THE TOV EQUATION APPLIED TO NEUTRON STARS:

PROJECT FOR THE COURSE OF NUMERICAL TECHNIQUES FOR MODELING RELATIVISTIC HYDRODYNAMICS

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ABSTRACT

I develop here a solver for the Tolman-Oppenheimer-Volkoff (TOV) equations. Then, I explore the interior of a neutron star for a given equation of state (EoS). In the end I show the dependence of the final radius and stellar mass on the central density.

INTRODUCTION

Neutron stars (NSs) are one of the most mysterious and fascinating objects in the universe. Their existence was first predicted in 1934 by Zwicky but we have to wait till the radio-pulsar observation in 1967 by Jocelyn Bell for the first discovery of a NS. If in all these years progresses has been made in the study and observations of these objects, it's still not clear their structure and EoS. Given the physical properties of typical NSs (a radius $R_\star \sim 10$ km for a mass of $M_\star \sim 2\ M_\odot$ is really close to Schwarzschild radius $R_S \sim 6$ km) they are extremely general-relativistic objects and they have to be treated accordingly. Einstein field equations are a system of 10 complex non-linear equations. But for a perfect fluid in a static spherical symmetric spacetime, the equations are reduced to TOV equations that admit numerical solution for a given EoS. The purpose of this project is to develop a numerical solver for the TOV equations and study the structure of NSs and the dependence of total mass and radius on central density.

1 TOV EQUATIONS AND NUMERICAL SOLUTION

NSs are physical objects governed by general relativity (GR). In order to study them we need a GR treatment: we solve here Einstein equation for a spherically symmetric star in static equilibrium in function of density, pressure and mass. Then we describe the numerical solution to these equations.

1.1 TOV equations

In the framework of GR we have to use Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

where R is the Ricci tensor and $T_{\mu\nu} = -P g_{\mu\nu} + (P + \rho) u_\mu u_\nu$ is the stress-energy tensor with P being the pressure and ρ the total energy density.

The metric describing a spherical symmetric time-invariant space-time is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 d\Omega^2 \tag{2}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Knowing the metric elements we can compute the components of the Ricci tensor. For definition:

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\nu\alpha}$$

where $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}[g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}]$ are the Christoffel symbols. The results are:

$$\begin{aligned}
R_{00} &= (-\nu'' + \nu'\lambda' - (\nu')^2 - \frac{2\nu'}{r}) e^{2(\nu-\lambda)} \\
R_{11} &= \nu'' - \nu'\lambda' + (\nu')^2 - \frac{2\lambda'}{r} \\
R_{22} &= (1 + r\nu' - r\lambda') e^{-2\lambda} - 1 \\
R_{33} &= R_{22} \sin^2\theta.
\end{aligned}$$

With the Ricci tensor we can compute the Ricci scalar $R = \gamma^{\mu\nu} R_{\mu\nu}$ easily. Now, under the assumption of a perfect static fluid the stress energy tensor $T_{\mu\nu}$ is reduced to:

$$\begin{aligned}
T^0_0 &= \rho(r) c^2 \\
T^j_i &= -P(r) \delta^i_j.
\end{aligned}$$

Now, putting the values of $T_{\mu\nu}$, $R_{\mu\nu}$, R, $g_{\mu\nu}$ in 1 we get:

$$\begin{aligned}
R^0_0 - \frac{1}{2} g^0_0 R &= e^{-2\lambda}(\frac{1}{r^2} - \frac{2\lambda'}{r}) - \frac{1}{r^2} = -\frac{8\pi G}{c^4}\rho \\
R^1_1 - \frac{1}{2} g^1_1 R &= e^{-2\lambda}(\frac{1}{r^2} - \frac{2\nu'}{r}) - \frac{1}{r^2} = \frac{8\pi G}{c^4} P \\
R^2_2 - \frac{1}{2} g^2_2 R &= e^{-2\lambda}(\nu'' + (\nu')^2 - \lambda'\nu' + \frac{\nu' - \lambda'}{r}) = \frac{8\pi G}{c^4} P \\
R^3_3 - \frac{1}{2} g^3_3 R &= R^2_2 - \frac{1}{2} g^2_2 R = \frac{8\pi G}{c^4} P
\end{aligned}$$

