NOTE ON SEQUENCES OF INTEGERS NO ONE OF WHICH IS DIVISIBLE BY ANY OTHER

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Let $a_1, a_2, a_3, \ldots$ be a sequence of integers, say (A), such that a_m is not a divisor of a_n unless $m = n$. Chowla, Davenport, and I proposed the question whether the density of every sequence (A) is zero. Besicovitch† proved that this was not so by showing that, if d_a is the density of integers having a divisor between a and $2a$, then $\liminf_{a\to\infty} d_a = 0$.

We can easily prove that the upper density of any sequence (A) does not exceed $\frac{1}{2}$. In fact, (A) cannot contain $n+1$ elements $a_1, a_2, \ldots, a_{n+1}$ at most equal to $2n$. For, if $a_m = 2^{\alpha_m} b_m$, where b_m is odd, and so has at most n different values, two of the b's must be equal. If these correspond to indices m_1, m_2, clearly a_{m_1} is divisible by a_{m_2} if $m_1 > m_2$‡.

We prove now that the lower density of (A) is zero§. This follows from the

THEOREM. $\sum_{n=1}^{\infty} \frac{1}{a_n \log a_n}$ *converges.*

More generally, we show that if p_n denotes the greatest prime factor of a_n, then

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \prod_{p \leqslant p_n} \left(1 - \frac{1}{p}\right) \leqslant 1, \tag{1}$$

where the product refers to the primes not greater than p_n. It follows that

$$\sum_{n=1}^{\infty} \frac{1}{a_n \log a_n} < c,$$

where c is a constant independent of the sequence, since

$$\prod_{p \leqslant p_n} \left(1 - \frac{1}{p}\right) > \frac{c}{\log p_n} \geqslant \frac{c}{\log a_n}.$$

For suppose that (1) is not true; then, for some integer N,

$$\sum_{n=1}^{N} \frac{1}{a_n} \prod_{p \leqslant p_n} \left(1 - \frac{1}{p}\right) > 1.$$

* Received 30 November, 1934; read 13 December, 1934.

† " On the density of certain sequences ", *Math. Annalen*, 110 (1934), 336–341.

‡ This proof is due to M. Wachsberger and E. Weissfeld.

§ A different proof has been given by Behrend, *Journal London Math. Soc.*. 10 (1935). 42–44.

Consider the a's to be arranged according to the magnitude of their greatest prime factors. Let n be a sufficiently large integer, and denote by $n(a)$ the number of integers not greater than n divisible by at least one of the a's, and by $n(a_k)$ the number of integers not greater than n divisible by a_k but by no a_i with $i<k$. Clearly

$$n(a)=\sum_{k=1}^{\infty} n(a_k)>\sum_{k=1}^{N} n(a_k). \tag{2}$$

The $n(a_k)$ integers include among their number the integers not greater than n of the form $a_k x$, where all the prime factors of x are greater than p_k. The number of integers $m \leqslant n/a_k$, not divisible by any prime $p \leqslant p_k$, is, by the usual argument based upon the sieve of Eratosthenes, at least

$$n(a_k) \geqslant \frac{n}{a_k} \prod_{p \leqslant p_k} \left(1-\frac{1}{p}\right)-2^k,$$

and this is, *a fortiori*, a lower bound for $n(a_k)$. Hence, from (2),

$$n>n(a)>\sum_{k=1}^{N} \frac{n}{a_k} \prod_{p \leqslant p_k} \left(1-\frac{1}{p}\right)-\sum_{k=1}^{N} 2^k.$$

This gives a contradiction for large n since N is independent of n.

We conclude by proving that in Besicovitch's theorem lim inf may be replaced by lim, *i.e.* for every $\epsilon>0$, and $a>a(\epsilon)$ say, the density of integers having a divisor between a and $2a$ is less than ϵ.

We require the following lemma, easily proved by the method of Turán*.

Lemma. *The normal number of prime factors less than a of an integer is* $\log\log a$.

This means that, for arbitrary $\epsilon>0$, $\delta>0$, and $a>a(\epsilon, \delta)$, $n \geqslant a$, the number of integers not greater than n having either more or less respectively than $(1+\epsilon)\log\log a$, $(1-\epsilon)\log\log a$ prime factors less than a is less than δn.

We divide the integers lying between a and $2a$ into two classes. Put in the first the integers $b_1, b_2, \ldots, b_y$ having at most $\frac{2}{3}\log\log a$ prime factors and in the second those, say $c_1, c_2, \ldots, c_z$, having more than $\frac{2}{3}\log\log a$ prime factors.

* "On a theorem of Hardy and Ramanujan", *Journal London Math. Soc.*, 9 (1934), 274–276.

The number of integers not greater than n divisible by a b is less than

$$\sum_{i=1}^{y} \frac{n}{b_i} < \frac{n}{a} y < \tfrac{1}{3}\epsilon n,$$

for from the lemma, replacing a and n by $2a$, we have $y < \frac{1}{3}\epsilon a$.

The integers divisible by a c can be arranged in two sets. In the first are those of the form $c_i x$, where $x < n/c_i$ and has at most $\frac{2}{3} \log\log a$ prime factors less than a. The number in this set is less than

$$\tfrac{1}{3}\epsilon \sum_{i=1}^{z} \frac{n}{c_i} < \tfrac{1}{3}\epsilon \frac{nz}{a} < \tfrac{1}{3}\epsilon n.$$

The second set includes the integers of the form $c_i x$, where x has more than $\frac{2}{3} \log\log a$ prime factors less than a. Hence these integers have more than $\frac{4}{3} \log\log a$ prime factors less than a and so the number of them is less than $\frac{1}{3}\epsilon n$. This proves the theorem.

I have since proved the following generalisation of Besicovitch's theorem:

The density of the integers having a divisor between n and $n^{1+\epsilon_n}$, where $\epsilon_n \to 0$ as $n \to \infty$, tends to zero as n tends to infinity.

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