MAT 245 Lab 4

Oct. 16, 2017

Ill-conditioned matrices

The condition number. Recall that the condition number of an $n \times n$ matrix A is defined to be

$$\kappa(A) := \left(\sup_{e \neq 0} \frac{\|A^{-1}e\|_2}{\|e\|_2}\right)\left(\sup_{b \neq 0} \frac{\|b\|_2}{\|A^{-1}b\|_2}\right) = \left\|A^{-1}\right\|_2 \|A\|_2 .$$

If we are studying the linear system $Ax = b$, one can intuitively think of the condition number as measuring the rate of change solution of the solution x as we perturb b. When we working with the Frobenius matrix norm, as above, there is a particularly simple formula for the condition number. If $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ denote the maximum and minimum singular values of A respectively, then

$$\kappa(A) = \frac{\sigma_{\max}}{\sigma_{\min}}.$$

The Hilbert matrix. The $n \times n$ Hilbert matrix is the matrix $H(n)$ defined by

$$(H(n))_{ij} = \frac{1}{i + j - 1}.$$

In other words, it is the matrix whose ij^{th} entry is $\frac{1}{i+j-1}$. For example:

$$H(5) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}$$

Goals

1. Write a `python` function that returns the Hilbert matrix $H(n)$ given $n \in \mathbb{N}$.
2. Write a `python` function that takes a `numpy` 2D-array A and computes the condition number $\kappa(A)$.
3. Use your implementation of the condition number function to compute $\kappa(H(n))$ for $n = 10, 15, 20, 25$. Compare these values with those computed by the University of Kyoto at: `http://bit.ly/2x6jIP7`. Do they match?
4. Plot the graph of the function $n \mapsto \kappa(H(n))$ for $1 \leq n \leq 10$.
5. Define random two vectors $t_0, t_1 \in \mathbb{R}^{20}$ using the `numpy.random.rand` function. Set

$$b_0 := H(20) \cdot t_0 \quad \text{and} \quad b_1 := H(20) \cdot t_1$$

   Now compute x_0 and x_1 by

$$x_0 = H(20)^{-1} \cdot b_0 \quad \text{and} \quad x_1 := H(20)^{-1} \cdot b_1$$

   Clearly $x_i = t_i$ since

$$x_i = H(20)^{-1} H(20) t_i \quad i = 0, 1.$$

   Use `numpy.linalg.norm` to compute the norm of $\|x_i - t_i\|_2$ for $i = 0, 1$. Do you obtain the expected result?

6. Now let's try computing the x_i in a different way. This time use `numpy.linalg.solve` to solve the linear systems

$$H(20)x_0 = b_0 \quad \text{and} \quad H(20)x_1 = b_1.$$

   By the definition of b_i, we should again have $x_i = t_i$. Compute $\|x_i - t_i\|_2$ as above to see whether this is the case.

Computing the variance

Consider the following two formulas for computing the variance of a vector in $\mathbb{R}^n$:

(a)

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

(b)

$$S^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right).$$

One of these algorithms is numerically unstable.

1. Write `python` implementations of both formulas.

2. Can you determine which one is more numerically stable? Use examples to demonstrate the issues with the unstable algorithm. (Hint: Try sample data whose variance is very small relative to the mean and consider the placement of the ^{2}).

3. Compare both algorithms to the results obtained from `numpy`'s own variance function `numpy.var`. If v_1 and v_2 are the results of any two variance computations, try computing $\frac{|v_1 - v_2|}{|v_1|}$.

Approximating the exponential function

1. Implement a `python` function that computes the N^{th} partial sum of $\exp(x)$. In other words, implement the mapping

$$(x, N) \longmapsto \sum_{k=0}^{N} \frac{x^k}{k!}.$$

2. Compute approximate values for $\exp(-5.5)$ for $N = 5, 10, 15$ in three ways:

   (a) directly with the function you implemented in the first step.

   (b) with $\exp(-5.5) = 1/\exp(5.5)$, and your implementation for $\exp(5.5)$.

   (c) with $\exp(-5.5) = (\exp(0.5))^{-11}$, and your implementation for $\exp(0.5)$.

   How can the results be interpreted?