Now, demanding the continuity of the stress-energy tensor (i.e. $\nabla_\mu T^{\mu\nu} = 0$) and the condition $e^{-\lambda} = 1 - \frac{2Gm(r)}{rc^2}$ (in order to have a metric that is continuous with the vacuum solution at the boundary), through algebraic manipulation we get:

$$\frac{dP(r)}{dr} = -\frac{G}{r^2}\frac{(\rho + \frac{P}{c^2})(m(r) + 4\pi r^3 \frac{P}{c^2})}{1 - \frac{2Gm(r)}{c^2 r}}. \tag{3}$$

Equation 3 is called Tolman-Oppenheimer-Volkoff equation (shortly TOV). The TOV equation combined with the mass continuity equation:

$$\frac{dm}{dr} = 4\pi r^2 \rho, \tag{4}$$

gives us a system of 2 equation and 3 unknows. To solve the system we need a barotropic equation of state (i.e. $P(\rho)$) like the polytropic one:

$$P = K\rho_0^\gamma. \tag{5}$$

Setting the boundary conditions we can solve for $P(r)$ and find the radial profile of $m(r)$ and $\rho(r)$ as well.

An important remark: the density ρ that appears in 4 is the total mass-energy density i.e. the rest-mass density ρ_0 in addition to the internal energy density per unit rest-mass.

1.2 The Numerical Method

Excluding the non-physical case of ρ costant, TOV equation 5 can only be solved numerically. To numerically integrate the system of 3 and 4 we must first set a value for $\rho(0) = \rho_c$, K and γ. The first solver I developed was a simple Euler Method: The solver integrate the TOV equation each time using the previous value P_n like this: $P_{n+1} = P_n + RHS(r_n)dr$ for a given choice of dr where the RHS is the one specified in eq. 3. I found out that a value of $dr = 10^{-4}$ is the best compromise between a fast computational time and a small grid interval. With that a value of dr we get a typical value of 93945 iterations. Usually, models with a value of $\rho_c \sim 10^{13}$ g cm^{-3}, yield radii ~ 10 km. So our stepsize corresponds in physical units to $\sim 1cm$.

Again we simply integrate the TOV equation $P_{n+1} = P_n + dr\, RHS(r_n)$ evaluating each time $m(r)$ as 4 with the same procedure. In order to do that, we must first compute the central pressure P_c from ρ_c using 5.

To simplify the equations I worked in $c = G = M_\odot = 1$ units. The TOV equation results in:

$$\frac{dP}{dr} = \frac{m}{r^2}(\rho + 4\pi P\frac{r^3}{m})(\frac{1}{1 - 2\frac{m}{r}})$$

The problem rise at the center of the star $r = 0$ because we need to regularize the singular terms of the form $\propto m/r, m/r^2, m/r^3$ for $r \to 0$. In order to stabilize the equation I use l'Hôpital's rule expanding in Taylor series

$m(r) \sim \frac{1}{6}m'''(0)r^3$ for $r < 10^{-3}$ (remembering the definition of ρ). Instead for $r > 10^{-3}$ I simply use the value of cumulative mass till that point.

Now given the EoS we don't have any problem to compute density in function of pressure in order to eliminate ρ from the system as $(P/K)^{1/\gamma} + P/(\gamma-1)$[1].

After some experimentation with a simple Euler Method, ispired by Ott notes[2] I choose to adopt a 4th-order Runge-Kutta (RK4) method:

This method is specified by:

$$\begin{aligned}
k_1 &= RHS_n \\
k_2 &= RHS(r_n + \frac{\Delta r}{2}, u_n + \frac{\Delta r}{2}k_1) \\
k_3 &= RHS(r_n + \frac{\Delta r}{2}, u_n + \frac{\Delta r}{2}k_2) \\
k_4 &= RHS(r_n + \Delta r, u_n + \Delta r k_3) \\
u_{n+1} &= u_n + \frac{\Delta r}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

where u_n can be both P_r or m_r in our case.

k_1 is the increment based on the slope at the start of the interval using the RHS (exactly like Euler Method). k_2 is the increment based on the slope at half of the interval using k_1 instead k_3 is increment using k_2. Finally k_4 is the increment based on the slope at the end of the interval.

