(t) = 1.1547 e^{-\frac{1}{2}t} \cos(0.87t - 1.57)$$

$$x_2(t) = 1.1547 e^{-\frac{1}{2}t} \cos(0.87t + 0.52)$$

$$y(t) = 1.1547 e^{-\frac{1}{2}t} \cos(0.87t - 1.57)$$

$$X(t) = \begin{pmatrix} 1.15 e^{-\frac{1}{2}t} \cos(0.87t - 1.57) \\ 1.15 e^{-\frac{1}{2}t} \cos(0.87t + 0.52) \end{pmatrix} \varepsilon(t)$$

$$y(t) = \left(1.1547 e^{-\frac{1}{2}t} \cos(0.87t - 1.57)\right)\varepsilon(t)$$

Problem 3

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix} x(t)$$

(a) compute the analytical expressions of the output response $y(t)$, with initial conditions $x(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^\top$ and input $u(t) = (1+t)\,\varepsilon(t)$;

(b) compute the state response $x(t)$ with initial conditions $x(0) = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}^\top$ and null input $u(t) = 0$.

A) $U(s) = \varepsilon(t) + t\,\varepsilon(t) = \frac{1}{s} + \frac{1}{s^2}$

$$res\_y = \begin{bmatrix} -0.16 \\ -0.5 + 0.28875j \\ -0.5 - 0.28875j \\ 1.16 \\ 1.2 \end{bmatrix} \quad pol\_y = \begin{bmatrix} -5 \\ -0.5 + 0.875j \\ -0.5 - 0.875j \\ 0 \\ 0 \end{bmatrix}$$

when two consecutive numbers we add t^n where $n = 0, 1, 2 \ldots$

$$y(t) = -0.16e^{-5t} + 1.16 + 1.2t + 1.15\,e^{-\frac{1}{2}t}\cos(0.87t + 2.62)$$

B)

res X1 = 0 pol X1 = 0

res X2 = 0 pol X2 = 0

res X3 = 1 pol X3 = -10

res X4 = 1 pol X4 = -5

$$\boxed{x(t) = \begin{bmatrix} 0 & 0 & e^{-10t} & e^{-5t} \end{bmatrix} \varepsilon(t)}$$

Problem 4 (**)

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x(t)$$

(a) compute the analytical expressions of the output response, with initial conditions $x(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^\top$ and input $u(t) = \sin(t\sqrt{2})$.
(b) repeat the computation for $u(t) = \sin(t)$.
Hints:
$\mathcal{L}(\sin(\omega_0 t)) = \frac{\omega_0}{s^2+\omega_0^2}$

$\sqrt{2} = 1.4142\ldots, \frac{\sqrt{2}}{2} = 0.7071\ldots$

$\mathcal{L}^{-1}\left(\frac{R}{(s-\sigma_0-j\omega_0)^2} + \frac{R^\star}{(s-\sigma_0+j\omega_0)^2}\right) = 2|R|e^{\sigma_0 t} t\cos(\omega_0 t + \arg(R))\varepsilon(t).$

a) $\mathcal{L}\{u(t)\} = U(s) = \boxed{\frac{\sqrt{2}}{s^2+2}}$

$\text{res}\,y = \begin{bmatrix} -0.71 \\ -0.71 \\ 0.71 \\ 0.71 \end{bmatrix} \qquad \text{pol}\,y = \begin{bmatrix} 1.415 \\ -1.415 \\ j \\ -j \end{bmatrix}$

$$y(s) = \frac{-0.71}{s-1.415} - \frac{0.71}{s+1.415} + \frac{0.71}{s+j} + \frac{0.71}{s-j}$$

$-0.71\, e^{-1.415t} - 0.71 e^{1.415}$

$-0.71\left(\cos(-1.41t) + j\cancel{\sin(1.41t)} + \cos(1.41t) + j\cancel{\sin(1.41t)}\right)$

$-0.71 \times 2\cos(1.41t)$

$= -\sqrt{2}\cos(\sqrt{2}t)$

$0.71\left(e^{-jt} + e^{jt}\right)$

$2 \times 0.71\cos(t) = \sqrt{2}\cos(t)$

$y(t) = \sqrt{2}\left(-\cos(\sqrt{2}t) + \cos(t)\right)$

6)

$$res\,y = \begin{bmatrix} 0 \\ -0.25j \\ 0 \\ 0.25j \end{bmatrix} \qquad poly = \begin{bmatrix} j \\ j \\ -j \\ -j \end{bmatrix}$$

$$= \frac{-0.25j}{s-j} + \frac{0.25j}{s+j}$$

$$2\,|0.25|\,e^{0}t\cos\left(t+\frac{\pi}{2}\right) = \frac{1}{2}\,t\sin(t)$$