Automatic Control
Laboratory practice 2

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Objectives: study of the natural modes, internal stability and BIBO stability of LTI continuous-time dynamical systems

Problem 1

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 4 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

(a) study the internal stability;
(b) perform the modal analysis, i.e., classify the natural modes;
(c) study the BIBO stability;
(d) repeat (b) and (c) with $A = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
(e) if possible, compute the time constant for $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ and for $A = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Which system has natural modes with faster convergence rate?

Solution
(a) The system is internally asymptotically stable.
(b) The natural modes are of kind $e^{-\frac{1}{2}t}\cos(\dots) \rightarrow$ (exponentially) convergent.
(c) The system is BIBO stable.
(d) The system is internally asymptotically stable ($\Rightarrow$ BIBO stable); the natural modes are of kind $e^{-\frac{1}{6}t}\cos(\dots) \rightarrow$ (exponentially) convergent.
(e) the time constants are $\tau = 2$ and $\tau = 6$; the natural modes of the first

1)

a) $\text{eig}(A) = \begin{pmatrix} -0.5 + 0.87j \\ -0.5 - 0.87j \end{pmatrix}$

$\lambda_1 = (0.87j - 0.5) \quad M_1 = 1$

$\lambda_2 = (0.87j - 0.5) \quad M_1 = 1$

$\mathbb{R}e(\lambda_1) = -0.5 \; ; \; \mathbb{R}e(\lambda_2) = -0.5$

Convergent $\Rightarrow$ internally asymptotically stable

b) $\underbrace{e^{-\frac{1}{2}t} \cos(\ldots)}_{\text{convergent}} + \underbrace{e^{-1/2 t} \cos(\ldots)}_{\text{convergent}}$

c) $H(s) = \left(\frac{2(s+3)}{s^2 + s + 2} \right) \Rightarrow$ $z = -3$, $\lambda_1, \lambda_2 < 0$ hence BIBO stable

d) this time eign values are $\lambda_1 = -0.17 + 0.23j$
$\lambda_2 = -0.17 - 0.23j$

the answers are same for B, C because $\mathbb{R}e(\lambda_1, \lambda_2) < 0$

e) $\tau_1 = \left| \frac{1}{Re(\lambda_1)} \right| \qquad \tau_2 = \left| \frac{1}{Re(\lambda_2)} \right|$

$= 2 \qquad\qquad = 6$

Problem 2

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 1 & 3 \\ 6 & 4 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 4 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

(a) study the internal stability;
(b) perform the modal analysis, i.e., classify the natural modes;
(c) study the BIBO stability.

a) $eig(A) = \begin{pmatrix} -2 \\ 7 \end{pmatrix} \Rightarrow$ unstable

b) $y(t) = \underbrace{e^{-2t}}_{\text{conv.}} + \underbrace{e^{7t}}_{\text{div.}}$

c) $H(s) = \dfrac{2}{s-7} \Rightarrow$ unstable because of th pole $\lambda_2 = 7$

$$\boxed{H = C \times (sI - A)^{-1} \times B + D}$$

Problem 3

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 02 & \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) - 2u(t)$$

(a) study the internal stability;
(b) perform the modal analysis, i.e., classify the natural modes;
(c) study the BIBO stability.

a) $\mathrm{eig}(A) = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{array}{lll} \lambda_1 = -2 & ; & \lambda_2 = 0 \\ \mu_1 = 1 & ; & \mu_2 = 2 \end{array}\right\} \text{unstable}$$

b) $y(t) = \underbrace{e^{-2t}}_{\text{conv}} + \underbrace{t}_{\text{div}} + \underbrace{1}_{\text{bounded}}$

c) $H(s) = \dfrac{-2(s+2.5)}{s+2} \Rightarrow$ BIBO stable

Problem 4

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 5 & -1 & 2 \\ 3 & 1 & 0 \\ -5 & 4 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} u(t)$$

$$y(t) = \frac{1}{2} \begin{pmatrix} 1 & -1 & 3 \end{pmatrix} x(t)$$

(a) study the internal stability;
(b) perform the modal analysis, i.e., classify the natural modes;
(c) study the BIBO stability.

a) $\mathrm{eig}(A) = \begin{pmatrix} 1.19 + 2.8j \\ 1.19 - 2.8j \\ 3.6 \end{pmatrix} \Rightarrow$ internally unstable

b) $y(t) = \underbrace{e^{1.19t} \cos(\ldots)}_{D} + \underbrace{e^{1.19t} \cos(\ldots)}_{D} + \underbrace{e^{3.6t}}_{D}$

c) poles of $H(s) = \begin{pmatrix} 3.62 \\ 1.19 + 2.8j \\ 1.19 - 2.8j \end{pmatrix}$

BIBO unstable

Problem 5

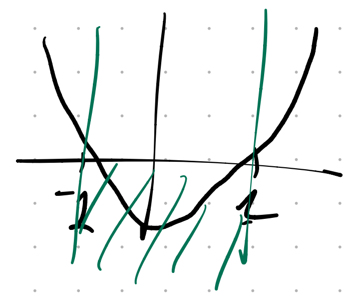
Given $p \in \mathbb{R}$, study the internal stability of an LTI system with

$$A = \begin{pmatrix} p^2-1 & 0 & 0 \\ 0 & p-3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(Hint: A is diagonal)

$\lambda_1 = -1 \qquad \lambda_2 = p-3 \qquad \lambda_3 = p^2-1$

$\lambda_1 = -1 \qquad \lambda_2 = p-3 \qquad \lambda_3 = (p-1)(p+1)$



$y(t) = e^{-t} + e^{(p-3)t} + e^{(p^2-1)t}$

when $p < 3 \to$ convergent

" $p = 3 \to$ bounded

Divergent

when $|p| < 1 \to$ convergent

conv

" $|p| = 1 \to$ convergent

Bounded

when $|p| \leq 1$; internally stable

Given $p \in \mathbb{R}$, study the BIBO stability of an LTI system with tf

$$H(s) = \frac{4}{s^2 + (p+1)s + 4p - 2}$$

$1 \qquad (p+1) \qquad (4p-2)$

for stability $\lambda < 0$; 1 change of sign

either $\quad p+1 < 0 \quad ; \quad 4p-2 < 0$

or $\quad p+1 > 0 \quad ; \quad 4p-2 < 0$

not possible

hence $\quad \boxed{p < -\frac{1}{2}}$