At start, we don't really know when to stop the computation so we integrate until the surface of the star: this one can be in first approximation set for $r = R_\star$ such that $P(R_\star) = 0$. Numerically we need to stop at some finite value (I decide $P \leqslant 10^{-12}$ in c=G=$M_\odot$=1 units).

The code in the end will return the radial profile of $P(r), m(r), \rho(r)$ and the radius of the star $R_\star$ and the mass of the star $m(R_\star) = M$.

2 RESULTS

I run my code with different values of ρ_c and EoS. For every choice of ρ_c I report the final mass M and radius $R_\star$.

2.1 Plots

The plots shows the pressure, mass, density profile for a given choice of ρ_c and $\gamma = 2.5$ and $K = 1.982 \times 10^{-6}$. I report here only the one for $\rho_c = 5.0 \times 10^{14}$ g/cm^3 in fig. 1 and the one for $\rho_c = 8.0 \times 10^{14}$ g/cm^3 in fig. 2. Plots for different central density follow the same trend. A comparison between pressure, mass, density profile for different central densities are shown in fig. 3, 4 (linear scale), 5.

1 The ρ in the EoS 5 is the rest-mass density ρ_0, we still have to add the internal energy density per unit mass to get the total mass-energy density.

2 http://www.tapir.caltech.edu/~cott/CGWAS2013/TOV_Notes.pdf

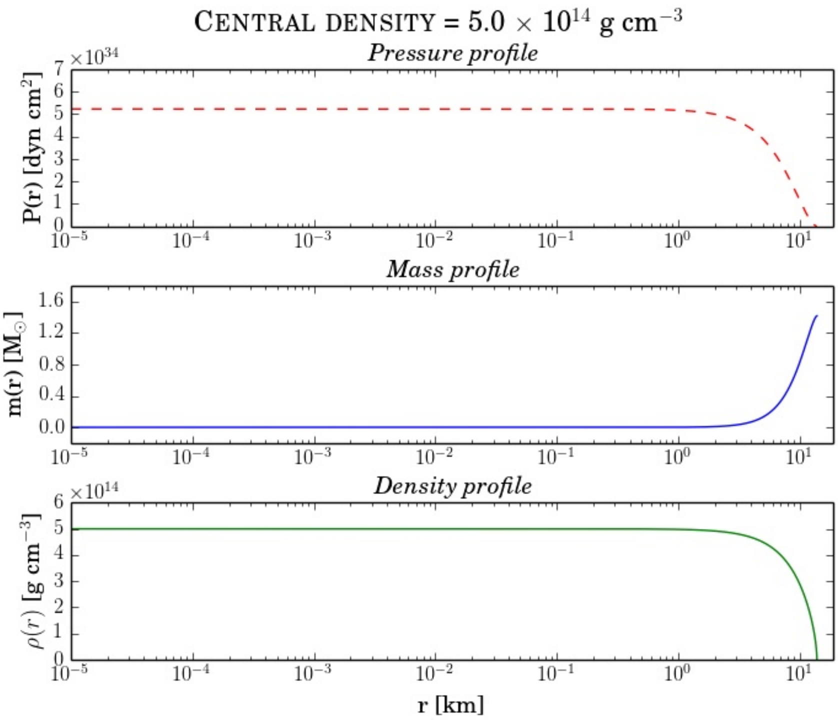

Figure 1: Pressure, mass, density profile (respectively top, middle and bottom panel) for $\rho_c = 5.0 \times 10^{14}$ g/cm^3.

I found out that I get the same values using the Euler Method or RK4 method. That's due to the small stepsize chosen.

As expected model with lower central densities decline more smoothly near the surface compared to the high density ones.

In table 1 I report the total mass and final radii of the NS for a given ρ_c (always for $\gamma = 2.75$, $K = 1.982 \times 10^{-6}$). The model that yields a NS mass of 1.5 $M_\odot$ is the one corresponding to $\rho_c = 5.25 \times 10^{14}$ g/cm^3 and a stellar radii of 13.9 km.

In fig.6 I show the dependance of NS mass on central density ρ_c (for a given EoS).

In fig.7 I show the dependance of final NS radii on central density ρ_c (for a given EoS).

On the other hand, I noticed how the models are really sensible to the choice of the polytropic index γ: a small variation leads to complete different values of the stellar radii and mass. For example, a model with $\gamma = 2.5$ $\rho_c = 5 \times 10^{14}$ yields $M = 4.11\ M_\odot$ and a stellar radii $R = 21.06$ km. That's completely different from the one reported in tab.1.

Only this little example shows how the models are sensible to the EoS chosen and the importance to costraint the latter through simultaneous observations of mass and radii.

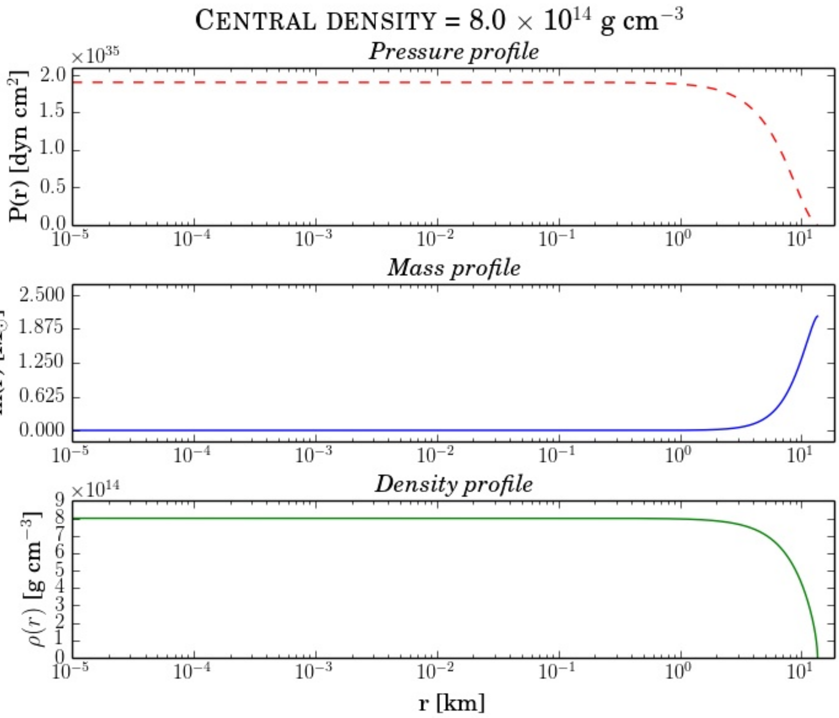

Figure 2: Pressure, mass, density profile (respectively top, middle and bottom panel) for ρ_c = 8.0 × 1014 g/cm^3.

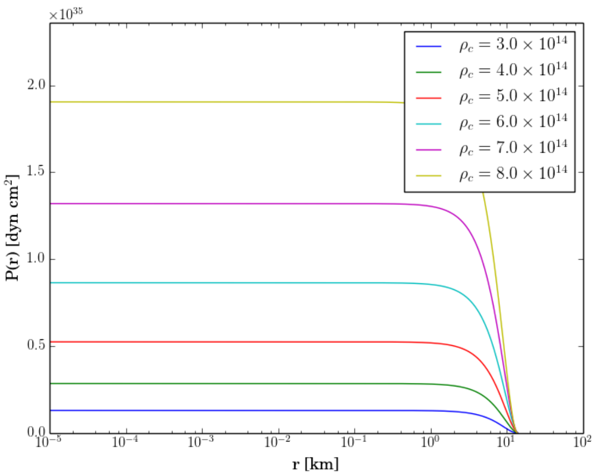

Figure 3: Pressure profiles for central densities in the interval [3.0-8.0] g/cm^3

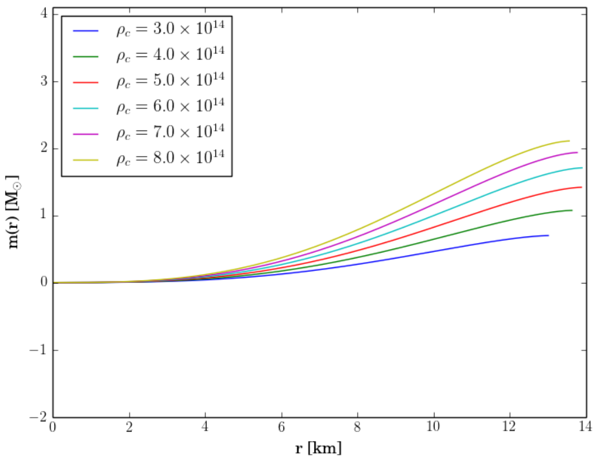

Figure 4: Mass profiles for central densities in the interval [3.0-8.0] g/cm^3

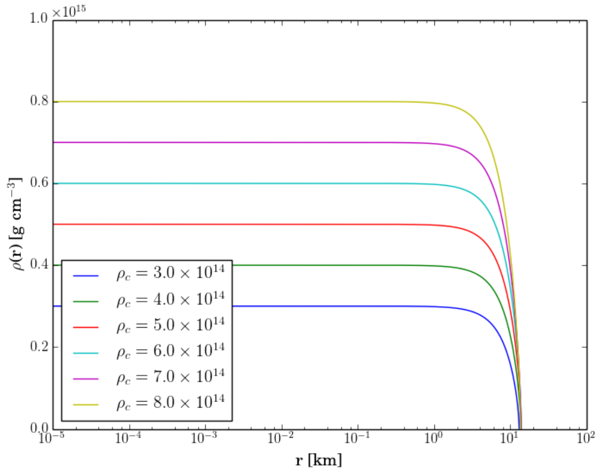

Figure 5: Density profiles for central densities in the interval [3.0-8.0] g/cm^3

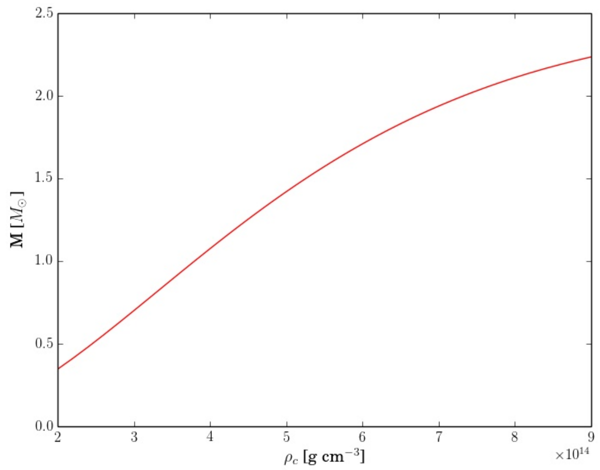

Figure 6: NS mass in function of central density ρ_c for polytropic index $\gamma = 2.75$.

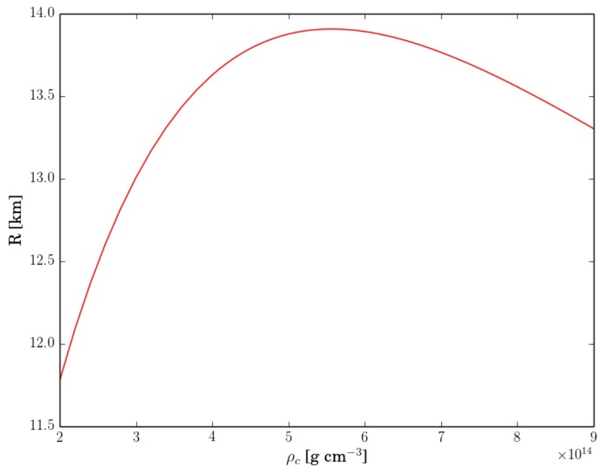

Figure 7: NS radii in function of central density ρ_c for polytopic index $\gamma = 2.75$

Table 1: NS mass and radius for a given central density ρ_c

ρ_c (10^{14}g/cm^3)	M ($M_\odot$)	$R_\star$ (km)
2.0	0.35	11.78
2.5	0.52	12.50
3.0	0.70	13.00
3.5	0.89	13.38
4.0	1.07	13.63
4.5	1.25	13.79
5.0	1.42	13.88
5.25	1.50	13.90
5.5	1.57	13.91
6.0	1.71	13.90
6.5	1.83	13.84
7.0	1.94	13.76
7.5	2.03	13.67
8.0	2.11	13.56
8.5	2.18	13.43
9.0	2.23	13.30

3 CONCLUSION

To summarize, I used a Runge-Kutta 4 method to resolve TOV equations. The results are sensible to the central density input ρ_c and the choice of the EoS. The mass and radius output depends strongly on ρ_c as shown in fig. 6 and fig. 7.